DERIVATIVES AND ASYMPTOTICS OF WHITTAKER FUNCTIONS

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ABSTRACT. Let F be a p -adic field and G_n be one of the groups $GL(n, F)$, $GSO(2n-1, F)$, $GSp(2(n-1), F)$, or $GSO(2(n-1), F)$. Using the mirabolic subgroup or analogues of it, and related “derivative” functors, we give an asymptotic expansion of functions in the Whittaker model of generic representations of G_n , with respect to a minimal set of characters of subgroups of the maximal torus. Denoting by Z_n the center of G_n and by N_n the unipotent radical of its standard Borel subgroup, we characterize generic representations occurring in $L^2(Z_n N_n \backslash G_n)$ in terms of these characters.

This is related to a conjecture of Lapid and Mao for general split groups, asserting that the generic representations occurring in $L^2(Z_n N_n \backslash G_n)$ are the generic discrete series; we prove it for the group G_n .

INTRODUCTION

Let G_n be the points of one of the groups $GL(n)$, $GSO(2n-1)$, $GSp(2n)$, or $GSO(2(n-1))$ over a p -adic field F . The main result (Theorem 2.1) of this work describes the asymptotic behaviour of the restriction of Whittaker functions to the standard maximal torus in terms of a family of characters which is minimal in some sense. From results of [L-M], this restriction can be described for split reductive groups in terms of cuspidal exponents.

Here, after having defined analogues of the mirabolic subgroup for the groups G_n and the corresponding derivative functors, following [C-P] (where the case of completely reducible derivatives is treated for $GL(n)$), we choose to describe the restriction of Whittaker functions to the torus in terms of central exponents of the derivatives.

This description, inspired by [B], is better adapted to understanding when the Whittaker model of a unitary generic representation is a subspace of $L^2(Z_n N_n \backslash G_n)$ (these representations are conjectured to be generic discrete series by Lapid and Mao).

In the first section, we review the groups in question and define their mirabolic subgroups. We also give a decomposition of the unipotent radical of the standard Borel subgroup and a description of how nondegenerate characters of this radical behave with respect to this decomposition.

In Section 2, we give properties of the derivative functors and use them to prove our asymptotic expansion of Whittaker functions, which is Theorem 2.1.

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In Section 3, we characterize generic representations with Whittaker model included in $L^2(Z_n N_n \backslash G_n)$ in terms of central exponents of the derivatives, in Corollary 3.1. We then prove in Theorem 3.2 the conjecture 3.5 of [L-M].

After the writing of this paper, Patrick Delorme informed the author that he obtained the proof of the conjecture of Lapid and Mao in the general case (Theorem 8 of [D2]).

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1. MIRABOLIC SUBGROUP AND NONDEGENERATE CHARACTERS

Let F be a nonarchimedean local field. We denote by $\mathfrak{O}_F$ its ring of integer and by $\mathfrak{P}_F = \varpi_F \mathfrak{O}_F$ the maximal ideal of this ring, where ϖ_F is a uniformiser of F .

We give a list of groups and describe some of their properties which will be used in the sequel:

• Case A:

The group G_0 is trivial and, for $n \geq 1$, the group G_n is $GL(n, F)$.

We consider the maximal torus of G_n consisting of diagonal matrices; it is isomorphic to $(F^*)^n$.

For $n \geq 2$, the simple roots of this group can be chosen to be the characters

$$\alpha_i(\text{diag}(x_1, \dots, x_n)) = x_i x_{i+1}^{-1}$$

for i between 1 and $n - 1$.

The root subgroup U_{α_i} is given by matrices of the form $I_n + x E_{i, i+1}$ for x in F .

Standard Levi subgroups of G_n are given by matrices of the form $\text{diag}(a_1, \dots, a_r)$ where a_i belongs to $GL(n_i, F)$, with $n_1 + \dots + n_r = n$. We denote the preceding group by $M_{(n_1, \dots, n_r)}$, and the corresponding standard parabolic subgroup is denoted by $P_{(n_1, \dots, n_r)}$, with unipotent radical $U_{(n_1, \dots, n_r)}$.

For $n \geq 2$ we denote by U_n the group $U_{(n-1, 1)}$ of matrices of the form

$$\begin{bmatrix} I_{n-1} & V \\ & 1 \end{bmatrix}.$$

It is isomorphic to F^{n-1} .

For $n > k \geq 1$, the group G_k embeds naturally in G_n and is given by matrices of the form $\text{diag}(g, I_{n-k})$; we denote by Z_k its center.

We denote by P_n the mirabolic subgroup $G_{n-1} \ltimes U_n$ of G_n .

• Case B:

The group G_0 is trivial, and the group G_1 is F^* .

For $n \geq 2$, the group $G_n = GO(2n - 1, F)$ is the group of matrices g in $GL(2n - 1, F)$ such that ${}^t g J g$ belongs to $F^* J$, where J is the antidiagonal matrix of $GL(2n - 1, F)$ with ones on the second diagonal. We call the multiplier of an element g in G_n the scalar $\mu(g)$ such that ${}^t g J g$ is equal to $\mu(g) J$ (one checks that for this group, the multiplier actually belongs to F^{*^2}). It is the direct product of $SO(2n - 1, F)$ with F^* , more precisely, of $SO(2n - 1, F)$ and the group $I(F^*)$ of matrices $I(t) = t I_{2n-1}$ for t in F^* .

The maximal torus of G_n is equal to the product of the torus of matrices of the form $\text{diag}(x_{n-1}^{-1}, \dots, x_1^{-1}, 1, x_1, \dots, x_{n-1})$ with $I(F^*)$ and is isomorphic to $(F^*)^n$.

For $n \geq 3$, the simple roots of this group can be chosen to be the characters

$$\alpha_{i+1}(\text{diag}(tx_{n-1}^{-1}, \dots, tx_1^{-1}, t, tx_1, \dots, tx_{n-1})) = x_i x_{i+1}^{-1},$$

for i between 1 and $n-2$, and $\alpha_1(\text{diag}(tx_{n-1}^{-1}, \dots, tx_1^{-1}, t, tx_1, \dots, tx_{n-1})) = x_1^{-1}$.

The root subgroups U_{α_i} are given by matrices of the form

$$\text{diag}(1, \dots, 1, u, 1, \dots, 1, u^{-1}, 1, \dots, 1)$$

for matrices u in the unipotent radical of the Borel of $GL(2, F)$.

For $n > k \geq 1$, the group G_k embeds naturally in G_n and is given by matrices of the form $\text{diag}(\mu(g)I_{n-k}, g, I_{n-k})$, where $\mu(g)$ is the multiplier of the element g of G_k .

Its center Z_k is given by matrices of the form

$$z_k(t) = \text{diag}(t^2 I_{n-k}, t I_{2k-1}, I_{n-k})$$

for t in K^* .

For $n \geq 1$, the standard Levi subgroups of G_n are given by matrices of the form

$$\text{diag}(\mu(g)^{\tau} a_{r-1}^{-1}, \dots, \mu(g)^{\tau} a_1^{-1}, g, a_1, \dots, a_r),$$

where a_i belongs to $GL(n_i, F)$, τa is the transpose of a with respect to the second diagonal, and g belongs to G_m , with $2m-1+2n_1+\dots+2n_r=2n-1$. We denote the preceding group by $M_{(m;n_1, \dots, n_r)}$, and the corresponding standard parabolic subgroup consisting of block upper triangular matrices is denoted by $P_{(m;n_1, \dots, n_r)}$, with unipotent radical $U_{(m;n_1, \dots, n_r)}$.

For $n \geq 2$ we denote by U_n the group $U_{(n-1;1)}$, of matrices of the form

$$\begin{bmatrix} 1 & -{}^{\tau}V & -{}^{\tau}VV/2 \\ & I_{2n-3} & V \\ & & 1 \end{bmatrix}.$$

It is isomorphic to F^{2n-3} .

For $n \geq 2$, we denote by P_n the “mirabolic” subgroup $G_{n-1} \ltimes U_n$ of G_n .

• **Case C:**

The group G_0 is trivial, and the group G_1 is F^* .

For $n \geq 2$, the group $G_n = GSp(2(n-1), F)$, where $GSp(2(n-1), F)$ is the group of matrices g in $GL(2(n-1), F)$ such that ${}^t g J g$ belongs to $F^* J$, where $J = \begin{bmatrix} 0 & W \\ -W & 0 \end{bmatrix}$ and W is the antidiagonal matrix of $GL(n-1, F)$ with ones on the second diagonal. It is the semi-direct product of $Sp(2(n-1), F)$ with F^* , more precisely, of $Sp(2n, F)$ and the group $I(F^*)$ of matrices $I(t) = \text{diag}(tI_{n-1}, I_{n-1})$ for t in F^* .

The maximal torus of G_n is equal to the product of the torus of matrices of the form $\text{diag}(x_{n-1}^{-1}, \dots, x_1^{-1}, x_1, \dots, x_{n-1})$ with $I(F^*)$ and is isomorphic to $(F^*)^n$.

The simple roots of this group are the characters

$$\alpha_{i+1}(\text{diag}(tx_{n-1}^{-1}, \dots, tx_1^{-1}, x_1, \dots, x_{n-1})) = x_i x_{i+1}^{-1},$$

for i between 1 and $n-2$, and

$$\alpha_1(\text{diag}(tx_{n-1}^{-1}, \dots, tx_1^{-1}, x_1, \dots, x_{n-1})) = tx_1^{-2}.$$

For i more than 1, the root subgroup U_{α_i} is given by matrices of the form

$$\text{diag}(1, \dots, 1, u, 1, \dots, 1, u^{-1}, 1, \dots, 1),$$

for matrices u in the unipotent radical of the Borel of $GL(2, F)$, whereas U_{α_1} is given by matrices $\text{diag}(1, \dots, 1, u, 1, \dots, 1)$, for matrices u in the unipotent radical of the Borel of $GL(2, F)$.

For $n > k \geq 2$, the group G_k embeds naturally in G_n and is given by matrices of the form $\text{diag}(\mu(g)I_{n-k}, g, I_{n-k})$, where $\mu(g)$ is the multiplier of the element g of G_k . Its center Z_k is given by matrices of the form $z_k(t) = \text{diag}(t^2 I_{n-k}, t I_{2(k-1)}, I_{n-k})$ for t in F^* . The group G_1 , which is equal to its center Z_1 , embeds as $I(F^*)$.

For $n \geq 1$, the standard Levi subgroups of G_n are:

– either matrices

$$\text{diag}(\mu(g)^{\tau} a_{r-1}^{-1}, \dots, \mu(g)^{\tau} a_1^{-1}, g, a_1, \dots, a_r),$$

where a_i belongs to $GL(n_i, F)$ and g belongs to G_m with $m \geq 2$, with $2(m-1) + 2n_1 + \dots + 2n_r = 2(n-1)$. We denote the preceding group by $M_{(m; n_1, \dots, n_r)}$, and the corresponding standard parabolic subgroup consisting of block upper triangular matrices is denoted by $P_{(m; n_1, \dots, n_r)}$, with unipotent radical $U_{(m; n_1, \dots, n_r)}$.

– or the matrices

$$z_1 \cdot \text{diag}(a_{r-1}^{-1}, \dots, a_1^{-1}, a_1, \dots, a_r),$$

with a_i in $GL(n_i, F)$, $2n_1 + \dots + 2n_r = 2(n-1)$, and z_1 in G_1 .

For $n \geq 2$, we denote by U_n the group $U_{(n-1; 1)}$ of matrices of the form

$$\begin{bmatrix} 1 & -{}^{\tau}V_2 & {}^{\tau}V_1 & x \\ & I_{n-2} & & V_1 \\ & & I_{n-2} & V_2 \\ & & & 1 \end{bmatrix}.$$

For $n \geq 3$, it is an extension of $F^{2(n-2)}$ by F , which is the Heisenberg group corresponding to the alternating bilinear form on $F^{2(n-2)}$, given by $(W_1, W_2) \times (V_1, V_2) \mapsto -{}^{\tau}W_2 \cdot V_1 + {}^{\tau}W_1 \cdot V_2$.

It is a two-step nilpotent subgroup, with center equal to its derived subgroup, given by matrices with $V_1 = V_2 = 0$, and the maximal abelian quotient U_n^{ab} of U_n is $F^{2(n-2)}$.

The group U_2 is the unipotent radical of the standard Borel of $G_2 = GSp(2, F) = GL(2, F)$ and is isomorphic to $(F, +)$.

For $n \geq 2$, we denote by P_n the “mirabolic” subgroup $G_{n-1} \ltimes U_n$ of G_n .

• Case D:

We denote by G_0 the trivial group. The group G_1 is F^* .

For $n \geq 2$, the group $G_n = GSO(2(n-1), F)$ is the group of matrices g in $GL(2(n-1), F)$ satisfying the fact that ${}^t g J g$ belongs to $F^* J$, where

J is the antidiagonal matrix of $GL(2(n-1), F)$ with ones on the second diagonal. It is the semi-direct product of $SO(2(n-1), F)$ with the group $I(F^*)$, of the matrices $I(t) = \text{diag}(tI_{n-1}, I_{n-1})$ for t in F^* .

The maximal torus of G_n is equal to the product of the torus of matrices of the form $\text{diag}(x_{n-1}^{-1}, \dots, x_1^{-1}, x_1, \dots, x_{n-1})$ with $I(F^*)$, and it is isomorphic to $(F^*)^n$.

If n is 2, then G_2 is the diagonal torus of $GL(2, F)$.

For $n \geq 3$, the simple roots of this group can be chosen to be the characters

$$\alpha_{i+1}(\text{diag}(tx_{n-1}^{-1}, \dots, tx_1^{-1}, x_1, \dots, x_{n-1})) = x_i x_{i+1}^{-1},$$

for i between 1 and $n-2$, and $\alpha_1(\text{diag}(tx_n^{-1}, \dots, tx_1^{-1}, x_1, \dots, x_n)) = tx_1^{-1}x_2^{-1}$.

For $i \geq 1$, the root subgroup $U_{\alpha_{i+1}}$ is given by matrices of the form

$$\text{diag}(1, \dots, 1, u, 1, \dots, 1, u^{-1}, 1, \dots, 1)$$

for matrices u in the unipotent radical of the Borel of $GL(2, F)$, whereas U_{α_1} is given by matrices $\text{diag}(1, \dots, 1, u, 1, \dots, 1)$ for matrices u of the form

$$\begin{bmatrix} 1 & & y & \\ & 1 & & -y \\ & & 1 & \\ & & & 1 \end{bmatrix}, \text{ with } y \text{ in } F.$$

For $n > k \geq 2$, the group G_k embeds naturally in G_n and is given by matrices of the form $\text{diag}(\mu(g)I_{n-k}, g, I_{n-k})$, where $\mu(g)$ is the multiplier of the element g of G_k . For $k \geq 3$, its center Z_k is given by matrices of the form $z_k(t) = \text{diag}(t^2 I_{n-k}, tI_{2(k-1)}, I_{n-k})$ for t in F^* .

We denote by Z_2 the subgroup of the torus G_2 , given by matrices of the form $z_2(t) = \text{diag}(tI_{n-2}, 1, t, I_{n-2})$ for t in F^* .

The group G_1 which is equal to its center Z_1 embeds as $I(F^*)$.

The standard Levi subgroups of G_n are the following:

– The groups given by matrices of the form

$$\text{diag}(\mu(g)^{\tau} a_r^{-1}, \dots, \mu(g)^{\tau} a_1^{-1}, g, a_1, \dots, a_r),$$

where a_i belongs to $GL(n_i, F)$, g belongs to G_m with $m \geq 3$, with $2(m-1) + 2n_1 + \dots + 2n_r = 2(n-1)$. We denote the preceding group by $M_{(m; n_1, \dots, n_r)}$, and the corresponding standard parabolic subgroup is denoted by $P_{(m; n_1, \dots, n_r)}$, with unipotent radical $U_{(m; n_1, \dots, n_r)}$.

– The groups given by matrices of the form

$$g_2 \cdot \text{diag}(\tau a_r^{-1}, \dots, \tau a_1^{-1}, 1, 1, a_1, \dots, a_r),$$

with a_i in $GL(n_i, F)$, $2n_1 + \dots + 2n_r = 2(n-2)$, and g_2 in G_2 .

– The groups given by matrices of the form

$$z_1 \cdot \text{diag}(\tau a_r^{-1}, \dots, \tau a_1^{-1}, a_1, \dots, a_r),$$

with a_i in $GL(n_i, F)$, $2n_1 + \dots + 2n_r = 2(n-1)$, and z_1 in G_1 .

– The groups given by matrices of the form

$$z_2 \cdot \text{diag}(\tau a_r^{-1}, \dots, \tau a_1^{-1}, g, a_1, \dots, a_r),$$

where a_i belongs to $GL(n_i, F)$, and g in $GL(2(m-1), F)$ is of the form

$$\begin{bmatrix} A & & V & \\ & t' & & L' \\ L & & t & \\ & V' & & A' \end{bmatrix},$$

with

$$\begin{bmatrix} A & V \\ L & t \end{bmatrix} \in GL(m-1, F),$$

and A a $(m-2 \times m-2)$ -matrix,

$$\begin{bmatrix} {}^\tau A' & {}^\tau L' \\ {}^\tau V' & t' \end{bmatrix} = \begin{bmatrix} A & V \\ L & t \end{bmatrix}^{-1},$$

with $2m + 2n_1 + \dots + 2n_r = 2n$ and z_2 in Z_2 .

For $n \geq 3$ we denote by U_n the subgroup $U_{(n-1;1)}$ of G_n , of matrices of the form

$$\begin{bmatrix} 1 & -{}^\tau V & -{}^\tau V V / 2 \\ & I_{2(n-2)} & V \\ & & 1 \end{bmatrix}.$$

It is isomorphic to F^{2n-2} .

For $n \geq 3$ denote by P_n the “mirabolic” subgroup $G_{n-1} \ltimes U_n$ of G_n .

We denote by U_2 the group U_{α_1} , and by P_2 the group $G_1 \ltimes U_2$ of G_2 .

Lemma 1.1. *We denote by Z_i the center of G_i , except in Case D when $n = 2$, where we denote by Z_2 the subgroup $\text{diag}(1, t)$ with t in F^* of G_2 . In all cases, one checks that the maximal torus A_n of G_n is the direct product $Z_1 \cdot Z_2 \dots Z_{n-1} \cdot Z_n$, and each Z_i is isomorphic to F^* . Moreover, the i -th root has the property that $\alpha_i(z_1 \dots z_n) = z_i$; in other words, these coordinates parametrize the torus A_n such that simple roots become canonical projections.*

The unipotent radical N_{n+1} of the standard Borel subgroup of G_{n+1} is equal to $U_2 \dots U_{n+1}$. Let θ be a nondegenerate character of N_{n+1} (i.e. that restricts nontrivially to any of the simple root subgroups). We denote by θ_{i+1} the character $\theta|_{U_{i+1}}$, except in the Case D, for $i = 2$. In this case $U_3 = U_{\alpha_1} \times U_{\alpha_2}$, and we denote by θ_3 the character $\theta_3(u_{\alpha_1} u_{\alpha_2}) = \theta(u_{\alpha_2})$.

Because θ is trivial on U_{der} , and according to the description of U_{der} in Theorem 4.1 of [B-H], the character θ_{i+1} must be trivial on every root subgroup U_α contained in U_{i+1} such that α is not simple. Moreover, for Case D, $n = 2$, the character θ_3 is trivial on U_{α_1} .

Conversely, if a nontrivial character θ_{i+1} of U_{i+1} is trivial on every $U_\alpha \subset U_{i+1}$ which is not simple, and if, in Case D, $n = 2$, we impose in addition that θ_3 is trivial on U_{α_1} , then one checks that the normalizer of θ_{i+1} in the mirabolic subgroup P_{i+1} is $P_i U_{i+1}$. As the group $U_2 \dots U_i$ is a subgroup of P_i , a family of nontrivial characters θ_{i+1} of U_{i+1} , trivial on every $U_\alpha \subset U_{i+1}$ except U_{α_i} , defines a nondegenerate character of $N_{n+1} = U_2 \dots U_{n+1}$ by $\theta(u_2 \dots u_{n+1}) = \prod_{i=1}^n \theta_{i+1}(u_{i+1})$.

Now we fix such a nondegenerate character θ and write θ^k for the character $\theta_2 \dots \theta_k$ of N_k .

2. DERIVATIVES AND WHITTAKER FUNCTIONS

If G is an l -group, we denote by $Alg(G)$ the category of smooth complex G -modules. If (π, V) belongs to $Alg(G)$, H is a closed subgroup of G , and χ is a character of H , we denote by $V(H, \chi)$ the subspace of V generated by vectors of the form $\pi(h)v - \chi(h)v$ for h in H and v in V . This space is actually stable under the action of the subgroup $N_G(\chi)$ of the normalizer $N_G(H)$ of H in G , which fixes χ .

We denote by δ_H the positive character of $N_G(H)$ such that if μ is a right Haar measure on G and λ is the left translation of smooth functions with compact support on G , then $\mu \circ \lambda(n^{-1}) = \delta_H(n)\mu$ for n in N .

This gives the spaces $V(H, \chi)$ and $V_{H, \chi} = V/V(H, \chi)$ (that we simply denote by V_H when χ is trivial) a structure of smooth $N_G(\chi)$ -modules.

Notation being as in the first section, and for k an integer between 2 and n , we define the following functors:

- First we recall the definition of the Jacquet functors:
Let P be a parabolic subgroup of G_n , with Levi subgroup M , and unipotent radical U .
We denote by J_P the functor from $Alg(G_n)$ to $Alg(M)$ such that if (π, V) is a smooth G_n -module, we have $J_P(V) = V_U$, and M acts on $J_P(V)$ by $J_P\pi(m)(v + V(U, 1)) = \delta_U(m)^{-1/2}\pi(m)v + V(U, 1)$.
- With the same notation, we denote by i_P^G the functor from $Alg(M)$ to $Alg(G_n)$ such that if ρ is a smooth M -module, and $\bar{\rho}$ is the corresponding P -module obtained by inflation of ρ to P , then $i_P^G(\rho)$ is the G_n -module $ind_P^{G_n}(\bar{\rho})$, where ind is the usual normalized compact induction.
- The functor $\Phi_{\theta_k}^-$ (denoted r_{U_k, θ_k} in section 1 of [B-Z.2]) from $Alg(P_k)$ to $Alg(P_{k-1})$ such that if (π, V) is a smooth P_k -module, $\Phi_{\theta_k}^- V = V_{U_k, \theta_k}$, and P_{k-1} acts on $\Phi_{\theta_k}^-(V)$ by $\Phi_{\theta_k}^- \pi(p)(v + V(U_k, \theta_k)) = \delta_{U_k}(p)^{-1/2}\pi(p)(v + V(U_k, \theta_k))$.
- The functor $\Phi_{\theta_k}^+$ (denoted i_{U_k, θ_k} in section 1 of [B-Z.2]) from $Alg(P_{k-1})$ to $Alg(P_k)$ such that for π in $Alg(P_{k-1})$, one has $\Phi_{\theta_k}^+ \pi = ind_{P_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \theta_k)$, where ind is the usual compact induction.
- The functor $\hat{\Phi}_{\theta_k}^+$ (I_{U_k, θ_k} in section 1 of [B-Z.2]) from $Alg(P_{k-1})$ to $Alg(P_k)$ such that for π in $Alg(P_{k-1})$, one has $\hat{\Phi}_{\theta_k}^+ \pi = Ind_{P_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \theta_k)$, where Ind is the usual induction.
- The functor Ψ^- is the Jacquet functor J_{U_k} (denoted $r_{U_k, 1}$ in section 1 of [B-Z.2]) from $Alg(P_k)$ to $Alg(G_{k-1})$ such that if (π, V) is a smooth P_k -module, $\Psi^- V = V_{U_k, 1}$, and G_{k-1} acts on $\Psi^-(V)$ by $\Psi^- \pi(g)(v + V(U_k, 1)) = \delta_{U_k}(g)^{-1/2}\pi(g)(v + V(U_k, 1))$.
- The functor Ψ^+ (denoted $i_{U_k, 1}$ in section 1 of [B-Z.2]) from $Alg(G_{k-1})$ to $Alg(P_k)$, such that for π in $Alg(G_{k-1})$, one has $\Psi^+ \pi = ind_{G_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes 1) = \delta_{U_k}^{1/2}\pi \otimes 1$.

As we already fixed the character θ of N_n , we will most of the time forget the dependence in θ_k of $\Phi_{\theta_k}^-$ and $\Phi_{\theta_k}^+$, and we will write these functors Φ^- and Φ^+ . These functors have the following properties which follow (except for c) and d), which are trivial) from Proposition 1.9 of [B-Z.2]:

- Proposition 2.1.** a) The functors Φ^- , Φ^+ , Ψ^- , and Ψ^+ are exact.
 b) Ψ^- is left adjoint to Ψ^+ .
 b') Φ^- is left adjoint to $\hat{\Phi}^+$.
 c) $\Phi^-\Psi^+ = 0$
 d) $\Psi^-\Psi^+ = Id$.

Now we want to know how these functors restrict to smooth P_k -modules which are submodules of the space $C^\infty(N_k \backslash P_k, \theta^k) = Ind_{N_k}^{P_k}(\theta^k)$ of functions on P_k , fixed by some open subgroup of P_k under right translation, and which transform by θ^k under left translation by elements of N_k .

The next proposition shows the stability of this type of module under Φ^- and Φ^+ .

Proposition 2.2. For any submodule τ of $C^\infty(N_k \backslash P_k, \theta^k)$, the P_{k-1} -module $\Phi^-\tau$ is a submodule of $C^\infty(N_{k-1} \backslash P_{k-1}, \theta^{k-1})$, with model given by restriction of functions $\delta_{U_k}^{-1/2}W$ in τ to P_{k-1} , and such that we have $\Phi^-\tau(p)W = \rho(p)W$ for p in P_{k-1} , where ρ is the action by right translation.

Conversely, the P_{k+1} -module $\Phi^+\tau$ can be identified with a submodule of $C^\infty(N_{k+1} \backslash P_{k+1}, \theta^{k+1})$, with the natural action of P_{k+1} by right translation.

Proof. The first property will hold if we show that $C^\infty(N_k \backslash P_k, \theta^k)(U_k, \theta_k)$ is the kernel of the restriction map to $C^\infty(N_{k-1} \backslash P_{k-1}, \theta^{k-1})$. This is a straightforward adaptation of the proof of Proposition 2.1 of [C-P].

The second property is a consequence of the following equalities and inclusions:

$$\begin{aligned} \Phi^+(C^\infty(N_k \backslash P_k, \theta^k)) &= ind_{P_k U_{k+1}}^{P_{k+1}}(\delta_{U_{k+1}}^{1/2} \cdot Ind_{N_k}^{P_k}(\theta^k) \otimes \theta_{k+1}) \\ &\subset Ind_{P_k U_{k+1}}^{P_{k+1}}(\delta_{U_{k+1}}^{1/2} \cdot Ind_{N_k}^{P_k}(\theta^k) \otimes \theta_{k+1}). \end{aligned}$$

Then

$$\delta_{U_{k+1}}^{1/2} \cdot Ind_{N_k}^{P_k}(\theta^k) \simeq Ind_{N_k}^{P_k}(\theta^k)$$

because the character $\delta_{U_{k+1}}^{1/2}$ of P_k is trivial on N_k .

Finally,

$$Ind_{P_k U_{k+1}}^{P_{k+1}}(Ind_{N_k}^{P_k}(\theta^k) \otimes \theta_{k+1}) \simeq Ind_{N_{k+1}}^{P_{k+1}}(\theta^{k+1}). \quad \square$$

More can be said about smooth P_k -submodules of the space $C^\infty(N_k \backslash P_k, \theta^k) = Ind_{N_k}^{P_k}(\theta^k)$. If τ is a P_k -submodule of $C^\infty(N_k \backslash P_k, \theta^k)$, then the derived subgroup of U_k (which is trivial except in Case D) acts trivially.

To see this, take W in $C^\infty(N_k \backslash P_k, \theta^k)$. We claim that if u belongs to the derived subgroup U_k^{der} of U_k , then $\tau(u)W$ and W are equal. Hence let p belong to P_k ; one has $\tau(u)W(p) = W(pu) = W(pup^{-1}p) = \theta^k(pup^{-1})W(p)$. However, P_k normalizes U_k (so $\theta^k(pup^{-1}) = \theta_k(pup^{-1})$) so that it normalizes its derived subgroup as well; as θ_k is trivial on this subgroup, this proves our claim.

For such modules being P_k -modules, there is a nice interpretation of $V(U_k, 1)$ in terms of the analytic behaviour of Whittaker functions. First, we make the following observation.

Remark 2.1. For $k \geq 3$, as a consequence of the Iwasawa decomposition, any element g of G_{k-1} can be written in the form pzc with p in P_{k-1} , z in Z_{k-1} , and c in $K = G_{k-1}(\mathfrak{O}_F)$, and the absolute value of z depends only on g , so we can write it $|z(g)|_F$.

If a function W is in the space of $C^\infty(N_k \backslash P_k, \theta^k)$, then for g in G_{k-1} we show that $W(g)$ vanishes whenever $|z(g)|_F$ is large enough.

Indeed if we take the “natural” group isomorphism u from $(F^m, +)$ to U_k^{ab} , for some positive integer m , and recalling that it is in fact U_k^{ab} that acts on V , then $u(x)$ will fix W for x near zero in F^m .

But then, for g in G_{k-1} of the form pzc , one has $W(g) = W(gu(x)) = \theta_k(gu(x)g^{-1})W(g)$, which is equal to $\theta_k(zcu(x)(cz)^{-1})W(g)$ because P_{k-1} normalizes θ_k . This implies the equality $[\theta_k(zcu(x)(cz)^{-1}) - 1]W(g) = [\theta_k(u(zcx)) - 1]W(g) = 0$ for any x in a neighbourhood of zero depending only on W . The assertion follows easily.

Proposition 2.3. *Let (τ, V) be a P_k -submodule of $C^\infty(N_k \backslash P_k, \theta^k)$. Then the space $V(U_k, 1)$ is the subspace of V , of functions W such that there exists an integer N_W with $W(g) = 0$, for any g satisfying $|z(g)|_F \leq q_F^{-N_W}$.*

Proof. Suppose first that a function W is in $V(U_k, 1)$, so we can write it $\pi(u)W' - W'$ for some u in U_k^{ab} and some W' in V . Then, writing g as pzc and u as $u(x)$ for x in F^m , we have $[\pi(u)W' - W'](g) = [\theta_k(u(zcx)) - 1]W'(g)$, which will be zero to 0 when $|z|_F$ is close to zero.

Conversely, we use the characterization of Jacquet and Langlands, asserting that the elements W of $V(U_k, 1) = V(U_k^{ab}, 1)$ are those such that $\int_U \tau(u)W du$ is zero as soon as the open compact subgroup U of U_k^{ab} contains some compact open subgroup U_W . Therefore, suppose W is in V and that it vanishes on elements g of $G_{n-1}(F)$ satisfying $|z(g)|_F \leq q_F^{-N_W}$.

Let U be any open compact subgroup of U_k^{ab} that we identify with a subgroup of F^m . The integral $\int_U \tau(u)W du$ evaluated at $g = pzc$ is equal to $\int_{x \in U} \theta_k(zcx)W(g)dx$. Hence this integral is always zero for $|z|_F \leq q_F^{-N_W}$ because $W(g)$ is as well.

We now recall that as θ_k is a nontrivial character of U_k^{ab} , there exists a compact open ball U_0 of $U_k^{ab} \simeq F^n$ such that the integral $\int_{x \in U} \theta_k(x)dx$ is zero whenever the compact open subgroup U of U_k^{ab} contains U_0 . However, then for $|z|_F \geq q_F^{-N_W}$, if t_W is an element of F^* of absolute value $q_F^{N_W}$, the integral $\int_{x \in U} \theta_k(zcx)W(g)dx$ is also zero as soon as U contains $t_W U_0$. Hence $\int_U \tau(u)W du$ is zero when U is a compact open subgroup of U_k^{ab} containing $U_W = t_W U_0$ and when W belongs to $V(U_k, 1)$. $\square$

For any smooth P_n -module τ and any integer $k \geq 1$, we denote by $\tau_{(k)}$ the representation of P_{n-k+1} equal to $(\Phi^-)^{k-1}\tau$, and by $\tau^{(k)}$ the representation of G_{n-k} equal to $\Psi^-(\Phi^-)^{k-1}\tau = \Psi^-\tau_{(k)}$.

We say that a smooth irreducible representation π of G_n is θ^n -generic if it is isomorphic to a submodule of the induced representation $\text{Ind}_{N_n}^{G_n}(\theta^n)$. If it is the case, the submodule of $\text{Ind}_{N_n}^{G_n}(\theta^n)$ isomorphic to π is unique, and it is called the Whittaker model of π and is denoted by $W(\pi, \theta^n)$.

Now we let (π, V) be a θ^n -generic representation of G_n (hence a smooth P_n -module as well). We denote by (π', V') the representation of P_n obtained on the space of restrictions of functions in $W(\pi, \theta)$ to P_n , it is a quotient of π as a P_n -module, and restriction to P_n is known to be an isomorphism in Case A.

The following proposition follows from repeatedly applying Proposition 2.2, and from Proposition 2.3.

Proposition 2.4. *Let τ be a smooth P_n -submodule of $C^\infty(N_n \backslash P_n, \theta^n)$, and let $k \geq 0$ be an integer. Then the P_{k+1} -module $\tau_{(n-k-1)}$ is a submodule of $C^\infty(N_{k+1} \backslash P_{k+1}, \theta^{k+1})$, with model given by restriction of functions $[\delta_{U_{k+2}} \cdots \delta_{U_n}]^{-1/2} W$ in τ to P_{k+1} . In this realisation, one has $\tau_{(n-k-1)}(p)W = \rho(p)W$ for p in P_{k+1} , where ρ is the action by right translation.*

The next proposition asserts amongst other things that for every $k \geq 1$, the G_{n-k} -module $\pi^{(k)}$ has finite length.

Proposition 2.5. *If (π, V) is a smooth representation of G_n of finite length, then for k between 1 and $n-1$, the G_k -module $\pi^{(n-k)}$ (hence its quotient $\pi^{(n-k)}$) has finite length.*

Proof. For $k \geq 1$, except in Case D where $k = 2$, we denote by $U_{k,n-k}$ the unique standard unipotent radical (denoted by $U_{(k;n-k)}$ in the previous section) containing U_{α_k} as only simple root subgroup.

In Case D, for $k = 2$, we denote by $U_{2,n-2}$ the unique standard unipotent radical containing U_{α_1} and U_{α_2} as only simple root subgroups.

In all cases, the corresponding Levi $M_{k,n-k}$ is the direct product of G_k with $GL(n-k, F)$.

Now the G_k -module $\pi^{(n-k)}$ is a quotient of the Jacquet $G_k \times GL(n-k)$ -module

$$(\pi_{U_{k,n-k}}, V/V(U_{k,n-k}, 1)),$$

as the kernel of the surjective map $\pi \rightarrow \pi^{(n-k)}$ contains $V(U_{k,n-k}, 1)$. More precisely, let $N_{n-k,A}$ be the unipotent radical of the standard Borel subgroup of $GL(n-k, F)$. The group $U_{k+1} \cdots U_n$ is the semidirect product $N_{n-k,A} \ltimes U_{k,n-k}$ so that the space $V^{(n-k)}$ of $\pi^{(n-k)}$ is equal to the quotient

$$V/V(N_{n-k,A} \ltimes U_{k,n-k}, \theta_{|N_{n-k,A}}^n \otimes 1_{U_{k,n-k}}),$$

where V is the space of π .

We denote by I_k the surjection obtained by factorisation from $V_{U_{k,n-k}}$ onto $V^{(n-k)}$. From Lemma 2.32 of [B-Z], the map I_k identifies with the projection

$$V_{U_{k,n-k}} \twoheadrightarrow (V_{U_{k,n-k}})_{N_{n-k,A}, \theta_{|N_{n-k,A}}^n} = V_{U_{k,n-k}} / V_{U_{k,n-k}}(N_{n-k,A}, \theta_{|N_{n-k,A}}^n).$$

The map I_k is in fact a G_k -module morphism because of the equality of modulus characters

$$(\delta_{U_{k,n-k}})_{|G_k} = (\delta_{U_{k+1}} \cdots \delta_{U_n})_{|G_k},$$

which is itself a consequence of the decomposition

$$U_{k,n-k} = \prod_{i=k+1}^n (U_{k,n-k} \cap U_i).$$

The group $N_{A,n-k}$ being a union of compact subgroups, the map I_k preserves exact sequences.

As the Jacquet module functor preserves finite length, the $G_k \times GL(n-k, F)$ -module $\pi_{U_{k,n-k}}$ has a finite composition series $0 \subset (\pi_{U_{k,n-k}})_1 \subset \cdots \subset (\pi_{U_{k,n-k}})_{r_k} = \pi_{U_{k,n-k}}$. We put $\pi_i^{(n-k)} = I_k[(\pi_{U_{k,n-k}})_i]$.

Hence $\pi_i^{(n-k)} / \pi_{i-1}^{(n-k)}$ is equal to $[(\pi_{U_{k,n-k}})_i / (\pi_{U_{k,n-k}})_{i-1}]_{N_{n-k,A}, \theta_{|N_{n-k,A}}}$, but as a $G_k \times GL(n-k, F)$ -module, the quotient $(\pi_{U_{k,n-k}})_i / (\pi_{U_{k,n-k}})_{i-1}$ is isomorphic to $\rho_1 \otimes \rho_2$ for irreducible representations ρ_1 and ρ_2 of G_k and $GL(n-k, F)$, respectively.

Because the character θ^n restricts to $N_{n-k,A}$ as a nondegenerate character, the quotient $\pi_i^{(n-k)}/\pi_{i-1}^{(n-k)}$ is equal to $\rho_1 \otimes (\rho_2)_{N_{n-k,A}, \theta|_{N_{n-k,A}}}$. Thus it is zero unless ρ_2 is generic, in which case it is equal to the irreducible representation ρ_1 .

Therefore we proved that $\pi^{(n-k)}$ has finite length as a G_k -module, smaller than the length of the Jacquet module $\pi_{U_{k,n-k}}$ as a $G_k \times GL(n-k, F)$ -module. $\square$

There is another property of the maps I_k defined in the proof of the preceding proposition that is worth mentioning, that being that their restriction to generalised characteristic subspaces is nonzero. More formally, let G be an l -group, and T be a closed abelian subgroup of G . If V is a smooth G -module, following [C], we define for each character χ of T the T -submodule

$$V_{\chi, \infty} = \{v \in V \mid \exists n \in \mathbb{N}, \forall t \in T, (\tau(t) - \chi(t)Id)^n(v) = 0\}.$$

If V is T -finite (i.e. every vector in V generates a finite dimensional T -module), then it is the finite direct sum of its (nonzero by definition) generalised characteristic subspaces, and every such (nonzero) $V_{\chi, \infty}$ contains the nonzero generalised eigenspace

$$V_{\chi} = \{v \in V \mid \exists n \in \mathbb{N}, \forall t \in T, (\tau(t) - \chi(t)Id)(v) = 0\}.$$

First recall that smooth $(F^*)^r$ -modules E , with a filtration $0 = E_0 \subset E_1 \subset \dots \subset E_{r-1} \subset E_r = E$ such that $(F^*)^r$ acts by a character on each quotient, are $(F^*)^r$ -finite.

Lemma 2.1. *Let E be a smooth $(F^*)^r$ -module E , with a filtration $0 = E_0 \subset E_1 \subset \dots \subset E_{r-1} \subset E_r = E$ such that $(F^*)^r$ acts by a character c_{i+1} on each quotient E_{i+1}/E_i . Then any vector of E lies in a finite dimensional $(F^*)^r$ -submodule.*

Proof. One proves this by induction on the smallest i such that E_i contains v . If this i is 1, the group $(F^*)^r$ only multiplies v by a scalar, and we are done.

Suppose that the result is known for E_i , and take v in E_{i+1} but not in E_i . Then for every t in $(F^*)^r$, the vector $\tau(t)v - c_{i+1}(t)v$ belongs to E_i . By smoothness, the set $\{\tau(u)v \mid u \in (U_F)^r\}$ is actually equal to $\{\tau(u)v \mid u \in P\}$ for P a finite set of $(U_F)^r$. The vector space generated by this set is stabilized by $(U_F)^r$ and has a finite basis $v_1, \dots, v_m$. Now the vectors

$$\tau(1, \dots, 1, \varpi_F, 1, \dots, 1)v_l - c_{i+1}(1, \dots, 1, \varpi_F, 1, \dots, 1)v_l$$

belong to E_i , hence by induction hypothesis to a finite dimensional $(F^*)^r$ -submodule V_l of E_i . Finally the finite dimensional space $Vect(v_1, \dots, v_m) + V_1 + \dots + V_m$ is stable under $(U_F)^r$ and the elements $(1, \dots, 1, \varpi_F, 1, \dots, 1)$, hence $(F^*)^r$, and contains v . $\square$

This in particular applies to the $Z_k Z_n$ -module $V_{U_{k,n-k}}$ and the Z_k -module V described in the proof of Proposition 2.5, as both are respectively $G_k \times GL(n-k, F)$ and G_k -modules of finite length.

Now we can prove the following property of the maps I_k :

Proposition 2.6. *Let (π, V) be a θ^n -generic representation of G_n , and for $k \geq 1$, let $U_{k,n-k}$ and $M_{k,n-k} \simeq G_k \times GL(n-k, F)$ be the subgroups of G_n defined in the proof of Proposition 2.5. Let χ be a character of the central subgroup $Z_k Z_n$ of $M_{k,n-k}$, and denote by the same letter its restriction to the central subgroup Z_k of G_k . If the generalised characteristic subspace $(V_{U_{k,n-k}})_{\chi, \infty}$ is nonzero, then*

the map I_k restricts nontrivially to $(V_{U_{k,n-k}})_\chi$. In particular, the space $V_\chi^{(n-k)}$ is nonzero.

Proof. Suppose that the subspace $(V_{U_{k,n-k}})_{\chi,\infty}$ of $V_{U_{k,n-k}}$ is nonzero; hence $(V_{U_{k,n-k}})_\chi$ is nonzero. The space $(V_{U_{k,n-k}})_\chi$ is an $M_{k,n-k}$ -submodule of $V_{U_{k,n-k}}$, so it has finite length. Hence it contains some irreducible representation $\rho_1 \otimes \rho_2$ of $M_{k,n-k}$. Hence $\text{Hom}_{M_{k,n-k}}(\rho_1 \otimes \rho_2, V_{U_{k,n-k}})$ is nonzero, but then from Bernstein's second adjointness theorem (see [Bu], Theorem 3), we deduce that V is a quotient of the representation $\rho_1 \times \rho_2$ parabolically induced from $\rho_1 \otimes \rho_2$. As V admits a nonzero Whittaker form, so does $\rho_1 \times \rho_2$, and from a classical result of Rodier (Theorem 7 of [R]) this implies that ρ_1 and ρ_2 are generic with respect to some nondegenerate character. As genericity doesn't depend on the character for $GL(n-k, F)$, we deduce that $I_k(\rho_1 \otimes \rho_2) = \rho_1$. Hence I_k restricts nontrivially to $(V_{U_{k,n-k}})_\chi$, and the image $I_k[(V_{U_{k,n-k}})_\chi]$ contains ρ_1 , which is a nonzero submodule of $V_\chi^{(n-k)}$. $\square$

We will also need to know that if k is an integer between 1 and $n-1$ and χ is a character of Z_k , then the Z_k -modules $(V^{(n-k)})_\chi$ and $(V'^{(n-k)})_\chi$ are nonzero at the same time. We already know from the previous proposition that this is equivalent to the fact that the Z_k -modules $(V'^{(n-k)})_\chi$ and $(V_{U_{k,n-k}})_\chi$ are nonzero at the same time, and that $(V'^{(n-k)})_\chi$ nonzero implies that $(V_{U_{k,n-k}})_\chi$ is nonzero.

Proposition 2.7. *If (π, V) is a θ^n -generic representation of G_n , and for $k \geq 1$, let $U_{k,n-k}$ and $M_{k,n-k} \simeq G_k \times GL(n-k, F)$ be the subgroups of G_n defined in the proof of Proposition 2.5. Let χ be a character of the central subgroup $Z_k Z_n$ of $M_{k,n-k}$, and denote by the same letter its restriction to the central subgroup Z_k of G_k . Then the space $(V_{U_{k,n-k}})_\chi$ is nonzero if and only if the space $(V'^{(n-k)})_\chi$ is nonzero.*

Proof. We only need to prove that if $(V_{U_{k,n-k}})_\chi$ is nonzero, then the space $(V'^{(n-k)})_\chi$ is nonzero.

Suppose that the $M_{k,n-k}$ -module $(V_{U_{k,n-k}})_\chi$ is nonzero, it is of finite length, and hence it contains an irreducible $M_{k,n-k}$ -submodule ρ . Call $P_{k,n-k}$ the parabolic subgroup $M_{k,n-k}U_{k,n-k}$ and $P_{k,n-k}^-$ its opposite parabolic subgroup (with unipotent radical $(U_{k,n-k})^-$). We already saw that by Bernstein's second adjointness theorem the induced representation $i_{P_{k,n-k}}^{G_n}(\rho)$ has π as a quotient, and therefore $i_{P_{k,n-k}}^{G_n}(\rho)$ is θ^n -generic. Then from Theorem 7 of [R], the $M_{k,n-k}$ -module ρ is θ -generic, where θ is the restriction of θ^n to the unipotent radical of the Borel of $M_{k,n-k}$. Both have the same Whittaker model $W(\pi, \theta^n)$, i.e. the (unique up to scalar) Whittaker form on the space of $i_{P_{k,n-k}}^{G_n}(\rho)$ factorises through the projection from $i_{P_{k,n-k}}^{G_n}(\rho)$ to π . Let L^- be a nonzero θ -Whittaker form on the space of ρ . By Theorems 1.4 and 1.6 of [C-S], there is a nonzero Whittaker form L on the space of $i_{P_{k,n-k}}^{G_n}(\rho)$ whose restriction to the subspace $C_c^\infty(P_{k,n-k}^- \backslash P_{k,n-k}^- U_{k,n-k}, (\delta_{U_k}^-)^{1/2} \rho)$ of functions with support in $P_{k,n-k}^- U_{k,n-k}$ is given by

$$f \mapsto \int_{U_{k,n-k}} L^-(f(u)) \theta^{-1}(u) du.$$

We denote by $\bar{L}$ the Whittaker form on the space of π which lifts to L . In particular, for any $\bar{f}$ in the space of π , which is the image of f in $C_c^\infty(P_{k,n-k}^- \backslash P_{k,n-k}^- U_{k,n-k})$, one has

$$\bar{L}(\bar{f}) = \int_{U_{k,n-k}} L^-(f(u))\theta^{-1}(u)du.$$

Let v be a vector in the space of ρ such that $L^-(v)$ is nonzero. Let K' be a compact subgroup of G_n , with Iwahori decomposition with respect to $P_{k,n-k}$, and such that $K' \cap M_{k,n-k}$ fixes v . Then the function α equal to $u^-mu \mapsto \rho(m)v$ on $(U_{k,n-k})^- M_{k,n-k} (U_{k,n-k} \cap K')$, and zero outside, is well defined and belongs to the space $C_c^\infty(P_{k,n-k}^- \backslash P_{k,n-k}^- U_{k,n-k}; \rho)$. We denote by W_α the corresponding Whittaker function $g \mapsto \bar{L}(\pi(g)\bar{\alpha})$. If a belongs to the group Z_k , one has

$$\begin{aligned} W_\alpha(a) &= \int_{U_{k,n-k}} L^-(\alpha(ua))\theta^{-1}(u)du \\ &= \chi(a)\delta_{U_{k,n-k}}^{-1/2}(a) \int_{U_{k,n-k}} L^-(\alpha(a^{-1}ua))\theta^{-1}(u)du \\ &= \chi(a)\delta_{U_{k,n-k}}^{1/2}(a) \int_{U_{k,n-k}} L^-(\alpha(u))\theta^{-1}(a u a^{-1})du \\ &= \chi(a)\delta_{U_{k,n-k}}^{1/2}(a) L^-(v) \int_{U_{k,n-k}} \theta^{-1}(a u a^{-1})du. \end{aligned}$$

In this last integral, u stays in the compact set $U_{k,n-k} \cap K'$. Hence there is a (punctured) neighbourhood of zero in $\text{Lie}(Z_k) = F$ such that $a(U_{k,n-k} \cap K')a^{-1}$ is a subset of $\text{Ker}(\theta)$ when a belongs to this neighbourhood. Finally, up to multiplication of the function α by a scalar, one has

$$W_\alpha(a) = \chi(a)\delta_{U_{k,n-k}}^{1/2}(a) L^-(v)$$

whenever a is in this neighbourhood of zero.

A similar computation gives the equality

$$W_\alpha(zg) = \chi(z)\delta_{U_{k,n-k}}(z)^{1/2}c_g$$

for z in Z_k in a neighbourhood of zero and g in G_k , where c_g is the constant $\int_{U_{k,n-k}} L^-(\alpha(ug))du$. Hence from Propositions 2.3, 2.4, and the equality $(\delta_{U_{k,n-k}})|_{G_k} = (\delta_{U_{k+1}} \dots \delta_{U_n})|_{G_k}$, we deduce that the vector $(\delta_{U_{k+2}} \dots \delta_{U_n})^{-1/2}W_\alpha$ in the space of $\pi'_{(n-k+1)}$ is such that its image in $\pi'^{(n-k)}$ is nonzero and belongs to the space $(\pi'^{(n-k)})_\chi$. This proves the proposition. $\square$

A straightforward generalisation of the proof of the preceding proposition gives the following corollary.

Corollary 2.1. *Let G be the F -points of a quasi-split reductive group defined over F . Let P be a parabolic subgroup of G with a Levi subgroup M , and let P^- be its opposite subgroup with $P \cap P^- = M$. Let (π, V) be a smooth θ -generic representation of G , for some nondegenerate character θ of the unipotent radical U of a Borel subgroup of G contained in P . We denote by j_{P^-} the map defined in Theorem 3.4 of [D] from $(V^*)^{U_\theta}$ to $(J_P(V)^*)^{M \cap U_\theta}$. If L is a nonzero vector of the line $(V^*)^{U_\theta}$, then the linear form $j_{P^-}(L)$ restricts nontrivially to any irreducible M -submodule of $J_P(V)$ whenever the Jacquet module $J_P(V)$ is nonzero.*

We now return to the study of F^* -modules E with finite factor series such that F^* acts by a character on each quotient. From Lemma 2.1, any vector of E will belong to a finite dimensional F^* -submodule E' as in:

Proposition 2.8. *If E' is a nonzero finite dimensional F^* -submodule of E , then E' has a basis B in which the action of F^* is given by a block diagonal matrix $\text{Mat}_B(\tau(t))$ with each block of the form*

$$\begin{pmatrix} c(t) & c(t)P_{1,2}(v_F(t)) & c(t)P_{1,3}(v_F(t)) & \dots & c(t)P_{1,q}(v_F(t)) \\ & c(t) & c(t)P_{2,3}(v_F(t)) & \dots & c(t)P_{2,q}(v_F(t)) \\ & & \ddots & & \vdots \\ & & & c(t) & c(t)P_{q-1,q}(v_F(t)) \\ & & & & c(t) \end{pmatrix},$$

with c one of the c_i 's, q a positive integer depending on the block, and the $P_{i,j}$'s being polynomials with no constant term of degree at most $j - i$.

Proof. First we decompose E' as a direct sum under the action of the compact abelian group U_F . Because E' has a filtration by the spaces $E' \cap E_i$ and F^* acts on each subfactor as one of the c_i 's, the group U_F acts on each weight space as the restriction of one of the c_i 's. Now each weight space is stable under F^* by commutativity, and so we can restrict ourselves to the case where E' is a weight space of U_F .

Again E' has a filtration such that F^* acts on each subfactor as one of the c_i 's (with all these characters having the same restriction to U_F), say $c_{i_1}, \dots, c_{i_k}$. In particular, we deduce that the endomorphism $\tau(\varpi_F)$ has a triangular matrix in a basis adapted to this filtration, with eigenvalues $c_{i_1}(\varpi_F), \dots, c_{i_k}(\varpi_F)$. As $\tau(\varpi_F)$ is trigonalisable, the space E' is the direct sum its characteristic subspaces, and again these characteristic subspaces are stable under F^* .

Finally, one can assume that E' is a characteristic subspace for some eigenvalue $c(\pi)$ of $\tau(\varpi_F)$, on which U_F acts as the character c , where c is one of the c_i 's.

Hence there is a basis B of E' such that

$$\text{Mat}_B(c^{-1}(t)\tau(t)) = \begin{pmatrix} 1 & A_{1,2}(t) & A_{1,3}(t) & \dots & A_{1,q}(t) \\ & 1 & A_{2,3}(t) & \dots & A_{2,q}(t) \\ & & \ddots & & \vdots \\ & & & 1 & A_{q-1,q}(t) \\ & & & & 1 \end{pmatrix}$$

for any t in F^* , where the $A_{i,j}$'s are smooth functions on F^* . Therefore, we only have to prove that the $A_{i,j}$'s are polynomials of the valuation of F with no constant term.

We do this by induction on q . It is obvious when $q = 1$. Suppose the statement holds for $q - 1$, and suppose that E' is of dimension q , with basis $B = (v_1, \dots, v_q)$. Considering the two $c^{-1}\tau(F^*)$ -modules $\text{Vect}(v_1, \dots, v_{q-1})$ and $\text{Vect}(v_1, \dots, v_q)/\text{Vect}(v_1)$ of dimension $q - 1$, we deduce that for every couple (i, j) different from $(1, q)$, there is a polynomial with no constant term $P_{i,j}$ of degree at most $j - i$ such that $A_{i,j} = P_{i,j} \circ v_F$. Now because $c^{-1}\tau$ is a representation of F^* and because the $P_{i,j} \circ v_F$'s vanish on U_F for $(i, j) \neq (1, q)$, we deduce that $A_{1,q}$ is a smooth morphism from $(U_F, \times)$ to $(\mathbb{C}, +)$, which must be zero because $(\mathbb{C}, +)$ has

no nontrivial compact subgroups. From this we deduce that $A_{1,q}$ is invariant under translation by elements of U_F (i.e. $A_{1,q}(\varpi_F^k u) = A_{1,q}(\varpi_F^k)$ for every U in U_F).

Denote by $M(k)$ the matrix $\text{Mat}_B(c^{-1}\tau(\varpi_F^k))$ for k in $\mathbb{Z}$. One has $M(k) = M(1)M(k-1)$ for $k \geq 1$, which implies $A_{1,q}(\varpi_F^k) = \sum_{j=2}^{q-1} P_{1,j}(1)P_{j,q}(k-1) + A_{1,q}(\varpi_F^{k-1}) + A_{1,q}(\varpi_F) = Q(k) + A_{1,q}(\varpi_F^{k-1}) + A_{1,q}(\varpi_F)$ for Q a polynomial of degree at most $q-2$. This in turn implies that $A_{1,q}(\varpi_F^k) = \sum_{l=1}^{k-1} Q(l) + kA_{1,q}(\varpi_F) = R(k)$ for R a polynomial of degree at most $q-1$, according to the theory of Bernoulli polynomials, for any $k \geq 0$. The same reasoning for $k \leq 0$ implies $A_{1,q}(\varpi_F^k) = R'(k)$ for R' a polynomial of degree at most $q-1$, for any $k \leq 0$. We need to show that $R = R'$ to conclude.

We know that $M(k)$ is a matrix whose coefficients are polynomials in k for $k > 0$ of degree at most $q-1$; we denote it by $P(k)$. The matrix $M(k)$ has the same property for $k < 0$; we denote it by $P'(k)$. Moreover, for any $k \geq 0$ and $k' \leq 0$, with $k+k' \geq 0$, one has $P(k+k') = P(k)P'(k')$. Fix $k > q-1$; then the matrices $P(k+k')$ and $P(k)P'(k')$ are equal for k' in $[1-q, 0]$. As their coefficients are polynomials in k' with degree at most $q-1$, the equality $P(k+z') = P(k)P'(z')$ holds for any complex number z' . Now fix such a complex number z' . The equality $P(k+z')$ and $P(k)P'(z')$ holds for any integer $k > q-1$, and as both matrices have coefficients which are polynomials in k , this equality actually holds for any complex number z , so that $P(z+z')$ equals $P(z)P'(z')$ for any complex numbers z and z' .

As $P(0) = I_q$, we deduce that P and P' are equal on $\mathbb{C}$, and this implies that R is equal to R' . $\square$

From this we deduce the following theorem, giving an expansion at infinity of Whittaker functions of generic representations of G_n . For $GL(n)$, the statement holds for any smooth P_n -submodule of finite length $C^\infty(N_n \backslash P_n)$:

Theorem 2.1. *Let θ be a nondegenerate character of the group N_n , let π be a θ -generic representation of G_n , and let $c_{1,n-k}, \dots, c_{r_k,n-k}$ be the characters of Z_k appearing in a composition series of $\tau = \pi^{(n-k)}$. Then, for any function W in the space of π , the function*

$$W(z_1, z_2, \dots, z_{n-1}) = W(z_1 z_2 \dots z_{n-1})$$

is a linear combination of functions of the form

$$\prod_{k=1}^{n-1} [c_{i_k,k} \delta_{U_{k+1}}^{1/2} \dots \delta_{U_n}^{1/2}](z_k) v_F(z_k)^{m_k} \phi_k(z_k)$$

for i_k between 1 and r_k , positive integers m_k , and functions ϕ_k in $C_c^\infty(F)$.

Proof. Actually we prove the following stronger statement, which is satisfied by $\pi_{(0)}$ according to Proposition 2.5:

Let π be a submodule of $C^\infty(N_n \backslash P_n, \theta)$ such that for every k between 1 and $n-1$, the G_k -module $\tau = \pi^{(n-k)} = \Psi^-(\Phi^-)^{n-k-1}(\pi)$ has a composition series such that on each respective quotient, the central subgroup Z_k acts by the characters $c_{1,n-k}, \dots, c_{r_k,n-k}$.

Then, for any function W in the space of π , the function $W(z_1, z_2, \dots, z_{n-1}) = W(z_1 z_2, \dots, z_{n-1})$ is a linear combination of functions of the form

$$\prod_{k=1}^{n-1} [c_{i_k, k} \delta_{U_{k+1}}^{1/2} \dots \delta_{U_n}^{1/2}](z_k) v_F(z_k)^{m_k} \phi_k(z_k)$$

for i_k between 1 and r_k , positive integers m_k , and functions ϕ_k in $C_c^\infty(F)$.

We first give the principle of the proof for the group G_2 , which already contains the main ideas. We give the general proof after.

Let π be a submodule of $C^\infty(N_2 \backslash P_2, \theta)$ such that $\pi^{(1)}$ has a finite composition series with the group $G_1 = Z_1$ acting by the character $c_{i,1}$ on the i -th quotient. Let W belong to the space of π and v be its image in the space E of $\pi^{(1)}$. It belongs to a finite dimensional Z_1 -submodule E' of E , on which Z_1 acts by a matrix of the form determined in Proposition 2.8. Take a basis $B = (e_1, \dots, e_q)$ of E' , and denote by $M(a)$ the matrix $M_B(\tau(a))$ (with a in Z_1 and $\tau(a) = \pi^{(1)}(a)$). Hence we have $\tau(a)e_l = \sum_{k=1}^q M(a)_{k,l} e_k$ for each l between 1 and q .

Taking preimages $\tilde{E}_1, \dots, \tilde{E}_q$ of $e_1, \dots, e_q$ in $\pi_{(0)}$, we denote by $\tilde{E}$ the function vector

$$\begin{pmatrix} \tilde{E}_1 \\ \vdots \\ \tilde{E}_q \end{pmatrix}.$$

If the image v of W in $\pi^{(1)}$ is equal to $x_1 e_1 + \dots + x_q e_q$, there is an integer M such that for every z_1 in Z_1 , the function

$$W(z_1) - (x_1, \dots, x_q) \tilde{E}(z_1)$$

vanishes for $|z_1|_F \leq q_F^{-M}$ according to Proposition 2.3. We denote by S the function $(x_1, \dots, x_q) \tilde{E}$.

Because of Remark 2.1, there is an integer M' such that for any z_1 in Z_1 of absolute value greater than $q_F^{M'}$, both $W(z_1)$ and $S(z_1)$ are zero, so that the difference $D(z_1)$ of the two functions is a smooth function which vanishes whenever z_1 has absolute value outside $[q_F^{-M}, q_F^{M'}]$.

On the other hand, for any a in Z_1 , one has $\rho(a)p(\tilde{E}_l) = \sum_{k=1}^q M(a)_{k,l} p(\tilde{E}_k)$ for p the projection from $\pi_{(0)}$ to $\pi^{(1)}$. As we identified the kernel of this projection in Proposition 2.3, we deduce (see the general proof after) that there is a z_0 in Z_1 such that the function $\mathbf{1}_{\{|z_1| \leq 1\}} S(z_1)$ is equal to

$$(x_1, \dots, x_q)^t M(z_1) [\delta_{U_2}^{-1/2}(z_0)^t M(z_0^{-1})] \tilde{E}(z_1) \delta_{U_2}^{1/2}(z_1) \mathbf{1}_{\{|z_1| \leq 1\}}.$$

As for the function D , one then proves that there exist two positive integers M_1 and M'_1 such that the function $\mathbf{1}_{\{|z_1| > 1\}} S(z_1)$ has its support contained in an interval of the form $[q_F^{-M_1}, q_F^{M'_1}]$.

Finally the function

$$W(z_1) = \delta_{U_2}^{1/2}(z_1) [\delta_{U_2}^{-1/2}(z_1) D(z_1)] + \mathbf{1}_{\{|z_1| \leq 1\}} S(z_1) + \delta_{U_2}^{1/2}(z_1) [\delta_{U_2}^{-1/2}(z_1) \mathbf{1}_{\{|z_1| > 1\}} S(z_1)]$$

has the desired form because the left and right terms between $[\cdot]$ of the sum have compact support in Z_1 and the middle term is of the requested form according to Proposition 2.8.

The general proof is by induction on n . For $n = 1$, the group G_1 is isomorphic to F^* , the group P_1 is trivial, and the statement is obvious. Suppose now that n is

greater than or equal to 2. Let W belong to the space of π . We denote by v its image in the space E of $\pi^{(1)}$. The vector v belongs to a finite dimensional Z_{n-1} -submodule E' of E , on which Z_{n-1} acts by a matrix of the form determined in Proposition 2.8. We fix a basis $B = (e_1, \dots, e_q)$ of E' and denote by $M(a)$ the matrix $M_B(\tau(a))$ (with a in Z_{n-1} and $\tau(a) = \pi^{(1)}(a)$). Hence we have $\tau(a)e_l = \sum_{k=1}^q M(a)_{k,l}e_k$ for each l between 1 and q .

Taking preimages $\tilde{E}_1, \dots, \tilde{E}_q$ of $e_1, \dots, e_q$ in $\pi_{(0)}$, we denote by $\tilde{E}$ the function vector

$$\begin{pmatrix} \tilde{E}_1 \\ \vdots \\ \tilde{E}_q \end{pmatrix}.$$

If the image v of W in $\pi^{(1)}$ is equal to $x_1e_1 + \dots + x_qe_q$, there is an integer M such that for every $(z_1, \dots, z_{n-2})$ in $Z_1 \times \dots \times Z_{n-2}$, the function

$$W(z_1, \dots, z_{n-1}) - (x_1, \dots, x_q)\tilde{E}(z_1, \dots, z_{n-1})$$

vanishes for $|z_{n-1}|_F \leq q_F^{-M}$. We denote by S the function $(x_1, \dots, x_q)\tilde{E}$.

Because of Remark 2.1, there is an integer M' such that for any $(z_1, \dots, z_{n-2})$ in $Z_1 \times \dots \times Z_{n-2}$ and any z_{n-1} in Z_{n-1} of absolute value greater than $q_F^{M'}$, both $W(z_1, \dots, z_{n-1})$ and $S(z_1, \dots, z_{n-1})$ are zero, so that the difference $D(z_1, \dots, z_{n-1})$ of the two functions is a smooth function which vanishes whenever z_{n-1} has absolute value outside $[q_F^{-M}, q_F^{M'}]$. Moreover, there is a compact subgroup U of $Z_{n-1}(\mathfrak{O}_F)$ independent of $(z_1, \dots, z_{n-1})$ such that both functions (hence D) are invariant when z_{n-1} is multiplied by an element of U . Denoting by $(z_\alpha)_\alpha \in A$ a finite set of representatives for

$$\{z \mid |q_F^{-M}| \leq |z_{n-1}|_F \leq q_F^{M'}\}/U,$$

this implies that $D(z_1, \dots, z_{n-1})$ is equal to $\sum_{\alpha \in A} D(z_1, \dots, z_{n-2}, z_\alpha) \mathbf{1}_{z_\alpha U}(z_{n-1})$, which we can always write as $\sum_{\alpha \in A} D(z_1, \dots, z_{n-2}, z_\alpha) \delta_{U_n}^{1/2}(z_{n-1}) D_\alpha(z_{n-1})$ with $D_\alpha = \delta_{U_n}^{-1/2} \mathbf{1}_{z_\alpha U}$ in $C_c^\infty(\text{Lie}(Z_{n-1}))$.

Each function $D(z_1, \dots, z_{n-2}, z_\alpha)$ is equal to $W(z_1, \dots, z_\alpha) - S(z_1, \dots, z_\alpha)$, and the restrictions to P_{n-1} of the functions $\delta_{U_n}^{-1/2}[\pi(z_\alpha)D]$ belong to the smooth submodule $\Phi^-(\pi)$ of $C^\infty(N_{n-1} \backslash P_{n-1}, \theta)$, which still satisfies the hypothesis of the statement.

Hence, by induction hypothesis, the function D is a sum of functions of the form

$$\prod_{k=1}^{n-1} [c_{i_k, k} \delta_{U_{k+1}}^{1/2} \dots \delta_{U_n}^{1/2}](z_k) v_F(z_k)^{m'_k} \phi'_k(z_k)$$

for i_k between 1 and r_k , null or positive integers or integer vectors m'_k , and functions ϕ'_k in $C_c^\infty(\text{Lie}(Z_k))$.

Now call p the projection from $\pi_{(0)}$ to $\pi^{(1)}$. Then for any a in Z_{n-1} , one has $\rho(a)p(\tilde{E}_l) = \sum_{k=1}^q M(a)_{k,l}p(\tilde{E}_k)$. Hence as $\rho(a)p(\tilde{E}_l)$ equals $p(\delta_{U_n}^{-1/2}(a)\pi_{(0)}(a)\tilde{E}_l)$, we deduce that there is a punctured neighbourhood of zero in Z_{n-1} such that for each l , the function $\delta_{U_n}^{-1/2}(a)\pi_{(0)}(a)\tilde{E}_l - \sum_{k=1}^q M(a)_{k,l}\tilde{E}_k$ vanishes on elements $g = pac$ of G_{n-1} (p in P_{n-1} , a in Z_{n-1} , and c in $G_{n-1}(\mathfrak{O}_F)$) such that a is in this neighbourhood.

In particular, there exists N_a such that for every $(z_1, \dots, z_{n-1})$, the vector function

$$\delta_{U_n}^{-1/2}(a)\pi_{(0)}(a)\tilde{E}(z_1, \dots, z_{n-1}) - {}^tM(a)\tilde{E}(z_1, \dots, z_{n-1})$$

vanishes when we have $|z_{n-1}|_F \leq q_F^{-N_a}$.

This implies, as in the proof of Proposition 2.6 of [C-P], the following claim:

Claim. There is actually an M'' such that for every z in Z_{n-1} , with $|z_{n-1}|_F \leq q_F^{-M''}$, and every a in Z_{n-1} , with $|a|_F \leq 1$, the function $\tilde{E}(z_1, \dots, z_{n-1}a)$ is equal to $\delta_{U_n}^{1/2}(a){}^tM(a)\tilde{E}(z_1, \dots, z_{n-1})$.

Proof of the Claim. We denote $(z_1, \dots, z_{n-2})$ by x and z_{n-1} by z .

If U is an open compact subgroup of $Z_{n-1}(\mathfrak{O}_F)$, such that $\tilde{E}$ and the homomorphism $a \in Z_{n-1} \mapsto M(a) \in G_q(\mathbb{C})$ are U invariant, we denote by $u_1, \dots, u_s$ the representatives of $Z_{n-1}(\mathfrak{O}_F)/U$ and by ω the canonical generator of $Z_{n-1}/Z_{n-1}(\mathfrak{O}_F)$. We put $M'' = \max_{i,j}(N_{u_i}, N_\omega)$.

Then for z in $\{z \in Z_{n-1}, |z|_F \leq q_F^{-M''}\}$ and $a = \omega^r u_i u$ in $\{z \in Z_{n-1}, |z|_F \leq 1\}$ (with u in U , and $r \in \mathbb{N}$), we have

$$\tilde{E}(x, za) = \tilde{E}(x, z\omega^r u_i) = \delta_{U_n}^{1/2}(u_i){}^tM(u_i)\tilde{E}(x, z\omega^r)$$

because $z\omega^r$ belongs to $\{z \in Z_{n-1}, |z|_F \leq q_F^{-M''}\} \subset \{z \in Z_{n-1}, |z|_F \leq q_F^{-N_{u_i}}\}$. However, if $r \geq 1$, again one has

$$\tilde{E}(x, z\omega^r) = \delta_{U_n}^{1/2}(\omega){}^tM(\omega)(\omega_i)\tilde{E}(x, z\omega^{r-1})$$

and $z\omega^{r-1}$ belongs to

$$\{z \in Z_{n-1}, |z|_F \leq q_F^{-N_2}\} \subset \{z \in Z_{n-1}, |z|_F \leq q_F^{-N_\omega}\},$$

and repeating this step, we deduce the equality $\tilde{E}(x, za) = \delta_{U_n}^{1/2}(a){}^tM(a)\tilde{E}(x, z)$. $\square$

Hence there is an element z_0 in Z_{n-1} with $|z_0|_F = q_F^{-M''}$ such that for every $(z_1, \dots, z_{n-2})$ in $Z_1 \times \dots \times Z_{n-2}$, the vector $\tilde{E}(z_1, \dots, z_{n-1})$ is equal to

$$\delta_{U_n}^{1/2}(z_{n-1}){}^tM(z_{n-1})[\delta_{U_n}^{-1/2}(z_0){}^tM(z_0^{-1})]\tilde{E}(z_1, \dots, z_{n-2}, z_0)$$

for any z_{n-1} with $|z_{n-1}|_F \leq 1$.

Hence the function $\mathbf{1}_{\{|z_{n-1}| \leq 1\}}S(z_1, \dots, z_{n-1})$ is equal to

$$(x_1, \dots, x_q){}^tM(z_{n-1})[\delta_{U_n}^{-1/2}(z_0){}^tM(z_0^{-1})] \times \tilde{E}(z_1, \dots, z_{n-2}, z_0)\delta_{U_n}^{1/2}(z_{n-1})\mathbf{1}_{\{|z_{n-1}| \leq 1\}}.$$

One proves as for the function D that function $\mathbf{1}_{\{|z_{n-1}| > 1\}}(z_{n-1})S(z_1, \dots, z_{n-1})$ is of the form

$$\sum_{\beta \in B} S(z_1, \dots, z_{n-2}, z_\beta)\delta_{U_n}^{1/2}(z_{n-1})S_\beta(z_{n-1})$$

with S_β in $C_c^\infty(F)$ for some finite set B .

By induction hypothesis again, applied to the function $(\delta_{U_n}^{-1/2}\tilde{E}_i)(z_1, \dots, z_{n-2}, z_0)$ and the function $(\delta_{U_n}^{-1/2}S)(z_1, \dots, z_{n-2}, z_\beta)$, we deduce that the function

$$S = \mathbf{1}_{\{|z_{n-1}| \leq 1\}}S + \mathbf{1}_{\{|z_{n-1}| > 1\}}S$$

is a sum of functions of the form $\prod_{k=1}^{n-1} c_{i_k, n-k}(z_k)\delta_{U_{k+1}}^{1/2} \dots \delta_{U_n}^{1/2}(z_k)v_F(z_k)m_j''\phi_k''(z_k)$ for i_k between 1 and r_k , null or positive integers or integer vectors m_k'' , and functions ϕ_k'' in $C_c^\infty(F)$.

The statement follows as the function W equals $D + S$. $\square$

3. $L^2(Z_n N_n \backslash G_n)$ AND DISCRETE SERIES

First we characterise the Whittaker functions which belong to $L^2(Z_n N_n \backslash G_n)$ in terms of exponents of the “shifted derivatives” (see [B, 7.2]). This result has been used in [M].

We say that a character of an F^* is positive if its (complex) absolute value is of the form $| \cdot |_F^r$ for some positive real r .

Theorem 3.1. *Let θ be a nondegenerate character of the group N_n , let π be a θ -generic representation of G_n , and let $c_{1,n-k}, \dots, c_{r_k,n-k}$ be the characters of Z_k appearing in a composition series of $\tau = \pi^{(n-k)}$. Then the integral*

$$\int_{N_n \backslash P_n} |W(p)|^2 dp$$

converges for any W in π if and only if all the characters $c_{i_k,k} \delta_{U_{k+1}}^{1/2}$ are positive for k between 1 and $n-1$.

Proof. Again we prove the stronger statement:

Let (π, V) be a P_n -submodule of $C^\infty(N_n \backslash P_n, \theta)$ such that for every k between 1 and $n-1$, the G_k -module $\tau = \pi^{(n-k)} = \Psi^-(\Phi^-)^{n-k-1}(\pi)$ has a composition series such that, on each respective quotient, the central subgroup Z_k acts by the characters $c_{1,n-k}, \dots, c_{r_k,n-k}$. Then the integral

$$\int_{N_n \backslash P_n} |W(p)|^2 dp$$

converges for any W in π if and only if all the characters $c_{i_k,k} \delta_{U_{k+1}}^{1/2}$ are positive for k between 1 and $n-1$.

Let's first work out the proof for G_2 , which already gives a good idea of the general case.

Suppose first that the characters $c_{i_1,1} \delta_{U_1}^{1/2}$ are positive because of the equality $\int_{N_2 \backslash P_2} |W(p)|^2 dp = \int_{G_1} |W(g)|^2 dg$, and, as G_1 is Z_1 , the convergence of the integral is a straightforward consequence of Theorem 2.1.

Conversely suppose that the integral $\int_{N_2 \backslash P_2} |W(p)|^2 dp = \int_{Z_1} |W(z)|^2 dz$ converges for every W in V . Take a nonzero vector v in $\Psi^-(V)$ that transforms by $c_{i_1,1}$ under Z_1 (see Proposition 2.8). Then take a preimage W of v in V . According to Proposition 2.3, we deduce that for every a in Z_1 , the function $\delta_{U_2}^{-1/2}(a) \pi(a) W(z) - c_{i_1,1}(a) W(z)$ vanishes when z is in a punctured neighbourhood of zero depending on a . As in Claim 2 again, this implies that there exists an integer N large enough such that $W(az)$ equals $\delta_{U_2}^{1/2}(a) c_{i_1,1}(a) W(z)$ for $|z|_F \leq q_F^{-N}$ and $|a|_F \leq 1$.

As v is nonzero, the vector W is not in the kernel of the projection from V to $\Psi^-(V)$, so that according to Proposition 2.3, there is a z_0 of absolute value less than q_F^{-N} such that $W(z_0)$ is not zero. Calling W_0 the function $\pi(z_0)W$, we deduce, as $W_0(z)$ equals $\delta_{U_2}^{1/2}(z) c_{i_1,1}(z) W(z_0)$ for z small enough, that the character $\delta_{U_2}^{1/2} c_{i_1,1}$ must be positive in order to make the integral $\int_{Z_1} |W_0(z)|^2 dz$ converge.

The general proof is by induction on n . As before, for $n=1$, the group G_1 is isomorphic to F^* , the groups P_1 and N_1 are trivial, and the statement is obvious.

Now for $n \geq 2$, suppose first that all the characters $c_{i_k,k}\delta_{U_{k+1}}^{1/2}$ are positive. Let W belong to the space of π ; first we notice the equality

$$\int_{N_n \backslash P_n} |W(p)|^2 dp = \int_{N_{n-1} \backslash G_{n-1}} |W(g)|^2 dg.$$

Now the Iwasawa decomposition reduces the convergence of this integral to that of

$$\int_{A_{n-1}} |W(a)|^2 \delta_{N_{n-1}}^{-1}(a) d^*a.$$

Using coordinates $(z_1, \dots, z_{n-1})$ (see Lemma 1.1) of A_{n-1} , the function $\delta_{N_{n-1}}^{-1}(z_1, \dots, z_{n-1})$ is equal to $\prod_{k=1}^{n-2} (\delta_{U_{k+1}} \dots \delta_{U_{n-1}})^{-1}(z_k)$.

According to Theorem 2.1 the function $|W(z_1, \dots, z_{n-1})|^2$ is bounded by a sum of functions of the form

$$\prod_{k=1}^{n-1} |c_{i_k,k}|(z_k) |c_{l_k,k}|(z_j) (\delta_{U_{k+1}} \dots \delta_{U_n})(z_k) v_F(z_k)^{m_k} \phi_k(z_k).$$

Hence our integral will converge if the same is true of the integrals

$$\int_{A_{n-1}} \prod_{k=1}^{n-1} |c_{i_k,k}|(z_k) |c_{l_k,k}|(z_k) \delta_{U_n}(z_k) v_F(t_k)^{m_k} \phi_k(z_k) dz_1 \dots dz_n,$$

i.e. if the integrals $\int_{Z_k} |c_{i_k,k}|(z_k) |c_{l_k,k}|(z_k) \delta_{U_n}(z_k) v_F(t_k)^{m_k} \phi_k(t_k) dz_k$ converge for any k between 1 and $n-1$.

But the restriction of δ_{U_n} to Z_k is equal to $\delta_{U_{k+1}}$, so the convergence follows from our assertion on the characters $c_{i_k,k}\delta_{U_n}^{1/2}$.

Conversely, suppose that every W in $\pi_{(0)}$ belongs to the space $L^2(N_n \backslash P_n)$ corresponding to a right invariant measure on $N_n \backslash P_n$.

By Iwasawa decomposition, one obtains that

$$\int_{N_n \backslash P_n} |W(p)|^2 dp = \int_{N_{n-1} \backslash G_{n-1}} |W(g)|^2 dg$$

is equal to $\int_{A_{n-1} \times F_n} |W(ak)|^2 \delta_{N_{n-1}}^{-1}(a) d^*a dk$ which is greater than $dk(U) \int_{A_{n-1}} |W(a)|^2 \delta_{N_{n-1}}^{-1}(a) d^*a$ for some compact open subgroup U fixing W . In particular the integral $\int_{A_{n-1}} |W(a)|^2 \delta_{N_{n-1}}^{-1}(a) d^*a$ converges for any W in π .

This, by Fubini's theorem and the smoothness of W , implies that $\int_{A_{n-2}} |W(a)|^2 \delta_{N_{n-1}}^{-1}(a) d^*a$ is finite for any W in π . However, the restriction of $\delta_{N_{n-1}}$ to A_{n-2} is equal to $\delta_{N_{n-2}} \delta_{U_{n-1}}$, so that the integral $\int_{A_{n-2}} |W(a)|^2 \delta_{U_{n-1}}^{-1} \delta_{N_{n-2}}^{-1}(a) d^*a$ is finite for W in π , which by Iwasawa decomposition again implies that $\int_{N_{n-1} \backslash P_{n-1}} |\delta_{U_{n-1}}^{-1/2} W(p)|^2 dp$ is finite.

The functions $\delta_{U_{n-1}}^{-1/2} W$ belong to the space of $\phi^-(\pi)$. Hence by induction, all the characters $c_{i_k,k}\delta_{U_{k+1}}^{1/2}$ are positive for $k \leq n-2$, so we only need to check that the characters $c_{i_{n-1},n-1}\delta_{U_n}^{1/2}$ are positive. Suppose that one of them, $c_{1,n-1}\delta_{U_n}^{1/2}$ for instance, wasn't positive.

Then, taking v nonzero in $\Psi^-(\pi)$ such that Z_{n-1} multiplies v by $c_{1,n-1}$, according to Proposition 2.3 and taking W as a preimage of v in π , there is a positive integer N_a such that $\delta_{U_n}^{-1/2}(a)\pi(a)W(g) - c_{1,n-1}(a)W(g)$ is zero whenever there is any g

in G_{n-1} with $|z(g)|_F \leq q_F^{-Na}$. As in Claim 2, this implies that there is a positive integer N such that $W(ag)$ is equal to $\delta_{U_n}^{1/2}(a)c_{1,n-1}(a)W(g)$ whenever $|z(g)|_F \leq q_F^{-N}$ and $|a|_F \leq 1$. We recall that W doesn't belong to $V(U_n, 1)$ (otherwise v would be zero), hence according to Proposition 2.3, there is g_0 in G_{n-1} with $|z(g_0)|_F \leq q_F^{-N}$ such that $W(g_0)$ is nonzero. We denote by W_0 the function $\pi(g_0)W$, and we recall that the integral

$$\begin{aligned} & \int_{A_{n-1}} |W_0(a)|^2 \delta_{N_{n-1}}^{-1}(a) d^*a \\ &= \int_{Z_1 \times \cdots \times Z_{n-1}} |W_0(z_1 \cdots z_{n-1})|^2 \delta_{N_{n-1}}^{-1}(z_1 \cdots z_{n-1}) dz_1 \cdots dz_{n-1} \end{aligned}$$

is finite. Hence the smoothness of W_0 and Fubini's theorem imply that the integral

$$\int_{Z_{n-1}} |W_0(z_{n-1})|^2 \delta_{N_{n-1}}^{-1}(z_{n-1}) dz_{n-1} = \int_{Z_{n-1}} |W_0(z_{n-1})|^2 dz_{n-1}$$

is finite. However, for $|z_{n-1}|_F \leq 1$, the function $W_0(z_{n-1})$ is equal to $\delta_{U_n}^{1/2}(z_{n-1})c_{1,n-1}(z_{n-1})W(g_0)$ with $W(g_0)$ nonzero. Hence it is square integrable at zero if and only if $\delta_{U_n} c_{1,n-1}^2$, thus $\delta_{U_n}^{1/2} c_{1,n-1}$ is positive. $\square$

Remark 3.1. The last proof more or less contains the following fact (which is more precisely a consequence of an induction, and the last step of the proof):

For every character $c_{i_k, n-k}$ appearing in a factor series of $\pi^{(n-k)}$, there is W in V such that $W(z_k)$ is equal to $[c_{i_k, n-k} \delta_{U_{k+1}}^{1/2} \cdots \delta_{U_n}^{1/2}](z_k)$ near zero. Hence this family of characters is minimal in the sense that each of them must occur in the expansion given in Proposition 2.1 of some W in V .

From this we deduce a characterization of the Whittaker functions in $L^2(Z_n N_n \backslash G_n)$.

Corollary 3.1. *Let θ be a nondegenerate character of the group N_n , let π be a θ -generic representation of G_n with unitary central character, and let $c_{1, n-k}, \dots, c_{r_k, n-k}$ be the central characters appearing in the factor series of $\tau = \pi^{(n-k)}$. Then the integral*

$$\int_{Z_n N_n \backslash G_n} |W(g)|^2 dg$$

converges for any W in π if and only if all the characters $c_{i_k, k}$ are positive for k between 1 and $n-1$.

Proof. By the Iwasawa decomposition, the integral $\int_{Z_n N_n \backslash G_n} |W(g)|^2 dg$ converges for every W in $W(\pi, \theta)$ if and only if $\int_{A_{n-1}} |W(a)|^2 \delta_{N_n}^{-1}(a) da$ converges for every W in $W(\pi, \theta)$.

As the character δ_{N_n} restricts to G_{n-1} as $\delta_{N_{n-1}} \delta_{U_n}$, this integral is equal to

$$\int_{A_{n-1}} |\delta_{U_n}^{-1/2} W(a)|^2 \delta_{N_{n-1}}^{-1}(a) da.$$

However, this integral converges for any W in $W(\pi, \theta)$ if and only if so does the integral

$$\int_{N_n \backslash P_n} |\delta_{U_n}^{-1/2} W(p)|^2 dp$$

for any W in $W(\pi, \theta)$.

By the statement in the proof of Theorem 3.1 applied to $\delta_{U_n}^{-1/2} \otimes \pi'$, this is the case if and only all the characters $c_{i_k, k}$ are positive for k between 1 and $n-1$. $\square$

Let P be a standard proper parabolic subgroup of G_n , U its unipotent radical, and M its standard Levi subgroup. If (π, V) is a smooth irreducible representation of G_n , one refers to a cuspidal exponent of π with respect to P as a character χ of the center of M such that the characteristic space of the Jacquet module $(V_U)_{\chi, \infty}$ is nonzero. Denoting by Δ the set of simple roots $\{\alpha_1, \dots, \alpha_n\}$ of G_n , we denote by $P^{\{i_1, \dots, i_t\}}$ the standard parabolic subgroup associated with the set of positive roots $\Delta - \{\alpha_{i_1}, \dots, \alpha_{i_t}\}$, by $U^{\{i_1, \dots, i_t\}}$ its unipotent radical, and by $M^{\{i_1, \dots, i_t\}}$ its standard Levi subgroup, which admits as a central subgroup the product $Z_{i_1} \dots Z_{i_t}$.

Notice that except for Case D, for $\{i_1, \dots, i_t\} = \{2\}$, where we used the notation $U_{2, n-2}$ for $U^{\{1, 2\}}$, the group $U^{\{k\}}$ is what we already denoted by $U_{k, n-k}$.

We denote by $A_{i_1, \dots, i_t}^-$ the set

$$\{z_{i_1} \dots z_{i_t} \in Z_{i_1} \dots Z_{i_t}, |z_{i_k}|_F \leq 1, \text{ and } |z_{i_1} \dots z_{i_t}|_F < 1\}.$$

Theorem 4.4.6 of [C] then asserts that π with unitary central character is a discrete series representation if and only if, for every standard parabolic subgroup $P^{\{i_1, \dots, i_t\}}$, if χ is a cuspidal exponent of π with respect to $P^{\{i_1, \dots, i_t\}}$, the restriction of χ to $A_{i_1, \dots, i_t}^-$ is less than 1, or equivalently if χ restricted to $Z_{i_1} \dots Z_{i_t}$ is positive.

We also notice that for any k , the Jacquet module $V_{U^{\{i_k\}}}$ surjects onto $V_{U^{\{i_1, \dots, i_t\}}}$ and the character $\delta_{U^{\{i_1, \dots, i_t\}}}$ restricts to Z_{i_k} as $\delta_{U^{\{i_k\}}}$. Hence if χ is a cuspidal exponent of π with respect to $P^{\{i_1, \dots, i_t\}}$, then $\chi|_{Z_{i_k}}$ is the restriction to Z_{i_k} of a cuspidal exponent of π with respect to $P^{\{i_k\}}$. This implies that π irreducible with unitary central character is a discrete series representation if and only if the cuspidal exponents of π with respect to maximal parabolic subgroups $P^{\{i_k\}}$ have positive restriction to Z_{i_k} .

We call a character χ of Z_k , such that $(V^{(n-k)})_{\chi}$ (or equivalently $(V^{(n-k)})_{\chi, \infty}$) is nonzero, an exponent of the derivative $(\pi^{(n-k)}, V^{(n-k)})$. Now we recall that we showed in Proposition 2.7 that the Z_k modules $V_{U_{k, n-k}}$ and $V^{(n-k)}$ have the same nonzero weight subspaces. This allows us to prove in our four cases the following conjecture of Lapid and Mao ([L-M, Conjecture 3.5]).

Patrick Delorme proved this conjecture for general split groups ([D2, Theorem 8]).

Theorem 3.2. *Let π be a generic representation of G_n with unitary central character and with Whittaker model $W(\pi, \theta)$. Then the following statements are equivalent:*

i) *The integral*

$$\int_{N_n Z_n \backslash G_n} |W(g)|^2 dg$$

converges for any W in $W(\pi, \theta)$.

- ii) *All the exponents of the derivatives of π are positive.*
- iii) *the representation π is square-integrable.*

Proof. By assumption, the exponents of the derivatives of π are the characters $c_{i_k, k}$ of Corollary 3.1. Hence i) $\Leftrightarrow$ ii) is Corollary 3.1.

ii) $\Leftrightarrow$ iii): We treat Case D separately, so assume first that G_n is not $GSO(2(n-1), F)$.

By Proposition 2.6, every cuspidal exponent of π corresponding to $V_{U^{(k)}}$ is positive if and only if every exponent of the derivative $\pi^{(n-k)}$ is positive. However, we have already seen that this implies that π is a discrete series representation.

For Case D, we could have reversed the roles of the roots α_1 and α_2 (which correspond to the two symmetric roots at the end of the Dynkin diagram). The only effect it would have is to change the definition of the derivative functors $\pi^{(n-2)}$ and $\pi^{(n-1)}$. Indeed U_2 would become U_{α_2} , and Z_1 and Z_2 would be exchanged. The character θ_3 would have to be trivial on U_{α_2} instead of being trivial on U_{α_1} . But i) and ii) would still be equivalent in this case, and i) is independent of these choices.

In both cases, the maps I_k from $V_{U_{k, n-k}}$ to $V^{(n-k)}$ take nonzero weight subspaces to nonzero weight subspaces. For $n \geq 3$, the space $V_{U_{k, n-k}}$ is equal to $V_{U^{(k)}}$. In the first case, $V_{U_{1, n-1}}$ is equal to $V_{U^{(1)}} = V_{U_{\alpha_1}}$, and it is equal to $V_{U^{(2)}} = V_{U_{\alpha_2}}$ in the second case. This implies that all the exponents of the derivatives of π are positive if and only if all cuspidal exponents of π with respect to maximal parabolic subgroups are positive. Again this proves ii) $\Leftrightarrow$ iii). $\square$

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