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REPRESENTATIONS ASSOCIATED TO SMALL NILPOTENT ORBITS FOR COMPLEX SPIN GROUPS

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Abstract. This paper provides a comparison between the K-structure of unipotent representations and regular sections of bundles on nilpotent orbits for complex groups of type D. Precisely, let $G_0 = \mathrm{Spin}(2n, \mathbb{C})$ be the Spin complex group as a real group, and let $K \cong G_0$ be the complexification of the maximal compact subgroup of G_0. We compute K-spectra of the regular functions on some small nilpotent orbits $\mathcal{O}$ transforming according to characters ψ of $C_K(\mathcal{O})$ trivial on the connected component of the identity $C_K(\mathcal{O})^0$. We then match them with the K-types of the genuine (i.e., representations which do not factor to $\mathrm{SO}(2n, \mathbb{C})$) unipotent representations attached to $\mathcal{O}$.

1. Introduction

Let G be a connected complex semisimple group (viewed as a real group), and let $\mathfrak{g}, \mathfrak{g}^\vee, \mathfrak{g}_\mathbb{C}$ be its Lie algebra, dual Lie algebra, and complexified Lie algebra, respectively. Special unipotent representations of G were introduced in [BV85]. To each nilpotent orbit $\mathcal{O}^\vee \subset \mathfrak{g}^\vee$ an infinitesimal character $\lambda_{\mathcal{O}^\vee}$ is associated via the Jacobson–Morozov theorem; the orbit is associated to a Lie triple $\{e^\vee, h^\vee, f^\vee\}$, and $\lambda_{\mathcal{O}^\vee} = h^\vee/2$ determines an infinitesimal character. *Special unipotent representations* are defined as the irreducible $(\mathfrak{g}_\mathbb{C}, K)$-modules Ξ satisfying

(1) the infinitesimal character is $(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee})$ (see Section 4 for the parametrization of $(\mathfrak{g}_\mathbb{C}, K)$-modules) and
(2) the annihilator of Ξ in the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$, denoted $Ann_{U(\mathfrak{g}_\mathbb{C})}\Xi$, is the unique maximal primitive ideal of $U(\mathfrak{g}_\mathbb{C})$ with infinitesimal character $(\lambda_\mathcal{O}, \lambda_\mathcal{O})$.

Denote by $\mathcal{U}_G(\mathcal{O}^\vee, \lambda_{\mathcal{O}^\vee})$ the set of unipotent representations of G associated to $\mathcal{O}^\vee$. In [Bar89], the unitarity of these representations is established for the case of classical groups, and the whole unitary dual for such groups is determined. In the process, a larger set of representations is introduced which are called unipotent. A finite set of infinitesimal characters $\lambda_{\mathcal{O}^\vee, s^\vee}$ is associated to each $\mathcal{O}^\vee$, and an irreducible module Ξ is *unipotent* if it satisfies (1) and (2) with the more general $\lambda_{\mathcal{O}^\vee, s^\vee}$ instead. The results in [Bar89] can be viewed as proving that the modules in $\mathcal{U}_G(\mathcal{O}^\vee, \lambda_{\mathcal{O}^\vee, s^\vee})$ are the building blocks of the unitary dual.

In [Bar17] a different viewpoint is taken. Instead of parametrizing by $\mathcal{O}^\vee$, the unipotent representations are parametrized by nilpotent orbits $\mathcal{O} \subset \mathfrak{g}$. The precise setting is as follows. Let $G_0 \subset G$ be the real points of a complex linear reductive

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algebraic group G with Lie algebra $\mathfrak{g}_0$ and maximal compact subgroup K_0. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ be the Cartan decomposition, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the complexification. Let K be the complexification of K_0. Then K acts on $\mathfrak{s}$ by the adjoint action. We denote by $C_K(e) = \{g \in K \mid Ad(g)e = e\}$ the centralizer of $e \in \mathfrak{s}$ in K.

For each irreducible admissible representation Ξ of G_0, [Vog91] defines the associated cycle $AC(\Xi)$ to be a union of nilpotent K-orbits in $\mathfrak{s}$, and an algebraic representation of each centralizer $C_K(e)$ of a representative e for each orbit in $AC(\Xi)$.

Assume that G_0 is a connected complex group viewed as a real Lie group. Then $G \cong G_0 \times G_0$, and $K \cong G_0$ as complex groups. Furthermore $\mathfrak{s} \cong \mathfrak{g}_0$ as complex vector spaces. In this case there is only one K-orbit $\mathcal{O} \subset \mathfrak{s} \cong \mathfrak{g}_0$ in $AC(\Xi)$. The main results in [Vog00, Chapter 7] and [Vog91, Theorem 4.11] imply that, in the case of a complex group,

$$\Xi \mid_K = R(\mathcal{O}, \psi) - Y,$$

with ψ an algebraic representation, $R(\mathcal{O}, \psi)$ as defined in equation (1.1.1) below, and Y an $S(\mathfrak{g}/\mathfrak{k})$-module supported on orbits of strictly smaller dimension.

Definition 1.1. Let $e \in \mathfrak{s}$ be a nilpotent element, and let $\mathcal{O} := K \cdot e \subset \mathfrak{s}$ be the K-adjoint orbit of e. We say that an irreducible admissible representation Ξ is *associated* to $\mathcal{O}$ if $\mathcal{O}$ occurs with nonzero multiplicity in the associated cycle. We will write $\mathcal{U}_{G_0}(\mathcal{O}, \lambda)$ for the set of irreducible representations of G_0 with maximal annihilator with infinitesimal character λ, and associated to $\mathcal{O}$.

Let $C_K(\mathcal{O}) := C_K(e)$ denote the centralizer of e in K, with e a representative of $\mathcal{O}$, and let $A_K(\mathcal{O}) := C_K(\mathcal{O})/C_K(\mathcal{O})^0$ be the component group. In the case of G_0 being a connected complex group viewed as a real group, it is conjectured that there exists an infinitesimal character $\lambda_{\mathcal{O}}$ such that in addition, we have the following.

Conjecture. *For each $\mathcal{O}$ there is $\lambda_{\mathcal{O}}$ such that there is a* 1-1 *correspondence*

$$\psi \in \widehat{A_K(\mathcal{O})} \longleftrightarrow \Xi(\mathcal{O}, \psi) \in \mathcal{U}_{G_0}(\mathcal{O}, \lambda_{\mathcal{O}})$$

satisfying the additional condition

$$\Xi(\mathcal{O}, \psi) \Big|_K \cong R(\mathcal{O}, \psi),$$

where

$$\begin{aligned} R(\mathcal{O}, \psi) &= \mathrm{Ind}_{C_K(\mathcal{O})}^K(\psi) \\ &= \{f : K \to V_\psi \mid f(gx) = \psi(x^{-1})f(g)\ \forall g \in K,\ x \in C_K(\mathcal{O})\} \end{aligned} \tag{1.1.1}$$

is the ring of regular functions on $\mathcal{O}$ transforming according to ψ.

It is also natural to conjecture that there is a choice of $\lambda_{\mathcal{O}}$ such that the representations are unitary. The results in [Bar89] and [Bre99] establish the unitarity of the modules considered in this paper.

Candidates for parameters $\lambda_{\mathcal{O}}$ satisfying the Conjecture above are listed in [Bar17]. Essentially they coincide with the $\lambda_{\mathcal{O}^\vee, s^\vee}$ introduced in [Bar89]. The validity of the conjecture is established for large classes of nilpotent orbits in the classical complex groups. Such parameters $\lambda_{\mathcal{O}}$ are available for the exceptional groups as well, [Bar17] for F_4, and to appear elsewhere for type E.

This conjecture cannot be valid for all nilpotent orbits in the case of real groups; $AC(\Xi)$ is supported on several K-orbits which are the *components* of the intersection of a complex nilpotent orbit $\mathcal{O}_c$ with $\mathfrak{s}$. The $R(\mathcal{O}, \psi)$ can be the same for different components of a particular $\mathcal{O}_c$. The representations with associated cycle containing a given component have drastically different K-structures. Examples can be found in [Vog00]. In addition, many examples are known (e.g., the case of the minimal orbit in certain real forms of type D) for which there are no representations with associated cycle supported on $\mathcal{O}$ or any real form of its complexification. The analogues of the results in this paper in the case of the real *Spin* groups are studied in [BT18].

In the case when ψ is the trivial representation, our results provide irreducible (unitary) $(\mathfrak{g}, K)$-modules with K-structure $R(\mathcal{O})$. By the Kraft–Procesi classification [KP82] of nilpotent orbits whose closures are normal, the orbits we consider are normal, so also $R(\mathcal{O}) = R(\overline{\mathcal{O}})$.

In this paper we investigate this conjecture for *small* orbits in the complex case using different techniques than in [Bar17]; [BT18] investigates the analogue for the real *Spin* groups. For the condition of *small* we require that

$$[\mu : R(\mathcal{O}, \psi)] \leq c_{\mathcal{O}},$$

i.e., that the multiplicity of any $\mu \in \widehat{K}$ be uniformly bounded. This puts a restriction on $\dim \mathcal{O}$:

$$\dim \mathcal{O} \leq \operatorname{rank}(\mathfrak{k}) + |\Delta^+(\mathfrak{k}, \mathfrak{t})|, \tag{1.1.2}$$

where $\mathfrak{t} \subset \mathfrak{k}$ is a Cartan subalgebra and $\Delta^+(\mathfrak{k}, \mathfrak{t})$ is a positive system. The reason for this restriction is as follows. Let (Π, X) be an admissible representation of G_0, and let μ be the highest weight of a representation $(\pi, V) \in \widehat{K}$ which is dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Assume that $\dim \operatorname{Hom}_K[\pi, \Pi] \leq C$ and Π has associated variety (cf. [Vog91]). Then

$$\dim\{v : v \in X \text{ belongs to an isotypic component with } ||\mu|| \leq t\} \leq Ct^{|\Delta^+(\mathfrak{k},\mathfrak{t})|+\dim \mathfrak{t}}.$$

The dimension of (π, V) grows like $t^{|\Delta^+(\mathfrak{k},\mathfrak{t})|}$, the number of representations with highest weight $||\mu|| \leq t$ grows like $t^{\dim \mathfrak{t}}$, and the multiplicities are assumed uniformly bounded. On the other hand, considerations involving primitive ideals imply that the dimension of this set grows like $t^{\dim G \cdot e/2}$ with $e \in \mathcal{O}$, and half the dimension of (the complex orbit) $G \cdot e$ is the dimension of the (K-orbit) $K \cdot e \in \mathfrak{s}$. In the case of type D, condition (1.1.2) coincides with being spherical; see [Pan94]. Since we only deal with characters of $C_K(\mathcal{O})$, multiplicity ≤ 1 is guaranteed.

In the case of the complex groups of type D_n, we consider $G_0 = \operatorname{Spin}(2n, \mathbb{C})$ viewed as a real group, and hence $K \cong G_0$ is the complexification of the maximal compact subgroup $K_0 = \operatorname{Spin}(2n)$ of G. In Section 2 we list all small nilpotent orbits satisfying (1.1.2) and describe (the component groups of) their centralizers. In Section 3, we compute $R(\mathcal{O}, \psi)$ for each $\mathcal{O}$ in Subsection 2.1 and $\psi \in \widehat{A_K(\mathcal{O})}$. In Section 4 we associate to each $\mathcal{O}$ an infinitesimal character $\lambda_{\mathcal{O}}$ by [Bar17]. The fact is that $\mathcal{O}$ is the minimal orbit which can be the associated variety of a $(\mathfrak{g}, K)$-module with infinitesimal character (λ_L, λ_R), with λ_L and λ_R both conjugate to $\lambda_{\mathcal{O}}$. We make a complete list of irreducible modules $\overline{X}(\lambda_L, \lambda_R)$ (in terms of Langlands classification) which are attached to $\mathcal{O}$. Then we match the K-structure of these representations with $R(\mathcal{O},\psi)$. This demonstrates the conjecture we state at the beginning of the introduction. The following theorem summarizes this.

Theorem 1.2. *With notation as above, view $G_0 = \mathrm{Spin}(2n,\mathbb{C})$ as a real group. The K-structure of each representation in $\mathcal{U}_{G_0}(\mathcal{O},\lambda_{\mathcal{O}})$ is calculated explicitly and matches the K-structure of the $R(\mathcal{O},\psi)$ with $\psi \in \widehat{A_K(\mathcal{O})}$. That is, there is a* 1-1 *correspondence $\psi \in \widehat{A_K(\mathcal{O})} \longleftrightarrow \Xi(\mathcal{O},\psi) \in \mathcal{U}_{G_0}(\mathcal{O},\lambda_{\mathcal{O}})$ satisfying*

$$\Xi(\mathcal{O},\psi)\mid_K \cong R(\mathcal{O},\psi).$$

For the case $O(2n,\mathbb{C})$ (rather than $\mathrm{Spin}(2n,\mathbb{C})$), the K-structure of the representations studied in this paper were considered earlier in [McG94] and [BP11]. The unitarity of modules for the Spin groups (in particular the ones in this paper) is established in [Bre99].

2. Preliminaries

2.1. **Nilpotent orbits.** The complex nilpotent orbits of type D_n are parametrized by partitions of $2n$, with even blocks occurring with even multiplicities, and with I, II in the *very even* case (see [CM93]). The small nilpotent orbits satisfying (1.1.2) are those $\mathcal{O}$ with $\dim\mathcal{O} \le n^2$.

We list them out as the following four cases:

Case 1: $n = 2p$	$\mathcal{O} = [3\ 2^{n-2}\ 1]$	$\dim\mathcal{O} = n^2$,
Case 2: $n - 2p$ or $2p+1$	$\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$ $0 \le k \le p-1$	$\dim\mathcal{O} = 4nk - 4k^2 + 4n - 8k - 4$,
Case 3: $n = 2p$	$\mathcal{O} = [2^n]_{I,II}$	$\dim\mathcal{O} = n^2 - n$,
Case 4: $n = 2p$ or $2p+1$	$\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$ $0 \le k < n/2$	$\dim\mathcal{O} = 4nk - 4k^2 - 2k$.

Note that these are the orbits listed in [McG94]. The proof of the next proposition, and the details about the nature of the component groups, are in Section 5.

Proposition 2.2 (Corollary 5.4)**.**

Case 1: *If $\mathcal{O} = [3\ 2^{2p-2}\ 1]$, then $A_K(\mathcal{O}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*
Case 2: *If $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$ with $2n-4k-3 > 1$, then $A_K(\mathcal{O}) \cong \mathbb{Z}_2$.*
Case 3: *If $\mathcal{O} = [2^{2p}]_{I,II}$, then $A_K(\mathcal{O}) \cong \mathbb{Z}_2$.*
Case 4: *If $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$ with $2k < n$, then $A_K(\mathcal{O}) \cong 1$.*

In all cases $C_K(\mathcal{O}) = Z(K)\cdot C_K(\mathcal{O})^0$.

3. Regular sections

We use the notation introduced in Sections 1 and 2. We compute the centralizers needed for $R(\mathcal{O},\psi)$ in $\mathfrak{k}$ and in K. We use the standard roots and basis for $\mathfrak{so}(2n,\mathbb{C})$. A basis for the Cartan subalgebra is given by $H(\epsilon_i)$; the root vectors are $X(\pm\epsilon_i \pm \epsilon_j)$. Realizations in terms of the Clifford algebra and explicit calculations are in Section 5.

Let e be a representative of the orbit $\mathcal{O}$, and let $\{e, h, f\}$ be the corresponding Lie triple. Let

- $C_{\mathfrak{k}}(h)_i$ be the i-eigenspace of $\mathrm{ad}(h)$ in $\mathfrak{k}$,
- $C_{\mathfrak{k}}(e)_i$ be the i-eigenspace of $\mathrm{ad}(h)$ in the centralizer of e in $\mathfrak{k}$,
- $C_{\mathfrak{k}}(h)^+ := \sum_{i>0} C_{\mathfrak{k}}(h)_i$ and $C_{\mathfrak{k}}(e)^+ := \sum_{i>0} C_{\mathfrak{k}}(e)_i$.

3.1. We describe the centralizer for $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$ in detail. These are Cases 1 and 2. Representatives for e and h are

$$e = X(\epsilon_1 - \epsilon_{2k+2}) + X(\epsilon_1 + \epsilon_{2k+2}) + \sum_{2\le i\le 2k+1} X(\epsilon_i + \epsilon_{k+i}),$$

$$h = 2H(\epsilon_1) + \sum_{2\le i\le 2k+1} H(\epsilon_i) = H(2, \underbrace{1, \dots, 1}_{2k}, \underbrace{0, \dots, 0}_{n-1-2k}).$$

Then

$$\begin{aligned}
&C_{\mathfrak{k}}(h)_0 = \mathfrak{gl}(1) \times \mathfrak{gl}(2k) \times \mathfrak{so}(2n-2-4k),\\
&C_{\mathfrak{k}}(h)_1 = \mathrm{Span}\{X(\epsilon_1 - \epsilon_i),\ X(\epsilon_i \pm \epsilon_j),\ 2 \le i \le 2k+1 < j \le n\},\\
&C_{\mathfrak{k}}(h)_2 = \mathrm{Span}\{X(\epsilon_1 \pm \epsilon_j),\ X(\epsilon_i + \epsilon_l), 2 \le i \ne l \le 2k+1 < j \le n\},\\
&C_{\mathfrak{k}}(h)_3 = \mathrm{Span}\{X(\epsilon_1 + \epsilon_i),\ 2 \le i \le 2k+1\}.
\end{aligned} \tag{3.1.1}$$

Similarly

(3.1.2)

$$\begin{aligned}
&C_{\mathfrak{k}}(e)_0 \cong \mathfrak{sp}(2k) \times \mathfrak{so}(2n-3-4k),\\
&C_{\mathfrak{k}}(e)_1 = \mathrm{Span}\{X(\epsilon_1 - \epsilon_i) - X(\epsilon_{k+i} \pm \epsilon_{2k+2}),\ X(\epsilon_1 - \epsilon_{k+i}) - X(\epsilon_i \pm \epsilon_{2k+2}),\\
&\qquad 2 \le i \le k+1, X(\epsilon_j \pm \epsilon_l),\ 2 \le j \le 2k+1,\ 2k+3 \le l \le n\},\\
&C_{\mathfrak{k}}(e)_2 = C_{\mathfrak{k}}(h)_2,\\
&C_{\mathfrak{k}}(e)_3 = C_{\mathfrak{k}}(h)_3.
\end{aligned}$$

We denote by χ the trivial character of $C_{\mathfrak{k}}(e)$. A representation of K will be denoted by its highest weight:

$$V = V(a_1, \dots, a_n), \quad a_1 \ge \dots \ge |a_n|,$$

with all $a_i \in \mathbb{Z}$ or all $a_i \in \mathbb{Z} + 1/2$.

We will compute

$$\mathrm{Hom}_{C_{\mathfrak{k}}(e)}[V^*, \chi] = \mathrm{Hom}_{C_{\mathfrak{k}}(e)_0}\left[V^*/(C_{\mathfrak{k}}(e)^+V^*), \chi\right] := \left[V^*/(C_{\mathfrak{k}}(e)^+V^*)\right]^{\chi}. \tag{3.1.3}$$

3.2. **Case 1.** $n = 2p$, $\mathcal{O} = [3\ 2^{n-2}\ 1]$.

In this case $C_{\mathfrak{k}}(h)_0 = \mathfrak{gl}(1) \times \mathfrak{gl}(n-2) \times \mathfrak{so}(2), C_{\mathfrak{k}}(e)_0 = \mathfrak{sp}(n-2)$.

Consider the parabolic $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ determined by h,

$$\begin{aligned}
&\mathfrak{l} = C_{\mathfrak{k}}(h)_0 \cong \mathfrak{gl}(1) \times \mathfrak{gl}(n-2) \times \mathfrak{so}(2),\\
&\mathfrak{n} = C_{\mathfrak{k}}(h)^+.
\end{aligned} \tag{3.2.1}$$

We denote by V^* the dual of V. Since $n = 2p$, $V^* \cong V$. If V^* is a representation such that $\mathrm{Hom}_{C_{\mathfrak{k}}(e)}[V^*, \chi]$ in (3.1.3) is nonzero, then V^* is a quotient of a generalized Verma module $M(\lambda) = U(\mathfrak{k}) \otimes_{U(\overline{\mathfrak{p}})} F(\lambda)$, where λ is a weight of V^* which is dominant for $\overline{\mathfrak{p}}$. This is

$$\lambda = (-a_1; -a_{n-1}, \dots, -a_2; -a_n).$$

The ; denotes the fact that this is a (highest) weight of $\mathfrak{l} \cong \mathfrak{gl}(1) \times \mathfrak{gl}(n-2) \times \mathfrak{so}(2)$.

We choose the standard positive root system $\triangle^+(\mathfrak{l})$ for $\mathfrak{l}$. As a $C_{\mathfrak{k}}(e)_0$-module,

$$\mathfrak{n} = C_{\mathfrak{k}}(e)^+ \oplus \mathfrak{n}^\perp,$$

where we can choose $\mathfrak{n}^\perp = \mathrm{Span}\{X(\epsilon_1 - \epsilon_j),\ 2 \le j \le n-1\}$. This complement is $\mathfrak{l}$-invariant. It restricts to the standard module of $C_{\mathfrak{k}}(e)_0 = \mathfrak{sp}(n-2)$.

The generalized Bernstein–Gelfand–Gelfand resolution is

(3.2.2)

$$0 \cdots \longrightarrow \bigoplus_{w\in W^+,\ \ell(w)=k} M(w\cdot\lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{w\in W^+,\ \ell(w)=1} M(w\cdot\lambda) \longrightarrow M(\lambda) \longrightarrow V^* \longrightarrow 0,$$

with $w\cdot\lambda := w(\lambda+\rho(\mathfrak{k})) - \rho(\mathfrak{k})$, and $w \in W^+$, the $W(\mathfrak{l})$-coset representatives that make $w\cdot\lambda$ dominant for $\Delta^+(\mathfrak{l})$. This is a free $C_{\mathfrak{k}}(e)^+$-resolution so we can compute cohomology by considering

(3.2.3)

$$0 \cdots \longrightarrow \bigoplus_{w\in W^+,\ \ell(w)=k} \overline{M(w\cdot\lambda)} \longrightarrow \cdots \longrightarrow \bigoplus_{w\in W^+,\ \ell(w)=1} \overline{M(w\cdot\lambda)} \longrightarrow \overline{M(\lambda)} \longrightarrow \overline{V^*} \longrightarrow 0,$$

where $\overline{X}$ denotes $X/[C_{\mathfrak{k}}(e)^+X]$.

Note that in the sequences, $M(w\cdot\lambda) \cong S(\mathfrak{n}) \otimes_{\mathbb{C}} F(w\cdot\lambda)$ and $\overline{M(w\cdot\lambda)} \cong S(\mathfrak{n}^\perp) \otimes_{\mathbb{C}} F(w\cdot\lambda)$. As an $\mathfrak{l}$-module, $\mathfrak{n}^\perp$ has highest weight $(1;0,\dots,0,-1;0)$. Then $S^k(\mathfrak{n}^\perp) \cong F(k;0,\dots,0,-k;0)$ as an $\mathfrak{l}$-module.

Let $\mu := (-\alpha_1; -\alpha_{n-1},\dots,-\alpha_2; -\alpha_n)$ be the highest weight of an $\mathfrak{l}$-module. By the Pieri's rule,

$$S^k(\mathfrak{n}^\perp)\otimes F_\mu = \sum V(-\alpha_1+k; -\alpha_{n-1}-k_{n-1},\dots,-\alpha_3-k_3,-\alpha_2-k_2;-\alpha_n). \tag{3.2.4}$$

The sum is taken over

$$\{k_i \mid k_i \ge 0,\ \sum k_i = k,\ 0 \le k_i \le \alpha_{i-1}-\alpha_i,\ 3 \le i \le n-1\}.$$

Lemma 3.3. $\mathrm{Hom}_{C_{\mathfrak{k}}(e)_0}[S(\mathfrak{n}^\perp)\otimes F_\mu : \chi] \ne 0$ *for every* μ*. The multiplicity is* 1.

Proof. Since $(\mathfrak{gl}(n-2), \mathfrak{sp}(n-2))$ is a hermitian symmetric pair, the theorem of Cartan and Helgason (cf. Theorem 3.3.1.1 in [War72]) implies that a composition factor in the terms of $S(\mathfrak{n}^\perp)\otimes F_\mu$ in (3.2.4) admits $C_{\mathfrak{k}}(e)_0$-fixed vectors only if

$$-\alpha_{n-1}-k_{n-1} = -\alpha_{n-2}-k_{n-2},\ -\alpha_{n-3}-k_{n-3} = -\alpha_{n-4}-k_{n-4},\dots,-\alpha_3-k_3 = -\alpha_2-k_2.$$

The conditions $0 \le k_i \le \alpha_{i-1}-\alpha_i$ imply

$$\begin{aligned} k_{n-2} &= 0, \quad k_{n-1} = \alpha_{n-2}-\alpha_{n-1},\\ &\vdots\\ k_4 &= 0, \quad k_5 = \alpha_4-\alpha_5,\\ k_2 &= 0, \quad k_3 = \alpha_2-\alpha_3. \end{aligned} \tag{3.3.1}$$

Therefore, given μ, the weight of the $C_{\mathfrak{k}}(e)_0$-fixed vector in $S(\mathfrak{n}^\perp)\otimes F_\mu$ is

$$(-\alpha_1+\alpha_2-\alpha_3+\alpha_4-\alpha_5+\cdots+\alpha_{n-2}-\alpha_{n-1}; -\alpha_{n-2},-\alpha_{n-2},\dots,-\alpha_2,\alpha_2;-\alpha_n),$$

and the multiplicity is 1. □

Corollary 3.4. *For every* $V(a_1,\dots,a_n) \in \widehat{K}$, $\mathrm{Hom}_{C_{\mathfrak{k}}(e)}[V,\chi] = 0$ *or* 1*. The action of* $\mathrm{ad}\,h$ *is* $-2\sum_{1\le i\le p} a_{2i-1}$.

Proof. The first statement follows from Lemma 3.3 and the surjection

$$\overline{M(\lambda)} \cong S(\mathfrak{n}^{\perp}) \otimes_{\mathbb{C}} F(\lambda) \longrightarrow \overline{V^*} \longrightarrow 0.$$

The action of $\operatorname{ad} h$ is computed from the module

$$V(-a_1+k; -a_{n-2}, -a_{n-2}, \dots, -a_2, -a_2; -a_n) \tag{3.4.1}$$

with $k = a_2 - a_3 + a_4 - a_5 + \dots + a_{n-2} - a_{n-1}$. The value is $-2\sum_{1\le i\le p} a_{2i-1}$. □

$\ell(\mathbf{w}) = \mathbf{1}$. To show that the weights in (3.4.1) actually occur, it is enough to show that these weights do not occur in the term in the BGG resolution (3.2.3) with $\ell(w) = 1$.

We calculate $w \cdot \lambda$:

$$\rho = \rho(\mathfrak{k}) = (-(n-1); -1, -2, \dots, -(n-2); 0)$$

is dominant for $\overline{\mathfrak{p}}$, and

$$\lambda + \rho = (-a_1 - n + 1; -a_{n-1} - 1, -a_{n-2} - 2, \dots, -a_2 - n + 2; -a_n).$$

There are three elements $w \in W^+$ of length 1. They are the left $W(\mathfrak{l})$-cosets of

$$w_1 = s_{\epsilon_1 - \epsilon_{n-1}},\ w_2 = s_{\epsilon_2 - \epsilon_n}, w_3 = s_{\epsilon_2 + \epsilon_n}.$$

So

$$\begin{aligned} w_1 \cdot \lambda &= (-a_2 + 1; -a_{n-1}, -a_{n-2}, \dots, -a_4, -a_3, -a_1 - 1; -a_n), \\ w_2 \cdot \lambda &= (-a_1; -a_n + 1, -a_{n-2}, -a_{n-3}, \dots, -a_3, -a_2; -a_{n-1} - 1), \\ w_3 \cdot \lambda &= (-a_1; a_n + 1, -a_{n-2}, -a_{n-3}, \dots, -a_3, -a_2; a_{n-1} + 1). \end{aligned} \tag{3.4.2}$$

Lemma 3.5. *For all* λ, $\operatorname{Hom}_{C_{\mathfrak{k}}(e)}[\overline{M(w_i \cdot \lambda)}, \chi] = 1$. *The eigenvalues of* $\operatorname{ad} h$ *are different from* $-2\sum_{1\le i\le p} a_{2i-1}$ *for each* w_i.

Proof. The $\mathfrak{sp}(n-2)$-fixed weights coming from $S(\mathfrak{n}^{\perp}) \otimes F(w_i \cdot \lambda)$, $i = 1, 2, 3$, are

$$\begin{aligned} w_1 \longleftrightarrow &(a_1 - a_2 - a_3 + a_4 - a_5 + \dots + a_{n-2} - a_{n-1} + 2; \\ &\quad - a_{n-2}, -a_{n-2}, \dots, -a_4, -a_4, -a_1 - 1, -a_1 - 1; -a_n) \\ w_2 \longleftrightarrow &(-a_1 + a_2 - a_3 + \dots + a_{n-4} - a_{n-3} + a_{n-2} - a_n + 1; \\ &\quad - a_{n-2}, -a_{n-2}, \dots, -a_4, -a_4, -a_2, -a_2; -a_{n-1} - 1) \\ w_3 \longleftrightarrow &(-a_1 + a_2 - a_3 + \dots + a_{n-4} - a_{n-3} + a_{n-2} + a_n + 1; \\ &\quad - a_{n-2}, -a_{n-2}, \dots, -a_4, -a_4, -a_2, -a_2; a_{n-1} - 1) \end{aligned} \tag{3.5.1}$$

The negatives of the weights of h are

$$\begin{aligned} w_0 = 1 &\quad\longleftrightarrow\quad 2(a_1 + a_3 + \dots + a_{n-1}), \\ w_1 &\quad\longleftrightarrow\quad 2(a_2 + a_3 + a_5 \dots + a_{n-1} - 1), \\ w_2 &\quad\longleftrightarrow\quad 2(a_1 + a_3 + \dots + a_{n-3} + a_n - 1), \\ w_3 &\quad\longleftrightarrow\quad 2(a_1 + a_3 + \dots + a_{n-3} - a_n - 1). \end{aligned} \tag{3.5.2}$$

The last three weights are not equal to the first one. This completes the proof. □

Theorem 3.6. *Let $K=\mathrm{Spin}(2n,\mathbb{C})$ with $n=2p$. Every representation $V(a_1,\dots,a_n)$ has $C_{\mathfrak{k}}(e)$ fixed vectors and the multiplicity is 1. We write $C_K(\mathcal{O}) := C_K(e)$. In summary,*

$$\mathrm{Ind}_{C_K(\mathcal{O})^0}^K(Triv) = \bigoplus_{a\in\widehat{K}} V(a_1,\dots,a_n).$$

Theorem 3.6 can be interpreted as computing regular functions on the universal cover $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ transforming trivially under $C_{\mathfrak{k}}(e)_0$. We decompose it further:

(3.6.1) $R(\widetilde{\mathcal{O}},Triv) := \mathrm{Ind}_{C_K(\mathcal{O})^0}^K(Triv) = \mathrm{Ind}_{C_K(\mathcal{O})}^K\left[\mathrm{Ind}_{C_K(\mathcal{O})^0}^{C_K(\mathcal{O})}(Triv)\right].$

The inner induced module splits into

$$\mathrm{Ind}_{C_K(\mathcal{O})^0}^{C_K(\mathcal{O})}(Triv) = \sum \psi, \tag{3.6.2}$$

where ψ are the irreducible representations of $C_K(\mathcal{O})$ trivial on $C_K(\mathcal{O})^0$. Thus, the sum in (3.6.2) is taken over $\widehat{A_K(\mathcal{O})}$.

Then

$$R(\widetilde{\mathcal{O}},Triv) = \mathrm{Ind}_{C_K(\mathcal{O})^0}^K(Triv) = \sum_{\psi\in\widehat{A_K(\mathcal{O})}} R(\mathcal{O},\psi). \tag{3.6.3}$$

We will decompose $R(\mathcal{O},\psi)$ explicitly as a representation of K.

Lemma 3.7. *Let $K=\mathrm{Spin}(2n,\mathbb{C})$ with $n=2p$. Let μ_i, $1\le i\le 4$, be the following K-types parametrized by their highest weights:*

$$\mu_1=(0,\dots,0), \mu_2=(1,0,\dots,0),$$
$$\mu_3=(\tfrac12,\dots,\tfrac12), \mu_4=(\tfrac12,\dots,\tfrac12,-\tfrac12).$$

Let ψ_i be the restriction of the highest weight of μ_i to $C_K(\mathcal{O})$, respectively. Then

$$\mathrm{Ind}_{C_K(\mathcal{O})^0}^{C_K(\mathcal{O})}(Triv) = \sum_{i=1}^4 \psi_i.$$

Proposition 3.8. *Let $K=\mathrm{Spin}(2n,\mathbb{C})$ with $n=2p$. The induced representation* (3.6.3) *decomposes as*

$$\mathrm{Ind}_{C_K(\mathcal{O})}^K(Triv) = \sum_{i=1}^4 R(\mathcal{O},\psi_i),$$

where

$$R(\mathcal{O},\psi_1) = \mathrm{Ind}_{C_K(\mathcal{O})}^K(\psi_1) = \bigoplus V(a_1,\dots,a_n) \text{ with } a_i\in\mathbb{Z}, \sum a_i\in 2\mathbb{Z},$$
$$R(\mathcal{O},\psi_2) = \mathrm{Ind}_{C_K(\mathcal{O})}^K(\psi_2) = \bigoplus V(a_1,\dots,a_n) \text{ with } a_i\in\mathbb{Z}, \sum a_i\in 2\mathbb{Z}+1,$$
$$R(\mathcal{O},\psi_3) = \mathrm{Ind}_{C_K(\mathcal{O})}^K(\psi_3) = \bigoplus V(a_1,\dots,a_n) \text{ with } a_i\in\mathbb{Z}+1/2, \sum a_i\in 2\mathbb{Z}+p,$$
$$R(\mathcal{O},\psi_4) = \mathrm{Ind}_{C_K(\mathcal{O})}^K(\psi_4) = \bigoplus V(a_1,\dots,a_n) \text{ with } a_i\in\mathbb{Z}+1/2, \sum a_i\in 2\mathbb{Z}+p+1.$$

3.9. **Case 2:** $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$, $0 \le k \le p-1$.

Consider the parabolic $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ determined by h:

$$\mathfrak{l} = C_{\mathfrak{k}}(h)_0 \cong \mathfrak{gl}(1) \times \mathfrak{gl}(2k) \times \mathfrak{so}(2n-2-4k),$$
$$\mathfrak{n} = C_{\mathfrak{k}}(h)^+.$$

In this section, let $\epsilon = -1$ when n is even; $\epsilon = 1$ when n is odd. The dual of V, denoted V^*, has lowest weight $(\epsilon a_n, -a_{n-1}, \dots, -a_2, -a_1)$. It is therefore a quotient of a generalized Verma module $M(\lambda) = U(\mathfrak{k}) \otimes_{U(\overline{\mathfrak{p}})} F(\lambda)$, where λ is dominant for $\overline{\mathfrak{p}}$ and dominant for the standard positive system for $\mathfrak{l}$:

$$\lambda = (-a_1; \underbrace{-a_{2k+1}, \dots, -a_3, -a_2}_{2k}; \underbrace{a_{2k+2}, \dots, a_{n-1}, \epsilon a_n}_{n-1-2k}).$$

$\mathfrak{n} = C_{\mathfrak{k}}(e)^+ \oplus \mathfrak{n}^{\perp}$ as a module for $C_{\mathfrak{k}}(e)_0$. A basis for $\mathfrak{n}^{\perp} \subset C_{\mathfrak{k}}(h)_1$ is given by

$$\{X(\epsilon_1 - \epsilon_{2k+2})\}, \quad 2 \le i \le 2k+1.$$

This is the standard representation of $\mathfrak{sp}(2k)$, trivial for $\mathfrak{so}(2n-4-4k)$. We write its highest weight as

$$(1; 0, \dots, 0, -1; 0, \dots, 0).$$

We can now repeat the argument for the case $k = p$; there is an added constraint that $a_{2k+3} = \cdots = a_n = 0$ because the representation with highest weight $(a_{2k+2}, \dots, a_{n-1}, \epsilon a_n)$ of $\mathfrak{so}(2n-2-4k)$ must have fixed vectors for $\mathfrak{so}(2n-3-4k)$.

Then the next theorem follows.

Theorem 3.10. *A representation $V(a_1, \dots, a_n)$ has $C_{\mathfrak{k}}(e)$ fixed vectors if and only if*

$$a_{2k+3} = \cdots = a_n = 0,$$

and the multiplicity is 1. *In summary,*

$$\mathrm{Ind}^K_{C_K(\mathcal{O})^0}(Triv) = \bigoplus V(a_1, \dots, a_{2k+2}, 0 \dots, 0), \text{ with } a_1 \ge \dots \ge a_{2k+2} \ge 0,\ a_i \in \mathbb{Z}.$$

As in (3.6.3), we decompose $\mathrm{Ind}^K_{C_K(\mathcal{O})^0}(Triv)$ further into the sum of $R(\mathcal{O}, \psi)$ with $\psi \in \widehat{A_K(\mathcal{O})}$.

Lemma 3.11. *Let μ_1, μ_2 be the following K-types parametrized by their highest weights:*

$$\mu_1 = (0, \dots, 0), \mu_2 = (1, 0, \dots, 0).$$

Let ψ_i be the restriction of the highest weight of μ_i to $C_G(\mathcal{O})$, respectively. Then

$$\mathrm{Ind}^{C_K(\mathcal{O})}_{C_K(\mathcal{O})^0}(Triv) = \psi_1 + \psi_2.$$

Proposition 3.12. *The induced representation* (3.6.3) *decomposes as*

$$\mathrm{Ind}^K_{C_K(\mathcal{O})^0}(Triv) = R(\mathcal{O}, \psi_1) + R(\mathcal{O}, \psi_2),$$

where

$$R(\mathcal{O}, \psi_1) = \mathrm{Ind}^K_{C_K(\mathcal{O})}(\psi_1) = \bigoplus V(a_1, \dots, a_{2k+2}, 0, \dots, 0) \text{ with } a_i \in \mathbb{Z},\ \sum a_i \in 2\mathbb{Z},$$
$$R(\mathcal{O}, \psi_2) = \mathrm{Ind}^K_{C_K(\mathcal{O})}(\psi_2) = \bigoplus V(a_1, \dots, a_{2k+2}, 0, \dots, 0) \text{ with } a_i \in \mathbb{Z}, \sum a_i \in 2\mathbb{Z}+1.$$

3.13. Now we treat $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$ with $0 \le k \le p$. These are Cases 3 and 4. When $k = p$ (and hence $n = 2p$), the orbit is labeled by I, II. The computation is similar and easier than the previous two cases. We state the results for $R(\widetilde{O}, Triv)$ as follows.

Theorem 3.14.

Case 3: *For $k = p$, so $n = 2p$,*

$$\begin{aligned} \mathcal{O}_I = [2^n]_I, \qquad R(\widetilde{\mathcal{O}_I}, Triv) &= \operatorname{Ind}^K_{C_K(\mathcal{O}_I)^0}(Triv) \\ &= \bigoplus V(a_1, a_1, a_3, a_3, \ldots, a_{n-1}, a_{n-1}), \\ \mathcal{O}_{II} = [2^n]_{II}, \qquad R(\widetilde{\mathcal{O}_{II}}, Triv) &= \operatorname{Ind}^K_{C_K(\mathcal{O}_{II})^0}(Triv) \\ &= \bigoplus V(a_1, a_1, a_3, a_3, \ldots, a_{n-1}, -a_{n-1}). \end{aligned}$$

Case 4: *For $k \le p - 1$,*

$$\begin{aligned} \mathcal{O} = [2^{2k}\ 1^{2n-4k}], \qquad R(\widetilde{\mathcal{O}}, Triv) &= \operatorname{Ind}^K_{C_K(\mathcal{O})^0}(Triv) \\ &= \bigoplus V(a_1, a_1, a_3, a_3, \ldots, a_{2k-1}, a_{2k-1}, 0, \ldots, 0), \end{aligned}$$

satisfying $a_1 \ge a_3 \ge \cdots \ge a_{2k-1} \ge 0$.

Proof. We treat the case $n = 2p$ and $k \le p - 1$; $n = 2p + 1$ is similar. A representative of $\mathcal{O}$ is $e = X(\epsilon_1 + \epsilon_2) + \cdots + X(\epsilon_{2k-1} + \epsilon_{2k})$, and the corresponding middle element in the Lie triple is $h = H(\underbrace{1, \ldots, 1}_{2k}, \underbrace{0, \ldots, 0}_{n-2k})$. Thus

$$\begin{aligned} C_{\mathfrak{k}}(h)_0 &= \mathfrak{gl}(2k) \times \mathfrak{so}(2n - 4k), \\ C_{\mathfrak{k}}(h)_1 &= \operatorname{Span}\{X(\epsilon_i \pm \epsilon_j)\}, \qquad 1 \le i \le 2k < j \le n, \\ C_{\mathfrak{k}}(h)_2 &= \operatorname{Span}\{X(\epsilon_l + \epsilon_m)\}, \qquad 1 \le l \ne m \le 2k. \end{aligned} \tag{3.14.1}$$

and

$$\begin{aligned} C_{\mathfrak{k}}(e)_0 &= \mathfrak{sp}(2k) \times \mathfrak{so}(2n - 4k), \\ C_{\mathfrak{k}}(e)_1 &= C_{\mathfrak{k}}(h)_1, \\ C_{\mathfrak{k}}(e)_2 &= C_{\mathfrak{k}}(h)_2. \end{aligned} \tag{3.14.2}$$

As before, let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ be the parabolic subalgebra determined by h, and let $V = V(a_1, \ldots, a_n)$ be an irreducible representation of K. Since we assumed $n = 2p$, $V = V^*$. In this case $C_{\mathfrak{k}}(e)^+ = \mathfrak{n}$, so Kostant's theorem implies $V/[C_{\mathfrak{k}}(e)^+ V] = V_{\mathfrak{l}}(a_1, \ldots a_{2k}; a_{2k+1}, \ldots, a_n)$ as a $\mathfrak{gl}(2k) \times \mathfrak{so}(2n - 4k)$-module. Since we want $\mathfrak{sp}(2k) \times \mathfrak{so}(2n - 4k)$-fixed vectors, $a_{2k+1} = \cdots = a_n = 0$, and Cartan and Helgason's theorem implies $a_1 = a_2, a_3 = a_4, \ldots, a_{2k-1} = a_{2k}$.

When $n = 2p$ and $\mathcal{O} = [2^n]_{I,II}$, the calculations are similar to $k \le p - 1$. The choices I, II are

$$e_I = X(\epsilon_1 - \epsilon_2) + X(\epsilon_3 - \epsilon_4) + \cdots + X(\epsilon_{n-1} - \epsilon_n), \qquad h_I = H(1, \ldots, 1),$$

$$e_{II} = X(\epsilon_1 - \epsilon_2) + X(\epsilon_3 - \epsilon_4) + \cdots + X(\epsilon_{n-3} - \epsilon_{n-2}) + X(\epsilon_{n-1} + \epsilon_n), \qquad h_{II} = H(1, \ldots, 1, -1).$$

These orbits are induced from the two nonconjugate maximal parabolic subalgebras with $\mathfrak{gl}(n)$ as Levi components, and $R(\widetilde{\mathcal{O}_{I,II}}, Triv)$ are just the induced modules from the trivial representation on the Levi component. □

We aim at decomposing $R(\widetilde{\mathcal{O}}, Triv) = \sum R(\mathcal{O}, \psi)$ with $\psi \in \widehat{A_K(\mathcal{O})}$ as before.

Lemma 3.15.

Case 3: *$n = 2p$, $\mathcal{O} = [2^n]_{I,II}$. Let μ_1, μ_2, ν_1, ν_2, be*

$$\mu_1 = (1, \dots, 1), \mu_2 = (\tfrac{1}{2}, \dots \tfrac{1}{2}),$$
$$\nu_1 = (1, \dots, 1, -1), \nu_2 = (\tfrac{1}{2}, \dots, \tfrac{1}{2}, -\tfrac{1}{2}).$$

Let ψ_i be the restriction of the highest weight of μ_i to $C_K(e)$, and let ϕ_i be the restriction of the highest weight of ν_i, respectively. Then

$$\begin{aligned}\mathrm{Ind}_{C_K(\mathcal{O}_I)^0}^{C_K(\mathcal{O}_I)}(Triv) &= \psi_1 + \psi_2,\\ \mathrm{Ind}_{C_K(\mathcal{O}_{II})^0}^{C_K(\mathcal{O}_{II})}(Triv) &= \phi_1 + \phi_2.\end{aligned}$$

The ψ_i, ϕ_i are viewed as representations of $\widehat{A_K(\mathcal{O}_{I,II})}$, and ψ_1 and ϕ_1 are $Triv$, and ψ_2, ϕ_2 are Sgn.

Case 4: *$\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$, $0 \le k \le p-1$.*

$$\mathrm{Ind}_{C_K(\mathcal{O})^0}^{C_K(\mathcal{O})}(Triv) = Triv.$$

Then we are able to split up $R(\widetilde{\mathcal{O}}, Triv)$ as a sum of $R(\mathcal{O}, \psi)$ as in (3.6.3).

Proposition 3.16.

Case 3: *$n = 2p$, $\mathcal{O} = [2^n]_{I,II}$: $R(\widetilde{\mathcal{O}}_{I,II}) = R(\mathcal{O}_{I,II}, Triv) + R(\mathcal{O}_{I,II}, Sgn)$ with*

$$\begin{aligned}
R(\mathcal{O}_I, Triv) &= \mathrm{Ind}_{C_K(\mathcal{O}_I)}^K(Triv)\\
&= \bigoplus V(a_1, a_1, a_3, a_3, \dots, a_{n-1}, a_{n-1}), \quad \text{with } a_i \in \mathbb{Z},\\
R(\mathcal{O}_I, Sgn) &= \mathrm{Ind}_{C_K(\mathcal{O}_I)}^K(Sgn)\\
&= \bigoplus V(a_1, a_1, a_3, a_3, \dots, a_{n-1}, a_{n-1}), \quad \text{with } a_i \in \mathbb{Z} + 1/2,\\
R(\mathcal{O}_{II}, Triv) &= \mathrm{Ind}_{C_K(\mathcal{O}_{II})}^K(Triv)\\
&= \bigoplus V(a_1, a_1, a_3, a_3, \dots, a_{n-1}, -a_{n-1}), \quad \text{with } a_i \in \mathbb{Z},\\
R(\mathcal{O}_{II}, Sgn) &= \mathrm{Ind}_{C_K(\mathcal{O}_{II})}^K(Sgn)\\
&= \bigoplus V(a_1, a_1, a_3, a_3, \dots, a_{n-1}, -a_{n-1}), \quad \text{with } a_i \in \mathbb{Z} + 1/2,
\end{aligned}$$

satisfying $a_1 \ge a_3 \ge \dots \ge a_{n-1} \ge 0$.

Case 4: *$\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$, $0 \le k \le p-1$:*

$$\begin{aligned}
R(\widetilde{\mathcal{O}}, Triv) = R(\mathcal{O}, Triv) &= \mathrm{Ind}_{C_K(\mathcal{O})}^K(Triv)\\
&= \bigoplus V(a_1, a_1, a_3, a_3, \dots, a_{2k-1}, a_{2k-1}, 0, \dots, 0), \quad \text{with } a_i \in \mathbb{Z},
\end{aligned}$$

satisfying $a_1 \ge a_3 \ge \dots \ge a_{2k-1} \ge 0$.

4. Representations with small support

4.1. **Langlands classification.** Let G be a complex linear algebraic reductive group viewed as a real Lie group. Let θ be a Cartan involution with fixed points K. Let $G \supset B = HN \supset H = TA$ be a Borel subgroup containing a fixed θ-stable Cartan subalgebra H, with

$$T = \{h \in H \mid \theta(h) = h\},$$
$$A = \{h \in H \mid \theta(h) = h^{-1}\}.$$

The Langlands classification is as follows. Let $\chi \in \widehat{H}$. Denote by

$$X(\chi) := \mathrm{Ind}_B^G[\chi \otimes \mathbb{1}]_{K\text{-finite}}$$

the corresponding admissible standard module (Harish-Chandra induction). Let (μ, ν) be the differentials of $\chi\mid_T$ and $\chi\mid_A$, respectively. Let $\lambda_L = (\mu+\nu)/2$, and let $\lambda_R = (\mu - \nu)/2$. We write $X(\mu,\nu) = X(\lambda_L, \lambda_R) = X(\chi)$.

Theorem 4.2.

(1) $X(\mu,\nu)$ *has a unique irreducible subquotient denoted* $\overline{X}(\mu,\nu)$ *which contains the* K*-type with extremal weight* μ *occurring with multiplicity one in* $X(\mu,\nu)$.

(2) $\overline{X}(\mu,\nu)$ *is the unique irreducible quotient when* $\langle Re\nu, \alpha\rangle > 0$ *for all* $\alpha \in \Delta(\mathfrak{n},\mathfrak{h})$, *and the unique irreducible submodule when* $\langle Re\nu,\alpha\rangle < 0$.

(3) $\overline{X}(\mu,\nu) \cong \overline{X}(\mu',\nu')$ *if and only if there is* $w \in W$ *such that* $w\mu = \mu', w\nu = \nu'$. *Similarly for* (λ_L, λ_R).

Assume λ_L, λ_R are both dominant integral. Write $F(\lambda)$ to be the finite-dimensional representation of G with infinitesimal character λ. Then $\overline{X}(\lambda_L, -\lambda_R)$ is the finite-dimensional representation $F(\lambda_L) \otimes F(-w_0\lambda_R)$ where $w_0 \in W$ is the longest element in the Weyl group. The lowest K-type has extremal weight $\lambda_L - \lambda_R$. Weyl's character formula implies

$$\overline{X}(\lambda_L, -\lambda_R) = \sum_{w\in W} \epsilon(w) X(\lambda_L, -w\lambda_R). \tag{4.2.1}$$

In the following contents in this section, we use different notation as follows. We write $(G,K) = (\mathrm{Spin}(2n,\mathbb{C}), \mathrm{Spin}(2n))$ and $(\underline{G}, \underline{K}) = (\mathrm{SO}(2n,\mathbb{C}), \mathrm{SO}(2n))$.

4.3. **Infinitesimal characters.** From [Bar17], we can associate to each $\mathcal{O}$ in Section 2.1 an infinitesimal character $\lambda_{\mathcal{O}}$. The fact is that $\mathcal{O}$ is the minimal orbit which can be the associated variety of a $(\mathfrak{g}, K)$-module with infinitesimal character (λ_L, λ_R), with λ_L and λ_R both conjugate to $\lambda_{\mathcal{O}}$. The $\lambda_{\mathcal{O}}$ are listed below.

Case 1: $n = 2p$, $\mathcal{O} = [3\ 2^{n-2}\ 1]$,

$$\lambda_{\mathcal{O}} = \rho/2 = (p - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \mid p-1, \dots, 1, 0).$$

Case 2: $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$, $0 \le k \le p-1$,

$$\lambda_{\mathcal{O}} = (k + \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \mid n-k-2, \dots, 1, 0).$$

Case 3: $n = 2p$, $\mathcal{O}_{I,II} = [2^n]_{I,II}$,

$$\lambda_{\mathcal{O}_I} = \left(\frac{2n-1}{4}, \frac{2n-5}{4}, \dots, \frac{-(2n-7)}{4}, \frac{-(2n-3)}{4}\right),$$
$$\lambda_{\mathcal{O}_{II}} = \left(\frac{2n-1}{4}, \frac{2n-5}{4}, \dots, \frac{-(2n-7)}{4}, \frac{(2n-3)}{4}\right).$$

Case 4: $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$, $0 \le k \le p-1$,

$$\lambda_{\mathcal{O}} = (k, k-1, \dots, 1; n-k-1, \dots, 1, 0).$$

Notice that the infinitesimal characters in Cases 1 and 2 are nonintegral. For instance, in Case 1, $\lambda_{\mathcal{O}} = \rho/2$, where ρ is a half sum of the positive roots of type D_{2p}. The integral system is of type $D_p \times D_p$. The notation $|$ separates the coordinates of the two D_p.

4.4. We define the following irreducible modules in terms of Langlands classification:

Case 1: $n = 2p$, $\mathcal{O} = [3\ 2^{n-1}\ 1]$.
(i) $\Xi_1 = \overline{X}(\lambda_{\mathcal{O}}, -\lambda_{\mathcal{O}})$;
(ii) $\Xi_2 = \overline{X}(\lambda_{\mathcal{O}}, -w_1\lambda_{\mathcal{O}})$, where $w_1\lambda_{\mathcal{O}} = (p-\frac{1}{2}, \dots, \frac{3}{2}, -\frac{1}{2} \mid p-1, \dots, 1, 0)$;
(iii) $\Xi_3 = \overline{X}(\lambda_{\mathcal{O}}, -w_2\lambda_{\mathcal{O}})$, where $w_2\lambda_{\mathcal{O}} = (p-1, \dots, 1, 0 \mid p-\frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2})$;
(iv) $\Xi_4 = \overline{X}(\lambda_{\mathcal{O}}, -w_3\lambda_{\mathcal{O}})$, where $w_3\lambda_{\mathcal{O}} = (p-1, \dots, 1, 0 \mid p-\frac{1}{2}, \dots, \frac{3}{2}, -\frac{1}{2})$.
Case 2: $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$, $0 \le k \le p-1$.
(i) $\Xi_1 = \overline{X}(\lambda_{\mathcal{O}}, -\lambda_{\mathcal{O}})$;
(ii) $\Xi_2 = \overline{X}(\lambda_{\mathcal{O}}, -w_1\lambda_{\mathcal{O}})$, $w_1\lambda_{\mathcal{O}} = (k+\frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \mid n-k-2, \dots, 1, 0)$.
Case 3: $n = 2p$, $\mathcal{O}_{I,II} = [2^n]_{I,II}$.
(i) $\Xi_I = \overline{X}(\lambda_{\mathcal{O}_I}, -\lambda_{\mathcal{O}_I})$;
(i′) $\Xi_I = \overline{X}(\lambda_{\mathcal{O}_I}, -w\lambda_{\mathcal{O}_I})$, $w\lambda_{\mathcal{O}_I} = \left(\frac{2n-3}{4}, \frac{2n-7}{4}, \dots, \frac{-(2n-5)}{4}, \frac{-(2n-1)}{4}\right)$;
(ii) $\Xi_{II} = \overline{X}(\lambda_{\mathcal{O}_{II}}, -\lambda_{\mathcal{O}_{II}})$;
(ii′) $\Xi'_{II} = \overline{X}(\lambda_{\mathcal{O}_{II}}, -w\lambda_{\mathcal{O}_{II}})$, $w\lambda_{\mathcal{O}_{II}} = \left(\frac{2n-3}{4}, \frac{2n-7}{4}, \dots, \frac{-(2n-5)}{4}, \frac{2n-1}{4}\right)$.
Case 4: $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$, $0 \le k \le p-1$.
(i) $\Xi = \overline{X}(\lambda_{\mathcal{O}}, -\lambda_{\mathcal{O}})$.

Remark 4.5. The representations introduced above form the set $\mathcal{U}_G(\mathcal{O}, \lambda_{\mathcal{O}})$. The integral systems are of type $D_p \times D_p$ in Case 1, $D_{k+1} \times D_{n-k-1}$ in Case 2, A_n in Case 3, and D_n in Case 4. It is then a matter of computing the multiplicity of the sgn representation in the corresponding primitive ideal double cells for these integral systems. We omit the details.

Notation. We write $F(\lambda)$ for the finite-dimensional representation of the appropriate SO or Spin group with infinitesimal character λ. Write $V(\mu)$ for the finite-dimensional representation of the appropriate SO or Spin group with highest weight μ.

4.6. K**-structure.** We compute the K-types of each representation listed in Subsection 4.4.

Case 1: The arguments are refinements of those in [McG94]. Let H be the image of $\mathrm{Spin}(2p,\mathbb{C}) \times \mathrm{Spin}(2p,\mathbb{C})$ in $\mathrm{Spin}(4p,\mathbb{C})$, and U the image of the maximal compact subgroup $\mathrm{Spin}(2p) \times \mathrm{Spin}(2p)$ in K. Irreducible representations of U can be viewed as $\mathrm{Spin}(2p) \times \mathrm{Spin}(2p)$-representations such that $\pm(I, I)$ acts trivially.

Cases (i) and (ii) factor to representations of $SO(2n,\mathbb{C})$, and (iii) and (iv) are genuine for $\mathrm{Spin}(2n,\mathbb{C})$.

The Kazhdan–Lusztig conjectures for a nonintegral infinitesimal character (cf. [ABV92, Chapters 16 and 17]) together with Weyl's formula for the character of a finite-dimensional module (see (4.2.1)) imply that

$$\overline{X}(\rho/2, -w_i\rho/2) = \sum_{w\in W(D_p\times D_p)} \epsilon(w) X(\rho/2, -ww_i\rho/2), \tag{4.6.1}$$

since $W(\lambda_{\mathcal{O}}) = W(D_p \times D_p)$.

Restricting (4.6.1) to K, and using Frobenius reciprocity, we get

$$\overline{X}(\rho/2, -w_i\rho/2) \mid_K = \mathrm{Ind}_U^K [F_1(\rho/2) \otimes F_2(-w_i\rho/2)], \tag{4.6.2}$$

where $F_{1,2}$ are finite dimensional representations of the two factors $\mathrm{Spin}(2p,\mathbb{C}) \times \mathrm{Spin}(2p,\mathbb{C})$ with infinitesimal character $\rho/2$ and $-w_i\rho/2$, respectively. The terms $[F_1(\rho/2)\otimes F_2(-w_i\rho/2)]$ are

- **(i)** $V(1/2,\dots,1/2) \otimes V(1/2,\dots,1/2) \boxtimes V(0,\dots,0) \otimes V(0,\dots,0)$,
- **(ii)** $V(1/2,\dots,-1/2) \otimes V(1/2,\dots,1/2) \boxtimes V(0,\dots,0) \otimes V(0,\dots,0)$,
- **(iii)** $V(1/2,\dots,1/2) \otimes V(0,\dots,0) \boxtimes V(0,\dots,0) \otimes V(1/2,\dots,1/2)$,
- **(iv)** $V(1/2,\dots,1/2) \otimes V(0,\dots,0) \boxtimes V(0,\dots,0) \otimes V(1/2,\dots,-1/2)$

as $\mathrm{Spin}(n) \times \mathrm{Spin}(n)$-representations (see Subsection 4.4 for the notation).

Lemma 4.7. *Let* $SPIN_+ = V(\frac{1}{2},\dots,\frac{1}{2})$ *and* $SPIN_- = V(\frac{1}{2},\dots,\frac{1}{2},-\frac{1}{2}) \in \widehat{\mathrm{Spin}(n)}$. *Then*

$$\begin{aligned} SPIN_+ \otimes SPIN_+ &= \bigoplus_{0\le k\le[\frac{p}{2}]} V(\underbrace{1\dots1}_{2k},\underbrace{0\dots0}_{p-2k}),\\ SPIN_+ \otimes SPIN_- &= \bigoplus_{0\le k\le[\frac{p-1}{2}]} V(\underbrace{1\dots1}_{2k+1}.\underbrace{0\dots0}_{p-2k-1}). \end{aligned} \tag{4.7.1}$$

Proof. The proof is straightforward. □

Lemma 4.7 implies that (4.6.2) becomes

(4.7.2)

$$\begin{aligned}
&\text{(i) } \overline{X}(\rho/2,-\rho/2)\mid_K &&= \mathrm{Ind}_U^K\left[\bigoplus_{0\le k\le[\frac{p}{2}]} V(\underbrace{1,\dots,1}_{2k},0,\dots,0)\boxtimes V(0,\dots,0)\right],\\
&\text{(ii) } \overline{X}(\rho/2,-w_1\rho/2)\mid_K &&= \mathrm{Ind}_U^K\left[\bigoplus_{0\le k\le[\frac{p-1}{2}]} V(\underbrace{1,\dots,1}_{2k+1},0,\dots,0)\boxtimes V(0,\dots,0)\right],\\
&\text{(iii) } \overline{X}(\rho/2,-w_2\rho/2)\mid_K &&= \mathrm{Ind}_U^K\left[V(1/2,\dots,1/2)\boxtimes V(1/2,\dots,1/2)\right],\\
&\text{(iv) } \overline{X}(\rho/2,-w_3\rho/2)\mid_K &&= \mathrm{Ind}_U^K\left[V(1/2,\dots,1/2)\boxtimes V(1/2,\dots,-1/2)\right].
\end{aligned}$$

Proposition 4.8.
(4.8.1)

$$\overline{X}(\rho/2,-\rho/2)|_K=\bigoplus V(a_1,\dots,a_n),\qquad \textit{with } a_i\in\mathbb{Z},\ \sum a_i\in 2\mathbb{Z},$$

$$\overline{X}(\rho/2,-w_1\rho/2)|_K=\bigoplus V(a_1,\dots,a_n),\qquad \textit{with } a_i\in\mathbb{Z},\ \sum a_i\in 2\mathbb{Z}+1,$$

$$\overline{X}(\rho/2,-w_2\rho/2)|_K=\bigoplus V(a_1,\dots,a_n),\qquad \textit{with } a_i\in\mathbb{Z}+1/2,\ \sum a_i\in 2\mathbb{Z}+p,$$

$$\overline{X}(\rho/2,-w_3\rho/2)|_K=\bigoplus V(a_1,\dots,a_n),\qquad \textit{with } a_i\in\mathbb{Z}+1/2,\ \sum a_i\in 2\mathbb{Z}+p+1.$$

Proof. In the first two cases we can substitute $\big(G^{split},K^{split}\big):=\big(\mathrm{SO}(2p,2p),S[O(2p)\times O(2p)]\big)$ for $\big(K,U\big)$, and $\big(\mathrm{Spin}(2p,2p),\mathrm{Spin}(2p)\times\mathrm{Spin}(2p)/\{\pm(I,I)\}\big)$ for the last two cases. The problem of computing the K-structure of $\overline{X}$ reduces to finding the finite-dimensional representations of G^{split} which contain factors of $F(\rho/2)\otimes F(-w_i\rho/2)$. Any finite-dimensional representation of G^{split} is a Langlands quotient of a principal series. Principal series have fine lowest K-types (see [Vog81]). Let MA be a split Cartan subgroup of G^{split}. A principal series is parametrized by a $(\delta,\nu)\in\widehat{MA}$. The δ are called fine, and each fine K^{split}-type μ is a direct sum of a Weyl group orbit of a *fine* δ. This implies that the multiplicities in (4.7.2) are all one, and all the finite-dimensional representations occur in (i),(ii),(iii),(iv). The four formulas correspond to the various orbits of the δ. ☐

Case 2: $\mathcal{O}=[3\ 2^{2k}\ 1^{2n-4k-3}]$**,** $0\le k\le p-1$**.** Recall that

$$\lambda_{\mathcal{O}}=(k+\frac{1}{2},\dots,\frac{3}{2},\frac{1}{2}\mid n-k-2,\dots,1,0),$$

and the integral system is $D_{k+1}\times D_{n-k-1}$. The irreducible modules are of the form $\overline{X}(\lambda_L,-w\lambda_R)$ such that $\lambda_{\mathcal{O}}$ is dominant, $w_i\lambda_{\mathcal{O}}$ is antidominant for $D_{k+1}\times D_{n-k-1}$, and they factor to $\mathrm{SO}(2n,\mathbb{C})$. These representations are listed in Subsection 4.4.

We need to work with the real form $\big(\mathrm{SO}(r,s),S[O(r)\times O(s)]\big)$. A representation of $O(n)$, $r=2m+\eta$ with $\eta=0$ or 1, will be denoted by $V(a_1,\dots,a_m;\epsilon)$, with $\epsilon=\pm1,1/2$ according to Weyl's convention, and $a_1\ge a_2\ge\dots\ge a_m\ge 0$. If $a_m=0$, there are two inequivalent representations with this highest weight, one for $\epsilon=1$ and one for $\epsilon=-1$. Each restricts irreducibly to $\mathrm{SO}(r)$ as the representation $V(a_1,\dots,a_m)\in\widehat{\mathrm{SO}(r)}$. When $a_m\ne 0$, there is a unique representation with this highest weight, $\epsilon=1/2$, or ϵ is suppressed altogether. The restriction of this representation to $\mathrm{SO}(r)$ is a sum of two representations, $V(a_1,\dots,a_m)$ and $V(a_1,\dots,a_{m-1},-a_m)$.

Representations of $Pin(s)$ are parametrized in the same way, with $a_1\ge\dots\ge a_m\ge 0$ allowed to be nonnegative decreasing half-integers.

Representations of $S[O(r)\times O(s)]$ are parametrized by restrictions of $V(a;\epsilon_1)\boxtimes V(b;\epsilon_2)$ with the following equivalences:

(1) If one of $\epsilon_i=\frac{1}{2}$, say, $\epsilon_1=\frac{1}{2}$, then $V(a;\epsilon_1)\boxtimes V(b;\epsilon_2)=V(a';\delta_1)\boxtimes V(b';\delta_2)$ if and only if $a=a',b=b',\epsilon_1=\delta_1,\epsilon_2=\delta_2$.
(2) If $\epsilon_1,\epsilon_2,\delta_1,\delta_2\in\{\pm1\}$, then $V(a;\epsilon_1)\boxtimes V(b;\epsilon_2)=V(a';\delta_1)\boxtimes V(b';\delta_2)$ iff $a=a',b=b',\epsilon_1\epsilon_2=\delta_1\delta_2$.

Lemma 4.9. *Let* $PIN = V(\frac{1}{2}\dots,\frac{1}{2}) \in \widehat{Pin(s)}$, $s = 2m+\eta$ *with* $\eta = 0$ *or* 1. *Then*

$$PIN \otimes PIN = \sum_{\ell=0}^{m-1} V(\underbrace{1\dots1}_{\ell},\underbrace{0\dots0}_{m-\ell};\epsilon) + V(1,\dots,1;1/2), \tag{4.9.1}$$

where the sum is over $\epsilon = 1$ *and* -1.

Proof. Omitted. □

We will use the groups $\underline{U} = S[O(2k+2)\times O(2n-2k-2)] \subset \underline{K} = \mathrm{SO}(2n)$. Again, the representations that we want are in Subsection 4.4. As before,

$$\overline{X}(\lambda_{\mathcal{O}}, -w_i\lambda_{\mathcal{O}}) = \sum_{w\in W(D_{k+1}\times D_{n-k-1})} \epsilon(w)X(\lambda_{\mathcal{O}}, -ww_i\lambda_{\mathcal{O}}). \tag{4.9.2}$$

Restricting to K, and using Frobenius reciprocity, (4.9.2) implies

$$\overline{X}(\lambda_{\mathcal{O}}, -w_i\lambda_{\mathcal{O}})\mid_{\underline{K}} = \mathrm{Ind}_{\underline{U}}^{\underline{K}}[F_1(\lambda_{\mathcal{O}})\otimes F_2(-w_i\lambda_{\mathcal{O}})]. \tag{4.9.3}$$

The terms $[F_1(\lambda_{\mathcal{O}})\otimes F_2(-w_i\lambda_{\mathcal{O}})]$ are

- **(i)** $V(1/2,\dots,1/2)\otimes V(0,\dots,0) \boxtimes V(1/2,\dots,1/2)\otimes V(0,\dots,0)$,
- **(ii)** $V(1/2,\dots,1/2,-1/2)\otimes V(0,\dots,0) \boxtimes V(1/2,\dots,1/2,-1/2)\otimes V(0,\dots,0)$.

Lemma 4.10.
(4.10.1)

$$\overline{X}(\lambda_{\mathcal{O}}, -\lambda_{\mathcal{O}}) = \mathrm{Ind}_{\underline{U}}^{\underline{K}}\Big[\sum_{0\le 2\ell\le k+1} V(\underbrace{1,\dots,1}_{2\ell},0,\dots,0;1)\boxtimes V(0,\dots,0;1)$$

$$+ \sum_{0\le 2\ell\le k+1} V(\underbrace{1,\dots,1}_{2\ell},0,\dots,0;1)\boxtimes V(0,\dots,0;-1)\Big],$$

$$\overline{X}(\lambda_{\mathcal{O}}, -w_1\lambda_{\mathcal{O}}) = \mathrm{Ind}_{\underline{U}}^{\underline{K}}\Big[\sum_{0\le 2\ell+1\le k+1} V(\underbrace{1,\dots,1}_{2l+1},0,\dots,0;1)\boxtimes V(0,\dots,0;1)$$

$$+ \sum_{0\le 2\ell+1\le k+1} V(\underbrace{1,\dots,1}_{2\ell+1},0,\dots,0;1)\boxtimes V(0,\dots,0;-1)\Big].$$

Proof. This follows from Lemma 4.9. □

Proposition 4.11.
(4.11.1)

$$\overline{X}(\lambda_{\mathcal{O}}, -\lambda_{\mathcal{O}})|_K = \bigoplus V(a_1,\dots,a_k,0,\dots,0), \quad \textit{with } a_i\in\mathbb{Z}, \sum a_i \in 2\mathbb{Z},$$

$$\overline{X}(\lambda_{\mathcal{O}}, -w_1\lambda_{\mathcal{O}})|_K = \bigoplus V(a_1,\dots,a_k,0,\dots,0), \quad \textit{with } a_i\in\mathbb{Z}. \sum a_i \in 2\mathbb{Z}+1.$$

Proof. The proof is almost identical to that of Proposition 4.8. When $k = p-1$, the group G^{split} in the proof of Proposition 4.8 is replaced by $G^{qs} = \mathrm{SO}(2p, 2p+2)$ and K^{split} is replaced by $K^{qs} = S[O(2p)\times O(2p+2)]$. When $k < p-1$, the group G^{split} is replaced by $G^{k+1,n-k-1} = SO(2k+2, 2n-2k-2)$ and K^{split} is replaced by $K^{k+1,n-k-1} = S[O(2k+2)\times O(2n-2k-2)]$. We follow [Vog81]. The K-types μ in (4.10.1) have $\mathfrak{q}(\lambda_L)$ as the θ-stable parabolic $\mathfrak{q} = \mathfrak{l}+\mathfrak{u}$ determined by

$\xi = (0, \ldots, 0; \underbrace{1, \ldots, 1}_{n-2k-2}, 0 \ldots, 0)$. The Levi component is $S[O(2k) \times O(2k+2)]$. The resulting $\mu_L = \mu - 2\rho(\mathfrak{u} \cap \mathfrak{s})$ are fine $U \cap L$-types. A bottom layer argument reduces the proof to the quasisplit case $n = 2p + 1$. □

Cases 3, 4: We use the infinitesimal characters in 4.3 and the representations are from Subsection 4.4 again.

In Case 4, $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$ with $k < p$. There is a unique irreducible representation with associated support $\mathcal{O}$, and it is spherical. It is a special unipotent representation with character given by [BV85].

When $n = 2p$ and $k = p$, there are two nilpotent orbits $\mathcal{O}_{I,II} = [2^n]_{I,II}$. The representations $\Xi_{I,II}$ in Subsection 4.4 are spherical representations, one each for $\mathcal{O}_{I,II}$ that are not genuine. The two representations are induced irreducibly from the trivial representation of the parabolic subgroups with Levi components $\mathrm{GL}(n)_{I,II}$. On the other hand, the representations $\Xi'_{I,II}$ are induced irreducibly from the character $Det^{1/2}$ of the parabolic subgroups with Levi components $\mathrm{GL}(n)_{I,II}$. All of these are unitary.

Proposition 4.12. *The K-types of these representations are:*

Case 3: $\mathcal{O}_{I,II} = [2^{2p}]_{I,II}$:

$$
\begin{aligned}
\Xi_I|_K &= \bigoplus V(a_1, a_1, a_3, a_3, \ldots, a_{n-1}, a_{n-1}), \ \textit{with } a_i \in \mathbb{Z},\\
\Xi'_I|_K &= \bigoplus V(a_1, a_1, a_3, a_3, \ldots, a_{n-1}, a_{n-1}), \ \textit{with } a_i \in \mathbb{Z} + 1/2,\\
\Xi_{II}|_K &= \bigoplus V(a_1, a_1, a_3, a_3, \ldots, a_{n-1}, -a_{n-1}), \ \textit{with } a_i \in \mathbb{Z},\\
\Xi'_{II}|_K &= \bigoplus V(a_1, a_1, a_3, a_3, \ldots, a_{n-1}, -a_{n-1}), \ \textit{with } a_i \in \mathbb{Z} + 1/2,
\end{aligned}
\tag{4.12.1}
$$

satisfying $a_1 \geq a_3 \geq \cdots \geq a_{n-1} \geq 0$

Case 4: $\mathcal{O} = [2^{2k}\ 1^{2n-4k}],\ 0 \leq k < n/2$:

$$\Xi|_K = \bigoplus V(a_1, a_1, \ldots, a_k, a_k, 0, \ldots, 0), \quad \textit{with} \ \ a_i \in \mathbb{Z},$$

satisfying $a_1 \geq a_3 \geq \cdots \geq a_k \geq 0$.

Proof. These are well known. The cases $[2^n]_{I,II}$ follow by Cartan and Helgason's theorem since (D_n, A_{n-1}) is a symmetric pair (for the real form $\mathrm{SO}^*(2n)$). They also follow by the method outlined below for the other cases.

For $2k < n$, the methods outlined in [BP15] combined with [Bar17] give the answer; the representations are Θ-lifts of the trivial representation of $\mathrm{Sp}(2k, \mathbb{C})$. More precisely $\overline{X}(\lambda_{\mathcal{O}}, -\lambda_{\mathcal{O}})$ is $\Omega/[\mathfrak{sp}(2k, \mathbb{C})\Omega]$, where Ω is the oscillator representation for the pair $O(2n, \mathbb{C}) \times \mathrm{Sp}(2k, \mathbb{C})$. The K-structure can then be computed using seesaw pairs, namely Ω is also the oscillator representation for the pair $O(2n) \times \mathrm{Sp}(4k, \mathbb{R})$. □

4.13. We resume the notation used in Section 3. Let $(G_0, K) = (\mathrm{Spin}(2n, \mathbb{C}), \mathrm{Spin}(2n, \mathbb{C}))$. By comparing Propositions 3.8, 3.12, 3.16 and the K-structure of representations listed in this section, we have the following matchup:

Case 1: $\Xi_i|_K = R(\mathcal{O}, \psi_i),\ 1 \leq i \leq 4$;
Case 2: $\Xi_i|_K = R(\mathcal{O}, \psi_i),\ i = 1, 2$;
Case 3: $\Xi_I|_K = R(\mathcal{O}_I, Triv)$, $\Xi'_I|_K = R(\mathcal{O}_I, Sgn)$,
$\Xi_{II}|_K = R(\mathcal{O}_{II}, Triv)$, $\Xi'_{II}|_K = R(\mathcal{O}_{II}, Sgn)$;
Case 4: $\Xi|_K = R(\mathcal{O}, Triv)$.

Then the following theorem follows.

Theorem 4.14. *Attain the notation above. Let* $G_0 = \mathrm{Spin}(2n,\mathbb{C})$ *be viewed as a real group. The* K*-structure of each representations in* $\mathcal{U}_{G_0}(\mathcal{O},\lambda_{\mathcal{O}})$ *is calculated explicitly and matches the* K*-structure of the* $R(\mathcal{O},\psi)$ *with* $\psi \in \widehat{A_K(\mathcal{O})}$*. That is, there is a* 1-1 *correspondence* $\psi \in \widehat{A_K(\mathcal{O})} \longleftrightarrow \Xi(\mathcal{O},\psi) \in \mathcal{U}_{G_0}(\mathcal{O},\lambda_{\mathcal{O}})$ *satisfying*

$$\Xi(\mathcal{O},\psi)\mid_K \cong R(\mathcal{O},\psi).$$

5. Clifford algebras and Spin groups

Since the main interest is in the case of $\mathrm{Spin}(V)$, the simply connected groups of type D, we realize everything in the context of the Clifford algebra.

5.1. Let (V,Q) be a quadratic space of even dimension $2n$, with a basis $\{e_i, f_i\}$ with $1 \le i \le n$, satisfying $Q(e_i,f_j)=\delta_{ij}$, $Q(e_i,e_j)=Q(f_i,f_j)=0$. Occasionally we will replace e_j, f_j by two orthogonal vectors v_j, w_j satisfying $Q(v_j,v_j)=Q(w_j,w_j)=1$, and orthogonal to the e_i, f_i for $i \neq j$. Precisely they will satisfy $v_j = (e_j+f_j)/\sqrt{2}$ and $w_j = (e_j - f_j)/(i\sqrt{2})$ (where $i := \sqrt{-1}$, not an index). Let $C(V)$ be the Clifford algebra with automorphisms α defined by $\alpha(x_1\cdots x_r) = (-1)^r x_1\cdots x_r$ and $\star$ given by $(x_1\cdots x_r)^\star = (-1)^r x_r \cdots x_1$, subject to the relation $xy+yx = 2Q(x,y)$ for $x, y \in V$. The double cover of $O(V)$ is

$$Pin(V) := \{x \in C(V) \mid x\cdot x^\star = 1,\ \alpha(x)Vx^\star \subset V\}.$$

The double cover $\mathrm{Spin}(V)$ of $\mathrm{SO}(V)$ is given by the elements in $Pin(V)$ which are in $C(V)^{even}$, i.e., $\mathrm{Spin}(V) := Pin(V) \cap C(V)^{even}$. For Spin, α can be suppressed from the notation since it is the identity.

The action of $Pin(V)$ on V is given by $\rho(x)v = \alpha(x)vx^\star$. The element $-I \in \mathrm{SO}(V)$ is covered by

$$\pm\mathcal{E}_{2n} = \pm i^{n-1} v_n w_n \prod_{1\le j\le n-1}[1-e_jf_j] = \pm i^n \prod_{1\le j \le n}[1-e_jf_j]. \tag{5.1.1}$$

These elements satisfy

$$\mathcal{E}_{2n}^2 = \begin{cases} +Id & \text{if } n \in 2\mathbb{Z},\\ -Id & \text{otherwise.}\end{cases}$$

The center of $\mathrm{Spin}(V)$ is

$$Z(\mathrm{Spin}(V)) = \{\pm I, \pm\mathcal{E}_{2n}\} \cong \begin{cases} \mathbb{Z}_2\times\mathbb{Z}_2 & \text{if } n \text{ is even,}\\ \mathbb{Z}_4 & \text{if } n \text{ is odd.}\end{cases}$$

The Lie algebra of $Pin(V)$ as well as $\mathrm{Spin}(V)$ is formed of elements of even order ≤ 2 satisfying

$$x + x^\star = 0.$$

The adjoint action is $\operatorname{ad} x(y) = xy - yx$. A Cartan subalgebra and the root vectors corresponding to the usual basis in Weyl normal form are formed of the elements

$$\begin{aligned} (1-e_if_i)/2 &\longleftrightarrow H(\epsilon_i),\\ e_ie_j/2 &\longleftrightarrow X(-\epsilon_i-\epsilon_j),\\ e_if_j/2 &\longleftrightarrow X(-\epsilon_i+\epsilon_j),\\ f_if_j/2 &\longleftrightarrow X(\epsilon_i+\epsilon_j). \end{aligned} \tag{5.1.2}$$

5.2. **Nilpotent orbits.** We write $\widetilde{K} = \mathrm{Spin}(V) = \mathrm{Spin}(2n,\mathbb{C})$ and $K = \mathrm{SO}(V) = \mathrm{SO}(2n,\mathbb{C})$. A nilpotent orbit of an element e will have Jordan blocks denoted by

(5.2.1)
$$e_1 \longrightarrow e_2 \longrightarrow \cdots \longrightarrow e_k \longrightarrow v \longrightarrow -f_k \longrightarrow f_{k-1} \longrightarrow -f_{k-2} \longrightarrow \cdots \longrightarrow \pm f_1 \longrightarrow 0,$$

$$\begin{array}{ccccccccc} e_1 & \longrightarrow & e_2 & \longrightarrow & \dots & \longrightarrow & e_{2\ell} & \longrightarrow 0, \\ f_{2\ell} & \longrightarrow & -f_{2\ell-1} & \longrightarrow & \dots & \longrightarrow & -f_1 & \longrightarrow 0 \end{array}$$

with the conventions about the e_i, f_j, v (equal to some appropriate v_m) as before. Every block is realized by a representative E, with the arrow in the block standing for the map ad_E. More precisely, a realization of the odd block is given by the representative $\frac{1}{2}\left(\sum_{i=1}^{k-1} e_{i+1}f_i + vf_k\right)$, and a realization of the even blocks by $\frac{1}{2}\left(\sum_{i=1}^{2\ell-1} e_{i+1}f_i\right)$. When there are only even blocks, there are two orbits; one block of the form $\frac{1}{2}\left(\sum_{i=1}^{2\ell-1} e_{i+1}f_i + e_{2\ell}f_{2\ell-1}\right)$ is replaced by $\frac{1}{2}\left(\sum_{i=1}^{2\ell-1} e_{i+1}f_i + f_{2\ell}f_{2\ell-1}\right)$. Since the sizes of all blocks sum up to $2n$, there is an even number of odd sized blocks; any two blocks of equal odd size $2k+1$ can be replaced by a pair of blocks of the same form as the even ones.

The centralizer of e in $\mathfrak{so}(V)$ has a Levi component isomorphic to a product of $\mathfrak{so}(r_{2k+1})$ and $\mathfrak{sp}(2r_{2\ell})$, where r_j is the number of blocks of size j. The centralizer of e in $\mathrm{SO}(V)$ has Levi component $\prod \mathrm{Sp}(2r_{2\ell}) \times S[\prod O(r_{2k+1})]$. For each odd sized block define

$$\mathcal{E}_{2k+1} = i^k v \prod (1 - e_j f_j). \tag{5.2.2}$$

This is an element in $Pin(V)$, and acts by $-Id$ on the block. Even products of $\pm\mathcal{E}_{2k+1}$ belong to $\mathrm{Spin}(V)$ and represent the connected components of $C_{\widetilde{K}}(e)$.

Proposition 5.3. *Let m be the number of distinct odd blocks. Then*

$$A_K(\mathcal{O}) \cong \begin{cases} \mathbb{Z}_2^{m-1} & \text{if } m > 0, \\ 1 & \text{if } m = 0. \end{cases}$$

Furthermore,

(1) *If e has an odd block of size $2k+1$ with $r_{2k+1} > 1$, then $A_{\widetilde{K}}(\mathcal{O}) \cong A_K(\mathcal{O})$.*
(2) *If all $r_{2k+1} \leq 1$, then there is an exact sequence*

$$1 \longrightarrow \{\pm I\} \longrightarrow A_{\widetilde{K}}(\mathcal{O}) \longrightarrow A_K(\mathcal{O}) \longrightarrow 0.$$

Proof. Assume that there is an $r_{2k+1} > 1$. Let

$$\begin{array}{ccccccc} e_1 & \to & \dots & \to & e_{2k+1} & \to 0, \\ f_{2k+1} & \to & \dots & \to & -f_1 & \to 0 \end{array}$$

be two of the blocks. In the Clifford algebra this element is $e = (e_2f_1 + \cdots + e_{2k+1}f_{2k})/2$. The element $\sum_{j=1}^{2k+1}(1 - e_jf_j)$ in the Lie algebra commutes with e. So its exponential

$$\prod \exp\left(i\theta(1 - e_jf_j)/2\right) = \prod [\cos\theta/2 + i\sin\theta/2(1 - e_jf_j)] \tag{5.3.1}$$

also commutes with e. At $\theta = 0$, the element in (5.3.1) is I; at $\theta = 2\pi$, it is $-I$. Thus $-I$ is in the connected component of the identity of $A_{\widetilde{K}}(\mathcal{O})$ (when $r_{2k+1} > 1$), and therefore $A_{\widetilde{K}}(\mathcal{O}) = A_K(\mathcal{O})$.

Assume there are no blocks of odd size. Then $C_K(\mathcal{O}) \cong \prod \mathrm{Sp}(r_{2l})$ is simply connected, so $C_{\widetilde{K}}(\mathcal{O}) \cong C_K(\mathcal{O}) \times \{\pm I\}$. Therefore $A_{\widetilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.

Assume there are m distinct odd blocks with $m \in 2\mathbb{Z}_{>0}$ and $r_{2k_1+1} = \cdots = r_{2k_m+1} = 1$. In this case, $C_K(\mathcal{O}) \cong \prod \mathrm{Sp}(r_{2l}) \times S[\underbrace{O(1) \times \cdots \times O(1)}_{m}]$, and hence $A_K(\mathcal{O}) \cong \mathbb{Z}_2^{m-1}$. Even products of $\{\pm\mathcal{E}_{2k_j+1}\}$ are representatives of elements in $A_{\widetilde{K}}(\mathcal{O})$. They satisfy

$$\mathcal{E}_{2k+1} \cdot \mathcal{E}_{2\ell+1} = \begin{cases} -\mathcal{E}_{2\ell+1} \cdot \mathcal{E}_{2k+1} & k \neq \ell, \\ (-1)^k I & k = \ell. \end{cases}$$

□

Corollary 5.4.

(1) *If* $\mathcal{O} = [3\ 2^{n-2}\ 1]$, *then* $A_{\widetilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\pm\mathcal{E}_3 \cdot \mathcal{E}_1, \pm I\}$.
(2) *If* $\mathcal{O} = [3\ 2^{2k}\ 1^{2n-4k-3}]$ *with* $2n-4k-3 > 1$, *then* $A_{\widetilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.
(3) *If* $\mathcal{O} = [2^n]_{I,II}$ *(n even), then* $A_{\widetilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.
(4) *If* $\mathcal{O} = [2^{2k}\ 1^{2n-4k}]$ *with* $2k < n$, *then* $A_{\widetilde{K}}(\mathcal{O}) \cong 1$.

In all cases $C_{\widetilde{K}}(\mathcal{O}) = Z(\widetilde{K}) \cdot C_{\widetilde{K}}(\mathcal{O})^0$.

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