REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 22, Pages 119–159 (October 15, 2018)
https://doi.org/10.1090/ert/518

ON PRO-p-IWAHORI INVARIANTS OF R-REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS

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Abstract. Let F be a locally compact field with residue characteristic p, and let $\mathbf{G}$ be a connected reductive F-group. Let $\mathcal{U}$ be a pro-p Iwahori subgroup of $G = \mathbf{G}(F)$. Fix a commutative ring R. If π is a smooth $R[G]$-representation, the space of invariants $\pi^{\mathcal{U}}$ is a right module over the Hecke algebra $\mathcal{H}$ of $\mathcal{U}$ in G.

Let P be a parabolic subgroup of G with a Levi decomposition $P = MN$ adapted to $\mathcal{U}$. We complement a previous investigation of Ollivier-Vignéras on the relation between taking $\mathcal{U}$-invariants and various functor like Ind_P^G and right and left adjoints. More precisely the authors' previous work with Herzig introduced representations $I_G(P, \sigma, Q)$ where σ is a smooth representation of M extending, trivially on N, to a larger parabolic subgroup $P(\sigma)$, and Q is a parabolic subgroup between P and $P(\sigma)$. Here we relate $I_G(P, \sigma, Q)^{\mathcal{U}}$ to an analogously defined $\mathcal{H}$-module $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$, where $\mathcal{U}_M = \mathcal{U} \cap M$ and $\sigma^{\mathcal{U}_M}$ is seen as a module over the Hecke algebra $\mathcal{H}_M$ of $\mathcal{U}_M$ in M. In the reverse direction, if $\mathcal{V}$ is a right $\mathcal{H}_M$-module, we relate $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes \text{c-Ind}_{\mathcal{U}}^G \mathbf{1}$ to $I_G(P, \mathcal{V} \otimes_{\mathcal{H}_M} \text{c-Ind}_{\mathcal{U}_M}^M \mathbf{1}, Q)$. As an application we prove that if R is an algebraically closed field of characteristic p, and π is an irreducible admissible representation of G, then the contragredient of π is 0 unless π has finite dimension.

Contents

1. Introduction

1.1. The present paper is a companion to [AHV] and is similarly inspired by the classification results of [AHHV17]; however it can be read independently. We recall the setting. We have a non-archimedean locally compact field F of residue characteristic p and a connected reductive F-group $\mathbf{G}$. We fix a commutative ring R and study the smooth R-representations of $G = \mathbf{G}(F)$.

Received by the editors March 14, 2018, and, in revised form, June 17, 2018.
2010 *Mathematics Subject Classification.* Primary 20C08; Secondary 11F70.
Key words and phrases. Parabolic induction, pro-p Iwahori Hecke algebra.
The first-named author was supported by JSPS KAKENHI Grant Number 26707001.

In [AHHV17] the irreducible admissible R-representations of G are classified in terms of supersingular ones when R is an algebraically closed field of characteristic p. That classification is expressed in terms of representations $I_G(P,\sigma,Q)$, which make sense for any R. In that notation, P is a parabolic subgroup of G with a Levi decomposition $P = MN$ and σ a smooth R-representation of the Levi subgroup M; there is a maximal parabolic subgroup $P(\sigma)$ of G containing P to which σ inflated to P extends to a representation $e_{P(\sigma)}(\sigma)$, and Q is a parabolic subgroup of G with $P \subset Q \subset P(\sigma)$. Then

$$I_G(P,\sigma,Q) = \mathrm{Ind}_{P(\sigma)}^G(e_{P(\sigma)}(\sigma) \otimes \mathrm{St}_Q^{P(\sigma)}),$$

where Ind stands for parabolic induction and $\mathrm{St}_Q^{P(\sigma)} = \mathrm{Ind}_Q^{P(\sigma)} R/\sum \mathrm{Ind}_{Q'}^{P(\sigma)} R$, the sum being over parabolic subgroups Q' of G with $Q \subsetneq Q' \subset P(\sigma)$. Alternatively, $I_G(P,\sigma,Q)$ is the quotient of $Ind_Q^G(e_Q(\sigma))$ by $\sum \mathrm{Ind}_{Q'}^G e_{Q'}(\sigma)$ with Q' as above, where $e_Q(\sigma)$ is the restriction of $e_{P(\sigma)}(\sigma)$ to Q, similarly for Q'.

In [AHV] we mainly studied what happens to $I_G(P,\sigma,Q)$ when we apply to it, for a parabolic subgroup P_1 of G, the left adjoint of $\mathrm{Ind}_{P_1}^G$, or its right adjoint. Here we tackle a different question. We fix a pro-p Iwahori subgroup $\mathcal{U}$ of G in good position with respect to P, so that in particular $\mathcal{U}_M = \mathcal{U} \cap M$ is a pro-p Iwahori subgroup of M. One of our main goals is to identify the R-module $I_G(P,\sigma,Q)^{\mathcal{U}}$ of $\mathcal{U}$-invariants, as a right module over the Hecke algebra $\mathcal{H} = \mathcal{H}_G$ of $\mathcal{U}$ in G - the convolution algebra on the double coset space $\mathcal{U}\backslash G/\mathcal{U}$ - in terms of the module $\sigma^{\mathcal{U}_M}$ over the Hecke algebra $\mathcal{H}_M$ of $\mathcal{U}_M$ in M. That goal is achieved in section 4, Theorem 4.17.

1.2. The initial work has been done in [OV17, §4] where $(\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ is identified. Precisely, writing M^+ for the monoid of elements $m \in M$ with $m(\mathcal{U}\cap N)m^{-1} \subset \mathcal{U}\cap N$, the subalgebra $\mathcal{H}_{M^+}$ of $\mathcal{H}_M$ with support in M^+, has a natural algebra embedding θ into $\mathcal{H}$ and [OV17, Proposition 4.4] identifies $(\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ with $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\sigma^{\mathcal{U}_M}) = \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+}} \mathcal{H}$. So in a sense, this paper is a sequel to [OV17] although some of our results here are used in [OV17, §5].

As $I_G(P,\sigma,Q)$ is only a subquotient of $\mathrm{Ind}_P^G \sigma$ and taking $\mathcal{U}$-invariants is only left exact, it is not straightforward to describe $I_G(P,\sigma,Q)^{\mathcal{U}}$ from the previous result. However, that takes care of the parabolic induction step, so in a first approach we may assume $P(\sigma) = G$ so that $I_G(P,\sigma,Q) = e_G(\sigma) \otimes \mathrm{St}_Q^G$. The crucial case is when moreover σ is e-minimal, that is, not an extension $e_M(\tau)$ of a smooth R-representation τ of a proper Levi subgroup of M. That case is treated first and the general case in section 4 only.

1.3. To explain our results, we need more notation. We choose a maximal F-split torus T in G and a minimal parabolic subgroup $B = ZU$ with Levi component Z the G-centralizer of T. We assume that $P = MN$ contains B and M contains Z, and that $\mathcal{U}$ corresponds to an alcove in the apartment associated to T in the adjoint building of G. It turns out that when σ is e-minimal and $P(\sigma) = G$, the set Δ_M of simple roots of T in Lie$(M \cap U)$ is orthogonal to its complement in the set Δ of simple roots of T in Lie U. We assume until the end of this section that Δ_M and $\Delta_2 = \Delta \setminus \Delta_M$ are orthogonal. If M_2 is the Levi subgroup - containing Z - corresponding to Δ_2, both M and M_2 are normal in G, $M \cap M_2 = Z$ and $G = MM_2$. Moreover the normal subgroup M_2' of G generated by N is included in M_2 and $G = MM_2'$.

We say that a right $\mathcal{H}_M$-module $\mathcal{V}$ is extensible to $\mathcal{H}$ if T_z^M acts trivially on $\mathcal{V}$ for $z \in Z \cap M_2'$ (section 3.3). In this case, we show that there is a natural structure of right $\mathcal{H}$-module $e_{\mathcal{H}}(\mathcal{V})$ on $\mathcal{V}$ such that $T_g \in \mathcal{H}$ corresponding to $\mathcal{U}g\mathcal{U}$ for $g \in M_2'$ acts as in the trivial character of G (section 3.4). We call $e_{\mathcal{H}}(\mathcal{V})$ the extension of $\mathcal{V}$ to $\mathcal{H}$ though $\mathcal{H}_M$ is not a subalgebra of $\mathcal{H}$. That notion is already present in [Abe] in the case where R has characteristic p. Here we extend the construction to any R and prove some more properties. In particular we produce an $\mathcal{H}$-equivariant embedding $e_{\mathcal{H}}(\mathcal{V})$ into $\operatorname{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \mathcal{V}$ (Lemma 3.10). If Q is a parabolic subgroup of G containing P, we go further and put on $e_{\mathcal{H}}(\mathcal{V}) \otimes_R (\operatorname{Ind}_Q^G R)^{\mathcal{U}}$ and $e_{\mathcal{H}}(\mathcal{V}) \otimes_R (\operatorname{St}_Q^G)^{\mathcal{U}}$ structures of $\mathcal{H}$-modules (Proposition 3.15 and Corollary 3.17) - note that $\mathcal{H}$ is not a group algebra and there is no obvious notion of tensor product of $\mathcal{H}$-modules.

If σ is an R-representation of M extensible to G, then its extension $e_G(\sigma)$ is simply obtained by letting M_2' act trivially on the space of σ; moreover it is clear that $\sigma^{\mathcal{U}_M}$ is extensible to $\mathcal{H}$, and one shows easily that $e_G(\sigma)^{\mathcal{U}} = e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})$ as an $\mathcal{H}$-module (section 3.5). Moreover, the natural inclusion of $e_G(\sigma)$ into $\operatorname{Ind}_P^G \sigma$ induces on taking pro-p Iwahori invariants an embedding $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \to (\operatorname{Ind}_P^G \sigma)^{\mathcal{U}}$ which, via the isomorphism of [OV17], yields exactly the above embedding of $\mathcal{H}$-modules of $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})$ into $\operatorname{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\sigma^{\mathcal{U}_M})$.

Then we show the $\mathcal{H}$-modules $(e_G(\sigma) \otimes_R \operatorname{Ind}_Q^G R)^{\mathcal{U}}$ and $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\operatorname{Ind}_Q^G R)^{\mathcal{U}}$ are equal, and similarly $(e_G(\sigma) \otimes_R \operatorname{St}_Q^G)^{\mathcal{U}}$and $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\operatorname{St}_Q^G)^{\mathcal{U}}$ are equal (Theorem 4.9).

1.4. We turn back to the general case where we do not assume that Δ_M and $\Delta \setminus \Delta_M$ are orthogonal. Nevertheless, given a right $\mathcal{H}_M$-module $\mathcal{V}$, there exists a largest Levi subgroup $M(\mathcal{V})$ of G - containing Z - corresponding to $\Delta_M \cup \Delta_1$ where Δ_1 is a subset of $\Delta \setminus \Delta_M$ orthogonal to Δ_M, such that $\mathcal{V}$ extends to a right $\mathcal{H}_{M(\mathcal{V})}$-module $e_{M(\mathcal{V})}(\mathcal{V})$ with the notation of section 1.3. For any parabolic subgroup Q between P and $P(\mathcal{V}) = M(\mathcal{V})U$ we put (Definition 4.12)

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) = \operatorname{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(e_{M(\mathcal{V})}(\mathcal{V}) \otimes_R (\operatorname{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}).$$

We refer to Theorem 4.17 for the description of the right $\mathcal{H}$-module $I_G(P, \sigma, Q)^{\mathcal{U}}$ for any smooth R-representation σ of $\mathcal{U}$. As a special case, it says that when σ is e-minimal then $P(\sigma) \supset P(\sigma^{\mathcal{U}_M})$ and if moreover $P(\sigma) = P(\sigma^{\mathcal{U}_M})$, then $I_G(P, \sigma, Q)^{\mathcal{U}}$ is isomorphic to $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$.

Remark 1.1. In [Abe] are attached similar $\mathcal{H}$-modules to $(P, \mathcal{V}, Q)$; here we write them as $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ because their definition uses, instead of $\operatorname{Ind}_{\mathcal{H}_M}^{\mathcal{H}}$ a different kind of induction, which we call coinduction. In [Abe] those modules are used to give, when R is an algebraically closed field of characteristic p, a classification of simple $\mathcal{H}$-modules in terms of supersingular modules - that classification is similar to the classification of irreducible admissible R-representations of G in [AHHV17]. Using the comparison between induced and coinduced modules established in [Vig15b, 4.3] for any R, our Corollary 4.24 expresses $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ as a module $I_{\mathcal{H}}(P_1, \mathcal{V}_1, Q_1)$; consequently we show in section 4.5 that the classification of [Abe] can also be expressed in terms of modules $I_{\mathcal{H}}(P, \mathcal{V}, Q)$.

1.5. In a reverse direction one can associate to a right $\mathcal{H}$-module $\mathcal{V}$ a smooth R-representation $\mathcal{V} \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$ of G (seeing $\mathcal{H}$ as the endomorphism ring of the $R[G]$-module $R[\mathcal{U} \backslash G]$).

If $\mathcal{V}$ is a right $\mathcal{H}_M$-module, we construct, again using [OV17], a natural $R[G]$-map

$$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}} R[\mathcal{U}\backslash G] \to \mathrm{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\mathcal{V}\otimes_{\mathcal{H}_M} R[\mathcal{U}_M\backslash M])\otimes_R \mathrm{St}_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})}),$$

with the notation of section 1.4. We show in section 5 that it is an isomorphism under a mild assumption on the $\mathbb{Z}$-torsion in $\mathcal{V}$; in particular it is an isomorphism if $p=0$ in R.

1.6. In the final section 6, we turn back to the case where R is an algebraically closed field of characteristic p. We prove that the smooth dual of an irreducible admissible R-representation V of G is 0 unless V is finite dimensional - that result is new if F has positive characteristic, a case where the proof of Kohlhaase [Koh] for char$(F)=0$ does not apply. Our proof first reduces to the case where V is supercuspidal (by [AHHV17]) then uses again the $\mathcal{H}$-module $V^{\mathcal{U}}$.

2. Notation, useful facts, and preliminaries

2.1. **The group G and its standard parabolic subgroups $P=MN$.** In all that follows, p is a prime number and F is a local field with finite residue field k of characteristic p. We denote an algebraic group over F by a bold letter, like $\mathbf{H}$, and use the same ordinary letter for the group of F-points, $H=\mathbf{H}(F)$. We fix a connected reductive F-group $\mathbf{G}$. We fix a maximal F-split subtorus $\mathbf{T}$ and write $\mathbf{Z}$ for its $\mathbf{G}$-centralizer; we also fix a minimal parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ with Levi component $\mathbf{Z}$, so that $\mathbf{B}=\mathbf{ZU}$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. Let $X^*(\mathbf{T})$ be the group of F-rational characters of $\mathbf{T}$ and let Φ be the subset of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{G}$. Then $\mathbf{B}$ determines a subset Φ^+ of positive roots - the roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{U}$- and a subset of simple roots Δ. The $\mathbf{G}$-normalizer $\mathbf{N_G}$ of $\mathbf{T}$ acts on $X^*(\mathbf{T})$ and through that action, $\mathbf{N_G}/\mathbf{Z}$ identifies with the Weyl group of the root system Φ. Set $\mathcal{N}:=\mathbf{N_G}(F)$ and note that $\mathbf{N_G}/\mathbf{Z}\simeq\mathcal{N}/Z$; we write $\mathbb{W}$ for $\mathcal{N}/Z$.

A standard parabolic subgroup of $\mathbf{G}$ is a parabolic F-subgroup containing $\mathbf{B}$. Such a parabolic subgroup $\mathbf{P}$ has a unique Levi subgroup $\mathbf{M}$ containing $\mathbf{Z}$, so that $\mathbf{P}=\mathbf{MN}$ where $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$ - we also call $\mathbf{M}$ standard. By a common abuse of language to describe the preceding situation, we simply say "let $P=MN$ be a standard parabolic subgroup of G"; we sometimes write N_P for N and M_P for M. The parabolic subgroup of G opposite to P will be written $\overline{P}$ and its unipotent radical $\overline{N}$, so that $\overline{P}=M\overline{N}$, but beware that $\overline{P}$ is not standard! We write $\mathbb{W}_M$ for the Weyl group $(M\cap\mathcal{N})/Z$.

If $\mathbf{P}=\mathbf{MN}$ is a standard parabolic subgroup of G, then $\mathbf{M}\cap\mathbf{B}$ is a minimal parabolic subgroup of $\mathbf{M}$. If Φ_M denotes the set of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{M}$, with respect to $\mathbf{M}\cap\mathbf{B}$ we have $\Phi_M^+=\Phi_M\cap\Phi^+$ and $\Delta_M=\Phi_M\cap\Delta$. We also write Δ_P for Δ_M as P and M determine each other, $P=MU$. Thus we obtain a bijection $P\mapsto\Delta_P$ from standard parabolic subgroups of G to subsets of Δ, with B corresponding to Φ and G to Δ. If I is a subset of Δ, we sometimes denote by $P_I=M_IN_I$ the corresponding standard parabolic subgroup of G. If $I=\{\alpha\}$ is a singleton, we write $P_\alpha=M_\alpha N_\alpha$. We note a few useful properties. If P_1 is another standard parabolic subgroup of G, then $P\subset P_1$ if and only if $\Delta_P\subset\Delta_{P_1}$; we have $\Delta_{P\cap P_1}=\Delta_P\cap\Delta_{P_1}$ and the parabolic subgroup corresponding to $\Delta_P\cup\Delta_{P_1}$ is the subgroup $\langle P,P_1\rangle$ of G generated by P and P_1. The standard parabolic subgroup of M associated to $\Delta_M\cap\Delta_{M_1}$ is $M\cap P_1=(M\cap M_1)(M\cap N_1)$ [Car85, Proposition 2.8.9]. It is convenient to write G' for the subgroup of G generated by the unipotent radicals of the parabolic subgroups; it is also the normal subgroup of G generated by U, and we have $G = ZG'$. For future reference, we now give a useful lemma extracted from [AHHV17].

Lemma 2.1. *The group $Z \cap G'$ is generated by the $Z \cap M'_\alpha$, α running through Δ.*

Proof. Take $I = \emptyset$ in [AHHV17, II.6.Proposition]. □

Let v_F be the normalized valuation of F. For each $\alpha \in X^*(T)$, the homomorphism $x \mapsto v_F(\alpha(x)) : T \to \mathbb{Z}$ extends uniquely to a homomorphism $Z \to \mathbb{Q}$ that we denote in the same way. This defines a homomorphism $Z \xrightarrow{v} X_*(T) \otimes \mathbb{Q}$ such that $\alpha(v(z)) = v_F(\alpha(z))$ for $z \in Z, \alpha \in X^*(T)$.

An interesting situation occurs when $\Delta = I \sqcup J$ is the union of two orthogonal subsets I and J. In that case, $G' = M'_I M'_J$, M'_I and M'_J commute with each other, and their intersection is finite and central in G [AHHV17, II.7 Remark 4].

2.2. $I_G(P, \sigma, Q)$ **and minimality.** We recall from [AHHV17] the construction of $I_G(P, \sigma, Q)$, our main object of study.

Let σ be an R-representation of M and let P_σ be the standard parabolic subgroup with $\Delta_{P_\sigma} = \Delta_\sigma$ where

$$\Delta_\sigma = \{\alpha \in \Delta \setminus \Delta_P \mid Z \cap M'_\alpha \text{ acts trivially on } \sigma\}.$$

We also let $P(\sigma)$ be the standard parabolic subgroup with

$$\Delta_{P(\sigma)} = \Delta_\sigma \cup \Delta_P.$$

This is the largest parabolic subgroup $P(\sigma)$ containing P to which σ extends, here $N \subset P$ acts on σ trivially. Clearly when $P \subset Q \subset P(\sigma)$, σ extends to Q and the extension is denoted by $e_Q(\sigma)$. The restriction of $e_{P(\sigma)}(\sigma)$ to Q is $e_Q(\sigma)$. If there is no risk of ambiguity, we write

$$e(\sigma) = e_{P(\sigma)}(\sigma).$$

Definition 2.2. An $R[G]$**-triple** is a triple (P, σ, Q) made out of a standard parabolic subgroup $P = MN$ of G, a smooth R-representation of M, and a parabolic subgroup Q of G with $P \subset Q \subset P(\sigma)$. To an $R[G]$-triple (P, σ, Q) is associated a smooth R-representation of G:

$$I_G(P, \sigma, Q) = \mathrm{Ind}_{P(\sigma)}^G (e(\sigma) \otimes \mathrm{St}_Q^{P(\sigma)}),$$

where $\mathrm{St}_Q^{P(\sigma)}$ is the quotient of $\mathrm{Ind}_Q^{P(\sigma)} \mathbf{1}$, $\mathbf{1}$ denoting the trivial R-representation of Q, by the sum of its subrepresentations $\mathrm{Ind}_{Q'}^{P(\sigma)} \mathbf{1}$, the sum being over the set of parabolic subgroups Q' of G with $Q \subsetneq Q' \subset P(\sigma)$.

Note that $I_G(P, \sigma, Q)$ is naturally isomorphic to the quotient of $\mathrm{Ind}_Q^G(e_Q(\sigma))$ by the sum of its subrepresentations $\mathrm{Ind}_{Q'}^G(e_{Q'}(\sigma))$ for $Q \subsetneq Q' \subset P(\sigma)$ by [AHV, Lemma 2.5].

It might happen that σ itself has the form $e_P(\sigma_1)$ for some standard parabolic subgroup $P_1 = M_1 N_1$ contained in P and some R-representation σ_1 of M_1. In that case, $P(\sigma_1) = P(\sigma)$ and $e(\sigma) = e(\sigma_1)$. We say that σ is e**-minimal** if $\sigma = e_P(\sigma_1)$ implies $P_1 = P, \sigma_1 = \sigma$.

Lemma 2.3 ([AHV, Lemma 2.9]). *Let $P = MN$ be a standard parabolic subgroup of G and let σ be an R-representation of M. There exists a unique standard parabolic subgroup $P_{\min,\sigma} = M_{\min,\sigma}N_{\min,\sigma}$ of G and a unique e-minimal representation of $\sigma_{\min}$ of $M_{\min,\sigma}$ with $\sigma = e_P(\sigma_{\min})$. Moreover $P(\sigma) = P(\sigma_{\min})$ and $e(\sigma) = e(\sigma_{\min})$.*

Lemma 2.4. *Let $P = MN$ be a standard parabolic subgroup of G and let σ be an e-minimal R-representation of M. Then Δ_P and $\Delta_{P(\sigma)} \setminus \Delta_P$ are orthogonal.*

That comes from [AHHV17, II.7 Corollary 2]. That corollary of [AHHV17] also shows that when R is a field and σ is supercuspidal, then σ is e-minimal. Lemma 2.4 shows that $\Delta_{P_{\min,\sigma}}$ and $\Delta_{P(\sigma_{\min})} \setminus \Delta_{P_{\min,\sigma}}$ are orthogonal.

Note that when Δ_P and Δ_σ are orthogonal of union $\Delta = \Delta_P \sqcup \Delta_\sigma$, then $G = P(\sigma) = MM'_\sigma$ and $e(\sigma)$ is the R-representation of G simply obtained by extending σ trivially on M'_σ.

Lemma 2.5 ([AHV, Lemma 2.11]). *Let (P, σ, Q) be an $R[G]$-triple. Then we have that $(P_{\min,\sigma}, \sigma_{\min}, Q)$ is an $R[G]$-triple and $I_G(P, \sigma, Q) = I_G(P_{\min,\sigma}, \sigma_{\min}, Q)$.*

2.3. **Pro-p Iwahori Hecke algebras.** We fix a special parahoric subgroup $\mathcal{K}$ of G fixing a special vertex x_0 in the apartment $\mathcal{A}$ associated to T in the Bruhat-Tits building of the adjoint group of G. We let $\mathcal{B}$ be the Iwahori subgroup fixing the alcove $\mathcal{C}$ in $\mathcal{A}$ with vertex x_0 contained in the Weyl chamber (of vertex x_0) associated to B. We let $\mathcal{U}$ be the pro-p radical of $\mathcal{B}$ (the pro-p Iwahori subgroup). The pro-p Iwahori Hecke ring $\mathcal{H} = \mathcal{H}(G, \mathcal{U})$ is the convolution ring of compactly supported functions $G \to \mathbb{Z}$ constant on the double classes of G modulo $\mathcal{U}$. We denote by $T(g)$ the characteristic function of $\mathcal{U}g\mathcal{U}$ for $g \in G$, seen as an element of $\mathcal{H}$. Let R be a commutative ring. The pro-p Iwahori Hecke R-algebra $\mathcal{H}_R$ is $R \otimes_{\mathbb{Z}} \mathcal{H}$. We will follow the custom to still denote by h the natural image $1 \otimes h$ of $h \in \mathcal{H}$ in $\mathcal{H}_R$.

For $P = MN$ a standard parabolic subgroup of G, the similar objects for M are indexed by M, we have $\mathcal{K}_M = \mathcal{K} \cap M, \mathcal{B}_M = \mathcal{B} \cap M, \mathcal{U}_M = \mathcal{U} \cap M$, the pro-$p$ Iwahori Hecke ring $\mathcal{H}_M = \mathcal{H}(M, \mathcal{U}_M)$, $T^M(m)$ the characteristic function of $\mathcal{U}_M m \mathcal{U}_M$ for $m \in M$, seen as an element of $\mathcal{H}_M$. The pro-p Iwahori subgroup $\mathcal{U}$ of G satisfies the Iwahori decomposition with respect to P:

$$\mathcal{U} = \mathcal{U}_N \mathcal{U}_M \mathcal{U}_{\overline{N}},$$

where $\mathcal{U}_N = \mathcal{U} \cap N, \mathcal{U}_{\overline{N}} = \mathcal{U} \cap \overline{N}$. The linear map

$$\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}, \quad \theta(T^M(m)) = T(m) \quad (m \in M) \tag{2.1}$$

does not respect the product. But if we introduce the monoid M^+ of elements $m \in M$ contracting $\mathcal{U}_N$, meaning $m\mathcal{U}_N m^{-1} \subset \mathcal{U}_N$, and the submodule $\mathcal{H}_{M^+} \subset \mathcal{H}_M$ of functions with support in M^+, we have [Vig15b, Theorem 1.4]:

$\mathcal{H}_{M^+}$ is a subring of $\mathcal{H}_M$ and $\mathcal{H}_M$ is the localization of $\mathcal{H}_{M^+}$ at an element $\tau^M \in \mathcal{H}_{M^+}$ central and invertible in $\mathcal{H}_M$, meaning $\mathcal{H}_M = \bigcup_{n \in \mathbb{N}} \mathcal{H}_{M^+}(\tau^M)^{-n}$. The map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ is injective and its restriction $\theta|_{\mathcal{H}_{M^+}}$ to $\mathcal{H}_{M^+}$ respects the product.

These properties are also true when (M^+, τ^M) is replaced by its inverse $(M^-, (\tau^M)^{-1})$ where $M^- = \{m^{-1} \in M \mid m \in M^+\}$.

3. Pro-p Iwahori invariants of $I_G(P, \sigma, Q)$

3.1. **Pro-p Iwahori Hecke algebras: Structures.** Here we supplement the notation of sections 2.1 and 2.3. The subgroups $Z^0 = Z \cap \mathcal{K} = Z \cap \mathcal{B}$ and $Z^1 = Z \cap \mathcal{U}$

are normal in $\mathcal{N}$ and we put

$$W = \mathcal{N}/Z^0,\ W(1) = \mathcal{N}/Z^1,\ \Lambda = Z/Z^0,\ \Lambda(1) = Z/Z^1,\ Z_k = Z^0/Z^1.$$

We have $\mathcal{N} = (\mathcal{N} \cap \mathcal{K})Z$ so that we see the finite Weyl group $\mathbb{W} = \mathcal{N}/Z$ as the subgroup $(\mathcal{N} \cap \mathcal{K})/Z^0$ of W; in this way W is the semidirect product $\Lambda \rtimes \mathbb{W}$. We put $\mathcal{N}_{G'} = \mathcal{N} \cap G'$. The image $W_{G'} = W'$ of $\mathcal{N}_{G'}$ in W is an affine Weyl group generated by the set S^{aff} of affine reflections determined by the walls of the alcove $\mathcal{C}$. The group W' is normal in W and W is the semidirect product $W' \rtimes \Omega$ where Ω is the image in W of the normalizer $\mathcal{N}_{\mathcal{C}}$ of $\mathcal{C}$ in $\mathcal{N}$. The length function ℓ on the affine Weyl system (W', S^{aff}) extends to a length function on W such that Ω is the set of elements of length 0. We also view ℓ as a function of $W(1)$ via the quotient map $W(1) \to W$. We write

(3.1)

$(\hat{w}, \tilde{w}, w) \in \mathcal{N} \times W(1) \times W$ corresponding via the quotient maps $\mathcal{N} \to W(1) \to W$.

When $w = s$ in S^{aff} or more generally w in $W_{G'}$, we will most of the time choose $\hat{w}$ in $\mathcal{N} \cap G'$ and $\tilde{w}$ in the image ${}_1W_{G'}$ of $\mathcal{N} \cap G'$ in $W(1)$.

We are now ready to describe the pro-p Iwahori Hecke ring $\mathcal{H} = \mathcal{H}(G, \mathcal{U})$ [Vig16]. We have $G = \mathcal{U}\mathcal{N}\mathcal{U}$ and for $n, n' \in \mathcal{N}$ we have $\mathcal{U}n\mathcal{U} = \mathcal{U}n'\mathcal{U}$ if and only if $nZ^1 = n'Z^1$. For $n \in \mathcal{N}$ of image $w \in W(1)$ and $g \in \mathcal{U}n\mathcal{U}$ we denote $T_w = T(n) = T(g)$ in $\mathcal{H}$. The relations among the basis elements $(T_w)_{w \in W(1)}$ of $\mathcal{H}$ are:

(1) Braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.
(2) Quadratic relations: $T_{\tilde{s}}^2 = q_s T_{\tilde{s}^2} + c_{\tilde{s}} T_{\tilde{s}}$

for $\tilde{s} \in W(1)$ lifting $s \in S^{\mathrm{aff}}$, where $q_s = q_G(s) = |\mathcal{U}/\mathcal{U} \cap \hat{s}\mathcal{U}(\hat{s})^{-1}|$ depends only on s, and $c_{\tilde{s}} = \sum_{t \in Z_k} c_{\tilde{s}}(t) T_t$ for integers $c_{\tilde{s}}(t) \in \mathbb{N}$ summing to $q_s - 1$.

We shall need the basis elements $(T_w^*)_{w \in W(1)}$ of $\mathcal{H}$ defined by:

(1) $T_w^* = T_w$ for $w \in W(1)$ of length $\ell(w) = 0$.
(2) $T_{\tilde{s}}^* = T_{\tilde{s}} - c_{\tilde{s}}$ for $\tilde{s} \in W(1)$ lifting $s \in S^{\mathrm{aff}}$.
(3) $T_{ww'}^* = T_w^* T_{w'}^*$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.

We need more notation for the definition of the admissible lifts of S^{aff} in $\mathcal{N}_G$. Let $s \in S^{\mathrm{aff}}$ fixing a face $\mathcal{C}_s$ of the alcove $\mathcal{C}$ and $\mathcal{K}_s$ the parahoric subgroup of G fixing $\mathcal{C}_s$. The theory of Bruhat-Tits associates to $\mathcal{C}_s$ a certain root $\alpha_s \in \Phi^+$ [Vig16, §4.2]. We consider the group G'_s generated by $U_{\alpha_s} \cup U_{-\alpha_s}$ where $U_{\pm\alpha_s}$ the root subgroup of $\pm\alpha_s$ (if $2\alpha_s \in \Phi$, then $U_{2\alpha_s} \subset U_{\alpha_s}$) and the group $\mathcal{G}'_s$ generated by $\mathcal{U}_{\alpha_s} \cup \mathcal{U}_{-\alpha_s}$ where $\mathcal{U}_{\pm\alpha_s} = U_{\pm\alpha_s} \cap \mathcal{K}_s$. When $u \in \mathcal{U}_{\alpha_s} - \{1\}$, the intersection $\mathcal{N}_G \cap \mathcal{U}_{-\alpha_s} u \mathcal{U}_{-\alpha_s}$ (equal to $\mathcal{N}_G \cap U_{-\alpha_s} u U_{-\alpha_s}$ [BT72, 6.2.1 (V5)], [Vig16, §3.3 (19)]) possesses a single element $n_s(u)$. The group $Z'_s = Z \cap \mathcal{G}'_s$ is contained in $Z \cap \mathcal{K}_s = Z^0$; its image in Z_k is denoted by $Z'_{k,s}$.

The elements $n_s(u)$ for $u \in \mathcal{U}_{\alpha_s} - \{1\}$ are the admissible lifts of s in $\mathcal{N}_G$; their images in $W(1)$ are the admissible lifts of s in $W(1)$. By [Vig16, Theorem 2.2, Proposition 4.4], when $\tilde{s} \in W(1)$ is an admissible lift of s, $c_{\tilde{s}}(t) = 0$ if $t \in Z_k \setminus Z'_{k,s}$, and

$$c_{\tilde{s}} \equiv (q_s - 1)|Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} T_t \mod p. \tag{3.2}$$

The admissible lifts of S in $\mathcal{N}_G$ are contained in $\mathcal{N}_G \cap \mathcal{K}$ because $\mathcal{K}_s \subset \mathcal{K}$ when $s \in S$.

Definition 3.1. An admissible lift of the finite Weyl group $\mathbb{W}$ in $\mathcal{N}_G$ is a map $w \mapsto \hat{w} : \mathbb{W} \to \mathcal{N}_G \cap \mathcal{K}$ such that $\hat{s}$ is admissible for all $s \in S$ and $\hat{w} = \hat{w}_1 \hat{w}_2$ for $w_1, w_2 \in \mathbb{W}$ such that $w = w_1 w_2$ and $\ell(w) = \ell(w_1) + \ell(w_2)$.

Any choice of admissible lifts of S in $\mathcal{N}_G \cap \mathcal{K}$ extends uniquely to an admissible lift of $\mathbb{W}$ ([AHHV17, IV.6], [OV17, Proposition 2.7]).

Let $P = MN$ be a standard parabolic subgroup of G. The groups $Z, Z^0 = Z \cap \mathcal{K}_M = Z \cap \mathcal{B}_M, Z^1 = Z \cap \mathcal{U}_M$ are the same for G and M, but $\mathcal{N}_M = \mathcal{N} \cap M$ and $M \cap G'$ are subgroups of $\mathcal{N}$ and G'. The monoid M^+ (section 2.3) contains $(\mathcal{N}_M \cap \mathcal{K})$ and is equal to $M^+ = \mathcal{U}_M \mathcal{N}_{M^+} \mathcal{U}_M$ where $\mathcal{N}_{M^+} = \mathcal{N} \cap M^+$. An element $z \in Z$ belongs to M^+ if and only if $v_F(\alpha(z)) \geq 0$ for all $\alpha \in \Phi^+ \setminus \Phi^+_M$ (see [Vig15b, Lemme 2.2]). Put $W_M = \mathcal{N}_M / Z^0$ and $W_M(1) = \mathcal{N}_M / Z^1$.

Let $\epsilon = +$ or $\epsilon = -$. We denote by $W_{M^\epsilon}, W_{M^\epsilon}(1)$ the images of $\mathcal{N}_{M^\epsilon}$ in $W_M, W_M(1)$. We see the groups $W_M, W_M(1), {}_1W_{M'}$ as subgroups of $W, W(1), {}_1W_{G'}$. As θ (section 2.3), the linear injective map

$$\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}, \quad \theta^*(T^{M,*}_w) = T^*_w, \quad (w \in W_M(1)), \tag{3.3}$$

respects the product on the subring $\mathcal{H}_{M^\epsilon}$. Here $T^{M,*}_w \in \mathcal{H}_M$ is defined in the same way as T^*_w for $\mathcal{H}_M$. Note that θ and θ^* satisfy the obvious transitivity property with respect to a change of parabolic subgroups.

3.2. **Orthogonal case.** Let us examine the case where Δ_M and $\Delta \setminus \Delta_M$ are orthogonal, writing $M_2 = M_{\Delta \setminus \Delta_M}$ as in section 1.3.

From $M \cap M_2 = Z$ we get $W_M \cap W_{M_2} = \Lambda, W_M(1) \cap W_{M_2}(1) = \Lambda(1)$, the semisimple building of G is the product of those of M and M_2. The set S^{aff} is the disjoint union of S^{aff}_M and $S^{\mathrm{aff}}_{M_2}$, the group $W_{G'}$ is the direct product of $W_{M'}$ and $W_{M'_2}$. For $\tilde{s} \in W_M(1)$ lifting $s \in S^{\mathrm{aff}}_M$, the elements $T^M_{\tilde{s}} \in \mathcal{H}_M$ and $T_{\tilde{s}} \in \mathcal{H}$ satisfy the same quadratic relations. A word of caution is necessary for the lengths ℓ_M of W_M and ℓ_{M_2} of W_{M_2} different from the restrictions of the length ℓ of W_M, for example $\ell_M(\lambda) = 0$ for $\lambda \in \Lambda \cap W_{M'_2}$.

Lemma 3.2. *We have* $\Lambda = (W_{M^\epsilon} \cap \Lambda)(W_{M'_2} \cap \Lambda)$.

Proof. We prove the lemma for $\epsilon = -$. The case $\epsilon = +$ is similar. The map $v : Z \to X_*(T) \otimes \mathbb{Q}$ defined in section 2.1 is trivial on Z^0 and we also write v for the resulting homomorphism on Λ. For $\lambda \in \Lambda$ there exists $\lambda_2 \in W_{M'_2} \cap \Lambda$ such that $\lambda\lambda_2 \in W_{M^-}$, or equivalently $\alpha(v(\lambda\lambda_2)) \leq 0$ for all $\alpha \in \Phi^+ \setminus \Phi^+_M = \Phi^+_{M_2}$. It suffices to have the inequality for $\alpha \in \Delta_{M_2}$. The matrix $(\alpha(\beta^\vee))_{\alpha,\beta \in \Delta_{M_2}}$ is invertible, hence there exists $n_\beta \in \mathbb{Z}$ such that $\sum_{\beta \in \Delta_{M_2}} n_\beta \alpha(\beta^\vee) \leq -\alpha(v(\lambda))$ for all $\alpha \in \Delta_{M_2}$. As $v(W_{M'_2} \cap \Lambda)$ contains $\bigoplus_{\alpha \in \Delta_{M_2}} \mathbb{Z}\alpha^\vee$ where $\alpha^\vee$ is the coroot of α [Vig16, after formula (71)], there exists $\lambda_2 \in W_{M'_2} \cap \Lambda$ with $v(\lambda_2) = \sum_{\beta \in \Delta_{M_2}} n_\beta \beta^\vee$. $\square$

The groups $\mathcal{N} \cap M'$ and $\mathcal{N} \cap M'_2$ are normal in $\mathcal{N}$, and

$$\mathcal{N} = (\mathcal{N} \cap M')\mathcal{N}_{\mathcal{C}}(\mathcal{N} \cap M'_2) = Z(\mathcal{N} \cap M')(\mathcal{N} \cap M'_2),$$

and

$$W = W_{M'} \Omega W_{M'_2} = W_M W_{M'_2} = W_{M^+} W_{M'_2} = W_{M^-} W_{M'_2}.$$

The first two equalities are clear, the equality $W_M W_{M'_2} = W_{M^\epsilon} W_{M'_2}$ follows from $W_M = \mathbb{W}_M \Lambda$, $\mathbb{W}_M \subset W_{M^\epsilon}$ and the lemma. The inverse image in $W(1)$ of these groups are

(3.4)
$$W(1) = {}_1W_{M'}\,\Omega(1)\,{}_1W_{M_2'} = W_M(1)\,{}_1W_{M_2'} = W_{M^+}(1)\,{}_1W_{M_2'} = W_{M^-}(1)\,{}_1W_{M_2'}.$$

We recall the function $q_G(n) = q(n) = |\mathcal{U}/(\mathcal{U}\cap n^{-1}\mathcal{U}n)|$ on $\mathcal{N}$ [Vig16, Proposition 3.38] and we extend to $\mathcal{N}$ the functions q_M on $\mathcal{N}\cap M$ and q_{M_2} on $\mathcal{N}\cap M_2$:

$$(3.5)\qquad q_M(n) = |\mathcal{U}_M/(\mathcal{U}_M\cap n^{-1}\mathcal{U}_M n)|,\quad q_{M_2}(n) = |\mathcal{U}_{M_2}/(\mathcal{U}_{M_2}\cap n^{-1}\mathcal{U}_{M_2}n)|.$$

The functions q, q_M, q_{M_2} descend to functions on $W(1)$ and on W, also denoted by q, q_M, q_{M_2}.

Lemma 3.3. *Let $n\in\mathcal{N}$ of image $w\in W$. We have*

(1) $q(n) = q_M(n)q_{M_2}(n)$.
(2) $q_M(n) = q_M(n_M)$ *if* $n = n_M n_2$, $n_M\in\mathcal{N}\cap M, n_2\in\mathcal{N}\cap M_2'$ *and similarly when M and M_2 are permuted.*
(3) $q(w) = 1 \Leftrightarrow q_M(\lambda w_M) = q_{M_2}(\lambda w_{M_2}) = 1$, *if* $w = \lambda w_M w_{M_2}, (\lambda, w_M, w_{M_2})\in \Lambda\times\mathbb{W}_M\times\mathbb{W}_{M_2}$.
(4) *On the coset $(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}n$, q_M is constant equal to $q_M(n_{M'})$ for any element $n_{M'}\in M'\cap(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}n$. A similar result is true when M and M_2 are permuted.*

Proof. We put $\mathcal{U}_{M'} = \mathcal{U}\cap M'$ and $\mathcal{U}_{M_2'} = \mathcal{U}\cap M_2'$. The product map

$$(3.6)\qquad Z^1\prod_{\alpha\in\Phi_{M,red}}\mathcal{U}_\alpha\prod_{\alpha\in\Phi_{M_2,red}}\mathcal{U}_\alpha\to\mathcal{U}$$

with $\mathcal{U}_\alpha = U_\alpha\cap\mathcal{U}$, is a homeomorphism. We have $\mathcal{U}_M = Z^1\mathcal{Y}_{M'}$, $\mathcal{U}_{M'} = (Z^1\cap M')\mathcal{Y}_{M'}$ where $\mathcal{Y}_{M'} = \prod_{\alpha\in\Phi_{M,red}}\mathcal{U}_\alpha$ and $\mathcal{N}\cap M_2'$ commutes with $\mathcal{Y}_{M'}$, in particular $\mathcal{N}\cap M_2'$ normalizes $\mathcal{Y}_{M'}$. Similar results are true when M and M_2 are permuted, and $\mathcal{U} = \mathcal{U}_{M'}\mathcal{U}_{M_2} = \mathcal{U}_M\mathcal{U}_{M_2'}$.

Writing $\mathcal{N} = Z(\mathcal{N}\cap M')(\mathcal{N}\cap M_2')$ (in any order), we see that the product map

$$(3.7)\qquad Z^1(\mathcal{Y}_{M'}\cap n^{-1}\mathcal{Y}_{M'}n)(\mathcal{Y}_{M_2'}\cap n^{-1}\mathcal{Y}_{M_2'}n)\to\mathcal{U}\cap n^{-1}\mathcal{U}n$$

is a homeomorphism. The inclusions induce bijections

$$(3.8)\quad \mathcal{Y}_{M'}/(\mathcal{Y}_{M'}\cap n^{-1}\mathcal{Y}_{M'}n)\simeq\mathcal{U}_{M'}/(\mathcal{U}_{M'}\cap n^{-1}\mathcal{U}_{M'}n)\simeq\mathcal{U}_M/(\mathcal{U}_M\cap n^{-1}\mathcal{U}_M n),$$

similarly for M_2, and also a bijection

$$(3.9)\quad \mathcal{U}/(\mathcal{U}\cap n^{-1}\mathcal{U}n)\simeq(\mathcal{Y}_{M_2'}/(\mathcal{Y}_{M_2'}\cap n^{-1}\mathcal{Y}_{M_2'}n))\times(\mathcal{Y}_{M'}/(\mathcal{Y}_{M'}\cap n^{-1}\mathcal{Y}_{M'}n)).$$

From (3.8) and (3.9), we get

$$(3.10)\quad \mathcal{U}/(\mathcal{U}\cap n^{-1}\mathcal{U}n)\simeq(\mathcal{U}_{M_2'}/(\mathcal{U}_{M_2'}\cap n\mathcal{U}_{M_2'}n^{-1}))\times(\mathcal{U}_{M'}/(\mathcal{U}_{M'}\cap n\mathcal{U}_{M'}n^{-1}))$$

which implies the assertion (1) in the lemma.

The assertion (2) follows from (3.7) since $\mathcal{N}\cap M_2'$ normalizes $\mathcal{Y}_{M'}$; with (1), it implies the assertion (3).

A subgroup of $\mathcal{N}$ normalizes $\mathcal{U}_M$ if and only if it normalizes $\mathcal{Y}_{M'}$ by (3.8) if and only if $q_M = 1$ on this group. The group $\mathcal{N}\cap M_2'$ normalizes $\mathcal{Y}_{M'}$. Therefore the group $(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}$ normalizes $\mathcal{U}_M$. The coset $(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}n$ contains an element $n_{M'}\in M'$. For $x\in(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}$, $(xn_{M'})^{-1}\mathcal{U}xn_{M'} = n_{M'}^{-1}\mathcal{U}n_{M'}$ hence $q_M(xn_{M'}) = q_M(n_{M'})$. ☐

3.3. **Extension of an $\mathcal{H}_M$-module to $\mathcal{H}$.** This section is inspired by similar results for the pro-p Iwahori Hecke algebras over an algebraically closed field of characteristic p [Abe, Proposition 4.16]. We keep the setting of section 3.2 and we introduce ideals:

- $\mathcal{J}_\ell$ (resp., $\mathcal{J}_r$) the left (resp., right) ideal of $\mathcal{H}$ generated by $T^*_w - 1_{\mathcal{H}}$ for all $w \in {}_1W_{M'_2}$,
- $\mathcal{J}_{M,\ell}$ (resp., $\mathcal{J}_{M,r}$) the left (resp., right) ideal of $\mathcal{H}_M$ generated by $T^{M,*}_\lambda - 1_{\mathcal{H}_M}$ for all λ in ${}_1W_{M'_2} \cap W_M(1) = {}_1W_{M'_2} \cap \Lambda(1)$.

The next proposition shows that the ideals $\mathcal{J}_\ell = \mathcal{J}_r$ are equal and similarly $\mathcal{J}_{M,\ell} = \mathcal{J}_{M,r}$. After the proposition, we will drop the indices ℓ and r.

Proposition 3.4. *The ideals $\mathcal{J}_\ell$ and $\mathcal{J}_r$ are equal to the submodule $\mathcal{J}'$ of $\mathcal{H}$ generated by $T^*_w - T^*_{ww_2}$ for all $w \in W(1)$ and $w_2 \in {}_1W_{M'_2}$.*

The ideals $\mathcal{J}_{M,\ell}$ and $\mathcal{J}_{M,r}$ are equal to the submodule $\mathcal{J}'_M$ of $\mathcal{H}_M$ generated by $T^{M,*}_w - T^{M,*}_{w\lambda_2}$ for all $w \in W_M(1)$ and $\lambda_2 \in \Lambda(1) \cap {}_1W_{M'_2}$.

Proof.

(1) We prove $\mathcal{J}_\ell = \mathcal{J}'$. Let $w \in W(1), w_2 \in {}_1W_{M'_2}$. We prove by induction on the length of w_2 that $T^*_w(T^*_{w_2} - 1) \in \mathcal{J}'$. This is obvious when $\ell(w_2) = 0$ because $T^*_w T^*_{w_2} = T^*_{ww_2}$. Assume that $\ell(w_2) = 1$ and put $s = w_2$. If $\ell(ws) = \ell(w) + 1$, as before $T^*_w(T^*_s - 1) \in \mathcal{J}'$ because $T^*_w T^*_s = T^*_{ws}$. Otherwise $\ell(ws) = \ell(w) - 1$ and $T^*_w = T^*_{ws^{-1}} T^*_s$ hence

$$T^*_w(T^*_s - 1) = T^*_{ws^{-1}}(T^*_s)^2 - T^*_w = T^*_{ws^{-1}}(q_s T^*_{s^2} - T^*_s c_s) - T^*_w = q_s T^*_{ws} - T^*_w(c_s + 1).$$

Since $c_s + 1 = \sum_{t \in Z'_k} c_s(t) T_t$ with $c_s(t) \in \mathbb{N}$ and $\sum_{t \in Z'_k} c_s(t) = q_s$ [Vig16, Proposition 4.4],

$$q_s T^*_{ws} - T^*_w(c_s + 1) = \sum_{t \in Z'_k} c_s(t)(T^*_{ws} - T^*_w T^*_t) = \sum_{t \in Z'_k} c_s(t)(T^*_{ws} - T^*_{wss^{-1}t}) \in \mathcal{J}'.$$

Assume now that $\ell(w_2) > 1$. Then, we factorize $w_2 = xy$ with $x, y \in {}_1W_{M_2}$ of length $\ell(x), \ell(y) < \ell(w_2)$ and $\ell(w_2) = \ell(x) + \ell(y)$. The element $T^*_w(T^*_{w_2} - 1) = T^*_w T^*_x(T^*_y - 1) + T^*_w(T^*_x - 1)$ lies in $\mathcal{J}'$ by induction.

Conversely, we prove $T^*_{ww_2} - T^*_w \in \mathcal{J}_\ell$. We factorize $w = xy$ with $y \in {}_1W_{M'_2}$ and $x \in {}_1W_{M'}\Omega(1)$. Then, we have $\ell(w) = \ell(x) + \ell(y)$ and $\ell(ww_2) = \ell(x) + \ell(yw_2)$. Hence

$$T^*_{ww_2} - T^*_w = T^*_x(T^*_{yw_2} - T^*_y) = T^*_x(T^*_{yw_2} - 1) - T^*_x(T^*_y - 1) \in \mathcal{J}_\ell.$$

This ends the proof of $\mathcal{J}_\ell = \mathcal{J}'$.

By the same argument, the right ideal $\mathcal{J}_r$ of $\mathcal{H}$ is equal to the submodule of $\mathcal{H}$ generated by $T^*_{w_2 w} - T^*_w$ for all $w \in W(1)$ and $w_2 \in {}_1W_{M'_2}$. But this latter submodule is equal to $\mathcal{J}'$ because ${}_1W_{M'_2}$ is normal in $W(1)$. Therefore we proved $\mathcal{J}' = \mathcal{J}_r = \mathcal{J}_\ell$.

(2) Proof of the second assertion. We prove $\mathcal{J}_{M,\ell} = \mathcal{J}'_M$. The proof is easier than in (1) because for $w \in W_M(1)$ and $\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)$, we have $\ell(w\lambda_2) = \ell(w) + \ell(\lambda_2)$ hence $T^{M,*}_w(T^{M,*}_{\lambda_2} - 1) = T^{M,*}_{w\lambda_2} - T^{M,*}_w$. We have also $\ell(\lambda_2 w) = \ell(\lambda_2) + \ell(w)$ hence $(T^{M,*}_{\lambda_2} - 1)T^{M,*}_w = T^{M,*}_{\lambda_2 w} - T^{M,*}_w$ hence $\mathcal{J}_{M,r}$ is equal to the submodule of $\mathcal{H}_M$ generated by $T^{M,*}_{\lambda_2 w} - T^{M,*}_w$ for all $w \in W_M(1)$ and $\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)$. This latter submodule is $\mathcal{J}'_M$, as ${}_1W_{M'_2} \cap \Lambda(1) = {}_1W_{M'_2} \cap W_M(1)$ is normal in $W_M(1)$. Therefore $\mathcal{J}'_M = \mathcal{J}_{M,r} = \mathcal{J}_{M,\ell}$. ∎

By Proposition 3.4, a basis of $\mathcal{J}$ is $T^*_w - T^*_{ww_2}$ for w in a system of representatives of $W(1)/{}_1W_{M'_2}$, and $w_2 \in {}_1W_{M'_2} \setminus \{1\}$. Similarly a basis of $\mathcal{J}_M$ is $T^{M,*}_w - T^{M,*}_{w\lambda_2}$ for w in a system of representatives of $W_M(1)/(\Lambda(1) \cap {}_1W_{M'_2})$. and $\lambda_2 \in (\Lambda(1) \cap {}_1W_{M'_2}) \setminus \{1\}$.

Proposition 3.5. *The natural ring inclusion of $\mathcal{H}_{M^-}$ in $\mathcal{H}_M$ and the ring inclusion of $\mathcal{H}_{M^-}$ in $\mathcal{H}$ via θ^* induce ring isomorphisms*

$$\mathcal{H}_M/\mathcal{J}_M \xleftarrow{\sim} \mathcal{H}_{M^-}/(\mathcal{J}_M \cap \mathcal{H}_{M^-}) \xrightarrow{\sim} \mathcal{H}/\mathcal{J}.$$

Proof.

(1) The left map is obviously injective. We prove the surjectivity. Let $w \in W_M(1)$. Let $\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)$ such that $w\lambda_2^{-1} \in W_{M^-}(1)$ (see (3.4)). We have $T^{M,*}_{w\lambda_2^{-1}} \in \mathcal{H}_{M^-}$ and $T^{M,*}_w = T^{M,*}_{w\lambda_2^{-1}} T^{M,*}_{\lambda_2} = T^{M,*}_{w\lambda_2^{-1}} + T^{M,*}_{w\lambda_2^{-1}}(T^{M,*}_{\lambda_2} - 1)$. Therefore $T^{M,*}_w \in \mathcal{H}_{M^-} + \mathcal{J}_M$. As w is arbitrary, $\mathcal{H}_M = \mathcal{H}_{M^-} + \mathcal{J}_M$.

(2) The right map is surjective: let $w \in W(1)$ and $w_2 \in {}_1W_{M'_2}$ such that $ww_2^{-1} \in W_{M^-}(1)$ (see (3.4)). Then $T^*_w - T^*_{ww_2^{-1}} \in \mathcal{J}$ with the same arguments as in (1), using Proposition 3.4. Therefore $\mathcal{H} = \theta^*(\mathcal{H}_{M^-}) + \mathcal{J}$.

We prove the injectivity: $\theta^*(\mathcal{H}_{M^-}) \cap \mathcal{J} = \theta^*(\mathcal{H}_{M^-} \cap \mathcal{J}_M)$. Let $\sum_{w \in W_{M^-}(1)} c_w T^{M,*}_w$, with $c_w \in \mathbb{Z}$, be an element of $\mathcal{H}_{M^-}$. Its image by θ^* is $\sum_{w \in W(1)} c_w T^*_w$ where we have set $c_w = 0$ for $w \in W(1) \setminus W_{M^-}(1)$. We have $\sum_{w \in W(1)} c_w T^*_w \in \mathcal{J}$ if and only if $\sum_{w_2 \in {}_1W_{M'_2}} c_{ww_2} = 0$ for all $w \in W(1)$. If $c_{ww_2} \neq 0$, then $w_2 \in {}_1W_{M'_2} \cap W_M(1)$, that is, $w_2 \in {}_1W_{M'_2} \cap \Lambda(1)$. The sum $\sum_{w_2 \in {}_1W_{M'_2}} c_{ww_2}$ is equal to $\sum_{\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)} c_{w\lambda_2}$. By Proposition 3.4, $\sum_{w \in W(1)} c_w T^*_w \in \mathcal{J}$ if and only if $\sum_{w \in W_{M^-}(1)} c_w T^{M,*}_w \in \mathcal{J}_M$. ☐

We construct a ring isomorphism

$$e^* : \mathcal{H}_M/\mathcal{J}_M \xrightarrow{\sim} \mathcal{H}/\mathcal{J}$$

by using Proposition 3.5. For any $w \in W(1)$, $T^*_w + \mathcal{J} = e^*(T^{M,*}_{w_{M^-}} + \mathcal{J}_M)$ where $w_{M^-} \in W_{M^-}(1) \cap w\, {}_1W_{M'_2}$ (see (3.4)), because by Proposition 3.4, $T^*_w + \mathcal{J} = T^*_{w_{M^-}} + \mathcal{J}$ and $T^*_{w_{M^-}} + \mathcal{J} = e^*(T^{M,*}_{w_{M^-}} + \mathcal{J}_M)$ by construction of e^*. We check that e^* is induced by θ^*.

Theorem 3.6. *The linear map $\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}$ induces a ring isomorphism*

$$e^* : \mathcal{H}_M/\mathcal{J}_M \xrightarrow{\sim} \mathcal{H}/\mathcal{J}.$$

Proof. Let $w \in W_M(1)$. We have to show that $T^*_w + \mathcal{J} = e^*(T^{M,*}_w + \mathcal{J}_M)$. We saw above that $T^*_w + \mathcal{J} = e^*(T^{M,*}_{w_{M^-}} + \mathcal{J}_M)$ with $w = w_{M^-}\lambda_2$ with $\lambda_2 \in {}_1W_{M'_2} \cap W_M(1)$. As $\ell_M(\lambda_2) = 0$, $T^{M,*}_w = T^{M,*}_{w_{M^-}} T^{M,*}_{\lambda_2} \in T^{M,*}_{w_{M^-}} + \mathcal{J}_M$. Therefore $T^{M,*}_{w_{M^-}} + \mathcal{J}_M = T^{M,*}_w + \mathcal{J}_M$. This ends the proof of the theorem. ☐

We now wish to compute e^* in terms of the T_w instead of the T^*_w.

Proposition 3.7. *Let $w \in W(1)$. Then, $T_w + \mathcal{J} = e^*(T^M_{w_M} q_{M_2}(w) + \mathcal{J}_M)$ for any $w_M \in W_M(1) \cap w\, {}_1W_{M'_2}$.*

Proof. The element w_M is unique modulo right multiplication by an element $\lambda_2 \in W_M(1) \cap {}_1W_{M'_2}$ of length $\ell_M(\lambda_2) = 0$ and $T^M_{w_M} q_{M_2}(w) + \mathcal{J}_M$ does not depend on the choice of w_M. We choose a decomposition (see (3.4)):

$$w = \tilde{s}_1 \dots \tilde{s}_a u \tilde{s}_{a+1} \dots \tilde{s}_{a+b}, \quad \ell(w) = a + b,$$

for $u \in \Omega(1)$, $\tilde{s}_i \in {}_1W_{M'}$ lifting $s_i \in S^{\mathrm{aff}}_M$ for $1 \leq i \leq a$ and $\tilde{s}_i \in {}_1W_{M'_2}$ lifting $s_i \in S^{\mathrm{aff}}_{M_2}$ for $a+1 \leq i \leq a+b$, and we choose $u_M \in W_M(1)$ such that $u \in u_M\, {}_1W_{M'_2}$. Then

$$w_M = \tilde{s}_1 \dots \tilde{s}_a u_M \in W_M(1) \cap w\ {}_1W_{M'_2}$$

and $q_{M_2}(w) = q_{M_2}(\tilde{s}_{a+1} \dots \tilde{s}_{a+b})$ (Lemma 3.3 (4)). First we check the proposition in three simple cases:

Case 1. Let $w = \tilde{s} \in {}_1W_{M'}$ lifting $s \in S^{\mathrm{aff}}_M$; we have $T_{\tilde{s}} + \mathcal{J} = e^*(T^M_{\tilde{s}} + \mathcal{J}_M)$ because $T^*_{\tilde{s}} - e^*(T^{M,*}_{\tilde{s}}) \in \mathcal{J}$, $T_{\tilde{s}} = T^*_{\tilde{s}} + c_{\tilde{s}}$, $T^M_{\tilde{s}} = T^{M,*}_{\tilde{s}} + c_{\tilde{s}}$ and $1 = q_{M_2}(\tilde{s})$.

Case 2. Let $w = u \in W(1)$ of length $\ell(u) = 0$ and $u_M \in W_M(1)$ such that $u \in u_M\, {}_1W_{M'_2}$. We have $\ell_M(u_M) = 0$ and $q_{M_2}(u) = 1$ (Lemma 3.3). We deduce $T_u + \mathcal{J} = e^*(T^M_{u_M} + \mathcal{J}_M)$ because $T^*_u + \mathcal{J} = T^*_{u_M} + \mathcal{J} = e^*(T^{M,*}_{u_M} + \mathcal{J}_M)$, and $T_u = T^*_u, T^M_{u_M} = T^{M,*}_{u_M}$.

Case 3. Let $w = \tilde{s} \in {}_1W_{M'_2}$ lifting $s \in S^{\mathrm{aff}}_{M_2}$; we have $T_{\tilde{s}} + \mathcal{J} = e^*(q_{M_2}(\tilde{s}) + \mathcal{J}_M)$ because $T^*_{\tilde{s}} - 1, c_{\tilde{s}} - (q_s - 1) \in \mathcal{J}$, $T_{\tilde{s}} = T^*_{\tilde{s}} + c_{\tilde{s}} \in q_s + \mathcal{J}$ and $q_s = q_{M_2}(\tilde{s})$.

In general, the braid relations $T_w = T_{\tilde{s}_1} \dots T_{\tilde{s}_a} T_u T_{\tilde{s}_{a+1}} \dots T_{\tilde{s}_{a+b}}$ give a similar product decomposition of $T_w + \mathcal{J}$, and the simple cases 1, 2, 3 imply that $T_w + \mathcal{J}$ is equal to

$$\begin{aligned} e^*(T^M_{\tilde{s}_1} + \mathcal{J}_M) \dots e^*(T^M_{\tilde{s}_a} + \mathcal{J}_M) e^*(T^M_{u_M} + \mathcal{J}_M) e^*(q_{M_2}(\tilde{s}_{a+1}) \\ + \mathcal{J}_M) \dots e^*(q_{M_2}(\tilde{s}_{a+b}) + \mathcal{J}_M) \\ = e^*(T^M_{w_M} q_{M_2}(w) + \mathcal{J}_M). \end{aligned}$$

The proposition is proved. ☐

Propositions 3.4, 3.5, 3.7, and Theorem 3.6 are valid over any commutative ring R (instead of $\mathbb{Z}$).

The two-sided ideal of $\mathcal{H}_R$ generated by $T^*_w - 1$ for all $w \in {}_1W_{M'_2}$ is $\mathcal{J}_R = \mathcal{J} \otimes_{\mathbb{Z}} R$, the two-sided ideal of $\mathcal{H}_{M,R}$ generated by $T^*_\lambda - 1$ for all $\lambda \in {}_1W_{M'_2} \cap \Lambda(1)$ is $\mathcal{J}_{M,R} = \mathcal{J}_M \otimes_{\mathbb{Z}} R$, and we get as in Proposition 3.5 isomorphisms

$$\mathcal{H}_{M,R}/\mathcal{J}_{M,R} \xleftarrow{\sim} \mathcal{H}_{M^-,R}/(\mathcal{J}_{M,R} \cap \mathcal{H}_{M^-,R}) \xrightarrow{\sim} \mathcal{H}_R/\mathcal{J}_R,$$

giving an isomorphism $\mathcal{H}_{M,R}/\mathcal{J}_{M,R} \to \mathcal{H}_R/\mathcal{J}_R$ induced by θ^*. Therefore, we have an isomorphism from the category of right $\mathcal{H}_{M,R}$-modules where $\mathcal{J}_M$ acts by 0 onto the category of right $\mathcal{H}_R$-modules where $\mathcal{J}$ acts by 0.

Definition 3.8. A right $\mathcal{H}_{M,R}$-module $\mathcal{V}$ where $\mathcal{J}_M$ acts by 0 is called extensible to $\mathcal{H}$. The corresponding $\mathcal{H}_R$-module where $\mathcal{J}$ acts by 0 is called its extension to $\mathcal{H}$ and denoted by $e_{\mathcal{H}}(\mathcal{V})$ or $e(\mathcal{V})$.

With the element basis T^*_w, $\mathcal{V}$ is extensible to $\mathcal{H}$ if and only if

$$vT^{M,*}_{\lambda_2} = v \text{ for all } v \in \mathcal{V} \text{ and } \lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1). \tag{3.11}$$

The $\mathcal{H}$-module structure on the R-module $e(\mathcal{V}) = \mathcal{V}$ is determined by

$$vT^*_{w_2} = v, \quad vT^*_w = vT^{M,*}_w \quad \text{for all } v \in \mathcal{V}, w_2 \in {}_1W_{M'_2}, w \in W_M(1). \tag{3.12}$$

It is also determined by the action of T^*_w for $w \in {}_1W_{M'_2} \cup W_{M^+}(1)$ (or $w \in {}_1W_{M'_2} \cup W_{M^-}(1)$). Conversely, a right $\mathcal{H}$-module $\mathcal{W}$ over R is extended from an $\mathcal{H}_M$-module if and only if

$$vT^*_{w_2} = v \quad \text{for all} \quad v \in \mathcal{W}, w_2 \in {}_1W_{M'_2}. \tag{3.13}$$

In terms of the basis elements T_w instead of T^*_w, this says the following.

Corollary 3.9. *A right $\mathcal{H}_M$-module $\mathcal{V}$ over R is extensible to $\mathcal{H}$ if and only if*

$$vT^M_{\lambda_2} = v \text{ for all } v \in \mathcal{V} \text{ and } \lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1). \tag{3.14}$$

Then, the structure of an $\mathcal{H}$-module on the R-module $e(\mathcal{V}) = \mathcal{V}$ is determined by

$$vT_{w_2} = vq_{w_2}, \quad vT_w = vT^M_w q_{M_2}(w) \quad \text{for all} \quad v \in \mathcal{V}, w_2 \in {}_1W_{M'_2}, w \in W_M(1). \tag{3.15}$$

($W_{M^+}(1)$ or $W_{M^-}(1)$ instead of $W_M(1)$ is enough.) A right $\mathcal{H}$-module $\mathcal{W}$ over R is extended from an $\mathcal{H}_M$-module if and only if

$$vT_{w_2} = vq_{w_2} \quad \text{for all} \quad v \in \mathcal{W}, w_2 \in {}_1W_{M'_2}. \tag{3.16}$$

3.4. **$\sigma^{\mathcal{U}_M}$ is extensible to $\mathcal{H}$ of extension $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$.** Let $P = MN$ be a standard parabolic subgroup of G such that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal, and let σ be a smooth R-representation of M extensible to G. Let $P_2 = M_2N_2$ denote the standard parabolic subgroup of G with $\Delta_{P_2} = \Delta \setminus \Delta_P$.

Recall that $G = MM'_2$, that $M \cap M'_2 = Z \cap M'_2$ acts trivially on σ, $e(\sigma)$ is the representation of G equal to σ on M and trivial on M'_2. We will describe the $\mathcal{H}$-module $e(\sigma)^{\mathcal{U}}$ in this section. We first consider $e(\sigma)$ as a subrepresentation of $\mathrm{Ind}_P^G \sigma$. For $v \in \sigma$, let $f_v \in (\mathrm{Ind}_P^G \sigma)^{M'_2}$ be the unique function with value v on M'_2. Then, the map

$$v \mapsto f_v : \sigma \to \mathrm{Ind}_P^G \sigma \tag{3.17}$$

is the natural G-equivariant embedding of $e(\sigma)$ in $\mathrm{Ind}_P^G \sigma$. As $\sigma^{\mathcal{U}_M} = e(\sigma)^{\mathcal{U}}$ as R-modules, the image of $e(\sigma)^{\mathcal{U}}$ in $(\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ is made out of the f_v for $v \in \sigma^{\mathcal{U}_M}$.

We now recall the explicit description of $(\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$. For each $d \in \mathbb{W}_{M_2}$, we fix a lift $\hat{d} \in {}_1W_{M'_2}$ and for $v \in \sigma^{\mathcal{U}_M}$ let $f_{P\hat{d}\mathcal{U},v} \in (\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ for the function with support contained in $P\hat{d}\mathcal{U}$ and value v on $\hat{d}\mathcal{U}$. As $Z \cap M'_2$ acts trivially on σ, the function $f_{P\hat{d}\mathcal{U},v}$ does not depend on the choice of the lift $\hat{d} \in {}_1W_{M'_2}$ of d. By [OV17, Lemma 4.5], recalling that $w \in \mathbb{W}_{M_2}$ is of minimal length in its coset $w\mathbb{W}_M = \mathbb{W}_M w$ as Δ_M and Δ_{M_2} are orthogonal to each other:

The map $\bigoplus_{d \in \mathbb{W}_{M_2}} \sigma^{\mathcal{U}_M} \to (\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ given on each d-component by $v \mapsto f_{P\hat{d}\mathcal{U},v}$, is an $\mathcal{H}_{M^+}$-equivariant isomorphism where $\mathcal{H}_{M^+}$ is seen as a subring of $\mathcal{H}$ via θ, and induces an $\mathcal{H}_R$-module isomorphism

$$v \otimes h \mapsto f_{P\mathcal{U},v}h : \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} \to (\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}. \tag{3.18}$$

In particular for $v \in \sigma^{\mathcal{U}_M}$, $v \otimes T(\hat{d})$ does not depend on the choice of the lift $\hat{d} \in {}_1W_{M'_2}$ of d and

$$f_{P\hat{d}\mathcal{U},v} = f_{P\mathcal{U},v}T(\hat{d}). \tag{3.19}$$

As G is the disjoint union of $Pd\hat{}\mathcal{U}$ for $d \in \mathbb{W}_{M_2}$, we have $f_v = \sum_{d\in\mathbb{W}_{M_2}} f_{P\hat{d}\mathcal{U},v}$ and f_v is the image of $v \otimes e_{M_2}$ in (3.18), where

$$e_{M_2} = \sum_{d\in\mathbb{W}_{M_2}} T(\hat{d}). \tag{3.20}$$

Recalling (3.17) we get the following.

Lemma 3.10. *The map $v \mapsto v \otimes e_{M_2} : e(\sigma)^{\mathcal{U}} \to \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ is an $\mathcal{H}_R$-equivariant embedding.*

Remark 3.11. The trivial map $v \mapsto v \otimes 1_{\mathcal{H}}$ is not an $\mathcal{H}_R$-equivariant embedding.

We describe the action of $T(n)$ on $e(\sigma)^{\mathcal{U}}$ for $n \in \mathcal{N}$. By definition for $v \in e(\sigma)^{\mathcal{U}}$,

$$vT(n) = \sum_{y\in\mathcal{U}/(\mathcal{U}\cap n^{-1}\mathcal{U}n)} yn^{-1}v. \tag{3.21}$$

Proposition 3.12. *We have $vT(n) = vT^M(n_M)q_{M_2}(n)$ for any $n_M \in \mathcal{N} \cap M$ is such that $n = n_M(\mathcal{N} \cap M_2')$.*

Proof. The description (3.10) of $\mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n)$ gives

$$vT(n) = \sum_{y_1\in\mathcal{U}_M/(\mathcal{U}_M\cap n^{-1}\mathcal{U}_M n)} y_1 \sum_{y_2\in\mathcal{U}_{M_2'}/(\mathcal{U}_{M_2'}\cap n^{-1}\mathcal{U}_{M_2'} n)} y_2 n^{-1}v.$$

As M_2' acts trivially on $e(\sigma)$, we obtain

$$vT(n) = q_{M_2}(n) \sum_{y_1\in\mathcal{U}_M/(\mathcal{U}_M\cap n^{-1}\mathcal{U}_M n)} y_1 n_M^{-1} v = q_{M_2}(n)\, vT^M(n_M).$$

□

Theorem 3.13. *Let σ be a smooth R-representation of M. If $P(\sigma) = G$, then $\sigma^{\mathcal{U}_M}$ is extensible to $\mathcal{H}$ of extension $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$. Conversely, if $\sigma^{\mathcal{U}_M}$ is extensible to $\mathcal{H}$ and generates σ, then $P(\sigma) = G$.*

Proof.

(1) The $\mathcal{H}_M$-module $\sigma^{\mathcal{U}_M}$ is extensible to $\mathcal{H}$ if and only if $Z \cap M_2'$ acts trivially on $\sigma^{\mathcal{U}_M}$. Indeed, for $v \in \sigma^{\mathcal{U}_M}, z_2 \in Z \cap M_2'$,

$$vT^M(z_2) = \sum_{y\in\mathcal{U}_M/(\mathcal{U}_M\cap z_2^{-1}\mathcal{U}_M z_2)} yz_2^{-1}v = \sum_{y\in\mathcal{Y}_{M'}/(\mathcal{Y}_{M'}\cap z_2^{-1}\mathcal{Y}_{M'} z_2)} yz_2^{-1}v = z_2^{-1}v,$$

by (3.21), then (3.8), then the fact that z_2^{-1} commutes with the elements of $\mathcal{Y}_M$.

(2) $P(\sigma) = G$ if and only if $Z \cap M_2'$ acts trivially on σ (the group $Z \cap M_2'$ is generated by $Z \cap M_\alpha'$ for $\alpha \in \Delta_{M_2}$ by Lemma 2.1). The R-submodule $\sigma^{Z\cap M_2'}$ of elements fixed by $Z \cap M_2'$ is stable by M, because $M = ZM'$, the elements of M' commute with those of $Z \cap M_2'$ and Z normalizes $Z \cap M_2'$.

(3) Apply (1) and (2) to get the theorem except the equality $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$ when $P(\sigma) = G$ which follows from Propositions 3.12 and 3.7. □

Let $\mathbf{1}_M$ denote the trivial representation of M over R (or $\mathbf{1}$ when there is no ambiguity on M). The right $\mathcal{H}_R$-module $(\mathbf{1}_G)^{\mathcal{U}} = \mathbf{1}_{\mathcal{H}}$ (or $\mathbf{1}$ if there is no ambiguity) is the trivial right $\mathcal{H}_R$-module: for $w \in W_M(1)$, $T_w = q_w\mathrm{id}$ and $T_w^* = \mathrm{id}$ on $\mathbf{1}_{\mathcal{H}}$.

Example 3.14. The $\mathcal{H}$-module $(\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ is the extension of the $\mathcal{H}_{M_2}$-module $(\mathrm{Ind}_{M_2\cap B}^{M_2} \mathbf{1})^{\mathcal{U}_{M_2}}$. Indeed, the representation $\mathrm{Ind}_P^G \mathbf{1}$ of G is trivial on N_2, as $G = MM_2'$ and $N_2 \subset M'$ (as $\Phi = \Phi_M \cup \Phi_{M_2}$). For $g = mm_2'$ with $m \in M, m_2' \in M_2'$ and $n_2 \in N_2$, we have $Pgn_2 = Pm_2'n_2 = Pn_2m_2' = Pm_2' = Pg$. The group $M_2 \cap B = M_2 \cap P$ is the standard minimal parabolic subgroup of M_2 and $(\mathrm{Ind}_P^G \mathbf{1})|_{M_2} = \mathrm{Ind}_{M_2\cap B}^{M_2} \mathbf{1}$. Apply Theorem 3.13 as follows.

3.5. **The $\mathcal{H}_R$-module $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$.** Let $P = MN$ be a standard parabolic subgroup of G such that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal, let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module which is extensible to $\mathcal{H}_R$ of extension $e(\mathcal{V})$, and let Q be a parabolic subgroup of G containing P. Let $P_2 = M_2N_2$ denote the standard parabolic subgroup of G with $\Delta_{P_2} = \Delta \setminus \Delta_P$.

We define on the R-module $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ a structure of a right $\mathcal{H}_R$-module as follows.

Proposition 3.15.

(1) *The diagonal action of T_w^* for $w \in W(1)$ on $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ defines a structure of a right $\mathcal{H}_R$-module.*

(2) *The action of the T_w is also diagonal and satisfies:*

$$((v \otimes f)T_w, (v \otimes f)T_w^*) = (vT_{uw_{M'}} \otimes fT_{uw_{M_2'}}, vT_{uw_{M'}}^* \otimes fT_{uw_{M_2'}}^*),$$

where $w = uw_{M'}w_{M_2'}$ with $u \in W(1), \ell(u) = 0, w_{M'} \in {}_1W_{M'}, w_{M_2'} \in {}_1W_{M_2'}$.

Proof. If the lemma is true for P it is also true for Q, because the R-module $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ naturally embedded in $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ is stable by the action of $\mathcal{H}$ defined in the lemma. So, we suppose $Q = P$.

For each element in ${}_1S^{\mathrm{aff}}$ we fix an admissible lift and denote the set of admissible lifts by ${}_1S^{\mathrm{aff}}$. We also use the obvious notation ${}_1S_M^{\mathrm{aff}}$ and ${}_1S_{M_2}^{\mathrm{aff}}$. Suppose that T_w^* for $w \in W(1)$ acts on $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ as in (1). The braid relations obviously hold. The quadratic relations hold because T_s^* with $s \in {}_1S^{\mathrm{aff}}$, acts trivially either on $e(\mathcal{V})$ or on $(\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$. Indeed, ${}_1S^{\mathrm{aff}} = {}_1S_M^{\mathrm{aff}} \cup {}_1S_{M_2}^{\mathrm{aff}}$, T_s^* for $s \in {}_1S_M^{\mathrm{aff}}$, acts trivially on $(\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ which is extended from an $\mathcal{H}_{M_2}$-module (Example 3.14), and T_s^* for $s \in {}_1S_{M_2}^{\mathrm{aff}}$, acts trivially on $e(\mathcal{V})$ which is extended from an $\mathcal{H}_M$-module. This proves (1).

We describe now the action of T_w instead of T_w^* on the $\mathcal{H}$-module $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$. Let $w \in W(1)$. We write $w = uw_{M'}w_{M_2'} = uw_{M_2'}w_{M'}$ with $u \in W(1), \ell(u) = 0, w_{M'} \in {}_1W_{M'}, w_{M_2'} \in {}_1W_{M_2'}$. We have $\ell(w) = \ell(w_{M'}) + \ell(w_{M_2'})$ hence $T_w = T_uT_{w_{M'}}T_{w_{M_2'}}$.

For $w = u$, we have $T_u = T_u^*$ and $(v\otimes f)T_u = (v\otimes f)T_u^* = vT_u^*\otimes fT_u^* = vT_u\otimes fT_u$.

For $w = w_{M'}$, $(v \otimes f)T_w^* = vT_w^* \otimes f$; for $s \in {}_1S_M^{\mathrm{aff}}$, $c_s = \sum_{t\in Z_k\cap {}_1W_{M'}} c_s(t)T_t^*$ in particular, we have $(v\otimes f)T_s = (v\otimes f)(T_s^* + c_s) = v(T_s^* + c_s)\otimes f = vT_s\otimes f$. Hence $(v \otimes f)T_w = vT_w \otimes f$.

For $w = w_{M_2'}$, we have similarly $(v\otimes f)T_w^* = v\otimes fT_w^*$ and $(v\otimes f)T_w = v\otimes fT_w$. □

Example 3.16. Let $\mathcal{X}$ be a right $\mathcal{H}_R$-module. Then $\mathbf{1}_{\mathcal{H}} \otimes_R \mathcal{X}$ where the T_w^* acts diagonally is an $\mathcal{H}_R$-module isomorphic to $\mathcal{X}$. But the action of the T_w on $\mathbf{1}_{\mathcal{H}} \otimes_R \mathcal{X}$ is not diagonal.

It is known [Ly15] that $(\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}}$ and $(\mathrm{St}_Q^G)^{\mathcal{U}}$ are free R-modules and that $(\mathrm{St}_Q^G)^{\mathcal{U}}$ is the cokernel of the natural $\mathcal{H}_R$-map

$$\bigoplus_{Q\subsetneq Q'} (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}} \to (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \tag{3.22}$$

although the invariant functor $(-)^{\mathcal{U}}$ is only left exact.

Corollary 3.17. *The diagonal action of T_w^* for $w \in W(1)$ on $e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ defines a structure of a right $\mathcal{H}_R$-module satisfying Proposition* 3.15(2)*.*

4. Hecke module $I_{\mathcal{H}}(P, \mathcal{V}, Q)$

4.1. **Case $\mathcal{V}$ extensible to $\mathcal{H}$.** Let $P = MN$ be a standard parabolic subgroup of G such that Δ_P and $\Delta\backslash\Delta_P$ are orthogonal, let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module extensible to $\mathcal{H}_R$ of extension $e(\mathcal{V})$, and let Q be a parabolic subgroup of G containing P. As Q and M_Q determine each other: $Q = M_Q U$, we denote also $\mathcal{H}_{M_Q} = \mathcal{H}_Q$ and $\mathcal{H}_{M_Q,R} = \mathcal{H}_{Q,R}$ when $Q \neq P, G$. When $Q = G$ we drop G and we denote $e_{\mathcal{H}}(\mathcal{V}) = e(\mathcal{V})$.

Lemma 4.1. *$\mathcal{V}$ is extensible to an $\mathcal{H}_{Q,R}$-module $e_{\mathcal{H}_Q}(\mathcal{V})$.*

Proof. This is straightforward. By Corollary 3.9, $\mathcal{V}$ extensible to $\mathcal{H}$ means that $T^M(z)$ acts trivially on $\mathcal{V}$ for all $z \in \mathcal{N}_{M_2'} \cap Z$. We have $M_Q = MM'_{2,Q}$ with $M'_{2,Q} \subset M_Q \cap M'_2$ and $\mathcal{N}_{M'_{2,Q}} \subset \mathcal{N}_{M'_2}$; hence $T^M(z)$ acts trivially on $\mathcal{V}$ for all $z \in \mathcal{N}_{M'_{2,Q}} \cap Z$ meaning that $\mathcal{V}$ is extensible to $\mathcal{H}_Q$. □

Remark 4.2. We cannot say that $e_{\mathcal{H}_Q}(\mathcal{V})$ is extensible to $\mathcal{H}$ of extension $e(\mathcal{V})$ when the set of roots Δ_Q and $\Delta \setminus \Delta_Q$ are not orthogonal (Definition 3.8).

Let Q' be an arbitrary parabolic subgroup of G containing Q. We are going to define an $\mathcal{H}_R$-embedding $\mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+},\theta} \mathcal{H}$ defining an $\mathcal{H}_R$-homomorphism

$$\oplus_{Q\subsetneq Q'\subset G} \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$$

of cokernel isomorphic to $e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$. In the extreme case $(Q, Q') = (P, G)$, the $\mathcal{H}_R$-embedding $e(\mathcal{V}) \xrightarrow{\iota(P,G)} \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$ is given in the following lemma where f_G and $f_{P\mathcal{U}} \in (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ denote the characteristic functions of G and $P\mathcal{U}$, $f_G = f_{P\mathcal{U}} e_{M_2}$ (see (3.20)).

Lemma 4.3. *There is a natural $\mathcal{H}_R$-isomorphism*

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{P\mathcal{U}} : \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} \xrightarrow{\kappa_P} e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}},$$

and compatible $\mathcal{H}_R$-embeddings

$$v \mapsto v \otimes f_G : e(\mathcal{V}) \to e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}, \tag{4.1}$$

$$v \mapsto v \otimes e_{M_2} : e(\mathcal{V}) \xrightarrow{\iota(P,G)} \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}). \tag{4.2}$$

Proof. We show first that the map

$$v \mapsto v \otimes f_{P\mathcal{U}} : \mathcal{V} \to e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}} \tag{4.3}$$

is $\mathcal{H}_{M^+}$-equivariant. Let $w \in W_{M^+}(1)$. We write $w = uw_{M'}w_{M'_2}$ as in Proposition 3.15 (2), so that $f_{P\mathcal{U}}T_w = f_{P\mathcal{U}}T_{uw_{M'_2}}$. We have $f_{P\mathcal{U}}T_{uw_{M'_2}} = f_{P\mathcal{U}}$ because ${}_1W_{M'} \subset W_{M^+}(1) \cap W_{M^-}(1)$ hence $uw_{M'_2} = ww_{M'}^{-1} \in W_{M^+}(1)$ and in $\mathbf{1}_{\mathcal{H}_M} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ we have $(1 \otimes 1_{\mathcal{H}})T_{uw_{M'_2}} = 1T^M_{uw_{M'_2}} \otimes 1_{\mathcal{H}}$, and $T^M_{uw_{M'_2}}$ acts trivially in $\mathbf{1}_{\mathcal{H}_M}$ because $\ell_M(uw_{M'_2}) = 0$. We deduce $(v \otimes f_{P\mathcal{U}})T_w = vT_w \otimes f_{P\mathcal{U}}T_w = vT^M_w \otimes f_{P\mathcal{U}}$.

By adjunction (4.3) gives an $\mathcal{H}_R$-equivariant linear map

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{P\mathcal{U}} : \mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} \xrightarrow{\kappa_P} e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}. \tag{4.4}$$

We prove that κ_P is an isomorphism. Recalling $\hat{d} \in \mathcal{N} \cap M'_2, \tilde{d} \in {}_1W_{M'_2}$ lift d, one knows that

$$\mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} = \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes T_{\tilde{d}}, \quad e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}} = \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes f_{P\hat{d}\mathcal{U}}, \tag{4.5}$$

where each summand is isomorphic to $\mathcal{V}$. The left equality follows from section 4.1 and Remark 3.7 in [Vig15b] recalling that $w \in \mathbb{W}_{M_2}$ is of minimal length in its coset $\mathbb{W}_M w = w\mathbb{W}_M$ as Δ_M and Δ_{M_2} are orthogonal; for the second equality see section 3.4 (3.19). We have $\kappa_P(v \otimes T_{\tilde{d}}) = (v \otimes f_{P\mathcal{U}})T_{\tilde{d}} = v \otimes f_{P\mathcal{U}}T_{\tilde{d}}$ (Proposition 3.15). Hence κ_P is an isomorphism.

We consider the composite map

$$v \mapsto v \otimes 1 \mapsto v \otimes f_{P\mathcal{U}}e_{M_2} : e(\mathcal{V}) \to e(\mathcal{V}) \otimes_R \mathbf{1}_{\mathcal{H}} \to e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}},$$

where the right map is the tensor product $e(\mathcal{V}) \otimes_R -$ of the $\mathcal{H}_R$-equivariant embedding $\mathbf{1}_{\mathcal{H}} \to (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ sending 1_R to $f_{P\mathcal{U}}e_{M_2}$ (Lemma 3.10); this map is injective because $(\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}/\mathbf{1}$ is a free R-module; it is $\mathcal{H}_R$-equivariant for the diagonal action of the T^*_w on the tensor products (Example 3.16 for the first map). By compatibility with (4.4), we get the $\mathcal{H}_R$-equivariant embedding $v \mapsto v \otimes e_{M_2} : e(\mathcal{V}) \xrightarrow{\iota(P,G)} \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$. ☐

For a general (Q, Q') the $\mathcal{H}_R$-embedding $\mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$ is given in the next proposition generalizing Lemma 4.3. The element e_{M_2} of $\mathcal{H}_R$ appearing in the definition of $\iota(P, G)$ is replaced in the definition of $\iota(Q, Q')$ by an element $\theta_{Q'}(e_Q^{Q'}) \in \mathcal{H}_R$ that we define first.

Until the end of section 4, we fix an admissible lift $w \mapsto \hat{w} : \mathbb{W} \to \mathcal{N} \cap \mathcal{K}$ (Definition 3.1) and $\tilde{w}$ denotes the image of $\hat{w}$ in $W(1)$. We denote $\mathbb{W}_{M_Q} = \mathbb{W}_Q$ and by ${}^{\mathbb{W}_Q}\mathbb{W}$ the set of $w \in \mathbb{W}$ of minimal length in their coset $\mathbb{W}_Q w$. The group G is the disjoint union of $Q\hat{d}\mathcal{U}$ for d running through ${}^{\mathbb{W}_Q}\mathbb{W}$ [OV17, Lemma 2.15 (2)]: $G = \bigsqcup_{d \in {}^{\mathbb{W}_Q}\mathbb{W}} Q\hat{d}\mathcal{U}$. Since $Q\hat{d}\mathcal{U} \subset Q'\mathcal{U}$ if and only if $\hat{d} \in Q'$, namely $d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}$, we have

$$Q'\mathcal{U} = \bigsqcup_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} Q\hat{d}\mathcal{U}. \tag{4.6}$$

Set

$$e_Q^{Q'} = \sum_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} T_{\tilde{d}}^{M_{Q'}}. \tag{4.7}$$

We write $e_Q^G = e_Q$. We have $e_P^Q = \sum_{d \in \mathbb{W}_{M_{2,Q}}} T_{\tilde{d}}^{M_Q}$.

Remark 4.4. Note that ${}^{\mathbb{W}_M}\mathbb{W} = \mathbb{W}_{M_2}$ and $e_P = e_{M_2}$, where M_2 is the standard Levi subgroup of G with $\Delta_{M_2} = \Delta \setminus \Delta_M$, as Δ_M and $\Delta \setminus \Delta_M$ are orthogonal. More generally, ${}^{\mathbb{W}_Q}\mathbb{W}_{M_{Q'}} = {}^{\mathbb{W}_{M_{2,Q}}}\mathbb{W}_{M_{2,Q'}}$ where $M_{2,Q'} = M_2 \cap M_{Q'}$.

Note that $e_Q^{Q'} \in \mathcal{H}_{M^+} \cap \mathcal{H}_{M^-}$. We consider the linear map

$$\theta_Q^{Q'} : \mathcal{H}_Q \to \mathcal{H}_{Q'} \quad T_w^{M_Q} \mapsto T_w^{M_{Q'}} \quad (w \in W_{M_Q}(1)).$$

We write $\theta_Q^G = \theta_Q$ so that $\theta_Q(T_w^{M_Q}) = T_w$. When $Q = P$ this is the map θ defined earlier. Similarly we denote by $\theta_Q^{Q',*}$ the linear map sending the $T_w^{M_Q,*}$ to $T_w^{M_{Q'},*}$ and $\theta_Q^{G,*} = \theta_Q^*$. We have

$$\theta_{Q'}(e_Q^{Q'}) = \sum_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} T_{\tilde{d}}, \quad \theta_{Q'}(e_P^{Q'}) = \theta_Q(e_P^Q)\theta_{Q'}(e_Q^{Q'}). \tag{4.8}$$

Proposition 4.5. *There exists an $\mathcal{H}_R$-isomorphism*
(4.9)

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{Q\mathcal{U}} : \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+},\theta} \mathcal{H} \xrightarrow{\kappa_Q} e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}},$$

and compatible $\mathcal{H}_R$-embeddings

$$v \otimes f_{Q'\mathcal{U}} \mapsto v \otimes f_{Q'\mathcal{U}} : e_{\mathcal{H}_{Q'}}(\mathcal{V}) \otimes_R (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}} \to e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}, \tag{4.10}$$

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_{Q'}(e_Q^{Q'}) : \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})). \tag{4.11}$$

Proof. We have the $\mathcal{H}_{M_Q,R}$-embedding

$$v \mapsto v \otimes e_P^Q : e_{\mathcal{H}_Q}(\mathcal{V}) \to \mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}_Q = \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}_Q}(\mathcal{V})$$

by Lemma 4.3 (4.2) as Δ_M is orthogonal to $\Delta_{M_Q} \setminus \Delta_M$. Applying the parabolic induction which is exact, we get the $\mathcal{H}$-embedding

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes e_P^Q \otimes 1_{\mathcal{H}} : \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}_Q}(\mathcal{V})).$$

Note that $T_{\tilde{d}}^{M_Q} \in \mathcal{H}_{M_Q^+}$ for $d \in \mathbb{W}_{M_Q}$. By transitivity of the parabolic induction, it is equal to the $\mathcal{H}_R$-embedding

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_Q(e_P^Q) : \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}). \tag{4.12}$$

On the other hand we have the $\mathcal{H}_R$-embedding

$$v \otimes f_{Q\mathcal{U}} \mapsto v \otimes \theta_Q(e_P^Q) : e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \to \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) \tag{4.13}$$

given by the restriction to $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ of the $\mathcal{H}_R$-isomorphism given in Lemma 4.3 (4.1), from $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ to $\mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ sending $v \otimes f_{P\mathcal{U}}$ to $v \otimes 1_{\mathcal{H}}$, noting that $v \otimes f_{Q\mathcal{U}} = (v \otimes f_{P\mathcal{U}})\theta_Q(e_P^Q)$ by Proposition 3.15, $f_{Q\mathcal{U}} = f_{P\mathcal{U}}\theta_Q(e_P^Q)$ and $\theta_Q(e_P^Q)$ acts trivially on $e(\mathcal{V})$ (this is true for $T_{\tilde{d}}$ for $\tilde{d} \in {}_1W_{M_2'}$). Comparing the embeddings (4.12) and (4.13), we get the $\mathcal{H}_R$-isomorphism (4.9).

We can replace Q by Q' in the $\mathcal{H}_R$-homomorphisms (4.9), (4.12), and (4.13). With (4.12) we see $\mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ and $\mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$ as $\mathcal{H}_R$-submodules of $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$. As seen in (4.8) we have $\theta_{Q'}(e_P^{Q'}) = \theta_Q(e_P^Q)\theta_{Q'}(e_Q^{Q'})$. We deduce the $\mathcal{H}_R$-embedding (4.11).

By (3.19) for Q and (4.6),

$$f_{Q'\mathcal{U}} = \sum_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} f_{Q\mathcal{U}} T_{\tilde{d}} = f_{Q\mathcal{U}} \theta_{Q'}(e_Q^{Q'})$$

in $(\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$. We deduce that the $\mathcal{H}_R$-embedding corresponding to (4.11) via κ_Q and $\kappa_{Q'}$ is the $\mathcal{H}_R$-embedding (4.10). □

We recall that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal and that $\mathcal{V}$ is extensible to $\mathcal{H}$ of extension $e(\mathcal{V})$.

Corollary 4.6. *The cokernel of the $\mathcal{H}_R$-map*

$$\bigoplus_{Q \subsetneq Q' \subset G} \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the $\iota(Q,Q')$, is isomorphic to $e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ via κ_Q.

4.2. **Invariants in the tensor product.** We return to the setting where $P = MN$ is a standard parabolic subgroup of G, σ is a smooth R-representation of M with $P(\sigma) = G$ of extension $e(\sigma)$ to G, and Q a parabolic subgroup of G containing P. We still assume that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal.

The $\mathcal{H}_R$-modules $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$ are equal (Theorem 3.13). We compute $I_G(P,\sigma,Q)^{\mathcal{U}} = (e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$.

Theorem 4.7. *The natural linear maps $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \to (e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ and $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}} \to (e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$ are isomorphisms.*

Proof. We need some preliminaries. In [GK14, Ly15], are introduced a finite free $\mathbb{Z}$-module $\mathfrak{M}$ (depending on Δ_Q) and a $\mathcal{B}$-equivariant embedding $\mathrm{St}_Q^G\mathbb{Z} \xrightarrow{\iota} C_c^\infty(\mathcal{B},\mathfrak{M})$ (we indicate the coefficient ring in the Steinberg representation) which induces an isomorphism $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}} \simeq C_c^\infty(\mathcal{B},\mathfrak{M})^{\mathcal{B}}$.

Lemma 4.8.

(1) $(\mathrm{Ind}_Q^G \mathbb{Z})^{\mathcal{B}}$ *is a direct factor of* $\mathrm{Ind}_Q^G \mathbb{Z}$.

(2) $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}$ *is a direct factor of* $\mathrm{St}_Q^G\mathbb{Z}$.

Proof.

(1) [AHV, Example 2.2].

(2) As $\mathfrak{M}$ is a free $\mathbb{Z}$-module, $C_c^\infty(\mathcal{B},\mathfrak{M})^{\mathcal{B}}$ is a direct factor of $C_c^\infty(\mathcal{B},\mathfrak{M})$. Consequently, $\iota((\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}) = C_c^\infty(\mathcal{B},\mathfrak{M})^{\mathcal{B}}$ is a direct factor of $\iota(\mathrm{St}_Q^G\mathbb{Z})$. As ι is injective, we get (2). □

We now prove Theorem 4.7. We may and do assume that σ is e-minimal (because $P(\sigma) = P(\sigma_{\min}), e(\sigma) = e(\sigma_{\min})$) so that Δ_M and $\Delta \setminus \Delta_M$ are orthogonal and we use the same notation as in section 3.2 in particular $M_2 = M_{\Delta \setminus \Delta_M}$. Let V be the space of $e(\sigma)$ on which M_2' acts trivially. The restriction of $\mathrm{Ind}_Q^G \mathbb{Z}$ to M_2 is $\mathrm{Ind}_{Q\cap M_2}^{M_2} \mathbb{Z}$, that of $\mathrm{St}_Q^G\mathbb{Z}$ is $\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z}$.

As in [AHV, Example 2.2], $((\mathrm{Ind}_{Q\cap M_2}^{M_2} \mathbb{Z}) \otimes V)^{\mathcal{U}_{M_2'}} \simeq (\mathrm{Ind}_{Q\cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2'}} \otimes V$. We have

$$(\mathrm{Ind}_{Q\cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2'}} = (\mathrm{Ind}_{Q\cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}} = (\mathrm{Ind}_Q^G \mathbb{Z})^{\mathcal{U}}.$$

The first equality follows from $M_2 = (Q \cap M_2)\mathbb{W}_{M_2}\mathcal{U}_{M_2}$, $\mathcal{U}_{M_2} = Z^1\mathcal{U}_{M_2'}$ and Z^1 normalizes $\mathcal{U}_{M_2'}$ and is normalized by $\mathbb{W}_{M_2}$. The second equality follows from $\mathcal{U} =$

$\mathcal{U}_{M'}\mathcal{U}_{M_2}$ and $\mathrm{Ind}_Q^G\,\mathbb{Z}$ is trivial on M'. Therefore $((\mathrm{Ind}_Q^G\,\mathbb{Z})\otimes V)^{\mathcal{U}_{M_2'}} \simeq (\mathrm{Ind}_Q^G\,\mathbb{Z})^{\mathcal{U}}\otimes V$. Now taking fixed points under $\mathcal{U}_M$, as $\mathcal{U}=\mathcal{U}_{M_2'}\mathcal{U}_M$,

$$((\mathrm{Ind}_Q^G\,\mathbb{Z})\otimes V)^{\mathcal{U}} \simeq ((\mathrm{Ind}_Q^G\,\mathbb{Z})^{\mathcal{U}}\otimes V)^{\mathcal{U}_M} = (\mathrm{Ind}_Q^G\,\mathbb{Z})^{\mathcal{U}}\otimes V^{\mathcal{U}_M}.$$

The equality uses that the $\mathbb{Z}$-module $\mathrm{Ind}_Q^G\,\mathbb{Z}$ is free. We get the first part of the theorem as $(\mathrm{Ind}_Q^G\,\mathbb{Z})^{\mathcal{U}}\otimes V^{\mathcal{U}_M} \simeq (\mathrm{Ind}_Q^G\,R)^{\mathcal{U}}\otimes_R V^{\mathcal{U}_M}$.

Tensoring with R the usual exact sequence defining $\mathrm{St}_Q^G\mathbb{Z}$ gives an isomorphism $\mathrm{St}_Q^G\mathbb{Z}\otimes R\simeq \mathrm{St}_Q^G R$ and in [GK14,Ly15], it is proved that the resulting map $\mathrm{St}_Q^G R\xrightarrow{\iota_R} C^\infty(\mathcal{B},\mathfrak{M}\otimes R)$ is also injective. Their proof in no way uses the ring structure of R, and for any $\mathbb{Z}$-module V, tensoring with V gives a $\mathcal{B}$-equivariant embedding $\mathrm{St}_Q^G\mathbb{Z}\otimes V\xrightarrow{\iota_V} C_c^\infty(\mathcal{B},\mathfrak{M}\otimes V)$. The natural map $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}\otimes V\to \mathrm{St}_Q^G\mathbb{Z}\otimes V$ is also injective by Lemma 4.8 (2). Taking $\mathcal{B}$-fixed points we get inclusions

$$(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}\otimes V\to(\mathrm{St}_Q^G\mathbb{Z}\otimes V)^{\mathcal{B}}\to C_c^\infty(\mathcal{B},\mathfrak{M}\otimes V)^{\mathcal{B}}\simeq \mathfrak{M}\otimes V. \tag{4.14}$$

The composite map is surjective, so the inclusions are isomorphisms. The image of ι_V consists of functions which are left Z^0-invariant, and $\mathcal{B}=Z^0\mathcal{U}'$ where $\mathcal{U}'=G'\cap\mathcal{U}$. It follows that ι yields an isomorphism $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}'}\simeq C_c^\infty(Z^0\backslash\mathcal{B},\mathfrak{M})^{\mathcal{U}'}$ again consisting of the constant functions. So that in particular $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}'}=(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}$ and reasoning as previously we get isomorphisms

$$(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}'}\otimes V\simeq(\mathrm{St}_Q^G\mathbb{Z}\otimes V)^{\mathcal{U}'}\simeq\mathfrak{M}\otimes V. \tag{4.15}$$

The equality $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}'}=(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}$ and the isomorphisms remain true when we replace $\mathcal{U}'$ by any group between $\mathcal{B}$ and $\mathcal{U}'$. We apply these results to $\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z}\otimes V$ to get that the natural map $(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z})^{\mathcal{U}_{M_2'}}\otimes V\to(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z}\otimes V)^{\mathcal{U}_{M_2'}}$ is an isomorphism and also that $(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z})^{\mathcal{U}_{M_2'}}=(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z})^{\mathcal{U}_{M_2}}$. We have $\mathcal{U}=\mathcal{U}_{M'}\mathcal{U}_{M_2}$ so $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}=(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z})^{\mathcal{U}_{M_2}}$ and the natural map $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}\otimes V\to(\mathrm{St}_Q^G\mathbb{Z}\otimes V)^{\mathcal{U}_{M_2'}}$ is an isomorphism. The $\mathbb{Z}$-module $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}$ is free and the $V^{\mathcal{U}_M}=V^{\mathcal{U}}$, so taking fixed points under $\mathcal{U}_M$, we get $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}\otimes V^{\mathcal{U}}\simeq(\mathrm{St}_Q^G\mathbb{Z}\otimes V)^{\mathcal{U}}$. We have $\mathrm{St}_Q^G\mathbb{Z}\otimes V=\mathrm{St}_Q^G R\otimes_R V$ and $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}\otimes V^{\mathcal{U}}=(\mathrm{St}_Q^G R)^{\mathcal{U}}\otimes_R V^{\mathcal{U}}$. This ends the proof of the theorem. □

Theorem 4.9. *The $\mathcal{H}_R$-modules $(e(\sigma)\otimes_R\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}=e(\sigma)^{\mathcal{U}}\otimes_R(\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}$ are equal. The $\mathcal{H}_R$-modules $(e(\sigma)\otimes_R\mathrm{St}_Q^G)^{\mathcal{U}}=e(\sigma)^{\mathcal{U}}\otimes_R(\mathrm{St}_Q^G)^{\mathcal{U}}$ are also equal.*

Proof. We already know that the R-modules are equal (Theorem 4.7). We show that they are equal as $\mathcal{H}$-modules. The $\mathcal{H}_R$-modules $e(\sigma)^{\mathcal{U}}\otimes_R(\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}=e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})\otimes_R(\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}$ are equal (Theorem 3.13), they are isomorphic to $\mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\sigma^{\mathcal{U}_M}))$ (Proposition 4.5), to $(\mathrm{Ind}_Q^G(e_Q(\sigma)))^{\mathcal{U}}$ [OV17, Proposition 4.4], and to $(e(\sigma)\otimes_R\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}$ [AHV, Lemma 2.5]). We deduce that the $\mathcal{H}_R$-modules $e(\sigma)^{\mathcal{U}}\otimes_R(\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}=(e(\sigma)\otimes_R\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}$ are equal. The same is true when Q is replaced by a parabolic subgroup Q' of G containing Q. The representation $e(\sigma)\otimes_R\mathrm{St}_Q^G$ is the cokernel of the natural $R[G]$-map

$$\bigoplus_{Q\subsetneq Q'} e(\sigma)\otimes_R\mathrm{Ind}_{Q'}^G\,\mathbf{1}\xrightarrow{\alpha_Q} e(\sigma)\otimes_R\mathrm{Ind}_Q^G\,\mathbf{1}$$

and the $\mathcal{H}_R$-module $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ is the cokernel of the natural $\mathcal{H}_R$-map

$$\bigoplus_{Q \subsetneq Q'} e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}} \xrightarrow{\beta_Q} e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$$

obtained by tensoring (3.22) by $e(\sigma)^{\mathcal{U}}$ over R, because the tensor product is right exact. The maps $\beta_Q = \alpha_Q^{\mathcal{U}}$ are equal and the R-modules $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}} = (e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$ are equal. This implies that the $\mathcal{H}_R$-modules $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}} = (e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$ are equal. □

Remark 4.10. The proof shows that the representations $e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}$ and $e(\sigma) \otimes \mathrm{St}_Q^G$ of G are generated by their $\mathcal{U}$-fixed vectors if the representation σ of M is generated by its $\mathcal{U}_M$-fixed vectors. Indeed, the R-modules $e(\sigma)^{\mathcal{U}} = \sigma^{\mathcal{U}_M}$, $(\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}_{M'_2}} = (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ are equal. If $\sigma^{\mathcal{U}_M}$ generates σ, then $e(\sigma)$ is generated by $e(\sigma)^{\mathcal{U}}$. The representation $\mathrm{Ind}_Q^G \mathbf{1}|_{M'_2}$ is generated by $(\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ (this follows from the lemma below), we have $G = MM'_2$ and M'_2 acts trivially on $e(\sigma)$. Therefore the $R[G]$-module generated by $\sigma^{\mathcal{U}} \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ is $e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}$. As $e(\sigma) \otimes_R \mathrm{St}_Q^G$ is a quotient of $e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}$, the $R[G]$-module generated by $\sigma^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ is $e(\sigma) \otimes_R \mathrm{St}_Q^G$.

Lemma 4.11. *For any standard parabolic subgroup P of G, the representation $\mathrm{Ind}_P^G \mathbf{1}|_{G'}$ is generated by its $\mathcal{U}$-fixed vectors.*

Proof. Because $G = PG'$ it suffices to prove that if J is an open compact subgroup of $\overline{N}$ the characteristic function 1_{PJ} of PJ is a finite sum of translates of $1_{P\mathcal{U}} = 1_{P\mathcal{U}_{\overline{N}}}$ by G'. For $t \in T$ we have $P\mathcal{U}t = Pt^{-1}\mathcal{U}_{\overline{N}}t$ and we can choose $t \in T \cap J'$ such that $t^{-1}\mathcal{U}_{\overline{N}}t \subset J$. □

4.3. **General triples.** Let $P = MN$ be a standard parabolic subgroup of G. We now investigate situations where Δ_P and $\Delta \setminus \Delta_P$ are not necessarily orthogonal. Let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module.

Definition 4.12. Let $P(\mathcal{V}) = M(\mathcal{V})N(\mathcal{V})$ be the standard parabolic subgroup of G with $\Delta_{P(\mathcal{V})} = \Delta_P \cup \Delta_{\mathcal{V}}$ and

$$\Delta_{\mathcal{V}} = \{\alpha \in \Delta \text{ orthogonal to } \Delta_M,\, T^M(z) \text{ acts trivially on } \mathcal{V} \text{ for all } z \in Z \cap M'_\alpha\}.$$

If Q is a parabolic subgroup of G between P and $P(\mathcal{V})$, the triple $(P, \mathcal{V}, Q)$ called an $\mathcal{H}_R$-triple, defines a right $\mathcal{H}_R$-module $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ equal to

$$\mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes_R (\mathrm{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) = (e(\mathcal{V}) \otimes_R (\mathrm{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V})^+,R},\theta} \mathcal{H}_R,$$

where $e(\mathcal{V})$ is the extension of $\mathcal{V}$ to $\mathcal{H}_{M(\mathcal{V})}$.

This definition is justified by the fact that $M(\mathcal{V})$ is the maximal standard Levi subgroup of G such that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}_{M(\mathcal{V})}$.

Lemma 4.13. *$\Delta_{\mathcal{V}}$ is the maximal subset of $\Delta \setminus \Delta_P$ orthogonal to Δ_P such that $T_\lambda^{M,*}$ acts trivially on $\mathcal{V}$ for all $\lambda \in \Lambda(1) \cap {}_1W_{M'_{\mathcal{V}}}$.*

Proof. For $J \subset \Delta$ let M_J denote the standard Levi subgroup of G with $\Delta_{M_J} = J$. The group $Z \cap M'_J$ is generated by the $Z \cap M'_\alpha$ for all $\alpha \in J$ (Lemma 2.1). When J is orthogonal to Δ_M and $\lambda \in \Lambda_{M'_J}(1)$, $\ell_M(\lambda) = 0$ where ℓ_M is the length associated to S_M^{aff}, and the map $\lambda \mapsto T_\lambda^{M,*} = T_\lambda^M : \Lambda_{M'_J}(1) \to \mathcal{H}_M$ is multiplicative. □

The following is the natural generalization of Proposition 4.5 and Corollary 4.6. Let Q' be a parabolic subgroup of G with $Q \subset Q' \subset P(\mathcal{V})$. Applying the results of section 4.1 to $M(\mathcal{V})$ and its standard parabolic subgroups $Q \cap M(\mathcal{V}) \subset Q' \cap M(\mathcal{V})$, we have an $\mathcal{H}_{M(\mathcal{V}),R}$-isomorphism

$$\begin{aligned}&\mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_Q}(\mathcal{V}))\\ &\quad = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+},\theta} \mathcal{H}_{M(\mathcal{V}),R} \xrightarrow{\kappa_{Q\cap M(\mathcal{V})}} e(\mathcal{V}) \otimes_R (\mathrm{Ind}_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})} \mathbf{1})^{\mathcal{U}_{M(\mathcal{V})}}\\ &\qquad\qquad v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{Q\mathcal{U}\cap M(\mathcal{V})}:\end{aligned}$$

and an $\mathcal{H}_{M(\mathcal{V}),R}$-embedding

$$\begin{aligned}&\mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q\cap M(\mathcal{V}),Q'\cap M(\mathcal{V}))} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_Q}(\mathcal{V}))\\ &\qquad v \otimes 1_{\mathcal{H}_{M(\mathcal{V})}} \mapsto v \otimes \theta_{Q'}^{P(\mathcal{V})}(e_Q^{Q'}).\end{aligned}$$

Applying the parabolic induction $\mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}$ which is exact and transitive, we obtain an $\mathcal{H}_R$-isomorphism $\kappa_Q = \mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(\kappa_{Q\cap M(\mathcal{V})})$,

$$\begin{aligned}&\mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \xrightarrow{\kappa_Q} \mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes_R (\mathrm{Ind}_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})} \mathbf{1}_{M_Q})^{\mathcal{U}_{M(\mathcal{V})}})\\ &\qquad v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{Q\mathcal{U}_{M(\mathcal{V})}} \otimes 1_{\mathcal{H}}\end{aligned} \tag{4.16}$$

and an $\mathcal{H}_R$-embedding $\iota(Q,Q') = \mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(\iota(Q,Q')^{M(\mathcal{V})})$

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_{Q'}(e_Q^{Q'}) : \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})). \tag{4.17}$$

Applying Corollary 4.6 we obtain:

Theorem 4.14. *Let $(P,\mathcal{V},Q)$ be an $\mathcal{H}_R$-triple. Then, the cokernel of the $\mathcal{H}_R$-map*

$$\oplus_{Q\subsetneq Q'\subset P(\mathcal{V})} \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})),$$

defined by the $\iota(Q,Q')$ is isomorphic to $I_{\mathcal{H}}(P,\mathcal{V},Q)$ via the $\mathcal{H}_R$-isomorphism κ_Q.

Let σ be a smooth R-representation of M and let Q be a parabolic subgroup of G with $P \subset Q \subset P(\sigma)$.

Remark 4.15. The $\mathcal{H}_R$-module $I_{\mathcal{H}}(P,\sigma^{\mathcal{U}_M},Q)$ is defined if $\Delta_Q \setminus \Delta_P$ and Δ_P are orthogonal because $Q \subset P(\sigma) \subset P(\sigma^{\mathcal{U}_M})$ (Theorem 3.13).

We denote here by $P_{\min} = M_{\min}N_{\min}$ the minimal standard parabolic subgroup of G contained in P such that $\sigma = e_P(\sigma|_{M_{\min}})$ (Lemma 2.3, we drop the index σ). The sets of roots $\Delta_{P_{\min}}$ and $\Delta_{P(\sigma|_{M_{\min}})} \setminus \Delta_{P_{\min}}$ are orthogonal (Lemma 2.4). The groups $P(\sigma) = P(\sigma|_{M_{\min}})$, the representations $e(\sigma) = e(\sigma|_{M_{\min}})$ of $M(\sigma)$, the representations $I_G(P,\sigma,Q) = I_G(P_{\min},\sigma|_{M_{\min}},Q) = \mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \mathrm{St}_Q^{P(\sigma)})$ of G, and the R-modules $\sigma^{\mathcal{U}_{M_{\min}}} = \sigma^{\mathcal{U}_M}$ are equal. From Theorem 3.13,

$$P(\sigma) \subset P(\sigma^{\mathcal{U}_{M_{\min}}}), \quad e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) = e(\sigma)^{\mathcal{U}_{M(\sigma)}},$$

and $P(\sigma^{\mathcal{U}_{M_{\min}}}) = P(\sigma)$ if $\sigma^{\mathcal{U}_{M_{\min}}}$ generates the representation $\sigma|_{M_{\min}}$. The $\mathcal{H}_R$-module

$$I_{\mathcal{H}}(P_{\min},\sigma^{\mathcal{U}_{M_{\min}}},Q) = \mathrm{Ind}_{\mathcal{H}_{M(\sigma^{\mathcal{U}_{M_{\min}}})}}^{\mathcal{H}}(e(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\mathrm{St}_Q^{P(\sigma^{\mathcal{U}_{M_{\min}}})})^{\mathcal{U}_{M(\sigma^{\mathcal{U}_{M_{\min}}})}})$$

is defined because $\Delta_{P_{\min}}$ and $\Delta_{P(\sigma^{\mathcal{U}_{M_{\min}}})} \setminus \Delta_{P_{\min}}$ are orthogonal and $P \subset Q \subset P(\sigma) \subset P(\sigma^{\mathcal{U}_{M_{\min}}})$.

Remark 4.16. If $\sigma^{\mathcal{U}_{M_{\min}}}$ generates the representation $\sigma|_{M_{\min}}$ (in particular if R is an algebraically closed field of characteristic p and σ is irrreducible), then $P(\sigma) = P(\sigma^{\mathcal{U}_{M_{\min}}})$ hence

$$I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) = \mathrm{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}(e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}).$$

Applying Theorem 4.9 to $(P_{\min} \cap M(\sigma), \sigma|_{M_{\min}}, Q\cap M(\sigma))$, the $\mathcal{H}_{M(\sigma),R}$-modules

$$e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}} = (e_{M(\sigma)}(\sigma) \otimes_R \mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}} \tag{4.18}$$

are equal. We have the $\mathcal{H}_R$-isomorphism [OV17, Proposition 4.4]:

$$\begin{aligned} I_G(P, \sigma, Q)^{\mathcal{U}} &= (\mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \mathrm{St}_Q^{P(\sigma)}))^{\mathcal{U}} \\ &\xrightarrow{ov} \mathrm{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}((e(\sigma) \otimes_R \mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}) \\ &f_{P(\sigma)\mathcal{U},x} \mapsto x \otimes 1_{\mathcal{H}} \quad (x \in (e(\sigma) \otimes_R \mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}). \end{aligned}$$

We deduce the following.

Theorem 4.17. *Let (P, σ, Q) be an $R[G]$-triple. Then, we have the $\mathcal{H}_R$-isomorphism*

$$I_G(P, \sigma, Q)^{\mathcal{U}} \xrightarrow{ov} \mathrm{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}(e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}).$$

In particular,

$$I_G(P, \sigma, Q)^{\mathcal{U}} \simeq \begin{cases} I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) & \text{if } P(\sigma) = P(\sigma^{\mathcal{U}_{M_{\min}}}), \\ I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q) & \text{if } P = P_{\min}, P(\sigma) = P(\sigma^{\mathcal{U}_M}). \end{cases}$$

4.4. **Comparison of the parabolic induction and coinduction.** Let $P = MN$ be a standard parabolic subgroup of G, let $\mathcal{V}$ be a right $\mathcal{H}_R$-module, and let Q be a parabolic subgroup of G with $Q \subset P(\mathcal{V})$. When R is an algebraically closed field of characteristic p, in [Abe], we associated to $(P, \mathcal{V}, Q)$ an $\mathcal{H}_R$-module using the parabolic coinduction

$$\mathrm{Coind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(-) = \mathrm{Hom}_{\mathcal{H}_{M_Q^-,\theta^*}}(\mathcal{H}, -) : \mathrm{Mod}_R(\mathcal{H}_{M_Q}) \to \mathrm{Mod}_R(\mathcal{H})$$

instead of the parabolic induction $\mathrm{Ind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(-) = - \bigotimes_{\mathcal{H}_{M_Q^+,\theta}} \mathcal{H}$. The index θ^* in the parabolic coinduction means that $\mathcal{H}_{M_Q^-}$ embeds in $\mathcal{H}$ by θ_Q^*. Our terminology is different from the one in [Abe] where the parabolic coinduction is called induction. For a parabolic subgroup Q' of G with $Q \subset Q' \subset P(\mathcal{V})$, there is a natural inclusion of $\mathcal{H}_R$-modules

$$\mathrm{Hom}_{\mathcal{H}_{M_{Q'}^-,\theta^*}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{i(Q,Q')} \mathrm{Hom}_{\mathcal{H}_{M_Q^-,\theta^*}}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})) \tag{4.19}$$

because $\theta^*(\mathcal{H}_{M_Q^-}) \subset \theta^*(\mathcal{H}_{M_{Q'}^-})$ as $W_{M_Q^-}(1) \subset W_{M_{Q'}^-}(1)$, and $vT_w^{M_{Q'},*} = vT_w^{M_Q,*}$ for $w \in W_{M_Q^-}(1)$ and $v \in \mathcal{V}$. (This is [Abe, Proposition 4.19] when R is an algebraically closed field of characteristic p. This follows from our formulation of the extension for any R.)

Definition 4.18. Let $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ denote the cokernel of the map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \mathrm{Hom}_{\mathcal{H}_{M_{Q'}^-,\theta^*}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Hom}_{\mathcal{H}_{M_Q^-,\theta^*}}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the $\mathcal{H}_R$-embeddings $i(Q, Q')$.

When R is an algebraically closed field of characteristic p, we showed that the $\mathcal{H}_R$-module $CI_{\mathcal{H}}(P,\mathcal{V},Q)$ is simple when $\mathcal{V}$ is simple and supersingular (Definition 4.25), and that any simple $\mathcal{H}_R$-module is of this form for an $\mathcal{H}_R$-triple $(P,\mathcal{V},Q)$ where $\mathcal{V}$ is simple and supersingular, P,Q and the isomorphism class of $\mathcal{V}$ are unique [Abe]. The aim of this section is to compare the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(P,\mathcal{V},Q)$ with the $\mathcal{H}_R$-modules $CI_{\mathcal{H}}(P,\mathcal{V},Q)$ and to show that the classification is also valid with the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(P,\mathcal{V},Q)$.

It is already known that a parabolically coinduced module is a parabolically induced module and vice versa [Abe, Proposition 4.15], [Vig15b, Theorem 1.8]. To make it more precise we need to introduce notation.

We lift the elements w of the finite Weyl group $\mathbb{W}$ to $\hat{w}\in\mathcal{N}_G\cap\mathcal{K}$ as in [AHHV17, IV.6], [OV17, Proposition 2.7]: they satisfy the braid relations $\hat{w}_1\hat{w}_2=(w_1w_2)\hat{}$ when $\ell(w_1)+\ell(w_2)=\ell(w_1w_2)$ and when $s\in S$, $\hat{s}$ is admissible, in particular lies in ${}_1W_{G'}$.

Let $\mathbf{w},\mathbf{w}_M,\mathbf{w}^M$ denote, respectively, the longest elements in $\mathbb{W},\mathbb{W}_M$ and $\mathbf{w}\mathbf{w}_M$. We have $\mathbf{w}=\mathbf{w}^{-1}=\mathbf{w}^M\mathbf{w}_M, \mathbf{w}_M=\mathbf{w}_M^{-1}$, $\hat{\mathbf{w}}=\hat{\mathbf{w}}^M\hat{\mathbf{w}}_M$,

$$\mathbf{w}^M(\Delta_M)=-\mathbf{w}(\Delta_M)\subset\Delta,\quad \mathbf{w}^M(\Phi^+\setminus\Phi^+_M)=\mathbf{w}(\Phi^+\setminus\Phi^+_M).$$

Let $\mathbf{w}.M$ be the standard Levi subgroup of G with $\Delta_{\mathbf{w}.M}=\mathbf{w}^M(\Delta_M)$ and $\mathbf{w}.P$ the standard parabolic subgroup of G with Levi $\mathbf{w}.M$. We have

$$\mathbf{w}.M=\hat{\mathbf{w}}^M M(\hat{\mathbf{w}}^M)^{-1}=\hat{\mathbf{w}}M(\hat{\mathbf{w}})^{-1},\quad \mathbf{w}^{\mathbf{w}.M}=\mathbf{w}_M\mathbf{w}=(\mathbf{w}^M)^{-1}.$$

The conjugation $w\mapsto\mathbf{w}^M w(\mathbf{w}^M)^{-1}$ in W gives a group isomorphism $W_M\to W_{\mathbf{w}.M}$ sending S_M^{aff} onto $S_{\mathbf{w}.M}^{\mathrm{aff}}$, respecting the finite Weyl subgroups $\mathbf{w}^M\mathbb{W}_M(\mathbf{w}^M)^{-1}=\mathbb{W}_{\mathbf{w}.M}=\mathbf{w}\mathbb{W}_M\mathbf{w}^{-1}$, and exchanging W_{M^+} and $W_{(\mathbf{w}.M)^-}=\mathbf{w}W_{M^+}\mathbf{w}^{-1}$. The conjugation by $\tilde{\mathbf{w}}^M$ restricts to a group isomorphism $W_M(1)\to W_{\mathbf{w}.M}(1)$ sending $W_{M^+}(1)$ onto $W_{(\mathbf{w}.M)^-}(1)$. The linear isomorphism

$$\mathcal{H}_M\xrightarrow{\iota(\tilde{\mathbf{w}}^M)}\mathcal{H}_{\mathbf{w}.M}\quad T^M_w\mapsto T^{\mathbf{w}.M}_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}\text{ for }w\in W_M(1),\tag{4.20}$$

is a ring isomorphism between the pro-p-Iwahori Hecke rings of M and $\mathbf{w}.M$. It sends the positive part $\mathcal{H}_{M^+}$ of $\mathcal{H}_M$ onto the negative part $\mathcal{H}_{(\mathbf{w}.M)^-}$ of $\mathcal{H}_{\mathbf{w}.M}$ [Vig15b, Proposition 2.20]. We have $\tilde{\mathbf{w}}=\tilde{\mathbf{w}}_M\tilde{\mathbf{w}}^{\mathbf{w}.M}=\tilde{\mathbf{w}}^M\tilde{\mathbf{w}}_M$, $(\tilde{\mathbf{w}}^M)^{-1}=\tilde{\mathbf{w}}^{\mathbf{w}.M}t_M$ where $t_M=\tilde{\mathbf{w}}^2\tilde{\mathbf{w}}_M^{-2}\in Z_k$.

Definition 4.19. The **twist $\tilde{\mathbf{w}}^M.\mathcal{V}$ of $\mathcal{V}$ by $\tilde{\mathbf{w}}^M$** is the right $\mathcal{H}_{\mathbf{w}.M}$-module deduced from the right $\mathcal{H}_M$-module $\mathcal{V}$ by functoriality: as R-modules $\tilde{\mathbf{w}}^M.\mathcal{V}=\mathcal{V}$ and for $v\in\mathcal{V},w\in W_M(1)$ we have $vT^{\mathbf{w}.M}_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}=vT^M_w$.

We can define the twist $\tilde{\mathbf{w}}^M.\mathcal{V}$ of $\mathcal{V}$ with the $T^{M,*}_w$ instead of T^M_w.

Lemma 4.20. *For $v\in\mathcal{V},w\in W_M(1)$ we have $vT^{\mathbf{w}.M,*}_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}=vT^{M,*}_w$ in $\tilde{\mathbf{w}}^M.\mathcal{V}$.*

Proof. By the ring isomorphism $\mathcal{H}_M\xrightarrow{\iota(\tilde{\mathbf{w}}^M)}\mathcal{H}_{\mathbf{w}.M}$, we have $c^{\mathbf{w}.M}_{\tilde{\mathbf{w}}^M\tilde{s}(\tilde{\mathbf{w}}^M)^{-1}}=c^M_{\tilde{s}}$ when $\tilde{s}\in W_M(1)$ lifts $s\in S^{\mathrm{aff}}_M$. So the equality of the lemma is true for $w=\tilde{s}$. Apply the braid relations to get the equality for all $w\in W_M(1)$. ☐

We return to the $\mathcal{H}_R$-module $\mathrm{Hom}_{\mathcal{H}_{M^-,\theta^*}}(\mathcal{H},V)$ parabolically coinduced from $\mathcal{V}$. It has a natural direct decomposition indexed by the set $\mathbb{W}^{\mathbb{W}_M}$ of elements d in the finite Weyl group $\mathbb{W}$ of minimal length in the coset $d\mathbb{W}_M$. Indeed it is known that the linear map

$$f \mapsto (f(T_{\tilde{d}}))_{d\in\mathbb{W}^{\mathbb{W}_M}} : \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},\mathcal{V}) \to \bigoplus_{d\in\mathbb{W}^{\mathbb{W}_M}} \mathcal{V}$$

is an isomorphism. For $v \in \mathcal{V}$ and $d \in \mathbb{W}^{\mathbb{W}_M}$, there is a unique element

$$f_{\tilde{d},v} \in \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},\mathcal{V}) \ \text{ satisfying } f(T_{\tilde{d}}) = v \text{ and } f(T_{\tilde{d}'}) = 0 \text{ for } d' \in \mathbb{W}^{\mathbb{W}_M} \setminus \{d\}.$$

It is known that the map $v \mapsto f_{\tilde{\mathbf{w}}^M,v} : \tilde{\mathbf{w}}^M.\mathcal{V} \to \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},\mathcal{V})$ is $\mathcal{H}_{(\mathbf{w}.M)^+}$-equivariant: $f_{\tilde{\mathbf{w}}^M,vT_w^{\mathbf{w}.M}} = f_{\tilde{\mathbf{w}}^M,v}T_w$ for all $v \in \mathcal{V}, w \in W_{\mathbf{w}.M^+}(1)$. By adjunction, this $\mathcal{H}_{(\mathbf{w}.M)^+}$-equivariant map gives an $\mathcal{H}_R$-homomorphism from an induced module to a coinduced module:

$$v \otimes 1_{\mathcal{H}} \mapsto f_{\tilde{\mathbf{w}}^M,v} : \tilde{\mathbf{w}}^M.\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^+},\theta} \mathcal{H} \xrightarrow{\mu_P} \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},\mathcal{V}). \tag{4.21}$$

This is an isomorphism [Abe, Proposition 4.15], [Vig15b, Theorem 1.8].

The naive guess that a variant μ_Q of μ_P induces an $\mathcal{H}_R$-isomorphism between the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q)$ and $CI_{\mathcal{H}}(P,\mathcal{V},Q)$ turns out to be true. The proof is the aim of the rest of this section.

The $\mathcal{H}_R$-module $I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q)$ is well defined because the parabolic subgroups of G containing $\mathbf{w}.P$ and contained in $P(\tilde{\mathbf{w}}^M.\mathcal{V})$ are $\mathbf{w}.Q$ for $P \subset Q \subset P(\mathcal{V})$, as follows from Lemma 4.21.

Lemma 4.21. $\Delta_{\tilde{\mathbf{w}}^M.\mathcal{V}} = -\mathbf{w}(\Delta_{\mathcal{V}})$.

Proof. Recall that $\Delta_{\mathcal{V}}$ is the set of simple roots $\alpha \in \Delta \setminus \Delta_M$ orthogonal to Δ_M and $T^M(z)$ acts trivially on $\mathcal{V}$ for all $z \in Z \cap M'_\alpha$, and the corresponding standard parabolic subgroup $P_{\mathcal{V}} = M_{\mathcal{V}}N_{\mathcal{V}}$. The $Z \cap M'_\alpha$ for $\alpha \in \Delta_{\mathcal{V}}$ generate the group $Z \cap M'_{\mathcal{V}}$. A root $\alpha \in \Delta \setminus \Delta_M$ orthogonal to Δ_M is fixed by $\mathbf{w}_M$ so $\mathbf{w}^M(\alpha) = \mathbf{w}(\alpha)$ and

$$\hat{\mathbf{w}}^M M_{\mathcal{V}} (\hat{\mathbf{w}}^M)^{-1} = \hat{\mathbf{w}} M_{\mathcal{V}} (\hat{\mathbf{w}})^{-1}.$$

The proof of Lemma 4.21 is straightforward as $\Delta = -\mathbf{w}(\Delta)$, $\Delta_{\mathbf{w}.M} = -\mathbf{w}(\Delta_M)$. ☐

Before going further, we check the commutativity of the extension with the twist. As $Q = M_QU$ and M_Q determine each other we denote $\mathbf{w}_{M_Q} = \mathbf{w}_Q, \mathbf{w}^{M_Q} = \mathbf{w}^Q$ when $Q \neq P, G$.

Lemma 4.22. $e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V}) = \tilde{\mathbf{w}}^Q.e_{\mathcal{H}_Q}(\mathcal{V})$.

Proof. As R-modules $\mathcal{V} = e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V}) = \tilde{\mathbf{w}}^Q.e_{\mathcal{H}_Q}(\mathcal{V})$. A direct computation shows that the Hecke element $T_w^{\mathbf{w}.Q,*}$ acts in the $\mathcal{H}_R$-module $e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V})$, by the identity if $w \in \tilde{\mathbf{w}}^Q{}_1W_{M'_2}(\mathbf{w}^Q)^{-1}$ and by $T^{M,*}_{(\tilde{\mathbf{w}}^Q)^{-1}w\tilde{\mathbf{w}}^Q}$ if $w \in \tilde{\mathbf{w}}^Q{}_1W_{M'_2}(\mathbf{w}^Q)^{-1}$ where M_2 denotes the standard Levi subgroup with $\Delta_{M_2} = \Delta_Q \setminus \Delta_P$. Whereas in the $\mathcal{H}_R$-module $\tilde{\mathbf{w}}^Q.e_{\mathcal{H}_Q}(\mathcal{V})$, the Hecke element $T_w^{\mathbf{w}.Q,*}$ acts by the identity if $w \in {}_1W_{\mathbf{w}.M'_2}$ and by $T^{M,*}_{(\tilde{\mathbf{w}}^M)^{-1}w\tilde{\mathbf{w}}^M}$ if $w \in W_{\mathbf{w}.M}(1)$. So the lemma means that

$${}_1W_{\mathbf{w}.M'_2} = \tilde{\mathbf{w}}^Q{}_1W_{M'_2}(\mathbf{w}^Q)^{-1}, \quad (\tilde{\mathbf{w}}^Q)^{-1}w\tilde{\mathbf{w}}^Q = (\tilde{\mathbf{w}}^M)^{-1}w\tilde{\mathbf{w}}^M \text{ if } w \in W_{\mathbf{w}.M}(1).$$

These properties are easily proved using that ${}_1W_{G'}$ is normal in $W(1)$ and that the sets of roots Δ_P and $\Delta_Q \setminus \Delta_P$ are orthogonal: $\mathbf{w}_Q = \mathbf{w}_{M_2}\mathbf{w}_M$, the elements $\mathbf{w}_{M_2}$ and $\mathbf{w}_M$ normalize W_M and W_{M_2}, the elements of $\mathbb{W}_{M_2}$ commutes with the elements of $\mathbb{W}_M$. ☐

We return to our guess. The variant μ_Q of μ_P is obtained by combining the commutativity of the extension with the twist and the isomorphism (4.21) applied to $(Q, e_{\mathcal{H}_Q}(\mathcal{V}))$ instead of $(P, \mathcal{V})$. The $\mathcal{H}_R$-isomorphism μ_Q is

$$v \otimes 1_{\mathcal{H}} \mapsto f_{\tilde{\mathbf{w}}^M, v} : \mathrm{Ind}_{\mathcal{H}_{\mathbf{w}.M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V})) \xrightarrow{\mu_Q} \mathrm{Hom}_{\mathcal{H}_{M_Q^-}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})). \tag{4.22}$$

Our guess is that μ_Q induces an $\mathcal{H}_R$-isomorphism from the cokernel of the $\mathcal{H}_R$-map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \mathrm{Ind}_{\mathcal{H}_{\mathbf{w}.Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_{\mathbf{w}.Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V}))$$

defined by the $\mathcal{H}_R$-embeddings $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$, isomorphic to $I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M \mathcal{V}, \mathbf{w}.\overline{Q})$ via $\kappa_{\mathbf{w}.Q}$ (Theorem 4.14), onto the cokernel $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ the $\mathcal{H}_R$-map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \mathrm{Hom}_{\mathcal{H}_{M_{Q'}^-}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Hom}_{\mathcal{H}_{M_Q^-}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the $\mathcal{H}_R$-embeddings $i(Q, Q')$. This is true if $i(Q, Q')$ corresponds to $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$ via the isomorphisms $\mu_{Q'}$ and μ_Q. This is the content of the next proposition.

Proposition 4.23. *For all $Q \subsetneq Q' \subset P(\mathcal{V})$ we have*

$$i(Q, Q') \circ \mu_{Q'} = \mu_Q \circ \iota(\mathbf{w}.Q, \mathbf{w}.Q').$$

We postpone to section 4.6 the rather long proof of the proposition.

Corollary 4.24. *The $\mathcal{H}_R$-isomorphism $\mu_Q \circ \kappa_{\mathbf{w}.Q}^{-1}$ induces an $\mathcal{H}_R$-isomorphism*

$$I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M \mathcal{V}, \mathbf{w}.\overline{Q}) \to CI_{\mathcal{H}}(P, \mathcal{V}, Q).$$

4.5. **Supersingular $\mathcal{H}_R$-modules, classification of simple $\mathcal{H}_R$-modules.** We recall first the notion of supersingularity based on the action of the center of $\mathcal{H}$.

The center of $\mathcal{H}$ [Vig14, Theorem 1.3] contains a subalgebra $\mathcal{Z}_{T^+}$ isomorphic to $\mathbb{Z}[T^+/T_1]$ where T^+ is the monoid of dominant elements of T and T_1 is the pro-p-Sylow subgroup of the maximal compact subgroup of T.

Let $t \in T$ of image $\mu_t \in W(1)$ and let $(E_o(w))_{w \in W(1)}$ denote the alcove walk basis of $\mathcal{H}$ associated to a closed Weyl chamber o of $\mathbb{W}$. The element

$$E_o(C(\mu_t)) = \sum_{\mu'} E_o(\mu')$$

is the sum over the elements in μ' in the conjugacy class $C(\mu_t)$ of μ_t in $W(1)$. It is a central element of $\mathcal{H}$ and does not depend on the choice of o. We write also $z(t) = E_o(C(\mu_t))$.

Definition 4.25. A non-zero right $\mathcal{H}_R$-module $\mathcal{V}$ is called supersingular when, for any $v \in \mathcal{V}$ and any non-invertible $t \in T^+$, there exists a positive integer $n \in \mathbb{N}$ such that $v(z(t))^n = 0$. If one can choose n independent on (v, t), then $\mathcal{V}$ is called uniformly supersingular.

Remark 4.26. One can choose n independent on (v, t) when $\mathcal{V}$ is finitely generated as a right $\mathcal{H}_R$-module. If R is a field and $\mathcal{V}$ is simple we can take $n = 1$.

When G is compact modulo the center, $T^+ = T$, and any non-zero $\mathcal{H}_R$-module is supersingular.

The induction functor $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}: \mathrm{Mod}(\mathcal{H}_{M,R}) \to \mathrm{Mod}(\mathcal{H}_R)$ has a left adjoint $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}$ and a right adjoint $\mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}$ [Vig15b]: for $\mathcal{V} \in \mathrm{Mod}(\mathcal{H}_R)$,
(4.23)

$$\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \tilde{\mathbf{w}}^{\mathbf{w}.M} \circ (\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M}), \quad \mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, \mathcal{V}).$$

In the left adjoint, $\mathcal{V}$ is seen as a right $\mathcal{H}_{(\mathbf{w}.M)^-}$-module via the ring homomorphism $\theta^*_{\mathbf{w}.M}: \mathcal{H}_{(\mathbf{w}.M)^-} \to \mathcal{H}$; in the right adjoint, $\mathcal{V}$ is seen as a right $\mathcal{H}_{M^+}$-module via the ring homomorphism $\theta_M: \mathcal{H}_{M^+} \to \mathcal{H}$ (section 2.3).

Proposition 4.27. *Assume that $\mathcal{V}$ is a supersingular right $\mathcal{H}_R$-module and that p is nilpotent in $\mathcal{V}$. Then $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$, and if $\mathcal{V}$ is uniformly supersingular $\mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$.*

Proof. This is a consequence of three known properties:

(1) $\mathcal{H}_M$ is the localization of $\mathcal{H}_{M^+}$ (resp., $\mathcal{H}_{M^-}$) at T_μ^M for any element $\mu \in \Lambda_T(1)$, central in $W_M(1)$ and strictly N-positive (resp., N-negative), and $T_\mu^M = T_\mu^{M,*}$. See [Vig15b, Theorem 1.4].
(2) When o is anti-dominant, $E_o(\mu) = T_\mu$ if $\mu \in \Lambda^+(1)$ and $E_o(\mu) = T^*_\mu$ if $\mu \in \Lambda^-(1)$.
(3) Let an integer $n > 0$ and $\mu \in \Lambda(1)$ such that the $\mathbb{W}$-orbit of $v(\mu) \in X_*(T) \otimes \mathbb{Q}$ (definition in section 2.1) and of μ have the same number of elements. Then

$$(E_o(C(\mu)))^n E_o(\mu) - E_o(\mu)^{n+1} \in p\mathcal{H}.$$

See [Vig15a, Lemma 6.5], where the hypotheses are given in the proof (but not written in the lemma).

Let $\mu \in \Lambda_T^+(1)$ satisfying (1) for M^+ and (3), similarly let $\mathbf{w}.\mu \in \Lambda_T^-(1)$ satisfying (1) for $(\mathbf{w}.M)^-$ and (3). For $(R, \mathcal{V})$ as in the proposition, let $v \in \mathcal{V}$ and $n > 0$ such that $vE_o(C(\mu))^n = vE_o(C(\mathbf{w}.\mu))^n = 0$. Multiplying by $E_o(\mu)$ or $E_o(\mathbf{w}.\mu)$, and applying (3) and (2) for o anti-dominant we get:

$$vE_o(\mu^{n+1}) = vT_\mu^{n+1} \in p\mathcal{V}, \quad vE_o((\mathbf{w}.\mu)^{n+1}) = v(T^*_{\mathbf{w}.\mu})^{n+1} \in p\mathcal{V}.$$

The proposition follows from: $vT_\mu^{n+1}, v(T^*_{\mathbf{w}.\mu})^{n+1}$ in $p\mathcal{V}$ (as explained in [Abe16, Proposition 5.17] when $p = 0$ in R). From $v(T^*_{\mathbf{w}.\mu})^{n+1}$ in $p\mathcal{V}$, we get $v \otimes (T^{\mathbf{w}.M,*}_{\mathbf{w}.\mu})^{n+1}$ $= v(T^*_{\mathbf{w}.\mu})^{n+1} \otimes 1_{\mathcal{H}_{\mathbf{w}.M}}$ in $p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M}$. As $T^{\mathbf{w}.M,*} = T^{\mathbf{w}.M}$ is invertible in $\mathcal{H}_{\mathbf{w}.M}$ we get $v \otimes 1_{\mathcal{H}_{\mathbf{w}.M}}$ in $p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M}$. As v was arbitrary, $\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M} \subset p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M}$. If p is nilpotent in $\mathcal{V}$, then $\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M} = 0$. Suppose now that there exists $n > 0$ such that $\mathcal{V}(z(t))^n = 0$ for any non-invertible $t \in T^+$; then $\mathcal{V}T_\mu^{n+1} \subset p\mathcal{V}$ where $\mu = \mu_t$ and hence $\varphi(h) = \varphi(hT^M_{\mu^{-n-1}})T_\mu^{n+1}$ in $p\mathcal{V}$ for an arbitrary $\varphi \in \mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, \mathcal{V})$ and an arbitrary $h \in \mathcal{H}_M$. We deduce $\mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, \mathcal{V}) \subset \mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, p\mathcal{V})$. If p is nilpotent in $\mathcal{V}$, then $\mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, \mathcal{V}) = 0$. ☐

Recalling that $\tilde{\mathbf{w}}^M.\mathcal{V}$ is obtained by functoriality from $\mathcal{V}$ and the ring isomorphism $\iota(\tilde{\mathbf{w}}^M)$ defined in (4.20), the equivalence between $\mathcal{V}$ supersingular and $\tilde{\mathbf{w}}^M\mathcal{V}$ supersingular follows from Lemma 4.28

Lemma 4.28.

(1) *Let $t \in T$. Then t is dominant for U_M if and only if $\hat{\mathbf{w}}^M t(\hat{\mathbf{w}}^M)^{-1} \in T$ is dominant for $U_{\mathbf{w}.M}$.*

(2) *The R-algebra isomorphism* $\mathcal{H}_{M,R} \xrightarrow{\iota(\tilde{\mathbf{w}}^M)} \mathcal{H}_{\mathbf{w}.M,R}$, $T^M_w \mapsto T^{\mathbf{w}.M}_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}$ *for* $w \in W_M(1)$ *sends* $z^M(t)$ *to* $z^{\mathbf{w}.M}(\hat{\mathbf{w}}^M t(\hat{\mathbf{w}}^M)^{-1})$ *for* $t \in T$ *dominant for* U_M.

Proof. The conjugation by $\hat{\mathbf{w}}^M$ stabilizes T, sends U_M to $U_{\mathbf{w}.M}$, and sends the $\mathbb{W}_M$-orbit of $t \in T$ to the $\mathbb{W}_{\mathbf{w}.M}$-orbit of $\hat{\mathbf{w}}^M t(\hat{\mathbf{w}}^M)^{-1}$, as $\mathbf{w}^M \mathbb{W}_M (\mathbf{w}^M)^{-1} = \mathbb{W}_{\mathbf{w}.M}$. It is known that $\iota(\tilde{\mathbf{w}}^M)$ respects the anti-dominant alcove walk bases [Vig15b, Proposition 2.20]: it sends $E^M(w)$ to $E^{\mathbf{w}.M}(\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1})$ for $w \in W_M(1)$. □

We deduce the following.

Corollary 4.29. *Let* $\mathcal{V}$ *be a right* $\mathcal{H}_{M,R}$*-module. Then* $\mathcal{V}$ *is supersingular if and only if the right* $\mathcal{H}_{\mathbf{w}.M,R}$*-module* $\tilde{\mathbf{w}}^M\mathcal{V}$ *is supersingular.*

Assume R is an algebraically close field of characteristic p. The supersingular simple $\mathcal{H}_{M,R}$-modules are classified in [Vig15a]. By Corollaries 4.24 and 4.29, the classification of the simple $\mathcal{H}_R$-modules in [Abe] remains valid with the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(P,\mathcal{V},Q)$ instead of $CI_{\mathcal{H}}(P,\mathcal{V},Q)$:

Corollary 4.30 (Classification of simple $\mathcal{H}_R$-modules)**.** *Assume* R *is an algebraically closed field of characteristic* p*. Let* $(P,\mathcal{V},Q)$ *be an* $\mathcal{H}_R$*-triple where* $\mathcal{V}$ *is simple and supersingular. Then, the* $\mathcal{H}_R$*-module* $I_{\mathcal{H}}(P,\mathcal{V},Q)$ *is simple. A simple* $\mathcal{H}_R$*-module is isomorphic to* $I_{\mathcal{H}}(P,\mathcal{V},Q)$ *for an* $\mathcal{H}_R$*-triple* $(P,\mathcal{V},Q)$ *where* $\mathcal{V}$ *is simple and supersingular,* P, Q *and the isomorphism class of* $\mathcal{V}$ *are unique.*

4.6. **A commutative diagram.** We prove in this section Proposition 4.23. For $Q \subset Q' \subset P(\mathcal{V})$ we show by an explicit computation that

$$\mu_Q^{-1} \circ i(Q,Q') \circ \mu_{Q'}\ \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_{\mathbf{w}.Q'}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V})) \to \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_{\mathbf{w}.Q}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V}))$$

is equal to $\iota(\mathbf{w}.Q,\mathbf{w}.Q')$. The R-module $e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V}) \otimes 1_{\mathcal{H}}$ generates the $\mathcal{H}_R$-module $e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V}) \otimes_{\mathcal{H}_{\mathbf{w}.Q',R},\theta^+} \mathcal{H}_R = \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_{\mathbf{w}.Q'}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V}))$ and by (4.17)

$$\iota(\mathbf{w}.Q,\mathbf{w}.Q')(v \otimes 1_{\mathcal{H}}) = v \otimes \sum_{d \in {}^{\mathbb{W}_{M_{\mathbf{w}.Q}}}\mathbb{W}_{M_{\mathbf{w}.Q'}}} T_{\tilde{d}} \tag{4.24}$$

for $v \in \mathcal{V}$ seen as an element of $e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V})$ in the LHS and an element of $e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V})$ in the RHS.

Lemma 4.31. $(\mu_Q^{-1} \circ i(Q,Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}}) = v \otimes \sum_{d \in \mathbb{W}^{\mathbb{W}_{M_Q}}_{M_{Q'}}} q_d T^*_{\tilde{\mathbf{w}}^Q(\tilde{\mathbf{w}}^{Q'}\tilde{d})^{-1}}$.

Proof. $\mu_{Q'}(v \otimes 1_{\mathcal{H}})$ is the unique homomorphism $f_{\tilde{\mathbf{w}}^{M_{Q'}},v} \in \mathrm{Hom}_{\mathcal{H}_{M^-_{Q'}},\theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ sending $T_{\tilde{\mathbf{w}}^{Q'}}$ to v and vanishing on $T_{\tilde{d}'}$ for $d' \in \mathbb{W}^{\mathbb{W}_{M_{Q'}}} \setminus \{\mathbf{w}^{Q'}\}$ by (4.22). By (4.19), $i(Q,Q')$ is the natural embedding of $\mathrm{Hom}_{\mathcal{H}_{M^-_{Q'}},\theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ in $\mathrm{Hom}_{\mathcal{H}_{M^-_{Q}},\theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q}}(\mathcal{V}))$ therefore $i(Q,Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}},v})$ is the unique homomorphism $\mathrm{Hom}_{\mathcal{H}_{M^-_{Q}},\theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q}}(\mathcal{V}))$ sending $T_{\tilde{\mathbf{w}}^{Q'}}$ to v and vanishing on $T_{\tilde{d}'}$ for $d' \in \mathbb{W}^{\mathbb{W}_{M_{Q'}}} \setminus \{\mathbf{w}^{Q'}\}$. As $\mathbb{W}^{\mathbb{W}_{M_Q}} = \mathbb{W}^{\mathbb{W}_{Q'}} \mathbb{W}^{\mathbb{W}_{M_Q}}_{M_{Q'}}$, this homomorphism vanishes on $T_{\tilde{w}}$ for w not in $\mathbf{w}^{M_{Q'}}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. By [Abe16, Lemma 2.22], the inverse of μ_Q is the $\mathcal{H}_R$-isomorphism

$$\text{(4.25)}\qquad \mathrm{Hom}_{\mathcal{H}_{M_Q^-},\theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})) \xrightarrow{\mu_Q^{-1}} \mathrm{Ind}_{\mathcal{H}_{\mathbf{w}.M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V}))$$
$$f \mapsto \sum_{d\in \mathbb{W}^{\mathbb{W}_M}} f(T_{\tilde d}) \otimes T^*_{\tilde{\mathbf{w}}^M \tilde d^{-1}},$$

where $\mathbb{W}^{\mathbb{W}_M}$ is the set of $d \in \mathbb{W}$ with minimal length in the coset $d\mathbb{W}_M$. We deduce the explicit formula

$$(\mu_Q^{-1}\circ i(Q,Q')\circ \mu_{Q'})(v\otimes 1_{\mathcal{H}}) = \sum_{w\in \mathbb{W}^{\mathbb{W}_{M_Q}}} i(Q,Q')(f^{Q'}_{\tilde{\mathbf{w}}^{M_{Q'}},v})(T_{\tilde w})\otimes T^*_{\tilde{\mathbf{w}}^{M_Q}\tilde w^{-1}}.$$

Some terms are zero: the terms for $w \in \mathbb{W}^{\mathbb{W}_{M_Q}}$ not in $\mathbf{w}^{M_{Q'}}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. We analyze the other terms for w in $\mathbb{W}^{\mathbb{W}_{M_Q}} \cap \mathbf{w}^{M_{Q'}}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$; this set is $\mathbf{w}^{M_{Q'}}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. Let $w = \mathbf{w}^{M_{Q'}}d, d\in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$, and $\tilde w = \tilde{\mathbf{w}}^{M_{Q'}}\tilde d$ with $\tilde d \in {}_1W_{G'}$ lifting d. By the braid relations $T_{\tilde w} = T_{\tilde{\mathbf{w}}^{M_{Q'}}}T_{\tilde d}$. We have $T_{\tilde d} = \theta^*(T^{M_{Q'}}_{\tilde d})$ by the braid relations because $d\in \mathbb{W}_{M_{Q'}}$, $S_{M_{Q'}} \subset S^{\text{aff}}$ and $\theta^*(c^{M_{Q'}}_{\tilde s}) = c_{\tilde s}$ for $s \in S_{M_{Q'}}$. As $\mathbb{W}_{M_{Q'}} \subset W_{M^-_{Q'}} \cap W_{M^+_{Q'}}$, we deduce:

$$\begin{aligned} i(Q,Q')(f^{Q'}_{\tilde{\mathbf{w}}^{M_{Q'}},v})(T_{\tilde w}) &= i(Q,Q')(f^{Q'}_{\tilde{\mathbf{w}}^{M_{Q'}},v})(T_{\tilde{\mathbf{w}}^{M_{Q'}}}T_{\tilde d}) \\ &= i(Q,Q')(f^{Q'}_{\tilde{\mathbf{w}}^{M_{Q'}},v})(T_{\tilde{\mathbf{w}}^{M_{Q'}}})T^{M_{Q'}}_{\tilde d} \\ &= vT^{M_{Q'}}_{\tilde d} = q_d v. \end{aligned}$$

Corollary 3.9 gives the last equality. □

The formula for $(\mu_Q^{-1}\circ i(Q,Q')\circ\mu_{Q'})(v\otimes 1_{\mathcal{H}})$ given in Lemma 4.31 is different from the formula (4.24) for $\iota(\mathbf{w}.Q,\mathbf{w}.Q')(v\otimes 1_{\mathcal{H}})$. It needs some work to prove that they are equal.

A first reassuring remark is that ${}^{\mathbb{W}_{M_{\mathbf{w}.Q}}}\mathbb{W}_{M_{\mathbf{w}.Q'}} = \{\mathbf{w}d^{-1}\mathbf{w} \mid d\in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\}$, so the two summation sets have the same number of elements. But better,

$${}^{\mathbb{W}_{M_{\mathbf{w}.Q}}}\mathbb{W}_{M_{\mathbf{w}.Q'}} = \{\mathbf{w}^Q(\mathbf{w}^{Q'}d)^{-1} \mid d\in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\}$$

because $\mathbf{w}_{Q'}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\mathbf{w}_Q = \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. To prove the latter equality, we apply the criterion: $w\in \mathbb{W}_{M_{Q'}}$ lies in ${\mathbb{W}_{M_{Q'}}}^{\mathbb{W}_{M_Q}}$ if and only if $w(\alpha) > 0$ for all $\alpha\in\Delta_Q$ noticing that $d\in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ implies $\mathbf{w}_Q(\alpha)\in -\Delta_Q$, $d\mathbf{w}_Q(\alpha) \in -\Phi_{M_{Q'}}$, $\mathbf{w}_{Q'}d\mathbf{w}_Q(\alpha) > 0$. Let $x_d = \mathbf{w}^Q(\mathbf{w}^{Q'}d)^{-1}$. We have $\tilde{\mathbf{w}}^{M_Q}(\tilde{\mathbf{w}}^{M_{Q'}}\tilde d)^{-1} = \tilde x_d$ because the lifts $\tilde w$ of the elements $w\in\mathbb{W}$ satisfy the braid relations and $\ell(x_d) = \ell(\mathbf{w}_Q d^{-1}\mathbf{w}_{Q'}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q d^{-1}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q) - \ell(d^{-1}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q) - \ell(d) = -\ell(\mathbf{w}^{Q'}) + \ell(\mathbf{w}^Q) - \ell(d)$. We have $q_d = q_{\mathbf{w}_{\mathbf{w}.Q}x_d\mathbf{w}_{\mathbf{w}.Q'}}$ because $\mathbf{w}d^{-1}\mathbf{w} = \mathbf{w}_{\mathbf{w}.Q}x_d\mathbf{w}_{\mathbf{w}.Q'}$, and $q_d = q_{d^{-1}} = q_{\mathbf{w}d^{-1}\mathbf{w}}$. So

$$\sum_{d\in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}} q_d T^*_{\tilde{\mathbf{w}}^Q(\tilde{\mathbf{w}}^{Q'}\tilde d)^{-1}} = \sum_{x_d\in {}^{\mathbb{W}_{M_{\mathbf{w}.Q}}}\mathbb{W}_{M_{\mathbf{w}.Q'}}} q_{\mathbf{w}_{\mathbf{w}.Q}x_d\mathbf{w}_{\mathbf{w}.Q'}} T^*_{\tilde x_d}.$$

In the RHS, only $\tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q, \mathbf{w}.Q'$ appear. The same holds true in the formula (4.24). The map $(P, \mathcal{V}, Q, Q') \mapsto (\mathbf{w}.P, \tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q, \mathbf{w}.Q')$ is a bijection of the set of triples $(P, \mathcal{V}, Q, Q')$ where $P = MN, Q, Q'$ are standard parabolic subgroups of G, $\mathcal{V}$ a right $\mathcal{H}_R$-module, $Q \subset Q' \subset P(\mathcal{V})$ by Lemma 4.21. So we can replace $(\mathbf{w}.P, \tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q, \mathbf{w}.Q')$ by $(P, \mathcal{V}, Q, Q')$. Our task is reduced to prove in $e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+}, \theta} \mathcal{H}_R$:

$$v \otimes \sum_{d \in {}^{\mathbb{W}_{M_Q}}\mathbb{W}_{M_{Q'}}} T_{\tilde{d}} = v \otimes \sum_{d \in {}^{\mathbb{W}_{M_Q}}\mathbb{W}_{M_{Q'}}} q_{\mathbf{w}_Q d \mathbf{w}_{Q'}} T^*_{\tilde{d}}. \tag{4.26}$$

A second simplification is possible: we can replace $Q \subset Q'$ by the standard parabolic subgroups $Q_2 \subset Q_2'$ of G with $\Delta_{Q_2} = \Delta_Q \setminus \Delta_P$ and $\Delta_{Q_2'} = \Delta_{Q'} \setminus \Delta_P$, because Δ_P and $\Delta_{P(\mathcal{V})} \setminus \Delta_P$ are orthogonal. Indeed, $\mathbb{W}_{M_{Q'}} = \mathbb{W}_M \times \mathbb{W}_{M_{Q_2'}}$ and $\mathbb{W}_{M_Q} = \mathbb{W}_M \times \mathbb{W}_{M_{Q_2}}$ are direct products, the longest elements $\mathbf{w}_{Q'} = \mathbf{w}_M \mathbf{w}_{Q_2'}, \mathbf{w}_Q = \mathbf{w}_M \mathbf{w}_{Q_2}$ are direct products and

$${}^{\mathbb{W}_{M_Q}}\mathbb{W}_{M_{Q'}} = {}^{\mathbb{W}_{M_{Q_2}}}\mathbb{W}_{M_{Q_2'}}, \quad \mathbf{w}_Q d \mathbf{w}_{Q'} = \mathbf{w}_{Q_2} d \mathbf{w}_{Q_2'}.$$

Once this is done, we use the properties of $e_{\mathcal{H}_Q}(\mathcal{V})$: $vh \otimes 1_{\mathcal{H}} = v \otimes \theta_Q(h)$ for $h \in \mathcal{H}_{M_{Q_2}^+}$, and $T_w^{Q,*}$ acts trivially on $e_{\mathcal{H}_Q}(\mathcal{V})$ for $w \in {}_1W_{M'_{Q_2}} \cup (\Lambda(1) \cap {}_1W_{M'_{Q_2'}})$. Set ${}_1\mathbb{W}_{M'_{Q_2'}} = \{w \in {}_1W_{M'_{Q_2'}} \mid w \text{ is a lift of some element in } \mathbb{W}_{M_{Q_2'}}\}$ and ${}_1\mathbb{W}_{M'_{Q_2}}$ similarly. Then $Z_k \cap {}_1\mathbb{W}_{M'_{Q_2'}} \subset (\Lambda(1) \cap {}_1W_{M'_{Q_2'}}) \cap {}_1W_{M_{Q_2}^+}$ and ${}_1\mathbb{W}_{M'_{Q_2}} \subset {}_1W_{M'_{Q_2}} \cap {}_1W_{M_{Q_2}^+}$. This implies that (4.26) where $Q \subset Q'$ has been replaced by $Q_2 \subset Q_2'$ follows from a congruence

$$\sum_{d \in {}^{\mathbb{W}_{M_{Q_2}}}\mathbb{W}_{M_{Q_2'}}} T_{\tilde{d}} \equiv \sum_{d \in {}^{\mathbb{W}_{M_{Q_2}}}\mathbb{W}_{M_{Q_2'}}} q_{\mathbf{w}_{Q_2} d \mathbf{w}_{Q_2'}} T^*_{\tilde{d}} \tag{4.27}$$

in the finite subring $H({}_1\mathbb{W}_{M_{Q_2'}})$ of $\mathcal{H}$ generated by $\{T_w \mid w \in {}_1\mathbb{W}_{M'_{Q_2'}}\}$ modulo the right ideal $\mathcal{J}_2$ with generators $\{\theta_Q(T_w^{Q,*}) - 1 \mid w \in (Z_k \cap {}_1\mathbb{W}_{M'_{Q_2'}}) \cup {}_1\mathbb{W}_{M'_{Q_2}}\}$.

Another simplification concerns $T^*_{\tilde{d}}$ modulo $\mathcal{J}_2$ for $d \in \mathbb{W}_{M_{Q_2'}}$. We recall that for any reduced decomposition $d = s_1 \dots s_n$ with $s_i \in S \cap \mathbb{W}_{M_{Q_2'}}$ we have $T^*_{\tilde{d}} = (T_{\tilde{s}_1} - c_{\tilde{s}_1}) \dots (T_{\tilde{s}_n} - c_{\tilde{s}_n})$ where the $\tilde{s}_i$ are admissible. For $\tilde{s}$ admissible, by (3.2)

$$c_{\tilde{s}} \equiv q_s - 1.$$

Therefore

$$T^*_{\tilde{d}} \equiv (T_{\tilde{s}_1} - q_{s_1} + 1) \cdots (T_{\tilde{s}_n} - q_{s_n} + 1).$$

Let $\mathcal{J}' \subset \mathcal{J}_2$ be the ideal of $H({}_1\mathbb{W}_{M'_{Q_2'}})$ generated by $\{T_t - 1 \mid t \in Z_k \cap {}_1W_{M'_{Q_2'}}\}$. Then the ring $H({}_1\mathbb{W}_{M'_{Q_2'}})/\mathcal{J}'$ and its right ideal $\mathcal{J}_2/\mathcal{J}'$ are the specialization of the generic finite ring $H(\mathbb{W}_{M_{Q_2'}})^g$ over $\mathbb{Z}[(q_s)_{s \in S_{M_{Q_2'}}}]$ where the q_s for $s \in S_{M_{Q_2'}} = S \cap \mathbb{W}_{M_{Q_2'}}$ are indeterminates, and of its right ideal $\mathcal{J}_2^g$ with the same generators. The similar congruence modulo $\mathcal{J}_2^g$ in $H(\mathbb{W}_{M_{Q_2'}})^g$ (the generic congruence) implies the congruence (4.27) by specialization.

We will prove the generic congruence in a more general setting where H is the generic Hecke ring of a finite Coxeter system$(\mathbb{W}, S)$ and parameters $(q_s)_{s \in S}$ such that $q_s = q_{s'}$ when s, s' are conjugate in $\mathbb{W}$. The Hecke ring H is a $\mathbb{Z}[(q_s)_{s \in S}]$-free module of basis $(T_w)_{w\in\mathbb{W}}$ satisfying the braid relations and the quadratic relations $T_s^2 = q_s + (q_s-1)T_s$ for $s\in S$. The other basis $(T^*_w)_{w\in\mathbb{W}}$ satisfies the braid relations and the quadratic relations $(T^*_s)^2 = q_s-(q_s-1)T^*_s$ for $s\in S$, and is related to the first basis by $T^*_s = T_s-(q_s-1)$ for $s\in S$, and more generally $T_wT^*_{w^{-1}} = T^*_{w^{-1}}T_w = q_w$ for $w\in\mathbb{W}$ [Vig16, Proposition 4.13].

Let $J\subset S$ and $\mathcal{J}$ is the right ideal of H with generators T^*_w-1 for all w in the group $\mathbb{W}_J$ generated by J.

Lemma 4.32. *A basis of $\mathcal{J}$ is $(T^*_{w_1}-1)T^*_{w_2}$ for $w_1\in\mathbb{W}_J\setminus\{1\}, w_2\in {}^{\mathbb{W}_J}\mathbb{W}$, and adding $T^*_{w_2}$ for $w_2\in {}^{\mathbb{W}_J}\mathbb{W}$ gives a basis of H. In particular, $\mathcal{J}$ is a direct factor of $\mathcal{H}$.*

Proof. The elements $(T^*_{w_1}-1)T^*_w$ for $w_1\in\mathbb{W}_J, w\in\mathbb{W}$ generate $\mathcal{J}$. We write $w=u_1w_2$ with unique elements $u_1\in\mathbb{W}_J, w_2\in {}^{\mathbb{W}_J}\mathbb{W}$, and $T^*_w = T^*_{u_1}T^*_{w_2}$. Therefore, $(T^*_{w_1}-1)T^*_{u_1}T^*_{w_2}$. By an induction on the length of u_1, one proves that $(T^*_{w_1}-1)T^*_{u_1}$ is a linear combination of $(T^*_{v_1}-1)$ for $v_1\in\mathbb{W}_J$ as in the proof of Proposition 3.4. It is clear that the elements $(T^*_{w_1}-1)T^*_{w_2}$ and $T^*_{w_2}$ for $w_1\in\mathbb{W}_J\setminus\{1\}, w_2\in {}^{\mathbb{W}_J}\mathbb{W}$ form a basis of H. ☐

Let $\mathbf{w}_J$ denote the longest element of $\mathbb{W}_J$ and $\mathbf{w}=\mathbf{w}_S$.

Lemma 4.33. *In the generic Hecke ring H, the congruence modulo $\mathcal{J}$*

$$\sum_{d\in {}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv \sum_{d\in {}^{\mathbb{W}_J}\mathbb{W}} q_{\mathbf{w}_J d\mathbf{w}}T^*_d$$

holds true.

Proof.

Step 1. We show

$${}^{\mathbb{W}_J}\mathbb{W} = \mathbf{w}_J\,{}^{\mathbb{W}_J}\mathbb{W}\,\mathbf{w},\quad q_{\mathbf{w}_J}q_{\mathbf{w}_J d\mathbf{w}}T^*_d = T_{\mathbf{w}_J}T_{\mathbf{w}_J d\mathbf{w}}T^*_{\mathbf{w}}.$$

The equality between the groups follows from the characterization of ${}^{\mathbb{W}_J}\mathbb{W}$ in $\mathbb{W}$: an element $d\in\mathbb{W}$ has minimal length in $\mathbb{W}_J d$ if and only if $\ell(ud)=\ell(u)+\ell(d)$ for all $u\in\mathbb{W}_J$. An easy computation shows that $\ell(u\mathbf{w}_J d\mathbf{w}) = \ell(u)+\ell(\mathbf{w}_J d\mathbf{w})$ for all $u\in\mathbb{W}_J, d\in {}^{\mathbb{W}_J}\mathbb{W}$ (both sides are equal to $\ell(u)+\ell(\mathbf{w})-\ell(\mathbf{w}_J)-\ell(d)$). The second equality follows from $q_{\mathbf{w}_J}q_{\mathbf{w}_J d\mathbf{w}} = q_{d\mathbf{w}}$ because $(\mathbf{w}_J)^2=1$ and $\ell(\mathbf{w}_J)+\ell(\mathbf{w}_J d\mathbf{w}) = \ell(d\mathbf{w})$ (both sides are $\ell(\mathbf{w})-\ell(d)$) and from $q_{d\mathbf{w}}T^*_d = T_{d\mathbf{w}}T^*_{\mathbf{w}d^{-1}}T^*_d = T_{d\mathbf{w}}T^*_{\mathbf{w}}$. We also have $T_{d\mathbf{w}} = T_{\mathbf{w}_J}T_{\mathbf{w}_J d\mathbf{w}}$.

Step 2. The multiplication by $q_{\mathbf{w}_J}$ on the quotient $H/\mathcal{J}$ is injective (Lemma 4.32) and $q_{\mathbf{w}_J}\equiv T_{\mathbf{w}_J}$. By Step 1, $q_{\mathbf{w}_J d\mathbf{w}}T^*_d \equiv T_{\mathbf{w}_J d\mathbf{w}}T^*_{\mathbf{w}}$ and

$$\sum_{d\in {}^{\mathbb{W}_J}\mathbb{W}} q_{\mathbf{w}_J d\mathbf{w}}T^*_d \equiv \sum_{d\in {}^{\mathbb{W}_J}\mathbb{W}} T_dT^*_{\mathbf{w}}.$$

The congruence

$$\sum_{d\in {}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv \sum_{d\in {}^{\mathbb{W}_J}\mathbb{W}} T_dT^*_s \tag{4.28}$$

for all $s\in S$ implies the lemma because $T^*_{\mathbf{w}} = T^*_{s_1}\dots T^*_{s_n}$ for any reduced decomposition $\mathbf{w}=s_1\dots s_n$ with $s_i\in S$.

Step 3. When $J = \emptyset$, the congruence (4.28) is an equality

$$\sum_{w \in \mathbb{W}} T_w = \sum_{w \in \mathbb{W}} T_w T_s^*. \tag{4.29}$$

It holds true because $\sum_{w \in \mathbb{W}} T_w = \sum_{w < ws} T_w(T_s+1)$ and $(T_s+1)T_s^* = T_s T_s^* + T_s^* = q_s + T_s^* = T_s + 1$.

Step 4. Conversely the congruence (4.28) follows from (4.29) because

$$\sum_{w \in \mathbb{W}} T_w = (\sum_{u \in W_J} T_u) \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv (\sum_{u \in W_J} q_u) \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d$$

(recall $q_u = T^*_{u^{-1}} T_u \equiv T_u$) and we can simplify by $\sum_{u \in W_J} q_u$ in $H/\mathcal{J}$. □

This ends the proof of Proposition 4.23.

5. Universal representation $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$

The invariant functor $(-)^{\mathcal{U}}$ by the pro-p-Iwahori subgroup $\mathcal{U}$ of G has a left adjoint

$$- \bigotimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G] : \mathrm{Mod}_R(\mathcal{H}) \to \mathrm{Mod}_R^\infty(G).$$

The smooth R-representation $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G]$ of G constructed from the right $\mathcal{H}_R$-module $\mathcal{V}$ is called universal. We write

$$R[\mathcal{U} \backslash G] = \mathbb{X}.$$

Question 5.1. Does $\mathcal{V} \neq 0$ imply $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X} \neq 0$ or does $v \otimes 1_{\mathcal{U}} = 0$ for $v \in \mathcal{V}$ imply $v = 0$? We have no counterexample. If R is a field and the $\mathcal{H}_R$-module $\mathcal{V}$ is simple, the two questions are equivalent: $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X} \neq 0$ if and only if the map $v \mapsto v \otimes 1_{\mathcal{U}}$ is injective. When R is an algebraically closed field of characteristic p, $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X} \neq 0$ for all simple $\mathcal{H}_R$-modules $\mathcal{V}$ if this is true for $\mathcal{V}$ simple supersingular (this is a consequence of Corollary 5.13).

The functor $- \bigotimes_{\mathcal{H}_R} \mathbb{X}$ satisfies a few good properties: it has a right adjoint and is compatible with the parabolic induction and the left adjoint (of the parabolic induction). Let $P = MN$ be a standard parabolic subgroup and $\mathbb{X}_M = R[\mathcal{U}_M \backslash M]$. We have functor isomorphisms

$$(- \bigotimes_{\mathcal{H}_R} \mathbb{X}) \circ \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \to \mathrm{Ind}_P^G \circ (- \bigotimes_{\mathcal{H}_{M,R}} \mathbb{X}_M), \tag{5.1}$$

$$(-)_N \circ (- \bigotimes_{\mathcal{H}_R} \mathbb{X}) \to (- \bigotimes_{\mathcal{H}_{M,R}} \mathbb{X}_M) \circ \mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}. \tag{5.2}$$

The first one is [OV17, formula 4.15], the second one is obtained by left adjunction from the isomorphism $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \circ (-)^{\mathcal{U}_M} \to (-)^{\mathcal{U}} \circ \mathrm{Ind}_P^G$ [OV17, formula (4.14)]. If $\mathcal{V}$ is a right $\mathcal{H}_R$-supersingular module and p is nilpotent in $\mathcal{V}$, then $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$ if $M \neq G$ (Proposition 4.27). Applying (5.2) we deduce the following.

Proposition 5.2. *If p is nilpotent in $\mathcal{V}$ and $\mathcal{V}$ supersingular, then $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X}$ is left cuspidal.*

Remark 5.3. For a non-zero smooth R-representation τ of M, Δ_τ is orthogonal to Δ_P if τ is left cuspidal. Indeed, we recall from [AHHV17, II.7 Corollary 2] that Δ_τ is not orthogonal to Δ_P if and only if there exists a proper standard parabolic subgroup X of M such that σ is trivial on the unipotent radical of X; moreover τ is a subrepresentation of $\mathrm{Ind}_X^M(\tau|_X)$, so the image of τ by the left adjoint of Ind_X^M is not 0.

From now on, $\mathcal{V}$ is a non-zero right $\mathcal{H}_{M,R}$-module and

$$\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M.$$

In general, when $\sigma \neq 0$, let $P_\perp(\sigma)$ be the standard parabolic subgroup of G with $\Delta_{P_\perp(\sigma)} = \Delta_P \cup \Delta_{\perp,\sigma}$ where $\Delta_{\perp,\sigma}$ is the set of simple roots $\alpha \in \Delta_\sigma$ orthogonal to Δ_P.

Proposition 5.4.

(1) $P(\mathcal{V}) \subset P_\perp(\sigma)$ *if* $\sigma \neq 0$.
(2) $P(\mathcal{V}) = P_\perp(\sigma)$ *if the map* $v \mapsto v \otimes 1_{\mathcal{U}_M}$ *is injective.*
(3) $P(\mathcal{V}) = P(\sigma)$ *if the map* $v \mapsto v \otimes 1_{\mathcal{U}_M}$ *is injective,* p *nilpotent in* $\mathcal{V}$ *and* $\mathcal{V}$ *supersingular.*
(4) $P(\mathcal{V}) = P(\sigma)$ *if* $\sigma \neq 0$*,* R *is a field of characteristic* p *and* $\mathcal{V}$ *simple supersingular.*

Proof.
(1) $P(\mathcal{V}) \subset P_\perp(\sigma)$ means that $Z \cap M'_{\mathcal{V}}$ acts trivially on $\mathcal{V} \otimes 1_{\mathcal{U}_M}$, where $M_{\mathcal{V}}$ is the standard Levi subgroup such that $\Delta_{M_{\mathcal{V}}} = \Delta_{\mathcal{V}}$. Let $z \in Z \cap M'_{\mathcal{V}}$ and $v \in \mathcal{V}$. As Δ_M and $\Delta_{\mathcal{V}}$ are orthogonal, we have $T^{M,*}(z) = T^M(z)$ and $\mathcal{U}_M z \mathcal{U}_M = \mathcal{U}_M z$. We have $v \otimes 1_{\mathcal{U}_M} = vT^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes T^M(z) 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M z} = v \otimes z^{-1} 1_{\mathcal{U}_M} = z^{-1}(v \otimes 1_{\mathcal{U}_M})$.
(2) If $v \otimes 1_{\mathcal{U}_M} = 0$ for $v \in \mathcal{V}$ implies $v = 0$, then $\sigma \neq 0$ because $\mathcal{V} \neq 0$. By (1) $P(\mathcal{V}) \subset P_\perp(\sigma)$. As in the proof of (1), for $z \in Z \cap M'_{\perp,\sigma}$ we have $vT^{M,*}(z) \otimes 1_{\mathcal{U}_M} = vT^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M}$ and our hypothesis implies $vT^{M,*}(z) = v$ hence $P(\mathcal{V}) \supset P_\perp(\sigma)$.
(3) Proposition 5.2, Remark 5.3, and (2).
(4) Question 5.1 and (3). □

Let Q be a parabolic subgroup of G with $P \subset Q \subset P(\mathcal{V})$. In this chapter we will compute $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$ where $I_{\mathcal{H}}(P, \mathcal{V}, Q) = \mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes (\mathrm{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})$ (Theorem 5.11). The smooth R-representation $I_G(P, \sigma, Q)$ of G is well defined: it is 0 if $\sigma = 0$ and $\mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \mathrm{St}_Q^{P(\sigma)})$ if $\sigma \neq 0$ because (P, σ, Q) is an $R[G]$-triple by Proposition 5.4. We will show that the universal representation $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$ is isomorphic to $I_G(P, \sigma, Q)$, if $P(\mathcal{V}) = P(\sigma)$ and $p = 0$, or if $\sigma = 0$ (Corollary 5.12). In particular, $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G] \simeq I_G(P, \sigma, Q)$ when R is an algebraically closed field of characteristic p and $\mathcal{V}$ is supersingular.

5.1. $Q = G$**.** We consider first the case $Q = G$. We are in the simple situation where $\mathcal{V}$ is extensible to $\mathcal{H}$ and $P(\mathcal{V}) = P(\sigma) = G$, $I_{\mathcal{H}}(P, \mathcal{V}, G) = e(\mathcal{V})$ and $I_G(P, \sigma, G) = e(\sigma)$. We recall that $\Delta \setminus \Delta_P$ is orthogonal to Δ_P and that M_2 denotes the standard Levi subgroup of G with $\Delta_{M_2} = \Delta \setminus \Delta_P$.

The $\mathcal{H}_R$-morphism $e(\mathcal{V}) \to e(\sigma)^{\mathcal{U}} = \sigma^{\mathcal{U}_M}$ sending v to $v \otimes 1_{\mathcal{U}_M}$ for $v \in \mathcal{V}$, gives by adjunction an $R[G]$-homomorphism

$$v \otimes 1_{\mathcal{U}} \mapsto v \otimes 1_{\mathcal{U}_M} : e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi^G} e(\sigma).$$

If Φ^G is an isomorphism, then $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ is the extension to G of $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})|_M$, meaning that M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$. The converse is true.

Lemma 5.5. *If M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$, then Φ^G is an isomorphism.*

Proof. Suppose that M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$. Then $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ is the extension to G of $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})|_M$, and by Theorem 3.13, $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}}$ is the extension of $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}_M}$. Therefore, by (3.12),

$$(v \otimes 1_{\mathcal{U}})T^*_w = (v \otimes 1_{\mathcal{U}})T^{M,*}_w \quad \text{for all } v \in \mathcal{V}, w \in W_M(1).$$

As $\mathcal{V}$ is extensible to $\mathcal{H}$, the natural map $v \mapsto v \otimes 1_{\mathcal{U}} : \mathcal{V} \xrightarrow{\Psi} (e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}_M}$ is $\mathcal{H}_M$-equivariant, i.e.,

$$vT^{M,*}_w \otimes 1_{\mathcal{U}} = (v \otimes 1_{\mathcal{U}})T^{M,*}_w \quad \text{for all } v \in \mathcal{V}, w \in W_M(1)$$

because (3.12) $vT^{M,*}_w \otimes 1_{\mathcal{U}} = vT^*_w \otimes 1_{\mathcal{U}} = v \otimes T^*_w = (v \otimes 1_{\mathcal{U}})T^*_w$ in $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$.

We recall that $-\bigotimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$ is the left adjoint of $(-)^{\mathcal{U}_M}$. The adjoint $R[M]$-homomorphism $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M \to e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ sends $v \otimes 1_{\mathcal{U}_M}$ to $v \otimes 1_{\mathcal{U}}$ for all $v \in \mathcal{V}$. The $R[M]$-module generated by the $v \otimes 1_{\mathcal{U}}$ for all $v \in \mathcal{V}$ is equal to $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ because M'_2 acts trivially. Hence we obtained an inverse of Φ^G. □

Our next move is to determine if M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$. It is equivalent to see if M'_2 acts trivially on $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$ as this set generates the representation $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ of G and M'_2 is a normal subgroup of G as M'_2 and M commute and $G = ZM'M'_2$. Obviously, $\mathcal{U} \cap M'_2$ acts trivially on $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$. The group of double classes $(\mathcal{U} \cap M'_2)\backslash M'_2/(\mathcal{U} \cap M'_2)$ is generated by the lifts $\hat{s} \in \mathcal{N} \cap M'_2$ of the simple affine roots s of $W_{M'_2}$. Therefore, M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ if and only if for any simple affine root $s \in S^{\mathrm{aff}}_{M'_2}$ of $W_{M'_2}$, any $\hat{s} \in \mathcal{N} \cap M'_2$ lifting s acts trivially on $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$.

Lemma 5.6. *Let $v \in \mathcal{V}, s \in S^{\mathrm{aff}}_{M'_2}$ and $\hat{s} \in \mathcal{N} \cap M'_2$ lifting s. We have*

$$(q_s + 1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0.$$

Proof. We compute:

$$T_s(\hat{s}1_{\mathcal{U}}) = \hat{s}(T_s 1_{\mathcal{U}}) = 1_{\mathcal{U}\hat{s}\mathcal{U}(\hat{s})^{-1}} = \sum_u \hat{s}u(\hat{s})^{-1}1_{\mathcal{U}} = \sum_{u^{op}} u^{op}1_{\mathcal{U}},$$

$$T_s(\hat{s}^2 1_{\mathcal{U}}) = \hat{s}^2(T_s 1_{\mathcal{U}}) = 1_{\mathcal{U}\hat{s}\mathcal{U}(\hat{s})^{-2}} = 1_{\mathcal{U}(\hat{s})^{-1}\mathcal{U}} = \sum_u u\hat{s}1_{\mathcal{U}}$$

for u in the group $\mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U})$ and u^{op} in the group $\hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U})$; the reason is that $\hat{s}^2$ normalizes $\mathcal{U}$, $\mathcal{U}\hat{s}\mathcal{U}\hat{s}^{-1}$ is the disjoint union of the sets $\mathcal{U}\hat{s}u^{-1}(\hat{s})^{-1}$ and $\mathcal{U}(\hat{s})^{-1}\mathcal{U}$ is the disjoint union of the sets $\mathcal{U}(\hat{s})^{-1}u^{-1}$. We introduce now a natural bijection

$$u \to u^{op} : \mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U}) \to \hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U}) \tag{5.3}$$

which is not a group homomorphism. We recall the finite reductive group $G_{k,s}$ quotient of the parahoric subgroup $\mathfrak{K}_s$ of G fixing the face fixed by s of the alcove $\mathcal{C}$. The Iwahori groups $Z^0\mathcal{U}$ and $Z^0\hat{s}\mathcal{U}(\hat{s})^{-1}$ are contained in $\mathfrak{K}_s$ and their images in $G_{s,k}$ are opposite Borel subgroups $Z_k U_{s,k}$ and $Z_k U^{op}_{s,k}$. Via the surjective maps $u \mapsto \overline{u} : \mathcal{U} \to U_{s,k}$ and $u^{op} \mapsto \overline{u}^{op} : \hat{s}\mathcal{U}(\hat{s})^{-1} \to U^{op}_{s,k}$ we identify the groups $\mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U}) \simeq U_{s,k}$ and similarly $\hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U}) \simeq U^{op}_{s,k}$. Let $G'_{k,s}$ be the group generated by $U_{s,k}$ and $U^{op}_{s,k}$, and let $B'_{s,k} = G'_{k,s} \cap Z_k U_{s,k} = (G'_{k,s} \cap Z_k)U_{s,k}$. We suppose (as we can) that $\hat{s} \in \mathfrak{K}_s$ and that its image $\hat{s}_k$ in $G_{s,k}$ lies in $G'_{k,s}$. We have $\hat{s}_k U_{s,k}(\hat{s}_k)^{-1} = U^{op}_{s,k}$ and the Bruhat decomposition $G'_{k,s} = B'_{k,s} \sqcup U_{k,s}\hat{s}_k B'_{k,s}$ implies the existence of a canonical bijection $\overline{u}^{op} \to \overline{u} : (U^{op}_{k,s} - \{1\}) \to (U_{k,s} - \{1\})$ respecting the cosets $\overline{u}^{op} B'_{k,s} = \overline{u}\hat{s}_k B'_{k,s}$. Via the preceding identifications we get the wanted bijection (5.3).

For $v \in e(\mathcal{V})$ and $z \in Z^0 \cap M'_2$ we have $vT_z = v$, $z1_{\mathcal{U}} = T_z 1_{\mathcal{U}}$ and $v \otimes T_z 1_{\mathcal{U}} = vT_z \otimes 1_{\mathcal{U}}$ therefore $Z^0 \cap M'_2$ acts trivially on $\mathcal{V} \otimes 1_{\mathcal{U}}$. The action of the group $(Z^0 \cap M'_2)\mathcal{U}$ on $\mathcal{V} \otimes 1_{\mathcal{U}}$ is also trivial. As the image of $Z^0 \cap M'_2$ in $G_{s,k}$ contains $Z_k \cap G'_{s,k}$,

$$u\hat{s}(v \otimes 1_{\mathcal{U}}) = u^{op}(v \otimes 1_{\mathcal{U}})$$

when u and u^{op} are not units and correspond via the bijection (5.3). So we have

$$v \otimes T_s(\hat{s}1_{\mathcal{U}}) - (v \otimes 1_{\mathcal{U}}) = v \otimes T_s(\hat{s}^2 1_{\mathcal{U}}) - v \otimes \hat{s}1_{\mathcal{U}}. \tag{5.4}$$

We can move T_s on the other side of $\otimes$ and as $vT_s = q_s v$ (Corollary 3.9), we can replace T_s by q_s. We have $v \otimes \hat{s}^2 1_{\mathcal{U}} = v \otimes T_{s^{-2}} 1_{\mathcal{U}}$ because $\hat{s}^2 \in Z^0 \cap M'_2$ normalizes $\mathcal{U}$; as we can move $T_{s^{-2}}$ on the other side of $\otimes$ and as $vT_{s^{-2}} = v$ we can forget $\hat{s}^2$. So (5.4) is equivalent to $(q_s + 1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0$. ☐

Combining the two lemmas we obtain the following.

Proposition 5.7. *When $\mathcal{V}$ is extensible to $\mathcal{H}$ and has no $q_s + 1$-torsion for any $s \in S^{\mathrm{aff}}_{M'_2}$, then M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ and Φ^G is an $R[G]$-isomorphism.*

Proposition 5.7 for the trivial character $\mathbf{1}_{\mathcal{H}}$, says that $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X}$ is the trivial representation $\mathbf{1}_G$ of G when $q_s + 1$ has no torsion in R for all $s \in S^{\mathrm{aff}}$. This is proved in [OV17, Lemma 2.28] by a different method. The following counter-example shows that this is not true for all R.

Example 5.8. Let $G = GL(2, F)$ and let R be an algebraically closed field where $q_{s_0} + 1 = q_{s_1} + 1 = 0$ and $S_{\mathrm{aff}} = \{s_0, s_1\}$. (Note that $q_{s_0} = q_{s_1}$ is the order of the residue field of F.) Then the dimension of $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X}$ is infinite, in particular $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X} \neq \mathbf{1}_G$.

Indeed, the Steinberg representation $\mathrm{St}_G = (\mathrm{Ind}_B^G \mathbf{1}_Z)/\mathbf{1}_G$ of G is an indecomposable representation of length 2 containing an irreducible infinite dimensional representation π with $\pi^{\mathcal{U}} = 0$ of quotient the character $(-1)^{\mathrm{val} \circ \det}$. This follows from the proof of Theorem 3 and from Proposition 24 in [Vig89]. The kernel of the quotient map $\mathrm{St}_G \otimes (-1)^{\mathrm{val} \circ \det} \to \mathbf{1}_G$ is infinite dimensional without a non-zero $\mathcal{U}$-invariant vector. As the characteristic of R is not p, the functor of $\mathcal{U}$-invariants is exact hence $(\mathrm{St}_G \otimes (-1)^{\mathrm{val} \circ \det})^{\mathcal{U}} = \mathbf{1}_{\mathcal{H}}$. As $- \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ is the left adjoint of $(-)^{\mathcal{U}}$ there is a non-zero homomorphism

$$\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X} \to \mathrm{St}_G \otimes (-1)^{\mathrm{val} \circ \det}$$

with image generated by its $\mathcal{U}$-invariants. The homomorphism is therefore surjective.

5.2. $\mathcal{V}$ **extensible to** $\mathcal{H}$**.** Let $P = MN$ be a standard parabolic subgroup of G with Δ_P and $\Delta \setminus \Delta_P$ orthogonal. We still suppose that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}$, but now $P \subset Q \subset G$. So we have $I_{\mathcal{H}}(P, \mathcal{V}, Q) = e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ and $I_G(P, \sigma, Q) = e(\sigma) \otimes_R \mathrm{St}_Q^G$ where $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$. We compare the images by $-\bigotimes_{\mathcal{H}_R} \mathbb{X}$ of the $\mathcal{H}_R$-modules $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ and $e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ with the smooth R-representations $e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}$ and $e(\sigma) \otimes \mathrm{St}_Q^G$ of G.

As $-\bigotimes_{\mathcal{H}_R} \mathbb{X}$ is left adjoint of $(-)^{\mathcal{U}}$, the $\mathcal{H}_R$-homomorphism $v \otimes f \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f$: $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \to (e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ gives by adjunction an $R[G]$-homomorphism

$$v \otimes f \otimes 1_{\mathcal{U}} \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f : (e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi_Q^G} e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}.$$

When $Q = G$ we have $\Phi_G^G = \Phi^G$. By Remark 4.10, Φ_Q^G is surjective. Proposition 5.7 applies with M_Q instead of G and gives the $R[M_Q]$-homomorphism

$$v \otimes 1_{\mathcal{U}_{M_Q}} \mapsto v \otimes 1_{\mathcal{U}_M} : e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{Q,R}} \mathbb{X}_{M_Q} \xrightarrow{\Phi^Q} e_Q(\sigma).$$

Proposition 5.9. *The $R[G]$-homomorphism Φ_Q^G is an isomorphism if Φ^Q is an isomorphism, in particular if $\mathcal{V}$ has no $q_s + 1$-torsion for any $s \in S^{\mathrm{aff}}_{M_2' \cap M_Q}$.*

Proof. The proposition follows from another construction of Φ_Q^G that we now describe. Proposition 4.5 gives the $\mathcal{H}_R$-module isomorphism

$$v \otimes f_{Q\mathcal{U}} \mapsto v \otimes 1_{\mathcal{H}} : (e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_{Q,R}^+},\theta} \mathcal{H}.$$

We have the $R[G]$-isomorphism [OV17, Corollary 4.7]

$$v \otimes 1_{\mathcal{H}} \otimes 1_{\mathcal{U}} \mapsto f_{Q\mathcal{U}, v \otimes 1_{\mathcal{U}_{M_Q}}} : \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \otimes_{\mathcal{H}_R} \mathbb{X} \to \mathrm{Ind}_Q^G(e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{Q,R}} \mathbb{X}_{M_Q})$$

and the $R[G]$-isomorphism

$$f_{Q\mathcal{U}, v \otimes 1_{\mathcal{U}_M}} \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f_{Q\mathcal{U}} : \mathrm{Ind}_Q^G(e_Q(\sigma)) \to e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}.$$

From Φ^Q and these three homomorphisms, there exists a unique $R[G]$-homomorphism

$$(e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \to e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}$$

sending $v \otimes f_{Q\mathcal{U}} \otimes 1_{\mathcal{U}}$ to $v \otimes 1_{\mathcal{U}_M} \otimes f_{Q\mathcal{U}}$. We deduce: this homomorphism is equal to Φ_Q^G, $\mathcal{V} \otimes 1_{Q\mathcal{U}} \otimes 1_{\mathcal{U}}$ generates $(e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X}$, if Φ^Q is an isomorphism, then Φ_Q^G is an isomorphism. By Proposition 5.7, if $\mathcal{V}$ has no $q_s + 1$-torsion for any $s \in S^{\mathrm{aff}}_{M_2' \cap M_Q}$, then Φ^Q and Φ_Q^G are isomorphisms. ☐

We recall that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}$.

Proposition 5.10. *The $R[G]$-homomorphism Φ_Q^G induces an $R[G]$-homomorphism*

$$(e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \to e(\sigma) \otimes_R \mathrm{St}_Q^G,$$

It is an isomorphism if $\Phi_{Q'}^G$ is an $R[G]$-isomorphism for all parabolic subgroups Q' of G containing Q, in particular if $\mathcal{V}$ has no $q_s + 1$-torsion for any $s \in S^{\mathrm{aff}}_{M_2'}$.

Proof. The proof is straightforward, with the arguments already developed for Proposition 4.5 and Theorem 4.9. The representations $e(\sigma)\otimes_R \mathrm{St}_Q^G$ and $(e(\mathcal{V})\otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}$ of G are the cokernels of the natural $R[G]$-homomorphisms

$$\oplus_{Q\subsetneq Q'} e(\sigma)\otimes_R \mathrm{Ind}_{Q'}^G \mathbf{1} \xrightarrow{\mathrm{id}\otimes\alpha} e(\sigma)\otimes_R \mathrm{Ind}_Q^G \mathbf{1},$$

$$\oplus_{Q\subsetneq Q'}(e(\mathcal{V})\otimes_R (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X} \xrightarrow{\mathrm{id}\otimes\alpha^{\mathcal{U}}\otimes\mathrm{id}} (e(\mathcal{V})\otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}.$$

These $R[G]$-homomorphisms make a commutative diagram with the $R[G]$-homomorphisms $\bigoplus_{Q\subsetneq Q'}\Phi_{Q'}^G$ and Φ_Q^G going from the lower line to the upper line. Indeed, let $v\otimes f_{Q'\mathcal{U}}\otimes 1_{\mathcal{U}} \in (e(\mathcal{V})\otimes_R (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}$. On the one hand, it goes to $v\otimes f_{Q\mathcal{U}}\theta_{Q'}(e_Q^{Q'})\otimes 1_{\mathcal{U}} \in (e(\mathcal{V})\otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}$ by the horizontal map, and then to $v\otimes 1_{\mathcal{U}_M}\otimes f_{Q\mathcal{U}}\theta_{Q'}(e_Q^{Q'})$ by the vertical map. On the other hand, it goes to $v\otimes 1_{\mathcal{U}_M}\otimes f_{Q'\mathcal{U}}$ by the vertical map, and then to $v\otimes 1_{\mathcal{U}_M}\otimes f_{Q\mathcal{U}}\theta_{Q'}(e_Q^{Q'})$ by the horizontal map. One deduces that Φ_Q^G induces an $R[G]$-homomorphism $(e(\mathcal{V})\otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}\to e(\sigma)\otimes_R \mathrm{St}_Q^G$, which is an isomorphism if $\Phi_{Q'}^G$ is an $R[G]$-isomorphism for all $Q\subset Q'$. ☐

5.3. **General.** We consider now the general case: let $P=MN\subset Q$ be two standard parabolic subgroups of G and let $\mathcal{V}$ be a non-zero right $\mathcal{H}_{M,R}$-module with $Q\subset P(\mathcal{V})$. We recall $I_{\mathcal{H}}(P,\mathcal{V},Q)=\mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V})\otimes_R (\mathrm{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})$ and $\sigma=\mathcal{V}\otimes_{\mathcal{H}_{M,R}}\mathbb{X}_M$ (Proposition 5.4). There is a natural $R[G]$-homomorphism

$$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R}\mathbb{X}\xrightarrow{\Phi_I^G}\mathrm{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma)\otimes_R \mathrm{St}_Q^{P(\mathcal{V})})$$

obtained by composition of the $R[G]$-isomorphism [OV17, Corollary 4.7] (proof of Proposition 5.9):

$$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R}\mathbb{X}\to \mathrm{Ind}_{P(\mathcal{V})}^G((e(\mathcal{V})\otimes_R (\mathrm{St}_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})\otimes_{\mathcal{H}_{M(\mathcal{V}),R}}\mathbb{X}_{M(\mathcal{V})}),$$

with the $R[G]$-homomorphism

$$\mathrm{Ind}_{P(\mathcal{V})}^G((e(\mathcal{V})\otimes_R (\mathrm{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})\otimes_{\mathcal{H}_{M(\mathcal{V}),R}}\mathbb{X}_{M(\mathcal{V})})\to \mathrm{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma)\otimes_R \mathrm{St}_Q^{P(\mathcal{V})}),$$

image by the parabolic induction $\mathrm{Ind}_{P(\mathcal{V})}^G$ of the homomorphism

$$(e(\mathcal{V})\otimes_R (\mathrm{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})\otimes_{\mathcal{H}_{M(\mathcal{V}),R}}\mathbb{X}_{M(\mathcal{V})}\to e_{M(\mathcal{V})}(\sigma)\otimes_R \mathrm{St}_Q^{P(\mathcal{V})}$$

induced by the $R[M(\mathcal{V})]$-homomorphism $\Phi_Q^{P(\mathcal{V})}=\Phi_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})}$ of Proposition 5.10 applied to $M(\mathcal{V})$ instead of G.

This homomorphism Φ_I^G is an isomorphism if $\Phi_Q^{P(\mathcal{V})}$ is an isomorphism, in particular if $\mathcal{V}$ has no q_s+1-torsion for any $s\in S_{M_2'}^{\mathrm{aff}}$ where $\Delta_{M_2}=\Delta_{M(\mathcal{V})}\setminus\Delta_M$ (Proposition 5.10). We get the main theorem of this section.

Theorem 5.11. *Let $(P=MN,\mathcal{V},Q)$ be an $\mathcal{H}_R$-triple and $\sigma=\mathcal{V}\otimes_{\mathcal{H}_{M,R}}R[\mathcal{U}_M\backslash M]$. Then, (P,σ,Q) is an $R[G]$-triple. The $R[G]$-homomorphism*

$$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R}R[\mathcal{U}\backslash G]\xrightarrow{\Phi_I^G}\mathrm{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma)\otimes_R \mathrm{St}_Q^{P(\mathcal{V})})$$

is an isomorphism if $\Phi_Q^{P(\mathcal{V})}$ is an isomorphism. In particular Φ_I^G is an isomorphism if $\mathcal{V}$ has no q_s+1-torsion for any $s\in S_{M_2'}^{\mathrm{aff}}$.

Recalling $I_G(P,\sigma,Q) = \operatorname{Ind}_{P(\sigma)}^G(e(\sigma)\otimes_R \operatorname{St}_Q^{P(\sigma)})$ when $\sigma \neq 0$, we deduce the following.

Corollary 5.12. *We have the following:*

$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G] \simeq I_G(P,\sigma,Q)$, *if* $\sigma \neq 0$, $P(\mathcal{V}) = P(\sigma)$ *and* $\mathcal{V}$ *has no* q_s+1*-torsion for any* $s \in S^{\mathrm{aff}}_{M_2'}$.

$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G] = I_G(P,\sigma,Q) = 0$, *if* $\sigma = 0$.

Recalling $P(\mathcal{V}) = P(\sigma)$ if $\sigma \neq 0$, R is a field of characteristic p and $\mathcal{V}$ simple supersingular (Proposition 5.4 (4)), we deduce the following.

Corollary 5.13. $I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G] \simeq I_G(P,\sigma,Q)$ *if* R *is a field of characteristic* p *and* $\mathcal{V}$ *simple supersingular.*

6. Vanishing of the smooth dual

Let V be an $R[G]$-module. The dual $\operatorname{Hom}_R(V,R)$ of V is an $R[G]$-module for the contragredient action: $gL(gv) = L(v)$ if $g \in G$, $L \in \operatorname{Hom}_R(V,R)$ is a linear form and $v \in V$. When $V \in \operatorname{Mod}_R^\infty(G)$ is a smooth R-representation of G, the dual of V is not necessarily smooth. A linear form L is smooth if there exists an open subgroup $H \subset G$ such that $L(hv) = L(v)$ for all $h \in H, v \in V$; the space $\operatorname{Hom}_R(V,R)^\infty$of smooth linear forms is a smooth R-representation of G, called the **smooth dual** (or smooth contragredient) of V. The smooth dual of V is contained in the dual of V.

Example 6.1. When R is a field and the dimension of V over R is finite, the dual of V is equal to the smooth dual of V because the kernel of the action of G on V is an open normal subgroup $H \subset G$; the action of G on the dual $\operatorname{Hom}_R(V,R)$ is trivial on H.

We assume in this section that R is a field of characteristic p. Let $P = MN$ be a parabolic subgroup of G and $V \in \operatorname{Mod}_R^\infty(M)$. Generalizing the proof given in [Vig07, 8.1] when $G = GL(2,F)$ and the dimension of V is 1, we show the following.

Proposition 6.2. *If* $P \neq G$, *the smooth dual of* $\operatorname{Ind}_P^G(V)$ *is* 0.

Proof. Let L be a smooth linear form on $\operatorname{Ind}_P^G(V)$ and let K be an open pro-p-subgroup of G which fixes L. Let J be an arbitrary open subgroup of K, $g \in G$ and $f \in (\operatorname{Ind}_P^G(V))^J$ with support PgJ. We want to show that $L(f) = 0$. Let J' be any open normal subgroup of J and let φ denote the function in $(\operatorname{Ind}_P^G(V))^{J'}$ with support PgJ' and value $\varphi(g) = f(g)$ at g. For $j \in J$ we have $L(j\varphi) = L(\varphi)$, and the support of $j\varphi(x) = \varphi(xj)$ is $PgJ'j^{-1}$. The function f is the sum of translates $j\varphi$, where j ranges through the left cosets of the image X of $g^{-1}Pg \cap J$ in J/J', so that $L(f) = rL(\varphi)$ where r is the order of X in J/J'. We can certainly find J' such that $r \neq 1$, and then r is a positive power of p. As the characteristic of R is p we have $L(f) = 0$. □

The module $R[\mathcal{U}\backslash G]$ is contained in the module $R^{\mathcal{U}\backslash G}$ of functions $f : \mathcal{U}\backslash G \to R$. The actions of $\mathcal{H}$ and of G on $R[\mathcal{U}\backslash G]$ extend to $R^{\mathcal{U}\backslash G}$ by the same formulas. The pairing

$$(f,\varphi) \mapsto \langle f,\varphi\rangle = \sum_{g\in\mathcal{U}\backslash G} f(g)\varphi(g) : R^{\mathcal{U}\backslash G}\times R[\mathcal{U}\backslash G] \to R$$

identifies $R^{\mathcal{U}\backslash G}$ with the dual of $R[\mathcal{U}\backslash G]$. Let $h \in \mathcal{H}$ and $\check{h} \in \mathcal{H}$, $\check{h}(g) = h(g^{-1})$ for $g \in G$. We have

$$\langle f, h\varphi\rangle = \langle \check{h}f, \varphi\rangle.$$

Proposition 6.3. *When R is an algebraically closed field of characteristic p, G is not compact modulo the center and $\mathcal{V}$ is a simple supersingular right $\mathcal{H}_R$-module, the smooth dual of $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ is 0.*

Proof. Let $\mathcal{H}_R^{\mathrm{aff}}$ be the subalgebra of $\mathcal{H}_R$ of basis $(T_w)_{w\in W'(1)}$ where $W'(1)$ is the inverse image of W' in $W(1)$. The dual of $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ is contained in the dual of $\mathcal{V} \otimes_{\mathcal{H}_R^{\mathrm{aff}}} R[\mathcal{U}\backslash G]$; the $\mathcal{H}_R^{\mathrm{aff}}$-module $\mathcal{V}|_{\mathcal{H}_R^{\mathrm{aff}}}$ is a finite sum of supersingular characters [Vig15a]. Let $\chi : \mathcal{H}_R^{\mathrm{aff}} \to R$ be a supersingular character. The dual of $\chi \otimes_{\mathcal{H}_R^{\mathrm{aff}}} R[\mathcal{U}\backslash G]$ is contained in the dual of $R[\mathcal{U}\backslash G]$ isomorphic to $R^{\mathcal{U}\backslash G}$. It is the space of $f \in R^{\mathcal{U}\backslash G}$ with $\check{h}f = \chi(h)f$ for all $h \in \mathcal{H}_R^{\mathrm{aff}}$. The smooth dual of $\chi \otimes_{\mathcal{H}_R^{\mathrm{aff}}} R[\mathcal{U}\backslash G]$ is 0 if the dual of $\chi \otimes_{\mathcal{H}_R^{\mathrm{aff}}} R[\mathcal{U}\backslash G]$ has no non-zero element fixed by $\mathcal{U}$. Let us take $f \in R^{\mathcal{U}\backslash G/\mathcal{U}}$ with $\check{h}f = \chi(h)f$ for all $h \in \mathcal{H}_R^{\mathrm{aff}}$. We shall prove that $f = 0$. We have $\check{T}_w = T_{w^{-1}}$ for $w \in W(1)$.

Let $<$ denote the Bruhat order of $W(1)$ associated to S^{aff} [Vig16]. The elements $(T_t)_{t\in Z_k}$ and $(T_{\tilde{s}})_{s\in S^{\mathrm{aff}}}$ where $\tilde{s}$ is an admissible lift of s in $W^{\mathrm{aff}}(1)$, generate the algebra $\mathcal{H}_R^{\mathrm{aff}}$ and

$$T_tT_w = T_{tw}, \quad T_{\tilde{s}}T_w = \begin{cases} T_{\tilde{s}w}, & \tilde{s}w > w, \\ c_{\tilde{s}}T_w, & \tilde{s}w < w, \end{cases}$$

with $c_{\tilde{s}} = -|Z'_{k,s}| \sum_{t\in Z'_{k,s}} T_t$ because the characteristic of R is p [Vig16, Proposition 4.4]. Expressing $f = \sum_{w\in W(1)} a_wT_w$, $a_w \in R$, as an infinite sum, we have

$$T_tf = \sum_{w\in W(1)} a_{t^{-1}w}T_w, \quad T_{\tilde{s}}f = \sum_{w\in W(1), \tilde{s}w<w} (a_{(\tilde{s})^{-1}w} + a_wc_{\tilde{s}})T_w.$$

A character χ of $\mathcal{H}_R^{\mathrm{aff}}$ is associated to a character $\chi_k : Z_k \to R^*$ and a subset J of

$$S^{\mathrm{aff}}_{\chi_k} = \{s \in S^{\mathrm{aff}} \mid (\chi_k)|_{Z'_{k,s}} \ \text{trivial}\ \}$$

[Vig15a, Definition 2.7]. We have

$$\text{(6.1)} \qquad \begin{cases} \chi(T_t) = \chi_k(t), & t \in Z_k, \\ \chi(T_{\tilde{s}}) = \begin{cases} 0, & s \in S^{\mathrm{aff}} \setminus J, \\ -1, & s \in J. \end{cases} \end{cases} \qquad (\chi_k)(c_{\tilde{s}}) = \begin{cases} 0, & s \in S^{\mathrm{aff}} \setminus S^{\mathrm{aff}}_{\chi_k}, \\ -1, & s \in S^{\mathrm{aff}}_{\chi_k}. \end{cases}$$

Therefore $\chi_k(t)f = \check{T}_tf = T_{t^{-1}}f$ hence $\chi_k(t)a_w = a_{tw}$. We have $\chi(T_{\tilde{s}})f = \check{T}_{\tilde{s}}f = T_{(\tilde{s})^{-1}}f = T_{\tilde{s}}T_{(\tilde{s})^{-2}}f = \chi_k((\tilde{s})^2)T_{\tilde{s}}f$; as $(\tilde{s})^2 \in Z'_{k,s}$ [Vig16, three lines before Proposition 4.4] and $J \subset S^{\mathrm{aff}}_{\chi_k}$, we obtain

$$\text{(6.2)} \qquad T_{\tilde{s}}f = \begin{cases} 0, & s \in S^{\mathrm{aff}} \setminus J, \\ -f, & s \in J. \end{cases}$$

Introducing $\chi_k(t)a_w = a_{tw}$ in the formula for $T_{\tilde{s}}f$, we get

$$\begin{aligned}\sum_{w\in W(1),\tilde{s}w<w} a_w c_{\tilde{s}} T_w &= -|Z'_{k,s}|^{-1} \sum_{w\in W(1),\tilde{s}w<w,t\in Z'_{k,s}} a_w T_{tw} \\ &= -|Z'_{k,s}|^{-1} \sum_{w\in W(1),\tilde{s}w<w,t\in Z'_{k,s}} a_{t^{-1}w} T_w \\ &= -|Z'_{k,s}|^{-1} \sum_{t\in Z'_{k,s}} \chi_k(t^{-1}) \sum_{w\in W(1),\tilde{s}w<w} a_w T_w \\ &= \chi_k(c_{\tilde{s}}) \sum_{w\in W(1),\tilde{s}w<w} a_w T_w.\end{aligned}$$

$$\begin{aligned}T_{\tilde{s}}f &= \sum_{w\in W(1),\tilde{s}w<w} (a_{(\tilde{s})^{-1}w} + a_w \chi_k(c_{\tilde{s}}))T_w \\ &= \begin{cases}\sum_{w\in W(1),\tilde{s}w<w} a_{(\tilde{s})^{-1}w} T_w, & s\in S^{\mathrm{aff}}\setminus S^{\mathrm{aff}}_{\chi_k},\\ \sum_{w\in W(1),\tilde{s}w<w} (a_{(\tilde{s})^{-1}w} - a_w) T_w, & s\in S^{\mathrm{aff}}_{\chi_k}.\end{cases}\end{aligned}$$

From the last equality and (6.2) for $T_{\tilde{s}}f$, we get:

$$a_{\tilde{s}w} = \begin{cases}0, & s\in J\cup(S^{\mathrm{aff}}\setminus S^{\mathrm{aff}}_{\chi_k}), \tilde{s}w<w,\\ a_w, & s\in S^{\mathrm{aff}}_{\chi_k}\setminus J.\end{cases} \tag{6.3}$$

Assume that $a_w \neq 0$. By the first condition, we know that $w > \tilde{s}w$ for $s \in J\cup(S^{\mathrm{aff}}\setminus S^{\mathrm{aff}}_{\chi_k})$. The character χ is supersingular if for each irreducible component X of S^{aff}, the intersection $X\cap J$ is not empty and different from X [Vig15a, Definition 2.7, Theorem 6.18]. This implies that the group generated by the $s\in S^{\mathrm{aff}}_{\chi_k}\setminus J$ is finite. If χ is supersingular, by the second condition we can suppose $w > \tilde{s}w$ for any $s\in S^{\mathrm{aff}}$. But there is no such element if S^{aff} is not empty. □

Theorem 6.4. *Let π be an irreducible admissible R-representation of G with a non-zero smooth dual where R is an algebraically closed field of characteristic p. Then π is finite dimensional.*

Proof. Let (P,σ,Q) be an $R[G]$-triple with σ supercuspidal such that $\pi \simeq I_G(P,\sigma,Q)$. The representation $I_G(P,\sigma,Q)$ is a quotient of $\mathrm{Ind}_Q^G e_Q(\sigma)$ hence the smooth dual of $\mathrm{Ind}_Q^G e_Q(\sigma)$ is not zero. From Proposition 6.2, $Q = G$. We have $I_G(P,\sigma,G) = e(\sigma)$. The smooth dual of σ contains the smooth linear dual of $e(\sigma)$ hence is not zero. As σ is supercuspidal, the $\mathcal{H}_M$-module $\sigma^{\mathcal{U}_M}$ contains a simple supersingular submodule $\mathcal{V}$ [Vig15a, Proposition 7.10, Corollary 7.11]. The functor $-\bigotimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M\backslash M]$ being the right adjoint of $(-)^{\mathcal{U}_M}$, the irreducible representation σ is a quotient of $\mathcal{V}\otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M\backslash M]$, hence the smooth dual of $\mathcal{V}\otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M\backslash M]$ is not zero. By Proposition 6.3, $M = Z$. Hence σ is finite dimensional and the same is true for $e(\sigma) = I_G(B,\sigma,G)\simeq \pi$. □

Remark 6.5. When the characteristic of F is 0, Theorem 6.4 was proved by Kohlhaase for a field R of characteristic p. He gives two proofs [Koh, Proposition 3.9, Remark 3.10], but none of them extends to F of characteristic p. Our proof is valid without restriction on the characteristic of F and does not use the results of Kohlhaase. Our assumption that R is an algebraically closed field of characteristic p comes from the classification theorem in [AHHV17].

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