TWO-SIDED BGG RESOLUTIONS OF ADMISSIBLE REPRESENTATIONS

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ABSTRACT. We prove the conjecture of Frenkel, Kac and Wakimoto on the existence of two-sided BGG resolutions of G -integrable admissible representations of affine Kac-Moody algebras at fractional levels. As an application we establish the semi-infinite analogue of the generalized Borel-Weil theorem for minimal parabolic subalgebras which enables an inductive study of admissible representations.

1. INTRODUCTION

Wakimoto modules are representations of non-twisted affine Kac-Moody algebras introduced by Wakimoto [Wak] in the case of $\widehat{\mathfrak{sl}}_2$ and by Feigin and Frenkel [FF1] in the general case. Wakimoto modules have useful applications in representation theory and conformal field theory. In these applications it is important to have a resolution of an irreducible highest weight representation $L(\lambda)$ of an affine Kac-Moody algebra $\mathfrak{g}$ in terms of Wakimoto modules, that is, a complex

$$C^\bullet(\lambda) := C^{i-1}(\lambda) \xrightarrow{d_{i-1}} C^i(\lambda) \xrightarrow{d_i} C^{i+1}(\lambda) \rightarrow \dots$$

with a differential d_i which is a $\mathfrak{g}$ -module homomorphism such that $C^i(\lambda)$ is a direct sum of Wakimoto modules and

$$H^i(C^\bullet(\lambda)) = \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The existence of such a resolution has been proved by Feigin and Frenkel [FF2] for any integrable representations over arbitrary $\mathfrak{g}$ and by Bernard and Felder [BF] and Feigin and Frenkel [FF2] for any admissible representation [KW2] over $\widehat{\mathfrak{sl}}_2$. In their study of W -algebras Frenkel, Kac and Wakimoto [FKW, Conjecture 3.5.1] conjectured the existence of such a resolution for any principle admissible representations over arbitrary $\mathfrak{g}$. In this paper we prove the existence of a two-sided resolution in terms of Wakimoto modules for any $\overset{\circ}{\mathfrak{g}}$ -integrable admissible representations over arbitrary $\mathfrak{g}$ (Theorem 6.11), where $\overset{\circ}{\mathfrak{g}}$ is the classical part of $\mathfrak{g}$. For a general principal admissible representation of $\mathfrak{g}$ we obtain the two-sided resolution in terms of twisted Wakimoto modules (Theorem 6.15).

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Let us sketch the proof of our result briefly. By Fiebig’s equivalence [Fie] the block of the category $\mathcal{O}$ of $\mathfrak{g}$ containing an admissible representation $L(\lambda)$ is equivalent to the block containing an integrable representation¹. Therefore an admissible representation admits a usual BGG type resolution in terms of Verma modules by the result of [GL, RCW]. Hence the idea of Arkhipov [Ark1] is applicable in our situation: One can obtain a twisted BGG resolution of $L(\lambda)$ in terms of twisted Verma modules by applying the twisting functor T_w [Ark1] to the BGG resolution of $L(\lambda)$ as we have the “Borel-Weil-Bott” vanishing property [AS]

$$\mathcal{L}_i T_w L(\lambda) \cong \begin{cases} L(\lambda) & \text{if } i = \ell(w), \\ 0 & \text{otherwise} \end{cases}$$

for $w \in \mathcal{W}(\lambda)$, where $\mathcal{W}(\lambda)$ is the integral Weyl group of λ and $\ell : \mathcal{W}(\lambda) \rightarrow \mathbb{Z}_{\geq 0}$ is the length function; see Theorem 5.12. It remains to show that one can construct an inductive system of twisted BGG resolutions $\{B_w^\bullet(\lambda)\}$ of $L(\lambda)$ such that the complex $\varinjlim_w B_w^\bullet(\lambda)$ gives the required two-sided resolution of $L(\lambda)$; see §6 for the details.

We note that by applying the (generalized) quantum Drinfeld-Sokolov reduction functor [FKW, KRW] to the (duals of the) two-sided BGG resolutions of admissible representations we obtain resolutions of some of simple modules over W -algebras in terms of free field realizations due to the vanishing of the associated BRST cohomology [A1, A2, A3, A4, A5]. In particular we obtain two-sided resolutions of all the minimal series representations [FKW, A7] of the W -algebras associated with principal nilpotent elements in terms of free bosonic realizations.

As an application of the existence of two-sided BGG resolution for admissible representations we prove a semi-infinite analogue of the generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras (Theorem 7.7). This result is important since it enables an inductive study of admissible representations; see our subsequent paper [A6].

This paper is organized as follows. In §2 we collect and prove some basic results about semi-infinite cohomology [Feř] and semi-regular bimodules [Vor1] which are needed for later use. In particular we establish an important property of semi-regular bimodules in Proposition 2.1. In §2 we collect basic results on the semi-infinite Bruhat ordering (or the generic Bruhat ordering) of an affine Weyl group defined by Lusztig [Lus] and study the semi-infinite analogue of parabolic subgroups. Semi-infinite Bruhat ordering is important for us since it (conjecturally) describes the space of homomorphisms between Wakimoto modules; see Proposition 4.10 and Conjecture 4.11. The semi-infinite analogue of the minimal (or maximal) length representatives (Theorem 3.3) is important for describing the semi-infinite restriction functors studied in §7. In §4 we define Wakimoto modules and twisted Verma modules following [Vor2] and study some of their basic properties. In particular we prove the uniqueness of Wakimoto modules which was stated in [FF2] without a proof (Theorem 4.7). In §5 we generalize the Borel-Weil-Bott vanishing property of the twisting functor established in [AS] to the affine Kac-Moody algebra cases. In §6 we state and prove the main results of this paper. In §7 we study the semi-infinite restriction functor and establish the semi-infinite analogue of the

¹In the case $L(\lambda)$ is a non-principal G -integrable admissible representation this is a block of another Kac-Moody algebra.

generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras. This is a non-trivial fact since admissible representations are not unitarizable unless they are integrable.

2. SEMI-REGULAR BIMODULES AND SEMI-INFINITE COHOMOLOGY

2.1. Some notation. For $\mathbb{Z}$ -graded vector spaces $M = \bigoplus_{n \in \mathbb{Z}} M_n$, $N = \bigoplus_{n \in \mathbb{Z}} N_n$ with finite-dimensional homogeneous components let

$$\begin{aligned} \mathcal{H}om_{\mathbb{C}}(M, N) &= \bigoplus_{n \in \mathbb{Z}} \mathcal{H}om_{\mathbb{C}}(M, N)_n, \\ \mathcal{H}om_{\mathbb{C}}(M, N)_n &= \{f \in \mathcal{H}om_{\mathbb{C}}(M, N); f(M_i) \subset N_{i+n}\}, \\ \mathcal{E}nd_{\mathbb{C}}(M) &= \mathcal{H}om_{\mathbb{C}}(M, M). \end{aligned}$$

We denote by $M^* = \bigoplus_{n \in \mathbb{Z}} (M^*)_n$ the space $\mathcal{H}om_{\mathbb{C}}(M, \mathbb{C})$, where $\mathbb{C}$ is considered as a graded vector space concentrated in the degree 0 component. If M, N are module over an algebra A we denote by $\mathcal{H}om_A(M, N)$ the space of all A -homomorphisms in $\mathcal{H}om_{\mathbb{C}}(M, N)$.

2.2. Semi-infinite structure. Let $\mathfrak{g}$ be a complex Lie algebra. A *semi-infinite structure* [Vor1] of $\mathfrak{g}$ is the following data:

- (i) a $\mathbb{Z}$ -grading $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ of $\mathfrak{g}$ with finite-dimensional homogeneous components, $\dim_{\mathbb{C}} \mathfrak{g}_n < \infty$ for all n ,
- (ii) a *semi-infinite 1-cochain* $\gamma : \mathfrak{g} \rightarrow \mathbb{C}$.

Here by a semi-infinite 1-cochain we mean the following: Decompose $\mathfrak{g}$ into the direct sum of two subalgebras

$$\begin{aligned} (1) \quad \mathfrak{g} &= \mathfrak{g}_+ \oplus \mathfrak{g}_-, \\ (2) \quad \mathfrak{g}_+ &= \bigoplus_{i \geq 0} \mathfrak{g}_i, \quad \mathfrak{g}_- = \bigoplus_{i < 0} \mathfrak{g}_i. \end{aligned}$$

A linear map $\gamma : \mathfrak{g} \rightarrow \mathbb{C}$ is called a semi-infinite 1-cochain if γ satisfies

$$\gamma([x, y]) = \text{tr}((\text{ad } x)_{+-}(\text{ad } y)_{-+} - (\text{ad } y)_{+-}(\text{ad } x)_{-+}) \quad \text{for } x, y \in \mathfrak{g},$$

where $(\text{ad } x)_{\pm\mp}$ denotes the composition $\mathfrak{g}_{\mp} \xrightarrow{\text{ad } x} \mathfrak{g} \xrightarrow{\text{projection}} \mathfrak{g}_{\pm}$.

In the rest of this section we assume that $\mathfrak{g}$ is equipped with a semi-infinite structure such that $\gamma(\sum_{i \neq 0} \mathfrak{g}_i) = 0$.

We denote by U, U_-, U_+ , the enveloping algebras of $\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-$, respectively. These algebras inherit a $\mathbb{Z}$ -grading from the corresponding Lie algebras.

Let $\tilde{\mathcal{O}}^{\mathfrak{g}}$ be the category of $\mathbb{Z}$ -graded $\mathfrak{g}$ -modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$ with $\dim M_n < \infty$ for all m on which $\bigoplus_{j > 0} \mathfrak{g}_+$ acts locally nilpotently and $\mathfrak{g}_0$ acts locally finitely.

2.3. Semi-infinite cohomology. Choose a basis $\{x_i; i \in \mathbb{Z}\}$ of $\mathfrak{g}$ such that $\{x_i; i \geq 0\}$ and $\{x_i; i < 0\}$ are bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively, and let $\{c_{ij}^k\}$ be the structure constant: $[x_i, x_j] = \sum_k c_{ij}^k x_k$.

Denote by $\mathcal{C}l(\mathfrak{g})$ the Clifford algebra associated with $\mathfrak{g} \oplus \mathfrak{g}^*$, which has the following generators and relations:

$$\begin{aligned} \text{generators: } & \psi_i, \psi_i^* \quad \text{for } i \in \mathbb{Z}, \\ \text{relations: } & \{\psi_i, \psi_j^*\} = \delta_{i,j}, \quad \{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0. \end{aligned}$$

Here $\{X, Y\} = XY + YX$. The space of the semi-infinite forms $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$ of $\mathfrak{g}$ is by definition the irreducible representation of $\mathcal{Cl}(\mathfrak{g})$ generated by the vector $\mathbf{1}$ satisfying

$$\psi_i \mathbf{1} = 0 \quad \text{for } i \geq 0, \quad \psi_i^* \mathbf{1} = 0 \quad \text{for } i > 0.$$

It is graded by $\deg \mathbf{1} = 0$, $\deg \psi_i^* = 1$ and $\deg \psi_i = -1$: $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}) = \bigoplus_{p \in \mathbb{Z}} \bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g})$.

For $A \in \mathcal{E}nd_{\mathbb{C}}(\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}))$ of degree n set

$$(3) \quad : \psi_k A := \begin{cases} \psi_k A & \text{if } k < 0, \\ (-1)^n A \psi_k & \text{if } k \geq 0, \end{cases} \quad : \psi_k^* A := \begin{cases} \psi_k^* A & \text{if } k \leq 0, \\ (-1)^n A \psi_k^* & \text{if } k > 0. \end{cases}$$

The following defines a $\mathfrak{g}$ -module structure on $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$:

$$(4) \quad x_i \mapsto : \text{ad}(x_i) : + \gamma(x_i),$$

where

$$: \text{ad } x_i := \sum_{j,k} c_{ij}^k : \psi_k \psi_j^* :.$$

For $M \in \tilde{\mathcal{O}}^{\mathfrak{g}}$, define $d \in \text{End}(M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}))$ by

$$d = \sum_{i \in \mathbb{Z}} x_i \otimes \psi_i^* - 1 \otimes \frac{1}{2} \sum_{i,j,k \in \mathbb{Z}} c_{ij}^k : \psi_i^* (: \psi_j^* \psi_k :) : + 1 \otimes \sum_{i \in \mathbb{Z}} \gamma(x_i) \psi_i^*.$$

Then

$$d^2 = 0, \quad d(M \otimes \bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g})) \subset M \otimes \bigwedge^{\frac{\infty}{2}+p+1}(\mathfrak{g}).$$

The cohomology of the complex $(M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}), d)$ is called the *semi-infinite $\mathfrak{g}$ -cohomology* with coefficients in M and denoted by $H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, M)$ ([Feĭ, Vor1]).

2.4. Semi-regular bimodules. We consider the full dual space $\text{Hom}_{\mathbb{C}}(U, \mathbb{C})$ of U as a U -bimodule by $(Xf)(u) = f(uX)$, $(fX)(u) = f(Xu)$ for $X \in \mathfrak{g}$, $f \in \text{Hom}_{\mathbb{C}}(U, \mathbb{C})$, $u \in U$. The graded duals $U_{\pm}^*$ of $U_{\pm}$ are $\mathfrak{g}_{\pm}$ -submodule of $\text{Hom}_{\mathbb{C}}(U, \mathbb{C})$. By abuse of notation we denote by U^* the image of the embedding $U_+^* \otimes_{\mathbb{C}} U_-^* \hookrightarrow \text{Hom}_{\mathbb{C}}(U, \mathbb{C})$, $f_+ \otimes f_- \mapsto (u_- u_+ \mapsto f_+(u_+) f_-(u_-))$, $f_{\pm} \in U_{\pm}^*$, $u_{\pm} \in U$. Then U^* is a U -bisubmodule of $\text{Hom}_{\mathbb{C}}(U, \mathbb{C})$ and coincides with the image of the embedding $U_-^* \otimes_{\mathbb{C}} U_+^* \hookrightarrow \text{Hom}_{\mathbb{C}}(U, \mathbb{C})$, $f_- \otimes f_+ \mapsto (u_+ u_- \mapsto f_+(u_+) f_-(u_-))$.

Following [Vor2] define

$$US(\mathfrak{g}) = H^{\frac{\infty}{2}+0}(\mathfrak{g}, U^* \otimes_{\mathbb{C}} U),$$

where $\mathfrak{g}$ is given the opposite semi-infinite structure and the semi-infinite $\mathfrak{g}$ -cohomology is taken with respect to the diagonal left action on $U^* \otimes_{\mathbb{C}} U$. Here by the opposite semi-infinite structure we mean the one obtained by replacing $\mathfrak{g}_{\pm}$ with $\mathfrak{g}_{\mp}$ and γ with $-\gamma$. The space $US(\mathfrak{g})$ inherits the U -bimodule structure from $U^* \otimes U$ defined by

$$X(f \otimes u) = -(fX) \otimes u, \quad (f \otimes u)X = f \otimes (uX)$$

for $X \in \mathfrak{g}$, $f \in U^*$, $u \in U$. The U -bimodule $US(\mathfrak{g})$ is called the *semi-regular bimodule* of $\mathfrak{g}$. One has

$$(5) \quad US(\mathfrak{g}) \cong U_+^* \otimes_{U_+} U$$

as left $\mathfrak{g}_+$ -modules and right $\mathfrak{g}$ -modules, and the left $\mathfrak{g}$ -module structure of $US(\mathfrak{g})$ is defined through the isomorphism

$$(6) \quad U_+ \otimes_{U_-} U \cong \mathcal{H}om_{\mathbb{C}}(U_+, U) \cong \mathcal{H}om_{U_-}(U, U_- \otimes_{\mathbb{C}} \mathbb{C}_{-\gamma})$$

([Vor1, Soe2, Vor2]).

Let M be a $\mathfrak{g}$ -module and consider the following four left $\mathfrak{g}$ -module structures on $US(\mathfrak{g}) \otimes_{\mathbb{C}} M$:

$$(7) \quad \pi_1(X)(s \otimes m) = -(sX) \otimes m + s \otimes Xm, \quad \pi_2(X)(s \otimes m) = (Xs) \otimes m,$$

$$(8) \quad \pi'_1(X)(s \otimes m) = -(sX) \otimes m, \quad \pi'_2(X)(s \otimes m) = (Xs) \otimes m + s \otimes (Xm),$$

for $X \in \mathfrak{g}$, $s \in US(\mathfrak{g})$, $m \in M$. Clearly, the two actions π_1 and π_2 (resp. π'_1 and π'_2) commute.

Proposition 2.1 (cf. [AG, 6.4]). *For $M \in \tilde{\mathcal{O}}^{\mathfrak{g}}$ the two U -bimodule structures (π_1, π_2) and (π'_1, π'_2) on $US(\mathfrak{g}) \otimes_{\mathbb{C}} M$ are equivalent. Namely there exists a linear isomorphism $\Phi : US(\mathfrak{g}) \otimes_{\mathbb{C}} M \xrightarrow{\sim} US(\mathfrak{g}) \otimes_{\mathbb{C}} M$ such that $\Phi \circ \pi'_i(X) = \pi_i(X) \circ \Phi$ for $i = 1, 2$, $X \in \mathfrak{g}$.*

Proof. Define the linear isomorphism

$$\tilde{\Phi}_1 : U^* \otimes_{\mathbb{C}} U \otimes_{\mathbb{C}} M \xrightarrow{\sim} U^* \otimes_{\mathbb{C}} U \otimes_{\mathbb{C}} M$$

by $\tilde{\Phi}_1(f \otimes u \otimes m) = f \otimes (\Delta(u)(1 \otimes m))$ for $f \in U^*$, $u \in U$, $m \in M$, where $\Delta : U \rightarrow U \otimes_{\mathbb{C}} U$ is the coproduct. We have

$$\tilde{\Phi}_1 \circ \pi_{2,L}(X) = (\pi_{2,L}(X) + \pi_{3,L}(X)) \circ \tilde{\Phi}_1,$$

$$\tilde{\Phi}_1 \circ (\pi_{2,R}(X) + \pi_{3,R}(X)) = \pi_{2,R}(X) \circ \tilde{\Phi}_1,$$

where $\pi_{i,L}$ (resp. $\pi_{i,R}$) denotes the left action (resp. the right action) of $\mathfrak{g}$ on the i -th factor of $U^* \otimes U \otimes M$, and M is considered as a right $\mathfrak{g}$ -module by the action $mx = -xm$ for $m \in M$, $x \in \mathfrak{g}$.

Next consider the graded dual $M^* = \bigoplus_n (M^*)_n$ as a right module by the action $(fX)(m) = f(Xm)$. Let

$$\Psi : U^* \otimes_{\mathbb{C}} M \xrightarrow{\sim} U^* \otimes_{\mathbb{C}} M$$

be the linear isomorphism defined by $\Psi(f \otimes m)(u \otimes g) = (f \otimes m)((1 \otimes g)\Delta(u))$ for $f \in U^*$, $m \in M$, $u \in U$, $g \in M^*$, where M is identified with $(M^*)^*$. Extend this to the linear isomorphism

$$\tilde{\Phi}_2 : U^* \otimes_{\mathbb{C}} U \otimes_{\mathbb{C}} M \xrightarrow{\sim} U^* \otimes_{\mathbb{C}} U \otimes_{\mathbb{C}} M$$

by setting $\tilde{\Phi}_2(f \otimes u \otimes m) = \sum_i f_i \otimes u \otimes m_i$ if $\Psi(f \otimes m) = \sum_i f_i \otimes m_i$ with $f_i \in U^*$, $m_i \in M$. Then

$$\tilde{\Phi}_2 \circ \pi_{1,R}(X) = (\pi_{1,R}(X) + \pi_{3,R}(X)) \circ \tilde{\Phi}_2,$$

$$\tilde{\Phi}_2 \circ (\pi_{1,L}(X) + \pi_{3,L}(X)) = \pi_{1,L}(X) \circ \tilde{\Phi}_2.$$

Set

$$\tilde{\Phi} = \tilde{\Phi}_2 \circ \tilde{\Phi}_1 : U^* \otimes_{\mathbb{C}} U \otimes_{\mathbb{C}} M \xrightarrow{\sim} U^* \otimes_{\mathbb{C}} U \otimes_{\mathbb{C}} M.$$

Then

$$(9) \quad \begin{aligned} \tilde{\Phi} \circ (\pi_{1,L}(X) + \pi_{2,L}(X)) &= \tilde{\Phi}_2 \circ (\pi_{1,L}(X) + \pi_{2,L}(X) + \pi_{3,L}(X)) \circ \tilde{\Phi}_1 \\ &= (\pi_{1,L}(X) + \pi_{2,L}(X)) \circ \tilde{\Phi}, \end{aligned}$$

$$(10) \quad \tilde{\Phi} \circ (\pi_{2,R}(X) + \pi_{3,R}(X)) = \tilde{\Phi}_2 \circ \pi_{2,R}(X) \circ \tilde{\Phi}_1 = \pi_{2,R}(X) \circ \tilde{\Phi},$$

$$(11) \quad \tilde{\Phi} \circ \pi_{1,R}(X) = \tilde{\Phi}_2 \circ \pi_{1,R}(X) \circ \tilde{\Phi}_1 = (\pi_{1,R}(X) + \pi_{3,R}(X)) \circ \tilde{\Phi}.$$

By (9) and the definition of $US(\mathfrak{g})$, $\tilde{\Phi}$ gives rise to a linear isomorphism

$$\Phi : US(\mathfrak{g}) \otimes_{\mathbb{C}} M \xrightarrow{\sim} US(\mathfrak{g}) \otimes_{\mathbb{C}} M.$$

Moreover, Φ satisfies the required properties by (10) and (11). $\square$

2.5. Semi-infinite induction. Let $\mathfrak{h} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{h}_n$ be a graded Lie subalgebra of $\mathfrak{g}$ such that $\gamma|_{\mathfrak{h}}$ is a semi-infinite 1-cochain of $\mathfrak{h}$. Following [Vor2] we define the *semi-induced $\mathfrak{g}$ -module* $S\text{-ind}_{\mathfrak{h}}^{\mathfrak{g}} M$ as

$$S\text{-ind}_{\mathfrak{h}}^{\mathfrak{g}} M := H^{\infty+0}(\mathfrak{h}, US(\mathfrak{g}) \otimes_{\mathbb{C}} M),$$

where $US(\mathfrak{g}) \otimes_{\mathbb{C}} M$ is considered as an $\mathfrak{h}$ -module by the action π_1 defined in (7). The space $S\text{-ind}_{\mathfrak{h}}^{\mathfrak{g}} M$ inherits the structure of a $\mathfrak{g}$ -module from the action π_2 defined in (7).

Lemma 2.2. *The assignment $S\text{-ind}_{\mathfrak{h}}^{\mathfrak{g}} : M \mapsto S\text{-ind}_{\mathfrak{h}}^{\mathfrak{g}} M$ defines an exact functor from $\tilde{\mathcal{O}}^{\mathfrak{h}}$ to $\tilde{\mathcal{O}}^{\mathfrak{g}}$.*

Proof. Clearly, $S\text{-ind } M$ is an object of $\tilde{\mathcal{O}}^{\mathfrak{g}}$ since $US(\mathfrak{g}) \otimes_{\mathbb{C}} M$ is. By Proposition 2.1 we may replace the actions of π_1 and π_2 on $US(\mathfrak{g}) \otimes_{\mathbb{C}} M$ with π'_1 and π'_2 , simultaneously. It follows that

$$(12) \quad H^{\infty+\bullet}(\mathfrak{h}, US(\mathfrak{g}) \otimes_{\mathbb{C}} M) \cong H^{\infty+\bullet}(\mathfrak{h}, US(\mathfrak{g})) \otimes_{\mathbb{C}} M.$$

Since $US(\mathfrak{g})$ is free over $\mathfrak{h}_-$ and cofree over $\mathfrak{h}_+$, $H^{\infty+i}(\mathfrak{h}, US(\mathfrak{g})) = 0$ for $i \neq 0$ by [Vor1, Theorem 2.1]. (Note that the spectral sequence on [Vor1] converges since the complex $US(\mathfrak{g}) \otimes \bigwedge^{\infty+\bullet}(\mathfrak{h})$ is a direct sum of finite-dimensional subcomplexes consisting of homogeneous vectors.) We have shown that the functor $S\text{-ind}_{\mathfrak{h}}^{\mathfrak{g}}$ is exact. $\square$

In the case that $\mathfrak{h} = \mathfrak{g}$ and $\gamma_0 = \gamma$, we have the following assertion.

Proposition 2.3 ([Vor2, (1.9)]). *The functor $S\text{-ind}_{\mathfrak{g}}^{\mathfrak{g}} : \tilde{\mathcal{O}}^{\mathfrak{g}} \rightarrow \tilde{\mathcal{O}}^{\mathfrak{g}}$ is isomorphic to the identity functor.*

Proof. As $H^{\infty+0}(\mathfrak{g}, US(\mathfrak{g}))$ is isomorphic to the trivial representation $\mathbb{C}$ of $\mathfrak{g}$ ([Vor1, Theorem 2.1]), (12) gives the $\mathfrak{g}$ -module isomorphism $S\text{-ind}_{\mathfrak{g}}^{\mathfrak{g}} M \cong M$ as required. $\square$

2.6. Suppose that $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$$

with graded subalgebras $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ such that the restrictions $\gamma|_{\mathfrak{a}}$ and $\gamma|_{\bar{\mathfrak{a}}}$ of γ are semi-infinite 1-cochains of $\mathfrak{a}$ and $\bar{\mathfrak{a}}$, respectively.

Lemma 2.4. *$US(\mathfrak{g}) \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} US(\bar{\mathfrak{a}})$ as left $\mathfrak{a}$ -modules and right $\bar{\mathfrak{a}}$ -modules.*

Proof. We have $U_+^* \cong U(\mathfrak{a}_+)^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_+)^*$ as left $\mathfrak{a}_+$ -modules and right $\bar{\mathfrak{a}}_+$ -modules. Consider the composition

$$\begin{aligned} US(\mathfrak{a}) \otimes_{\mathbb{C}} US(\bar{\mathfrak{a}}) &\xrightarrow{\sim} (U(\mathfrak{a}_-) \otimes_{\mathbb{C}} U(\mathfrak{a}_+)^*) \otimes_{\mathbb{C}} (U(\bar{\mathfrak{a}}_+)^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_-)) \\ &\xrightarrow{\sim} U(\mathfrak{a}_+) \otimes_{\mathbb{C}} U_+^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_+) \rightarrow US(\mathfrak{g}), \end{aligned}$$

where the last map is the multiplication map. From the description (5), (6) of the $\mathfrak{g}$ -bimodule structure of semi-regular bimodules one sees that the image of the above map is stable under the left and the right action of $\mathfrak{g}$ on $US(\mathfrak{g})$. Hence the image must coincide with $US(\mathfrak{g})$ since it contains U_+^* . By the equality of the graded dimensions it follows that the above map is an isomorphism. $\square$

Lemma 2.5. *For $M \in \tilde{\mathcal{O}}^{\bar{\mathfrak{a}}}$, $\text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}} M \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} M$ as $\mathfrak{a}$ -modules, where $\mathfrak{a}$ acts only on the first factor $US(\mathfrak{a})$ of $US(\mathfrak{a}) \otimes_{\mathbb{C}} M$.*

Proof. We have

$$\begin{aligned} \text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}}(M) &\cong H^{\frac{\infty}{2}+0}(\bar{\mathfrak{a}}, US(\mathfrak{a}) \otimes_{\mathbb{C}} US(\bar{\mathfrak{a}}) \otimes_{\mathbb{C}} M) \\ &\cong US(\mathfrak{a}) \otimes_{\mathbb{C}} \text{S-ind}_{\bar{\mathfrak{a}}}^{\bar{\mathfrak{a}}} M \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} M \end{aligned}$$

by Lemmas 2.3 and 2.4. $\square$

3. SEMI-INFINITE BRUHAT ORDERING

3.1. Affine Kac-Moody algebras and affine Weyl groups. We first fix some notation which are used for the rest of the paper.

Let $\mathring{\mathfrak{g}}$ be a finite-dimensional complex simple Lie algebra, and fix a Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$. Let $\mathring{\Delta} \subset \mathring{\mathfrak{h}}^*$ be the set of roots of $\mathring{\mathfrak{g}}$. Choose a subset $\mathring{\Delta}_+ \subset \mathring{\mathfrak{h}}^*$ of positive roots and the set $\mathring{\Pi} = \{\alpha_i; i \in I\} \subset \mathring{\Delta}_+$, $I = \{1, 2, \dots, l\}$, of simple roots. Let θ be the highest root, θ_s the highest short root, $\mathring{\Delta}_- = -\mathring{\Delta}_+$, $\mathring{\rho} = \frac{1}{2} \sum_{\alpha \in \mathring{\Delta}_+} \alpha$.

Let $\mathring{Q} = \sum_{\alpha \in \mathring{\Delta}} \mathbb{Z}\alpha \subset \mathring{\mathfrak{h}}^*$, the root lattice of $\mathring{\mathfrak{g}}$. Set $\mathring{\mathfrak{n}} = \bigoplus_{\alpha \in \mathring{\Delta}_+} \mathring{\mathfrak{g}}_{\alpha}$, $\mathring{\mathfrak{n}}_- = \bigoplus_{\alpha \in \mathring{\Delta}_-} \mathring{\mathfrak{g}}_{\alpha}$, where $\mathring{\mathfrak{g}}_{\alpha}$ is the root space of $\mathring{\mathfrak{g}}$ with root α . We have the triangular decomposition

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}.$$

Let $(\mid)$ be the normalized invariant bilinear form of $\mathring{\mathfrak{g}}$. We identify $\mathring{\mathfrak{h}}$ with $\mathring{\mathfrak{h}}^*$ using $(\mid)$. Let $\mathring{\Delta}^{\vee} = \{\alpha^{\vee} : \alpha \in \mathring{\Delta}\}$, the set of coroots, $\mathring{Q}^{\vee} = \sum_{\alpha \in \mathring{\Delta}} \mathbb{Z}\alpha^{\vee} \subset \mathring{\mathfrak{h}} = \mathring{\mathfrak{h}}^*$,

the coroot lattice of $\mathring{\mathfrak{g}}$, $\mathring{\rho}^{\vee} = \frac{1}{2} \sum_{\alpha \in \mathring{\Delta}_+} \alpha^{\vee}$, where $\alpha^{\vee} = 2\alpha/(\alpha|\alpha)$.

Let $\mathring{\mathcal{W}} \subset GL(\mathring{\mathfrak{h}}^*)$ be the Weyl group of $\mathring{\mathfrak{g}}$, and let $s_{\alpha} \in \mathring{\mathcal{W}}$ be the reflection corresponding to $\alpha \in \mathring{\Delta}$: $s_{\alpha}(\lambda) = \lambda - \lambda(\alpha^{\vee})\alpha$.

Let $\mathfrak{g}$ be the affine Kac-Moody algebra associated with $\mathring{\mathfrak{g}}$:

$$\mathfrak{g} = \mathring{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

The commutation relations of $\mathfrak{g}$ are given by

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K, \quad [K, \mathfrak{g}] = 0, \quad [D, xt^n] = nxt^n.$$

We consider $\overset{\circ}{\mathfrak{g}}$ as a subalgebra of $\mathfrak{g}$ by the natural embedding $\overset{\circ}{\mathfrak{g}} \hookrightarrow \mathfrak{g}$, $x \mapsto xt^0$. Let

$$\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

be the Cartan subalgebra of $\mathfrak{g}$. The bilinear form $(|)$ from $\overset{\circ}{\mathfrak{h}}$ to $\mathfrak{h}$ by letting $(K|\overset{\circ}{\mathfrak{h}}) = (D|\overset{\circ}{\mathfrak{h}}) = (K|K) = (D|D) = 0$ and $(D|K) = 1$. We identify $\overset{\circ}{\mathfrak{h}}^*$ with the subspace of $\mathfrak{h}^*$ consisting of elements which vanish on $\mathbb{C}K \oplus \mathbb{C}D$. Thus,

$$\mathfrak{h}^* = \overset{\circ}{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta,$$

where Λ_0 and δ are defined by $\Lambda_0(K) = \delta(D) = 1$, $\Lambda_0(\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}\delta) = \delta(\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}K) = 0$. The number $\langle \lambda, K \rangle$ is called the *level* of λ .

Let $\Delta^{re} = \overset{\circ}{\Delta}_+ \sqcup \{\alpha + n\delta; \alpha \in \overset{\circ}{\Delta}, n \in \mathbb{N}\}$, the set of positive real roots of $\mathfrak{g}$, $\Delta_-^{re} = -\Delta_+^{re}$, $\Delta^{re} = \Delta_+^{re} \sqcup \Delta_-^{re}$ the set of real roots, $\Pi = \{\alpha_0 = -\theta + \delta, \alpha_1, \dots, \alpha_\ell\}$ the set of simple roots.

Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}$, or the *affine Weyl group* of $\overset{\circ}{\mathcal{W}}$. We have

$$\mathcal{W} = \overset{\circ}{\mathcal{W}} \ltimes \overset{\circ}{Q}^\vee.$$

The *extended affine Weyl group* $\mathcal{W}^e$ of $\mathfrak{g}$ is the semidirect product

$$\mathcal{W}^e = \overset{\circ}{\mathcal{W}} \ltimes P^\vee$$

where $\overset{\circ}{P}^\vee = \{\lambda \in \overset{\circ}{\mathfrak{h}}; \langle \alpha, \lambda \rangle \in \mathbb{Z} \text{ for all } \alpha \in \overset{\circ}{\Delta}\}$, the coweight lattice of $\overset{\circ}{\mathfrak{g}}$. We have

$$\mathcal{W}^e = \mathcal{W}_+^e \ltimes \mathcal{W},$$

where $\mathcal{W}_+^e$ is the subgroup of $\mathcal{W}^e$ consisting of elements which fix the set Π .

We denote by t_α or simply by α for the element of $\mathcal{W}^e$ corresponding to $\alpha \in \overset{\circ}{P}^\vee$. The reflection s_α corresponding to $\alpha = \bar{\alpha} + n\delta \in \Delta^{re}$ is given by $s_\alpha = t_{-n\bar{\alpha}^\vee} s_{\bar{\alpha}}$. We set $s_i = s_{\alpha_i}$ for $i \in I := \{0, 1, \dots, \ell\}$, so that $\mathcal{W} = \langle s_i; i \in I \rangle$. The action of $\overset{\circ}{\mathcal{W}}$ on $\overset{\circ}{\mathfrak{h}}^*$ is extended to the action of $\mathcal{W}^e$ on $\mathfrak{h}^*$ by

$$\begin{aligned} w(\Lambda_0) &= \Lambda_0, w(\delta) = \delta \quad w \in \overset{\circ}{\mathcal{W}}, \\ t_\alpha(\lambda) &= \lambda + \langle \Lambda, K \rangle \alpha - (\langle \lambda, \alpha \rangle + \frac{(\alpha|\alpha)}{2} \langle \lambda, K \rangle) \delta, \quad \lambda \in \mathfrak{h}^*. \end{aligned}$$

For $\lambda \in \tilde{\mathfrak{h}}^*$ let $\bar{\lambda} \in \overset{\circ}{\mathfrak{h}}^*$ be its restriction to $\overset{\circ}{\mathfrak{h}}$.

3.2. Twisted Bruhat ordering. Let $\ell : \mathcal{W}^e \rightarrow \mathbb{Z}_{\geq 0}$ be the length function: $\ell(w) = \sharp(\Delta_+^{re} \cap w(\Delta_-^{re}))$. We have

$$(13) \quad \ell(t_\mu y) = \sum_{\alpha \in \Delta_+ \cap y(\Delta_+)} |(\alpha|\mu)| + \sum_{\alpha \in \Delta_+ \cap y(\Delta_-)} |1 - (\alpha|\mu)|$$

for $\mu \in \overset{\circ}{P}^\vee$, $y \in \overset{\circ}{\mathcal{W}}$.

The *twisted length function* [Ark1] $\ell^y : \mathcal{W}^e \rightarrow \mathbb{Z}$ with the twist $y \in \mathcal{W}^e$ is defined by

$$\ell^y(w) = \sharp(\Delta_+^{re} \cap w(\Delta_-^{re}) \cap y(\Delta_+^{re})) - \sharp(\Delta_+^{re} \cap w(\Delta_-^{re}) \cap y(\Delta_-^{re})).$$

Lemma 3.1. *Let $w, y \in \mathcal{W}^e$.*

(i) *Suppose that $\ell(ys_i) = \ell(y) + 1$ for $i \in I$. Then*

$$\ell^{ys_i}(w) = \begin{cases} \ell^y(w) & \text{if } w^{-1}y(\alpha_i) \in \Delta_+^{re}, \\ \ell^y(w) - 2 & \text{if } w^{-1}y(\alpha_i) \in \Delta_-^{re}. \end{cases}$$

(ii) $\ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1})$.

Proof. (i) The assertion follows from the definition and the fact that

$$\Delta_+^{re} \cap ys_i(\Delta_-^{re}) = \Delta_+^{re} \cap y(\Delta_-^{re}) \sqcup \{y(\alpha_i)\} \quad \text{if } \ell(ys_i) = \ell(y) + 1.$$

(ii) We prove by induction on $\ell(y)$. If $\ell(y) = 0$, then $\ell^y(w) = \ell(w) = \ell(y^{-1}w)$. Suppose that $\ell(ys_i) = \ell(y) + 1$. If $w^{-1}y(\alpha_i) \in \Delta_+^{re}$, then $\ell(s_iy^{-1}w) = \ell(y^{-1}w) + 1$. Hence by (i) and induction hypothesis,

$$\ell^{ys_i}(w) = \ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1}) = \ell(s_iy^{-1}w) - \ell(s_iy^{-1}).$$

If $w^{-1}y(\alpha_i) \in \Delta_-^{re}$, then $\ell(s_iy^{-1}w) = \ell(y^{-1}w) - 1$. Again by (i) and induction hypothesis,

$$\ell^{ys_i}(w) = \ell^y(w) - 2 = \ell(y^{-1}w) - 2 - \ell(y^{-1}) = \ell(s_iy^{-1}w) - \ell(s_iy^{-1}).$$

This completes the proof. $\square$

For $w_1, w_2, y \in \mathcal{W}$ and $\gamma \in \Delta^{re}$, write $w_1 \xrightarrow[y]{\gamma} w_2$ if $w_1 = s_\gamma w_2$ and $\ell^y(w_1) > \ell^y(w_2)$. Below, we shall often omit the symbol γ above the arrow. Also, we shall omit the symbol y under the arrow if $y = 1$. By Lemma 3.1(ii) we have

$$(14) \quad w_1 \xrightarrow[y]{y(\gamma)} w_2 \iff y^{-1}w_1 \xrightarrow{y^{-1}\gamma} y^{-1}w_2.$$

In particular for $\beta = y(\alpha_i) \in \Delta_+^{re}$, $\alpha_i \in \Pi$, and $w_1, w_2 \in \mathcal{W}$ such that $\ell^y(w_2) - \ell^y(w_1) = 1$ we have the equivalence

$$(15) \quad \begin{array}{ccc} & s_\beta w_1 & \\ \nearrow y & & \nwarrow y \\ w_1 & & s_\beta w_2 \\ \searrow y & & \nearrow y \\ & w_2 & \end{array} \iff \begin{array}{ccc} & s_\beta w_1 & \\ \nwarrow y & & \nearrow y \\ s_\beta w_2 & & w_2 \end{array}$$

by [BGG, Lemma 11.3].

Define $w \succeq_y w'$ if there exists a sequence $w_1, w_2, \dots, w_k$ of elements of $\mathcal{W}$ such that

$$w \xrightarrow[y]{y} w_1 \xrightarrow[y]{y} w_2 \xrightarrow[y]{y} \dots \xrightarrow[y]{y} w_k \xrightarrow[y]{y} w'.$$

Note that

$$(16) \quad w \succeq_y w' \iff y^{-1}w \succeq y^{-1}w',$$

by (14), where $\succeq = \succeq_1$, the usual Bruhat ordering of $\mathcal{W}$. It follows that $\succeq_y$ defines a partial ordering of $\mathcal{W}$.

We will use the symbol $w \triangleright_y w'$ to denote a covering in the twisted Bruhat order $\succeq_y$. Thus $w \triangleright_y w'$ means that $w \succeq_y w'$ and $\ell^y(w) = \ell^y(w') + 1$.

3.3. Semi-infinite Bruhat ordering. Define the *semi-infinite length* [FF2] $\ell^{\frac{\infty}{2}}(w)$ of $w \in \mathcal{W}^e$ by

$$\ell^{\frac{\infty}{2}}(w) = \sharp\{\alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}); \bar{\alpha} \in \overset{\circ}{\Delta}_+\} - \sharp\{\alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}); \bar{\alpha} \in \overset{\circ}{\Delta}_-\}.$$

We have

$$(17) \quad \ell^{\frac{\infty}{2}}(t_\lambda y) = \ell(y) - 2(\overset{\circ}{\rho}|\lambda)$$

for $\lambda \in \overset{\circ}{P}_+^\vee$, $w \in \overset{\circ}{\mathcal{W}}$.

Set

$$\overset{\circ}{P}_+^\vee = \{\lambda \in \overset{\circ}{P}_+^\vee; \alpha(\lambda) \geq 0 \text{ for all } \alpha \in \overset{\circ}{\Delta}_+\},$$

We say that $\lambda \in \overset{\circ}{P}_+^\vee$ is sufficiently large if $\alpha(\lambda)$ is sufficiently large for all $\alpha \in \overset{\circ}{\Delta}_+$.

By (13) and (17) we have the following assertion.

Lemma 3.2. $\ell^{\frac{\infty}{2}}(w) = \ell^\lambda(w) = -\ell^{-\lambda}(w)$ for a sufficiently large $\lambda \in \overset{\circ}{P}_+^\vee$.

We write

$$w_1 \xrightarrow[\frac{\infty}{2}]{\gamma} w_2$$

for $w_1, w_2 \in \mathcal{W}$ and $\gamma \in \Delta^{re}$ if $w_1 = w_2 s_\gamma$ and $\ell^{\frac{\infty}{2}}(w_1) < \ell^{\frac{\infty}{2}}(w_2)$. (We shall often omit the symbol γ above the arrow.) Define $w \succeq_{\frac{\infty}{2}} w'$ if there exists a sequence $w_1, w_2, \dots, w_k$ of elements of $\mathcal{W}$ such that

$$w \xrightarrow[\frac{\infty}{2}]{} w_1 \xrightarrow[\frac{\infty}{2}]{} w_2 \xrightarrow[\frac{\infty}{2}]{} \dots \xrightarrow[\frac{\infty}{2}]{} w_k \xrightarrow[\frac{\infty}{2}]{} w'.$$

By Lemma 3.2

$$w \succeq_{\frac{\infty}{2}} w' \iff w' \succeq_{t_\lambda} w \quad \text{for a sufficiently large } \lambda \in \overset{\circ}{P}_+^\vee,$$

$$\iff w \succeq_{t_{-\lambda}} w' \quad \text{for a sufficiently large } \lambda \in \overset{\circ}{P}_+^\vee.$$

It follows that $\succeq_{\frac{\infty}{2}}$ defines a partial ordering of $\mathcal{W}$. Following Arkhipov [Ark1], we call it the *semi-infinite Bruhat ordering* on $\mathcal{W}$. By [Soe1, Claim 4.14] the semi-infinite Bruhat ordering coincides with the *generic Bruhat ordering* defined by Lusztig [Lus].

We will use the symbol $w \triangleright_{\frac{\infty}{2}} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\frac{\infty}{2}}$. Thus $w \triangleright_{\frac{\infty}{2}} w'$ means that $w \succeq_{\frac{\infty}{2}} w'$ and $\ell^{\frac{\infty}{2}}(w) = \ell^{\frac{\infty}{2}}(w') - 1$.

3.4. Semi-infinite analogue of parabolic subgroups and minimal (maximal) length representatives. Let S be a subset of $\overset{\circ}{\Pi}$, $\overset{\circ}{\Delta}_S$ the subroot system of $\overset{\circ}{\Delta}$ generated by $\alpha_i \in S$, $\overset{\circ}{\Delta}_S = \bigsqcup_{i=1}^r \overset{\circ}{\Delta}_{S,i}$ the decomposition into the simple subroot systems $\overset{\circ}{\Delta}_{1,S}, \dots, \overset{\circ}{\Delta}_{r,S}$. Let θ_i be the longest root of $\overset{\circ}{\Delta}_{S,i}$.

Set

$$\Delta_S = \{\alpha + n\delta \in \Delta^{re}; \alpha \in \overset{\circ}{\Delta}_S, n \in \mathbb{Z}\}, \quad \mathcal{W}_S = \langle s_\alpha; \alpha \in \Delta_S \rangle \subset \mathcal{W}.$$

Then Δ_S is a subroot system of Δ^{re} isomorphic to the affine root system associated with $\overset{\circ}{\Delta}_S$. Put $\Delta_{S,+} = \Delta_S \cap \Delta_+^{re}$, the set of positive root of Δ_S . Then $\Pi_S = S \sqcup \{-\theta_1 + \delta, \dots, -\theta_s + \delta\}$ is a set of simple roots of Δ_S . We have $\mathcal{W}_S = \overset{\circ}{\mathcal{W}}_S \ltimes t_{\overset{\circ}{Q}_S^\vee}$,

where $\mathring{Q}_S^\vee = \sum_{\alpha \in \mathring{\Delta}_S} \mathbb{Z}\alpha^\vee$. By (17), the restriction of the semi-infinite length function to $\mathcal{W}_S$ coincides with the semi-infinite length function of the affine Weyl group $\mathcal{W}_S$. Define

$$\mathcal{W}^S = \{w \in \mathcal{W}; w^{-1}(\Delta_{S,+}) \subset \Delta_+^{re}\}.$$

Theorem 3.3 ([Pet]). *The multiplication map $\mathcal{W}_S \times \mathcal{W}^S \rightarrow \mathcal{W}$, $(u, v) \mapsto uv$, is a bijection. Moreover, we have*

$$\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v) \quad \text{for } u \in \mathcal{W}_S, v \in \mathcal{W}^S.$$

Proof. First, we show the injectivity of the multiplication map. Suppose that $u_1v_1 = u_2v_2$ with $u_i \in \mathcal{W}_S$, $v_i \in \mathcal{W}^S$. Then $v_1 = uv_2$ with $u = u_1^{-1}u_2 \in \mathcal{W}_S$. If $u \neq 1$, then there exists $\alpha \in \Delta_{S,+}$ such that $u^{-1}(\alpha) \in -\Delta_{S,+}$. But then $v_2 \in \mathcal{W}^S$ implies that $v_1^{-1}(\alpha) = v_2^{-1}u^{-1}(\alpha) \in \Delta_-^{re}$, and this contradicts that $v_1 \in \mathcal{W}^S$. Hence $u_1 = u_2$, and so $v_1 = v_2$.

Second, we show that the multiplication map $\mathcal{W}_S \times \mathcal{W}^S \rightarrow \mathcal{W}$ is surjective. We will prove by induction on $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re})$ that there exists $u \in \mathcal{W}_S$ such that $u^{-1}w \in \mathcal{W}^S$. If $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re}) = 0$, $w \in \mathcal{W}^S$ there is nothing to show. Suppose that $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re}) > 0$. Then there exists $\beta \in \Pi_S$ such that $w^{-1}(\beta) \in \Delta_-^{re}$. Indeed, any element $\alpha \in \Delta_{S,+}$ is expressed as $\alpha = \sum_{\beta \in \Pi_S} n_\beta \beta$ with $n_\beta \in \mathbb{Z}_{\geq 0}$. Thus $w^{-1}(\alpha) = \sum_{\beta \in \Pi_S} n_\beta w^{-1}(\beta) \in \Delta_-^{re}$ implies that one of $w^{-1}(\beta)$ must belong to Δ_-^{re} . Now because $(s_\beta w)^{-1}(\Delta_{S,+}) = w^{-1}s_\beta(\Delta_{S,+}) = w^{-1}(\Delta_{S,+} \setminus \{\beta\} \sqcup \{-\beta\}) = w^{-1}(\Delta_{S,+}) \setminus \{w^{-1}(\beta)\} \sqcup \{-w^{-1}(\beta)\}$,

$$(s_\beta w)^{-1}(\Delta_{S,+}) \cap \Delta_-^{re} = w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re} \setminus \{w^{-1}(\beta)\}.$$

Hence by applying the induction hypothesis to $s_\beta w$ we find an element $u \in \mathcal{W}_S$ such that $u^{-1}s_\beta w \in \mathcal{W}^S$.

Finally, we prove the equality of the semi-infinite length. By (17), we have $\ell^{\frac{\infty}{2}}(t_\mu w) = \ell^{\frac{\infty}{2}}(t_\mu) + \ell^{\frac{\infty}{2}}(w)$ for any $\mu \in \mathring{Q}^\vee$. Hence we may assume that $u \in \mathring{\mathcal{W}}_S$. We will prove by induction on the length $\ell(u)$ of $u \in \mathring{\mathcal{W}}_S$ that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v)$ for any $v \in \mathcal{W}^S$. Suppose that $\ell(u) = 1$, so that $u = s_i$ for some $\alpha_i \in S$. Let $v = t_\mu y \in \mathcal{W}^S$ with $\mu \in \mathring{Q}^\vee$, $y \in \mathring{\mathcal{W}}$. Note that $v \in \mathcal{W}^S$ is equivalent to:

$$(18) \quad \text{if } \alpha \in \mathring{\Delta}_{S,+} \text{ then } \alpha(\mu) = \begin{cases} 0 & \text{if } y^{-1}(\alpha) \in \mathring{\Delta}_+, \\ 1 & \text{if } y^{-1}(\alpha) \in \mathring{\Delta}_-. \end{cases}$$

Since

$$\ell^{\frac{\infty}{2}}(s_i t_\mu y) = \ell(t_{s_i(\mu)} s_i y) = \ell(s_i y) - 2(\rho|\mu - \alpha_i(\mu)\alpha_i^\vee) = \ell(s_i y) - 2(\rho|\mu) + 2\alpha_i(\mu),$$

(18) implies that $\ell^{\frac{\infty}{2}}(s_i v) = \ell^{\frac{\infty}{2}}(v) + 1$. Next let $u = s_i u_1 \in \mathring{\mathcal{W}}_S$ with $u_1 \in \mathring{\mathcal{W}}_S$, $\alpha_i \in S$, $\ell(u) = \ell(u_1) + 1$, so that $u_1^{-1}(\alpha_i) \in \mathring{\Delta}_+$. Let $v = t_\mu y \in \mathcal{W}^S$ as above. We have

$$\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(t_{s_i u_1(\mu)} s_i u_1 y) = \ell(s_i u_1 y) - 2(\rho|s_i u_1(\mu)).$$

If $\ell(s_i u_1 y) = \ell(u_1 y) + 1$, then $\mathring{\Delta}_+ \ni (u_1 y)^{-1}(\alpha_i) = y^{-1}(u_1^{-1}(\alpha_i))$. Hence $(\mu|u_1^{-1}(\alpha_i)) = 0$ by (18), which means $s_i u_1(\mu) = u_1(\mu)$. By the induction hypothesis, this proves

that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v)$. If $\ell(s_i u_1 y) = \ell(u_1 y) - 1$, then $\overset{\circ}{\Delta}_- \ni (u_1 y)^{-1}(\alpha_i) = y^{-1}(u_1^{-1}(\alpha_i))$. So (18) gives $(\mu|u_1^{-1}(\alpha_i)) = 1$, which means $s_i u_1(\mu) = u_1(\mu) - \alpha_i^\vee$. By the induction hypothesis, this proves that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v)$ as required. $\square$

4. WAKIMOTO MODULES AND TWISTED VERMA MODULES

4.1. The category $\mathcal{O}$ of $\mathfrak{g}$. For any $\mathfrak{h}$ -module M we set $M_\mu = \{m \in M; hm = \mu(h)m \text{ for all } h \in \mathfrak{h}\}$.

Let $\mathcal{O}^\mathfrak{g}$ be the full subcategory of $\tilde{\mathcal{O}}^\mathfrak{g}$ consisting of modules on which $\mathfrak{h}$ acts semisimply. The formal character of $M \in \mathcal{O}^\mathfrak{g}$ is defined by

$$\text{ch } M = \sum_{\mu \in \mathfrak{h}^*} (\dim_{\mathbb{C}} M_\mu) e^\mu.$$

Let $\mathcal{O}_k^\mathfrak{g}$ be the full subcategory of $\mathcal{O}^\mathfrak{g}$ consisting of objects of level k , where a $\mathfrak{g}$ -module M is said to be of level k if K acts as the multiplication by k .

4.2. Twisting functors and twisted Verma modules. By abuse of notation we denote also by w a Tits lifting of $w \in \mathcal{W}^e$ to $\text{Aut}(\mathfrak{g})$.

For each $w \in \mathcal{W}$ the twisting functor $T_w : \mathcal{O}^\mathfrak{g} \rightarrow \mathcal{O}^\mathfrak{g}$ is defined as follows ([Ark1]): Let $\mathfrak{n}_w = \mathfrak{n}_- \cap w^{-1}(\mathfrak{n}_+)$ and set $N_w = U(\mathfrak{n}_w)$. Put

$$S_w = U \otimes_{N_w} N_w^*.$$

The space S_w has a U -bimodule structure, which is described as follows: Let $f \in \mathfrak{n}_- \setminus \{0\}$, and set $U_{(f)} = U \otimes_{\mathbb{C}[f]} \mathbb{C}[f, f^{-1}]$. Then $U_{(f)}$ is an associative algebra which contains U as a subalgebra. We set $S_f = U_{(f)}/U$. Choose a filtration $\mathfrak{n}_w = F^0 \supset F^1 \supset \dots \supset F^r \supset 0$, $r = \ell(w)$, consisting of ideals $F^p \subset \mathfrak{n}_w$ of codimension p . If $f_p \in F^{p-1} \setminus F^p$ we have an isomorphism of U -bimodules

$$(19) \quad S_w = S_{f_1} \otimes_U S_{f_2} \otimes_U \dots \otimes_U S_{f_r}.$$

We have

$$(20) \quad S_w \cong N_w^* \otimes_{N_w} U$$

as right U -modules and left N_w -modules. Put

$$\mathbf{1}_w^* = f_1^{-1} \otimes f_2^{-1} \otimes \dots \otimes f_r^{-1} \in S_w.$$

For $M \in \mathcal{O}^\mathfrak{g}$ define

$$T_w(M) = \phi_w(S_w \otimes_{U(\mathfrak{g})} M),$$

where ϕ_w means that the action of $\mathfrak{g}$ is twisted by the automorphism w of $\mathfrak{g}$. This defines a right exact functor $T_w : \mathcal{O}^\mathfrak{g} \rightarrow \mathcal{O}^\mathfrak{g}$ such that

$$(21) \quad T_w s_i \cong T_w T_i \quad \text{if } \alpha_i \in \Pi \text{ and } \ell(ws_i) = \ell(w) + 1,$$

where $T_i = T_{s_i}$.

The functor T_w admits a right adjoint functor G_w in the category $\mathcal{O}^\mathfrak{g}$ ([AS, §4]):

$$G_w(M) = \text{Hom}_U(S_w, \phi_w^{-1}(M)).$$

It is straightforward to extend the definition of T_w and G_w to $w \in \mathcal{W}^e$ ([A1]).

The following assertion follows in the same manner as [Soe2, Theorem 2.1].

Lemma 4.1. *Let $M \in \mathcal{O}^\mathfrak{g}$, $w \in \mathcal{W}^e$,*

- (i) *Suppose that M is free over $\mathfrak{n}_w$. Then $M \cong G_w T_w(M)$.*

(ii) Suppose that M is cofree over $w(\mathfrak{n}_w)$. Then $M \cong T_w G_w(M)$.

For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the Verma module of $\mathfrak{g}$ with highest weight λ . Set

$$M^w(\lambda) = T_w M(w^{-1} \circ \lambda).$$

The $\mathfrak{g}$ -module $M^w(\lambda) \in \mathcal{O}^{\mathfrak{g}}$ is called the *twisted Verma module* $M^w(\lambda)$ with highest weight λ and twist $w \in \mathcal{W}^e$. Note that by (20) we have

$$(22) \quad \begin{aligned} M^w(\lambda)_{\mu} &\cong \phi_w(N_w^* \otimes_{N_w} U(\mathfrak{n}_-))_{\mu-\lambda} \\ &\cong (U(w(\mathfrak{n}_-) \cap \mathfrak{n}_+)^* \otimes_{\mathbb{C}} U(w(\mathfrak{n}_-) \cap \mathfrak{n}_-))_{\mu-\lambda} \end{aligned}$$

as $\mathfrak{h}$ -modules. Hence

$$\text{ch } M^w(\lambda) = \text{ch } M(\lambda).$$

In particular $M^w(\lambda)$ is an object of $\mathcal{O}^{\mathfrak{g}}$.

By Lemma 4.1(i) we have

$$M(\mu) \cong G_w M^w(w \circ \mu).$$

Hence the functor T_w gives the isomorphism

$$(23) \quad \text{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu)) \xrightarrow{\sim} \text{Hom}_{\mathfrak{g}}(M^w(w \circ \lambda), M^w(w \circ \mu))$$

for $\lambda, \mu \in \mathfrak{h}^*$.

We have [AL, Proposition 6.3]

$$(24) \quad M^w(\lambda) \cong M(\lambda) \quad \text{if } \langle \lambda + \rho, \alpha^{\vee} \rangle \notin \mathbb{N} \quad \text{for all } \alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}).$$

4.3. Hom spaces between twisted Verma modules. For $\lambda \in \mathfrak{h}^*$ let $\Delta(\lambda)$ and $\mathcal{W}(\lambda)$ be its *integral root system* and *integral Weyl group*, respectively:

$$\begin{aligned} \Delta(\lambda) &= \{\alpha \in \Delta^{re}; \langle \lambda + \rho, \alpha^{\vee} \rangle \in \mathbb{Z}\}, \\ \mathcal{W}(\lambda) &= \langle s_{\alpha}; \alpha \in \Delta(\lambda) \rangle \subset \mathcal{W}. \end{aligned}$$

Let $\Delta(\lambda)_+ = \Delta(\lambda) \cap \Delta_+^{re}$ be the set of positive roots of $\Delta(\lambda)$, $\Pi(\lambda) \subset \Delta(\lambda)_+$ the set of simple roots of $\Delta(\lambda)$, and $\ell : \mathcal{W}(\lambda) \rightarrow \mathbb{Z}_{\geq 0}$ the length function.

For $y \in \mathcal{W}(\lambda)$ the twisted length function ℓ^y and the twisted Bruhat ordering $\succeq_{\lambda, y}$ are defined for $\mathcal{W}(\lambda)$. We will use the symbol $w \triangleright_{\lambda, y} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\lambda, y}$.

Recall that a weight $\lambda \in \mathfrak{h}^*$ is called *regular dominant* if

$$\langle \lambda + \rho, \alpha^{\vee} \rangle \notin \{0, -1, -2, \dots\}$$

for all $\alpha \in \Delta_+^{re}$. It is called *regular anti-dominant* if

$$\langle \lambda + \rho, \alpha^{\vee} \rangle \notin \{0, 1, 2, \dots\}$$

for all $\alpha \in \Delta_+^{re}$.

Theorem 4.2. Let $w, w', y \in \mathcal{W}(\lambda)$.

(i) If λ is regular dominant, then

$$\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(M^y(w \circ \lambda), M^y(w' \circ \lambda)) = \begin{cases} 1 & \text{if } w \succeq_{\lambda, y} w', \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If λ is regular anti-dominant, then

$$\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(M^y(w \circ \lambda), M^y(w' \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq_{\lambda, y} w', \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) By (23) the assertion follows from (16) and [KT, Proposition 2.5.5 (ii)]. Proof of (ii) is similar. $\square$

4.4. Wakimoto modules. Let $\mathfrak{g}, \mathfrak{h}$ be as in §3.1, and let us consider the $\mathbb{Z}$ -grading of $\mathfrak{g}$ with $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_1 = \bigoplus_{\alpha \in \Pi} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ of root α . Let $\rho = \check{\rho} + h^\vee \Lambda_0 \in \mathfrak{h}^*$, where $h^\vee$ is the dual Coxeter number of $\check{\mathfrak{g}}$. Then $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in \Pi$ and 2ρ define a semi-infinite 1-cochain of $\mathfrak{g}$ [Ark2].

Let $L\check{\mathfrak{n}}, L\check{\mathfrak{n}}_-, \mathfrak{a}$ and $\bar{\mathfrak{a}}$ be graded subalgebras of $\mathfrak{g}$ defined by

$$\begin{aligned} L\check{\mathfrak{n}} &= \check{\mathfrak{n}}[t, t^{-1}], & L\check{\mathfrak{n}}_- &= \check{\mathfrak{n}}_-[t, t^{-1}], \\ \mathfrak{a} &= L\check{\mathfrak{n}} \oplus \check{\mathfrak{h}}[t^{-1}]t^{-1}, & \bar{\mathfrak{a}} &= L\check{\mathfrak{n}}_- \oplus \check{\mathfrak{h}}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D. \end{aligned}$$

Then $0 = 2\rho|_{L\check{\mathfrak{n}}} = 2\rho|_{L\check{\mathfrak{n}}_-} = 2\rho|_{\mathfrak{a}}$ gives semi-infinite 1-cochains of $L\check{\mathfrak{n}}, L\check{\mathfrak{n}}_-, \mathfrak{a}$, and $2\rho|_{\bar{\mathfrak{a}}}$ gives a semi-infinite 1-cochain of $\bar{\mathfrak{a}}$.

Following [Vor2] we define the *Wakimoto module* $W(\lambda)$ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ by

$$W(\lambda) = \text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h}$ corresponding to λ regarded as a $\bar{\mathfrak{a}}$ -module by the natural projection $\bar{\mathfrak{a}} \twoheadrightarrow \mathfrak{h}$. By Lemma 2.5 we have

$$(25) \quad W(\lambda) \cong US(\mathfrak{a}) \text{ as } \mathfrak{a}\text{-modules},$$

and hence

$$(26) \quad H^{\frac{\infty}{2}+i}(\mathfrak{a}, W(\lambda)) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases} \text{ as } \mathfrak{h}\text{-modules},$$

$$(27) \quad \text{ch } W(\lambda) = \text{ch } M(\lambda).$$

In particular $W(\lambda)$ is an object of $\mathcal{O}^{\mathfrak{g}}$.

Theorem 4.7 below shows that the above definition of Wakimoto module coincides with that of Feigin and Frenkel [FF2, Fre2].

4.5. Wakimoto modules as inductive limits of twisted Verma modules.

Let $y, w, u \in \mathcal{W}$ such that $w = yu$ and $\ell(w) = \ell(y) + \ell(u)$. Then $T_w = T_y T_u$ and $S_w \cong S_y \otimes_U \phi_y(S_u)$. Let

$$j_{w,y} : S_y \longrightarrow S_w$$

be the homomorphism of left U -modules which maps $s \in S_y$ to $s \otimes \mathbf{1}_u^* \in S_y \otimes_U \phi_y(S_u) = S_w$. Define $\nu_{w,y}^\lambda \in \text{Hom}_{\mathfrak{g}}(M^y(\lambda), M^w(\lambda))$ by

$$\nu_{w,y}^\lambda(s \otimes v_{y^{-1} \circ \lambda}) = j_{w,y}(s) \otimes v_{w^{-1} \circ \lambda} \quad \text{for } s \in S_y,$$

where v_μ denotes the highest weight vector of $M(\mu)$ for $\mu \in \mathfrak{h}^*$. Then

$$\text{Hom}(M^y(\lambda), M^w(\lambda)) = \mathbb{C} \nu_{w,y}^\lambda$$

by (23). We have

$$(28) \quad \nu_{w_3, w_2}^\lambda \circ \nu_{w_2, w_1}^\lambda = \nu_{w_3, w_1}^\lambda$$

if $w_3 = w_2 u_2$, $w_2 = w_1 u_1$ with $\ell(w_1) = \ell(w_2) + \ell(u_2)$, $\ell(w_2) = \ell(w_1) + \ell(u_1)$.

Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\overset{\circ}{P}_+^\vee$ such that $\gamma_i - \gamma_{i-1} \in \overset{\circ}{P}_+^\vee$ and $\lim_{n \rightarrow \infty} \alpha(\gamma_n) = \infty$ for all $\alpha \in \overset{\circ}{\Delta}_+$. Then $t_{-\gamma_{i+1}} = t_{-\gamma_i} t_{-(\gamma_{i+1}-\gamma_i)}$ with $\ell(t_{-\gamma_{i+1}}) = \ell(t_{-\gamma_i}) + \ell(t_{-(\gamma_{i+1}-\gamma_i)})$ for all i . It follows that $\{M^{-\gamma_n}(\lambda) : \nu_{-\gamma_n, -\gamma_n}^\lambda\}$ forms an inductive system of $\mathfrak{g}$ -modules.

Proposition 4.3 ([Ark1, Lemma 6.1.7]). *There is an isomorphism of $\mathfrak{g}$ -modules*

$$W(\lambda) \cong \varinjlim_n M^{-\gamma_n}(\lambda).$$

Proof. For the reader's convenience we shall give a proof of Proposition 4.3 here. Set $W(\lambda)' = \varinjlim_n M^{-\gamma_n}(\lambda)$. First note that

$$\begin{aligned} t_{-\gamma_i}(\mathfrak{n}_{-\gamma_i}) &= t_{-\gamma_i}(\mathfrak{n}_-) \cap \mathfrak{n}_+ = \text{span}_{\mathbb{C}}\{x_\alpha t^n; \alpha \in \Delta_+, 0 \leq n < \alpha(\gamma_i)\}, \\ t_{-\gamma_i}(\mathfrak{n}_-) \cap \mathfrak{n}_- &= (\overset{\circ}{\mathfrak{h}} \oplus \overset{\circ}{\mathfrak{n}})[t^{-1}]t^{-1} \oplus \text{span}_{\mathbb{C}}\{x_{-\alpha} t^{-n}; \alpha \in \Delta_+, n > \alpha(\gamma_i)\}, \end{aligned}$$

where x_α is a root vector of $\overset{\circ}{\mathfrak{g}}$ of root α . Thus we have $t_{-\gamma_1}(\mathfrak{n}_{-\gamma_1}) \subset t_{-\gamma_2}(\mathfrak{n}_{-\gamma_2}) \subset \dots \subset \mathfrak{a}_+$ and $\mathfrak{a}_+ = \bigcup_{i \geq 1} t_{-\gamma_i}(\mathfrak{n}_{-\gamma_i})$. The map $j_{-\gamma_i, -\gamma_j} : S_{-\gamma_i} \rightarrow S_{-\gamma_j}$ restricts to the embedding $j_{-\gamma_i, -\gamma_j} : N_{-\gamma_i}^* \hookrightarrow N_{-\gamma_j}^*$ for $i < j$, and we have

$$U(\mathfrak{a}_+)^* \cong \varinjlim_i \phi_{-\gamma_i}(N_{-\gamma_i}^*)$$

as left $\mathfrak{a}_+$ -modules. Let $j_{-\gamma_i} : \phi_{-\gamma_i}(N_{-\gamma_i}^*) \hookrightarrow U(\mathfrak{a}_+)^*$ be the embedding of left $\phi_{-\gamma_i}(N_{-\gamma_i})$ -modules under the above identification.

Since $t_{-\gamma_i}(\mathfrak{n}_{-\gamma_i}) = \text{span}_{\mathbb{C}}\{x_\alpha t^{-n}; \alpha \in \Delta_+, 0 < n \leq \alpha(\gamma_i)\} \subset \mathfrak{a}$,

$$W(\lambda) \cong T_{-\gamma_i} G_{-\gamma_i}(W(\lambda))$$

by Lemma 4.1(ii). Hence

$$\text{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(\lambda), W(\lambda)) \cong \text{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i}(W(\lambda))).$$

As $\text{ch } G_{-\gamma_i}(W(\lambda)) = \text{ch } M(t_{\gamma_i} \circ \lambda)$, there exists a unique $\mathfrak{g}$ -module homomorphism $\psi_i : M(t_{\gamma_i} \circ \lambda) \rightarrow G_{-\gamma_i}(M)$ which sends $v_{t_{\gamma_i} \circ \lambda}$ to w_i , a vector of $G_{-\gamma_i}(W(\lambda))$ of weight $t_{\gamma_i} \circ \lambda$. Up to a non-zero constant multiplication, w_i is equal to the element of $G_{-\gamma_i}(W(\lambda)) = \text{Hom}_{N_{-\gamma_i}}(N_{-\gamma_i}^*, \phi_{-\gamma_i}^{-1}(W(\lambda)))$ which sends $f \in N_{-\gamma_i}^*$ to $j_{-\gamma_i}(f) \otimes 1_\lambda \in US(\mathfrak{a}) \otimes \mathbb{C}_\lambda = W(\lambda)$. The corresponding homomorphism $T_{-\gamma_i}(\psi_i) : M^{-\gamma_i}(\lambda) \rightarrow W(\lambda)$ is given by

$$(29) \quad T_{-\gamma_i}(\psi_i)(f \otimes v_{t_{\gamma_i} \circ \lambda}) = j_{-\gamma_i}(f) \otimes 1_\lambda \quad \text{for } f \in N_{-\gamma_i}^*.$$

It follows that $T_{-\gamma_i}(\psi_j) \circ \nu_{\gamma_j, \gamma_i}^\lambda = T_{-\gamma_i}(\psi_i)$ for $i < j$, and the sequence $\{T_{-\gamma_i}(\psi_j)\}$ yields a $\mathfrak{g}$ -module homomorphism

$$\Phi : W(\lambda)' = \varinjlim_i M^{-\gamma_i}(\lambda) \longrightarrow W(\lambda).$$

Fix $\mu \in \mathfrak{h}^*$. Since $W(\lambda) \cong US(\mathfrak{a})$ as an $\mathfrak{a}$ -module, it follows from (22) that $T_{-\gamma_i}$ restricts to the isomorphism $M^{-\gamma_i}(\lambda)_\mu \xrightarrow{\sim} W(\lambda)_\mu$ for a sufficiently large i . This completes the proof. $\square$

4.6. Endmorphisms of Wakimoto modules.

Proposition 4.4. *Let $\alpha \in \mathring{P}_+^\vee$, $\lambda \in \mathfrak{h}^*$.*

- (i) $T_{-\alpha}W(\lambda) \cong W(t_{-\alpha} \circ \lambda)$.
- (ii) $G_{-\alpha}W(\lambda) \cong W(t_\alpha \circ \lambda)$.

Proof. (i) Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\mathring{P}_+^\vee$ such that $\gamma_i - \gamma_{i-1} \in \mathring{P}_+^\vee$ and $\lim_{n \rightarrow \infty} \beta(\gamma_n) = \infty$ for all $\beta \in \mathring{\Delta}_+$. Set $\gamma'_i = \gamma_i + \alpha$. Then the sequence $\{\gamma'_1, \gamma'_2, \dots\}$ satisfies the same property. Hence by Proposition 4.3 and the fact that a homology functor commutes with inductive limits we have $T_{-\alpha}W(\lambda) \cong T_{-\alpha}(\varinjlim M^{-\gamma_i}(\lambda)) = \varinjlim T_{-\alpha}M^{-\gamma_i}(\lambda) = \varinjlim T_{-\alpha}T_{-\gamma_i}M(t_{\gamma_i} \circ \lambda) = \varinjlim T_{-\gamma'_i}M(t_{\gamma_i} \circ \lambda) = \varinjlim M^{-\gamma'_i}(t_\alpha \circ \lambda) \cong W(t_\alpha \circ \lambda)$. (ii) Since $\mathfrak{n}_{t_{-\alpha}} \subset \mathfrak{a}_-$, $W(\lambda)$ is free over $\mathfrak{n}_{t_{-\alpha}}$. Hence $W(t_\alpha \circ \lambda) = G_{-\alpha}T_{-\alpha}W(t_\alpha \circ \lambda) \cong G_{-\alpha}W(\lambda)$ by Lemma 4.1 and (i). $\square$

Corollary 4.5. *Let $\alpha \in \mathring{P}_+^\vee$. The functor $G_{-\alpha}$ gives the isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda), W(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(W(t_\alpha \circ \lambda), W(t_\alpha \circ \mu))$$

for $\lambda, \mu \in \mathfrak{h}^*$.

Proposition 4.6. *For $\lambda \in \mathfrak{h}^*$ we have $\mathrm{End}_{\mathfrak{g}}(W(\lambda)) = \mathbb{C}$.*

Proof. Let $\{\gamma_1, \gamma_2, \dots\}$ be in Subsection 4.5. Then

$$\begin{aligned} \mathrm{End}_{\mathfrak{g}}(W(\lambda)) &= \mathrm{Hom}_{\mathfrak{g}}(\varinjlim_i M^{-\gamma_i}(\lambda), W(\lambda)) \quad (\text{by Proposition 4.3}) \\ &= \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(\lambda), W(\lambda)) \cong \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i}W(\lambda)) \\ &\cong \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), W(t_{\gamma_i} \circ \lambda)) \quad (\text{by Proposition 4.4}). \end{aligned}$$

As we have seen in the proof of Proposition 4.3, the space $\mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), W(t_{\gamma_i} \circ \lambda))$ is one-dimensional and $\nu_{-\gamma_m, \gamma_n}^\lambda$ induces the isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_m}(\lambda), W(\lambda)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_n}(\lambda), W(\lambda)).$$

This completes the proof. $\square$

4.7. Uniqueness of Wakimoto modules. A finite filtration $0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = M$ of a $\mathfrak{g}$ -module M is called a *Wakimoto flag* if each successive quotient M_i/M_{i-1} is isomorphic to $W(\lambda_i)$ for some λ_i .

Theorem 4.7. *Suppose that k is non-critical, that is, $k \neq -h^\vee$. For an object M of $\mathcal{O}_k^\mathfrak{g}$ the following conditions are equivalent.*

- (i) M admits a Wakimoto flag.
- (ii) $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$ and $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ is finite-dimensional.

If this is the case the multiplicity $(M : W(\lambda))$ of $W(\lambda)$ in a Wakimoto flag of M equals to $\dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)_\lambda$. In particular if

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

as $\mathfrak{h}$ -modules, M is isomorphic to $W(\lambda)$.

The proof of Theorem 4.7 will be given in Subsection 4.8.

We put on record some consequences of Theorem 4.7:

Proposition 4.8. *A tilting module in $\mathcal{O}^{\mathfrak{g}}$ at a non-critical level admits a Wakimoto flag.*

Proof. By definition a tilting module M admits both a Verma flag and a dual Verma flag. It follows that M is free over $\mathfrak{n}_-$ and cofree over $\mathfrak{n}_+$. In particular M is free over $\mathring{\mathfrak{n}}[t^{-1}]t^{-1}$ and cofree over $\mathring{\mathfrak{n}}[t]$. Hence by [Vor1, Theorem 2.1], we have $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$. The fact that $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ is finite-dimensional follows from the Euler-Poincaré principle. $\square$

Proposition 4.9. *Suppose that $\langle \lambda + \rho, K \rangle \notin \mathbb{Q}_{\geq 0}$. Then $W(t_\alpha \circ \lambda) \cong M(t_\alpha \circ \lambda)$ for a sufficiently large $\alpha \in \mathring{P}_+^\vee$.*

Proof. Let α be sufficiently large. By the hypothesis $\langle t_\alpha(\lambda + \rho), \beta^\vee \rangle \notin \mathbb{N}$ for all $\beta \in \Delta_+^{re}$ such that $\bar{\beta} \in \mathring{\Delta}_+$. It follows from [A1, Theorem 3.1] that $M(t_\alpha \circ \lambda)$ is cofree over $\mathring{\mathfrak{n}}[t] = \mathfrak{a}_+$. Because $M(t_\alpha \circ \lambda)$ is obviously free over $\mathfrak{a}_-$ we have

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, M(t_\alpha \circ \lambda)) \cong \begin{cases} \mathbb{C}_{t_\alpha \circ \lambda} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

The following assertion follows from Proposition 4.9 and Corollary 4.5.

Proposition 4.10. *Let $\lambda, \mu \in \mathfrak{h}^*$ be of level k , and suppose that $k + h^\vee \notin \mathbb{Q}_{\geq 0}$. Then*

$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda), W(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(t_\alpha \circ \lambda), M(t_\alpha \circ \mu))$$

for a sufficiently large $\alpha \in \mathring{P}_+^\vee$. In particular if $\lambda \in \mathfrak{h}^*$ is integral, regular anti-dominant, then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(y \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq_{\frac{\infty}{2}} y, \\ 0 & \text{else,} \end{cases}$$

for $w, y \in \mathcal{W}$.

Conjecture 4.11. *Let $\lambda \in \mathfrak{h}^*$ be integral, regular dominant. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(y \circ \lambda)) = \begin{cases} 1 & \text{if } w \succeq_{\frac{\infty}{2}} y, \\ 0 & \text{else,} \end{cases}$$

for $w, y \in \mathcal{W}$.

In Theorem 6.11 below we prove Conjecture 4.11 in the case that $w \triangleright_{\frac{\infty}{2}} y$ (in a slightly more general setting).

4.8. Proof of Theorem 4.7. Let

$$\mathcal{H} = \mathring{\mathfrak{h}}[t, t^{-1}] \oplus \mathbb{C}K \subset \mathfrak{g},$$

the Heisenberg subalgebra. Denote by π_λ the irreducible representation of $\mathcal{H}$ with highest weight λ . We have $\pi_\lambda \cong U(\mathring{\mathfrak{h}}[t^{-1}]t^{-1})$ as a module over $\mathring{\mathfrak{h}}[t^{-1}]t^{-1} \subset \mathcal{H}$ provided that $\lambda(K) \neq 0$.

For $M \in \mathcal{O}_k^{\mathfrak{g}}$ one knows that $H^{\frac{\infty}{2}+\bullet}(\mathring{L}\mathfrak{n}, M)$ is naturally an $\mathcal{H}$ -module of level $k + h^\vee$ ([FF2]).

Lemma 4.12. *Let M be an object of $\mathcal{O}_k^{\mathfrak{g}}$ with $k \neq -h^\vee$. Then the following conditions are equivalent:*

- (i) $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$;
- (ii) $H^{\frac{\infty}{2}+i}(L\mathring{\mathfrak{n}}, M) = 0$ for $i \neq 0$.

Proof. The assumption that $k \neq -h^\vee$ implies that $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}, M)$ is semi-simple as an $\mathcal{H}$ -module and is a direct sum of π_μ s. Consider the Hochschild-Serre spectral sequence for the ideal $L\mathring{\mathfrak{n}} \subset \mathfrak{a}$ to compute $H^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, M)$. By definition, we have

$$E_2^{p,q} = \begin{cases} H_{-p}(\mathfrak{h}[t^{-1}]t^{-1}, H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, M)) & \text{for } p \leq 0, \\ 0 & \text{for } p > 0. \end{cases}$$

By the above mentioned fact $H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, M)$ is free over $U(\mathfrak{h}[t^{-1}]t^{-1})$. Hence

$$E_2^{p,q} = \begin{cases} H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, M)/\mathfrak{h}[t^{-1}]t^{-1}(H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, M)) & \text{for } p = 0, \\ 0 & \text{for } p \neq 0. \end{cases}$$

Therefore the spectral sequence collapses at $E_2 = E_\infty$, and $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$ if and only if $H^{\frac{\infty}{2}+i}(L\mathring{\mathfrak{n}}, M) = 0$ for $i \neq 0$. This completes the proof. $\square$

Proposition 4.13. *Let M be an object of $\mathcal{O}_k$ at a non-critical level k such that $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$. Then*

$$M \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$$

as $\mathfrak{a}$ -modules and $\mathfrak{h}$ -modules, where $\mathfrak{a}$ acts only on the first factor $US(\mathfrak{a})$ and $\mathfrak{h}$ acts as $h(s \otimes m) = \text{ad}(h)(s) \otimes m + s \otimes hm$.

Proof. By Proposition 2.3 it suffices to show that $\text{S-ind}_{\mathfrak{a}}^{\mathfrak{a}} M \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$. As in the proof of Lemma 4.12, we shall consider the Hochschild-Serre spectral sequence for the ideal $L\mathring{\mathfrak{n}} \subset \mathfrak{a}$ to compute $H^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, US(\mathfrak{a}) \otimes M)$. By definition we have

$$(30) \quad E_1^{\bullet,q} = H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, US(\mathfrak{a}) \otimes_{\mathbb{C}} M) \otimes_{\mathbb{C}} \bigwedge^{\bullet} (\mathfrak{h}[t^{-1}]t^{-1}),$$

$$(31) \quad E_2^{p,q} = H_{-p}(\mathfrak{h}[t^{-1}]t^{-1}, H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, US(\mathfrak{a}) \otimes_{\mathbb{C}} M)).$$

To compute the E_1 -term set

$$F^p US(\mathfrak{a}) = \bigoplus_{\langle \mu, \rho^\vee \rangle \geq p} US(\mathfrak{a})_\mu,$$

where $US(\mathfrak{a})$ is considered as an $\mathfrak{h}$ -module by the adjoint action. Then

$$US(\mathfrak{a}) = F^0 US(\mathfrak{a}) \supset F^1 US(\mathfrak{a}) \supset \dots, \quad \bigcap F^p US(\mathfrak{a}) = 0,$$

$$F^p US(\mathfrak{a}) \cdot L\mathring{\mathfrak{n}} \subset F^{p+1} US(\mathfrak{a}).$$

Define the filtration $F^\bullet(US(\mathfrak{a}) \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}))$ by setting

$$F^p(US(\mathfrak{a}) \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}})) = F^p US(\mathfrak{a}) \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}).$$

This defines a decreasing, weightwise regular filtration of the complex. Consider the associated spectral sequence $E'_r \Rightarrow H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}, US(\mathfrak{a}) \otimes_{\mathbb{C}} M)$. Because the associated

graded space $\text{gr } US(\mathfrak{a})$ with respect to this filtration is a trivial $L\check{\mathfrak{n}}^\circ$ -module the E_1 -term of the spectral sequence E'_r is isomorphic to $US(\mathfrak{a}) \otimes_{\mathbb{C}} H^{\frac{\infty}{2}+\bullet}(L\check{\mathfrak{n}}^\circ, M)$. Hence by the hypothesis and Lemma 4.12 the spectral sequence E'_r collapses at $E'_1 = E'_\infty$ and we obtain the isomorphism of $\mathfrak{h}$ -modules

$$(32) \quad H^{\frac{\infty}{2}+i}(L\check{\mathfrak{n}}^\circ, US(\mathfrak{a}) \otimes_{\mathbb{C}} M) \cong \begin{cases} US(\mathfrak{a}) \otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(L\check{\mathfrak{n}}^\circ, M) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

This is also an isomorphism of $\mathfrak{a}$ -modules since $US(\mathfrak{a}) \cong \text{gr } US(\mathfrak{a})$ as *left* $\mathfrak{a}$ -modules, where $x_\alpha t^n \in \mathfrak{a}$ is considered as an operator on $\text{gr } US(\mathfrak{a}) = \bigoplus_p F^p US(\mathfrak{a}) / F^{p+1} US(\mathfrak{a})$

which maps $F^p US(\mathfrak{a}) / F^{p+1} US(\mathfrak{a})$ to $F^{p+\alpha(\check{\rho}^\vee)} US(\mathfrak{a}) / F^{p+\alpha(\check{\rho}^\vee)+1} US(\mathfrak{a})$. We have computed the E_1 -term (30):

$$E_1^{\bullet,q} \cong \begin{cases} US(\mathfrak{a}) \otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(L\check{\mathfrak{n}}^\circ, M) \otimes_{\mathbb{C}} \bigwedge^\bullet(\mathfrak{h}[t^{-1}]t^{-1}) & \text{for } q = 0, \\ 0 & \text{for } q \neq 0. \end{cases}$$

It follows that

$$(33) \quad E_2^{p,q} \cong \begin{cases} US(\mathfrak{a}) \otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(\mathfrak{a}, M) & \text{for } p = q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

as $\mathfrak{h}$ -modules and $\mathfrak{a}$ -modules, see the proof of Lemma 4.12. The spectral sequence collapses at $E_2 = E_\infty$ and we obtain the required isomorphism. $\square$

Set

$$Q_{\frac{\infty}{2},+} = \sum_{\substack{\alpha \in \Delta^{\text{re}} \\ \check{\alpha} \in \check{\Delta}_-}} \mathbb{Z}_{\geq 0} \alpha + \mathbb{Z}_{\geq 0} \delta \subset \mathfrak{h}^*,$$

and define the partial ordering $\leq_{\frac{\infty}{2}}$ on $\mathfrak{h}^*$ by $\mu \leq_{\frac{\infty}{2}} \lambda \iff \lambda - \mu \in Q_{\frac{\infty}{2},+}$. Note that $\mu \leq_{\frac{\infty}{2}} \lambda$ if and only if $t_\alpha \circ \mu \leq t_\alpha \circ \lambda$ for a sufficiently large $\alpha \in Q_{\frac{\infty}{2},+}^\vee$.

Theorem 4.7. Since The direction (i) $\Rightarrow$ (ii) in Theorem 4.7 is obvious by (26), we shall prove that (ii) implies (i). Let $\{\lambda_1, \dots, \lambda_r\}$ be the set of weights of $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ with multiplicities counted, so that

$$(34) \quad M \cong \bigoplus_{i=1}^r US(\mathfrak{a}) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_i}$$

as $\mathfrak{a}$ -modules and $\mathfrak{h}$ -modules by Proposition 4.13. We may assume that if $\lambda_i \leq_{\frac{\infty}{2}} \lambda_j$, then $j < i$.

Set $\lambda = \lambda_1$. We shall show that there is a $\mathfrak{g}$ -module embedding $W(\lambda) \hookrightarrow M$. Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\check{P}_+^\vee$ such that $\gamma_i - \gamma_{i-1} \in \check{P}_+^\vee$ and $\lim_{n \rightarrow \infty} \alpha(\gamma_n) = \infty$ for all $\alpha \in \check{\Delta}_+$, so that $W(\lambda) = \varinjlim_n M^{-\gamma_n}(\lambda)$ by Proposition 4.3. By Lemma 4.1 (ii) we have $M \cong T_{-\gamma_i} G_{-\gamma_i}(M)$, and hence,

$$\text{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(\lambda), M) \cong \text{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i}(M)).$$

By (34), $\text{ch } G_{-\gamma_i}(M) = \sum_{i=1}^r \text{ch } M(t_{\gamma_i} \circ \lambda)$. Let i be sufficiently large so that $t_{\gamma_i} \circ \lambda$ is maximal in $G_{-\gamma_i}(M)$. Denote by Φ_i the $\mathfrak{g}$ -module homomorphism $\psi_i : M(t_{\gamma_i} \circ \lambda) \rightarrow G_{-\gamma_i}(M)$ which sends $v_{t_{\gamma_i} \circ \lambda}$ to a vector of $G_{-\gamma_i}(M)$ of weight $t_{\gamma_i} \circ \lambda$.

As in the proof of Proposition 4.3 $\{T_{-\gamma_i}(\psi_i) : M^{-\gamma_i}(\lambda) \mapsto M\}$ yield an injective $\mathfrak{g}$ -module homomorphism

$$\Phi : W(\lambda) = \varinjlim_i M^{-\gamma_i}(\lambda) \hookrightarrow M.$$

The map Φ induces the homomorphism $H^{\frac{\infty}{2}+0}(\mathfrak{a}, W(\lambda)) = \mathbb{C}_\lambda \rightarrow H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ which is certainly injective. It follows from the long exact sequence associated with the exact sequence $0 \rightarrow W(\lambda) \xrightarrow{\Phi} M \rightarrow M/W(\lambda) \rightarrow 0$ we obtain that $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M/W(\lambda)) = 0$ for $i \neq 0$ and $\dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M/W(\lambda)) = \dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M) - 1$. Theorem 4.7 follows by the induction on $\dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$. $\square$

4.9. Twisted Wakimoto modules. For $w \in \overset{\circ}{\mathcal{W}}$ we have the decomposition $\mathfrak{g} = w(\mathfrak{a}) \oplus w(\bar{\mathfrak{a}})$, and 2ρ defines a semi-infinite 1-cochain of the graded subalgebra $w(\bar{\mathfrak{a}})$. Hence we can define the *twisted Wakimoto module* $W^w(\lambda)$ with highest weight λ and twist $w \in \overset{\circ}{\mathcal{W}}$ by

$$W^w(\lambda) = \text{S-ind}_{w(\bar{\mathfrak{a}})}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h}$ corresponding to λ regarded as a $\bar{\mathfrak{a}}$ -module by the projection $\bar{\mathfrak{a}} \rightarrow \mathfrak{h}$. We have

$$W^w(\lambda) \cong US(w(\mathfrak{a})) \text{ as } w(\mathfrak{a})\text{-modules and } \text{ch } W^w(\lambda) = \text{ch } M(\lambda),$$

$$H^{\frac{\infty}{2}+i}(w(\mathfrak{a}), W^w(\lambda)) \cong \begin{cases} \mathbb{C}_\lambda & \text{for } i = 0, \\ 0 & \text{otherwise,} \end{cases} \text{ as } \mathfrak{h}\text{-modules.}$$

Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\overset{\circ}{P}_+^\vee$ such that $\gamma_i - \gamma_{i-1} \in \overset{\circ}{P}_+^\vee$ and $\lim_{n \rightarrow \infty} \alpha(\gamma_n) = \infty$ for all $\alpha \in \overset{\circ}{\Delta}_+$. The following assertion can be proved in the same manner as Proposition 4.3.

Proposition 4.14. *Let $\lambda \in \mathfrak{h}^*$, $w \in \overset{\circ}{\mathcal{W}}$. There is an isomorphism of $\mathfrak{g}$ -modules*

$$W^w(\lambda) \cong \varinjlim_n M^{-w(\gamma_n)}(\lambda).$$

The following assertion can be proved in the same manner as Theorem 4.7.

Theorem 4.15. *Let $\lambda \in \mathfrak{h}^*$ be non-critical, $w \in \overset{\circ}{\mathcal{W}}$. Let M be an object of $\mathcal{O}^{\mathfrak{g}}$ such that*

$$H^{\frac{\infty}{2}+i}(w(\mathfrak{a}), M) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

as $\mathfrak{h}$ -modules. Then M is isomorphic to $W^w(\lambda)$.

5. BOREL-WEIL-BOTT VANISHING PROPERTY OF TWISTING FUNCTORS

5.1. Left derived functors of twisting functors. The functor T_w , $w \in \mathcal{W}^e$, admits the left derived functor $\mathcal{L}_\bullet T_w$ in the category $\mathcal{O}^{\mathfrak{g}}$ since it is a Lie algebra homology functor:

$$\mathcal{L}_i T_w(M) = \phi_w(H_i(\mathfrak{g}, S_w \otimes_{\mathbb{C}} M)),$$

where $\mathfrak{g}$ acts on $N_w^* \otimes_{\mathbb{C}} M$ by $X(f \otimes m) = -fX \otimes m + f \otimes Xm$. Because

$$(35) \quad \mathcal{L}_i T_w(M) \cong \phi_w(H_i(\mathfrak{n}_w, N_w^* \otimes_{\mathbb{C}} M))$$

as $w(\mathfrak{n}_w)$ -modules, we have the following assertion.

Lemma 5.1. *Suppose $M \in \mathcal{O}^{\mathfrak{g}}$ is free over $\mathfrak{n}_w$. Then $\mathcal{L}_i T_w(M) = 0$ for $i \geq 1$.*

Let $\{e_i, h_i, f_i; i \in I\}$, $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$, be the Chevalley generators of $\mathfrak{g}$. For $i \in I$, let $\mathfrak{sl}_2^{(i)}$ denote the copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$ spanned by $\{e_i, h_i, f_i\}$

Proposition 5.2. *Let $M \in \mathcal{O}^{\mathfrak{g}}$, $i \in I$. Denote by N the largest $\mathfrak{sl}_2^{(i)}$ -integrable submodule of M . Then $T_i(M) \cong T_i(M/N)$, $\text{ch } \mathcal{L}_1 T_i(M) \cong \text{ch } N$ and $\mathcal{L}_p T_i(M) = 0$ for $p \geq 2$.*

Proof. Let $T_i^{(i)}$ denote the twisting functor for $\mathfrak{sl}_2^{(i)}$ corresponding to the reflection s_{α_i} . Because $T_i(M) \cong T_i^{(i)}(M)$ as $\mathfrak{sl}_2^{(i)}$ -modules and $\mathfrak{h}$ -modules, we have

$$(36) \quad \mathcal{L}_p T_i(M) \cong \mathcal{L}_p T_i^{(i)}(M) \quad \text{as } \mathfrak{sl}_2^{(i)}\text{-modules and } \mathfrak{h}\text{-modules.}$$

In particular $\mathcal{L}_p T_i(M) = 0$ for $p \geq 2$. It follows that the exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

yields the long exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{L}_1 T_i(N) \rightarrow \mathcal{L}_1 T_i(M) \rightarrow \mathcal{L}_1 T_i(M/N) \\ \rightarrow T_i(N) \rightarrow T_i(M) \rightarrow T_i(M/N) \rightarrow 0. \end{aligned}$$

Since M/N is free as $\mathbb{C}[f_i]$ -module $\mathcal{L}_1 T_i(M/N) = 0$ by Lemma 5.1. Also, $T_i(N) = 0$ and $\mathcal{L}_1 T_i(N) \cong N$ as $\mathfrak{h}$ -modules by [AS, Theorem 6.1] and (36). This completes the proof. $\square$

Let $L(\lambda) \in \mathcal{O}^{\mathfrak{g}}$ be the irreducible highest weight representation of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$.

Theorem 5.3 ([AS, Theorem 6.1]). *Let $\lambda \in \mathfrak{h}^*$ and suppose that $\langle \lambda, \alpha_i^{\vee} \rangle \in \mathbb{Z}_{\geq 0}$ with $i \in I$. Then*

$$\mathcal{L}_p T_i(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = 1, \\ 0 & \text{if } p \neq 1. \end{cases}$$

Proof. The hypothesis implies that $L(\lambda)$ is $\mathfrak{sl}_2^{(i)}$ -integrable. Therefore $\mathcal{L}_p T_i(L(\lambda)) = 0$ for $p \neq 1$ and $\text{ch } \mathcal{L}_1 T_i(L(\lambda)) = \text{ch } L(\lambda)$ by Proposition 5.2. $\square$

5.2. Twisting functors associated with integral Weyl group.

Lemma 5.4. *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi(\lambda)$. There exists $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ such that $s_{\alpha} = xs_i x^{-1}$, $\ell(s_{\alpha}) = 2\ell(x) + 1$ and $\Delta_+^{re} \cap x(\Delta_-^{re}) \cap \Delta(\lambda) = \emptyset$.*

Proof. Let $s_{\alpha} = s_{j_l} s_{j_{l-1}} \dots s_{j_1}$ be a reduced expression of s_{α} in $\mathcal{W}$. Then

$$\Delta_+^{re} \cap s_{\alpha}(\Delta_-^{re}) = \{\alpha_1, s_{j_1}(\alpha_{j_2}), \dots, s_{j_1} \dots s_{j_{l-1}}(\alpha_{j_l})\}$$

Since $\ell_{\lambda}(\alpha) = 1$, $\Delta_+^{re} \cap s_{\alpha}(\Delta_-^{re}) \cap \Delta(\lambda) = \{\alpha\}$. Thus there exists r such that $\alpha = s_{j_1} \dots s_{j_{r-1}}(\alpha_{j_r})$. Set $x = s_{j_1} \dots s_{j_{r-1}}$, $i = j_r$. Then $s_{\alpha} = s_{x(\alpha_i)} = xs_i x^{-1}$. It follows that $s_{j_l} \dots s_{j_{r+1}} = x$ and $\ell(s_{\alpha}) = 2\ell(x) + 1$. Also $\Delta_+^{re} \cap s_{\alpha}(\Delta_-^{re}) \cap \Delta(\lambda) = \{\alpha\}$ implies that $\Delta_+^{re} \cap x(\Delta_-^{re}) \cap \Delta(\lambda) = \emptyset$. $\square$

Note that if $\lambda, \alpha, \alpha_i, x$ are as in Lemma 5.4, then

$$T_\alpha = T_x \circ T_i \circ T_{x^{-1}}.$$

Let $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ be the block of $\mathcal{O}^{\mathfrak{g}}$ corresponding to λ , that is, the full subcategory of $\mathcal{O}^{\mathfrak{g}}$ consisting of objects M such that $[M : L(\mu)] \neq 0 \Rightarrow \mu \in \mathcal{W}(\lambda) \circ \mu$, where $[M : L(\mu)]$ is the multiplicity of $L(\mu)$ in the local composition factor of M .

Lemma 5.5. *Let $\lambda \in \mathfrak{h}^*$, $y \in \mathcal{W}$, and suppose that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Delta_+^{re} \cap y^{-1}(\Delta_-^{re})$. Then $T_y M(w \circ \lambda) \cong M(yw \circ \lambda)$, $T_y L(w \circ \lambda) \cong L(yw \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$. Moreover, T_w gives an equivalence of categories $\mathcal{O}_{[\lambda]}^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}_{[w \circ \lambda]}^{\mathfrak{g}}$. The same is true for G_w .*

Proof. First note that the assumption implies that $\mathcal{W}(y \circ \lambda) = y\mathcal{W}(\lambda)y^{-1}$.

We prove by induction on $\ell(y)$. Let $\ell(y) = 1$, so that $y = s_i$ for $i \in I$. Then the fact that $T_i M(w\lambda) \cong M(s_i w \circ \lambda)$ with $w \in \mathcal{W}(\lambda)$ follow from (24). By [A1, Theorems 3.1, 3.2] any object of $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ and $\mathcal{O}_{[s_i \circ \lambda]}^{\mathfrak{g}}$ is free over $\mathbb{C}[f_i]$ and cofree over $\mathbb{C}[e_i]$. Hence by Lemma 4.1 T_i gives an equivalence of categories $\mathcal{O}_{[\lambda]}^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}_{[s_i \circ \lambda]}^{\mathfrak{g}}$ with a quasi-inverse G_i . It follows that $T_i L(\lambda)$ is a simple $\mathfrak{g}$ -module which is a quotient of $T_i M(\lambda) = M(s_i \circ \lambda)$, and hence is isomorphic to $L(s_i \circ \lambda)$. Next let $y = s_i z$ with $z \in \mathcal{W}$, $\ell(y) = \ell(z) + 1$. Then $\Delta_+^{re} \cap y^{-1}(\Delta_-^{re}) = \{z^{-1}(\alpha_i)\} \sqcup (\Delta_+^{re} \cap z^{-1}\Delta_-^{re})$. The assertion follows from the induction hypothesis. $\square$

Corollary 5.6. *Let $\lambda, \alpha, \alpha_i, x$ be as in Lemma 5.4. Then T_x give an equivalence of categories $\mathcal{O}_{[x^{-1} \circ \lambda]}^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ such that $T_x M(\mu) \cong M(x \circ \mu)$, $T_x L(\mu) \cong L(x \circ \mu)$ for $\mu \in \mathcal{W}(x^{-1} \circ \lambda) \circ x^{-1}\lambda = x^{-1}\mathcal{W}(\lambda) \circ \lambda$.*

Lemma 5.7. *Let $\lambda \in \mathfrak{h}^*$, $\alpha_i \in \Pi$ such that $\langle \lambda + \rho, \alpha_i^\vee \rangle \notin \mathbb{Z}$. Then $T_i M^w(\lambda) \cong M^{s_i w s_i}(s_i \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$.*

Proof. By Lemma 5.5, $T_i M^w(\lambda) \cong T_i T_w M(w^{-1} \circ \lambda) \cong T_i T_w T_i M(s_i w^{-1} \circ \lambda) \cong T^{s_i w s_i} M(s_i w^{-1} s_i s_i \circ \lambda)$. $\square$

Lemma 5.8. *Let $\lambda \in \mathfrak{h}^*$, $\alpha_i \in \Pi$ such that $\langle \lambda + \rho, \alpha_i^\vee \rangle \notin \mathbb{Z}$. Then $T_i^2 : \mathcal{O}_{[\lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ is isomorphic to the identity functor, and so is $G_i^2 : \mathcal{O}_{[\lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$.*

Proof. By Lemma 5.5 T_i^2 induces an auto-equivalence of the category $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ such that $T_i^2 M(w \circ \lambda) \cong M(w \circ \lambda)$ and $T_i^2(L(w \circ \lambda)) \cong L(w \circ \lambda)$ for all $w \in \mathcal{W}(\lambda)$. The standard argument shows that such a functor must be isomorphic to the identity functor. $\square$

Corollary 5.9. *Let $\lambda \in \mathfrak{h}^*$, $w = s_\alpha y \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$, $y \in \mathcal{W}(\lambda)$, $\ell_\lambda(w) = \ell_\lambda(y) + 1$. Then $T_w : \mathcal{O}_{[\lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[w \circ \lambda]}^{\mathfrak{g}}$ is isomorphic to the functor $T_{s_\alpha} \circ T_y : \mathcal{O}_{[\lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[w \circ \lambda]}^{\mathfrak{g}}$.*

Proposition 5.10. *Let $\lambda \in \mathfrak{h}^*$, $w \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$ and suppose that $\langle w(\lambda + \rho), \alpha^\vee \rangle \notin \mathbb{N}$. Then the following sequence is exact:*

$$0 \rightarrow M(s_\alpha w \circ \lambda) \xrightarrow{\varphi_1} M(w \circ \lambda) \xrightarrow{\varphi_2} M^{s_\alpha}(w \circ \lambda) \xrightarrow{\varphi_3} M^{s_\alpha}(s_\alpha w \circ \lambda) \rightarrow 0,$$

where $\varphi_1, \varphi_2, \varphi_3$ are any non-trivial $\mathfrak{g}$ -homomorphisms.

Proof. First observe that $\text{Hom}_{\mathfrak{g}}(M(s_{\alpha}w \circ \lambda), M(w \circ \lambda)), \text{Hom}_{\mathfrak{g}}(M(w \circ \lambda), M^{s_{\alpha}}(w \circ \lambda))$ and $\text{Hom}_{\mathfrak{g}}(M^{s_{\alpha}}(w \circ \lambda), M^{s_{\alpha}}(s_{\alpha}w \circ \lambda))$ are all one-dimensional. (The first and the third are one-dimensional by Theorem 4.2.) By Lemma 5.4 there exists $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ such that $s_{\alpha} = xs_i x^{-1}$, $\ell(s_{\alpha}) = 2\ell(x) + 1$, and $\Delta_+^{re} \cap x(\Delta_-^{re}) \cap \Delta(\lambda) = \emptyset$. We have

$$\begin{aligned} M(y \circ \lambda) &\cong T_x M(x^{-1}y \circ \lambda), \\ M^{s_{\alpha}}(y \circ \lambda) &= T_x T_i T_{x^{-1}} M(xs_i x^{-1}y \circ \lambda) \\ &\cong T_x T_i M(s_i x^{-1}y \circ \lambda) \cong T_x M^{s_i}(x^{-1}y \circ \lambda) \end{aligned}$$

for $y \in \mathcal{W}(\lambda)$ by Lemma 5.5. Since $\langle x^{-1}w(\lambda + \rho), \alpha_i^{\vee} \rangle = \langle w(\lambda + \rho), \alpha^{\vee} \rangle \in \mathbb{N}$ there is an exact sequence

$$0 \rightarrow M(s_i x^{-1}w \circ \lambda) \rightarrow M(x^{-1}w \circ \lambda) \rightarrow M^{s_i}(x^{-1}w \circ \lambda) \rightarrow M^{s_i}(s_i x^{-1}w \circ \lambda) \rightarrow 0$$

by [AL, Proposition 6.2]. The required exact sequence is obtained by applying the exact functor $T_x : \mathcal{O}_{[x^{-1} \circ \lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ to the above. $\square$

Proposition 5.11. *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi(\lambda)$, $M \in \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$. Take $\alpha_i \in \Pi$, $x \in \mathcal{W}$ such that $\alpha = x(\alpha_i)$ and $x^{-1}\Delta(\lambda)_+ \subset \Delta_+^{re}$ as in Lemma 5.4. Let N' be the largest $\mathfrak{sl}_2^{(i)}$ -integrable submodule of $T_{x^{-1}}(M)$ and set $N = T_x(N') \subset M$. Then $T_{\alpha}(M) \cong T_{s_{\alpha}}(M/N)$, $\text{ch } \mathcal{L}_1 T_{s_{\alpha}}(M) = \text{ch } N$ and $\mathcal{L}_p T_{s_{\alpha}}(M) = 0$ for $p \geq 2$.*

Proof. We have $T_{\alpha} = T_x T_i T_{x^{-1}}$ and $T_{x^{-1}} : \mathcal{O}_{[\lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[x^{-1} \circ \lambda]}^{\mathfrak{g}}$, $T_x : \mathcal{O}_{x^{-1} \circ \lambda}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ are exact functors by Corollary 5.6. Therefore

$$(37) \quad \mathcal{L}_p T_{s_{\alpha}}(M) = T_x(\mathcal{L}_p T_i(T_{x^{-1}}M)).$$

Hence Proposition 5.2 gives that

$$\begin{aligned} T_{s_{\alpha}}(M) &= T_x T_i T_{x^{-1}}(M) \cong T_x T_i(T_{x^{-1}}(M)/N') \cong T_x T_i T_{x^{-1}}(M/N) = T_{s_{\alpha}}(M/N), \\ \text{ch } \mathcal{L}_1 T_{s_{\alpha}}(M) &= \text{ch } T_x T_{x^{-1}}(N) = \text{ch } N, \\ \mathcal{L}_p T_{s_{\alpha}}(M) &= 0 \quad \text{for } p \geq 0. \end{aligned}$$

This completes the proof. $\square$

Theorem 5.12. *Let $\lambda \in \mathfrak{h}^*$ be regular dominant weight, $w \in \mathcal{W}(\lambda)$. Then*

$$\mathcal{L}_p T_w(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\alpha \in \Pi(\lambda)$. Since $T_{x^{-1}}L(\lambda) = L(x^{-1} \circ \lambda)$ and $\langle x^{-1} \circ \lambda + \rho, \alpha_i^{\vee} \rangle = \langle \lambda + \rho, \alpha^{\vee} \rangle \in \mathbb{N}$, $T_{x^{-1}}L(\lambda)$ is $\mathfrak{sl}_2^{(i)}$ -integrable. Thus,

$$\mathcal{L}_p T_i T_{x^{-1}}L(\lambda) \cong \begin{cases} T_{x^{-1}}L(\lambda) & \text{if } p = 1, \\ 0 & \text{if } p \neq 0 \end{cases}$$

by Theorem 5.3. It follows from (37) that

$$(38) \quad \mathcal{L}_p T_{s_{\alpha}}(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the assertion follows in the same manner as in [AS, Corollary 6.2] by Corollary 5.9. $\square$

6. TWO-SIDED BGG RESOLUTIONS OF ADMISSIBLE REPRESENTATIONS

6.1. Admissible representations. A weight $\lambda \in \mathfrak{h}^*$ is called *admissible* if it is regular dominant and

$$\mathbb{Q}\Delta(\lambda) = \mathbb{Q}\Delta^{re}.$$

The irreducible representation $L(\lambda)$ is called admissible if λ is admissible. A complex number k is called an *admissible number* for $\mathfrak{g}$ if the weight $k\Lambda_0$ is admissible.

Let $r^\vee$ be the lacing number of $\mathring{\mathfrak{g}}$, that is, the maximal number of the edges of the Dynkin diagram of $\mathring{\mathfrak{g}}$. Also, let h be the Coxeter number of $\mathring{\mathfrak{g}}$.

Proposition 6.1 ([KW2, KW3]). *A complex number k is admissible if and only if*

$$(39) \quad k + h^\vee = \frac{p}{q} \quad \text{with } p, q \in \mathbb{N}, (p, q) = 1, p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) = r^\vee. \end{cases}$$

A complex number k of the form (39) is called an *admissible number with denominator q* . For an admissible number k with denominator q , we have

$$\Delta(k\Lambda_0) = \{\alpha + nq\delta; \alpha \in \Delta, n \in \mathbb{Z}\} \cong \Delta^{re} \text{ and } \mathcal{W}(k\Lambda_0) \cong \mathcal{W} \text{ if } (r^\vee, q) = 1, \\ \Delta(k\Lambda_0)^\vee = \{\alpha^\vee + nq\delta; \alpha \in \Delta, n \in \mathbb{Z}\} \cong {}^L\Delta^{re} \text{ and } \mathcal{W}(k\Lambda_0) \cong {}^L\mathcal{W} \text{ if } (r^\vee, q) = r^\vee,$$

where $\Delta(\lambda)^\vee = \{\alpha^\vee; \alpha \in \Delta(\lambda)\}$ and ${}^L\Delta^{re}$ and ${}^L\mathcal{W}$ are the real root system and the Weyl group of the non-twisted affine Kac-Moody algebra ${}^L\mathfrak{g}$ associated with the Langlands dual ${}^L\mathring{\mathfrak{g}}$ of $\mathring{\mathfrak{g}}$, respectively. Set

$$\dot{\alpha}_0 = \begin{cases} -\theta + q\delta & \text{if } (r^\vee, q) = 1, \\ -\theta_s + \frac{q}{r^\vee}\delta & \text{if } (r^\vee, q) = r^\vee. \end{cases}$$

Then $\Pi(k\Lambda_0) = \{\alpha_1, \dots, \alpha_\ell, \dot{\alpha}_0\}$. Put $\dot{s}_0 = s_{\dot{\alpha}_0} \in \mathcal{W}(k\Lambda_0)$, so that $\mathcal{W}(k\Lambda_0) = \langle s_1, \dots, s_\ell, \dot{s}_0 \rangle$.

For an admissible number k let Pr_k^+ be the set of admissible weights λ of level k such that $\lambda(\alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \mathring{\Delta}_+$. Then $\{L(\lambda); \lambda \in Pr_k^+\}$ is the set of irreducible admissible representations of level k which are integrable over $\mathring{\mathfrak{g}} \subset \mathfrak{g}$. We have $\Delta(\lambda) = \Delta(k\Lambda_0)$ for $\lambda \in Pr_k^+$.

For an admissible number k denote by Pr_k the set of admissible weights λ of level k such that $\Delta(\lambda) \cong \Delta(k\Lambda_0)$ as root systems. Then [KW2]

$$(40) \quad Pr_k = \bigcup_{\substack{y \in \mathcal{W}^e \\ y(\Delta(k\Lambda_0)) \subset \Delta_+^{re}}} Pr_{k,y}, \quad Pr_{k,y} = y \circ Pr_k^+.$$

Note that

$$(41) \quad \mathcal{W}(\lambda) = y\mathcal{W}(k\Lambda_0)y^{-1} \quad \text{for } \lambda \in Pr_{k,y}.$$

For $\lambda \in Pr_k$, let $\ell_\lambda^{\frac{\infty}{2}}(?)$ be the semi-infinite length function of the affine Weyl group $\mathcal{W}(\lambda)$. The semi-infinite Bruhat ordering $\preceq_{\lambda, \frac{\infty}{2}}$ are also defined for $\mathcal{W}(\lambda)$. We will use the symbol $w \triangleright_{\lambda, \frac{\infty}{2}} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\lambda, \frac{\infty}{2}}$.

Remark 6.2. The admissible weight $\lambda \in Pr_k$ is called the *principal admissible weight* [KW2] if $\Delta(\lambda) \cong \Delta^{re}$, that is, if the denominator q of k is prime to $r^\vee$.

6.2. Fiebig's equivalence and BGG resolution of admissible representations. The following theorem is the special case of a result of Fiebig [Fie, Theorem 11].

Theorem 6.3 ([Fie]). *Let λ be regular dominant. Suppose that there exists a symmetrizable Kac-Moody algebra $\mathfrak{g}'$ whose Weyl group $\mathcal{W}'$ is isomorphic to $\mathcal{W}(\lambda)$. Let λ' be an integral dominant weight of $\mathfrak{g}'$, $\mathcal{O}_{[\lambda']}$ the block of $\mathcal{O}^{\mathfrak{g}'}$ containing the irreducible highest weight representation $L^{\mathfrak{g}'}(\lambda')$ of $\mathfrak{g}'$ with highest weight λ' . Then there is an equivalence of categories*

$$\mathcal{O}_{[\lambda]}^{\mathfrak{g}} \cong \mathcal{O}_{[\lambda']}^{\mathfrak{g}'}$$

which maps $M(w \circ \lambda)$ and $L(w \circ \lambda)$, $w \in \mathcal{W}(\lambda)$, to $M^{\mathfrak{g}'}(\phi(w) \circ \lambda')$ and $L^{\mathfrak{g}'}(\phi(w) \circ \lambda')$, respectively. Here $M^{\mathfrak{g}'}(\lambda')$ is the Verma module of $\mathfrak{g}'$ with highest weight λ' and $\phi : \mathcal{W}(\lambda) \xrightarrow{\sim} \mathcal{W}'$ is the isomorphism.

Let k be an admissible number with denominator q , $\lambda \in Pr_k$. By Theorem 6.3 the block $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ is equivalent to a block of the category $\mathcal{O}$ of $\mathfrak{g}$ or ${}^L\mathfrak{g}$ containing an integrable representation. In particular the existence of a BGG resolution of an integrable representation of an affine Kac-Moody algebra [GL, RCW] implies the existence of a BGG resolution for $L(\lambda)$:

Theorem 6.4. *Let k be an admissible number, $\lambda \in Pr_k$. Then there exists a complex*

$$\mathcal{B}_{\bullet}(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}_2(\lambda) \xrightarrow{d_2} \mathcal{B}_1(\lambda) \xrightarrow{d_1} \mathcal{B}_0(\lambda) \xrightarrow{d_0} 0$$

of the form $\mathcal{B}_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_{\lambda}(w)=i}} M(w \circ \lambda)$, $d_i = \sum_{\substack{w, w' \in \mathcal{W}(\lambda) \\ \ell_{\lambda}(w)=i, \quad w \triangleright_{\lambda} w'}} d_{w', w}$, $d_{w', w} \in \text{Hom}_{\mathfrak{g}}(M(w \circ \lambda), M(w' \circ \lambda))$, such that

$$H_i(\mathcal{B}_{\bullet}(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The resolution of $L(\lambda)$ in Theorem 6.4 can be combinatorially constructed as follows [BGG]: Fix a $\mathfrak{g}$ -homomorphisms

$$i_{w', w}^{\lambda} : M(w \circ \lambda) \rightarrow M(w' \circ \lambda)$$

for $w, w' \in \mathcal{W}(\lambda)$ with $w \succeq_{\lambda} w'$ in such a way that $i_{w'', w'}^{\lambda} \circ i_{w', w}^{\lambda} = i_{w'', w}^{\lambda}$ if $w \succeq_{\lambda} w' \succeq_{\lambda} w$.

A quadruple (w_1, w_2, w_3, w_4) in $\mathcal{W}(\lambda)$ is called a *square* if $w_1 \triangleright_{\lambda} w_2 \triangleright_{\lambda} w_4$, $w_1 \triangleright_{\lambda} w_3 \triangleright_{\lambda} w_4$ and $w_2 \neq w_3$.

Theorem 6.5. *Let k be an admissible number, $\lambda \in Pr_k$. Assign $\epsilon_{w_2, w_1} \in \mathbb{C}^*$ for every pair (w_1, w_2) in $\mathcal{W}(\lambda)$ with $w_1 \triangleright_{\lambda} w_2$ in such a way that $\epsilon_{w_4, w_2} \epsilon_{w_2, w_1} + \epsilon_{w_4, w_3} \epsilon_{w_3, w_1} = 0$ for every square (w_1, w_2, w_3, w_4) of $\mathcal{W}(\lambda)$ (such an assignment is possible by [BGG]). Set $d_{w', w} = \epsilon_{w', w} i_{w', w}^{\lambda}$, $d_i = \sum_{\substack{w, w' \in \mathcal{W}(\lambda) \\ \ell_{\lambda}(w)=i, \quad w \triangleright_{\lambda} w'}} d_{w', w}$. Then*

$$\mathcal{B}_{\bullet}(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}_2(\lambda) \xrightarrow{d_2} \mathcal{B}_1(\lambda) \xrightarrow{d_1} \mathcal{B}_0(\lambda) \xrightarrow{d_0} 0,$$

where $\mathcal{B}_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_{\lambda}(w)=i}} M(w \circ \lambda)$, is a resolution of $L(\lambda)$.

6.3. Twisted BGG resolution. For $w_1, w_2, y \in \mathcal{W}(\lambda)$ with $w_1 \succeq_y w_2$, set

$$\varphi_{w_2, w_1}^{\lambda, y} = T_y(i_{y^{-1}w_2, y^{-1}w_1}^\lambda) : M^y(w_1 \circ \lambda) \rightarrow M^y(w_2 \circ \lambda).$$

A quadruple (w_1, w_2, w_3, w_4) in $\mathcal{W}(\lambda)$ is called a y -twisted square if $w_1 \triangleright_y w_2 \triangleright_y w_4$, $w_1 \triangleright_y w_3 \triangleright_y w_4$ and $w_2 \neq w_3$.

Theorem 6.6. *Let k be an admissible number, $\lambda \in \text{Pr}_k$, $y \in \mathcal{W}(\lambda)$. Assign $\epsilon_{w_2, w_1}^y \in \mathbb{C}^*$ for every pair (w_1, w_2) with $w_1 \triangleright_{\lambda, y} w_2$ in $\mathcal{W}(\lambda)$ in such a way that $\epsilon_{w_4, w_2}^y \epsilon_{w_2, w_1}^y + \epsilon_{w_4, w_3}^y \epsilon_{w_3, w_1}^y = 0$ for every y -twisted square (w_1, w_2, w_3, w_4) of $\mathcal{W}(\lambda)$. Set*

$$\begin{aligned} \mathcal{B}_i^y(\lambda) &= \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda^y(w) = i}} M^y(w \circ \lambda), d_{w', w}^y = \epsilon_{w', w}^y \varphi_{w', w}^{\lambda, y}, d_i \\ &= \sum_{\substack{w, w' \in \mathcal{W}(\lambda) \\ \ell_\lambda^y(w) = i, \quad w \triangleright_{\lambda, y} w'}} d_{w', w} : \mathcal{B}_i^y(\lambda) \rightarrow \mathcal{B}_{i-1}^y(\lambda). \end{aligned}$$

Then

$$\mathcal{B}_\bullet^y(\lambda) : \dots \xrightarrow{d_3} \mathcal{B}_2^y(\lambda) \xrightarrow{d_2} \mathcal{B}_1^y(\lambda) \xrightarrow{d_1} \mathcal{B}_0^y(\lambda) \xrightarrow{d_0} \mathcal{B}_{-1}^y(\lambda) \rightarrow \dots \rightarrow \mathcal{B}_{-\ell(y)}^y(\lambda) \rightarrow 0$$

is a complex of $\mathfrak{g}$ -modules such that

$$H_i(\mathcal{B}_\bullet^y(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Set $\epsilon_{y^{-1}w_1, y^{-1}w_2} = \epsilon_{w_1, w_2}^y$. Then $\{\epsilon_{w_1, w_2}^y\}$ satisfies the condition in Theorem 6.6 if and only if $\{\epsilon_{y^{-1}w_1, y^{-1}w_2}\}$ satisfies the condition in Theorem 6.4. In particular such an assignment is possible. Consider the BGG resolution $\mathcal{B}_\bullet(\lambda)$ of $L(\lambda)$ in Theorem 6.5 associated with this assignment. We have $\mathcal{B}_\bullet^y(\lambda) = T_y(\mathcal{B}_\bullet(\lambda))[-\ell(y)]$, where $[-\ell(y)]$ denotes the shift of the degree. Therefore the assertion follows from Theorem 5.12. $\square$

6.4. System of twisted BGG resolutions.

Proposition 6.7. *Let $\lambda \in \mathfrak{h}^*$ be regular dominant, $y = s_{\beta_1} s_{\beta_2} \dots s_{\beta_l}$ a reduced expression of $y \in \mathcal{W}(\lambda)$ with $\beta_i \in \Pi(\lambda)$. Set $y_i = s_{\beta_1} s_{\beta_2} \dots s_{\beta_i}$ for $i = 0, 1, \dots, l$ and fix a non-zero $\mathfrak{g}$ -homomorphism $\phi_w^{y_i} : M^{y_i}(w \circ \lambda) \rightarrow M^{y_{i+1}}(w \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$, $i = 1, \dots, l$. One can assign $\epsilon_{w_2, w_1}^i \in \mathbb{C}^*$ for each pair (w_1, w_2) with $w_1 \triangleright_{\lambda, y_i} w_2$ for all $i = 1, \dots, l$ in such a way that the following hold:*

- (i) $\epsilon_{w_4, w_2}^i \epsilon_{w_2, w_1}^i + \epsilon_{w_4, w_3}^i \epsilon_{w_3, w_1}^i = 0$ for every y_i -twisted square (w_1, w_2, w_3, w_4) of $\mathcal{W}(\lambda)$,
- (ii) If $w_1 \triangleright_{\lambda, y_i} w_2$, $w_1 \triangleright_{\lambda, y_{i-1}} w_2$, $\ell_\lambda^{y_i}(w_1) = \ell_\lambda^{y_{i-1}}(w_1)$ and $\ell_\lambda^{y_i}(w_2) = \ell_\lambda^{y_{i-1}}(w_2)$, then the following diagram commutes:

$$(42) \quad \begin{array}{ccc} M^{y_{i-1}}(w_1 \circ \lambda) & \xrightarrow{\epsilon_{w_2, w_1}^{i-1} \varphi_{w_2, w_1}^{\lambda, y_{i-1}}} & M^{y_{i-1}}(w_2 \circ \lambda) \\ \phi_{w_1}^{y_{i-1}} \downarrow & & \downarrow \phi_{w_2}^{y_{i-1}} \\ M^y(w_1 \circ \lambda) & \xrightarrow{\epsilon_{w_2, w_1}^i \varphi_{w_2, w_1}^{\lambda, y_i}} & M^y(w_2 \circ \lambda). \end{array}$$

Proposition 6.8. *Let $\lambda \in \mathfrak{h}^*$ be regular dominant, $y \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$ such that $\ell_\lambda(y s_\alpha) = \ell_\lambda(y) + 1$. Set $\beta = y(\alpha)$.*

- (i) Let $w_1, w_2 \in \mathcal{W}(\lambda)$. Suppose that $w_1 \triangleright_y w_2$, $w_1 \triangleright_{ys_\alpha} w_2$ and $\ell_\lambda^y(w_1) = \ell_\lambda^{ys_\alpha}(w_1)$. Then

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) = 1.$$

Moreover, either of the followings span the one-dimensional vector space $\operatorname{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda))$:

- (a) the composition $M^y(w_1 \circ \lambda) \rightarrow M^y(w_2 \circ \lambda) \rightarrow M^{ys_\alpha}(w_2 \circ \lambda)$ of any non-trivial $\mathfrak{g}$ -homomorphisms;
 (b) the composition $M^y(w_1 \circ \lambda) \rightarrow M^{ys_\alpha}(w_1 \circ \lambda) \rightarrow M^{ys_\alpha}(w_2 \circ \lambda)$ of any non-trivial $\mathfrak{g}$ -homomorphisms.
 (ii) Let $w_1, w_2 \in \mathcal{W}(\lambda)$. Suppose that $\ell_\lambda^y(w_1) = \ell_\lambda^y(w_2) + 2$ and $w_i^{-1}(\beta) \in \Delta_+^{re}$ for $i = 1, 2$. Then the composition $M^y(w_1 \circ \lambda) \rightarrow M^y(w_2 \circ \lambda) \rightarrow M^{ys_\alpha}(w_2 \circ \lambda)$ of any non-trivial homomorphisms is non-zero.
 (iii) Let $w \in \mathcal{W}(\lambda)$ and suppose that $s_\alpha w \triangleright_{\lambda, y} w$. Then the composition $M^y(s_\alpha w \circ \lambda) \rightarrow M^y(w \circ \lambda) \rightarrow M^{ys_\alpha}(w \circ \lambda)$ of any $\mathfrak{g}$ -homomorphisms is zero.

Proof. (i) Since $y^{-1}w_1 \triangleright y^{-1}w_2$, the Jantzen sum formula implies that

$$[M(y^{-1}w_2 \circ \lambda) : L(y^{-1}w_1 \circ \lambda)] = 1.$$

Hence $[M^{s_\alpha}(y^{-1}w_2 \circ \lambda) : L(y^{-1}w_1 \circ \lambda)] = 1$. As

$$\operatorname{Hom}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \cong \operatorname{Hom}(M(y^{-1}w_1 \circ \lambda), M^{s_\alpha}(y^{-1}w_2 \circ \lambda)),$$

it follows that

$$\dim_{\mathbb{C}} \operatorname{Hom}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \leq 1.$$

Now we have

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^y(w_2 \circ \lambda)) &\cong \operatorname{Hom}_{\mathfrak{g}}(M(y^{-1}w_1 \circ \lambda), M(y^{-1}w_2 \circ \lambda)), \\ \operatorname{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_1 \circ \lambda)) &\cong \operatorname{Hom}_{\mathfrak{g}}(M(y^{-1}w_1 \circ \lambda), M^{s_\alpha}(y^{-1}w_1 \circ \lambda)), \\ \operatorname{Hom}_{\mathfrak{g}}(M^y(w_2 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) &\cong \operatorname{Hom}_{\mathfrak{g}}(M(y^{-1}w_2 \circ \lambda), M^{s_\alpha}(y^{-1}w_2 \circ \lambda)), \\ \operatorname{Hom}_{\mathfrak{g}}(M^{ys_\alpha}(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) &\cong \operatorname{Hom}_{\mathfrak{g}}(M(s_\alpha y^{-1}w_1 \circ \lambda), M(s_\alpha y^{-1}w_2 \circ \lambda)). \end{aligned}$$

In particular they are all one-dimensional. Hence it remains to show that the compositions in (a) and (b) are non-trivial. This is equivalent to the non-triviality of the compositions

$$\begin{aligned} &M(y^{-1}w_1 \circ \lambda) \rightarrow M(y^{-1}w_2 \circ \lambda) \rightarrow M^{s_\alpha}(y^{-1}w_2 \circ \lambda) \\ &\text{and } M(y^{-1}w_1 \circ \lambda) \rightarrow M^{s_\alpha}(y^{-1}w_1 \circ \lambda) \rightarrow M^{s_\alpha}(y^{-1}w_2 \circ \lambda), \end{aligned}$$

respectively. Therefore we may assume that $y = 1$.

Since $\langle w_2(\lambda + \rho), \alpha^\vee \rangle \in \mathbb{N}$, we have the exact sequence

$$(43) \quad 0 \rightarrow M(s_\alpha w_2 \circ \lambda) \rightarrow M(w_2 \circ \lambda) \rightarrow M^{s_\alpha}(w_2 \circ \lambda) \rightarrow M^{s_\alpha}(s_\alpha w_2 \circ \lambda) \rightarrow 0$$

by Proposition 5.10. On the other hand,

$$(44) \quad w_1 \circ \lambda \not\leq_\lambda s_\alpha w_2 \circ \lambda$$

as we have the square $(s_\alpha w_1, w_1, s_\alpha w_2, w_2)$ by the assumption and (15). Hence (43) implies that the image of the highest weight vector of $M(w_1 \circ \lambda)$ in $M(w_2 \circ \lambda)$ does not lie in the kernel of the map $M(w_2 \circ \lambda) \rightarrow M^{s_\alpha}(w_2 \circ \lambda)$. This proves the

non-triviality of the composition map in (a) for $y = 1$, and thus, for all y . Next we show the non-triviality of the composition in (b). Consider the exact sequence

$$0 \rightarrow M(s_\alpha w_1 \circ \lambda) \rightarrow M(s_\alpha w_2 \circ \lambda) \rightarrow N \rightarrow 0$$

in the category $\mathcal{O}_{[\lambda]}^g$, where $N = M(s_\alpha w_2 \circ \lambda)/M(s_\alpha w_1 \circ \lambda)$. Applying the functor T_{s_α} we obtain the exact sequence

$$(45) \quad 0 \rightarrow \mathcal{L}_1 T_{s_\alpha} N \rightarrow M^{s_\alpha}(w_1 \circ \lambda) \rightarrow M^{s_\alpha}(w_2 \circ \lambda) \rightarrow T_i N \rightarrow 0.$$

By Proposition 5.11, the weights of $\mathcal{L}_1 T_{s_\alpha} N$ are contained in the set of weights of N , and hence of $M(s_\alpha w_2 \circ \lambda)$. Therefore (44) and (45) imply that the image of the highest weight vector of $M(w_1 \circ \lambda)$ in $M^{s_\alpha}(w_1 \circ \lambda)$ does not belong to the kernel of the map $M^{s_\alpha}(w_1 \circ \lambda) \rightarrow M^{s_\alpha}(w_2 \circ \lambda)$. This completes the proof of (i). (ii) Similarly, as above, the problem reduces to the case $y = 1$. By the assumption we have $s_\beta w_1 \triangleright_\lambda w_1$, $s_\beta w_2 \triangleright_\lambda w_2$. Thus $w_1 \not\triangleleft_\lambda s_\beta w_2$ because otherwise $(w_1, s_\beta w_1, s_\beta w_1, w_2)$ is a square. Hence (43) proves the assertion by the same argument as above. (iii) Again we may assume that $y = 1$ and the assertion follows from (43). $\square$

Proof of Proposition 6.7. We prove by induction on i that such an assignment is possible.

As we already remarked the case $i = 0$ is the well-known result of [BGG]. So let $i > 0$. Suppose that $w_1 \triangleright_{\lambda, y_i} w_2$. Set $\beta = y_{i-1}(\alpha_i) \in \Delta_+^{re}$. The following four cases are possible. (The case $w_1^{-1}(\beta) \in \Delta_+^{re}$, $w_2^{-1}(\beta) \in \Delta_-^{re}$ does not happen by [BGG, Lemma 11.3].)

I) $w_1^{-1}(\beta), w_2^{-1}(\beta) \in \Delta_+^{re}$. In this case $w_1 \triangleright_{\lambda, y_{i-1}} w_2$, $\ell_\lambda^{y_i}(w_1) = \ell_\lambda^{y_{i-1}}(w_1)$ and $\ell_\lambda^{y_i}(w_2) = \ell_\lambda^{y_{i-1}}(w_2)$. By Proposition 6.8 there exists a unique ϵ_{w_2, w_1}^i which makes the diagram (42) commute.

II) $w_1 = s_\beta w_2$. In this case $w_2 \triangleright_{\lambda, y_{i-1}} w_1$, $\ell_\lambda^{y_i}(w_1) = \ell_\lambda^{y_{i-1}}(w_1) - 2$ and $\ell_\lambda^{y_i}(w_2) = \ell_\lambda^{y_{i-1}}(w_2)$. We set $\epsilon_{w_2, w_1}^i = \epsilon_{w_1, w_2}^{i-1}$.

III) $w_1^{-1}(\beta), w_2^{-1}(\beta) \in \Delta_-^{re}$. In this case $w_1 \triangleright_{\lambda, y_{i-1}} w_2$, $\ell_\lambda^{y_i}(w_1) = \ell_\lambda^{y_{i-1}}(w_1) - 2$, $\ell_\lambda^{y_i}(w_2) = \ell_\lambda^{y_{i-1}}(w_2) - 2$, and we have the y_i -twisted square $(w_1, s_\beta w_1, w_2, s_\beta w_2)$. Note that $\epsilon_{s_\beta w_2, s_\beta w_1}^i$ is defined in I), and $\epsilon_{s_\beta w_1, w_1}^i, \epsilon_{s_\beta w_2, w_2}^i$ are defined in II). We set

$$(46) \quad \epsilon_{w_2, w_1}^i = -\frac{\epsilon_{s_\beta w_1, w_1}^i \epsilon_{s_\beta w_2, s_\beta w_1}^i}{\epsilon_{s_\beta w_2, w_2}^i}.$$

IV) $w_1^{-1}(\beta) \in \Delta_-^{re}$, $w_2^{-1}(\beta) \in \Delta_+^{re}$, $w_2 \neq s_\beta w_1$. In this case there exists a unique $w_3 \in \mathcal{W}$ such that $(s_\beta w_1, w_1, w_3, w_2)$ is a y_i -twisted square. Note that $w_3^{-1}(\beta) \in \Delta_+^{re}$ because $(w_3, w_2, s_\beta w_3, s_\beta w_2)$ is a y_i -twisted square by (15). Since $\epsilon_{w_3, s_\beta w_1}^i, \epsilon_{w_2, w_3}^i$ are defined in I) and $\epsilon_{w_1, s_\beta w_1}^i$ is defined in II), we can set

$$(47) \quad \epsilon_{w_1, w_1}^i = -\frac{\epsilon_{w_3, s_\beta w_1}^i \epsilon_{w_2, w_3}^i}{\epsilon_{w_1, s_\beta w_1}^i}.$$

Now let (w_1, w_2, w_3, w_4) be a y_i -twisted square. Set

$$A_i(w_1, w_2, w_3, w_4) = \frac{\epsilon_{w_4, w_2}^i \epsilon_{w_2, w_1}^i}{\epsilon_{w_4, w_3}^i \epsilon_{w_3, w_1}^i}.$$

We need to show that $A_i(w_1, w_2, w_3, w_4) = -1$.

The following four cases are possible.

1) $w_2 = s_\beta w_1$, $w_4 = s_\beta w_3$. In this case the assertion follows from the definition (46).

2) $w_2 = s_\beta w_1$, $w_4 \neq s_\beta w_3$. In this case $(s_\beta w)^{-1}(\beta) \in \Delta_-^{re}$, and $w_4^{-1}(\beta) \in \Delta_+^{re}$ because otherwise $w_3 = s_\beta w_4$. Hence the assertion follows from the definition (47).

3) $w_2 \neq s_\beta w_1$, $w_4 = s_\beta w_3$. In this case $(s_\beta w_1, w_1, s_\beta w_2, w_2)$, $(s_\beta w_1, w_1, s_\beta w_2, w_3)$, $(s_\beta w_2, w_2, s_\beta w_3, w_4)$ are y_i -twisted squares:

$$\begin{array}{ccccccc}
 s_\beta w_1 & \xrightarrow{y_i} & w_1 & \xrightarrow{y_i} & w_2 & & \\
 & \searrow y_i & & \nearrow y_i & & \searrow y_i & \\
 & & s_\beta w_2 & \xrightarrow{y_i} & w_3 & \xrightarrow{y_i} & s_\beta w_3
 \end{array}$$

We have by 1)

$$A_i(s_\beta w_1, w_1, s_\beta w_2, w_2) = A_i(s_\beta w_2, w_2, w_3, s_\beta w_3) = -1$$

and by 2)

$$A_i(s_\beta w_1, w_1, s_\beta w_2, w_3) = -1.$$

But

$$\begin{aligned}
 & A_i(w_1, w_2, w_3, s_\beta w_3) \\
 &= A_i(s_\beta w_1, w_1, s_\beta w_2, w_2) A_i(s_\beta w_2, w_2, w_3, s_\beta w_3) A_i(s_\beta w_1, s_\beta w_2, w_1, w_3).
 \end{aligned}$$

Hence the assertion follows.

4) $w_2 \neq s_\beta w_1$, $w_4 \neq s_\beta w_2$. We see as in [BGG, p. 57, c)] that $w_4 \neq s_\beta w_2, s_\beta w_3$, and hence as in [BGG, p. 56, 1)] we find that $(s_\beta w_1, s_\beta w_2, s_\beta w_3, s_\beta w_4)$ is also a y_i -twisted square. Hence a) $w_i^{-1}(\beta) \in \Delta_+^{re}$ for all i or b) $w_i^{-1}(\beta) \in \Delta_-^{re}$ for all i .

a) The case $w_i^{-1}(\beta) \in \Delta_+^{re}$ for all i : By the definition I) we have the commutative diagram

$$\begin{array}{ccc}
 M^{y_{i-1}}(w_1 \circ \lambda) & \xrightarrow{\epsilon_{w_4, w_a}^{i-1} \epsilon_{w_a, w_1}^{i-1} \varphi_{w_4, w_1}^{\lambda, y_{i-1}}} & M^{y_{i-1}}(w_4 \circ \lambda) \\
 \phi_{w_1}^{y_{i-1}} \downarrow & & \downarrow \phi_{w_4}^{y_{i-1}} \\
 M^y(w_1 \circ \lambda) & \xrightarrow{\epsilon_{w_4, w_a}^i \epsilon_{w_a, w_1}^i \varphi_{w_4, w_1}^{\lambda, y_i}} & M^y(w_4 \circ \lambda)
 \end{array}
 \tag{48}$$

for $a = 2, 3$. Since $\epsilon_{w_4, w_2}^{i-1} \epsilon_{w_2, w_1}^{i-1} = -\epsilon_{w_4, w_3}^{i-1} \epsilon_{w_3, w_1}^{i-1}$ by the induction hypothesis the commutativity of the above diagram implies that $\epsilon_{w_4, w_2}^i \epsilon_{w_2, w_1}^i = -\epsilon_{w_4, w_3}^i \epsilon_{w_3, w_1}^i$ by Proposition 6.8 (ii).

b) The case that $w_i^{-1}(\beta) \in \Delta_-^{re}$ for all i : We have that $(s_\beta w_1, w_1, s_\beta w_2, w_2)$, $(s_\beta w_1, w_1, s_\beta w_3, w_3)$, $(s_\beta w_1, s_\beta w_2, s_\beta w_3, s_\beta w_4)$, $(s_\beta w_2, w_2, s_\beta w_4, w_4)$ and $(s_\beta w_3, w_3, s_\beta w_4, w_4)$ are all y_i -twisted squares. Hence the assertion follows from the equality

$$\begin{aligned}
 & A_i(w_1, w_2, w_3, w_4) A_i(s_\beta w_1, s_\beta w_2, w_1, w_2) A_i(s_\beta w_1, w_1, s_\beta w_3, w_3) \\
 &= A_i(s_\beta w_1, s_\beta w_2, s_\beta w_3, s_\beta w_4) A_i(s_\beta w_2, w_2, s_\beta w_4, w_4) A_i(s_\beta w_3, s_\beta w_4, w_3, w_4).
 \end{aligned}$$

□

Let k be an admissible number, $\lambda \in Pr_k$. Let $y \in \mathcal{W}(\lambda)$, $\{y_i\}$, $\{\phi_w^{y_i}\}$, $\{\epsilon_{w_2, w_1}^i\}$ be as in Proposition 6.7. Because $\{\epsilon_{w_2, w_1}^i\}$ satisfies the condition in Theorem 6.6 there

is a corresponding twisted BGG resolution $\mathcal{B}_\bullet^{y_i}(\lambda)$ of $L(\lambda)$ for $i = 0, 1, \dots, l = \ell(y)$. Define

$$\Phi_p^{y_{i+1}, y_i} = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda^{y_i}(w) = \ell_\lambda^{y_{i+1}}(w) = p}} \phi_w^{y_{i+1}, y_i} : \mathcal{B}_p^{y_i}(w \circ \lambda) \rightarrow \mathcal{B}_p^{y_{i+1}}(w \circ \lambda).$$

Proposition 6.9. *In the above setting $\Phi_\bullet^{y_{i+1}, y_i}$ gives a quasi-isomorphism $\mathcal{B}_\bullet^{y_i}(\lambda) \sim \mathcal{B}_\bullet^{y_{i+1}}(\lambda)$ of complexes for each $i = 0, 1, \dots, l-1$.*

Lemma 6.10. *Let $\lambda \in \mathfrak{h}^*$, y, y_i be as in Proposition 6.7, $w_1, w_2 \in \mathcal{W}(\lambda)$.*

- (i) *Suppose that $w_1 \triangleright_{\lambda, y_i} w_2$, $\ell^{y_i}(w_1) = \ell^{y_{i+1}}(w_1)$. Then $w_1 \triangleright_{\lambda, y_{i+1}} w_2$.*
- (ii) *Suppose that $w_1 \triangleright_{\lambda, y_i} w_2$, $\ell^{y_i}(w_2) = \ell^{y_{i+1}}(w_2)$. Then either of the following two holds.*
 - (a) $w_2 = s_\beta w_1$ and $w_2 \triangleright_{\lambda, y_{i+1}} w_1$.
 - (b) $w_1 \triangleright_{\lambda, y_{i+1}} w_2$.

Proof. (1) By assumption $s_\beta w_1 \triangleright_{\lambda, y_i} w_2$. Therefore $(s_\beta w_1, w_1, s_\beta w_2, w_2)$ is a y_i -twisted square. (2) Similarly, if $w_2 \neq s_\beta w_1$, then $(s_\beta w_1, w_1, s_\beta w_2, w_2)$ y_i -twisted square. The $w_2 \neq s_\beta w_1$ case is obvious. $\square$

Proof of Proposition 6.9. The fact that $\Phi_\bullet^{y_i}$ defines a homomorphism of complexes follows from the commutativity of (42), Proposition 6.8 (iii), and Lemma 6.10. Since both complexes are quasi-isomorphic to $L(\lambda)$, to show that it defines a quasi-isomorphism it suffices to check that it defines a non-trivial homomorphism between the corresponding homology spaces. This follows from the fact that $\phi_1^{y_i} : M^{y_i}(\lambda) \rightarrow M^{y_{i+1}}(\lambda)$ sends the highest weight vector of $M^{y_i}(\lambda)$ to the highest weight vector of $M^{y_{i+1}}(\lambda)$. $\square$

6.5. Two-sided BGG resolutions of G -integrable admissible representations. For $\lambda \in Pr_k$ and $i \in \mathbb{Z}$ set

$$\mathcal{W}^i(\lambda) = \{w \in \mathcal{W}(\lambda); \ell_\lambda^{\frac{\infty}{2}}(w) = i\}.$$

We note that

$$\sharp \mathcal{W}^i(\lambda) = \begin{cases} 1 & \text{if } \mathring{\mathfrak{g}} = \mathfrak{sl}_2, \\ \infty & \text{else.} \end{cases}$$

Theorem 6.11. *Let k be an admissible number, $\lambda \in Pr_k^+$:*

- (i) *The space $\text{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda))$ is one-dimensional for $w, w' \in \mathcal{W}(\lambda)$ such that $w \triangleright_{\lambda, \frac{\infty}{2}} w'$.*
- (ii) *There exists a complex*

$$C^\bullet(\lambda) : \dots \rightarrow C^{-2}(\lambda) \xrightarrow{d_{-2}} C^{-1}(\lambda) \xrightarrow{d_{-1}} C^0(\lambda) \xrightarrow{d_0} C^1(\lambda) \xrightarrow{d_1} C^2(\lambda) \xrightarrow{d_2} \dots$$

in the category $\mathcal{O}$ of the form

$$C^i(\lambda) = \bigoplus_{w \in \mathcal{W}^i(\lambda)} W(w \circ \lambda), \quad d_i = \sum_{\substack{w \in \mathcal{W}^i(\lambda), \quad w' \in \mathcal{W}^{i+1}(\lambda) \\ w \triangleright_{\lambda, \frac{\infty}{2}} w'}} d_{w', w},$$

where $d_{w', w}$ is a non-trivial $\mathfrak{g}$ -homomorphism $W(w \circ \lambda) \rightarrow W(w' \circ \lambda)$, such that

$$H^i(C^\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

Proof. (ii) Let q be the denominator of k and set $M = q\overset{\circ}{Q}^\vee$ if $(r^\vee, q) = 1$ and $M = q\overset{\circ}{Q}$ if $(r^\vee, q) = r^\vee$, so that $\mathcal{W}(\lambda) = \overset{\circ}{W} \ltimes t_M$. Let $\gamma_1, \gamma_2, \dots$, be a sequence in $\overset{\circ}{P}_+^\vee \cap M$ such that $\gamma_i - \gamma \in \overset{\circ}{P}_+^\vee \cap M$, $\lim_{i \rightarrow \infty} \alpha(\gamma_i) = \infty$ for all $\alpha \in \overset{\circ}{\Delta}_+$.

By Proposition 6.9 there is an inductive system $\{\mathcal{B}_\bullet^{-\gamma_i}(\lambda)\}$ of twisted BGG resolutions. Let $\mathcal{B}_{-\gamma_i}^\bullet(\lambda)$ be the complex $\mathcal{B}_\bullet^{-\gamma_i}(\lambda)$ with the opposite homological grading. Thus it is a complex

$$B_{-\gamma_i}^\bullet(\lambda) \cdots \xrightarrow{d_{-2}} B_{\gamma_i}^{-1}(\lambda) \xrightarrow{d_{-1}} B_{-\gamma_i}^0(\lambda) \xrightarrow{d_0} B_{\gamma_i}^1(\lambda) \xrightarrow{d_1} \cdots$$

$$\text{of the form } \mathcal{B}_{\gamma_i}^p(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda^{-\gamma_i}(w) = -p}} M^{-\gamma_i}(w \circ \lambda), \quad d_p = \sum_{\substack{w, w' \in \mathcal{W}(\lambda) \\ \ell_\lambda^{-\gamma_i}(w) = -p, w \triangleright_\lambda, t_{-\gamma_i} w'}} d_{w', w}^{\gamma_i}, \quad d_{w', w}^{\gamma_i} :$$

$M^{-\gamma_i}(w \circ \lambda) \rightarrow M^{-\gamma_i}(w' \circ \lambda)$ such that

$$H^p(B_{-\gamma_i}^\bullet(\lambda)) = \begin{cases} L(\lambda) & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(C^\bullet(\lambda), d_\bullet)$ be the complex obtained as the inductive limit of complex $\mathcal{B}_{-\gamma_i}^\bullet(\lambda)$. By Lemma 3.2, Proposition 4.3 and Proposition 6.9 we have

$$C^p(\lambda) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} \lim_i M^{-\gamma_i}(w \circ \lambda) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} W(w \circ \lambda) \quad \text{for } p \in \mathbb{Z},$$

$$H^p(C^\bullet(\lambda)) = \varinjlim_i H^p(B_{-\gamma_i}^\bullet(\lambda)) = \begin{cases} L(\lambda) & \text{if } p = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the differential $d_p : C^p(\lambda) \rightarrow C^{p+1}(\lambda)$ has the form

$$d_p = \sum_{\substack{w \in \mathcal{W}^p(\lambda), \quad w' \in \mathcal{W}^{p+1}(\lambda) \\ w \triangleright_\lambda, \frac{\infty}{2} w'}} d_{w', w},$$

where $d_{w', w} : W(w \circ \lambda) \rightarrow W(w' \circ \lambda)$ is induced by the homomorphisms $d_{w', w}^{-\gamma_i} : M^{-\gamma_i}(w \circ \lambda) \rightarrow M^{-\gamma_i}(w' \circ \lambda)$ with $i = 1, 2, \dots$. To complete the proof of (ii) it remains to show that the map $d_{w', w}$ is nonzero for $w \triangleright_\lambda, \frac{\infty}{2} w'$.

Let $w', w \in \mathcal{W}(\lambda)$ such that $w \triangleright_\lambda, \frac{\infty}{2} w'$. We have the commutative diagram

$$\begin{array}{ccc} M^{-\gamma_i}(w' \circ \lambda) & \xrightarrow{d_{w, w'}^{-\gamma_i}} & M^{-\gamma_i}(w \circ \lambda) \\ \downarrow \phi_{-\gamma_i}^{w' \circ \lambda} & & \downarrow \phi_{-\gamma_i}^{w \circ \lambda} \\ W(w' \circ \lambda) & \xrightarrow{d_{w, w'}} & W(w \circ \lambda) \end{array}$$

for all i . By applying the functor $G_{-\gamma_i}$ we obtain the commutative diagram

$$\begin{array}{ccc} M(t_{\gamma_i} w' \circ \lambda) & \xrightarrow{G_{-\gamma_i}(d_{w, w'}^{-\gamma_i})} & M(t_{\gamma_i} w \circ \lambda) \\ \downarrow G_{-\gamma_i}(\phi_{-\gamma_i}^{w' \circ \lambda}) & & \downarrow G_{-\gamma_i}(\phi_{-\gamma_i}^{w \circ \lambda}) \\ W(t_{\gamma_i} w' \circ \lambda) & \xrightarrow{G_{-\gamma_i}(d_{w, w'})} & W(t_{\gamma_i} w \circ \lambda). \end{array}$$

By Corollary 4.5 $d_{w,w'} \neq 0$ if and only if $G_{-\gamma_i}(d_{w,w'}) \neq 0$. Therefore it is sufficient to show that $G_{-\gamma_i}(\phi_{-\gamma_i}^{w' \circ \lambda}) \circ G_{-\gamma_i}(d_{w,w'}^{-\gamma_i}) : M(t_{\gamma_i} w' \circ \lambda) \rightarrow W(t_{\gamma_i} w \circ \lambda)$ is non-zero for a sufficiently large i .

Write $w' = s_\alpha w$ with $\alpha \in \Delta^{re}$, $\bar{\alpha} \in \mathring{\Delta}_-$. (This is possible because $s_\alpha = s_{-\alpha}$.) Then, for a sufficiently large i , $\beta := t_{\gamma_i}(\alpha) \in \Delta_+^{re}$ and $t_{\gamma_i} s_\alpha w = s_\beta t_{\gamma_i} w \rightarrow t_{\gamma_i} w$. The determinant formula [Fre1, Proposition 2 (2)] shows that the image of the highest weight vector of $M(t_{\gamma_i} w' \circ \lambda) = M(s_\beta t_{\gamma_i} w \circ \lambda)$ in $M(t_{\gamma_i} w \circ \lambda)$ is not in the kernel of the map $G_{\gamma_i}(\phi_{\gamma_i}^{w', \lambda}) : M(t_{\gamma_i} w \circ \lambda) \rightarrow W(t_{\gamma_i} w \circ \lambda)$. Therefore $G_{\gamma_i}(\phi_{\gamma_i}^{w', \lambda}) \circ G_{\gamma_i}(d_{w,w'}^{\gamma_i})$ is non-zero, and hence so is $d_{w,w'}$.

Finally, we shall prove (i). Note that

$$\mathrm{Hom}_{\mathfrak{g}}(W(w' \circ \lambda), W(w \circ \lambda)) = \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(w' \circ \lambda), W(w \circ \lambda))$$

and that $\mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(w' \circ \lambda), W(w \circ \lambda))$ is at most one-dimensional by the Jantzen sum formula since $w' \triangleright_\lambda w$. It follows from (the proof of) (ii) that $\mathrm{Hom}_{\mathfrak{g}}(W(w' \circ \lambda), W(w \circ \lambda))$ is spanned by $d_{w,w'}$. This completes the proof. $\square$

Remark 6.12. By Theorem 6.11 (i) the resolution in Theorem 6.11 (ii) may be described in terms of screening operators as in [BF] provided that the existence of corresponding cycles is established; see e.g. [TK].

The following assertion is an immediate consequence of Theorem 6.11 which generalizes [FF2, Theorem 4.1].

Theorem 6.13. *Let k be an admissible number, $\lambda \in Pr_k^+$, $p \in \mathbb{Z}$. We have*

$$\begin{aligned} H^{\frac{\infty}{2}+p}(\mathfrak{a}, L(\lambda)) &= \bigoplus_{w \in \mathcal{W}^p(\lambda)} \mathbb{C}_{w \circ \lambda} \quad \text{as } \mathfrak{h}\text{-modules,} \\ H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{n}}, L(\lambda)) &= \bigoplus_{w \in \mathcal{W}^p(\lambda)} \pi_{w \circ \lambda + h \vee \Lambda_0} \quad \text{as } \mathcal{H}\text{-modules.} \end{aligned}$$

6.6. A description of vacuum admissible representation. Let $V^k(\mathring{\mathfrak{g}})$ be the universal affine vertex algebra associated with $\mathring{\mathfrak{g}}$ at level k :

$$V^k(\mathring{\mathfrak{g}}) = U(\mathfrak{g}) \otimes_{U(\mathring{\mathfrak{g}}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representations of $\mathring{\mathfrak{g}}[t] \oplus \mathbb{C}K$ on which $\mathring{\mathfrak{g}}[t]$ acts trivially and K acts as the multiplication by k . By [Fre2, Proposition 5.2] we have an injective homomorphism of vertex algebras

$$V^k(\mathring{\mathfrak{g}}) \hookrightarrow W(k\Lambda_0)$$

for all $k \in \mathbb{C}$. Hence $V^k(\mathring{\mathfrak{g}})$ may be regarded as a vertex subalgebra of $W(k\Lambda_0)$.

Note that $L(k\Lambda_0)$ is the unique simple quotient of $V^k(\mathring{\mathfrak{g}})$.

Proposition 6.14. *Let k be an admissible number, $\Psi : W(\mathring{s}_0 \circ k\Lambda_0) \rightarrow W(k\Lambda_0)$ a non-zero $\mathfrak{g}$ -homomorphism, which exists uniquely up to a nonzero constant multiplication by Theorem 6.11(i). Then the image of the highest weight vector of $W(\mathring{s}_0 \circ k\Lambda_0)$ generates the maximal submodule of $V^k(\mathring{\mathfrak{g}}) \subset W(k\Lambda_0)$.*

Proof. By [KW1] the maximal submodule of $V^k(\overset{\circ}{\mathfrak{g}})$ is generated by a singular vector v of weight $\dot{s}_0 \circ k\Lambda_0$. Consider the two-sided resolution $C^\bullet(k\Lambda_0)$ of $L(k\Lambda_0)$ in Theorem 6.11 (ii). Because it is a resolution of $L(k\Lambda_0)$ and $V^k(\overset{\circ}{\mathfrak{g}}) \subset W(k\lambda_0)$, the vector v must be in the image of $d_{1,w} : W(w \circ k\Lambda_0) \rightarrow W(k\lambda_0)$ for some $w \in \mathcal{W}^{-1}(k\Lambda_0)$. Since the weight $w \circ k\Lambda_0$ is strictly smaller than $\dot{s}_0 \circ k\Lambda_0$ for $w \in \mathcal{W}^{-1}(k\Lambda_0) \setminus \{\dot{s}_0\}$, the only possibility is that v is the image of the highest weight vector of $W(\dot{s}_0 \circ k\Lambda_0)$. $\square$

6.7. Two-sided BGG resolutions of more general admissible representations.

tions. Let $\lambda \in Pr_{k,y}$ with $y = \bar{y}t_\eta$, $\bar{y} \in \overset{\circ}{\mathcal{W}}$, $\eta \in \overset{\circ}{Q}^\vee$. Then there exists $\lambda_1 \in Pr_k^+$ such that $\lambda = y \circ \lambda_1$. Since $y(\Delta(\lambda_1)_+) \subset \Delta_+^{re}$, $T_y : \mathcal{O}_{[\lambda_1]}^\mathfrak{g} \rightarrow \mathcal{O}_{[\lambda]}^\mathfrak{g}$ is exact,

$$\begin{aligned} T_y L(\lambda_1) &\cong L(\lambda), \\ T_y W(w \circ \lambda_1) &\cong T_y \varinjlim_i M^{-\gamma_i}(w \circ \lambda_1) \cong \varinjlim_i T_y M^{-\gamma_i}(w \circ \lambda_1) \\ &\cong \varinjlim_i M^{-y(\gamma_i)}(ywy^{-1} \circ \lambda) \cong W^{\bar{y}}(ywy^{-1} \circ \lambda) \end{aligned}$$

for $w \in \mathcal{W}(\lambda_1) = y^{-1}\mathcal{W}(\lambda)y$ by Proposition 4.14, Lemmas 5.5 and 5.7, where $(\gamma_1, \gamma_2, \dots)$ is a sequence as in proof of Theorem 6.11. Therefore the following assertion follows immediately from Theorem 6.6.

Theorem 6.15. . Let k be an admissible number, $\lambda \in Pr_{k,y}$ with $y = \bar{y}t_\eta$, $\bar{y} \in \overset{\circ}{\mathcal{W}}$, $\eta \in \overset{\circ}{P}^\vee$. Then there exists a complex

$$C^\bullet(\lambda) : \dots \xrightarrow{d_{-3}} C^{-2}(\lambda) \xrightarrow{d_{-2}} C^{-1}(\lambda) \xrightarrow{d_{-1}} C^0(\lambda) \xrightarrow{d_0} C^1(\lambda) \xrightarrow{d_1} C^2(\lambda) \xrightarrow{d_2} \dots$$

in the category $\mathcal{O}$ of the form

$$C^i = \bigoplus_{w \in \mathcal{W}^i(\lambda)} W^{\bar{y}}(w \circ \lambda), d_i = \sum_{\substack{w \in \mathcal{W}^i(\lambda), \ w' \in \mathcal{W}^{i+1}(\lambda) \\ w \triangleright_{\lambda, \frac{\infty}{2}} w'}} d_{w', w}.$$

such that

$$H^i(C^\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

Remark 6.16. If $\lambda \in Pr_{k,y}$ and $\bar{y} = 1$ (that is, $y \in \overset{\circ}{P}^\vee$), then $W^{\bar{y}}(w \circ \lambda) = W(w \circ \lambda)$. Hence the above is the resolution of $L(\lambda)$ in terms of (non-twisted) Wakimoto modules as conjectured in [FKW].

7. SEMI-INFINITE RESTRICTION AND INDUCTION

7.1. Feigin-Frenkel parabolic induction. Let $\overset{\circ}{\mathfrak{p}}$ be a parabolic subalgebra of $\overset{\circ}{\mathfrak{g}}$ containing $\overset{\circ}{\mathfrak{b}}_-$, and let $\overset{\circ}{\mathfrak{p}} = \overset{\circ}{\mathfrak{l}} \oplus \overset{\circ}{\mathfrak{m}}_-$ be the direct sum decomposition of $\overset{\circ}{\mathfrak{p}}$ with the Levi subalgebra $\overset{\circ}{\mathfrak{l}}$ containing $\overset{\circ}{\mathfrak{h}}$ and the nilpotent radical $\overset{\circ}{\mathfrak{m}}_-$. Denote by $\overset{\circ}{\mathfrak{m}} \subset \overset{\circ}{\mathfrak{n}}$ the opposite algebra of $\overset{\circ}{\mathfrak{m}}_-$, so that $\overset{\circ}{\mathfrak{g}} = \overset{\circ}{\mathfrak{p}} \oplus \overset{\circ}{\mathfrak{m}}$. Let

$$\overset{\circ}{\mathfrak{l}} = \overset{\circ}{\mathfrak{l}}_0 \oplus \bigoplus_{i=1}^s \overset{\circ}{\mathfrak{l}}_i$$

be the decomposition of $\overset{\circ}{\mathfrak{l}}$ into direct sum of simple Lie subalgebras $\overset{\circ}{\mathfrak{l}}_i$, $i = 1, \dots, s$, and its center $\overset{\circ}{\mathfrak{l}}_0$ of $\overset{\circ}{\mathfrak{l}}$. Let $\overset{\circ}{\mathfrak{h}}_i = \overset{\circ}{\mathfrak{l}} \cap \overset{\circ}{\mathfrak{h}}$, the Cartan subalgebra of $\overset{\circ}{\mathfrak{l}}_i$, and denote by $\overset{\circ}{\Delta}_i \subset \overset{\circ}{\Delta}$ the subroot system of $\overset{\circ}{\mathfrak{g}}$ corresponding to $\overset{\circ}{\mathfrak{l}}_i$, $\overset{\circ}{\Pi}_i = \overset{\circ}{\Pi} \cap \overset{\circ}{\Delta}_i$. Let $h_i^\vee$ be the dual Coxeter number of $\overset{\circ}{\mathfrak{l}}_i$ (with a convention $h_0^\vee = 0$), θ_i the highest root of $\overset{\circ}{\Delta}_i$, $\theta_{i,s}$ the highest short roof of $\overset{\circ}{\Delta}_i$.

Let $\mathfrak{l}_i = \overset{\circ}{\mathfrak{l}}_i[t, t^{-1}] \oplus \mathbb{C}K \subset \mathfrak{g}$ for $i = 0, 1, \dots, s$. Set

$$K_i = \frac{2}{(\theta_i|\theta_i)}K,$$

and we consider K_i as an element of $\mathfrak{l}_i$. Thus,

$$\mathfrak{l}_i = \overset{\circ}{\mathfrak{l}}_i[t, t^{-1}] \oplus \mathbb{C}K_i,$$

and $\mathfrak{h}_i := \overset{\circ}{\mathfrak{h}}_i \oplus \mathbb{C}K_i$ is a Cartan subalgebra of $\overset{\circ}{\mathfrak{l}}_i$.

Define

$$\mathfrak{l} = \bigoplus_{i=0}^s \mathfrak{l}_i, \quad \mathfrak{t} = \bigoplus_{i=0}^s \mathfrak{h}_i.$$

The grading of $\mathfrak{l}_i$ induces the grading of $\mathfrak{l}$.

For $k \in \mathbb{C}$ define $k_0, \dots, k_s \in \mathbb{C}$ by

$$(49) \quad k_0 = k + h^\vee, \quad k_i + h_i^\vee = \frac{2}{(\theta_i|\theta_i)}(k + h^\vee) \quad \text{for } i = 1, \dots, s.$$

Lemma 7.1. *Let k be an admissible number for $\mathfrak{g}$. Then k_i , $i = 1, \dots, s$, is an admissible number for the Kac-Moody algebra $\mathfrak{l}_i$.*

Let $\mathcal{O}_{(k_0, \dots, k_s)}^{\mathfrak{l}}$ be the full subcategory of $\mathcal{O}^{\mathfrak{l}}$ consisting of objects on which K_i acts as the multiplication by k_i , $i = 0, 1, \dots, s$. Feigin and Frenkel [FF2, 5.2], [Fre2, §6] constructed a functor

$$\mathrm{F-ind}_{\mathfrak{l}}^{\mathfrak{g}} : \mathcal{O}_{(k_0, k_1, \dots, k_s)}^{\mathfrak{l}} \rightarrow \mathcal{O}_k^{\mathfrak{g}}, \quad M \rightarrow \mathrm{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(M),$$

which enjoys the property

$$(50) \quad \mathrm{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(M) \cong US(L\overset{\circ}{\mathfrak{m}}) \otimes_{\mathbb{C}} M$$

as modules over

$$L\overset{\circ}{\mathfrak{m}} = \overset{\circ}{\mathfrak{m}}[t, t^{-1}] \subset \mathfrak{g},$$

where $L\overset{\circ}{\mathfrak{m}}$ only on the first factor $US(L\overset{\circ}{\mathfrak{m}})$. In particular $\mathrm{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}$ is an exact functor.

Denote by $W_{\mathfrak{l}_i}(\lambda^{(i)})$ the Wakimoto module of the affine Kac-Moody algebra $\mathfrak{l}_i$ with highest weight $\lambda^{(i)} \in \mathfrak{h}_i^*$ and by $L_{\mathfrak{l}}(\lambda^{(i)})$ the irreducible highest weight representation of $\mathfrak{l}_i$ with highest weight $\lambda^{(i)}$ (with a convention that $W_{\mathfrak{l}_0}(\lambda^{(0)})$ is the irreducible representation of the Heisenberg algebra $\mathfrak{l}_0$ with highest weight $\lambda^{(0)}$). For $\lambda \in \mathfrak{t}^*$ let $W_{\mathfrak{l}}(\lambda)$ and $L_{\mathfrak{l}}(\lambda)$ be the Wakimoto module and the irreducible highest weight representation of $\mathfrak{l}$ with highest weight λ :

$$W_{\mathfrak{l}}(\lambda) = \bigotimes_{i=0}^s W_{\mathfrak{l}_i}(\lambda|_{\mathfrak{h}_i}), \quad L_{\mathfrak{l}}(\lambda) = \bigotimes_{i=0}^s L_{\mathfrak{l}_i}(\lambda|_{\mathfrak{h}_i}).$$

For $\lambda \in \mathfrak{h}^*$, define $\lambda_{\mathfrak{l}} \in \mathfrak{t}^*$ by

$$\lambda_{\mathfrak{l}}|_{\mathfrak{h}_i} = \lambda|_{\mathfrak{h}_i} \quad \text{and} \quad (\lambda_{\mathfrak{l}} + \rho_i)(K_i) = \frac{2}{(\theta_i|\theta_i)}(\lambda + \rho)(K)$$

for $i = 0, 1, \dots, s$.

Proposition 7.2 ([FF2]). *For $\lambda \in \mathfrak{h}^*$ we have $\mathrm{F}\text{-ind}_{\mathfrak{p}}^{\mathfrak{g}} W_{\mathfrak{l}}(\lambda_{\mathfrak{l}}) \cong W(\lambda)$.*

Proof. By using the Hochschild-Serre spectral sequence for $L\mathring{\mathfrak{m}} \subset \mathfrak{a}$ we see from (50) that

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, \mathrm{F}\text{-ind}_{\mathfrak{l}}^{\mathfrak{g}} W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})) \cong \begin{cases} \mathbb{C}_{\lambda} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the assertion follows from Theorem 4.7. $\square$

7.2. Semi-infinite restriction functors. Let $M \in \mathcal{O}_k^{\mathfrak{g}}$. Then $H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, M)$, $p \in \mathbb{Z}$, is naturally an $\mathfrak{l}$ -module on which K_i acts as the multiplication by k_i ; see e.g. [HT, Proposition 2.3]. Hence

$$\mathrm{S}\text{-res}_{\mathfrak{l}}^{\mathfrak{g}} := H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{m}}, ?)$$

defines a functor $\mathcal{O}_k^{\mathfrak{g}} \rightarrow \mathcal{O}_{(k_0, k_1, \dots, k_s)}^{\mathfrak{l}}$. We refer to $\mathrm{S}\text{-res}_{\mathfrak{l}}^{\mathfrak{g}}$ as the *semi-infinite restriction functor*.

The following assertion follows from Proposition 7.2.

Proposition 7.3. *For $\lambda \in \mathfrak{h}^*$ we have $H^{\frac{\infty}{2}+i}(L\mathring{\mathfrak{m}}, W(\lambda)) = 0$ for $i \neq 0$ and*

$$\mathrm{S}\text{-res}_{\mathfrak{l}}^{\mathfrak{g}} W(\lambda) \cong W_{\mathfrak{l}}(\lambda_{\mathfrak{l}}).$$

7.3. Decomposition of integral Weyl groups. Let k be an admissible number with denominator q , $\lambda \in Pr_k^+$. Let $\mathring{\mathcal{W}}_{S_i}$ be the parabolic subgroup of $\mathring{\mathcal{W}}$ corresponding to $\mathring{\mathfrak{l}}_i$, $\mathring{\mathcal{W}}_S = \mathring{\mathcal{W}}_{S_1} \times \mathring{\mathcal{W}}_{S_2} \times \dots \times \mathring{\mathcal{W}}_{S_s}$. Define $\dot{\alpha}_0^{(i)} \in \Delta(\lambda)$, $i = 1, \dots, s$, by

$$\begin{aligned} \dot{\alpha}_0^{(i)} &= -\theta_i + q\delta \quad \text{if } (r^{\vee}, q) = 1, \\ \text{and } (\dot{\alpha}_0^{(i)})^{\vee} &= -\theta_{i,s}^{\vee} + q\delta \quad \text{if } (r^{\vee}, q) = r^{\vee}. \end{aligned}$$

Set $\dot{s}_0^{(i)} = s_{\dot{\alpha}_0^{(i)}}$.

Let $\mathcal{W}(\lambda)_{S_i}$ be the subgroup of $\mathcal{W}(\lambda)$ generated by $\mathring{\mathcal{W}}_{S_i}$ and $\dot{s}_0^{(i)}$. Then

$$\mathcal{W}(\lambda)_S = \mathcal{W}(\lambda)_{S_1} \times \mathcal{W}(\lambda)_{S_2} \times \dots \times \mathcal{W}(\lambda)_{S_s}$$

is the subgroup corresponding to $\mathring{\mathcal{W}}_S$ described in §3.4. Let $\mathcal{W}(\lambda)^S \subset \mathcal{W}(\lambda)$ be as in Theorem 3.3 so that

$$(51) \quad \mathcal{W}(\lambda) = \mathcal{W}(\lambda)_S \times \mathcal{W}(\lambda)^S, \quad \ell_{\frac{\infty}{2}}^{\infty}(uv) = \ell_{\frac{\infty}{2}}^{\infty}(u) + \ell_{\frac{\infty}{2}}^{\infty}(v) \text{ for } u \in \mathcal{W}(\lambda)_S, v \in \mathcal{W}(\lambda)^S.$$

Let $w, w' \in \mathcal{W}(\lambda)_{S_i} \subset \mathcal{W}(\lambda)$ such that $w \triangleright_{\lambda, \frac{\infty}{2}} w'$. Then $w \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}^{(i)} = (w \circ \lambda)_{\mathfrak{l}}^{(i)}$, where $\circ_{\mathfrak{l}_i}$ is the dot action of $\mathcal{W}(\lambda)_{S_i}$ on $\mathfrak{h}_i^*$ and $\lambda_{\mathfrak{l}_i}^{(i)} = \lambda|_{\mathfrak{h}_i}$.

Proposition 7.4. *Let $\lambda \in Pr_k^+$, $w, w' \in \mathcal{W}(\lambda)_{S_i}$ with $i \in \{1, 2, \dots, s\}$ such that $w \triangleright_{\lambda, \frac{\infty}{2}} w'$. Then the correspondence $\Phi \mapsto \text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(\Phi)$ defines a linear isomorphism*

$$\text{Hom}_{\mathfrak{l}}(W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}}), W_{\mathfrak{l}}((w' \circ \lambda)_{\mathfrak{l}})) \xrightarrow{\sim} \text{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda)).$$

The inverse map is given by $\Psi \rightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(\Psi)$.

Proof. By Proposition 4.6 and Theorem 6.11 (i) both $\text{Hom}_{\mathfrak{l}}(W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}}), W_{\mathfrak{l}}((w' \circ \lambda)_{\mathfrak{l}}))$ and $\text{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda))$ are one-dimensional. The assertion follows since the correspondence $\Phi \mapsto \text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(\tilde{\Phi})$ is clearly injective and $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(\text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(\tilde{\Phi})) = \Phi$. $\square$

7.4. Semi-infinite restriction of admissible affine vertex algebras. Since it is defined by the semi-infinite cohomology the space $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(V^k(\overset{\circ}{\mathfrak{g}}))$ inherits a vertex algebra structure from $V^k(\overset{\circ}{\mathfrak{g}})$, and we have a natural vertex algebra homomorphism

$$\bigotimes_{i=0}^s V^{k_i}(\mathfrak{l}_i) \rightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(V^k(\overset{\circ}{\mathfrak{g}})),$$

where $V^{k_i}(\overset{\circ}{\mathfrak{l}}_i)$ denote the universal affine vertex algebra associated with $\mathfrak{l}_i$ at level k_i . By composing with the map $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(V^k(\overset{\circ}{\mathfrak{g}})) \rightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0))$ induced by the surjection $V^k(\overset{\circ}{\mathfrak{g}}) \twoheadrightarrow L(k\Lambda_0)$ this gives rise to a vertex algebra homomorphism

$$(52) \quad \bigotimes_{i=0}^s V^{k_i}(\mathfrak{l}_i) \rightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0)).$$

On the other hand, there is a natural surjective homomorphism

$$\bigotimes_{i=0}^s V^{k_i}(\mathfrak{l}_i) \twoheadrightarrow \bigotimes_{i=0}^s L_{\mathfrak{l}_i}(k_i\Lambda_0)$$

of vertex algebras, where $L_{\mathfrak{l}_i}(k_i\Lambda_0)$ is the unique simple quotient of $V^{k_i}(\mathfrak{l}_i)$.

Theorem 7.5. *Let k be an admissible number. The vertex algebra homomorphism (52) factors through the vertex algebra homomorphism*

$$\bigotimes_{i=0}^s L_{\mathfrak{l}_i}(k_i\Lambda_0) \hookrightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0)).$$

Proof. Put $\lambda = k\Lambda_0$ and let $C^{\bullet}(\lambda)$ be the two-sided BGG resolution of $L(k\Lambda_0)$ in Theorem 6.11. By the vanishing assertion of Proposition 7.3 the semi-infinite cohomology $H^{\frac{\infty}{2}+\bullet}(L(\overset{\circ}{\mathfrak{m}}, L(\lambda)))$ is isomorphic to the cohomology of the complex $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^{\bullet}(\lambda))$ obtained from $C^{\bullet}(\lambda)$ applying the functor $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}$. Thus $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0))$ is isomorphic to the zeroth cohomology of the complex $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^{\bullet}(\lambda))$.

Consider the map $C^{-1}(\lambda) \supset W(\dot{s}_0^{(i)} \circ \lambda) \xrightarrow{d_{1, \dot{s}_0^{(i)}}} W(\lambda) \subset C^0(\lambda)$ for $i = 1, \dots, s$. By applying the functor $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}$ this induces a non-zero homomorphism

$$W_{\mathfrak{l}}(\dot{s}_0^{(i)} \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}) \rightarrow W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})$$

by Proposition 7.4, and the image of the highest weight vector of $W_{\mathfrak{l}}(\dot{s}_0^{(i)} \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}})$ generates the maximal $\mathfrak{l}_i$ -submodule of $V^{k_i}(\mathfrak{l}_i) \subset W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})$ by Proposition 6.14. It follows that the maximal $\mathfrak{l}$ -submodule of $\bigotimes_{i=0}^s V^{k_i}(\mathfrak{l}_i) \subset W_{\mathfrak{l}}(\lambda)$ is in the image of $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(d_{-1}) : \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^{-1}(\lambda)) \rightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^0(\lambda))$. This completes the proof. $\square$

7.5. The case of minimal parabolic subalgebras. Consider the case that $\mathring{\mathfrak{p}}$ is generated by $\mathring{\mathfrak{b}}_-$ and e_i with $i \in \mathring{I}$. Then $\mathring{\mathfrak{l}} = \mathring{\mathfrak{l}}_0 \oplus \mathring{\mathfrak{l}}_1$, $\mathring{\mathfrak{l}}_1 = \mathfrak{sl}_2^{(i)}$ and $\mathfrak{l}_1 = \widehat{\mathfrak{sl}}_2^{(i)}$.

Theorem 7.6 ($\mathring{\mathfrak{p}}$ minimal). *Let k be an admissible number and let M be a module over the vertex algebra $L(k\Lambda_0)$. Then, for each $p \in \mathbb{Z}$, $H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, M)$ is a direct sum of admissible representations of level k_1 (see (49)) as $\widehat{\mathfrak{sl}}_2^{(i)}$ -modules.*

Proof. By Theorem 7.5, $L_{\mathfrak{l}_1}(k_1\Lambda_0)$ is a vertex subalgebra of $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0)) = H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{m}}, L(k\Lambda_0))$. If M is a module over $L(k\Lambda_0)$, then $H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, M)$ is naturally a module over $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0))$, and therefore, it is a module over $L_{\mathfrak{l}_1}(k_1\Lambda_0)$. The assertion follows since it is known by [AM] that any module over $L_{\mathfrak{l}_1}(k_1\Lambda_0)$ in the category $\mathcal{O}^{\mathfrak{l}_1}$ must be a direct sum of admissible representations of $\mathfrak{l}_1 \cong \mathfrak{sl}_2$. $\square$

The following assertion generalizes [HT, Theorem 3.8] in the case that $\mathring{\mathfrak{p}}$ is minimal.

Theorem 7.7 ($\mathring{\mathfrak{p}}$ minimal). *Let k be an admissible number, $\lambda \in Pr_k^+$. Then*

$$H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, L(\lambda)) \cong \bigoplus_{\substack{w \in \mathcal{W}(\lambda)^S \\ \ell \frac{\infty}{2}(w)=p}} L_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$$

as $\mathfrak{l}$ -modules.

Proof. It is known by [MF] (see also [FM]) that $L(\lambda)$ with $\lambda \in Pr_k^+$ is a module over $L(k\Lambda_0)$. Therefore $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ is a direct sum of irreducible admissible representations as $\widehat{\mathfrak{sl}}_2^{(i)}$ -modules by Theorem 7.6. Hence it is sufficient to determine the subspace $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))^{\mathfrak{l}_+}$ of the singular vectors of $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$. Clearly, any weight of $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))^{\mathfrak{l}_+}$ must be admissible for $\mathfrak{l}_1 = \widehat{\mathfrak{sl}}_2^{(i)}$.

As is remarked in the proof of Proposition 7.5, $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ is the cohomology of the complex $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^{\bullet}(\lambda))$ and we have $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^p(\lambda)) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$ by Proposition 7.3. Now Theorem 3.3 and Lemma 7.1 imply that

$$\begin{aligned} & \{(w \circ \lambda)_{\mathfrak{l}}; w \in \mathcal{W}(\lambda), (w \circ \lambda)_{\mathfrak{l}} \text{ is an admissible weight for } \widehat{\mathfrak{sl}}_2^{(i)}\} \\ &= \{(w \circ \lambda)_{\mathfrak{l}}; w \in \mathcal{W}(\lambda), (w \circ \lambda)_{\mathfrak{l}} \text{ is a dominant weight for } \widehat{\mathfrak{sl}}_2^{(i)}\} \\ &= \{(w \circ \lambda)_{\mathfrak{l}}; w \in \mathcal{W}(\lambda)^S\}. \end{aligned}$$

It follows that if a weight μ of $W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$ is admissible for $\widehat{\mathfrak{sl}}_2^{(i)}$, then $w \in \mathcal{W}(\lambda)^S$ and $\mu = (w \circ \lambda)_{\mathfrak{l}}$. Therefore the image $[[(w \circ \lambda)_{\mathfrak{l}}]]$ of the highest weight vector $|(w \circ \lambda)_{\mathfrak{l}}\rangle$ of $W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$ is non-zero in $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ and $\{[[(w \circ \lambda)_{\mathfrak{l}}]]; w \in \mathcal{W}(\lambda)^S\}$ forms a basis of $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))^{\mathfrak{l}_+}$. By Theorem 3.3, this completes the proof. $\square$

Remark 7.8. In the subsequent paper [A6] we prove that for an admissible number k any $L(k\Lambda_0)$ -module in the category $\mathcal{O}^{\mathfrak{g}}$ must be a direct sum of admissible representations. Hence it follows from the proof that the assertion of Theorem 7.7 is valid for any parabolic subalgebra of $\mathring{\mathfrak{g}}$.

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