ON THE CHARACTER OF CERTAIN IRREDUCIBLE MODULAR REPRESENTATIONS

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ABSTRACT. Let G be an almost simple, simply connected algebraic group over an algebraically closed field of characteristic $p > 0$. In this paper we restate our conjecture from 1979 on the characters of irreducible modular representations of G so that it is now directly applicable to any dominant highest weight.

1. Let G be an almost simple, simply connected algebraic group over $\mathbf{k}$, an algebraically closed field of characteristic $p > 1$. Let $\text{Rep}G$ be the category of finite dimensional $\mathbf{k}$ -vector spaces with a given rational linear action of G and let $\text{Irr}G$ be a set of representatives for the simple objects of $\text{Rep}G$. We fix a Borel subgroup B of G and a maximal torus T of B . Let $Y = \text{Hom}(\mathbf{k}^*, T)$, $X = \text{Hom}(T, \mathbf{k}^*)$ (with group operation written as $+$) and let $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbf{Z}$ be the obvious pairing. If $V \in \text{Irr}G$, then there is a well-defined $\lambda_V \in X$ with the following property: the T -action on the unique B -stable line in V is through λ_V ; according to Chevalley, $E \mapsto \lambda_E$ is a bijection from $\text{Irr}G$ to a subset of X of the form

$$X^+ = \{\lambda \in X; \langle \check{\alpha}_i, \lambda \rangle \in \mathbf{N} \quad \forall i \in I\}$$

for a well-defined basis $\{\check{\alpha}_i; i \in I\}$ of Y . For $\lambda \in X^+$ we shall denote by V_λ the object of $\text{Irr}G$ corresponding to λ . If $V \in \text{Rep}G$, then for any $\mu \in X$ we denote by $n_\mu(V)$ the multiplicity of μ in $V|_T$; we set $[V] = \sum_{\mu \in X} n_\mu(V)e^\mu \in \mathbf{Z}[X]$ where $\mathbf{Z}[X]$ is the group ring of X (the basis element of $\mathbf{Z}[X]$ corresponding to $\mu \in X$ is denoted by e^μ so that $e^\mu e^{\mu'} = e^{\mu+\mu'}$ for $\mu, \mu' \in X$). It is of considerable interest to compute explicitly the element $[V_\lambda] \in \mathbf{Z}[X]$ for any $\lambda \in X^+$. Let h be the Coxeter number of G . A conjectural formula for $[V_\lambda]$ (assuming that $p \geq c_G^0$ where c_G^0 is a constant depending only on the root datum of G) was stated in [L1, p. 316]. In the early 1990's it was proved (see [AJS] and the references there) that there exists a (necessarily unique) prime number $c_G \geq c_G^0$ depending only on the root datum of G such that the conjectural formula in [L1, p. 316] is true if $p \geq c_G$ and c_G is minimum possible (but c_G was not explicitly determined). In [F], Fiebig showed that $c_G \leq c'_G$ where c'_G is an explicitly known but very large constant. In [Wi], Williamson, partly in collaboration with Xuhua He, showed that for infinitely many G , c_G is much larger than c_G^0 . Now the conjecture in [L1, p. 316] had an unsatisfactory aspect: it applied only to a finite set of $\lambda \in X^+$ which, after application of Jantzen's results [J] on translation functors, becomes a larger but still finite set (including all λ in $X_{red}^+ = \{\lambda \in X^+; \langle \check{\alpha}_i, \lambda \rangle \leq p-1 \quad \forall i \in I\}$); then

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the case of a general $\lambda \in X^+$ had to be obtained by applying the Steinberg tensor product theorem [S]. In this note I want to offer a reformulation of the conjecture in [L1, p. 316] (now a known theorem for p large enough) which applies directly to any $\lambda \in X^+$; see 7(b).

2. Notation. Let NT be the normalizer of T in G and let $W = NT/T$ be the Weyl group. Note that W acts naturally on T hence on Y, X and $\mathbf{Z}[X]$. Let $\check{R} = \{y \in Y; y = w(\check{\alpha}_i) \text{ for some } w \in W, i \in I\}$ (the set of coroots). Define $\check{\alpha}_0 \in \check{R}$ by the condition that $\check{\alpha}_0 + \check{\alpha}_i \notin \check{R}$ for any $i \in I$. (Thus $\check{\alpha}_0$ is the highest coroot.) For $i \in I \cup \{0\}$ define $\alpha_i \in X$ by the condition that the map $X \rightarrow X$, $\lambda \mapsto \lambda - \langle \check{\alpha}_i, \lambda \rangle \alpha_i$, is induced by a (uniquely defined) element s_i of W . Let $w \mapsto \epsilon_w$ be the homomorphism $W \rightarrow \{1, -1\}$ such that $\epsilon_{s_i} = -1$ for any $i \in I$. Define $\rho \in X^+$ by $\langle \check{\alpha}_i, \rho \rangle = 1$ for any $i \in I$. Let $\leq$ be the partial order on X given by $\lambda \leq \lambda'$ whenever $\lambda' - \lambda \in \sum_{i \in I} \mathbf{N} \alpha_i$.

3. Let

$$\delta = \{\lambda \in X; \langle \check{\alpha}_i, \lambda + \rho \rangle \leq 0 \quad \forall i \in I, \langle \check{\alpha}_0, \lambda + \rho \rangle \geq -p\}.$$

For $i \in I$ we define $s'_i : X \rightarrow X$ by $s'_i(\lambda) = \lambda - \langle \check{\alpha}_i, \lambda + \rho \rangle \alpha_i$ (an affine reflection). We define $s'_0 : X \rightarrow X$ by $s'_0(\lambda) = \lambda - (\langle \check{\alpha}_0, \lambda + \rho \rangle + p) \alpha_0$ (an affine reflection). Let W_a be the subgroup of the group of permutations of X generated by $s'_i (i \in I \cup \{0\})$. Then W_a is a Coxeter group on the generators $s'_i (i \in I \cup \{0\})$, with length function $l : W_a \rightarrow \mathbf{N}$.

For $\lambda \in X$ we have $w^{-1}(\lambda) \in \delta$ for some $w \in W_a$ and among all such w there is a unique one, w_λ , of minimal length.

For $\lambda, \mu \in X^+$ we set $w = w_\lambda$ and

$$\mathbf{p}_{\mu, \lambda} = \sum_{y \in W_a; y^{-1}(\mu) = w^{-1}(\lambda)} (-1)^{l(yw)} P_{y, w}(1) \in \mathbf{Z}$$

where $P_{y, w}$ is the polynomial associated in [KL] to y, w in the Coxeter group W_a .

From the definitions we see that $\mathbf{p}_{\mu, \lambda} \neq 0 \implies \mu \leq \lambda$, $\mathbf{p}_{\lambda, \lambda} = 1$. Hence for $\lambda, \mu \in X^+$ we can define $\mathbf{q}_{\mu, \lambda} \in \mathbf{Z}$ by the requirement

$$\sum_{\nu \in X^+} \mathbf{p}_{\mu, \nu} \mathbf{q}_{\nu, \lambda} = \delta_{\mu, \lambda}$$

for any λ, μ in X^+ . We have $\mathbf{q}_{\mu, \lambda} \neq 0 \implies \mu \leq \lambda$ and $\mathbf{q}_{\lambda, \lambda} = 1$.

4. For any $\lambda \in X^+$ we can write uniquely $\lambda = \sum_{k \geq 0} p^k \lambda^k$ where $\lambda^k \in X_{red}^+$ for all $k \geq 0$ and $\lambda^k = 0$ for large k .

For any $\lambda \in X^+$ and any $k \in \mathbf{N}$ we define elements $E_\lambda^k \in \mathbf{Z}[X]$ by induction on k as follows:

$$E_\lambda^0 = \sum_{w \in W} \epsilon_w e^{w(\lambda + \rho)} / \sum_{w \in W} \epsilon_w e^{w(\rho)},$$

$$(a) \quad E_\lambda^k = \sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda^j} E_{\sum_{j; 0 \leq j \leq k-2} p^j \lambda^j + p^{k-1} \mu}^{k-1} \text{ if } k \geq 1.$$

Note that $E_\lambda^k \in \mathbf{Z}[X]^W$, the ring of W -invariants in $\mathbf{Z}[X]$.

We show that for $k \geq 0$ we have

$$(b) \quad E_\lambda^k = \sum_{\mu \in X^+} \mathbf{q}_{\mu, \sum_{j; j \geq k} p^{j-k} \lambda^j} E_{\sum_{j; 0 \leq j \leq k-1} p^j \lambda^j + p^k \mu}^{k+1}.$$

For any $\mu \in X^+$ and any $k, h \geq 0$,

$$\left(\sum_{j:0 \leq j \leq k-1} p^j \lambda^j + p^k \mu \right)^h$$

is equal to λ^h if $0 \leq h \leq k-1$ and to μ^{h-k} if $h \geq k$; hence, by (a), we have

$$E_{\sum_{j:0 \leq j \leq k-1} p^j \lambda^j + p^k \mu}^{k+1} = \sum_{\nu \in X^+} \mathbf{p}_{\nu, \sum_{h:h \geq k} p^{h-k} \mu^{h-k}} E_{\sum_{h:0 \leq h \leq k-1} p^h \lambda^h + p^k \nu}^k.$$

Thus the right-hand side of (b) is

$$\begin{aligned} & \sum_{\mu \in X^+} \mathbf{q}_{\mu, \sum_{j:j \geq k} p^{j-k} \lambda^j} \sum_{\nu \in X^+} \mathbf{p}_{\nu, \sum_{h:h \geq k} p^{h-k} \mu^{h-k}} E_{\sum_{h:0 \leq h \leq k-1} p^h \lambda^h + p^k \nu}^k \\ &= \sum_{\mu \in X^+} \mathbf{q}_{\mu, \sum_{j:j \geq k} p^{j-k} \lambda^j} \sum_{\nu \in X^+} \mathbf{p}_{\nu, \mu} E_{\sum_{h:0 \leq h \leq k-1} p^h \lambda^h + p^k \nu}^k \\ &= \sum_{\nu \in X^+} \delta_{\nu, \sum_{j:j \geq k} p^{j-k} \lambda^j} E_{\sum_{h:0 \leq h \leq k-1} p^h \lambda^h + p^k \nu}^k \\ &= E_{\sum_{h:0 \leq h \leq k-1} p^h \lambda^h + p^k \sum_{j:j \geq k} p^{j-k} \lambda^j}^k \\ &= E_{\lambda}^k. \end{aligned}$$

This proves (b).

By induction on k we see, using (a), that for any $k \geq 1$ we have

$$\begin{aligned} (c) \quad E_{\lambda}^k &= \sum_{\mu_0, \mu_1, \dots, \mu_{k-1} \text{ in } X^+} \mathbf{p}_{\mu_0, \lambda^0 + p\mu_1} \mathbf{p}_{\mu_1, \lambda^1 + p\mu_2} \cdots \mathbf{p}_{\mu_{k-2}, \lambda^{k-2} + p\mu_{k-1}} \\ &\quad \times \mathbf{p}_{\mu_{k-1}, \sum_{j \geq k-1} p^{j-k+1} \lambda^j} E_{\mu_0}^0. \end{aligned}$$

By induction on k we see, using (b), that for any $k \geq 1$ we have

$$\begin{aligned} (d) \quad E_{\lambda}^0 &= \sum_{\nu_0, \nu_1, \dots, \nu_{k-1} \text{ in } X^+} \mathbf{q}_{\nu_0, \lambda} \mathbf{q}_{\nu_1, (\nu_0 - \nu_0^0)/p} \mathbf{q}_{\nu_2, (\nu_1 - \nu_1^0)/p} \cdots \mathbf{q}_{\nu_{k-1}, (\nu_{k-2} - \nu_{k-2}^0)/p} \\ &\quad \times E_{\nu_0^0 + p\nu_1^0 + \cdots + p^{k-2}\nu_{k-2}^0 + p^{i-1}\nu_{k-1}}^k. \end{aligned}$$

5. Let $\lambda \in X^+$. We can find $n \geq 0$ such that $\lambda^n = \lambda^{n+1} = \cdots = 0$. If $k \geq n$ we have

$$E_{\lambda}^k = E_{\lambda}^{k+1}.$$

Indeed, if $\mu \in X^+$ and $\mathbf{q}_{\mu, \sum_{j:j \geq k} p^{j-k} \lambda^j} \neq 0$ we have $\mathbf{q}_{\mu, 0} \neq 0$ hence $\mu \leq 0$, $\mu = 0$ and $\mathbf{q}_{0,0} = 1$; it follows that

$$E_{\lambda}^k = E_{\sum_{j:0 \leq j \leq k-1} p^j \lambda^j}^{k+1} = E_{\lambda}^{k+1}.$$

Thus we can set $E_{\lambda}^{\infty} = E_{\lambda}^k$ for large k . Clearly, if $\lambda \in X_{red}^+$, then $E_{\lambda}^1 = E_{\lambda}^2 = \cdots = E_{\lambda}^{\infty}$.

Letting $k \rightarrow \infty$ in 4(c),(d), we deduce that

$$E_{\lambda}^{\infty} = \sum_{\mu_0, \mu_1, \mu_2, \dots \text{ in } X^+; \mu_h=0 \text{ for large } h} (\mathbf{p}_{\mu_0, \lambda^0 + p\mu_1} \mathbf{p}_{\mu_1, \lambda^1 + p\mu_2} \mathbf{p}_{\mu_2, \lambda^2 + p\mu_3} \cdots) E_{\mu_0}^0,$$

(note that for large h we have $\mathbf{p}_{\mu_h, \lambda^h + p\mu_{h+1}} = \mathbf{p}_{0,0} = 1$ so that the infinite product makes sense) and

$$E_\lambda^0 = \sum_{\nu_0, \nu_1, \nu_2, \dots \text{ in } X^+} (\mathbf{q}_{\nu_0, \lambda} \mathbf{q}_{\nu_1, (\nu_0 - \nu_1^0)/p} \mathbf{q}_{\nu_2, (\nu_1 - \nu_2^0)/p} \cdots) E_{\nu_0^0 + p\nu_1^0 + p^2\nu_2^0 + \dots}^\infty$$

(note that for large h we have $\nu_h = 0$ hence $\mathbf{q}_{\nu_{h+1}, (\nu_h - \nu_h^0)/p} = \mathbf{q}_{0,0} = 1$ so that the infinite product makes sense).

6. It is known since the early 1990's that, if p is not very small, then the conjecture 8.2 in [L2] on quantum groups at a p -th root of 1 holds. In particular for $\lambda \in X^+$, the element E_λ^1 describes the character of an irreducible finite dimensional representation of such a quantum group and the tensor product theorem [L2, 7.4] holds for it.

Thus, if for any $\xi = \sum_{\lambda \in X} c_\lambda e^\lambda \in \mathbf{Z}[X]$ (with $c_\lambda \in \mathbf{Z}$) and any $h \geq 0$ we set $\xi^{(h)} = \sum_{\lambda \in X} c_\lambda e^{p^h \lambda} \in \mathbf{Z}[X]$, then for any $\lambda \in X^+$ we have the equality

$$(a) \quad E_\lambda^1 = E_{\lambda^0}^1 (E_{\sum_{j \geq 1} p^{j-1} \lambda_j}^0)^{(1)}.$$

This can be also deduced from results in [K].

We show by induction on $k \geq 1$ that for any $\lambda \in X^+$ we have

$$(b) \quad E_\lambda^k = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \cdots (E_{\lambda^{k-1}}^1)^{(k-1)} (E_{\sum_{j \geq k} p^{j-k} \lambda_j}^0)^{(k)}.$$

By (a), we can assume that $k \geq 2$. Using 4(a), it is enough to show that

$$\begin{aligned} \sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda_j} E_{\sum_{j; 0 \leq j \leq k-2} p^j \lambda_j + p^{k-1} \mu}^{k-1} \\ = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \cdots (E_{\lambda^{k-1}}^1)^{(k-1)} (E_{\sum_{j \geq k} p^{j-k} \lambda_j}^0)^{(k)}. \end{aligned}$$

Replacing here

$$E_{\sum_{j; 0 \leq j \leq k-2} p^j \lambda_j + p^{k-1} \mu}^{k-1} = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \cdots (E_{\lambda^{k-2}}^1)^{(k-2)} (E_\mu^0)^{(k-1)},$$

which is known from the induction hypothesis, we see that it is enough to show that

$$\begin{aligned} \sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda_j} E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \cdots (E_{\lambda^{k-2}}^1)^{(k-2)} (E_\mu^0)^{(k-1)} \\ = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} \cdots (E_{\lambda^{k-1}}^1)^{(k-1)} (E_{\sum_{j \geq k} p^{j-k} \lambda_j}^0)^{(k)}. \end{aligned}$$

Thus, it is enough to show that

$$\sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda_j} (E_\mu^0)^{(k-1)} = (E_{\lambda^{k-1}}^1)^{(k-1)} (E_{\sum_{j \geq k} p^{j-k} \lambda_j}^0)^{(k)}$$

or that

$$\sum_{\mu \in X^+} \mathbf{p}_{\mu, \sum_{j; j \geq k-1} p^{j-k+1} \lambda_j} E_\mu^0 = E_{\lambda^{k-1}}^1 (E_{\sum_{j \geq k} p^{j-k} \lambda_j}^0)^{(1)}.$$

Using (a), the right-hand side is $E_{\sum_{j \geq k-1} p^{j-k+1} \lambda_j}^1$. This is equal to the left-hand side, by 4(a). This proves (b).

Letting $k \rightarrow \infty$ in (b) we obtain for any $\lambda \in X^+$:

$$(c) \quad E_\lambda^\infty = E_{\lambda^0}^1 (E_{\lambda^1}^1)^{(1)} (E_{\lambda^2}^1)^{(2)} \cdots$$

Note that for large h we have $\lambda^h = 0$ hence $E_{\lambda^h}^1 = 1$ so that the infinite product makes sense. (We have also used that $E_{\sum_{j \geq k} p^{j-k} \lambda^j}^0 = E_0^0 = 1$ for large h .)

7. We now assume that $p \geq c_G$. Then, by the first paragraph of section 6, we have

$$(a) \quad [V_\lambda] = E_\lambda^1 \text{ for any } \lambda \in X_{red}^+.$$

Using the Steinberg tensor product theorem [S] and (a), we see that for any $\lambda \in X^+$ we have

$$[V_\lambda] = [V_{\lambda^0}][V_{\lambda^1}]^{(1)}[V_{\lambda^2}]^{(2)} = \cdots = E_{\lambda^0}^1(E_{\lambda^1}^1)^{(1)}(E_{\lambda^2}^1)^{(2)} \dots$$

Using this and 6(c) we deduce

$$(b) \quad [V_\lambda] = E_\lambda^\infty.$$

8. We preserve the setup of section 7. In the identity

$$E_\lambda^0 = \sum_{\mu \in X^+} \mathbf{q}_{\mu, \lambda} E_\mu^1$$

(see 4(b)) the coefficient $\mathbf{q}_{\mu, \lambda}$ can be interpreted as the multiplicity of an irreducible representation of a quantum group at a p -th root of 1 in a not necessarily irreducible representation of that quantum group. In particular we have

$$(a) \quad \mathbf{q}_{\mu, \lambda} \in \mathbf{N}$$

for any λ, μ in X^+ . We show by descending induction on k that for any $\lambda \in X^+$ and any $k \geq 0$ we have

$$(b) \quad E_\lambda^k = [V_\lambda(k)]$$

for some $V_\lambda(k) \in \text{Rep}G$.

If k is large, we have $E_\lambda^k = E_\lambda^\infty$ and (b) follows from 7(b). Assume now that $k \geq 0$ and that (b) is known when k is replaced by $k+1$. Then (b) holds for k by (a), 4(b) and the induction hypothesis.

9. We now assume that $G = SL_2(\mathbf{k})$ and $p = 3$. Let $d_\lambda(k) = \dim V_\lambda(k)$ (see Sec. 8) where $\lambda \in X^+$ and $k \geq 0$. The numbers $d_\lambda(0)$ form a sequence

$$1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 18 \ 19 \dots$$

where X^+ is identified with $\mathbf{N}$ in an obvious way. We underline the numbers divisible by 3 in this sequence:

$$1 \ 2 \ \underline{3} \ 4 \ 5 \ \underline{6} \ 7 \ 8 \ \underline{9} \ 10 \ 11 \ \underline{12} \ 13 \ 14 \ \underline{15} \ 16 \ 17 \ \underline{18} \ 19 \dots$$

and regard the underlined numbers as mirrors. We now consider the reflections of the non-mirrors in the list in the mirrors to its left and replace them by the alternating sum of those reflections; we leave the mirrors unchanged. For example, 4 is replaced by $4 - 2 = 2$, 5 is replaced by $5 - 1 = 4$, 7 is replaced by $7 - 5 + 1 = 3$, 8 is replaced by $8 - 4 + 2 = 6$, etc. We obtain the sequence

$$1 \ 2 \ 3 \ 2 \ 4 \ 6 \ 3 \ 6 \ 9 \ 4 \ 8 \ 12 \ 5 \ 10 \ 15 \ 6 \ 12 \ 18 \ 7 \dots$$

which is the same as the sequence $d_\lambda(1)$. We arrange the numbers in the last sequence in groups of three and we underline every third group (in which all numbers are divisible by 3):

$$[1 \ 2 \ 3] \ [2 \ 4 \ 6] \ [\underline{3 \ 6 \ 9}] \ [4 \ 8 \ 12] \ [5 \ 10 \ 15] \ [\underline{6 \ 12 \ 18}] \ [7 \dots].$$

We regard the underlined groups as (new) mirrors. We consider the reflections of the non-(new) mirrors in the list in the (new) mirrors to its left (preserving the order in each group) and replace them by the alternating sum of those reflections; we leave the (new) mirrors unchanged. For example, $[4 \ 8 \ 12]$ is replaced by $[4 \ 8 \ 12] - [2 \ 4 \ 6] = [2 \ 4 \ 6]$, $[5 \ 10 \ 15]$ is replaced by $[5 \ 10 \ 15] - [1 \ 2 \ 3] = [4 \ 8 \ 12]$, $[7 \ 14 \ 21]$ is replaced by $[7 \ 14 \ 21] - [5 \ 10 \ 15] + [1 \ 2 \ 3] = [3 \ 6 \ 9]$, etc. We obtain the sequence

$$1 \ 2 \ 3 \ 2 \ 4 \ 6 \ 3 \ 6 \ 9 \ 2 \ 4 \ 6 \ 4 \ 8 \ 12 \ 6 \ 12 \ 18 \ 3 \dots$$

which is the same as the sequence $d_\lambda(2)$. We now arrange the numbers in the last sequence in groups of nine and we underline every third group (in which all numbers are divisible by 3). We repeat the previous procedure and we get the sequence $d_\lambda(3)$, then $d_\lambda(4)$, etc.

REFERENCES

- [AJS] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a p th root of unity and of semisimple groups in characteristic p : independence of p* , *Astérisque* **220** (1994), 321. MR1272539 (95j:20036)
- [F] P. Fiebig, *An upper bound on the exceptional characteristics for Lusztig's character formula*, *J. Reine Angew. Math.* **673** (2012), 1–31, DOI 10.1515/CRELLE.2011.170. MR2999126
- [J] J. C. Jantzen, *Moduln mit einem höchsten Gewicht* (German), *Lecture Notes in Mathematics*, vol. 750, Springer, Berlin, 1979. MR552943 (81m:17011)
- [K] S.-i. Kato, *On the Kazhdan-Lusztig polynomials for affine Weyl groups*, *Adv. in Math.* **55** (1985), no. 2, 103–130, DOI 10.1016/0001-8708(85)90017-9. MR772611 (86d:20048)
- [KL] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, *Invent. Math.* **53** (1979), no. 2, 165–184, DOI 10.1007/BF01390031. MR560412 (81j:20066)
- [L1] G. Lusztig, *Some problems in the representation theory of finite Chevalley groups*, *The Santa Cruz Conference on Finite Groups* (Univ. California, Santa Cruz, Calif., 1979), *Proc. Sympos. Pure Math.*, vol. 37, Amer. Math. Soc., Providence, RI, 1980, pp. 313–317. MR604598 (82i:20014)
- [L2] G. Lusztig, *Modular representations and quantum groups*, *Classical groups and related topics* (Beijing, 1987), *Contemp. Math.*, vol. 82, Amer. Math. Soc., Providence, RI, 1989, pp. 59–77, DOI 10.1090/conm/082/982278. MR982278 (90a:16008)
- [S] R. Steinberg, *Representations of algebraic groups*, *Nagoya Math. J.* **22** (1963), 33–56. MR0155937 (27 #5870)
- [Wi] G. Williamson, *Schubert calculus and torsion*, arxiv:1309.5055.

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