CORRIGENDUM TO “THE WEIL-STEINBERG CHARACTER OF FINITE CLASSICAL GROUPS”

G. HISS AND A. ZALESSKI

ABSTRACT. This paper corrects the statement and the proof of Theorem 1.5 of the paper quoted in the title (Represent. Theory 13 (2009), 427–459).

Theorem 1.5 of our paper [1] requires a correction. Below we provide a new statement of this theorem and correct the proof. A mistake in the original proof of Theorem 1.5 is due to missing the multiple 2 at a certain point of the proof (see [1, page 456, line 23]).

Let $\mathbf{G}$ be a simple algebraic group of type C_n defined over a field of characteristic 2 and $G = Sp(2n, q)$, $q = 2^k$. If μ is a dominant weight of $\mathbf{G}$ then φ_μ denotes the irreducible representation of $\mathbf{G}$ with highest weight μ , and Φ_μ is the representation of G afforded by the principal indecomposable module corresponding to $(\varphi_\mu)_G$ if μ is a q -restricted weight. In addition, $\omega := (\varphi_{(q-1)\lambda_n})_G$. Let st be the 2-modular Steinberg representation of G . Recall that $st = (\varphi_{(q-1)(\lambda_1 + \dots + \lambda_n)})_G = \Phi_{st}$, where $\lambda_1, \dots, \lambda_n$ are the fundamental weights of $\mathbf{G}$. The standard Frobenius endomorphism $\mathbf{G} \rightarrow \mathbf{G}$ is denoted by Fr_0 , and it acts on the representations and the weights of $\mathbf{G}$ (so $Fr_0(\mu) = 2\mu$).

Theorem 1.5 in [1] has to be corrected as follows:

Theorem 1.5. *Let $\lambda_1, \dots, \lambda_n$ be the fundamental weights of $\mathbf{G}$, and let $\tau = (q-1)(\lambda_1 + \dots + \lambda_{n-1})$. Then $\omega \otimes st = st \oplus \Phi_\tau$.*

Recall that $\varepsilon_1, \dots, \varepsilon_n$ denote the weights of $\mathbf{G}$ introduced in [2, Planchée III]. The following lemma is a refinement of [1, Lemma 7.2(1)].

Lemma. *$\varphi_{2\lambda_n}$ is the only composition factor of $\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}$ occurring with multiplicity 1.*

Proof. Let M be the $\mathbf{G}$ -module afforded by the representation $\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}$. Note that the weights of φ_{λ_n} and hence of M are known. In terms of ε_j the weights of φ_{λ_n} are $\pm\varepsilon_1 \pm \dots \pm \varepsilon_n$, so the weights of M are $\sum_{i \in N} \pm 2\varepsilon_i$, where N can be any subset of $\{1, \dots, n\}$ (possibly empty; in this case the weight in question is meant to be the zero weight). It follows that $2\lambda_i = 2\varepsilon_1 + \dots + 2\varepsilon_i$ ($i = 1, \dots, n$) occur as weights of M . $\square$

Let $\mathbf{H} = GL(2n, \overline{F}_2)$ and let $\varepsilon'_1, \dots, \varepsilon'_{2n}$ be the weights of the natural $\mathbf{H}$ -module V . One can embed $\mathbf{G}$ into $\mathbf{H}$ so that a maximal torus $\mathbf{T}$ of $\mathbf{G}$ is contained in a maximal torus $\mathbf{T}'$ of $\mathbf{H}$, and $\varepsilon_i = \varepsilon'_i|_{\mathbf{T}}$, $\varepsilon'_{n+i}|_{\mathbf{T}} = -\varepsilon_i$ for $i = 1, \dots, n$. Let V_i ($1 \leq i \leq 2n$) be the i -th exterior power of V , and V_0 the trivial $\mathbf{H}$ -module. Set

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$R = \bigoplus_{i=0}^{2n} V_i$. Then the weights of R are 0 and $\varepsilon'_{j_1} + \cdots + \varepsilon'_{j_i}$, where $1 \leq i \leq 2n$ and $0 < j_1 < j_2 < \cdots < j_i \leq 2n$. It follows that the weights of $R' := R_{\mathbf{G}}$ are 0 and $\pm \varepsilon_{j_1} \pm \cdots \pm \varepsilon_{j_i}$ where $1 \leq i \leq n$ and $0 < j_1 < j_2 < \cdots < j_i \leq n$. Therefore, the weights of $Fr_0(R')$ and M are the same.

Furthermore, 0 and $\lambda_1, \dots, \lambda_n$ are the only dominant weights of R' . Indeed, $\varepsilon_1 = \lambda_1$ and $\varepsilon_i = \lambda_i - \lambda_{i-1}$ for $i > 1$. Suppose that a non-zero weight $\sum_{i \in N} \pm \varepsilon_i = \sum_{i \in N} \pm (\lambda_i - \lambda_{i-1})$ is dominant. Then the coefficient of λ_i is non-negative for every i . It follows that $1 \in N$ and if $i \in N, i > 1$ then $i - 1 \in N$. So $\sum_{i \in N} \pm \varepsilon_i$ is dominant if and only if it is of shape $\sum_{i=1}^k \varepsilon_i$, $k = 1, \dots, n$, as claimed. Therefore, if μ is the highest weight of an irreducible constituent φ_μ of R' , then $\mu = \lambda_i$ for some i or 0.

Let m'_i ($i = 1, \dots, n$) be the multiplicity of $\varphi_{2\lambda_i}$ in $Fr_0(R')$ and m_i the multiplicity of $\varphi_{2\lambda_i}$ in M ; in addition, let m'_0, m_0 be the multiplicity of the trivial G -module in composition series of $Fr_0(R'), M$, respectively. We show that $m_i = m'_i$. As the restriction $\varphi_{2\lambda_i}$ to $Sp(2n, 2)$ is irreducible, and $(\varphi_{2\lambda_i})_{Sp(2n, 2)}$ and $(\varphi_{2\lambda_j})_{Sp(2n, 2)}$ are non-equivalent for $i \neq j$, it follows that m'_i , respectively, m_i is the multiplicity of the irreducible representation $(\varphi_{2\lambda_i})_{Sp(2n, 2)}$ in $Sp(2n, 2)$ -composition series of $Fr_0(R')$, respectively, M . Similarly, m'_0 and m_0 is the multiplicity of the trivial $Sp(2n, 2)$ -module in $Fr_0(R'), M$, respectively. By [1, Lemma 7.2], the composition factors of $\bigoplus_{i=0}^{2n} (V_i)_{Sp(2n, 2)}$ coincide with the composition factors of $(\varphi_{\lambda_n} \otimes \varphi_{\lambda_n})_{Sp(2n, 2)}$ with regarding their multiplicities. It follows that $m'_0 = m_0$ and $m'_i = m_i$ for $i = 1, \dots, n$.

Note that $(V_i)_{Sp(2n, \overline{F}_2)} \cong (V_{2n-i})_{Sp(2n, \overline{F}_2)}$ and $(V_i)_{Sp(2n, \overline{F}_2)}$ contains φ_{λ_i} for $i = 1, \dots, n$. In addition, $(V_0)_{Sp(2n, \overline{F}_2)} \cong (V_{2n})_{Sp(2n, \overline{F}_2)}$. It follows that φ_{λ_n} is the only composition factor of R' which may occur with multiplicity 1. By general theory, $\varphi_{2\lambda_n}$ does occur in M with multiplicity 1.

Proof of Theorem 1.5. Let $\nu = a_1\lambda_1 + \cdots + a_n\lambda_n$, where $0 \leq a_1, \dots, a_n \leq q - 1$, and $\nu' = a_1\lambda_1 + \cdots + a_{n-1}\lambda_{n-1}$. We show that Φ_ν is a direct summand of $\omega \cdot st$ if and only if $\nu = (q - 1)(\lambda_1 + \cdots + \lambda_n)$ or τ . By [1, Lemma 7.4], it suffices to show that st is an irreducible constituent of $(\varphi_\nu \otimes \varphi_{(q-1)\lambda_n})_G$ if and only if $\nu' = \tau$ and $a_n = 0$ or $q - 1$.

It can be deduced from Steinberg [3, Corollary to Theorem 41 and Theorem 43] that $\varphi_\nu = \varphi_{\nu'} \otimes \varphi_{a_n\lambda_n}$. In particular, $\varphi_{\nu'} \otimes \varphi_{(q-1)\lambda_n} = \varphi_{\nu' + (q-1)\lambda_n}$.

If $a_n = 0$, then $\nu = \nu'$ so the representation $\varphi_\nu \otimes \varphi_{(q-1)\lambda_n} = \varphi_{\nu' + (q-1)\lambda_n}$ is irreducible. As $\nu' + (q - 1)\lambda_n$ is a q -restricted dominant weight, $(\varphi_{\nu' + (q-1)\lambda_n})_G$ is irreducible, so it is not equal to st unless $\nu' = \tau$. If $\nu = \nu' = \tau$, then $(\varphi_{\nu' + (q-1)\lambda_n})_G = st$, so st is a direct summand of $(\varphi_\nu \otimes \varphi_{(q-1)\lambda_n})_G$ (when $a_n = 0$).

Suppose that $a_n > 0$. It follows from [1, Corollary 1.3] that every principal indecomposable module Φ_ν occurs as a direct summand of $(\varphi_{(q-1)\lambda_n})_G \otimes st$ with multiplicity at most 1; by [1, Lemma 7.4], this implies that st occurs as an irreducible constituent of $(\varphi_\nu \otimes \varphi_{(q-1)\lambda_n})_G$ with multiplicity at most 1. Therefore, the constituents occurring with multiplicity greater than 1 can be ignored.

We have

$$\varphi_\nu \otimes \varphi_{\lambda_{(q-1)\lambda_n}} = \varphi_{\nu'} \otimes \varphi_{a_n\lambda_n} \otimes \varphi_{(q-1)\lambda_n}.$$

We show that a composition factor of $(\varphi_{a_n\lambda_n} \otimes \varphi_{(q-1)\lambda_n})_G$ has multiplicity greater than 1, unless $a_n = q - 1$, and it is of form $(\varphi_{(q-1)\lambda_n})_G$.

Let $a_n = \sum_{i=0}^{k-1} 2^i b_i$ be the 2-adic expansion of a_n (so $0 \leq b_i \leq 1$). Then

$$\varphi_{a_n \lambda_n} \otimes \varphi_{(q-1)\lambda_n} = (\varphi_{b_0 \lambda_n} \otimes \varphi_{\lambda_n}) \otimes Fr_0(\varphi_{b_1 \lambda_n} \otimes \varphi_{\lambda_n}) \otimes \cdots \otimes Fr_0^{k-1}(\varphi_{b_{k-1} \lambda_n} \otimes \varphi_{\lambda_n}).$$

If $b_i = 0$, then $\varphi_{b_i \lambda_n} \otimes \varphi_{\lambda_n} = \varphi_{\lambda_n}$, otherwise $b_i = 1$ and then, by the lemma above, the only composition factor of $(\varphi_{b_i \lambda_n} \otimes \varphi_{\lambda_n})_G$ occurring with multiplicity 1 is $(\varphi_{2\lambda_j})_G$ for $1 \leq j \leq n$. Therefore, the composition factors of $\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}$ distinct from $\varphi_{2\lambda_n}$ can be ignored. This means that we have to decide whether st occurs as a composition factor of the representation obtained from

$$(\varphi_{\nu'} \otimes \varphi_{b_0 \lambda_n} \otimes \varphi_{\lambda_n} \otimes \cdots \otimes Fr_0^{k-1}(\varphi_{b_{k-1} \lambda_n} \otimes \varphi_{\lambda_n}))_G$$

by omitting $\varphi_{b_i \lambda_n}$ whenever $b_i = 0$, and replacing $\varphi_{b_i \lambda_n} \otimes \varphi_{\lambda_n}$ by $\varphi_{2\lambda_n}$ whenever $b_i = 1$. Let $B = \{i \in \{0, \dots, k-1\} : b_i = 1\}$.

Then we can write the resulting expression as

$$(\otimes_{i \in B} Fr_0^i(\varphi_{2\lambda_n}) \otimes_{i \notin B} Fr_0^i(\varphi_{\lambda_n}))_G = (\otimes_{i \in B} Fr_0^{i+1}(\varphi_{\lambda_n}) \otimes_{i \notin B} Fr_0^i(\varphi_{\lambda_n}))_G.$$

We first consider the set $B' := \{i \in B : i+1 \notin B\}$ (if $i = k-1$, then $i+1$ is regarded to be 0). Suppose that B' is non-empty (this means that $a_n \neq 0$ and $a_n \neq 2^k - 1$). For $i \in B'$ the lemma above applied to the term $(Fr_0^{i+1}(\varphi_{\lambda_n}) \otimes Fr_0^{i+1}(\varphi_{\lambda_n}))_G = (Fr_0^{i+1}(\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}))_G$ tells us that, by the above reason, we can replace $\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}$ by $\varphi_{2\lambda_n}$ so $(Fr_0^{i+1}(\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}))_G$ is replaced by $(Fr_0^{i+2}(\varphi_{\lambda_n}))_G$ ($i \in B'$). One can continue the analysis by repeating this reasoning, but it is more efficient to observe that the process is parallel to the addition of the residues of integers modulo $2^k - 1$. In order to justify this claim, we first compute $(\varphi_{a_n \lambda_n} \otimes Fr_0^i(\varphi_{\lambda_n}))_G$ for every $i = 0, \dots, k-1$, and then in general.

Note that

$$(\varphi_{a_n \lambda_n} \otimes \varphi_{2^i \lambda_n})_G = (Fr_0^i(\varphi_{b_i \lambda_n} \otimes \varphi_{\lambda_n}) \otimes_{j \neq i} \varphi_{2^j b_j \lambda_n})_G.$$

If $b_i = 0$, then $(\varphi_{b_i \lambda_n})_G = 1_G$ so $Fr_0^i(\varphi_{b_i \lambda_n} \otimes \varphi_{\lambda_n})_G = (Fr_0^i(\varphi_{\lambda_n}))_G$. Suppose that $b_i = 1$. In view of the lemma above, the composition factors of $\varphi_{b_i \lambda_n} \otimes \varphi_{\lambda_n}$ other than $\varphi_{2\lambda_n}$ occur with multiplicity greater than 1, so they are immaterial for our purpose. Therefore, we are left with $\varphi_{2\lambda_n}$, so it suffices to compute $(Fr_0^i(\varphi_{2\lambda_n}) \otimes_{j \neq i} \varphi_{2^j b_j \lambda_n})_G$. This is equal to $(Fr_0^{i+1}(\varphi_{\lambda_n}) \otimes_{j \neq i} \varphi_{2^j b_j \lambda_n})_G$. If $i+1 = k$, then $(Fr_0^k(\varphi_{\lambda_n}))_G = (\varphi_{2^k \lambda_n})_G \cong (\varphi_{\lambda_n})_G$, and the replacement of $(\varphi_{2^k \lambda_n})_G$ by $(\varphi_{\lambda_n})_G$ is parallel to taking the residue modulo $2^k - 1$. Next, if $b_{i+1} = 0$, then we stop, otherwise we repeat the same trick and obtain $(Fr_0^{i+2}(\varphi_{\lambda_n}) \otimes_{j \neq i, i+1} \varphi_{2^j b_j \lambda_n})_G$. The output of the procedure will be $\varphi_{a_n(i)}$, where $a_n(i) = (a_n + 2^i) \pmod{2^k - 1}$.

In general, applying this to $\varphi_{a_n \lambda_n} \otimes \varphi_{(q-1)\lambda_n} = (\varphi_{a_n \lambda_n} \otimes_{i=0}^{k-1} Fr_0^i(\varphi_{\lambda_n}))_G$, we obtain $(\varphi_{a'_n \lambda_n})_G$, where $a'_n = a_n + 1 + 2 + \cdots + 2^{k-1} \pmod{2^k - 1} = a_n$.

This is also true if B' is empty but $a_n = q - 1$. Indeed, the reasoning above for $i = 0, \dots, k-1$ remains valid, and we obtain $a_n(i) = (a_n + 2^i) \pmod{2^k - 1} = 2^i$. Then again $a'_n = a_n = q - 1$.

Thus, we conclude that it suffices to decide whether st is an irreducible constituent of $(\varphi_{\nu'} \otimes \varphi_{a_n \lambda_n})_G$. As mentioned above, $\varphi_{\nu'} \otimes \varphi_{a_n \lambda_n} = \varphi_{\nu' + a_n \lambda_n}$. Since $\nu' + a_n \lambda_n$ is a q -restricted dominant weight, $(\varphi_{\nu' + a_n \lambda_n})_G$ is irreducible. So st is an irreducible constituent of $(\varphi_{\nu' + a_n \lambda_n})_G$ if and only if $(\varphi_{\nu' + a_n \lambda_n})_G = st$, equivalently, $\nu' + a_n \lambda_n = (q-1)(\lambda_1 + \cdots + \lambda_n)$. This implies $\nu' = (q-1)(\lambda_1 + \cdots + \lambda_{n-1}) = \tau$ and $a_n = q - 1$.

Therefore, $\omega \otimes st = \Phi_\tau \oplus st$, as required. $\square$

The comments to Theorem 1.5 in [1, page 430, line -12] concerning the decomposition numbers of $\omega \cdot St$ cannot be applied to the new version of the theorem. In fact, we have:

Corollary. $\dim \Phi_\tau = |G|_2 \cdot (\dim \varphi_{(q-1)\lambda_n} - 1) = |G|_2 \cdot (q^n - 1)$, and the ordinary character corresponding to Φ_τ is multiplicity free.

Proof. $(\varphi_{(q-1)\lambda_n})_G \otimes st = \Phi_\tau \oplus st$, so $\dim \Phi_\tau = \dim st \cdot (\dim \varphi_{(q-1)\lambda_n} - 1)$. As $\dim st = |G|_2$ and $\dim \varphi_{(q-1)\lambda_n} = 2^{nk} = q^n$, the first claim follows. The second one follows from [1, Corollary 1.3]. $\square$

Example. Let $G = Sp(2, q)$, q even. Then $\tau = 0$; by the above corollary, $\dim \Phi_0 = q(q-1)$.

Example. Let $G = Sp(4, 4)$, so $n = 2$. Then $\tau = 3\lambda_1$ and $|G|_2 = 2^8 = 256$. So $\dim \varphi_{3\lambda_2} = 4^2 = 16$. By the corollary above, $\dim \Phi_\tau = 256 \cdot 15 = 3840$.

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LEHRSTUHL D FÜR MATHEMATIK, RWTH AACHEN UNIVERSITY, 52056 AACHEN, GERMANY
E-mail address: `gerhard.hiss@math.rwth-achen.de`

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA,
 VIA R. COZZI 53, 20126 MILANO, ITALY
E-mail address: `alexandre.zalesski@gmail.com`