VOGAN DUALITY FOR NONLINEAR TYPE B

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ABSTRACT. Let $\mathbb{G} = \text{Spin}[4n+1]$ be the connected, simply connected complex Lie group of type B_{2n} and let $G = \text{Spin}(p, q)$ ($p + q = 4n + 1$) denote a (connected) real form. If $q \notin \{0, 1\}$, G has a nontrivial fundamental group and we denote the corresponding nonalgebraic double cover by $\tilde{G} = \widetilde{\text{Spin}}(p, q)$. The main purpose of this paper is to describe a symmetry in the set of genuine parameters for the various $\tilde{G}$ at certain half-integral infinitesimal characters. This symmetry is used to establish a duality of the corresponding generalized Hecke modules and ultimately results in a character multiplicity duality for the genuine characters of $\tilde{G}$.

1. INTRODUCTION

1.1. Duality for linear groups. Before describing the main nonlinear result for which we are aiming (Theorem 7.6.5), we begin by recalling the linear version ([15]). Fix a real form G of a complex reductive group $\mathbb{G}$ and let $\Pi(G)$ denote the set of irreducible admissible representations of G . For simplicity, we restrict our attention to the (finite) set $\Pi_{\text{triv}}(G)$ of representations with trivial infinitesimal character. If $\bar{\pi} \in \Pi_{\text{triv}}(G)$, write π for a standard representation whose canonical irreducible subquotient is $\bar{\pi}$ (Section 2.1). We are interested in studying the composition factors of π via expressions of the form

$$\pi = \sum_{\bar{\eta} \in \Pi_{\text{triv}}(G)} m(\bar{\eta}, \pi) \bar{\eta} \quad m(\bar{\eta}, \pi) \in \{0, 1, 2, \dots\}$$

with the sum interpreted in an appropriate Grothendieck group. Similarly the inverse problem is to find an expression (again in a Grothendieck group) for $\bar{\pi}$ in terms of the standard representations

$$\bar{\pi} = \sum_{\eta \in \Pi_{\text{triv}}(G)} M(\eta, \bar{\pi}) \eta \quad M(\eta, \bar{\pi}) \in \mathbb{Z}.$$

Since it is well known that such expressions exist and are unique, it remains only to compute the numbers $m(\bar{\eta}, \pi)$ and $M(\eta, \bar{\pi})$.

This problem was solved by Vogan for reductive linear Lie groups ([12], [13], [16]) by extending the techniques used for Verma modules ([6], [5]). In this approach $\Pi_{\text{triv}}(G)$ is first parameterized by a collection of geometric parameters (roughly, irreducible equivariant local systems on the flag manifold). These parameters form a natural basis for a Hecke module whose combinatorics embody a generalized version of the original algorithm of Kazhdan and Lusztig ([8]). The resulting polynomials

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(sometimes called Kazhdan-Lusztig-Vogan polynomials) give the desired multiplicities through evaluation at one.

In a remarkable final paper ([15]), Vogan describes a symmetry in the geometric parameters for $\Pi_{\text{triv}}(G)$. Given a block $\mathcal{B} = \{\bar{\pi}_1, \dots, \bar{\pi}_n\} \subset \Pi_{\text{triv}}(G)$, Vogan constructs a real form $G^\vee \subset \mathbb{G}^\vee$ of the complex dual group, a block $\mathcal{B}^\vee = \{\bar{\eta}_1, \dots, \bar{\eta}_n\} \subset \Pi(G^\vee)$, and a bijection $\Psi : \mathcal{B} \rightarrow \mathcal{B}^\vee$. The map Ψ commutes with the combinatorics of the Hecke module for $\mathcal{B}$ allowing one to define a dual Hecke module ([15], Definition 13.3) isomorphic to the one for $\mathcal{B}^\vee$. On the level of multiplicities this implies

$$(1) \quad M(\pi_i, \bar{\pi}_j) = \pm m(\Psi(\bar{\pi}_j), \Psi(\pi_i))$$

([15], Theorem 1.15).

Theorem 7.6.5 below is a version of (1) for the nonalgebraic double covers of even rank spin groups in type B (see Definition 2.2.1 for a precise description of the groups being considered). Blocks are replaced with the (computationally easier) notion of a central character (Definition 6.2.1) and the dual groups are constructed explicitly (Definition 6.3.4). Moreover, unlike the type B linear groups in Vogan's theory, the groups of Definition 2.2.1 are preserved by the duality map.

Theorem 7.6.5 adds to the work of a number of people. The Kazhdan-Lusztig-Vogan algorithm has been generalized for a large class of nonlinear groups by Renard and Trapa [11]. Their method builds extended Hecke algebra structures to track nonintegral wall crossings and computes KLV-polynomials for a (potentially large) number of infinitesimal characters simultaneously. As in [15], it is ultimately a duality of these extended structures (Theorem 7.6.4) that is used to establish (1). Along these lines, Adams and Trapa have established a duality theory for nonalgebraic double covers whose corresponding root system is simply laced [4] (for technical reasons their results include G_2). Finally, Renard and Trapa build a general duality theory for the metaplectic group in [10] by directly extending ideas in [15].

1.2. Overview. We now give a brief outline of the above results and discuss their role in the current work. Beginning in [15], Vogan assigns each representation a list of abstract data that can be manipulated combinatorially, called a *(strong) bigrading*. The idea is to define Ψ on the level of bigradings in a way that is compatible with the KLV-algorithm. After the (difficult) construction of an appropriate dual group ([15], Theorem 11.1), Ψ is easily defined on the level of bigradings up to a small ambiguity. Once the ambiguity is dealt with, a proof of (1) follows formally.

Unfortunately, dealing with the ambiguity is a significant problem. A crude definition of Ψ leaves bigradings partitioned into small sets called *packets*. The key to defining Ψ completely is somehow distinguishing the elements of a packet. In [15] packets come in two flavors, *r-packets* and *c-packets*, and Vogan makes the key observation that the intersection of an *r*-packet and a *c*-packet can contain *at most one element* ([15], Proposition 8.3). In particular, bigradings can be manipulated (roughly speaking) as pairs consisting of one *r*-packet and one *c*-packet whose intersection is nonempty.

The results cited above for nonlinear groups take essentially the same approach. After appropriately defining a nonlinear bigrading, each one takes advantage (perhaps implicitly) of a nonlinear version of ([15], Proposition 8.3). For example, in [4] the nonempty packets are always singletons and thus the intersection property is hardly worth mentioning.

Sadly the intersection property does not hold for nonlinear type B. Although any packet can have at most two elements (Corollary 6.3.2), a two-element r-packet and a two-element c-packet can be *equal* (see Corollary 7.3.20). Therefore a new approach is needed to distinguish the elements in a packet. Doing this in a way that is compatible with duality is the main result of this paper (Section 7).

Before attacking this problem we must first isolate the appropriate packets of order two. This requires a fair amount of effort and is the main purpose of Sections 3, 4, 5, and 6. Each section discusses a new piece of structure that increases the precision with which we may specify a collection of representations. Although most of the statements are straightforward specializations of existing results to type B, we develop them somewhat carefully in preparation for Section 7.

Roughly speaking, here is the main idea. Fixing a Cartan subgroup of (even rank) $\tilde{G}$ (Section 3) and a half-integral infinitesimal character (Section 4) reduces the collection of representations to a finite set corresponding (approximately) to a conjugacy class of involutions (Remark 3.1.5). The theory of K -orbits (Section 5) is used to select an individual involution and (combined with the infinitesimal character) trims the problem to one of at most four (Theorem 5.4.5). A choice of central character cuts the problem in half, leaving the desired packets of size at most two (Corollary 6.3.2).

The dual group $\tilde{G}^\vee$ is easily determined from combinatorial data (Definition 6.3.4) and for duality we consider the same infinitesimal character and the opposite central character (Remark 6.2.2). On the level of packets the definition of Ψ is obvious (Theorem 6.3.8); however, defining Ψ on the level of representations is tricky. The key to this definition is the notion of a principal class (Definition 7.4.1) and the corresponding principal class map $\wp$ (Definition 7.4.5). The map $\wp$ assigns a principal class to those representations grouped in packets, and this assignment is bijective (Theorem 7.4.8). In particular, principal classes distinguish the remaining representations and allow us to unambiguously define Ψ . We then show Ψ has the desired properties (Theorems 7.5.3, 7.5.5, 7.5.6) and the main duality theorems (Theorems 7.6.4, 7.6.5) follow formally.

Initially, it may help to consider the special case where $\tilde{G} = \widetilde{\mathrm{Spin}}(2n+1, 2n)$ is split. In fact, if λ is a symmetric infinitesimal character (i.e. the integral Weyl group is of type $B_n \times B_n$), then $\tilde{G}^\vee = \tilde{G}$ and principal classes are simply genuine principal series representations. The general theory was very much developed with this case in mind.

1.3. Odd rank type B. The results discussed here beg the question of a duality theory for double covers of *odd rank* spin groups. To see that no such duality theory can exist, let $\tilde{G}$ be the nonlinear double cover of an odd rank spin group and suppose $\mathcal{B}$ is a block for $\tilde{G}$ that contains a discrete series representation. Theorem 5.3.2 (Case I) implies $\mathcal{B}$ must contain exactly two discrete series representations and thus any block dual to $\mathcal{B}$ must contain exactly two principal series representations. However, a careful analysis of the (odd rank) split Cartan subgroup shows this is impossible. In particular, Theorem 4.2.2 (Case I) implies the center of the split Cartan subgroup is generated by the center of the group and thus any block contains at most one principal series representation. Therefore no dual block exists (within the odd rank nonlinear spin groups).

One way to ‘fix’ this is to reduce the number of discrete series representations in any block. This is equivalent to reducing the number of elements in the trivial $\tilde{K}$ -orbit fiber (i.e. the number of closed $\mathbb{K}$ -orbits in the geometric picture). This requires making $\tilde{K}$ (and therefore $\tilde{G}$) disconnected. The author has made progress on this case and expects to return to it in a future paper.

2. PRELIMINARIES

2.1. The Langlands classification and $\text{Spin}(p, q)$. We begin with some notation.

Definition 2.1.1. Throughout this paper let $\mathbb{G} = \text{Spin}(2n+1, \mathbb{C})$ be the connected, simply connected complex Lie group of type B_n . Up to equivalence $\mathbb{G}$ has $n+1$ (connected) real forms, denoted $\text{Spin}(p, q)$, with $p+q = 2n+1$ and $p > q$ (by convention).

If $G = \text{Spin}(p, q)$, Lie algebras for $\mathbb{G}$ and G will be denoted by $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{R}}$, respectively, with analogous notation used for subgroups and subalgebras. If Θ is a Cartan involution for G , write $K = G^{\Theta}$ for the corresponding maximal compact subgroup. Given a Θ -stable Cartan subgroup $H \subset G$, let $\Delta(\mathfrak{g}, \mathfrak{h})$ be the induced root system and $W(\mathfrak{g}, \mathfrak{h})$ the algebraic Weyl group. Finally, the Killing form $(\cdot, \cdot)$ is a nondegenerate inner product on $\mathfrak{h}$ allowing us to identify $\mathfrak{h} \leftrightarrow \mathfrak{h}^*$.

We now recall the Langlands classification for algebraic groups. The discussion in this section is valid whenever G is a real reductive linear Lie group with abelian Cartan subgroups. Let $\mathcal{HC}(\mathfrak{g}, K)$ denote the category of Harish-Chandra modules for G . Write $\mathcal{HC}(\mathfrak{g}, K)_{\lambda}$ for the full subcategory of $\mathcal{HC}(\mathfrak{g}, K)$ consisting of modules with infinitesimal character $\lambda \in \mathfrak{h}^*$ and let $\mathcal{KHC}(\mathfrak{g}, K)_{\lambda}$ denote its Grothendieck group. The irreducible objects in $\mathcal{HC}(\mathfrak{g}, K)_{\lambda}$ (or equivalently their distribution characters) form a natural basis of $\mathcal{KHC}(\mathfrak{g}, K)_{\lambda}$ that we wish to understand. Typically this is achieved with the help of a second basis given by equivalence classes of *standard modules* (or equivalently their distribution characters) [14]. Loosely, standard modules are representations induced from discrete series representations on cuspidal parabolic subgroups of G and for our purposes we may assume these objects are known. Each standard module in $\mathcal{HC}(\mathfrak{g}, K)_{\lambda}$ has a canonical irreducible subquotient creating a one-to-one correspondence between the standard and irreducible modules (on the level of distribution characters).

Let $H \subset G$ be a Θ -stable Cartan subgroup and write $H = TA$ for its decomposition into compact and vector pieces. The main ingredients of the Langlands classification are given in the following definition.

Definition 2.1.2. Suppose $G = \text{Spin}(p, q)$ and let $\mathcal{D}_{\lambda}(p, q)$ be the set of K -conjugacy classes of triples (H, ϕ, Γ) , where $H = TA \subset G$ is a Θ -stable Cartan subgroup, ϕ is in the $W(\mathfrak{g}, \mathfrak{h})$ -orbit determined by λ (viewed in $\mathfrak{h}^*$), and Γ is a character of T whose differential is determined by ϕ . Specifically, we must have

$$d\Gamma = \phi|_{\mathfrak{t}} + \rho_i^{\phi} - 2\rho_{\text{ic}}^{\phi}$$

where ρ_i^{ϕ} and ρ_{ic}^{ϕ} are the half sums of positive (with respect to ϕ) imaginary and compact imaginary roots. We refer to ([17], Chapter 3) for more details. Although technically redundant, we will occasionally write $(H, \phi, \Gamma)_{\lambda}$ for a representative triple in $\mathcal{D}_{\lambda}(p, q)$ when we wish to emphasize the infinitesimal character λ .

Theorem 2.1.3 (Langlands). *The set $\mathcal{D}_\lambda(p, q)$ parameterizes the simple objects in $\mathcal{HC}(\mathfrak{g}, K)_\lambda$.*

In particular, $\mathcal{D}_\lambda(p, q)$ is a finite set that can be parameterized in terms of certain structure theoretic information for G . The set $\mathcal{D}_\lambda(p, q)$ will be referred to as the set of *linear parameters* for G at λ .

2.2. The Langlands classification and $\widetilde{\mathrm{Spin}}(p, q)$. Recall $G = \mathrm{Spin}(p, q)$ is a real form of $\mathbb{G}$ with $p + q = 2n + 1$ and $p > q$. Although $\mathbb{G}$ is simply connected, if $q \notin \{1, 0\}$, G has a nontrivial fundamental group. In particular, the (nonlinear) double cover $\widetilde{G}$ is a central extension of G and we have a short exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \widetilde{G} \rightarrow G \rightarrow 1.$$

Let $\pi : \widetilde{G} \rightarrow G$ be the projection map and follow the usual convention that preimages of subgroups under π are denoted by adding a tilde.

Definition 2.2.1. If $q \notin \{0, 1\}$, write $\widetilde{G} = \widetilde{\mathrm{Spin}}(p, q)$ for the nonlinear double cover of G . To simplify notation, set $\widetilde{G} = \widetilde{\mathrm{Spin}}(p, q) = \mathrm{Spin}(p, q) \times \{\pm 1\}$ whenever $q \in \{0, 1\}$ and take the map π to be trivial on $\{\pm 1\}$. A representation of $\widetilde{G}$ is said to be *genuine* if the action of -1 is nontrivial.

Definition 2.2.2. A *Cartan subgroup* $\widetilde{H} \subset \widetilde{G}$ is defined to be the centralizer in $\widetilde{G}$ of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. As the notation suggests, $\widetilde{H} = \pi^{-1}(H)$ whenever H is a Cartan subgroup of G .

Although H is always an abelian subgroup of G , it is not necessarily the case that $\widetilde{H} \subset \widetilde{G}$ is abelian (Proposition 3.4.5). This makes extending the triples of Definition 2.1.2 a slightly delicate problem. We begin with following observation.

Lemma 2.2.3 ([1], Section 3). *The identity component $(\widetilde{H})_0$ is central in $\widetilde{H}$.*

Suppose $\Pi_g(\widetilde{H})$ denotes the set of (equivalence classes of) irreducible genuine representations of $\widetilde{H}$ and $\Pi_g(Z(\widetilde{H}))$ denotes the same set for the center of $\widetilde{H}$. The following proposition relates these two sets.

Proposition 2.2.4 ([1], Proposition 2.2). *There is a bijection*

$$\psi : \Pi_g(Z(\widetilde{H})) \rightarrow \Pi_g(\widetilde{H})$$

sending an element $\chi \in \Pi_g(Z(\widetilde{H}))$ to $\psi(\chi) \in \Pi_g(\widetilde{H})$. Here $\psi(\chi)$ is the unique element of $\Pi_g(\widetilde{H})$ satisfying $\psi(\chi)|_{Z(\widetilde{H})} = n\chi$, where $n = \left| \widetilde{H}/Z(\widetilde{H}) \right|^{\frac{1}{2}}$ is the dimension of $\psi(\chi)$.

Definition 2.2.5. Suppose $\widetilde{G} = \widetilde{\mathrm{Spin}}(p, q)$ and $\widetilde{H} \subset \widetilde{G}$ is a Cartan subgroup. We say $(\widetilde{H}, \phi, \widetilde{\Gamma})$ is a *genuine triple* for $\widetilde{G}$ if $\widetilde{\Gamma}$ is an irreducible genuine representation of $\widetilde{H}$ and $\phi \in \mathfrak{h}^*$ determines the differential of $\psi^{-1}(\widetilde{\Gamma})$ (Proposition 2.2.4) on $\widetilde{T}$ as in Definition 2.1.2.

Proposition 2.2.6 ([11], Proposition 6.1). *Write $\mathcal{HC}(\mathfrak{g}, \widetilde{K})$ for the category of Harish-Chandra modules for $\widetilde{G}$ and let $\mathcal{HC}(\mathfrak{g}, \widetilde{K})^{\mathrm{gen}} \subset \mathcal{HC}(\mathfrak{g}, \widetilde{K})$ denote the full subcategory of genuine modules in $\mathcal{HC}(\mathfrak{g}, \widetilde{K})$. The genuine irreducible objects in $\mathcal{HC}(\mathfrak{g}, \widetilde{K})^{\mathrm{gen}}$ are parameterized by $\widetilde{K}$ -conjugacy classes of genuine triples $(\widetilde{H}, \phi, \widetilde{\Gamma})$ (Definition 2.2.5).*

Definition 2.2.7. In the setting of Definition 2.2.5, let $\tilde{\mathcal{D}}_\lambda(p, q)$ denote the set of $\tilde{K}$ -conjugacy classes of genuine triples $(\tilde{H}, \phi, \tilde{\Gamma})$ where the orbit of $\phi \in \mathfrak{h}^*$ is determined by λ . As before, $\tilde{\mathcal{D}}_\lambda(p, q)$ will be referred to as the set of *genuine parameters* for $\tilde{G}$ at infinitesimal character λ .

Definition 2.2.8. If $\gamma \in \tilde{\mathcal{D}}_\lambda(p, q)$ is a genuine parameter for $\tilde{G} = \widetilde{\text{Spin}}(p, q)$, denote by $\text{std}(\gamma)$ and $\text{irr}(\bar{\gamma})$ the corresponding standard and irreducible modules in $\mathcal{HC}(\mathfrak{g}, \tilde{K})_\lambda^{\text{gen}}$, respectively. Write $m(\bar{\gamma}, \delta) \in \mathbb{N}$ for the number of times $\text{irr}(\bar{\gamma})$ appears as a subquotient of $\text{std}(\delta)$. In $\mathcal{KHC}(\mathfrak{g}, \tilde{K})_\lambda^{\text{gen}}$ we have

$$\text{std}(\delta) = \sum_{\gamma \in \tilde{\mathcal{D}}_\lambda} m(\bar{\gamma}, \delta) \text{irr}(\bar{\gamma}).$$

Similarly, write $M(\gamma, \bar{\delta}) \in \mathbb{Z}$ for the multiplicity of $\text{std}(\gamma)$ in $\text{irr}(\bar{\delta})$ so that

$$\text{irr}(\bar{\delta}) = \sum_{\gamma \in \tilde{\mathcal{D}}_\lambda} M(\gamma, \bar{\delta}) \text{std}(\gamma)$$

in $\mathcal{KHC}(\mathfrak{g}, \tilde{K})_\lambda^{\text{gen}}$.

The integers $m(\bar{\gamma}, \delta)$ and $M(\gamma, \bar{\delta})$ play a fundamental role in the statement of our main results (Theorem 7.6.5).

3. CARTAN SUBGROUPS

A key ingredient in the Langlands classification (Sections 2.1, 2.2) for a real group is a concrete understanding of its Cartan subgroups. In this section we explicitly describe the structure of the Cartan subgroups in $G = \text{Spin}(p, q)$ and $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ (Definition 2.2.1). Although this material is well known, the particular form of the results stated here will be important in what follows.

3.1. Abstract involutions. We begin with a fundamental definition.

Definition 3.1.1. Recall $\mathfrak{g}$ denotes the complex Lie algebra of $\mathbb{G} = \text{Spin}(2n+1, \mathbb{C})$ and let $\mathfrak{h}^{\text{abs}} \subset \mathfrak{g}$ be a fixed, *abstract* Cartan subalgebra. Write $W = W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ for the *abstract* Weyl group and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}}) \subset (\mathfrak{h}^{\text{abs}})^*$ be the corresponding root system.

Abstractly, $W \cong \mathbb{Z}_2^n \rtimes S_n \cong \langle s_\alpha \mid \alpha \in \Pi \rangle$ where s_α is the root reflection in the simple root α . For $w \in W$, write $w = (\epsilon_1 \epsilon_2 \dots \epsilon_n, \sigma)$, where $\epsilon_i \in \{0, 1\}$ and $\sigma \in S_n$.

Definition 3.1.2. Given $w \in W$ define

$$\begin{aligned} n_s^w &= |\{\epsilon_i \mid \sigma(i) = i \text{ and } \epsilon_i = 0\}|, \\ n_r^w &= |\{\epsilon_i \mid \sigma(i) = i \text{ and } \epsilon_i = 1\}|, \\ n_c^w &= |\{\epsilon_i \mid \sigma(i) \neq i\}|. \end{aligned}$$

When the element $w \in W$ is clear from context, we may write n_s for n_s^w and so forth. It is often important to know when w possesses certain properties and for this purpose we define the *indicator bits*

$$\begin{aligned} \epsilon_s^w &= \begin{cases} 0 & n_s^w = 0 \\ 1 & n_s^w \neq 0 \end{cases} & \epsilon_r^w &= \begin{cases} 0 & n_r^w = 0, \\ 1 & n_r^w \neq 0, \end{cases} \\ \epsilon_p^w &= \begin{cases} 0 & n_s^w \text{ is even} \\ 1 & n_s^w \text{ is odd} \end{cases} & \epsilon_m^w &= \begin{cases} 0 & n_r^w \text{ is even,} \\ 1 & n_r^w \text{ is odd.} \end{cases} \end{aligned}$$

Let H be a Θ -stable Cartan subgroup of $G = \text{Spin}(p, q)$. The complexified Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is also Θ -stable and thus Θ acts on $\Delta(\mathfrak{g}, \mathfrak{h})$. Since G contains a compact Cartan subgroup, the action of Θ is equivalent to the regular action of an involution in $W(\mathfrak{g}, \mathfrak{h})$ ([2], Section 5).

Definition 3.1.3. Choose a conjugation map $i : \mathfrak{h}^{\text{abs}} \rightarrow \mathfrak{h}$. The *abstract* involution corresponding to $\mathfrak{h}$ and i is given by

$$\theta = i^{-1} \cdot \Theta \cdot i.$$

The involution θ depends on the choice of i up to conjugacy in $W = W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$.

Proposition 3.1.4 ([2], Proposition 12.9). *Let $H' \subset G$ be a Θ -stable Cartan subgroup, choose a conjugation map $j : \mathfrak{h}^{\text{abs}} \rightarrow \mathfrak{h}'$, and write $\theta' = j^{-1} \cdot \Theta \cdot j$ for the corresponding abstract involution. Then H and H' are K -conjugate in G if and only if θ and θ' are conjugate in W .*

Remark 3.1.5. Proposition 3.1.4 implies we have a well-defined injection

$$\left\{ \begin{array}{c} \Theta\text{-stable Cartan} \\ \text{subgroups} \end{array} \right\} / K \hookrightarrow \left\{ \begin{array}{c} \text{Involutions} \\ \text{in } W \end{array} \right\} / W.$$

This map is *bijective* if and only if G is split ([2], Proposition 12.12).

In particular, involutions in W play a fundamental role in the study of Cartan subgroups. For the remainder of this section we recall some related terminology and set up corresponding notation.

Definition 3.1.6. For $\theta \in W$ an involution let

$$\begin{aligned} \Delta_i^\theta &= \{ \alpha \in \Delta \mid \theta(\alpha) = \alpha \}, \\ \Delta_{\mathbb{R}}^\theta &= \{ \alpha \in \Delta \mid \theta(\alpha) = -\alpha \}, \\ \Delta_{\mathbb{C}}^\theta &= \{ \alpha \in \Delta \mid \theta(\alpha) \neq \pm\alpha \}, \end{aligned}$$

denote the *imaginary*, *real*, and *complex roots* for θ . The imaginary and real roots form subsystems of Δ and we denote their corresponding Weyl (sub)groups by $W_i(\theta)$ and $W_{\mathbb{R}}(\theta)$. The set $\Delta_{\mathbb{C}}^\theta$ is *not* a root system, however, if we write

$$\begin{aligned} \rho_i &= \frac{1}{2} \sum_{\alpha \in (\Delta_i^\theta)^+} \alpha, & \rho_r &= \frac{1}{2} \sum_{\alpha \in (\Delta_{\mathbb{R}}^\theta)^+} \alpha, \\ \Delta_{\mathbb{C}^\perp}^\theta &= \{ \alpha \in \Delta \mid (\alpha, \rho_i) = (\alpha, \rho_r) = 0 \}, \end{aligned}$$

then $\Delta_{\mathbb{C}^\perp}^\theta$ is a root system consisting of complex roots ([15], Definition 3.10).

In the setting of Definition 3.1.6, we can write $\Delta_{\mathbb{C}^\perp}^\theta = \Delta_1 \cup \Delta_2$ as an orthogonal disjoint union with $\theta(\Delta_1) = \Delta_2$. In particular,

$$W_{\mathbb{C}}^\theta = \{ (w, \theta w) \mid w \in W(\Delta_1) \}$$

is a Weyl subgroup of W isomorphic to $W(\Delta_1)$ ([15], Lemma 3.11).

Proposition 3.1.7. *Suppose $\theta \in W$ is an involution and let $k = \frac{n^\theta}{2}$. Then we have*

$$\begin{aligned} W_i(\theta) &\cong W(B_{n_s^\theta}) \times W(A_1)^k, \\ W_{\mathbb{R}}(\theta) &\cong W(B_{n_r^\theta}) \times W(A_1)^k, \\ W_{\mathbb{C}}^\theta &\cong W(A_{k-1}), \end{aligned}$$

where $W(A_i)$ and $W(B_i)$ are Weyl groups for root systems of type A_i and B_i , respectively.

Theorem 3.1.8 ([15], Proposition 3.12). *The subgroups $W_i(\theta)$ and $W_{\mathbb{R}}(\theta)$ are normal subgroups in W^θ (the centralizer of $\theta \in W$). Moreover, we have*

$$W^\theta \cong (W_i(\theta) \times W_{\mathbb{R}}(\theta)) \rtimes W_{\mathbb{C}}^\theta.$$

We conclude this section by describing a convenient way of representing involutions in terms of combinatorial objects called *diagrams*.

Definition 3.1.9. A *diagram* is a picture of the action of an involution on the standard basis for $(\mathfrak{h}^{\text{abs}})^*$ (equivalently the short roots in $\Delta^+(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$). Suppose $\theta \in W$ is an involution and let e_i denote the i th standard basis vector. The corresponding diagram D_θ is a sequence of symbols (read left to right) where the i th symbol represents the action of θ on e_i in an obvious way. For example the diagram of $\theta = (0001, (23))$ is given by

$$+ \ 3 \ 2 \ -.$$

Here the symbol $+$ represents that e_1 is fixed by θ and the symbol $-$ represents that e_4 is sent to $-e_4$ by θ . The vectors e_3 and e_2 are interchanged by θ and this is represented by placing the number 3 in position 2 and the number 2 in position 3.

It is also possible for θ to simultaneously interchange and negate two standard basis vectors. Negation of interchanged vectors will be represented in diagrams by placing parentheses around the corresponding numbers.

Remark 3.1.5 implies it will be important to understand W -conjugacy classes of involutions. To this end we have the following proposition whose proof is easy.

Proposition 3.1.10. *Two involutions $\theta_1 = (\epsilon_1 \epsilon_2 \dots \epsilon_n, \sigma_1)$ and $\theta_2 = (\epsilon'_1 \epsilon'_2 \dots \epsilon'_n, \sigma'_2)$ are conjugate in W if and only if σ_1 and σ_2 are conjugate in S_n and $n_s^{\theta_1} = n_s^{\theta_2}$ (equivalently $n_r^{\theta_1} = n_r^{\theta_2}$). In particular, the conjugacy class of θ is uniquely determined by the numbers n_c^θ and n_s^θ .*

In terms of diagrams, Proposition 3.1.10 implies the conjugacy class of θ is determined by the number of numbers (n_c^θ) and the number of $+$ signs (n_s^θ) that appear. Therefore, a general involution is conjugate to one whose diagram is of the form

$$2 \ 1 \ 4 \ 3 \cdots + \ + \cdots + \ - \ - \cdots -.$$

3.2. Abstract pairs. In this section we extend the notion of an abstract involution (Definition 3.1.3) to include data specific to the real form $G \subset \mathbb{G}$. Let H be Θ -stable Cartan subgroup of G . If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is imaginary (Definition 3.1.6), the corresponding root space $\mathfrak{g}_\alpha$ is fixed by Θ and thus entirely contained in the positive or negative eigenspace. We say α is *compact* if $\mathfrak{g}_\alpha$ is contained in the positive eigenspace and *noncompact* if it is contained in the negative eigenspace. Recall the map $\eta : \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}) \rightarrow \mathbb{Z}_2$ defined via

$$\eta(\alpha) = \begin{cases} 0 & \alpha \text{ compact,} \\ 1 & \alpha \text{ noncompact,} \end{cases}$$

is called a *grading* on $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ ([15], Definition 3.13).

Definition 3.2.1. Choose a conjugation map $i : \mathfrak{h}^{\text{abs}} \rightarrow \mathfrak{h}$. The *abstract pair* corresponding to $\mathfrak{h}$ and i is the ordered pair (θ, ε) , where θ is the abstract involution of Definition 3.1.3 and ε is the *abstract grading* on $\Delta_i^\theta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ defined via $\varepsilon(\beta) = \eta(\alpha)$, with $\beta = \alpha \cdot i$. We say an abstract imaginary root β is compact if $\varepsilon(\beta) = 0$ and noncompact if $\varepsilon(\beta) = 1$. Note (θ, ε) depends on the choice of i up to conjugation in $W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$.

Let (θ, ε) be an abstract pair and recall $\Delta_i^\theta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ is a root system of type $B_m \times A_1^k$, where $m = n_s^\theta$ and $k = \frac{n_\theta}{2}$ (Proposition 3.1.7). The long imaginary roots that form the A_1^k factor can each be written as a sum of two short complex roots interchanged by θ . It is easy to verify that such roots must be noncompact. In particular, not all possible abstract gradings arise from the construction in Definition 3.2.1.

It will be convenient to incorporate abstract gradings into our involution diagrams (Definition 3.1.9). The above discussion suggests we need only modify the portion of the diagram corresponding to the B_m factor of Δ_i^θ . In a root system of type B_m , any grading is determined by its values on a set of positive short roots. Since these are exactly the roots represented by $+$ signs in the diagram for θ , a grading can be described by indicating the $+$ signs that represent noncompact roots. We do this by drawing a circle around the corresponding $+$ signs.

Example 3.2.2. Suppose $m = 6$ and $\theta = 1$ so that all roots are imaginary. If

$$\varepsilon(e_i) = \begin{cases} 1 & 1 \leq i \leq 3, \\ 0 & 4 \leq i \leq 6, \end{cases}$$

is an abstract grading, the corresponding diagram $D_\theta(\varepsilon)$ is

$$\oplus \oplus \oplus + + +.$$

3.3. Cayley transforms. We now recall a fundamental tool for transferring information between conjugacy classes of Cartan subgroups. Suppose $H = TA$ is a Θ -stable Cartan subgroup of G with Lie algebra $\mathfrak{h}_\mathbb{R}$. Let $\Delta_\mathbb{R}^\Theta(\mathfrak{g}, \mathfrak{h})$ and $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ denote the roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ that are real and imaginary with respect to Θ .

Lemma 3.3.1 ([14], Lemma 4.3.7). *Suppose $\alpha \in \Delta_\mathbb{R}^\Theta(\mathfrak{g}, \mathfrak{h})$. Then there exists a root vector $X_\alpha \in \mathfrak{g}_\alpha \cap \mathfrak{g}_\mathbb{R}$ with the property*

$$[\Theta X_\alpha, X_\alpha] = h_\alpha,$$

where $h_\alpha \in \mathfrak{h}$ is the coroot for α . Moreover, the vector X_α is unique up to sign.

Lemma 3.3.2. *Suppose $\beta \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ is imaginary and noncompact. Then there exists a root vector $X_\beta \in \mathfrak{g}_\beta$ with the property*

$$[X_\beta, \overline{X_\beta}] = h_\beta,$$

where $h_\beta \in \mathfrak{h}$ is the coroot for β . Moreover, the vector X_β is unique up to sign.

Lemma 3.3.3. *Suppose $\gamma \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$ is imaginary and compact. Then there exists a root vector $X_\gamma \in \mathfrak{g}_\gamma$ with the property*

$$[\overline{X_\gamma}, X_\gamma] = h_\gamma,$$

where $h_\gamma \in \mathfrak{h}$ is the coroot for γ . Moreover, the vector X_γ is unique up to sign.

Definition 3.3.4 ([9], Chapter 6). Suppose $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$, $\beta \in \Delta_i^{\Theta}(\mathfrak{g}, \mathfrak{h})$ is noncompact, and choose nonzero root vectors $X_{\alpha} \in \mathfrak{g}_{\alpha}$ and $X_{\beta} \in \mathfrak{g}_{\beta}$ according to Lemmas 3.3.1 and 3.3.2. Let

$$\xi_{\alpha} = \frac{\pi i}{4} (\Theta X_{\alpha} - X_{\alpha}), \quad \zeta_{\beta} = \frac{\pi}{4} (\overline{X_{\beta}} - X_{\beta})$$

and define the corresponding *Cayley transform operators*

$$\mathcal{C}_{\alpha} = \text{Ad}(\exp(\xi_{\alpha})), \quad \mathcal{C}^{\beta} = \text{Ad}(\exp(\zeta_{\beta})).$$

The Cayley transform operators depend on the choices of X_{α} and X_{β} up to inverse. On the level of Cartan subalgebras we have

$$\begin{aligned} \mathcal{C}_{\alpha}(\mathfrak{h}_{\mathbb{R}}) &= \ker(\alpha|_{\mathfrak{h}_{\mathbb{R}}}) \oplus \mathbb{R}(X_{\alpha} + \Theta X_{\alpha}), \\ \mathcal{C}^{\beta}(\mathfrak{h}_{\mathbb{R}}) &= \ker(\beta|_{\mathfrak{h}_{\mathbb{R}}}) \oplus \mathbb{R}(X_{\beta} + \overline{X_{\beta}}). \end{aligned}$$

Let $(\mathfrak{h}_{\mathbb{R}})_{\alpha} = \mathcal{C}_{\alpha}(\mathfrak{h}_{\mathbb{R}})$ and $(\mathfrak{h}_{\mathbb{R}})^{\beta} = \mathcal{C}^{\beta}(\mathfrak{h}_{\mathbb{R}})$ denote the *Cayley transforms* of $\mathfrak{h}_{\mathbb{R}}$ with respect to the roots α and β , respectively. These are Θ -stable Cartan subalgebras that are *not* K -conjugate to $\mathfrak{h}_{\mathbb{R}}$ and we write $H_{\alpha} = T_{\alpha} A_{\alpha}$ and $H^{\alpha} = T^{\alpha} A^{\alpha}$ for the corresponding Cartan subgroups of G . It is a fundamental fact that we obtain any Θ -stable Cartan subgroup of G (up to K -conjugacy) by starting with H and performing a sequence of Cayley transforms with respect to real or noncompact imaginary roots.

Suppose $\mathfrak{h} \subset \mathfrak{g}$ is a Θ -stable Cartan subalgebra and choose a conjugation map $i : \mathfrak{h}^{\text{abs}} \rightarrow \mathfrak{h}$. It is easy to describe how the corresponding abstract pair (θ, ε) (Definition 3.2.1) is affected by Cayley transform.

Proposition 3.3.5 ([9], Proposition 6.72). *Suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a noncompact imaginary root and choose a conjugation map $j : \mathfrak{h}^{\text{abs}} \rightarrow \mathcal{C}^{\alpha}(\mathfrak{h})$. Then the abstract pair corresponding to $\mathcal{C}^{\alpha}(\mathfrak{h})$ and j is conjugate to $(s_{\alpha}\theta, \eta)$ where $\eta : \Delta_i^{s_{\alpha}\theta}(\mathfrak{g}, \mathfrak{h}^{\text{abs}}) \rightarrow \mathbb{Z}_2$ and*

$$\eta(\beta) = \begin{cases} \varepsilon(\beta) + 1 & \alpha + \beta \text{ is a root} \\ \varepsilon(\beta) & \alpha + \beta \text{ is not a root} \end{cases}.$$

In general, imaginary gradings are determined by the action of Θ on $\mathfrak{g}$ (not just $\mathfrak{h}$) and thus depend on the particular real form $G = \text{Spin}(p, q)$. In order to make this relationship precise, we need the following definition.

Definition 3.3.6. Let $\theta \in W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ be an involution and suppose ε is an abstract grading for $\Delta_i^{\theta}(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$. Set

$$\begin{aligned} n_{+}^{\theta}(\varepsilon) &= |\{\alpha \in \Delta^{+}(\mathfrak{g}, \mathfrak{h}^{\text{abs}}) \mid \alpha \text{ short and compact}\}|, \\ n_{\oplus}^{\theta}(\varepsilon) &= |\{\alpha \in \Delta^{+}(\mathfrak{g}, \mathfrak{h}^{\text{abs}}) \mid \alpha \text{ short and noncompact}\}|, \end{aligned}$$

and observe $n_s^{\theta} = n_{+}^{\theta}(\varepsilon) + n_{\oplus}^{\theta}(\varepsilon)$. As in Definition 3.1.2 we also define

$$\epsilon_{\oplus}^{\theta}(\varepsilon) = \begin{cases} 0 & n_{\oplus}^{\theta}(\varepsilon) = 0, \\ 1 & n_{\oplus}^{\theta}(\varepsilon) \neq 0. \end{cases}$$

Proposition 3.3.7. *Recall $G = \text{Spin}(p, q)$ with $p > q$, $p + q = 2n + 1$, and suppose $H \subset G$ is a Θ -stable Cartan subgroup. If (θ, ε) is an abstract pair representing H , then*

$$\begin{aligned} p &= 2 \cdot \max(n_{\oplus}^{\theta}(\varepsilon), n_{+}^{\theta}(\varepsilon)) + n_r^{\theta} + n_c^{\theta} + \begin{cases} 1 & n_{\oplus}^{\theta}(\varepsilon) \leq n_{+}^{\theta}(\varepsilon), \\ 0 & \text{else,} \end{cases} \\ q &= 2 \cdot \min(n_{\oplus}^{\theta}(\varepsilon), n_{+}^{\theta}(\varepsilon)) + n_r^{\theta} + n_c^{\theta} + \begin{cases} 1 & n_{\oplus}^{\theta}(\varepsilon) > n_{+}^{\theta}(\varepsilon), \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Remark 3.3.8. The formulas for p and q are independent of the choice of pair (θ, ε) representing H .

Proof. Up to conjugation, the diagram for the abstract pair (Example 3.2.2) of a maximally split Cartan subgroup of $G = \text{Spin}(p, q)$ is

$$\begin{array}{ccccccc} - & - & \cdots & - & + & + & \cdots + \\ 1 & 2 & & q & q+1 & & n \end{array}.$$

In particular, the real form is determined by the number of ‘ $-$ ’ signs appearing. Applying the above formulas to this case we have

$$\begin{aligned} p &= 2(n - q) + q + 0 + 1 = 2n - q + 1 = p, \\ q &= 2(0) + q + 0 + 0 = q, \end{aligned}$$

as desired. If H is not maximally split, it is easy to verify the value of q is unaffected by performing Cayley transforms in noncompact imaginary roots (Proposition 3.3.5). The formula for p follows from $p + q = 2n + 1$. $\square$

Definition 3.3.9. Suppose $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ and choose $X_{\alpha} \in \mathfrak{g}_{\alpha}$ according to Lemma 3.3.1. Let $Z_{\alpha} = X_{\alpha} + \Theta X_{\alpha}$ and define the elements

$$\begin{aligned} \sigma_{\alpha} &= \exp_G\left(\frac{\pi}{2}Z_{\alpha}\right) \in K, \\ \tilde{\sigma}_{\alpha} &= \exp_{\tilde{G}}\left(\frac{\pi}{2}Z_{\alpha}\right) \in \tilde{K}, \\ m_{\alpha} &= \exp_G(\pi Z_{\alpha}) = \exp(\pi i h_{\alpha}) \in H, \\ \tilde{m}_{\alpha} &= \exp_{\tilde{G}}(\pi Z_{\alpha}) \in \tilde{H}. \end{aligned}$$

Then $\sigma_{\alpha}, \tilde{\sigma}_{\alpha}$ are representatives of the root reflection s_{α} and we have $\sigma_{\alpha}^2 = m_{\alpha}$ and $\tilde{\sigma}_{\alpha}^2 = \tilde{m}_{\alpha}$ ([9], Proposition 6.52(c)).

Remark 3.3.10. The conditions in Lemma 3.3.1 determine the vector X_{α} (and thus Z_{α}) *only up to sign*. This ambiguity potentially affects the elements of Definition 3.3.9. We have

$$\exp_G(2\pi Z_{\alpha}) = m_{\alpha}^2 = 1,$$

so the element m_{α} is determined by the root α . However,

$$\exp_G\left(-\frac{\pi}{2}Z_{\alpha}\right) = \sigma_{\alpha}^{-1} = \sigma_{\alpha} m_{\alpha}$$

and conjugation in σ_{α} and σ_{α}^{-1} induces the same action on H or $\mathfrak{h}$, but not on G or $\mathfrak{g}$. In $\tilde{G}$, if α is short, then everything is the same as for G . If α is long,

$$\exp_{\tilde{G}}(2\pi Z_{\alpha}) = \tilde{m}_{\alpha}^2 = -1$$

and the element $\tilde{m}_{\alpha}$ is determined by the root α *only up to inverse*. Moreover,

$$\exp_{\tilde{G}}\left(-\frac{\pi}{2}Z_{\alpha}\right) = \tilde{\sigma}_{\alpha}^{-1} = -\tilde{\sigma}_{\alpha} \tilde{m}_{\alpha}$$

so conjugation in $\tilde{\sigma}_{\alpha}$ and $\tilde{\sigma}_{\alpha}^{-1}$ induces the same action on $\tilde{H}$ if and only if $\tilde{m}_{\alpha}$ is central in $\tilde{H}$.

In particular, some care is required when discussing the elements $\sigma_{\alpha}, \tilde{\sigma}_{\alpha}$, and $\tilde{m}_{\alpha}$ since these elements (and their corresponding actions) are not always determined by the root α . Depending on the context we may need to explicitly choose root vectors satisfying the condition in Lemma 3.3.1. One notable exception is when the root α is short, in which case $\tilde{m}_{\alpha}$ and the actions of $\sigma_{\alpha}, \tilde{\sigma}_{\alpha}$ are well defined.

Proposition 3.3.11 ([1], Lemma 5.1). *If $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ is short, then m_{α} is equal to the nontrivial element of $Z(G)$.*

Definition 3.3.12. Suppose $\beta \in \Delta_i^{\Theta}(\mathfrak{g}, \mathfrak{h})$ is imaginary and noncompact and choose a root vector X_{β} as in Lemma 3.3.2. Let $Z_{\beta} = X_{\beta} + \overline{X_{\beta}}$ and define

$$\begin{aligned} m_{\beta} &= \exp(\pi i Z_{\beta}) = \exp_G(\pi i h_{\beta}) \in H, \\ \tilde{m}_{\beta} &= \exp_{\tilde{G}}(-\pi i h_{\beta}) \in \tilde{H}. \end{aligned}$$

Similar caveats to Remark 3.3.10 apply here as well. The particular choice in the definition of $\tilde{m}_{\beta}$ is explained in the following proposition.

Proposition 3.3.13. *Let $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ be a real root and choose $X_{\alpha} \in \mathfrak{g}_{\alpha}$ according to Lemma 3.3.1. Then $\tilde{m}_{\alpha} = \tilde{m}_{\mathcal{C}_{\alpha}(\alpha)}$.*

Proof. The coroot for $\mathcal{C}_{\alpha}(\alpha)$ is $\mathcal{C}_{\alpha}(h_{\alpha}) = iZ_{\alpha} \in \mathfrak{h}_{\alpha}$ ([9], 6.68). Therefore, we have

$$\tilde{m}_{\mathcal{C}_{\alpha}(\alpha)} = \exp_{\tilde{G}}(-\pi i(iZ_{\alpha})) = \exp_{\tilde{G}}(\pi Z_{\alpha}) = \tilde{m}_{\alpha}$$

as desired. $\square$

3.4. Structure of $\pi_0(\tilde{H})$. In this section we describe the connected components of the Θ -stable Cartan subgroups of $\tilde{G} = \widetilde{\text{Spin}(p, q)}$. We begin with the following well-known proposition.

Proposition 3.4.1 ([7], Theorem 2). *If $H \subset G$ is a Θ -stable Cartan subgroup, then*

$$H \cong (S^1)^a \times (\mathbb{R}^{\times})^b \times (\mathbb{C}^{\times})^c$$

with $a + b + 2c = \frac{p+q-1}{2} = n$.

We now describe an effective algorithm for computing b . To begin, choose a conjugation map $i : \mathfrak{h}^{\text{abs}} \rightarrow \mathfrak{h}$ and suppose (θ, ε) denotes the corresponding abstract pair (Definition 3.2.1). Let $\Delta^{\vee} = \Delta^{\vee}(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ be the abstract coroot system and write $L(\Delta^{\vee})$ for the abstract coroot lattice.

Definition 3.4.2. In the above setting define

$$\begin{aligned} L(\Delta^{\vee})_+^{\theta} &= \{\alpha \in L(\Delta^{\vee}) \mid \theta(\alpha) = \alpha\}, \\ L(\Delta^{\vee})_-^{\theta} &= \{\alpha \in L(\Delta^{\vee}) \mid \theta(\alpha) = -\alpha\}, \\ M_+^{\theta} &= \kappa(L(\Delta^{\vee})_+^{\theta}), \\ M_-^{\theta} &= \kappa(L(\Delta^{\vee})_-^{\theta}), \end{aligned}$$

where $\kappa : L(\Delta^{\vee}) \rightarrow L(\Delta^{\vee})/2L(\Delta^{\vee})$ denotes the quotient map.

Theorem 3.4.3 ([7], Section 5). *In the setting of Definition 3.4.2 we have*

$$\pi_0(H) \cong M_-^{\theta}/M_-^{\theta} \cap M_+^{\theta}.$$

Proof. This is the main result of [7] applied to our setting. In particular, the coweight lattice is equal to the coroot lattice since $\mathbb{G}$ is simply connected. $\square$

Remark 3.4.4. We interpret Theorem 3.4.3 as follows. Using Definitions 3.3.9 and 3.3.12 it is easy to check

$$\begin{aligned} M_-^{\theta} &= \{m_{\alpha} \mid \alpha \in \Delta_{\mathbb{R}}^{\theta}(\mathfrak{g}, \mathfrak{h})\}, \\ M_+^{\theta} &= \{m_{\beta} \mid \beta \in \Delta_i^{\theta}(\mathfrak{g}, \mathfrak{h})\}, \end{aligned}$$

so that $\pi_0(H)$ is generated by the elements m_α , where α is a real root for H . We consider these elements modulo the ones in H_0 , the latter being determined by the imaginary roots of H . Since multiplication in M_-^θ is equivalent to addition in $\Delta^\vee$, we have completely described the structure of $\pi_0(H)$.

We now briefly recall how to describe $\pi_0(\tilde{H})$ where $\tilde{H} \subset \tilde{G} = \widetilde{\text{Spin}}(p, q)$ is a Cartan subgroup (see [3], Section 4). As in the previous section, choose a conjugation map $i : \mathfrak{h}^{\text{abs}} \rightarrow \mathfrak{h}$ and suppose (θ, ε) denotes the corresponding abstract pair (Definition 3.2.1). For each $m_\alpha \in M_-^\theta$ (Remark 3.4.4), choose once and for all an inverse image $\tilde{m}_\alpha$ under the projection map π and write

$$\pi^{-1}(m_\alpha) = \{\tilde{m}_\alpha, -\tilde{m}_\alpha\}.$$

The remaining (nonabelian) structure of $\tilde{H}$ is given by the following proposition (Lemma 2.2.3).

Proposition 3.4.5 ([3], Proposition 4.5). *The elements $\tilde{m}_\alpha$ above are subject to the following relations:*

$$\begin{aligned} \tilde{m}_\alpha^2 &= \begin{cases} -1 & \alpha \text{ is long,} \\ 1 & \alpha \text{ is short,} \end{cases} \\ [\tilde{m}_\alpha, \tilde{m}_\beta] &= \begin{cases} (-1)^{(\alpha, \beta^\vee)} & \alpha, \beta \text{ are both long,} \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

If $\{m_{\alpha_1}, \dots, m_{\alpha_k}\}$ is a set of representatives for the distinct connected components of H , then clearly $(\pm)\{\tilde{m}_{\alpha_1}, \dots, \tilde{m}_{\alpha_k}\}$ is a set of representatives for the distinct connected components of $\tilde{H}$. Moreover, their multiplicative structure is given by Proposition 3.4.5, and combined with Lemma 2.2.3, determines the multiplicative structure of $\tilde{H}$.

4. HALF-INTEGRAL INFINITESIMAL CHARACTERS

4.1. Abstract triples. Conjugation to $\mathfrak{h}^{\text{abs}}$ has been our main tool for relating algebraic structures associated with different Cartan subalgebras in $\mathfrak{g}$. To this point, maps relating Cartan subalgebras have been specified only up to Weyl group conjugation. From now on we will be working with finer structure and are thus forced to be more precise about our conjugation maps.

Definition 4.1.1. Fix a nonsingular element $\lambda \in (\mathfrak{h}^{\text{abs}})^*$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\phi \in \mathfrak{h}^*$, and suppose λ and ϕ define the same infinitesimal character (Section 2.1). Then there is an inner automorphism

$$i_{\lambda, \phi} : \mathbb{G} \rightarrow \mathbb{G}$$

whose differential induces a map (also denoted $i_{\lambda, \phi}$)

$$\begin{aligned} (\mathfrak{h}^{\text{abs}})^* &\rightarrow \mathfrak{h}^*, \\ \lambda &\rightarrow \phi. \end{aligned}$$

The map $i_{\lambda, \phi}$ is not unique; however, the restriction of any two such maps to $(\mathfrak{h}^{\text{abs}})^*$ is the same. We also write $i_{\lambda, \phi}$ for the induced maps on Weyl groups, root systems, and so forth. When the element λ is fixed or clear from context, we will often just write i_ϕ . Definition 4.1.1 leads to the following enhancement of Definition 3.2.1.

Definition 4.1.2. Fix an infinitesimal character $\lambda \in (\mathfrak{h}^{\text{abs}})^*$, a Θ -stable Cartan subgroup $\tilde{H} \subset \tilde{G}$, and suppose there exists a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ as in Definition 2.1.2. Write $\eta : \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}) \rightarrow \mathbb{Z}_2$ for the corresponding grading of $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$. The *abstract triple* associated to $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is the 3-tuple $(\theta, \varepsilon, \lambda)$, where

$$\begin{aligned}\theta &= i_{\lambda, \phi}^{-1} \cdot \Theta \cdot i_{\lambda, \phi}, \\ \varepsilon(\alpha) &= \eta(i_{\lambda, \phi}(\alpha))\end{aligned}$$

with α an abstract imaginary root for θ . In particular, θ is an involution of $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ corresponding to Θ and ε is a grading of $\Delta_i^\theta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$.

Fix a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ and recall $\tilde{\Gamma}$ is a genuine representation of $\tilde{H}$ compatible with ϕ . Technically, the abstract triple of Definition 4.1.2 is determined entirely by $i_{\lambda, \phi}$ and the pair $(\tilde{H}, \phi)$. However, the following proposition implies the existence of $\tilde{\Gamma}$ places restrictions on the abstract λ that are allowed.

Proposition 4.1.3 ([4], Section 6). *Let $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple and suppose $(\theta, \varepsilon, \lambda)$ is the corresponding abstract triple. If $\alpha \in \Delta$ is a long imaginary root for θ then*

$$(\lambda, \alpha^\vee) \in \begin{cases} \mathbb{Z} & \varepsilon(\alpha) = 0, \\ \mathbb{Z} + \frac{1}{2} & \varepsilon(\alpha) = 1. \end{cases}$$

If α is a long complex root for θ , then we have

$$(\lambda, \alpha^\vee + \theta(\alpha^\vee)) \in \begin{cases} \mathbb{Z} & (\alpha, \theta(\alpha^\vee)) = 0, \\ \mathbb{Z} + \frac{1}{2} & (\alpha, \theta(\alpha^\vee)) \neq 0. \end{cases}$$

Proof. Let $\tilde{\Upsilon}$ be the character of $Z(\tilde{H})$ corresponding to $\tilde{\Gamma}$ (Proposition 2.2.4) and suppose first that α is imaginary. Then

$$\begin{aligned}\tilde{\Upsilon}(\tilde{m}_\alpha^2) &= e^{(\text{d}\tilde{\Upsilon}, 2\pi i \alpha^\vee)}, \\ &= e^{(\lambda + \rho_i^\lambda - 2\rho_{ic}^\lambda, 2\pi i \alpha^\vee)}\end{aligned}$$

(Definition 2.1.2). Since α pairs integrally with ρ_i^λ and $2\rho_{ic}^\lambda$ we have

$$\tilde{\Upsilon}(\tilde{m}_\alpha^2) = e^{(\lambda, 2\pi i \alpha^\vee)} \cdot 1$$

and the result follows from

$$\tilde{m}_\alpha^2 = \begin{cases} 1 & \alpha \text{ compact}, \\ -1 & \alpha \text{ noncompact}. \end{cases}$$

The complex case is handled similarly ([4], Section 6). $\square$

Definition 4.1.4. An abstract triple satisfying the conditions of Proposition 4.1.3 is said to be *supportable*.

Definition 4.1.5. Define a *half-integer* to be a number x such that $2x \in \mathbb{Z}$ and a *strict half-integer* to be an element of $\mathbb{Z} + \frac{1}{2}$. A *regular* infinitesimal character $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ is said to be half-integral if its usual abstract coordinates are all half-integers.

Fix $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ half-integral and let $\tilde{H}$ be a Θ -stable Cartan subgroup of $\tilde{G}$. Suppose $\phi \in \mathfrak{h}^*$ such that λ and ϕ define the same infinitesimal character. We would like to know when there exists a genuine triple of the form $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$. Conjugating the pair $(\tilde{H}, \phi)$ by $i_{\lambda, \phi}^{-1}$ gives an abstract triple $(\theta, \varepsilon, \lambda)$ and Proposition 4.1.3 provides

a necessary condition on $(\theta, \varepsilon, \lambda)$. A wonderful fact is that this condition is also sufficient.

Proposition 4.1.6 ([4], Section 6). *In the situation above, a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ exists if and only if the abstract triple $(\theta, \varepsilon, \lambda)$ is supportable.*

Proof. According to Proposition 2.2.4, it is enough to construct a genuine character of $Z(\tilde{H})$ with differential as in Definition 2.1.2. If $\tilde{H} = \tilde{T}\tilde{A}$ it suffices to define a character of $(\tilde{T})_0$ whose differential is $\phi|_{\mathfrak{t}}$. We sketch how to do this and leave the details to the reader.

Choose a positive root system for $\Delta(\mathfrak{g}, \mathfrak{h})$ preserved by Θ and observe the simple imaginary and complex coroots span $\mathfrak{t}$. Suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is simple (and imaginary or complex) and define

$$\tilde{\Phi}_\alpha(x) = \begin{cases} e^{ix(\phi, \alpha^\vee)} & \alpha \text{ imaginary,} \\ e^{ix(\phi, \alpha^\vee + \Theta(\alpha^\vee))} & \alpha \text{ complex,} \end{cases}$$

for $x \in \mathbb{R}$. Using the conditions of Proposition 4.1.3 (and the fact that $\mathbb{G}$ is simply connected) one checks the product of the $\tilde{\Phi}_\alpha$ factors to a well-defined genuine character of $(\tilde{T})_0$. $\square$

4.2. Extensions of pairs. Let $\tilde{H} \subset \tilde{G}$ be a Θ -stable Cartan subgroup, fix a regular element $\phi \in \mathfrak{h}^*$, and suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple corresponding to the pair $(\tilde{H}, \phi)$. If $(\theta, \varepsilon, \lambda)$ is supportable (Definition 4.1.4), Proposition 4.1.6 implies there exists a genuine character $\tilde{\Gamma}$ such that $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple for $\tilde{G}$. In this section we determine how many such extensions exist.

Definition 4.2.1. Let $\left[(\tilde{H}, \phi)_\lambda\right]$ be the number of distinct (conjugacy classes of) genuine triples extending $(\tilde{H}, \phi)_\lambda$.

Suppose there exists a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ with infinitesimal character $\lambda \in (\mathfrak{h}^{\text{abs}})^*$. Proposition 2.2.4 implies the possibilities for $\tilde{\Gamma}$ are parameterized by (genuine) central characters of $\tilde{H}$ compatible with ϕ (Definition 2.1.2). However, $(\tilde{H})_0$ is central in $\tilde{H}$ by Lemma 2.2.3 and the behavior of $\tilde{\Gamma}$ on $(\tilde{H})_0$ is specified by ϕ . In particular, the distinct possibilities for $\tilde{\Gamma}$ are determined by the structure of the connected components of $\tilde{H}$ and this structure was determined in Section 3.4. Putting everything together we obtain the following result.

Theorem 4.2.2. *Let λ be a half-integral infinitesimal character, $H \subset G$ a Θ -stable Cartan subgroup, and suppose the abstract triple $(\theta, \varepsilon, \lambda)$ corresponding to the pair $(\tilde{H}, \phi)$ is supportable. Then*

$$\left[(\tilde{H}, \phi)_\lambda\right] \in \{1, 2, 4\}.$$

Proof. By the above remarks, it suffices to determine the components in $\pi_0(H)$ whose covers remain central in $\tilde{H}$. Generators for $\pi_0(H)$ depend only on the conjugacy class of θ (Remark 3.4.4) and their commutators are given by Proposition 3.4.5. We let

$$\begin{aligned} \alpha_i &= e_i - e_{i+1}, \\ \gamma &= e_n \end{aligned}$$

be roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ (in the usual coordinates) and proceed by cases depending on the conjugacy class of θ .

Case I. Suppose θ is given by the diagram (Definition 3.1.9)

$$D_\theta : \begin{array}{c} \text{---} \\ 1 \end{array} \text{---} \cdots \text{---} \begin{array}{c} \text{---} \\ n \end{array}$$

(i.e. $\theta = -1$) where n is the rank of $\tilde{G}$. If n is even, define $z = \tilde{m}_{\alpha_1} \tilde{m}_{\alpha_3} \cdots \tilde{m}_{\alpha_{n-1}}$. Using Proposition 3.3.11 and Proposition 3.4.5 it is easy to verify

$$Z(\tilde{H}) = \begin{cases} \langle (\tilde{H})_0, \pm 1, \tilde{m}_\gamma \rangle & n \text{ odd,} \\ \langle (\tilde{H})_0, \pm 1, \tilde{m}_\gamma, \tilde{z} \rangle & n \text{ even,} \end{cases}$$

so that

$$[(\tilde{H}, \phi)_\lambda] = \begin{cases} 2 & n \text{ is odd,} \\ 4 & n \text{ is even.} \end{cases}$$

Case II. Suppose θ is given by the diagram

$$\begin{array}{c} + \\ 1 \end{array} \cdots \begin{array}{c} + \\ s \end{array} \text{---} \cdots \text{---} \begin{array}{c} \text{---} \\ n \end{array}$$

with $0 < s < n$ and $r = n - s$. Then $\tilde{m}_\gamma \in (\tilde{H})_0$ and arguing as in Case I one verifies

$$[(\tilde{H}, \phi)_\lambda] = \begin{cases} 1 & r \text{ is odd,} \\ 2 & r \text{ is even.} \end{cases}$$

Case III. Suppose θ is given by the diagram

$$\begin{array}{c} (2) \\ 1 \end{array} \begin{array}{c} (1) \\ \end{array} \cdots \begin{array}{c} (k) \\ \end{array} \begin{array}{c} (k-1) \\ k \end{array} \text{---} \cdots \text{---} \begin{array}{c} \text{---} \\ n \end{array}$$

with $0 < k < n$ and $r = n - k$. Now $\tilde{m}_\gamma \notin (\tilde{H})_0$ and as in Case I it is easy to verify

$$[(\tilde{H}_c^s, \phi)_\lambda] = \begin{cases} 2 & r \text{ is odd,} \\ 4 & r \text{ is even.} \end{cases}$$

The remaining cases are handled similarly and left to the reader (Corollary 4.2.3). $\square$

Recall the indicator bits ϵ_s^θ , ϵ_r^θ , and ϵ_m^θ defined in Section 3.1. The results of Theorem 4.2.2 are summarized in the following corollary.

Corollary 4.2.3. *Let λ be a half-integral infinitesimal character, $H \subset G$ a Θ -stable Cartan subgroup, and suppose the abstract triple $(\theta, \varepsilon, \lambda)$ corresponding to the pair $(\tilde{H}, \phi)$ is supportable. Then*

$$[(\tilde{H}, \phi)_\lambda] = 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta (1-\epsilon_m^\theta)}.$$

5. K -ORBITS

Fix a nonsingular element $\lambda \in (\mathfrak{h}^{\text{abs}})^*$. We have seen that it is important to understand the K -conjugacy classes of pairs $(H', \phi')_\lambda$, where H' is a Θ -stable Cartan subgroup in $G = \text{Spin}(p, q)$ and ϕ' is a nonsingular element in $(\mathfrak{h}')^*$ such that ϕ' and λ define the same infinitesimal character. The K -conjugacy classes of such pairs will be referred to as K -orbits (for λ).

Definition 5.0.4 ([9], Chapter 7.8). Let $H \subset G$ be a Θ -stable Cartan subgroup. The *real Weyl group* for H is

$$W(G, H) = N_G(\mathfrak{h}_{\mathbb{R}})/Z_G(\mathfrak{h}_{\mathbb{R}}) = N_G(H)/H = N_K(\mathfrak{h}_{\mathbb{R}})/Z_K(\mathfrak{h}_{\mathbb{R}}) = N_K(H)/(H \cap K)$$

and we have $W(G, H) \subset W(\mathfrak{g}, \mathfrak{h})$. This definition also applies to $\tilde{G}$ and clearly $W(\tilde{G}, \tilde{H}) = W(G, H)$.

Fix a Θ -stable Cartan subgroup H and let Λ denote the $W(\mathfrak{g}, \mathfrak{h})$ -orbit of a nonsingular element $\phi \in (\mathfrak{h})^*$. Then $W(G, H)$ acts freely on Λ and thus freely on pairs of the form $(H, \phi)_\lambda$. Therefore the K -orbits whose first entry is conjugate to H are parameterized by $W(G, H)$ -orbits in Λ . For this reason $W(G, H)$ -orbits in Λ will also be referred to as K -orbits for (the conjugacy class of) H . Since $|W(\mathfrak{g}, \mathfrak{h})| = |\Lambda|$ the number of K -orbits for H is given by $|W(\mathfrak{g}, \mathfrak{h})/W(G, H)|$.

Remark 5.0.5. The (combinatorial) K -orbits discussed here are the same as the usual (geometric) $\mathbb{K}$ -orbits on the flag variety ([16], Corollary 2.2). While the geometric perspective is essential for all known proofs of the KLV-algorithm, only the combinatorics are important for our purposes. We therefore ignore the underlying geometry altogether and focus on a combinatorial description.

5.1. The cross action on K -orbits. Let $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ be a nonsingular element, H a Θ -stable Cartan subgroup of G , and suppose $\phi \in \mathfrak{h}^*$ and λ determine the same infinitesimal character. Let $\Lambda \subset \mathfrak{h}^*$ denote the $W(\mathfrak{g}, \mathfrak{h})$ -orbit of ϕ and suppose $\theta \in W(\mathfrak{g}, \mathfrak{h})$ is an involution. Recall Theorem 3.1.8 implies the centralizer of θ in $W(\mathfrak{g}, \mathfrak{h})$ has the form

$$W^\theta(\mathfrak{g}, \mathfrak{h}) \cong (W_i(\theta)(\mathfrak{g}, \mathfrak{h}) \times W_{\mathbb{R}}(\theta)(\mathfrak{g}, \mathfrak{h})) \rtimes W_{\mathbb{C}}^\theta(\mathfrak{g}, \mathfrak{h}).$$

Proposition 5.1.1 ([15], Proposition 4.16). *The real Weyl group $W(G, H)$ is a subgroup of $W^\theta(\mathfrak{g}, \mathfrak{h})$. Moreover,*

$$W(G, H) \cong (\mathcal{R} \times W_{\mathbb{R}}(\theta)(\mathfrak{g}, \mathfrak{h})) \rtimes W_{\mathbb{C}}^\theta(\mathfrak{g}, \mathfrak{h})$$

where $\mathcal{R} \subset W_i(\theta)(\mathfrak{g}, \mathfrak{h})$ with $\mathcal{R} \cong A \rtimes W_{ic}(\theta)(\mathfrak{g}, \mathfrak{h})$ and A is an elementary abelian two-group.

To simplify notation, recall the conjugation maps $\{i_{\lambda, \phi'} = i_{\phi'}\}_{\phi' \in \Lambda}$ (Definition 4.1.1) and set $W = W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$.

Proposition 5.1.2. *Let $(H, \phi)_\lambda$ be a pair as above and suppose $w \in W(G, H)$. Then the abstract triples associated with $(H, \phi)_\lambda$ and $(H, w\phi)_\lambda$ are the same.*

Proof. This follows easily from Proposition 5.1.1 and the fact that elements of $W(G, H)$ preserve the set of compact roots. $\square$

Proposition 5.1.2 implies the association of abstract triples to pairs $(H, \phi)_\lambda$ is defined on the level of K -orbits for H (viewed as $W(G, H)$ -orbits in $\mathfrak{h}^*$). We would

like to understand the extent to which this association is unique. We begin by recalling the familiar action of the abstract Weyl group on Λ .

Definition 5.1.3. Let $w \in W$ and $\phi \in \Lambda$. Define the *cross action* of w on ϕ as

$$w \times \phi = i_\phi \cdot w^{-1} \cdot i_\phi^{-1}(\phi).$$

Alternatively, we can define $w_\phi = i_\phi(w)$ so that

$$w \times \phi = w_\phi^{-1}(\phi).$$

If $(H, \phi)_\lambda$ is the corresponding pair, the cross action of w on $(H, \phi)_\lambda$ will be denoted $w \times (H, \phi)_\lambda = (H, w \times \phi)_\lambda$.

Proposition 5.1.4. *The cross action defines a left action of W on Λ . Moreover, the cross action commutes with the usual action of $W(\mathfrak{g}, \mathfrak{h})$ and thus descends to a transitive action at the level of K -orbits.*

Proof. This is well known but we prove it as an illustration of the above formalism. For $w_1, w_2 \in W$ and $\phi \in \Lambda$ we have

$$\begin{aligned} w_2 \times (w_1 \times \phi) &= i_{w_1 \times \phi} \cdot w_2^{-1} \cdot i_{w_1 \times \phi}^{-1}(w_1 \times \phi) \\ &= i_\phi \cdot w_1^{-1} \cdot w_2^{-1} \cdot w_1 \cdot i_\phi^{-1}(i_\phi \cdot w_1^{-1} \cdot i_\phi^{-1}(\phi)) \\ &= i_\phi \cdot w_1^{-1} \cdot w_2^{-1} \cdot i_\phi^{-1}(\phi) \\ &= (w_2 w_1) \times \phi. \end{aligned}$$

This proves the first claim. The second claim is left to the reader. $\square$

If $\phi \in \Lambda$, we denote the K -orbit of ϕ by $[\phi]$ and the corresponding cross action by $w \times [\phi]$. We conclude this section with a generalization of Proposition 5.1.2.

Proposition 5.1.5 ([15], Lemma 4.15). *Let $w \in W$ and suppose $(H, \phi)_\lambda$ is a pair with corresponding abstract triple $(\theta, \varepsilon, \lambda)$. Then the abstract triple associated to the pair $w \times (H, \phi)_\lambda$ is given by*

$$w \times (\theta, \varepsilon, \lambda) = (w \cdot \theta \cdot w^{-1}, w \times \varepsilon, \lambda)$$

where $w \times \varepsilon$ is the grading defined via

$$(w \times \varepsilon)(\alpha) = \varepsilon(w^{-1}\alpha).$$

In particular, the cross action on pairs induces an action (also called the cross action) on abstract triples.

5.2. The fiber over an involution. We begin with our usual setup. Let λ be a nonsingular element in $(\mathfrak{h}^{\text{abs}})^*$, H a Θ -stable Cartan subgroup of G , and $\phi \in \mathfrak{h}^*$ such that ϕ and λ define the same infinitesimal character. Let $\Lambda \subset \mathfrak{h}^*$ be the $W(\mathfrak{g}, \mathfrak{h})$ -orbit of ϕ and suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple corresponding to $(H, \phi)_\lambda$.

Definition 5.2.1. The *fiber over θ* (denoted $\theta_\lambda^\dagger$) is the set of K -orbits in Λ whose corresponding abstract triples begin with θ . Similarly, the *fiber over $(\theta, \varepsilon, \lambda)$* (denoted $(\theta, \varepsilon, \lambda)^\dagger$) is defined to be the set of K -orbits in Λ with associated abstract triple $(\theta, \varepsilon, \lambda)$.

The order of the fiber over a fixed involution is easy to describe.

Proposition 5.2.2. *Let $(H, \phi)_\lambda$ be a pair with corresponding abstract triple $(\theta, \varepsilon, \lambda)$. Then*

$$|\theta_\lambda^\dagger| = \left| \frac{W^\theta}{W(G, H)} \right|$$

where $W(G, H) \subset W$ denotes the image of the real Weyl group for H under the map i_ϕ^{-1} .

Proposition 5.2.2 implies the set of K -orbits for H can be viewed as a product of involutions and fibers. Numerically we have

$$\frac{\left| \frac{W(\mathfrak{g}, \mathfrak{h})}{W(G, H)} \right|}{\# \text{ of } K\text{-orbits for } H} = \frac{\left| \frac{W(\mathfrak{g}, \mathfrak{h})}{W^\theta} \right|}{\# \text{ of involutions}} \times \frac{\left| \frac{W^\theta}{W(G, H)} \right|}{\text{size of each fiber}}.$$

In addition to determining $|\theta_\lambda^\dagger|$, it will be important to understand the transitive action of W^θ on $\theta_\lambda^\dagger$. Since the situation is not changed by conjugation, it suffices to determine this action for a set of representative involutions in W . According to Proposition 5.1.1, the action of $w \in W^\theta$ on $\theta_\lambda^\dagger$ is trivial if w is an element of $W_{ic}(\theta)$, $W_{\mathbb{R}}(\theta)$, or $W_{\mathbb{C}}^\theta$. Hence it remains to understand the action of $W_i(\theta)$ on $\theta_\lambda^\dagger$, and specifically the action for (reflections corresponding to) noncompact imaginary roots.

Definition 5.2.3 ([14], Definition 8.3.4). A noncompact imaginary root in $\Delta(\mathfrak{g}, \mathfrak{h})$ is said to be of *type I* if the corresponding reflection is not induced by an element of K (i.e., by an element in $N_K(H)$). Otherwise, it is said to be of *type II*, with analogous terminology used for associated abstract roots. In particular, the cross action through an abstract noncompact imaginary root is nontrivial if and only if it is of type I.

Let $(H, \phi)_\lambda$ be a pair with corresponding abstract triple $(\theta, \varepsilon, \lambda)$ and suppose the imaginary root $\alpha \in \Delta_i^\theta$ is noncompact. Then α is real for $-\theta$ (i.e. $\alpha \in \Delta_{\mathbb{R}}^{-\theta}$) and recall the associated element in $M_-^{-\theta}$ (Definition 3.4.2) was denoted m_α (Remark 3.4.4).

Proposition 5.2.4 ([2], Lemma 14.11 (d,f)). *In the setting above, α is of type II if and only if m_α is trivial in $M_-^{-\theta}/M_-^{-\theta} \cap M_+^{-\theta}$.*

Remark 5.2.5. Theorem 3.4.3 implies $M_-^{-\theta}/M_-^{-\theta} \cap M_+^{-\theta}$ may be interpreted as the component group of a (dual) torus $H' \subset G'$ whose abstract conjugacy class contains $-\theta$ (Remark 3.1.5). Proposition 5.2.4 implies the type of α depends on the image of m_α in $\pi_0(H')$. In particular, $s_\alpha \times [\phi] = [\phi]$ if and only if $m_\alpha \in (H'_0)$. With some effort this result can be (inductively) extended to all of $W_i(\theta)$. In particular, for $w_1, w_2 \in W_i(\theta)$ we have

$$w_1 \times [\phi] = w_2 \times [\phi]$$

if and only if w_1 and w_2 (viewed as a product of reflections) have the same image in $\pi_0(H')$. Such a result is at the heart of the algorithm described in [2] (Section 15). Since we will need only an elementary version here, we omit the details.

It follows from the above discussion that the elements of $\theta_\lambda^\dagger$ are parameterized by distinct elements in a subset of $\pi_0(H')$. After fixing a basepoint $x \in \theta_\lambda^\dagger$ corresponding to H'_0 (an arbitrary choice), we can generate all of $\theta_\lambda^\dagger$ using only the grading of x ([2], Section 15).

Remark 5.3.1. The results of this section concern the *linear* group G and will be familiar to most readers. The detailed exposition is meant to set up the (possibly less familiar) nonlinear results of the next section.

Theorem 5.3.2. *Let $(H, \phi)_\lambda$ be a pair for G and suppose $(\theta, \varepsilon, \lambda)$ is the corresponding abstract triple. Then*

$$|(\theta, \varepsilon, \lambda)^\dagger| \in \{1, 2\}.$$

$$|(\theta, \varepsilon, \lambda)^\dagger| = 1.$$
$$D_\theta : \underset{1}{+} \quad + \quad \cdots \quad \underset{n}{+}$$

Suppose now $w = (\epsilon_1 \epsilon_2 \cdots \epsilon_n, 1)$ is a product of reflections in short noncompact imaginary roots. Since $m_{e_i} = m_{e_j}$ in $M_-^{-\theta}$ for all i and j , the cross action through any two such roots is the same (Remark 5.2.5). Moreover, this action is nontrivial since $M_+^{-\theta} = \{0\}$. Hence

$$|(\theta, \varepsilon, \lambda)^\dagger| = 2$$

$$|\theta_\lambda^\dagger| = 2 \binom{n}{n_{\oplus}^\theta(\varepsilon)}.$$

Case I'. Let $\theta \in W$ be given by the diagram

$$D_\theta : \begin{array}{ccccccc} + & \cdots & + & - \cdots & - \\ 1 & & n_s^\theta & & n \end{array}$$

with $n_r^\theta = n - n_s^\theta \neq 0$. The situation is the same as the previous case, except the elements m_{e_i} are now in $M_\pm^{-\theta}$. Hence all short noncompact imaginary roots are of type II and thus

$$\left| (\theta, \varepsilon, \lambda)^\dagger \right| = 1.$$

Therefore,

$$\left| \theta_\lambda^\dagger \right| = \binom{n_s^\theta}{n_\oplus^\theta(\varepsilon)}.$$

Case II. Let $\theta \in W$ be given by the diagram

$$D_\theta : \begin{array}{ccccccc} (2) & (1) & \cdots & (k) & (k-1) & + & \cdots + \\ 1 & & & k & k & k+1 & n \end{array}$$

where $k = n_c^\theta$, $n_s^\theta = n - k \neq 0$, and $n_r^\theta = 0$. Then

$$W_i(\theta) \cong W(A_1)^{k/2} \times W(B_{n-k})$$

and cross actions for elements of $W_i(\theta)$ that live in the $W(B_{n-k})$ factor behave as in Case I.

Suppose now that i is odd with $1 \leq i \leq k-1$ and define the roots

$$\begin{aligned} \alpha_i &= e_i - e_{i+1}, \\ \beta_i &= e_i + e_{i+1}. \end{aligned}$$

Then each root β_i is real for θ and imaginary for $-\theta$ and each root α_i is noncompact imaginary for θ and real for $-\theta$. Therefore it remains to determine the type of each α_i . Since $m_{\beta_i} = m_{\alpha_i} m_{e_j}$, we have

$$m_{\alpha_i} = m_{e_j}$$

in $M_\pm^{-\theta}/M_\pm^{-\theta}$, where $k+1 \leq j \leq n$. In particular, the cross actions with respect to s_{α_i} and s_{e_j} are the same, assuming e_j is noncompact (Remark 5.2.5). The analysis in Case I shows this action is nontrivial and thus each α_i is of type I. Since the actions are the same we conclude

$$\left| (\theta, \varepsilon, \lambda)^\dagger \right| = 2.$$

These elements are related by a cross action in any of the roots α_i or in any short noncompact imaginary root e_j . We then have

$$\left| \theta_\lambda^\dagger \right| = 2 \binom{n}{n_\oplus^\theta(\varepsilon)}.$$

The remaining cases are handled similarly and left to the reader (Corollary 5.3.3). $\square$

If we assume $G \subset \mathbb{G}$ is *not* the compact form, the results of Theorem 5.3.2 can be summarized as follows.

Corollary 5.3.3. *Let $(H, \phi)_\lambda$ be a pair for G and suppose $(\theta, \varepsilon, \lambda)$ is the corresponding abstract triple. Then $\left|(\theta, \varepsilon, \lambda)^\dagger\right| = 2$ if and only if $n_r^\theta = 0$. In other words, $\left|(\theta, \varepsilon, \lambda)^\dagger\right| = 2$ if and only if the diagram D_θ contains no $-$ signs. In particular, we have*

$$\left|\theta_\lambda^\dagger\right| = 2^{1-\epsilon_r^\theta} \binom{n_s^\theta}{n_\oplus^\theta(\varepsilon)}.$$

5.4. Fibers for $\tilde{G}$. Let $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ be a half-integral infinitesimal character. Recall the set $\tilde{\mathcal{D}}_\lambda(p, q)$ (Definition 2.2.7) is parameterized by the $\tilde{K}$ -conjugacy classes of genuine triples $(\tilde{H}, \phi, \tilde{\Gamma})$, where $\tilde{H}$ is a Θ -stable Cartan subgroup of $\tilde{G} = \widetilde{\text{Spin}}(p, q)$, $\phi \in \mathfrak{h}^*$ such that ϕ and λ define the same infinitesimal character, and $\tilde{\Gamma}$ is a genuine representation of $\tilde{H}$. Theorem 4.2.2 reduces the problem to understanding the $\tilde{K}$ -conjugacy classes of genuine pairs $(\tilde{H}, \phi)_\lambda$ whose corresponding abstract triples are supportable (Definition 4.1.4). A $\tilde{K}$ -conjugacy class of genuine pairs $(\tilde{H}, \phi)_\lambda$ will be called a $\tilde{K}$ -orbit for λ . We have seen the $\tilde{K}$ -orbits for λ correspond naturally to the K -orbits for λ , and these were parameterized in Theorem 5.3.2.

Definition 5.4.1. Let $(\tilde{H}, \phi)_\lambda$ be a genuine pair with corresponding abstract triple $(\theta, \varepsilon, \lambda)$. The *genuine fiber over θ* (denoted $\tilde{\theta}_\lambda^\dagger$) is the set of K -orbits in $\theta_\lambda^\dagger$ whose corresponding abstract triples are supportable. In particular, $\left[(\tilde{H}, \phi)_\lambda\right] \neq 0$ (Definition 4.2.1) if and only if $[\phi] \in \tilde{\theta}_\lambda^\dagger$.

Although the order of $\theta_\lambda^\dagger$ can be arbitrarily large (Corollary 5.3.3), Theorem 5.4.3 belows shows the order of $\tilde{\theta}_\lambda^\dagger$ is always small. Before stating the theorem we need the following definition.

Definition 5.4.2. Let $\theta \in W$ be an involution and suppose $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ is a fixed half-integral infinitesimal character. We say θ is *symmetric about λ* if $n_s^\theta > 0$ (Definition 3.1.2) and the number of integral short imaginary roots is equal to the number of (strictly) half-integral short imaginary roots. In terms of diagrams, θ is symmetric about λ if the number of $+$ signs whose coordinate is integral equals the number of $+$ signs whose coordinate is (strictly) half-integral. This is clearly possible only when n_s^θ is even. For convenience we also define

$$\epsilon_{\text{sym}}^\theta(\lambda) = \begin{cases} 1 & \theta \text{ is symmetric about } \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

As we see in the following theorem, the size of the genuine fiber $\tilde{\theta}_\lambda^\dagger$ increases when the symmetry condition of Definition 5.4.2 is satisfied.

Theorem 5.4.3. *Let $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ be a half-integral infinitesimal character and suppose $(\tilde{H}, \phi)_\lambda$ is a genuine pair for λ . Let $\theta \in W$ be the corresponding abstract involution and assume there exists a supportable abstract triple beginning with θ . Then*

$$\left|\tilde{\theta}_\lambda^\dagger\right| \in \{1, 2, 4\}.$$

Proof. We first determine the number of gradings ε that make the abstract triple $(\theta, \varepsilon, \lambda)$ supportable (by assumption there exists at least one). We then use Theorem 5.3.2 to determine the order of the fiber over each abstract triple. As usual we assume $\tilde{G}$ is *not* compact and proceed by cases for the diagram of θ .

Case I. Let θ be given by the diagram

$$D_\theta : \underset{1}{+} + \cdots + \underset{n}{+}$$

so that $n_s^\theta = n$. There are two possible gradings leading to supportable abstract triples if θ is symmetric about λ and only one otherwise. In either case, Theorem 5.3.2 implies the fiber over each abstract triple is two. Therefore we have

$$|\tilde{\theta}_\lambda^\dagger| = \begin{cases} 4 & \theta \text{ is symmetric about } \lambda, \\ 2 & \theta \text{ is not symmetric about } \lambda. \end{cases}$$

Note θ can be symmetric about λ only if n is even.

Case I'. Let θ be given by the diagram

$$D_\theta : \underset{1}{+} \cdots \underset{n_s^\theta}{+} - \cdots - \underset{n}{-}$$

with $n_r^\theta = n - n_s^\theta \neq 0$. Combining Case I above with Case I' of Theorem 5.3.2 gives

$$|\tilde{\theta}_\lambda^\dagger| = \begin{cases} 2 & \theta \text{ is symmetric about } \lambda, \\ 1 & \theta \text{ is not symmetric about } \lambda. \end{cases}$$

Case II. Let θ be given by the diagram

$$D_\theta : \underset{1}{(2)} \underset{1}{(1)} \cdots \underset{k}{(k)} \underset{k}{(k-1)} + \underset{k+1}{+} \cdots + \underset{n}{+}$$

where $k = n_c^\theta$, $n_s^\theta = n - k \neq 0$, and $n_r^\theta = 0$. Combining Case I above with Case II of Theorem 5.3.2 gives

$$|\tilde{\theta}_\lambda^\dagger| = \begin{cases} 4 & \theta \text{ is symmetric about } \lambda, \\ 2 & \theta \text{ is not symmetric about } \lambda. \end{cases}$$

The remaining cases are handled similarly and left to the reader (Corollary 5.4.4). $\square$

If we assume $\tilde{G}$ is *not* compact, the results of Theorem 5.4.3 can be summarized as follows.

Corollary 5.4.4. *Let $(\tilde{H}, \phi)_\lambda$ be a genuine pair for $\tilde{G}$ with corresponding abstract involution $\theta \in W$. If $|\tilde{\theta}_\lambda^\dagger| \neq 0$, we have*

$$|\tilde{\theta}_\lambda^\dagger| = 2^{1-\epsilon_r^\theta} 2^{\epsilon_{sym}^\theta(\lambda)}.$$

Fix a half-integral infinitesimal character $\lambda \in (\mathfrak{h}^{abs})^*$ and recall we have constructed an abstract triple $(\theta, \varepsilon, \lambda)$ for each element in $\tilde{\mathcal{D}}_\lambda(p, q)$. Let $\tilde{\mathcal{D}}_\lambda^\theta(p, q)$ denote the subset of $\tilde{\mathcal{D}}_\lambda(p, q)$ whose elements have abstract triples beginning with θ . We conclude this section with a description of the size of $\tilde{\mathcal{D}}_\lambda^\theta(p, q)$, which also turns out to be small.

Theorem 5.4.5. *Let $\lambda \in (\mathfrak{h}^{abs})^*$ be a half-integral infinitesimal character and let $\theta \in W$ be an involution. If $|\tilde{\theta}_\lambda^\dagger| \neq 0$, then*

$$\begin{aligned} |\tilde{\mathcal{D}}_\lambda^\theta(p, q)| &= 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} 2^{1-\epsilon_r^\theta} 2^{\epsilon_{sym}^\theta(\lambda)} \\ &= 2^{2+\epsilon_{sym}^\theta(\lambda)-\epsilon_s^\theta-\epsilon_r^\theta\epsilon_m^\theta}. \end{aligned}$$

In particular,

$$\left| \tilde{\mathcal{D}}_\lambda^\theta(p, q) \right| \in \{1, 2, 4\}.$$

Proof. Combine Corollary 4.2.3 with Corollary 5.4.4. For the last statement recall $\epsilon_{\text{sym}}^\theta(\lambda) = 1$ only if $\epsilon_s^\theta = 1$ (Definition 5.4.2). $\square$

6. CENTRAL CHARACTERS

Throughout this section let $G = \text{Spin}(p, q)$ with $p > q$, $p + q = 2n + 1$, and recall $\tilde{G}$ denotes the (connected) nonalgebraic double cover of G . Theorem 5.4.5 implies the elements of $\mathcal{HC}(\mathfrak{g}, \tilde{K})^{\text{gen}}$ can be naturally partitioned into sets of size at most four. In this section we use the notion of central characters to reduce this by a factor of two.

6.1. Abstract bigradings. Let $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ be a half-integral infinitesimal character and suppose $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple for λ . Recall $\tilde{\Gamma}$ is an irreducible genuine representation of $\tilde{H}$ whose differential is compatible with ϕ (Definition 2.1.2). Since the Cartan subgroup $\tilde{H}$ may not be abelian, the representation $\tilde{\Gamma}$ is not necessarily one-dimensional (Proposition 2.2.4). In this section we characterize the action of $\tilde{\Gamma}$ on certain finite order elements in $\tilde{H}$.

If α and β are short roots in $\Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$, Proposition 3.3.11 implies $\tilde{m}_\alpha = \pm \tilde{m}_\beta$. The following proposition shows we always have equality.

Proposition 6.1.1. *Let α and β be short orthogonal roots in $\Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$. Then $\tilde{m}_\beta = \tilde{m}_\alpha$.*

Proof. Let $\gamma = \alpha - \beta$ so that $s_\gamma(\alpha) = \beta$. Choose a root vector $X_\alpha \in \mathfrak{g}_\alpha$ according to Lemma 3.3.1. Then

$$\tilde{m}_\beta = \exp_{\tilde{G}}(\pi \text{Ad}(\tilde{\sigma}_\gamma)Z_\alpha) = \tilde{\sigma}_\gamma \exp_{\tilde{G}}(\pi Z_\alpha) \tilde{\sigma}_\gamma^{-1} = \tilde{\sigma}_\gamma \tilde{m}_\alpha \tilde{\sigma}_\gamma^{-1} = \tilde{m}_\alpha$$

by Proposition 3.3.11. $\square$

A similar result is true for imaginary roots whenever they are of the same type.

Proposition 6.1.2. *Let ε be the imaginary grading for $\Delta(\mathfrak{g}, \mathfrak{h})$ and suppose α and β are short orthogonal imaginary roots. Then $\tilde{m}_\alpha = \tilde{m}_\beta$ if and only if $\varepsilon(\alpha) = \varepsilon(\beta)$.*

Proof. Let $\gamma = \alpha - \beta$ and suppose $\varepsilon(\alpha) = \varepsilon(\beta)$. Then γ is compact and we can find an element $\tilde{\sigma}_\gamma \in \tilde{K}$ for which $\text{Ad}(\tilde{\sigma}_\gamma)h_\alpha = h_\beta$. We now proceed as in the proof of Proposition 6.1.1. Conversely, suppose $\varepsilon(\alpha) \neq \varepsilon(\beta)$ so that γ is noncompact. On the level of coroots we have $h_\alpha = h_\beta + 2h_\gamma$ and

$$\begin{aligned} \tilde{m}_\alpha &= -\exp_{\tilde{G}}(-\pi i h_\alpha) \\ &= \exp_{\tilde{G}}(-\pi i (h_\beta + 2h_\gamma)) \\ &= \exp_{\tilde{G}}(-\pi i h_\beta) \exp_{\tilde{G}}(-2\pi i h_\gamma) \\ &= \tilde{m}_\beta \tilde{m}_\gamma^2 = -\tilde{m}_\beta \end{aligned}$$

as desired. $\square$

We now construct a grading on the *real* roots of $\Delta(\mathfrak{g}, \mathfrak{h})$. We begin with the following lemma.

Lemma 6.1.3 ([11], Chapter 5). *Let $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ be a real root and choose a corresponding $\tilde{m}_{\alpha} \in \tilde{H}$. If α is long we have*

$$\tilde{\Gamma}(\tilde{m}_{\alpha}^2) = -I$$

and $\tilde{\Gamma}(\tilde{m}_{\alpha})$ has eigenvalues $\pm i$ occurring with equal multiplicity. If α is short we have

$$\tilde{\Gamma}(\tilde{m}_{\alpha}) = \pm I.$$

The data of Lemma 6.1.3 are conveniently packaged in the following definition.

Definition 6.1.4. Following [11], we say a long root $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ satisfies the *parity condition* if it is strictly half-integral with respect to ϕ (i.e., $(\phi, \alpha^{\vee}) \in \mathbb{Z} + \frac{1}{2}$). We say a short root $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ satisfies the parity condition if

$$\tilde{\Gamma}(\tilde{m}_{\alpha}) = (-1)(-1)^{(\phi, \alpha^{\vee})} \cdot I$$

([14], Section 8.3). Recall the element $\tilde{m}_{\alpha}$ is well defined for short roots and we always have $(\phi, \alpha^{\vee}) \in \mathbb{Z}$. Lemma 6.1.3 implies that this definition makes sense and gives a well-defined map

$$\eta : \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h}) \rightarrow \mathbb{Z}_2$$

where

$$\eta(\alpha) = \begin{cases} 0 & \alpha \text{ does not satisfy the parity condition,} \\ 1 & \alpha \text{ satisfies the parity condition.} \end{cases}$$

Proposition 6.1.5. *The map η defines a grading ([15], Definition 3.13) on $\Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$.*

Proof. This follows easily from Proposition 6.1.1 and the fact that the parity condition for a short root α is determined by the integer $(\phi, \alpha^{\vee})$. $\square$

Definition 6.1.6. In Section 3.2 we defined a grading ε on the imaginary root system $\Delta_i^{\Theta}(\mathfrak{g}, \mathfrak{h})$ for any Θ -stable Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Given a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_{\lambda}$, Proposition 6.1.5 defines a grading η on the real root system $\Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ with the same formal properties as ε . Applying the conjugation map $i_{\lambda, \phi}^{-1}$ (Definition 4.1.1) to $\{(\tilde{H}, \phi), \varepsilon, \eta\}$ gives an *abstract bigrading* $(\theta, \varepsilon, \eta, \lambda)$ for the abstract root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$. This extends the notion of an abstract triple from Definition 4.1.2.

It is easy to verify that conjugating $(\tilde{H}, \phi, \tilde{\Gamma})_{\lambda}$ by an element of $\tilde{K}$ does not change the abstract bigrading. In particular, abstract bigradings are defined on the level of genuine parameters $\tilde{\mathcal{D}}_{\lambda}(p, q)$ for $\tilde{G}$. Note that an abstract bigrading depends (potentially) on the full representative $(\tilde{H}, \phi, \tilde{\Gamma})_{\lambda}$ and not just the representative pair $(\tilde{H}, \phi)_{\lambda}$. The exact nature of this dependence is determined in the next section.

6.2. Central character. Let $\lambda \in (\mathfrak{h}^{\text{abs}})^{*}$ be a half-integral infinitesimal character and fix a Cartan subgroup $\tilde{H} \subset \tilde{G} = \widetilde{\text{Spin}}(p, q)$.

Definition 6.2.1. The *central character* of a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_{\lambda}$ is the genuine representation of $Z(\tilde{G})$ given by restricting $\tilde{\Gamma}$ to $Z(\tilde{G}) \subset \tilde{H}$.

Remark 6.2.2. The central character will turn out to be an important invariant of genuine triples. Since $\tilde{\Gamma}$ is assumed to be genuine, we automatically have $\tilde{\Gamma}(-1) = -I$. Therefore, the central character of $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is determined by the action of $\tilde{\Gamma}$ on a single nontrivial element in $Z(\tilde{G})$. Since $|Z(\tilde{G})| = 4$, there are only two such possibilities.

Remark 6.2.3. Given a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$, Proposition 2.2.4 implies

$$\tilde{\Gamma}|_{Z(\tilde{H})} = m\chi$$

where $m = \left| \tilde{H}/Z(\tilde{H}) \right|^{\frac{1}{2}}$ and χ is a genuine character of $Z(\tilde{H})$. In particular, central characters are not always one-dimensional representations of $Z(\tilde{G})$. This makes it technically incorrect to compare central characters for distinct genuine triples unless we happen to know the dimensions of their irreducible representations are equal. We remedy this by comparing the associated characters $\chi|_{Z(\tilde{G})}$ from Proposition 2.2.4. Unless otherwise stated, this convention is in effect whenever we compare central characters for different genuine triples.

Fix a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ and let η and ε be the corresponding real and imaginary gradings. In most cases, the central character of $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is determined by either η or ε .

Proposition 6.2.4. *In the above setting, suppose $\alpha \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$ is a short real root. Then $\tilde{m}_\alpha$ is well-defined and central in $\tilde{G}$ and we have*

$$\tilde{\Gamma}(\tilde{m}_\alpha) = \begin{cases} (-1)^{1-\eta(\alpha)} \cdot I & (\phi, \alpha) \in \mathbb{Z} + \frac{1}{2}, \\ (-1)^{\eta(\alpha)} \cdot I & (\phi, \alpha) \in \mathbb{Z}. \end{cases}$$

Proof. This follows from Definition 6.1.4 directly. $\square$

Proposition 6.2.5. *In the above setting, suppose $\alpha \in \Delta_i^{\Theta}(\mathfrak{g}, \mathfrak{h})$ is short. Then $\tilde{m}_\alpha$ is well defined and central in $\tilde{G}$ and we have*

$$\tilde{\Gamma}(\tilde{m}_\alpha) = \begin{cases} 1 \cdot I & (\phi, \alpha) \in \mathbb{Z} + \frac{1}{2}, \\ -1 \cdot I & (\phi, \alpha) \in \mathbb{Z}. \end{cases}$$

Proof. Recall $\tilde{m}_\alpha = \exp_{\tilde{G}}(-\pi i h_\alpha) = \exp_{\tilde{G}}(\pi i h_\alpha)$ is a central element in $\tilde{G}$ with $i h_\alpha \in \mathfrak{g}_{\mathbb{R}}$. It follows that $\tilde{m}_\alpha \in (\tilde{H})_0$ and it suffices to calculate $d\Gamma(-\pi i h_\alpha)$. From the identity $\alpha(h_\alpha) = 2$ and the definition of $d\tilde{\Gamma}$ in Section 2.1 we have

$$\begin{aligned} \tilde{\Gamma}(\tilde{m}_\alpha) &= \tilde{\Gamma}(\exp_{\tilde{G}}(-\pi i h_\alpha)) \\ &= e^{d\Gamma(-\pi i h_\alpha)} \cdot I \\ &= e^{-2\pi i((\phi, \alpha) + (\rho_i^\phi, \alpha) - (2\rho_{ic}^\phi, \alpha))} \cdot I. \end{aligned}$$

Now it is easy to check $(\rho_i^\phi, \alpha) \in \mathbb{Z} + \frac{1}{2}$ and $(2\rho_{ic}^\phi, \alpha) \in \mathbb{Z}$ so that

$$\tilde{\Gamma}(\tilde{m}_\alpha) = -e^{-2\pi i(\phi, \alpha)} \cdot I$$

and the result follows. $\square$

Remark 6.2.6. If $\Delta(\mathfrak{g}, \mathfrak{h})$ contains a short root that is not complex, we can use Proposition 6.2.4 or Proposition 6.2.5 to determine the central character of $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$. If this root is imaginary, the central character is determined by the

genuine pair $(\tilde{H}, \phi)_\lambda$ and all genuine triples extending $(\tilde{H}, \phi)_\lambda$ have the same central character.

It is important to observe there are (at most) two possibilities for the imaginary grading ε . In particular, if $\alpha, \beta \in \Delta_i^\theta(\mathfrak{g}, \mathfrak{h})$ are short, then

$$\varepsilon(\alpha) = \varepsilon(\beta) \iff (\phi, \alpha) \equiv (\phi, \beta) \pmod{\mathbb{Z}}.$$

Since the structure of Definition 6.1.4 is formally the same, a similar result holds for the real grading η as well. The following theorem describes the relationship between η and ε in terms of these possibilities.

Theorem 6.2.7. *Let $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple with corresponding real and imaginary gradings η and ε . Then η and ε are gradings of the opposite kind. In particular, if $\alpha \in \Delta_{\mathbb{R}}^\theta(\mathfrak{g}, \mathfrak{h})$ and $\beta \in \Delta_i^\theta(\mathfrak{g}, \mathfrak{h})$ are short, then*

$$\eta(\alpha) = \varepsilon(\beta) \iff (\phi, \alpha) \not\equiv (\phi, \beta) \pmod{\mathbb{Z}}.$$

Proof. Suppose $\eta(\alpha) = 1$. Proposition 6.2.4 and Proposition 6.2.5 give

$$\begin{aligned} \tilde{\Gamma}(\tilde{m}_\alpha) &= \begin{cases} 1 \cdot \mathbf{I} & (\phi, \alpha) \in \mathbb{Z} + \frac{1}{2}, \\ -1 \cdot \mathbf{I} & (\phi, \alpha) \in \mathbb{Z}, \end{cases} \\ \tilde{\Gamma}(\tilde{m}_\beta) &= \begin{cases} 1 \cdot \mathbf{I} & (\phi, \beta) \in \mathbb{Z} + \frac{1}{2}, \\ -1 \cdot \mathbf{I} & (\phi, \beta) \in \mathbb{Z}. \end{cases} \end{aligned}$$

and it remains to show

$$\tilde{m}_\alpha = \tilde{m}_\beta \iff \varepsilon(\beta) = 0.$$

If $\mathcal{C}_\alpha$ denotes the Cayley transform with respect to α , then $\tilde{m}_\alpha = \tilde{m}_{\mathcal{C}_\alpha(\alpha)}$ by Proposition 3.3.13. However,

$$\tilde{m}_\alpha = \tilde{m}_{\mathcal{C}_\alpha(\alpha)} = (-1)^{\varepsilon(\beta)} \tilde{m}_{\mathcal{C}_\alpha(\beta)} = (-1)^{\varepsilon(\beta)} \tilde{m}_\beta$$

by Proposition 6.1.2 and Proposition 3.3.5 and the result follows. $\square$

6.3. Numerical duality in even rank. Let $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ be a half-integral infinitesimal character. In this section we make our first attempt to define the map Ψ (Section 1) on the level of genuine parameters for the nonlinear groups $\widetilde{\text{Spin}}(p, q)$. To simplify things, we first fix a genuine central character (Definition 6.2.1).

Definition 6.3.1. Let $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ and suppose χ is a genuine character of $Z(\tilde{G})$. Let $\tilde{\mathcal{D}}_\lambda(\chi)(p, q) \subset \tilde{\mathcal{D}}_\lambda(p, q)$ denote the collection of genuine parameters whose central character is the same as χ (Remark 6.2.3). Similarly, if $\theta \in W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ is an abstract involution let $\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)$ be the subset of $\tilde{\mathcal{D}}_\lambda(\chi)(p, q)$ whose K -orbits are contained in $\tilde{\theta}_\lambda^\dagger$ (Definition 5.4.1).

If $\tilde{G}$ is *not* compact, we have the following analog of Theorem 5.4.5.

Corollary 6.3.2. *Let $\theta \in W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ be an involution with $|\tilde{\theta}_\lambda^\dagger| \neq 0$ and suppose χ is a genuine character of $Z(\tilde{G})$. Then if $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| \neq 0$ we have*

$$|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 2^{1-\epsilon_r^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} = 2^{1-\epsilon_r^\theta \epsilon_m^\theta}.$$

In particular,

$$|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| \in \{1, 2\}.$$

Proof. Fixing the genuine character χ eliminates the $2^{\epsilon_{\text{sym}}(\lambda)}$ factor in Corollary 5.4.4 (Proposition 6.2.5) and the $2^{1-\epsilon_s^\theta}$ factor of Corollary 4.2.3. $\square$

Let $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ and suppose $\tilde{H} \subset \tilde{G}$ is a Cartan subgroup. Fix a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ with corresponding abstract bigrading $(\theta, \varepsilon, \eta, \lambda)$ (Definition 6.1.6). The following definition is a dual version of Definition 3.3.6.

Definition 6.3.3. Set

$$\begin{aligned} n_-^\theta(\eta) &= |\{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}^{\text{abs}}) \mid \alpha \text{ short and } \eta(\alpha) = 0\}|, \\ n_\ominus^\theta(\eta) &= |\{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}^{\text{abs}}) \mid \alpha \text{ short and } \eta(\alpha) = 1\}|, \end{aligned}$$

and observe $n_r^\theta = n_-^\theta(\eta) + n_\ominus^\theta(\eta)$ (Definition 3.1.2).

Definition 6.3.4. Let $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ with $p + q = 2n + 1$ and $p > q$. Suppose $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple for $\tilde{G}$ with corresponding abstract bigrading $(\theta, \varepsilon, \eta, \lambda)$. The *dual bigrading* for $(\theta, \varepsilon, \eta, \lambda)$ is

$$\Psi(\theta, \varepsilon, \eta, \lambda) = (-\theta, \eta, \varepsilon, \lambda).$$

The *dual group* for $\tilde{G}$ and $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is $\tilde{G}^\vee = \widetilde{\text{Spin}}(p^\vee, q^\vee)$, where

$$\begin{aligned} p^\vee &= 2 \cdot \max(n_\ominus^\theta(\eta), n_-^\theta(\eta)) + n_s^\theta + n_c^\theta + \begin{cases} 1 & n_\ominus^\theta(\eta) \leq n_-^\theta(\eta), \\ 0 & \text{else,} \end{cases} \\ q^\vee &= 2 \cdot \min(n_\ominus^\theta(\eta), n_-^\theta(\eta)) + n_s^\theta + n_c^\theta + \begin{cases} 1 & n_\ominus^\theta(\eta) > n_-^\theta(\eta), \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Remark 6.3.5. In Definition 6.3.4, the dual group $\tilde{G}^\vee$ depends on both the group $\tilde{G}$ and the genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$. In particular, the infinitesimal and central characters of $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ play a role in determining $\tilde{G}^\vee$.

Proposition 6.3.6. *In the setting of Definition 6.3.4, there exists a genuine triple $(\tilde{H}^\vee, \phi^\vee, \tilde{\Gamma}^\vee)_\lambda$ for $\tilde{G}^\vee$ whose corresponding abstract bigrading is $(-\theta, \eta, \varepsilon, \lambda)$.*

Proof. Since η is a grading of $\Delta_\mathbb{R}^\theta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ satisfying the same formal properties as ε (Proposition 6.1.5), the abstract triple $(-\theta, \eta, \lambda)$ is supportable (Definition 4.1.4). The existence of $(\tilde{H}^\vee, \phi^\vee, \tilde{\Gamma}^\vee)_\lambda$ now follows from Proposition 4.1.6. The fact that $(\tilde{H}^\vee, \phi^\vee, \tilde{\Gamma}^\vee)_\lambda$ is a genuine triple for $\tilde{G}^\vee$ follows from Proposition 3.3.7 applied to $(-\theta, \eta)$. $\square$

Proposition 6.3.7. *Let $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple for $\tilde{G}$ with corresponding abstract bigrading $(\theta, \varepsilon, \eta, \lambda)$ and choose $(\tilde{H}^\vee, \phi^\vee, \tilde{\Gamma}^\vee)_\lambda$ for $\tilde{G}^\vee$ as in Proposition 6.3.6. Suppose χ is the central character of $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ and write $\chi^\vee$ for the opposite (genuine) central character (Remark 6.2.2). If $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ contains a short root that is not complex for θ , then the central character of $(\tilde{H}^\vee, \phi^\vee, \tilde{\Gamma}^\vee)_\lambda$ is $\chi^\vee$.*

Proof. This follows immediately from Remark 6.2.6 and Theorem 6.2.7. $\square$

We now come to the main theorem of this section (note the important restriction at the beginning). As usual we will assume G is not compact.

Theorem 6.3.8. *Let $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ with $p + q = 2n + 1$ and assume the rank of $\tilde{G}$ is even. Suppose $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple for $\tilde{G}$ with abstract bigrading*

$(\theta, \varepsilon, \eta, \lambda)$ and central character χ . If $\tilde{G}^\vee = \widetilde{\text{Spin}}(p^\vee, q^\vee)$ is the dual group for $\tilde{G}$ and $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ (Definition 6.3.4), then

$$\left| \tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q) \right| = \left| \tilde{\mathcal{D}}_\lambda^{-\theta}(\chi^\vee)(p^\vee, q^\vee) \right|.$$

Proof. First suppose $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ contains a short root that is not complex for θ . Since $\left| \tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q) \right| \neq 0$ by hypothesis, Corollary 6.3.2 implies

$$\left| \tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q) \right| = 2^{1-\epsilon_r^\theta \epsilon_m^\theta}.$$

By Propositions 6.3.6 and 6.3.7 we also have $\left| \tilde{\mathcal{D}}_\lambda^{-\theta}(\chi^\vee)(p, q) \right| \neq 0$ so that

$$\left| \tilde{\mathcal{D}}_\lambda^{-\theta}(\chi^\vee)(p, q) \right| = 2^{1-\epsilon_s^\theta \epsilon_p^\theta}$$

and it remains to show $2^{1-\epsilon_r^\theta \epsilon_m^\theta} = 2^{1-\epsilon_s^\theta \epsilon_p^\theta}$. Since n is even, $n - n_c^\theta = n_s^\theta + n_r^\theta$ is even and n_s^θ, n_r^θ must have the same parity (i.e. $\epsilon_p^\theta = \epsilon_m^\theta$). If $\epsilon_p^\theta = \epsilon_m^\theta = 0$, the result is obvious. If $\epsilon_p^\theta = \epsilon_m^\theta = 1$, then $\epsilon_s^\theta = \epsilon_r^\theta = 1$ and the result follows.

Now suppose all short roots in $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ are complex for θ . This is possible only if G is split, has even rank, and $p^\vee = p$ and $q^\vee = q$. Corollary 4.2.3 and Corollary 5.3.3 now imply

$$\left| \tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q) \right| = \left| \tilde{\mathcal{D}}_\lambda^{-\theta}(\chi^\vee)(p, q) \right| = 2$$

and the result follows. $\square$

7. DUALITY

Let $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ be a half-integral infinitesimal character and recall $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ denotes the (connected) nonalgebraic double cover of $G = \text{Spin}(p, q)$. If $\tilde{G}$ has even rank, Theorem 6.3.8 implies

$$\left| \tilde{\mathcal{D}}_\lambda(\chi)(p, q) \right| = \left| \tilde{\mathcal{D}}_\lambda(\chi^\vee)(p^\vee, q^\vee) \right|$$

for each central character χ . In this section we define a bijection

$$\Psi : \tilde{\mathcal{D}}_\lambda(\chi)(p, q) \rightarrow \tilde{\mathcal{D}}_\lambda(\chi^\vee)(p^\vee, q^\vee)$$

that commutes with the main operations of the nonlinear Kazhdan-Lusztig-Vogan (KLTV) algorithm. In particular, for $v \in \tilde{\mathcal{D}}_\lambda(\chi)(p, q)$ we require ([15], Theorem 11.9(c))

$$\begin{aligned} (2) \quad & \Psi(s_\alpha \times v) = s_\alpha \times \Psi(v), \\ (3) \quad & \Psi(\mathcal{C}^\alpha(v)) = \mathcal{C}_\alpha(\Psi(v)), \end{aligned}$$

where cross actions and Cayley transforms on the level of $\tilde{\mathcal{D}}_\lambda(\chi)(p, q)$ are defined in Sections 7.1 and 7.2. Although these definitions are well known, we recall them in preparation for proving the above equalities.

According to Corollary 6.3.2, defining Ψ is a matter of distinguishing the elements in a collection of sets of order two (called *packets* in the introduction). This requires understanding the centers of certain *even parity* (Definition 7.3.1) Cartan subgroups. Since these centers are not determined by central group elements, this is a delicate problem. We begin by defining an additional (almost central) element $z \in \tilde{G}$ and record how it is affected by cross actions and Cayley transforms. The details are in Section 7.3.

Since the element z is twisted by conjugation in $\tilde{K}$ (Proposition 7.3.14), it is difficult to use z to compare representations on the level of genuine triples. The *principal class map* $\wp$ (Definition 7.4.5) takes genuine triples to a pair of global objects that are used for comparison. Of course we must show $\wp$ is well defined (Proposition 7.4.7) and bijective (Theorem 7.4.8) on the appropriate sets $\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)$. The details are in Section 7.4.

With this additional structure at hand, we finally define the map Ψ and prove (2) and (3) (Theorems 7.5.3, 7.5.5, 7.5.6). As in [15], the proof of the main theorem follows formally (Section 7.6).

7.1. Extended cross action. Unlike the linear case, the nonlinear KLV-algorithm at a half-integral infinitesimal character requires an action of the *full* abstract Weyl group on genuine triples. In this section we briefly recall the definition from [11], Chapters 3 and 4.

Definition 7.1.1 ([14], Definition 7.2.16). Suppose $\phi \in \mathfrak{h}^*$ is a regular element and let

$$\Delta(\mathfrak{g}, \mathfrak{h})(\phi) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid (\phi, \alpha^\vee) \in \mathbb{Z}\}$$

denote the set of *integral roots* for ϕ . Then $\Delta(\mathfrak{g}, \mathfrak{h})(\phi)$ is a subroot system of $\Delta(\mathfrak{g}, \mathfrak{h})$ and we denote the corresponding *integral Weyl group* by $W(\mathfrak{g}, \mathfrak{h})(\phi) \subset W(\mathfrak{g}, \mathfrak{h})$.

Recall the abstract root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ and let $\mathcal{R} = L(\Delta) \subset (\mathfrak{h}^{\text{abs}})^*$ be the root lattice. Consider the quotient

$$\mathcal{Q} = (\mathfrak{h}^{\text{abs}})^* / \mathcal{R}$$

and observe the natural action of W on $(\mathfrak{h}^{\text{abs}})^*$ descends to $\mathcal{Q}$. Denote the image of λ in $\mathcal{Q}$ by $[\lambda]$.

Definition 7.1.2. A *family of infinitesimal characters* for $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ is a collection of dominant representatives for the W -orbit of $[\lambda]$ in $\mathcal{Q}$.

Remark 7.1.3. In [11], the authors define $\mathcal{Q}$ using the weight lattice $\mathcal{P} = X$ instead of the root lattice $\mathcal{R}$. Either choice will suffice; however, the root lattice is more convenient for our purposes.

The abstract Weyl group W acts via the extended cross action (Definition 7.1.4) on collections of genuine parameters whose infinitesimal characters live in families of the kind described above. In order to define this action, it is first necessary to specify how the infinitesimal characters in a family are related. To this end, suppose $\kappa \in \mathcal{F}(\lambda)$ and $w \in W$. Define the element $\mu_\kappa(w) \in \mathcal{R}$ by the requirement

$$\kappa + \mu_\kappa(w) \in w \cdot \mathcal{F}(\lambda)$$

with the convention that $\mu_\kappa(w) = w\kappa - \kappa$ for $w \in W(\kappa)$.

Definition 7.1.4 ([11], Definition 4.1). Let $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ be a half-integral infinitesimal character with corresponding family $\mathcal{F}(\lambda)$. Suppose $\kappa \in \mathcal{F}(\lambda)$ and $(\tilde{H}, \phi, \tilde{\Gamma})_\kappa$ is a genuine triple for κ . If $w \in W$ set

$$w \times \phi = \phi + i_\phi(\mu_\kappa(w^{-1}))$$

$$(\rho_i^{w \times \phi} - 2\rho_{\text{ic}}^{w \times \phi}) - (\rho_i^\phi - 2\rho_{\text{ic}}^\phi) = \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} m_\alpha \alpha \quad (m_\alpha \in \mathbb{Z}).$$

Then

$$\varphi = i_\phi(\mu_\kappa(w^{-1})) + \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} m_\alpha \alpha$$

determines a well-defined character Φ of $\tilde{H}$ ([14], Lemma 0.4.5). We define the (extended) cross action of $(\tilde{H}, \phi, \tilde{\Gamma})_\kappa$ by w to be

$$\begin{aligned} w \times \Gamma &= \Gamma \cdot \Phi, \\ w \times (\tilde{H}, \phi, \tilde{\Gamma})_\kappa &= (\tilde{H}, w \times \phi, w \times \tilde{\Gamma}). \end{aligned}$$

Remark 7.1.5. Definition 7.1.4 coincides with the usual cross action on triples ([14], Definition 8.3.1) whenever $w \in W(\kappa)$. This follows immediately from the convention $\mu_\kappa(w) = w\kappa - \kappa$ for w in $W(\kappa)$.

Remark 7.1.6. Definition 7.1.4 describes the (abstract) cross action on genuine triples for $\tilde{G}$. If $\tilde{H} \subset \tilde{G}$ is a Cartan subgroup, we will occasionally need the (regular) cross action of $W(\mathfrak{g}, \mathfrak{h})$ on genuine triples of the form $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$. The definition is obvious from Definition 7.1.4 and will be denoted the same way.

Proposition 7.1.7 ([11], Chapter 4). *In the above setting, let $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple and suppose $w \in W$. Then $w \times (\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is genuine triple and the extended cross action descends to a well-defined action on the level of genuine parameters for $\tilde{G}$.*

For $v \in \tilde{\mathcal{D}}_\lambda(p, q)$, we will write $w \times v \in \tilde{\mathcal{D}}_\lambda(p, q)$ for the cross action of $W(\lambda)$ on genuine parameters for $\tilde{G}$.

7.2. Cayley transforms in $\tilde{\mathcal{D}}_\lambda(p, q)$. The second operation needed in our discussion of the nonlinear KLV-algorithm is the Cayley transform on the level of $\tilde{\mathcal{D}}_\lambda(p, q)$. We refer the reader to [12] and [14] for more details.

To begin, let $\tilde{H} = \tilde{T}\tilde{A}$ be a Θ -stable Cartan subgroup of $\tilde{G}$ and fix a noncompact imaginary root $\alpha \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h})$. Choose a corresponding Cayley transform operator $\mathcal{C}^\alpha$ (Definition 3.3.4) and write

$$\begin{aligned} \mathcal{C}^\alpha(\tilde{H}) &= \tilde{H}^\alpha = \tilde{T}^\alpha \tilde{A}^\alpha, \\ \tilde{T}_1^\alpha &= \tilde{T} \cap \tilde{T}^\alpha, \\ \tilde{H}_1^\alpha &= \tilde{T}_1^\alpha \tilde{A}^\alpha. \end{aligned}$$

The following proposition describes the relationship between $\tilde{T}$ and $\tilde{T}^\alpha$.

Proposition 7.2.1 ([14], Lemma 8.3.5). *If α is of type I (Definition 5.2.3), then $\tilde{T}_1^\alpha = \tilde{T}^\alpha$. If α is of type II, s_α has a representative in $\tilde{T}^\alpha \setminus \tilde{T}_1^\alpha$ and $|\tilde{T}^\alpha / \tilde{T}_1^\alpha| = 2$. In particular, $\tilde{H}^\alpha = \tilde{H}_1^\alpha$ if and only if α is of type I.*

Definition 7.2.2 ([12], before Theorem 4.4). Fix a genuine triple $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ and write $\phi^\alpha = \mathcal{C}^\alpha(\phi)$ for the image of ϕ in $\mathfrak{h}^\alpha$. Let $\tilde{\Gamma}_1^\alpha$ be the irreducible representation of $\tilde{H}_1^\alpha$ satisfying

$$\begin{aligned} \tilde{\Gamma}_1^\alpha|_{\tilde{T}_1^\alpha} &= \tilde{\Gamma}|_{\tilde{T}^\alpha}, \\ \tilde{\Gamma}_1^\alpha|_{\tilde{A}^\alpha} &= \exp_{\tilde{G}}(\phi^\alpha|_{\mathfrak{a}_\mathbb{R}^\alpha}). \end{aligned}$$

According to Proposition 7.2.1, we define an irreducible representation of $\tilde{H}^\alpha$ via

$$\tilde{\Gamma}^\alpha = \begin{cases} \tilde{\Gamma}_1^\alpha & \alpha \text{ type I,} \\ \text{Ind}_{\tilde{H}_1^\alpha}^{\tilde{H}^\alpha} \tilde{\Gamma}_1^\alpha & \alpha \text{ type II.} \end{cases}$$

Proposition 7.2.3 ([12], before Theorem 4.4). *In the setting of Definition 7.2.2, suppose α is of type II. Then $\tilde{\Gamma}^\alpha = \text{Ind}_{\tilde{H}_1^\alpha}^{\tilde{H}^\alpha} \tilde{\Gamma}_1^\alpha$ is reducible if and only if the element $s_\alpha \in \tilde{T}^\alpha$ centralizes $\tilde{T}_1^\alpha$. In this case we write*

$$\tilde{\Gamma}^\alpha = \tilde{\Gamma}_+^\alpha \oplus \tilde{\Gamma}_-^\alpha$$

with $\tilde{\Gamma}_\pm^\alpha$ irreducible. In particular, $\tilde{\Gamma}^\alpha$ is always reducible if $\tilde{T}^\alpha$ is abelian.

Definition 7.2.4. Let $\Upsilon = (\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ and suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is noncompact and simple for ϕ . Define the *Cayley transform of Υ by α* to be

$$\mathcal{C}^\alpha(\Upsilon) = \begin{cases} \{(\tilde{H}^\alpha, \phi^\alpha, \tilde{\Gamma}^\alpha)_\lambda\} & \tilde{\Gamma}^\alpha \text{ irreducible,} \\ \{(\tilde{H}^\alpha, \phi^\alpha, \tilde{\Gamma}_+^\alpha)_\lambda, (\tilde{H}^\alpha, \phi^\alpha, \tilde{\Gamma}_-^\alpha)_\lambda\} & \tilde{\Gamma}^\alpha \text{ reducible.} \end{cases}$$

In particular, $\mathcal{C}^\alpha(\Upsilon)$ is double valued if and only if α is of type II and $\tilde{\Gamma}^\alpha$ is reducible (Definition 7.2.2). If $\beta \in \Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ is an abstract noncompact simple root for Υ , we define the *abstract Cayley transform of Υ by β* to be $\mathcal{C}^{i_\phi(\beta)}(\Upsilon)$.

Proposition 7.2.5 ([11], Lemma 6.14(g)). *In the setting of Definition 7.2.4, suppose α is long. Then $\mathcal{C}^\alpha(\Upsilon)$ is single valued and we have $\dim(\tilde{\Gamma}^\alpha) = m \cdot \dim(\tilde{\Gamma})$ where*

$$m = \begin{cases} 1 & \alpha \text{ type I,} \\ 2 & \alpha \text{ type II.} \end{cases}$$

Proof. The fact that $\mathcal{C}^\alpha(\Upsilon)$ is single valued is proven in [11]. The statement about dimensions is an easy consequence of Proposition 7.2.1 and Definition 7.2.2. $\square$

Proposition 7.2.6 ([12], Section 4). *The elements appearing in the definition of $\mathcal{C}^\alpha(\Upsilon)$ are genuine triples for $\tilde{G}$. Moreover, they have the same infinitesimal character as Υ and are well defined up to $\tilde{K}$ -conjugacy.*

For $v \in \tilde{\mathcal{D}}_\lambda(p, q)$, denote the Cayley transform on the level of genuine parameters for $\tilde{G}$ by $\mathcal{C}^\alpha(v) \in \tilde{\mathcal{D}}_\lambda(p, q)$. If $(\tilde{H}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple representing v , we will write $\dim(v) = \dim(\tilde{\Gamma})$. We will also need an inverse version of Definition 7.2.4. This is most easily stated in terms of abstract roots. We refer the reader to [11] or [12] for more details.

Definition 7.2.7. Let $v \in \tilde{\mathcal{D}}_\lambda(p, q)$ and suppose $\alpha \in \Delta_{\mathbb{R}}^\theta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ is a simple abstract root that is real for v . Define the *inverse Cayley transform of v by α* to be

$$\mathcal{C}_\alpha(v) = \{v' \in \tilde{\mathcal{D}}_\lambda(p, q) \mid \mathcal{C}^\alpha(v') = v\}.$$

More explicit definitions of inverse Cayley transforms appear in [12] and [14]. The following proposition gives a characterization of when the inverse Cayley transform is nontrivial.

Proposition 7.2.8 ([11], Section 5). *In the setting of Definition 7.2.7, $\mathcal{C}_\alpha(v)$ is nonempty if and only if α satisfies the parity condition (Definition 6.1.4).*

7.3. Even parity Cartan subgroups. Having defined the main operations of the nonlinear KLV-algorithm, we now turn to the definition of the map Ψ . Recall that this requires distinguishing elements in certain sets (packets) of order two (Corollary 6.3.2). It turns out nontrivial packets are associated with a particular subset of Cartan subgroups that we now describe.

Fix a half-integral infinitesimal character $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ and assume the rank of $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ is *even*. Let $\tilde{H} \subset \tilde{G}$ be a Θ -stable Cartan subgroup and suppose $\phi \in \mathfrak{h}^*$ is a regular element. Write $\theta = i_\phi^{-1}(\Theta)$ for the corresponding abstract involution and let $\epsilon_s^\theta, \epsilon_p^\theta, \epsilon_r^\theta, \epsilon_m^\theta$ be the indicator bits for θ (Definition 3.1.2). Note the indicator bits depend only on the W -conjugacy class of θ and therefore only on the $\tilde{K}$ -conjugacy class of $\tilde{H}$. Since the rank of $\tilde{G}$ is even, we always have $\epsilon_p^\theta = \epsilon_m^\theta$.

Definition 7.3.1. A Cartan subgroup for which $\epsilon_p^\theta = \epsilon_m^\theta = 0$ is said to be of *even parity*. In particular, an abstract involution corresponds to an even parity Cartan subgroup if and only if its diagram has an even number of both ‘+’ and ‘−’ signs.

Lemma 7.3.2. Let $\tilde{H} \subset \tilde{G}$ be a Θ -stable Cartan subgroup and suppose $\alpha \in \Delta_{\mathbb{R}}^\Theta(\mathfrak{g}, \mathfrak{h})$ is a real root. Then $\tilde{H}$ and $\tilde{H}_\alpha$ (Definition 3.3.4) have the same parity if and only if α is long.

Proof. It is easily verified that either $n_r^{s_\alpha \theta} = n_r^\theta$ or $n_r^{s_\alpha \theta} = n_r^\theta - 2$ and the result follows. $\square$

Definition 7.3.3. An even parity Cartan subgroup $\tilde{H}^{\text{es}} \subset \tilde{G}$ is said to be *evenly split* if the number of ‘−’ signs in any diagram for $\tilde{H}^{\text{es}}$ is maximal among even parity Cartan subgroups.

Remark 7.3.4. An evenly split Cartan subgroup is maximally split if and only if q is even. For any $\widetilde{\text{Spin}}(p, q)$ there is a unique conjugacy class of evenly split Cartan subgroups.

Remark 7.3.4 leads to some slightly messy expressions involving n and q . To simplify the notation we make the following definition.

Definition 7.3.5. For each $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ we set $\hat{q} = \lceil \frac{q-1}{2} \rceil$.

The major difficulties in the KLV-algorithm concern the delicate nature of even parity Cartan subgroups. To overcome these issues, we must construct our even parity Cartan subgroups in a regular way. To begin, fix an evenly split Cartan subgroup $\tilde{H}^{\text{es}} \subset \tilde{G}$ and choose a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h}^{\text{es}})$ such that either

$$\begin{array}{c} \frac{-}{1} \quad - \quad \cdots \quad \frac{-}{q} \quad + \quad + \quad \cdots \quad \frac{+}{n} \\ \text{or} \\ \frac{-}{1} \quad - \quad \cdots \quad \frac{-}{q-1} \quad \oplus \quad \oplus \quad \cdots \quad \oplus \quad \frac{+}{n} \end{array}$$

is the corresponding diagram. In the usual coordinates for $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{es}})$ set

$$\begin{aligned} \alpha_i &= e_{2i-1} - e_{2i}, \\ \beta_i &= e_{2i-1} + e_{2i}, \\ \gamma &= e_n. \end{aligned}$$

Choose root vectors $X_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$ and $X_\gamma \in \mathfrak{g}_\gamma$ according to Lemma 3.3.1 or Lemma 3.3.3 and define

$$z = \tilde{m}_{\alpha_1} \tilde{m}_{\alpha_2} \cdots \tilde{m}_{\alpha_{n/2}}.$$

The element z is almost central in $\tilde{G}$ and will play a critical role in our definition of Ψ . Using Lemma 2.2.3 and Proposition 3.4.5 it is easy to verify that

$$Z(\tilde{H}^{\text{es}}) = \begin{cases} \tilde{H}^{\text{es}} & q \in \{0, 1\}, \\ \langle (\tilde{H}^{\text{es}})_0, \pm z \rangle & 2 \leq q < n, \\ \langle \pm(\tilde{H}^{\text{es}})_0, \pm \tilde{m}_\gamma, \pm z \rangle & q = n, \end{cases}$$

and a genuine character of $Z(\tilde{H}^{\text{es}})$ is determined by its differential and its values on $\tilde{m}_\gamma$ and z .

Remark 7.3.6. Suppose $(\tilde{H}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple for $\tilde{G}$ with half-integral infinitesimal character λ . Proposition 2.2.4 implies the isomorphism class of $\tilde{\Gamma}$ is determined by its restriction to $Z(\tilde{H}^{\text{es}})$ and this restriction is given by

$$\tilde{\Gamma}|_{Z(\tilde{H}^{\text{es}})} = m\chi$$

where χ is a genuine character for $Z(\tilde{H}^{\text{es}})$. Therefore, we may treat $\tilde{\Gamma}$ as either a genuine representation of $\tilde{H}^{\text{es}}$ or a genuine character of $Z(\tilde{H}^{\text{es}})$.

Proposition 7.3.7. *Up to conjugacy, every even parity Cartan subgroup of $\tilde{G}$ can be obtained from $\tilde{H}^{\text{es}}$ through an iterative application of the operators $\mathcal{C}_{\alpha_i}$ and $\mathcal{C}_{\beta_i}$ (Definition 3.3.4).*

Proof. This is easily verified on the level of involutions using Proposition 3.3.5. $\square$

Proposition 7.3.7 implies we can associate a (nonunique) sequence in $\{\alpha_i, \beta_i\}$ to each conjugacy class of even parity Cartan subgroups in $\tilde{G}$. Since the corresponding operators $\mathcal{C}_{\alpha_i}$ and $\mathcal{C}_{\beta_i}$ commute, the ordering of the roots is unimportant. To eliminate the ambiguity, we make the following definition.

Definition 7.3.8. A sequence in $\{\alpha_i, \beta_i\}$ is called *standard* if it is of the form

$$\mathbf{c}_j^k = \beta_k \cdots \beta_2 \beta_1 \alpha_j \cdots \alpha_2 \alpha_1$$

with $0 \leq k \leq j \leq \hat{q}$.

There is a unique standard sequence associated to each conjugacy class of even parity Cartan subgroups in $\tilde{G}$. If $\mathbf{c}_j^k$ is a standard sequence we will write

$$\tilde{H}_{\mathbf{c}_j^k}^{\text{es}} = \mathcal{C}_j^k(\tilde{H}^{\text{es}}) = \mathcal{C}_{\beta_k} \cdots \mathcal{C}_{\beta_2} \mathcal{C}_{\beta_1} \mathcal{C}_{\alpha_j} \cdots \mathcal{C}_{\alpha_2} \mathcal{C}_{\alpha_1}(\tilde{H}^{\text{es}})$$

for the corresponding even parity Cartan subgroup.

Proposition 7.3.9. *Let $\mathbf{c}_j^k$ be a standard sequence and suppose $1 \leq i \leq \frac{n}{2}$. Then $\tilde{m}_{\alpha_i} \in \tilde{H}_{\mathbf{c}_j^k}^{\text{es}}$. In particular, the product $z = \tilde{m}_{\alpha_1} \tilde{m}_{\alpha_2} \cdots \tilde{m}_{\alpha_{n/2}} \in \tilde{H}^{\text{es}}$ is an element of $\tilde{H}_{\mathbf{c}_j^k}^{\text{es}}$.*

Proof. This follows from Theorem 3.4.3 (see also [14], Lemma 8.3.13). $\square$

Proposition 7.3.10. *Let $\mathbf{c}_j^k$ be a nonempty standard sequence and write $\tilde{m}_\gamma$ for the unique element corresponding to any short real root in $\tilde{H}^{\text{es}}$ (Proposition 6.1.1).*

Then the center of $\tilde{H}_{\mathfrak{c}_j^k}^{es}$ is given by

$$Z(\tilde{H}_{\mathfrak{c}_j^k}^{es}) = \begin{cases} \langle (\tilde{H}_{\mathfrak{c}_j^k}^{es})_0, \tilde{m}_{\alpha_{j+1}} \cdots \tilde{m}_{\alpha_{\hat{q}}}, \tilde{m}_\gamma \rangle & j < \hat{q}, k = 0, \\ \langle (\tilde{H}_{\mathfrak{c}_j^k}^{es})_0, \tilde{m}_{\alpha_{j+1}} \cdots \tilde{m}_{\alpha_{\hat{q}}} \rangle & j < \hat{q}, k > 0, \\ \langle (\tilde{H}_{\mathfrak{c}_j^k}^{es})_0, \tilde{m}_\gamma \rangle & j = \hat{q}, k = 0, \\ (\tilde{H}_{\mathfrak{c}_j^k}^{es})_0 & j = \hat{q}, k > 0. \end{cases}$$

In particular, $z \in Z(\tilde{H}_{\mathfrak{c}_j^k}^{es})$.

Proof. Since $\mathfrak{c}_j^k$ is nonempty, $-1 \in (\tilde{H}_{\mathfrak{c}_j^k}^{es})_0$. The first result follows from Theorem 3.4.3 and Proposition 3.4.5. The last statement follows by ([14], Lemma 8.3.13). $\square$

Corollary 7.3.11. *In the setting of Proposition 7.3.10, let $\mathfrak{c}_j^k$ be a standard sequence with $j < \hat{q}$. Suppose the genuine triples $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma}_1)_\lambda$ and $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma}_2)_\lambda$ are not $\tilde{K}$ -conjugate and have the same central character. Then $\tilde{\Gamma}_2(z) = -\tilde{\Gamma}_1(z)$.*

Proof. We have

$$\begin{aligned} \tilde{\Gamma}_1(\tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_j}) &= \tilde{\Gamma}_2(\tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_j}), \\ \tilde{\Gamma}_1(\tilde{m}_{\alpha_{\hat{q}+1}} \cdots \tilde{m}_{\alpha_{n/2}}) &= \tilde{\Gamma}_2(\tilde{m}_{\alpha_{\hat{q}+1}} \cdots \tilde{m}_{\alpha_{n/2}}), \end{aligned}$$

since $\tilde{m}_{\alpha_1} \cdots \tilde{m}_{\alpha_j}, \tilde{m}_{\alpha_{\hat{q}+1}} \cdots \tilde{m}_{\alpha_{n/2}} \in (\tilde{H}_{\mathfrak{c}_j^k}^{es})_0$. However, any genuine character extending $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi)_\lambda$ is determined by its restriction to $Z(\tilde{H}_{\mathfrak{c}_j^k}^{es})$. Since $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma}_1)_\lambda$ and $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma}_2)_\lambda$ have the same central characters we must have

$$\tilde{\Gamma}_1(\tilde{m}_{\alpha_{j+1}} \cdots \tilde{m}_{\alpha_{\hat{q}}}) \neq \tilde{\Gamma}_2(\tilde{m}_{\alpha_{j+1}} \cdots \tilde{m}_{\alpha_{\hat{q}}})$$

and the result follows. $\square$

Proposition 7.3.12. *Let $\mathfrak{c}_j^k$ be a standard sequence and suppose $\delta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{es})$. Then*

$$\delta(z) = \begin{cases} 1 & \delta \text{ is long,} \\ -1 & \delta \text{ is short.} \end{cases}$$

Proof. We have

$$\begin{aligned} \delta(z) &= \delta(m_{\alpha_1})\delta(m_{\alpha_2}) \cdots \delta(m_{\alpha_{n/2}}) \\ &= (-1)^{(\delta, \alpha_1^\vee)} (-1)^{(\delta, \alpha_2^\vee)} \cdots (-1)^{(\delta, \alpha_{n/2}^\vee)}. \end{aligned}$$

Suppose first that δ is long. If $\delta = \alpha_j$ or $\delta = \beta_j$ for some j , we have $(\delta, \alpha_i^\vee) \in \{0, 2\}$ for all i . Otherwise, $(\delta, \alpha_i^\vee) = -1$ for exactly two values of i and the result follows. If δ is short, then $(\delta, \alpha_i^\vee) = -1$ for exactly one value of i . $\square$

We will need an explicit description of the cross action (Definition 7.1.4) in real roots for even parity Cartan subgroups. This essentially requires understanding how conjugation affects the element z . We begin with the following general lemma.

Lemma 7.3.13. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a Θ -stable Cartan subalgebra and suppose $\alpha, \beta \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h})$. Choose root vectors $X_{\alpha} \in \mathfrak{g}_{\alpha}$ and $X_{\beta} \in \mathfrak{g}_{\beta}$ according to Lemma 3.3.1. Then we have*

$$\begin{aligned}\tilde{m}_{\alpha}\tilde{\sigma}_{\beta}\tilde{m}_{\alpha}^{-1} &= \tilde{\sigma}_{\beta}^{\beta(\tilde{m}_{\alpha})}, \\ \tilde{\sigma}_{\beta}\tilde{m}_{\alpha}\tilde{\sigma}_{\beta}^{-1} &= \tilde{\sigma}_{\beta}^{(1-\beta(\tilde{m}_{\alpha}))}\tilde{m}_{\alpha}.\end{aligned}$$

Proof. These statements are proved in the same way as [14], Lemma 4.3.19. $\square$

Proposition 7.3.14. *Fix a standard sequence $\mathfrak{c}_j^k$ and let $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_{\lambda}$ be a genuine triple. For $\delta \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{es})$ choose $X_{\delta} \in \mathfrak{g}_{\delta}$ according to Lemma 3.3.1 and write $\tilde{\sigma}_{\delta}$ for the corresponding root reflection in $\tilde{K}$. Then $\tilde{\sigma}_{\delta}$ acts on $\tilde{\Gamma}$ by conjugation and we have*

$$(\tilde{\sigma}_{\delta} \cdot \tilde{\Gamma})(z) = \begin{cases} \tilde{\Gamma}(z) & \delta \text{ is long,} \\ \tilde{\Gamma}(\tilde{m}_{\gamma})\tilde{\Gamma}(z) & \delta \text{ is short.} \end{cases}$$

Proof. This follows in the same fashion as Proposition 7.3.12 using Proposition 7.3.13. $\square$

Corollary 7.3.15. *Let λ be a half-integral infinitesimal character and fix a standard sequence $\mathfrak{c}_j^k$. If $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_{\lambda}$ is a genuine triple and $\delta \in \Delta_{\mathbb{R}}^{\Theta}(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{es})$ is a real root, then $s_{\delta} \times (\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_{\lambda}$ and $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_{\lambda}$ are $\tilde{K}$ -conjugate if and only if δ does not satisfy the parity condition (Definition 6.1.4).*

Proof. First suppose δ is long and does not satisfy the parity condition. Then δ is integral and Propositions 7.3.14 and 7.3.12 imply

$$(\tilde{\sigma}_{\delta} \cdot \tilde{\Gamma})(z) = \tilde{\Gamma}(z) = (s_{\delta} \times \tilde{\Gamma})(z).$$

In particular, $s_{\delta} \times (\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_{\lambda}$ and $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_{\lambda}$ are conjugate by an element of $\tilde{K}$. If δ satisfies the parity condition, then $s_{\delta} \times (\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_{\lambda}$ and $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_{\lambda}$ have different infinitesimal characters and cannot be $\tilde{K}$ -conjugate.

If δ is short (and thus integral) we have

$$\begin{aligned}(\tilde{\sigma}_{\delta} \cdot \tilde{\Gamma})(z) &= \tilde{\Gamma}(\tilde{m}_{\delta})\tilde{\Gamma}(z), \\ (s_{\delta} \times \tilde{\Gamma})(z) &= \delta(z)^{(\phi, \delta^{\vee})}\tilde{\Gamma}(z) = (-1)^{(\phi, \delta^{\vee})}\tilde{\Gamma}(z)\end{aligned}$$

so that $(\tilde{\sigma}_{\delta} \cdot \tilde{\Gamma})(z) = (s_{\delta} \times \tilde{\Gamma})(z)$ if and only if $\tilde{\Gamma}(\tilde{m}_{\delta}) = (-1)^{(\phi, \delta^{\vee})}$, i.e., δ does not satisfy the parity condition. $\square$

Corollary 7.3.16. *Let $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_{\lambda}$ be a genuine triple and suppose $\alpha \in \Delta_i^{\Theta}(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{es})(\phi)$ is an imaginary integral root. If $m = (\phi, \alpha^{\vee})$, then*

$$s_{\alpha} \times \tilde{\Gamma}(z) = \begin{cases} \tilde{\Gamma}(z) & \alpha \text{ long,} \\ (-1)^{m+1}\tilde{\Gamma}(z) & \alpha \text{ short.} \end{cases}$$

Proof. In the notation of Definition 7.1.4 the reader can verify

$$\varphi = \begin{cases} -m\alpha + \sum_{\substack{\beta \text{ long} \\ \beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^s)}} n_\beta \beta + \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^s)} 2 \cdot m_\beta \beta & \alpha \text{ long,} \\ -m\alpha - \alpha + \sum_{\substack{\beta \text{ long} \\ \beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^s)}} n_\beta \beta + \sum_{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^s)} 2 \cdot m_\beta \beta & \alpha \text{ short,} \end{cases}$$

for $n_\beta, m_\beta \in \mathbb{Z}$ and the result follows by Proposition 7.3.12. $\square$

Corollary 7.3.17. *Let $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple and suppose $\alpha \in \Delta_{\mathbb{C}}^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{es})(\phi)$ is a short complex integral root. If $m = (\phi, \alpha^\vee)$, then*

$$s_\alpha \times \tilde{\Gamma}(z) = (-1)^m \tilde{\Gamma}(z).$$

Proof. This follows from Proposition 7.3.12 in the same fashion as Corollary 7.3.16. $\square$

Corollary 7.3.18. *Let $\tilde{H}_{\mathfrak{c}_j^k}^{es} \subset \tilde{G}$ be an even parity Cartan subgroup with $j = \hat{q}$ and let $\alpha \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{es})$ be an imaginary root (not necessarily integral). Suppose there exist genuine triples of the form $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma}_1)_\lambda$ and $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, s_\alpha \cdot \phi, \tilde{\Gamma}_2)_\lambda$ with the same central character. If $m = (\phi, \alpha^\vee)$, then*

$$\tilde{\Gamma}_2(z) = \begin{cases} (-1)^{2m} \tilde{\Gamma}_1(z) & \alpha \text{ long,} \\ (-1)^{m+1} \tilde{\Gamma}_1(z) & \alpha \text{ short.} \end{cases}$$

Proof. Since z is an element of $(\tilde{H}_{\mathfrak{c}_j^k}^{es})_0$ (Proposition 7.3.10), it suffices to consider differentials. We leave the easy (but technical) details to the reader. $\square$

We conclude this section with an important result relating evenly split Cartan subgroups and Cayley transforms.

Proposition 7.3.19 ([11], Proposition 6.12). *Let $v \in \tilde{\mathcal{D}}_\lambda(p, q)$ and suppose $\alpha \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}^{abs})(\lambda)$ is an abstract integral noncompact imaginary root that is of type II for v . Then $s_\alpha \times v = v$ if and only $\dim(\mathcal{C}^\alpha(v)) = \dim(v)$.*

Corollary 7.3.20. *In the setting of Proposition 7.3.19, suppose $\tilde{G} = \widetilde{Spin}(p, q)$ has even rank and fix an evenly split Cartan subgroup $\tilde{H}^{es} \subset \tilde{G}$ (Definition 7.3.3). If v has a representative genuine triple of the form $(\tilde{H}_{\mathfrak{c}_j^k}^{es}, \phi, \tilde{\Gamma})_\lambda$ (Definition 7.3.8), then $s_\alpha \times v \neq v$.*

Proof. The existence of α implies we must have $j < \hat{q}$ (Remark 5.2.5). Propositions 2.2.4 and 7.3.10 and Theorem 3.4.3 imply

$$\dim(v) = \left(\frac{2^{n_r^\theta - 1}}{2} \right)^{\frac{1}{2}} = \left(2^{n_r^\theta - 2} \right)^{\frac{1}{2}}.$$

Similarly, we have

$$\begin{aligned}\dim(\mathcal{C}^\alpha(v)) &= \left(2^{n_r^\theta}\right)^{\frac{1}{2}}, \\ \frac{\dim(\mathcal{C}^\alpha(v))}{\dim(v)} &= \left(\frac{2^{n_r^\theta}}{2^{n_r^\theta-2}}\right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2,\end{aligned}$$

and the result follows from Proposition 7.3.19. $\square$

7.4. Principal classes. Fix a half-integral infinitesimal character $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ and suppose $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ has even rank and is not compact. Choose an evenly split Cartan subgroup $\tilde{H}^{\text{es}} \subset \tilde{G}$ (Definition 7.3.3) and let $\mathfrak{c}_j^k$ be a standard sequence (Definition 7.3.8). If $(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda$ is a genuine pair for which the corresponding abstract triple $(\theta, \varepsilon, \lambda)$ is supportable (Definition 4.1.4), Corollary 6.3.2 implies $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 2$ for some central character χ (Definition 6.2.1). In this section we use the element $z = \tilde{m}_{\alpha_1} \tilde{m}_{\alpha_2} \cdots \tilde{m}_{\alpha_{n/2}}$ of Section 7.3 to distinguish the two elements in $\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)$.

To avoid trivialities, we begin by assuming $\tilde{G} = \widetilde{\text{Spin}}(p, q)$ with $p > q \geq 2$. Let $v \in \tilde{\mathcal{D}}_\lambda(p, q)$ and suppose $(\tilde{H}^{\text{es}}, \varphi, \tilde{\Gamma})_\lambda$ is a genuine triple representing v . Since every short root for v is either real, compact imaginary, or noncompact imaginary and of type II (Proposition 5.2.4), v has a representative genuine triple $(\tilde{H}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$ with $(\phi, e_i) > 0$ for all i (recall our choice of coordinates in Section 7.3). Such a representative will be called *essentially dominant*.

Definition 7.4.1. Two essentially dominant genuine triples $(\tilde{H}^{\text{es}}, \phi_1, \tilde{\Gamma}_1)_\lambda$ and $(\tilde{H}^{\text{es}}, \phi_2, \tilde{\Gamma}_2)_\lambda$ are said to be in the same *principal class* if $\tilde{\Gamma}_1(z) = \tilde{\Gamma}_2(z)$. We denote the set of principal classes for $\tilde{G}$ with infinitesimal character λ by $\text{pc}_\lambda(p, q)$ and with central character χ by $\text{pc}_\lambda^\chi(p, q)$.

Proposition 7.4.2. Suppose $\Upsilon_1 = (\tilde{H}^{\text{es}}, \phi_1, \tilde{\Gamma}_1)_\lambda$ and $\Upsilon_2 = (\tilde{H}^{\text{es}}, \phi_2, \tilde{\Gamma}_2)_\lambda$ are essentially dominant, $\tilde{K}$ -conjugate genuine triples. Then Υ_1 and Υ_2 are in the same principal class.

Proof. Essential dominance implies ϕ_1 and ϕ_2 differ only by reflections in long roots. Since Υ_1 and Υ_2 are $\tilde{K}$ -conjugate, these roots must be either real or compact imaginary. The result now follows from Proposition 7.3.12 and Corollary 7.3.16. $\square$

Remark 7.4.3. Proposition 7.4.2 implies the notion of a principal class descends to the level of genuine parameters. Since a principal class does not see the corresponding $\tilde{K}$ -orbit, Corollary 6.3.2 implies $|\text{pc}_\lambda^\chi(p, q)| = 2$ whenever $|\text{pc}_\lambda^\chi(p, q)| \neq 0$. Note that if $\tilde{G}$ is split, a principal class is the same as a principal series representation.

In those cases where $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 2$, we will assign each element of $\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)$ a distinct principal class (Definition 7.4.5). This will complete the process of distinguishing the elements in $\tilde{\mathcal{D}}_\lambda(\chi)(p, q)$ and allow us to define the duality map Ψ (Definition 7.5.1). We begin with a definition.

Definition 7.4.4. Given an abstract involution $\theta \in W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$, let n_{cr}^θ denote half the number of abstract coordinates that are both interchanged and negated by θ . In

terms of diagrams, n_{cr}^θ is exactly half the number of parentheses appearing in D_θ (Section 3.1).

Definition 7.4.5. In the setting above, let $\mathfrak{c}_j^k$ be a standard sequence (Definition 7.3.8) and suppose $(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda$ is a genuine pair for which the corresponding abstract triple $(\theta, \varepsilon, \lambda)$ is supportable. The *principal class map*

$$\wp : \tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q) \rightarrow \text{pc}_\lambda^\chi(p, q)$$

is defined as follows. For $v \in \tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)$, choose a genuine triple $\Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi, \tilde{\Gamma}_1)_\lambda$ beginning with $\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}$ and representing v . Write $\tilde{\Gamma}_1^z$ for the unique genuine character of $Z(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}})$ corresponding to $\tilde{\Gamma}_1$ (Proposition 2.2.4). Then there is a unique genuine character $\tilde{\Gamma}_2^z$ of $Z(\tilde{H}^{\text{es}})$ with differential $(\mathcal{C}_j^k)^{-1}(\phi)$ and

$$\begin{aligned} \tilde{\Gamma}_2^z(\tilde{m}_\gamma) &= \tilde{\Gamma}_1^z(\tilde{m}_\gamma), \\ \tilde{\Gamma}_2^z(z) &= \tilde{\Gamma}_1^z(z) \tilde{\Gamma}_1^z(\tilde{m}_\gamma)^{n_{cr}^\theta}. \end{aligned}$$

Here $\tilde{m}_\gamma$ denotes the nontrivial central element of $\tilde{G}$ corresponding to the short real roots in $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{es}})$ (Proposition 6.1.1). Let $\tilde{\Gamma}_2$ be a genuine representation of $\tilde{H}^{\text{es}}$ corresponding to $\tilde{\Gamma}_2^z$ and define

$$\begin{aligned} \wp(\tilde{\Gamma}_1) &= \tilde{\Gamma}_2, \\ \wp(\Upsilon) &= (\tilde{H}^{\text{es}}, (\mathcal{C}_j^k)^{-1}(\phi), \tilde{\Gamma}_2)_\lambda, \\ \wp(v) &= [(\tilde{H}^{\text{es}}, (\mathcal{C}_j^k)^{-1}(\phi), \tilde{\Gamma}_2)_\lambda] \in \text{pc}_\lambda(p, q). \end{aligned}$$

In other words, $\wp(v)$ is defined to be the principal class represented by the genuine triple $(\tilde{H}^{\text{es}}, (\mathcal{C}_j^k)^{-1}(\phi), \tilde{\Gamma}_2)_\lambda$. In particular, v and $\wp(v)$ have the same central characters (Remark 6.2.3).

Remark 7.4.6. The above definition will ultimately be used to induce the duality map Ψ (Definition 7.5.1). Although the ‘twist’ factor $\tilde{\Gamma}_1^z(\tilde{m}_\gamma)^{n_{cr}^\theta}$ is not essential for the results of this section, it (or something like it) seems to be required for Ψ to have the desired properties. In particular, Theorem 7.5.3 (Case III) does not hold if the twist factor is eliminated. The author was annoyed by this to no end.

Proposition 7.4.7. *The map $\wp$ is well defined.*

Proof. The issue is the choice of genuine triple $\Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi, \tilde{\Gamma}_1)_\lambda$ representing v . It suffices to show $\wp(\Upsilon)$ is equivalent to $\wp(w \cdot \Upsilon)$ with

$$w \cdot \Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, w \cdot \phi, w \cdot \tilde{\Gamma}_1)_\lambda$$

and $w \in N_{\tilde{G}}(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}})/\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}} = W(\tilde{G}, \tilde{H}_{\mathfrak{c}_j^k}^{\text{es}})$. Note that $w \cdot \tilde{\Gamma}_1$ is defined only up to $\tilde{K}$ -conjugacy; however, the corresponding character of $Z(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}})$ is well defined. Since it is the character that matters for Definition 7.4.5, this level of precision is sufficient for our purposes.

As noted above, the differentials of $\wp(\Upsilon)$ and $\wp(w \cdot \Upsilon)$ will have essentially dominant representatives in $(\mathfrak{h}^{\text{es}})^*$. However, any genuine representation of $\tilde{H}^{\text{es}}$ with a fixed differential is determined by its values on $\tilde{m}_\gamma$ and z . Since conjugation

and cross action cannot affect $\tilde{m}_\gamma$, we simply need to understand their effect on z . Moreover, conjugating by w does not change the value of n_{cr}^θ , so we may effectively ignore the $\tilde{\Gamma}_1^z(\tilde{m}_\gamma)^{n_{cr}^\theta}$ term in the definition of $\tilde{\Gamma}_2^z(z)$.

Proposition 5.1.1 implies

$$W(\tilde{G}, \tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}) \cong W_{\mathbb{C}}^\theta \ltimes ((A \ltimes W_{ic}(\theta) \times W_{\mathbb{R}}(\theta)).$$

Our proof is by cases based on the element w .

Case I. Suppose $w \in W_{\mathbb{R}}(\theta)$. Then conjugation commutes with $(\mathcal{C}_j^k)^{-1}$ and $\wp(\Upsilon)$ is conjugate to $\wp(w \cdot \Upsilon)$.

Case II. Suppose $w \in W_{ic}(\theta)$. We may assume $w = s_\alpha$ for $\alpha \in \Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{\text{es}})(\phi)$ imaginary and compact. Write $\bar{\alpha} = (\mathcal{C}_j^k)^{-1}(\alpha)$ for the image of α in $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{es}})$. If $\bar{\alpha}$ remains imaginary and compact in $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{es}})$, then conjugation commutes with $(\mathcal{C}_j^k)^{-1}$ and we are done.

If $\bar{\alpha}$ is short and real in $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{es}})$, set $m = (\phi, \alpha^\vee) \in \mathbb{Z}$. Proposition 7.3.14 implies

$$(s_{\bar{\alpha}} \cdot \wp(\tilde{\Gamma}_1))(z) = \wp(\tilde{\Gamma}_1)(\tilde{m}_\gamma) \wp(\tilde{\Gamma}_1)(z)$$

and Corollary 7.3.16 gives

$$s_\alpha \cdot \tilde{\Gamma}_1(z) = (-1)^{(m+1)} \tilde{\Gamma}_1(z).$$

Therefore, $\wp(\Upsilon)$ is conjugate to $\wp(s_\alpha \cdot \Upsilon)$ if and only if

$$\wp(\tilde{\Gamma}_1)(\tilde{m}_\gamma) = (-1)^{(m+1)} \cdot \text{I},$$

or if and only if $\bar{\alpha}$ satisfies the parity condition for $\wp(\Upsilon)$. Since α is assumed to compact, this follows from Theorem 6.2.7.

If $\bar{\alpha}$ is long and real in $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{es}})$, Proposition 7.3.14 implies $(s_{\bar{\alpha}} \cdot \wp(\tilde{\Gamma}_1))(z) = \wp(\tilde{\Gamma}_1)(z)$. In particular, $\wp(\Upsilon)$ is conjugate to $\wp(s_\alpha \cdot \Upsilon)$ if and only if $\tilde{\Gamma}_1(z) = s_\alpha \cdot \tilde{\Gamma}_1(z)$ and this follows easily from Corollary 7.3.16.

Finally, if $\bar{\alpha}$ is long and complex in $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{es}})$, then the $\tilde{K}$ -orbits for $\wp(\Upsilon)$ and $\wp(w \cdot \Upsilon)$ differ by cross action in $\bar{\alpha}$ (Remark 7.1.6). If $s_{\bar{\alpha}}$ preserves the (fixed) set of short positive roots, then the same sequence of short reflections will produce essentially dominant representatives for $\wp(\Upsilon)$ and $\wp(s_\alpha \cdot \Upsilon)$. Since $\tilde{\Gamma}_1(z) = s_\alpha \cdot \tilde{\Gamma}_1(z)$, $\wp(\Upsilon)$ and $\wp(s_\alpha \cdot \Upsilon)$ represent the same principal class. Otherwise, $s_{\bar{\alpha}}$ will negate (and interchange) a short imaginary root and a short real root. If the imaginary root is compact, the real root will satisfy the parity condition (Theorem 6.2.7) and the composition of the two reflections changes $\wp(s_\alpha \cdot \tilde{\Gamma}_1)(z)$ by

$$(-1)^{m+1} \wp(\tilde{\Gamma}_1)(\tilde{m}_\gamma) = (-1)^{m+1} (-1)^{m+1} = 1$$

(for some integer m). If the imaginary root is noncompact, then the real root will not satisfy the parity condition and the composition of the two reflections changes $\wp(s_\alpha \cdot \tilde{\Gamma}_1)(z)$ by a factor of

$$(-1)^m \wp(\tilde{\Gamma}_1)(\tilde{m}_\gamma) = (-1)^m (-1)^m = 1.$$

Therefore, $\wp(\Upsilon)$ and $\wp(s_\alpha \cdot \Upsilon)$ represent the same principal class as in the previous case.

Case III. Suppose $w \in A$. Then

$$w \in W_i(\Theta)(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{\text{es}}) \cong W(B_{n_s^\Theta}) \times W(A_1)^k$$

(Proposition 3.1.7) and w can be expressed as a product of orthogonal reflections in noncompact imaginary roots ([15], Corollary 5.14).

If $j < \hat{q}$, the group A is generated by reflections in noncompact imaginary roots of type II. Suppose first that α is long and let $w = s_\alpha$. Then α is an element of the $(A_1)^k$ factor of $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{\text{es}})$ and we must have $\alpha = \mathcal{C}_j^k(\alpha_i)$ for some i . In particular, there exists a Cartan subgroup $\tilde{H}_\mathfrak{c}^{\text{es}} = \mathcal{C}^\alpha(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}})$ containing z and a representative of s_α (Proposition 7.2.1). Since z is central in $\tilde{H}_\mathfrak{c}^{\text{es}}$, z commutes with s_α and we have $\tilde{\Gamma}_1(z) = s_\alpha \cdot \tilde{\Gamma}_1(z)$. The result now follows as in Case II with $\bar{\alpha}$ long and real.

Suppose α is short and of type II. Corollary 7.3.16 gives

$$s_\alpha \times \tilde{\Gamma}_1(z) = (-1)^{(m+1)} \tilde{\Gamma}_1(z)$$

and Corollary 7.3.20 implies $s_\alpha \times v \neq v$. Therefore, we must have

$$s_\alpha \cdot \tilde{\Gamma}_1(z) = (-1)^{(m)} \tilde{\Gamma}_1(z)$$

by Corollary 7.3.11. In particular, $\wp(\Upsilon)$ and $\wp(s_\alpha \cdot \Upsilon)$ are conjugate if and only if $\bar{\alpha}$ does not satisfy the parity condition for $\wp(\Upsilon)$. Since α is assumed to be noncompact, this follows from Theorem 6.2.7.

Finally let $j = \hat{q}$ and $w = s_\epsilon s_\delta$, where δ and ϵ are orthogonal imaginary roots in $\Delta_i^\Theta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{\text{es}})$ that are noncompact and of type I. Set $m_1 = (\phi, \delta^\vee)$, $m_2 = (\phi, \epsilon^\vee)$, and suppose first that δ and ϵ are long. Then $m_1, m_2 \in \mathbb{Z} + \frac{1}{2}$ and Corollary 7.3.18 implies

$$\begin{aligned} s_\delta \cdot \tilde{\Gamma}_1(z) &= (-1) \cdot \tilde{\Gamma}_1(z), \\ s_\epsilon \cdot \tilde{\Gamma}_1(z) &= (-1) \cdot \tilde{\Gamma}_1(z), \\ s_\epsilon s_\delta \cdot \tilde{\Gamma}_1(z) &= (-1)^2 \cdot \tilde{\Gamma}_1(z) = \tilde{\Gamma}_1(z), \end{aligned}$$

as desired. If δ is long and ϵ is short, then $m_1 \in \mathbb{Z} + \frac{1}{2}$ and $m_2 \in \mathbb{Z}$. Corollary 7.3.18 implies

$$\begin{aligned} s_\epsilon \cdot \tilde{\Gamma}_1(z) &= (-1)^{m_2+1} \cdot \tilde{\Gamma}_1(z), \\ s_\epsilon s_\delta \cdot \tilde{\Gamma}_1(z) &= (-1)^{m_2} \cdot \tilde{\Gamma}_1(z). \end{aligned}$$

This is the desired result since ϵ is noncompact. Finally, if both δ and ϵ are short we have

$$\begin{aligned} s_\delta \cdot \tilde{\Gamma}_1(z) &= (-1)^{m_1+1} \cdot \tilde{\Gamma}_1(z), \\ s_\epsilon \cdot \tilde{\Gamma}_1(z) &= (-1)^{m_2+1} \cdot \tilde{\Gamma}_1(z), \\ s_\epsilon s_\delta \cdot \tilde{\Gamma}_1(z) &= (-1)^{m_1+m_2} \cdot \tilde{\Gamma}_1(z), \\ &= \tilde{\Gamma}_1(z), \end{aligned}$$

since $m_1 + m_2 \in 2\mathbb{Z}$. The result follows from Proposition 7.3.12.

Case IV. Suppose $w \in W_\mathbb{C}^\theta$. This case is handled in the same fashion as the previous cases. The reader is spared the details. $\square$

Theorem 7.4.8. *In the setting of Definition 7.4.5, the map*

$$\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q) \rightarrow pc_\lambda^\chi(p, q)$$

is a bijection.

Proof. Corollary 6.3.2 implies $\wp$ is, at worst, 2 to 1. In particular, if $\omega, v \in \tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)$, it remains to show $\wp(\omega) \neq \wp(v)$. Recall θ is a supportable abstract involution for the genuine pair $(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda$ and let $\epsilon_s^\theta, \epsilon_r^\theta, \epsilon_p^\theta = \epsilon_m^\theta = 0$ be the corresponding indicator bits (Definition 3.1.2). The proof is by cases for the standard sequence $\mathfrak{c}_j^k$.

Case I. Suppose $j < \hat{q}$ and $k = 0$ so that $\epsilon_r^\theta = 1$. Corollary 4.2.3 implies

$$\begin{aligned} \left[(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda \right] &= 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} \\ &= 2 \cdot 2^{1-\epsilon_s^\theta}. \end{aligned}$$

Therefore, each element of $\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)$ has a representative beginning with $(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda$ and the result follows from Corollary 7.3.11.

Case II. Suppose $j < \hat{q}$ and $k > 0$ so that $\epsilon_s^\theta = \epsilon_r^\theta = 1$. Corollary 4.2.3 implies

$$\left[(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda \right] = 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} = 2$$

and there are two $\tilde{K}$ -orbits in the genuine fiber $\tilde{\theta}_\lambda^\dagger$ of θ (Section 5.4). Since $\tilde{m}_\gamma \in (\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}})_0$ by Proposition 7.3.10, both genuine triples extending $(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda$ have the same central character and the result follows by Corollary 7.3.11.

Case III. Suppose $j = \hat{q}$ and $k > 0$ so that $\epsilon_s^\theta = 1$ and $\epsilon_r^\theta = 0$. Corollary 4.2.3 implies

$$\left[(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda \right] = 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} = 1.$$

Then $\tilde{m}_\gamma \in (\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}})_0 = \tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}$ by Proposition 7.3.10 and there are two $\tilde{K}$ -orbits in $\tilde{\theta}_\lambda^\dagger$ with a fixed abstract grading and central character (Proposition 6.2.5). Theorem 5.3.2 implies these orbits differ by cross action in any short noncompact root. Let $\Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple extending $(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda$ and let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ be a short noncompact root. Set $m = (\lambda, \alpha^\vee)$ and write $\bar{\alpha}$ for the image of α in $\Delta(\mathfrak{g}, \mathfrak{h}^{\text{es}})$. Corollary 7.3.16 and Proposition 7.3.14 imply

$$\begin{aligned} s_\alpha \times \tilde{\Gamma}(z) &= (-1)^{m+1} \tilde{\Gamma}(z), \\ s_{\bar{\alpha}} \cdot \wp(\tilde{\Gamma})(z) &= \wp(\tilde{\Gamma})(\tilde{m}_{\bar{\alpha}}) \wp(\tilde{\Gamma})(z). \end{aligned}$$

In particular, $\wp(\Upsilon) \neq \wp(s_\alpha \times \Upsilon)$ if and only if α does not satisfy the parity condition (Definition 6.1.4) for $\wp(\Upsilon)$. Since α is noncompact, this follows by Theorem 6.2.7.

Case IV. Suppose $j = \hat{q}$ and $k = 0$ so that $\epsilon_r^\theta = 0$. Corollary 4.2.3 implies

$$\begin{aligned} \left[(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda \right] &= 2^{1-\epsilon_s^\theta} 2^{\epsilon_r^\theta(1-\epsilon_m^\theta)} \\ &= 2^{1-\epsilon_s^\theta} \end{aligned}$$

and $z \in (\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}})_0$. Let $\Upsilon = (\tilde{H}_{\mathfrak{c}^k}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$ be a genuine triple extending $(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi)_\lambda$ and suppose α is a long noncompact imaginary root in $\Delta(\mathfrak{g}, \mathfrak{h}_{\mathfrak{c}_j^k}^{\text{es}})$. Then $(\tilde{H}_{\mathfrak{c}^k}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$ and $(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, s_\alpha \cdot \phi, s_\alpha \cdot \tilde{\Gamma})_\lambda$ are *not* conjugate in $\tilde{K}$. If $m = (\phi, \alpha^\vee)$, then $m \in \mathbb{Z} + \frac{1}{2}$ and Corollary 7.3.18 implies

$$\begin{aligned} s_\alpha \cdot \tilde{\Gamma}(z) &= (-1)^{2m} \cdot \tilde{\Gamma}(z) \\ &= (-1) \cdot \tilde{\Gamma}(z). \end{aligned}$$

Since α is long, $\wp(\Upsilon)$ and $\wp(s_\alpha \cdot \Upsilon)$ are not conjugate in $\tilde{K}$ by Proposition 7.3.14. $\square$

7.5. Definition of $\tilde{G}$. Let $\tilde{G} = \text{Spin}(p, q)$ with $p + q = 2n + 1$, $p > q$, and recall the rank of $\tilde{G}$ is even. Fix a half-integral infinitesimal character $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ and an evenly split Cartan subgroup $\tilde{H}^{\text{es}} \subset \tilde{G}$. Suppose $(\tilde{H}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple for $\tilde{G}$ with central character χ (Definition 6.2.1) and let $\tilde{G}^\vee = \widetilde{\text{Spin}}(p^\vee, q^\vee)$ be the dual group (Definition 6.3.4) for $\tilde{G}$ and $(\tilde{H}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$. Define maps $\wp, \wp^\vee$ for $\tilde{G}$ and $\tilde{G}^\vee$, respectively, (Definition 7.4.5) and choose a bijection

$$\tau : \text{pc}_\lambda^\chi(p, q) \rightarrow \text{pc}_\lambda^{\chi^\vee}(p^\vee, q^\vee)$$

on the corresponding principal classes (Definition 7.4.1). Let $\theta \in W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ be an involution and suppose $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| \neq 0$. Corollary 6.3.2 and Theorem 6.3.8 imply

$$|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = |\tilde{\mathcal{D}}_\lambda^{-\theta}(\chi^\vee)(p^\vee, q^\vee)| \in \{1, 2\}.$$

Definition 7.5.1. In the above setting, let $v \in \tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)$ and suppose $(\theta, \varepsilon, \eta, \lambda)$ is the corresponding abstract bigrading. If $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 1$, define $\Psi(v) = \omega$, where ω is the unique element in $\tilde{\mathcal{D}}_\lambda^{-\theta}(\chi^\vee)(p^\vee, q^\vee)$. If $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 2$, define $\Psi(v) = \omega$, where ω is the unique element in $\tilde{\mathcal{D}}_\lambda^{-\theta}(\chi^\vee)(p^\vee, q^\vee)$ with $\tau(\wp(v)) = \wp^\vee(\omega)$ (Theorem 7.4.8). In either case the abstract bigrading of $\Psi(v)$ is $(-\theta, \eta, \varepsilon, \lambda)$ (Proposition 6.3.6).

Remark 7.5.2. Definition 7.5.1 is technically incomplete when q or $q^\vee$ is less than or equal to one. If $q = 1$, then $\widetilde{\text{Spin}}(p, q)$ has representations corresponding to an even parity Cartan subgroup if and only if $\widetilde{\text{Spin}}(p, q)$ has discrete series representations. In this case it has exactly two of them and we define these to be the two principal classes for $\tilde{G}$. If $q = 0$, then $\widetilde{\text{Spin}}(p, q)$ is compact and $|\tilde{\mathcal{D}}_\lambda(\chi)(p, q)| \in \{0, 1\}$. To simplify the following formalism, we would like to have $|\tilde{\mathcal{D}}_\lambda(\chi)(p, q)| \in \{0, 2\}$ for any half-integral λ . Therefore, if $|\tilde{\mathcal{D}}_\lambda(\chi)(p, q)| \neq 0$, we create a formal ‘copy’ of the element in $\tilde{\mathcal{D}}_\lambda(\chi)(p, q)$ and consider the two corresponding elements to be the principal classes for $\widetilde{\text{Spin}}(p, q)$.

We now prove that Ψ has the desired properties with respect to the operations of the KLV-algorithm (equations ((2)) and ((3)) from the beginning of this section).

Theorem 7.5.3. *In the setting of Definition 7.5.1, let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})(\lambda)$ be an abstract integral root. If $v \in \tilde{\mathcal{D}}_\lambda(\chi)(p, q)$, then*

$$\Psi(s_\alpha \times v) = s_\alpha \times \Psi(v).$$

Proof. Let $(\theta, \varepsilon, \eta, \lambda)$ be the abstract bigrading for v . We have

$$-(s_\alpha \theta s_\alpha^{-1}) = s_\alpha(-\theta)s_\alpha^{-1}$$

and the result holds on the level of involutions (Proposition 5.1.5). If $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 1$, the elements v and $\Psi(v)$ are determined by their involutions and we are done.

If $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 2$, we need to show $\tau(\wp(s_\alpha \times v)) = \wp^\vee(s_\alpha \times \Psi(v))$. Since $\tau(\wp(v)) = \wp^\vee(\Psi(v))$ by definition, it suffices to show

$$\wp(s_\alpha \times v) = \wp(v) \iff \wp(s_\alpha \times \Psi(v)) = \wp(\Psi(v)).$$

The proof is by cases (for a change) based on the type of α for v .

Case I. Suppose α is imaginary and compact for v . Then α is real for $\Psi(v)$ and does not satisfy the parity condition. Therefore,

$$\begin{aligned} s_\alpha \times v &= v, \\ s_\alpha \times \Psi(v) &= \Psi(v), \end{aligned}$$

and we have

$$\begin{aligned} \wp(s_\alpha \times v) &= \wp(v), \\ \wp(s_\alpha \times \Psi(v)) &= \wp(\Psi(v)), \end{aligned}$$

by Proposition 7.4.7.

Case II. Suppose α is imaginary and noncompact for v . Proposition 4.1.3 implies α is short and we know α is real for $\Psi(v)$ and satisfies the parity condition. Then

$$\begin{aligned} s_\alpha \times v &\neq v, \\ s_\alpha \times \Psi(v) &\neq \Psi(v), \end{aligned}$$

by Corollaries 7.3.20 and 7.3.15. Therefore

$$\begin{aligned} \wp(s_\alpha \times v) &\neq \wp(v), \\ \wp(s_\alpha \times \Psi(v)) &\neq \wp(\Psi(v)), \end{aligned}$$

by Theorem 7.4.8.

Case III. Suppose α is short and complex for v . Choose a genuine triple $(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$ representing v and recall the number n_{cr}^θ from Definition 7.4.4. It is easy to check

$$n_{cr}^{s_\alpha \times \theta} = n_{cr}^\theta \pm 1.$$

If $m = (\lambda, \alpha^\vee)$, Corollary 7.3.17 and Proposition 7.3.14 imply

$$\begin{aligned} \wp(s_\alpha \times v) &= \wp(v) \\ \iff (-1)^m \tilde{\Gamma}(z) \tilde{\Gamma}(\tilde{m}_\gamma)^{n_{cr}^\theta + 1} &= \tilde{\Gamma}(z) \tilde{\Gamma}(\tilde{m}_\gamma)^{n_{cr}^\theta} \tilde{\Gamma}(\tilde{m}_\gamma) \\ \iff (-1)^m &= 1. \end{aligned}$$

By the same argument we also have $\wp(s_\alpha \times \Psi(v)) = \wp(\Psi(v)) \iff (-1)^m = 1$ and the result follows.

Case IV. Suppose α is long and complex for v . This is handled in the same fashion as Case III and is left to the reader. $\square$

Remark 7.5.4. As we noted in Remark 7.4.6, Case III in the above proof explains the need for the twist factor n_{cr}^θ in Definition 7.4.5; without it the statement becomes

$$\wp(s_\alpha \times v) = \wp(v) \iff (-1)^m = \tilde{\Gamma}(\tilde{m}_\gamma)$$

which depends on the relation between the infinitesimal and central characters. Since this relation is reversed by duality, the desired result fails in this case. The term $\tilde{\Gamma}(\tilde{m}_\gamma)^{n_{cr}^\theta}$ detects this on the level of involutions and alters exactly one side of the equation.

Theorem 7.5.5. *Let $v \in \tilde{\mathcal{D}}_\lambda(\chi)(p, q)$ and suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^{abs})$ is an abstract simple root that is imaginary and noncompact. Then α is real for $\Psi(v)$ and we have*

$$\Psi(\mathcal{C}^\alpha(v)) = \mathcal{C}_\alpha(\Psi(v)).$$

Note this is an equality of sets (Definitions 7.2.4 and 7.2.7).

Proof. Let $(\theta, \varepsilon, \eta, \lambda)$ be the abstract bigrading for v . Since $(-\theta, \eta, \varepsilon, \lambda)$ is the abstract bigrading for $\Psi(v)$, α is real for $\Psi(v)$ and satisfies the parity condition.

Case I. Suppose $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 1$ and α is long. Then $\mathcal{C}^\alpha(v)$ must be the unique element of $\tilde{\mathcal{D}}_\lambda^{s_\alpha \theta}(\chi)(p, q)$. Similarly, $\mathcal{C}_\alpha(\Psi(v))$ is the unique element of $\tilde{\mathcal{D}}_\lambda^{s_\alpha(-\theta)}(\chi)(p, q) = \tilde{\mathcal{D}}_\lambda^{-s_\alpha(\theta)}(\chi)(p, q)$ and the result follows.

Case II. Suppose $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 1$ and α is short. Using Proposition 5.2.4, it is easy to check that α must be of type II. Since $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 1$, we must have $s_\alpha \times v = v$ and Proposition 7.3.19 implies

$$\mathcal{C}^\alpha(v) = \{\omega_+, \omega_-\} \subset \tilde{\mathcal{D}}_\lambda^{s_\alpha(\theta)}(\chi)(p, q)$$

is double valued. In particular, $\Psi(\omega_+)$ and $\Psi(\omega_-)$ are exactly the elements in $\tilde{\mathcal{D}}_\lambda^{s_\alpha(-\theta)}(\chi)(p, q)$ with the same central character as $\Psi(v)$. Therefore, it suffices to show $\mathcal{C}_\alpha(\Psi(v))$ is double valued and this follows immediately from Corollary 7.3.20.

Case III. Suppose $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 2$ and α is short. Then both $\mathcal{C}^\alpha(v)$ and $\mathcal{C}_\alpha(\Psi(v))$ are single valued and the result follows for the same reasons as in Case I.

Case IV. Suppose $|\tilde{\mathcal{D}}_\lambda^\theta(\chi)(p, q)| = 2$ and α is long. Choose a representative $\Upsilon = (\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$ for v with $\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}} \subset \tilde{G}$ an even parity Cartan subgroup. Proposition 7.2.5 implies $\mathcal{C}^\alpha(v)$ is single valued and we have

$$\begin{aligned} W_i(\theta) &\cong W(B_{n_s^\theta}) \times W(A_1)^l, \\ W(G, \tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}) &\cong ((A \ltimes W_{ic}(\theta)) \times W_{\mathbb{R}}(\theta)) \rtimes W_{\mathbb{C}}^\theta. \end{aligned}$$

Suppose first that α is an element of the A_1^l factor of $\Delta_i^\theta(\mathfrak{g}, \mathfrak{h}^{abs})$. Then $j > k$ and conjugation in $W_{\mathbb{C}}^\theta$ allows us to choose the representative Υ such that

$$\bar{\alpha} = i_\phi(\alpha) = \mathcal{C}_j^k(\pm \alpha_j).$$

In particular, $\mathcal{C}^{\overline{\alpha}}(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}) = \tilde{H}_{\mathfrak{c}_{j-1}^k}^{\text{es}}$ is an even parity Cartan subgroup of $\tilde{G}$. Since α is long, Definition 7.2.2 directly implies

$$\wp(\Upsilon) = \wp(\mathcal{C}^{\overline{\alpha}}(\Upsilon))$$

so that

$$\wp(v) = \wp(\mathcal{C}^{\alpha}(v)).$$

A similar argument holds for $\Psi(v)$ and ultimately gives

$$\wp(\Psi(v)) = \wp(\mathcal{C}_{\alpha}(\Psi(v)))$$

and the result follows.

Now suppose α is an element of the $W(B_{n_s^{\theta}})$ factor of $\Delta_i^{\theta}(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$. Then $k > 0$ and conjugation in $W_{ic}(\theta)$ allows us to choose the representative Υ such that

$$\overline{\alpha} = i_{\phi}(\alpha) = \mathcal{C}_j^k(\pm\alpha_k).$$

In particular, $\mathcal{C}^{\overline{\alpha}}(\tilde{H}_{\mathfrak{c}_j^k}^{\text{es}}) = \tilde{H}_{\mathfrak{c}_{j-1}^k}^{\text{es}}$ is an even parity Cartan subgroup of $\tilde{G}$ and we proceed as above. $\square$

We will need an extension of Theorem 7.5.3 to the full abstract Weyl group $W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$. The precise statement requires a bit of setup. Let $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ be a half-integral infinitesimal character and suppose $\mathcal{F}(\lambda)$ is a family for λ (Definition 7.1.2). For each $\kappa \in \mathcal{F}(\lambda)$ we define a map $\wp_{\kappa} : \tilde{\mathcal{D}}_{\kappa}(\chi)(p, q) \rightarrow \text{pc}_{\kappa}^{\chi}(p, q)$ as in Definition 7.4.5 and let

$$\wp' = \coprod_{\kappa \in \mathcal{F}(\lambda)} \wp_{\kappa} : \coprod_{\kappa \in \mathcal{F}(\lambda)} \tilde{\mathcal{D}}_{\kappa}(\chi)(p, q) \rightarrow \coprod_{\kappa \in \mathcal{F}(\lambda)} \text{pc}_{\kappa}^{\chi}(p, q).$$

Identifying $\text{pc}_{\kappa}^{\chi}(p, q)$ with $\text{pc}_{\lambda}^{\chi}(p, q)$ in the obvious way gives a map

$$\wp : \coprod_{\kappa \in \mathcal{F}(\lambda)} \tilde{\mathcal{D}}_{\kappa}(\chi)(p, q) \rightarrow \text{pc}_{\lambda}^{\chi}(p, q).$$

Construct a map $\wp^{\vee}$ for $\tilde{G}^{\vee}$ in a similar fashion and fix a bijection

$$\tau : \text{pc}_{\lambda}^{\chi}(p, q) \rightarrow \text{pc}_{\lambda}^{\chi^{\vee}}(p^{\vee}, q^{\vee})$$

as above. Define maps $\Psi_{\kappa} : \tilde{\mathcal{D}}_{\kappa}(\chi)(p, q) \rightarrow \tilde{\mathcal{D}}_{\kappa}(\chi^{\vee})(p^{\vee}, q^{\vee})$ for $\kappa \in \mathcal{F}(\lambda)$ (Definition 7.5.1) and set

$$\Psi = \coprod_{\kappa \in \mathcal{F}(\lambda)} \Psi_{\kappa} : \coprod_{\kappa \in \mathcal{F}(\lambda)} \tilde{\mathcal{D}}_{\kappa}(\chi)(p, q) \rightarrow \coprod_{\kappa \in \mathcal{F}(\lambda)} \tilde{\mathcal{D}}_{\kappa}(\chi^{\vee})(p^{\vee}, q^{\vee}).$$

Theorem 7.5.6. *In the notation above, let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ be a long abstract root and suppose $s_{\alpha} \notin W(\lambda)$. If $\kappa \in \mathcal{F}(\lambda)$ and $v \in \tilde{\mathcal{D}}_{\kappa}(\chi)(p, q)$, then*

$$\Psi(s_{\alpha} \times v) = s_{\alpha} \times \Psi(v).$$

Proof. Let $(\theta, \varepsilon, \eta, \kappa)$ be the abstract bigrading for v . On the level of involutions, the result follows as in Theorem 7.5.3. If $|\tilde{\mathcal{D}}_{\kappa}^{\theta}| = 1$, the elements v and $\Psi(v)$ are determined by their involutions and we are done.

If $|\tilde{\mathcal{D}}_{\kappa}(\chi)(p, q)| = 2$, we again need to show

$$\wp(s_{\alpha} \times v) = \wp(v) \iff \wp(s_{\alpha} \times \Psi(v)) = \wp(\Psi(v))$$

and Proposition 4.1.3 implies it suffices to check this for complex and noncompact imaginary roots. The details are as in Theorem 7.5.3 and are omitted. $\square$

7.6. Character multiplicity duality. Let $\lambda \in (\mathfrak{h}^{\text{abs}})^*$ be a half-integral infinitesimal character. We begin with one final definition.

Definition 7.6.1 ([11], Definition 6.7). Let θ be an involution in $W(\mathfrak{g}, \mathfrak{h}^{\text{abs}})$ and recall that Δ^+ denotes the set of abstract roots that are positive for λ . The *length* of θ is defined to be

$$\ell(\theta) = \frac{1}{2} |\{\alpha \in \Delta^+ \mid \theta(\alpha) \notin \Delta^+\}| + \frac{1}{2} \dim(\theta_{-1})$$

where θ_{-1} is the negative eigenspace for θ . If $v \in \tilde{\mathcal{D}}_\lambda(p, q)$ and $(\theta, \varepsilon, \lambda)$ is the corresponding abstract triple, we define $\ell(v) = \ell(\theta)$.

Proposition 7.6.2. *In the setting of Definition 7.6.1, $\ell(v) \in \mathbb{N}^+$.*

Proof. It suffices to consider the case where $\tilde{G}$ is split. Suppose v is a principal series for $\tilde{G}$ so that $\theta = -1$ and let $n > 0$ denote the rank of $\tilde{G}$. One easily verifies

$$\begin{aligned} \ell(v) &= \frac{1}{2}n^2 + \frac{1}{2}n \\ &= \frac{n(n+1)}{2} = \binom{n+1}{2} \end{aligned}$$

and the result follows for v . We now proceed as in [14], Lemma 8.6.13. $\square$

Let $\tilde{G} = \text{Spin}(p, q)$ with $p + q = 2n + 1$, $p > q$, and recall the rank of $\tilde{G}$ is even. Suppose $(\tilde{H}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$ is a genuine triple for $\tilde{G}$ with central character χ (Definition 6.2.1) and let $\tilde{G}^\vee = \text{Spin}(p^\vee, q^\vee)$ be the dual group (Definition 6.3.4) for $\tilde{G}$ and $(\tilde{H}^{\text{es}}, \phi, \tilde{\Gamma})_\lambda$. Define the map

$$\Psi : \tilde{\mathcal{D}}_\lambda(\chi)(p, q) \rightarrow \tilde{\mathcal{D}}_\lambda(\chi^\vee)(p^\vee, q^\vee)$$

as in Definition 7.5.1. The following proposition describes the effect of Ψ on length.

Proposition 7.6.3. *Let $v \in \tilde{\mathcal{D}}_\lambda(\chi)(p, q)$ and suppose $(\theta, \varepsilon, \lambda)$ is the abstract triple for v . Then $\ell(\Psi(v)) = \binom{n+1}{2} - \ell(v)$.*

Proof. The proof is a straightforward calculation and is left to the reader. $\square$

Fix a family $\mathcal{F}(\lambda)$ of infinitesimal characters for λ and let $\mathcal{B} = \{\gamma_1, \dots, \gamma_r\} = \tilde{\mathcal{D}}_\lambda(\chi)(p, q)$. If $\delta_i = \Psi(\gamma_i)$, write $\mathcal{B}' = \{\delta_1, \dots, \delta_r\} = \tilde{\mathcal{D}}_\lambda(\chi^\vee)(p^\vee, q^\vee)$ and extend the map Ψ (and thus the sets $\mathcal{B}$ and $\mathcal{B}'$) as in Theorem 7.5.6. Let $\mathcal{M}$ (respectively $\mathcal{M}'$) denote the free $\mathbb{Z}[q, q^{-1}]$ module with basis $\mathcal{B}$ (respectively $\mathcal{B}'$). As in [11], we view $\mathcal{M}$ and $\mathcal{M}'$ as Hecke modules for the extended action of the Hecke algebra $\mathcal{H}(W)$ ([11], Definition 9.4). The integer matrix M (Definition 2.2.8) for $\mathcal{B}$ (respectively $\mathcal{B}'$) is then determined from the combinatorics of $\mathcal{M}$ (respectively $\mathcal{M}'$). The interested reader is referred to [17] for a reasonably concise account of this process.

For our purposes, only the following formalism is important. Define the dual $\mathbb{Z}[q, q^{-1}]$ module

$$\mathcal{M}^* = \text{Hom}_{\mathbb{Z}[q, q^{-1}]}(\mathcal{M}, \mathbb{Z}[q, q^{-1}])$$

and extend $\mathcal{M}^*$ to an $\mathcal{H}(W)$ -module as in [11], Theorem 11.1. Write $\check{\mathcal{B}} = \{\check{\gamma}_1, \dots, \check{\gamma}_r\}$ for the dual basis of $\mathcal{M}^*$ and define the $\mathbb{Z}[q, q^{-1}]$ -linear map

$$\begin{aligned} \Phi : \mathcal{M}^* &\longrightarrow \mathcal{M}' \\ \check{\gamma}_i &\longmapsto (-1)^{\ell(\gamma_i)} \delta_i. \end{aligned}$$

Theorem 7.6.4. *In the above setting, Φ is an isomorphism of $\mathcal{H}(W)$ -modules.*

Proof. It suffices to check the equivariance of the operators in $\mathcal{H}(W)$ corresponding to simple roots. Depending on the root type and length, there are many cases to consider. For integral roots, the details are as in [15], Proposition 13.10. For strictly half-integral roots, the details are as in [11], Theorem 11.1. In each case the result is a formal consequence of Theorems 7.5.3, 7.5.5, and 7.5.6. $\square$

Theorem 7.6.5. *In the above setting*

$$(2) \quad M(\gamma_i, \overline{\gamma_j}) = (-1)^{\ell(\gamma_j) - \ell(\gamma_i)} m(\overline{\delta_j}, \delta_i).$$

Proof. This follows immediately from Theorem 7.6.4 and Lemma 13.7 of [15]. $\square$

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