REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 24, Pages 178–209 (May 27, 2020)
https://doi.org/10.1090/ert/542

THE GROTHENDIECK GROUP OF UNIPOTENT REPRESENTATIONS: A NEW BASIS

G. LUSZTIG

Abstract. Let $G(F_q)$ be the group of rational points of a simple algebraic group defined and split over a finite field F_q. In this paper we define a new basis for the Grothendieck group of unipotent representations of $G(F_q)$.

Introduction

0.1. Let G be an adjoint simple algebraic group defined and split over a finite field $\mathbf{F}_q$, and let $G(\mathbf{F}_q)$ be the finite group of $\mathbf{F}_q$-rational points of G. Let W be the Weyl group of G. We fix a family c (in the sense of [L1]) in the set of irreducible representations of W. (This is the same as fixing a two-sided cell of W.) To c we associate a finite group $\mathcal{G}_c$ and an imbedding $c \subset M(\mathcal{G}_c)$ (with image $M_0(\mathcal{G}_c)$) as in [L1], [L3]. Here for any finite group Γ, $M(\Gamma)$ consists of pairs (x, ρ), where $x \in \Gamma$ and ρ is an irreducible representation of the centralizer of x; these pairs are taken up to Γ-conjugacy. Let $\mathbf{C}[M(\Gamma)]$ be the $\mathbf{C}$-vector space with basis $M(\Gamma)$, and let $A_\Gamma : \mathbf{C}[M(\Gamma)] \to \mathbf{C}[M(\Gamma)]$ be the "non-abelian Fourier transform" (as in [L1]). An element $f \in \mathbf{C}[M(\Gamma)]$ is said to be ≥ 0 if f is a linear combinations of basis elements $(x, \rho) \in M(\Gamma)$ with all coefficients in $\mathbf{R}_{\geq 0}$. As in [L5] we say that $f \in \mathbf{C}[M(\Gamma)]$ is *bipositive* if $f \geq 0$ and $A_\Gamma(f) \geq 0$.

Taking $\Gamma = \mathcal{G}_c$, we denote by $\mathbf{C}[M_0(\mathcal{G}_c)]$ the subspace of $\mathbf{C}[M(\mathcal{G}_c)]$ spanned by $M_0(\mathcal{G}_c)$. In this paper we construct a new basis $\tilde{\mathbf{B}}_c$ of $\mathbf{C}[M(\mathcal{G}_c)]$. Here are some of the properties of $\tilde{\mathbf{B}}_c$:

(I) All elements of $\tilde{\mathbf{B}}_c$ are bipositive.

(II) There is a unique bijection $M(\mathcal{G}_c) \xrightarrow{\sim} \tilde{\mathbf{B}}_c$, $(x, \rho) \mapsto \widehat{(x, \rho)}$, such that any (x, ρ) appears with non-zero coefficient in $\widehat{(x, \rho)}$; this coefficient is actually 1.

(III) Let $\leq$ be a transitive relation on $M(\mathcal{G}_c)$ generated by the relation for which $(x, \rho), (x', \rho')$ are related if (x, ρ) appears with non-zero coefficient in $\widehat{(x', \rho')}$. Then $\leq$ is a partial order on $M(\mathcal{G}_c)$ in which $(1, 1)$ is the unique minimal element. In particular, the basis $\tilde{\mathbf{B}}_c$ is related to the basis $M(\mathcal{G}_c)$ of $\mathbf{C}[M(\mathcal{G}_c)]$ by an upper triangular matrix with 1 on the diagonal and with integer entries.

(IV) $(1, 1)$ appears with coefficient 1 in any element of $\tilde{\mathbf{B}}_c$.

(V) The intersection $\tilde{\mathbf{B}}_c \cap \mathbf{C}[M_0(\mathcal{G}_c)]$ is the basis $\mathbf{B}_c$ of $\mathbf{C}[M_0(\mathcal{G}_c)]$ defined in [L5].

Note that (III) and (V) imply 0.5(i) of [L5] which was stated there without proof.

Received by the editors October 11, 2019, and, in revised form, March 25, 2020.
2020 *Mathematics Subject Classification.* Primary 20G99.
The author was supported by NSF grant DMS-1855773.

0.2. Let $H \subset H'$ be subgroups of $\mathcal{G}_c$ with H normal in H'. In 3.1 we define a linear map $\mathbf{s}_{H,H'} : \mathbf{C}[M(H'/H)] \to \mathbf{C}[M(\Gamma)]$ which commutes with the non-abelian Fourier transform and takes bipositive elements to bipositive elements.

In the case where G is of exceptional type, our basis $\tilde{\mathbf{B}}_c$ is obtained by applying $\mathbf{s}_{H,H'}$ to a very restricted set of bipositive elements (said to be *primitive*) of $\mathbf{C}[M(H'/H)]$, where H, H' are in the set of subgroups of $\mathcal{G}_c$ which are either $\{1\}$ or are associated in [L4] to the various left cells of W corresponding to c. This generalizes the definition of $\mathbf{B}_c$ given in [L5] where the linear map $\mathbf{s}_{H,H'}$ was applied only to $(1,1)$. In this case, our results can be interpreted as giving a new parametrization of $M(\mathcal{G}_c)$ by triples (H, H', Ξ), where H, H' are as above and Ξ runs through the primitive bipositive elements of $\mathbf{C}[M(H'/H)]$. (In each case H'/H is a symmetric group of small order.)

In the case where G is of classical type our basis $\tilde{\mathbf{B}}_c$ will be defined using a somewhat different approach. We will show elsewhere (based on results in [L5, §2]) that the approach described above for exceptional types works also for classical types, leading to the same $\tilde{\mathbf{B}}_c$.

0.3. Let Irr_c be the set of isomorphism classes of irreducible complex representations of $G(\mathbf{F}_q)$ which are unipotent and are associated to c as in [L3]. Let $\mathcal{U}_c$ be the (abelian) category of finite-dimensional complex representations of $G(\mathbf{F}_q)$ which are direct sums of representations in Irr_c, and let K_c be the Grothendieck group of $\mathcal{U}_c$. In [L3], a bijection $M(\mathcal{G}_c) \xrightarrow{\sim} \mathrm{Irr}_c$ is established. Via this bijection we can identify $\mathbf{C} \otimes K_c = \mathbf{C}[M(\mathcal{G}_c)]$ so that the basis Irr_c of K_c becomes the basis $M[\mathcal{G}_c]$ of $\mathbf{C}[M(\mathcal{G}_c)]$. Then the new basis $\tilde{\mathbf{B}}_c$ of $\mathbf{C}[M(\mathcal{G}_c)]$ becomes a new basis of $\mathbf{C} \otimes K_c$ (it is also a $\mathbf{Z}$-basis of K_c). The elements in this new $\mathbf{Z}$-basis of K_c represent objects of $\mathcal{U}_c$ which are called the new (unipotent) representations of $G(\mathbf{F}_q)$. They are in bijection with Irr_c. Note that by taking the disjoint union over the various families of W we obtain a new basis for the Grothendieck group of unipotent representations of $G(\mathbf{F}_q)$.

In type A_n we have $|c| = 1$ and we can take $\tilde{\mathbf{B}}_c$ to consist of $(1,1)$; then the desired properties of $\tilde{\mathbf{B}}_c$ are trivial. The properties above of $\tilde{\mathbf{B}}_c$ are verified in type B_n, C_n, D_n in §1. Another approach in type D_n is sketched in §2. The exceptional types are considered in §3.

0.4. **Notation.** For a, b in $\mathbf{Z}$ we set $[a,b] = \{z \in \mathbf{Z}; a \le z \le b\}$. For a, b in $\mathbf{Z}$ we write $a =_2 b$ instead of $a = b \mod 2$ and $a \ne_2 b$ instead of $a \ne b \mod 2$. For a finite set Y let $|Y|$ be the cardinal of Y.

1. The set $\mathbf{S}_D$

1.1. Let $D \in \mathbf{N}$. A subset I of $[1, D]$ is said to be an *interval* if $I = [a, b]$ for some $a \le b$ in $[1, D]$. Let $\mathcal{I}_D$ be the set of intervals of $[1, D]$. For $I = [a, b], I' = [a', b']$ in $\mathcal{I}_D$ we write $I \prec I'$ whenever $a' < a \le b < b'$. We say that I, I' are non-touching (and we write $I \spadesuit I'$) if $a' - b \ge 2$ or $a - b' \ge 2$. Let R_D be the set whose elements are the subsets of $\mathcal{I}_D$. Let $\emptyset \in R_D$ be the empty subset of $\mathcal{I}_D$. For $B \in R_D$ and $h \in \{0, 1\}$ we set $B^h = \{I \in B; |I| =_2 h\}$.

For $B \in R_D$ and $[a, b] \in \mathcal{I}_D$ we define $\mathcal{X}_B[a, b] = \bigcup_{I \in B^1; I \subset [a,b]} I$.

Let $I \in \mathcal{I}_D$. A subset E of I is said to be *discrete* if $i \ne j$ in E implies $i - j \ne \pm 1$. Such E is said to be maximal if $|E| = |I|/2$ (with $|I|$ even) or $|E| = (|I| + 1)/2$ (with $|I|$ odd). A maximal discrete subset of I exists; it is unique if $|I|$ is odd.

When $D\ge 2$ and $i\in[1,D]$ we define an (injective) map $\xi_i:\mathcal{I}_{D-2}\to\mathcal{I}_D$ by

$$\xi_i([a',b'])=[a'+2,b'+2] \text{ if } i\le a', \quad \xi_i([a',b'])=[a',b'] \text{ if } i\ge b'+2,$$

(a) $$\xi_i([a',b'])=[a',b'+2] \text{ if } a'<i<b'+2.$$

We define $t_i:R_{D-2}\to R_D$ by $B'\mapsto\{\xi_i(I');I'\in B'\}\sqcup\{i\}$. We have $|t_i(B')|=|B'|+1$.

1.2. We now assume that D is even. We say that $B\in R_D$ is *primitive* if it is of the form

(a) $B=\{[1,D],[2,D-1],\dots,[k,D+1-k]\}$ *for some* $k\in\mathbf{N}$, $k\le D/2$.

For example, $B=\emptyset\in R_D$ is primitive (with $k=0$). We define a subset $\mathbf{S}_D$ of R_D by induction on D as follows.

If $D=0$, $\mathbf{S}_D$ consists of a single element, namely $\emptyset\in R_D$. If $D\ge 2$ we say that $B\in R_D$ is in $\mathbf{S}_D$ if either B is primitive, or

(b) *there exist* $i\in[1,D]$ *and* $B'\in\mathbf{S}_{D-2}$ *such that* $B=t_i(B')$.

(This generalizes the definition of the set S_D in [L5, 1.2] which can be viewed as a subset of $\mathbf{S}_D$.)

Let $\tau_D:[1,D]\to[1,D]$ be the involution $i\mapsto D+1-i$. It induces an involution $I\mapsto\tau_D(I)$ of $\mathcal{I}_D$. One can verify that $I\mapsto\tau_D(I)$ defines an involution $\mathbf{S}_D\to\mathbf{S}_D$; we denote it again by τ_D.

1.3. For $D\ge 0$, let $\mathbf{S}_D^{prim}=\{B\in\mathbf{S}_D;B \text{ primitive}\}$.

Let $B\in R_D$. We consider the following properties $(P_0),(P_1),(P_2)$ that B may or may not have.

(P_0) *If* $I\in B$, $\tilde I\in B$, *then either* $I=\tilde I$, *or* $I♠\tilde I$, *or* $I\prec\tilde I$, *or* $\tilde I\prec I$.

(P_1) *If* $[a,b]\in B^1$ *and* $b-a\ge 2$, *then* $\mathcal{X}_B[a+1,b-1]$ *contains the unique maximal discrete subset of* $[a+1,b-1]$, *that is,* $\{a+1,a+3,a+5,\dots,b-1\}$.

(P_2) *Let* $k=|B^0|\in\mathbf{N}$. *There exists a (necessarily unique) sequence of integers* $0=h_0<h_1<h_2<\dots<h_{2k}<h_{2k+1}=D+1$ *such that* B^0 *consists of* $[h_1,h_{2k}]$, $[h_2,h_{2k-1}]$, $\dots$, $[h_k,h_{k+1}]$. *We have* $h_j=_2 j$ *for* $j\in[0,2k+1]$. *Assume now that* $k\ge 1$ *and that* $j\in[0,2k]-\{k\}$ *satisfies* $h_{j+1}\ge h_j+3$. *If* $j\in[0,k-1]$, *then* $\mathcal{X}_B[h_j+1,h_{j+1}-2]$ *contains the unique maximal discrete subset of* $[h_j+1,h_{j+1}-2]$; *if* $j\in[k+1,2k]$, *then* $\mathcal{X}_B[h_j+2,h_{j+1}-1]$ *contains the unique maximal discrete subset of* $[h_j+2,h_{j+1}-1]$.

Assume now that $D\ge 2$, $i\in[1,D]$, $B'\in R_{D-2}$, $B=t_i(B')\in R_D$. From the definitions we see that the following holds.

(a) B' *satisfies* $(P_0),(P_1),(P_2)$ *if and only if* B *satisfies* $(P_0),(P_1),(P_2)$.

Let $\mathbf{S}'_D$ be the set of all $B\in R_D$ which satisfy $(P_0),(P_1),(P_2)$. (This generalizes the definition of the set S'_D in [L5, 1.3]. Properties like $(P_0),(P_1)$ appeared in [L5, 1.3].)

In the setup of (a) we have the following consequence of (a).

(b) *We have* $B'\in\mathbf{S}'_{D-2}$ *if and only if* $B\in\mathbf{S}'_D$.

We show (extending [L5, 1.3(c)]):

(c) *We have* $\mathbf{S}_D=\mathbf{S}'_D$. *In particular any* $B\in\mathbf{S}_D$ *satisfies* $(P_0),(P_1),(P_2)$.

We argue by induction on D. If $D=0$, $\mathbf{S}'_D$ consists of the empty set hence (c) holds in this case. Assume now that $D\ge 2$. Let $B\in\mathbf{S}_D$. We show that $\mathbf{B}\in\mathbf{S}'_D$. If $B\in\mathbf{S}_D^{prim}$, then B clearly is in $\mathbf{S}'_D$. If $B\notin\mathbf{S}_D^{prim}$, then $B=t_i(B')$ for some $i,B'\in S_{D-2}$ as in 1.2(b). By the induction hypothesis we have $B'\in\mathbf{S}'_{D-2}$. By (b) we have $B\in\mathbf{S}'_D$. We see that $B\in\mathbf{S}_D\implies B\in\mathbf{S}'_D$. Conversely, let $B\in\mathbf{S}'_D$.

We show that $B \in \mathbf{S}_D$. If $B \in \mathbf{S}_D^{prim}$ this is obvious. Thus we can assume that $B \notin \mathbf{S}_D^{prim}$. From (P_2) we see that $B^1 \neq \emptyset$. Let $[a,b] \in B^1$ be such that $b-a$ is minimum. If $a < z < b$, $z =_2 a+1$, then by (P_1) we have $z \in [a',b']$ with $[a',b'] \in B^1$, $b'-a' < b-a$, contradicting the minimality of $b-a$. We see that no z as above exists. Thus, $[a,b] = \{i\}$ for some $i \in [1,D]$. Using (P_0) and $\{i\} \in B$, we see that B does not contain any interval of the form $[a,i]$ with $a < i$, or $[i,b]$ with $i < b$, or $[a,i-1]$ with $a < i$, or $[i+1,b]$ with $i < b$; hence any interval of B other than $\{i\}$ is of the form $\xi_i[a',b']$, where $[a',b'] \in \mathcal{I}_{D-2}$. Thus we have $B = t_i(B')$ for some $B' \in R_{D-2}$. From (a) we deduce that $B' \in \mathbf{S}'_{D-2}$. Using the induction hypothesis we deduce that $B' \in \mathbf{S}_{D-2}$. By the definition of $\mathbf{S}_D$, we have $B \in \mathbf{S}_D$. This completes the proof of (c).

We show:

(d) *Let $B \in \mathbf{S}_D$. If $I \in B^1, J \in B^0$, then $J \not\subset I$.*

We argue by induction on $|I|$. Let $I = [a,b]$, $J = [a',b']$. Assume that $J \subset I$. By (P_0) we have $J \prec I$. Since $b'-a'$ is odd, then either $x = a'$ or $x = b'$ satisfies $x =_2 a+1$. By (P_1) we can find $I' \in B^1$ such that $I' \prec I$, $x \in I'$. We have $|I'| < |I|$. By the induction hypothesis we have $J \not\subset I'$. We have $I' \cap J \neq \emptyset$ and $I' \neq J$ hence $I' \prec J$ so that $x \notin I'$, a contradiction. This proves (d).

Let $B \in \mathbf{S}_D$, and let $h_0 < h_1 < \cdots < h_{2k+1}$ be attached to B as in (P_2). We show:

(e) *If $[a,b] \in B^1$, then for some $j \in [0,2k]$ we have $h_j < a \le b < h_{j+1}$.*

We can find $j \in [0,2k]$ such that $h_j \le a \le h_{j+1}$. Assume first that $j \in [0,k-1]$. Then $[h_j, h_{2k+1-j}] \cap [a,b] \neq \emptyset$ and $[h_j, h_{2k+1-j}] \neq [a,b]$ (one is in B^0, the other in B^1). Using (P_0), we deduce $[h_j, h_{2k+1-j}] \prec [a,b]$ (which contradicts (d)) or $[a,b] \prec [h_j, h_{2k+1-j}]$ so that $h_j < a$. If $b \ge h_{2k-j}$, then $[h_{j+1}, h_{2k-j}] \subset [a,b]$ contradicting (d). Thus we have $b < h_{2k-j}$. If $b \ge h_{j+1}$, then $[h_{j+1}, h_{2k-j}] \cap [a,b] \neq \emptyset$ (it contains b) and $[h_{j+1}, h_{2k-j}] \neq [a,b]$. Hence, by (P_0), we have either $[h_{j+1}, h_{2k-j}] \prec [a,b]$ (which again contradicts (d)) or $[a,b] \prec [h_{j+1}, h_{2k-j}]$ hence $a > h_{j+1}$, contradicting our assumption. We see that $b < h_{j+1}$.

Assume next that $j \in [k+1, 2k]$. Then $[h_{2k-j}, h_{j+1}] \cap [a,b] \neq \emptyset$ (it contains a) and $[h_{2k-j}, h_{j+1}] \neq [a,b]$ (one is in B^0, the other in B^1). Using (P_0), we deduce $[h_{2k-j}, h_{j+1}] \prec [a,b]$ (which contradicts (d)) or $[a,b] \prec [h_{2k-j}, h_{j+1}]$, so that $b < h_{j+1}$. If $a = h_j$, then $[h_{2k+1-j}, h_j] \cap [a,b] \neq \emptyset$ (it contains a) and $[h_{2k+1-j}, h_j] \neq [a,b] \neq \emptyset$. Using (P_0) we deduce $[h_{2k+1-j}, h_j] \prec [a,b]$ (which contradicts (d)) or $[a,b] \prec [h_{2k+1-j}, h_j]$ hence $a < h_j$, a contradiction. We see that $a > h_j$.

Finally, we assume that $j = k$. Then $[h_k, h_{k+1}] \cap [a,b] \neq \emptyset$ and $[h_k, h_{k+1}] \neq [a,b]$ (one is in B^0, the other in B^1). Using (P_0), we deduce $[h_k, h_{k+1}] \prec [a,b]$ (which contradicts (d)) or $[a,b] \prec [h_k, h_{k+1}]$, so that $h_k < a$ and $b < h_{k+1}$. This proves (e).

The following result has already been proved as a part of the proof of (c).

(f) *Assume that $D \ge 2$, $i \in [1,D]$. Let $B \in \mathbf{S}_D$ be such that $\{i\} \in B$. Then there exists $B' \in \mathbf{S}_{D-2}$ such that $B = t_i(B')$.*

Let $B \in \mathbf{S}_D$, and let $I = [a,b] \in B^1$. Let $\mathcal{X}(I) = \{I' \in B^1; I' \subset I\}$. We show:

(g) $|\mathcal{X}(I)| = (b-a+2)/2$.

We argue by induction on $|I|$. If $|I| = 1$, then $\mathcal{X}(I) = \{I\}$ and the result is clear. Assume now that $|I| \ge 3$. By $(P_0), (P_1)$ we can find $a = z_0 < z_1 < \cdots < z_r = b$ $(r \ge 0)$ such that $z_0, z_1, \ldots, z_r$ are all congruent to $a \mod 2$ and $[z_0+1, z_1-1] \in$

$B^1, [z_1+1, z_2-1] \in B^1, \dots, [z_{r-1}+1, z_r-1] \in B^1$; moreover, any $I' \in B^1$ such that $I' \prec I$ is contained in exactly one of $[z_0+1, z_1-1], [z_1+1, z_2-1], \dots, [z_{r-1}+1, z_r-1]$. It follows that $|\mathcal{X}(I)| = 1 + \sum_{j \in [0,r-1]} |\mathcal{X}([z_j+1, z_{j+1}-1])|$. Using the induction hypothesis we can rewrite the last equality as

$$|\mathcal{X}(I)| = 1 + \sum_{j \in [0,r-1]} ((z_{j+1}-1) - (z_j+1) + 2)/2 = 1 + (b-a)/2.$$

This proves (g).

1.4. For $B \in \mathbf{S}_D$, $h \in \{0,1\}$, $j \in [1,D]$ we set $B_j^h = \{I \in B^h; j \in I\}$. From the definitions we deduce:

(a) *Assume that* $D \geq 2$, $i \in [1,D]$ *and that* $B' \in \mathbf{S}_{D-2}$. *Let* $B = t_i(B') \in \mathbf{S}_D$. *Then* $|B^0| = |B'^0|$. Moreover, for $h \in \{0,1\}$ and $r \in [1, D-2]$ *we have:*
$|{B'_r}^h| = |B_r^h|$ *if* $r \leq i-2$, $|{B'_r}^h| = |B_{r+2}^h|$ *if* $r \geq i$,
$|B_{i-1}^h| = |B_{i+1}^h| = |{B'_{i-1}}^h|$, $|B_i^h| = |{B'_{i-1}}^h| + h$ *if* $1 < i < D$,
$|B_{i-1}^h| = 0$ *if* $i = D$, $|B_{i+1}^h| = 0$ *if* $i = 1$.
This extends [L5, 1.4(a)].

1.5. Let $B \in \mathbf{S}_D - \mathbf{S}_S^{prim}$. As we noted in the proof of 1.3(c), in this case we must have $B^1 \neq \emptyset$ and we have $\{j\} \in B^1$ for some $j \in [1,D]$; we assume that j is as small as possible (then it is uniquely determined). As in that proof we have $B = t_j(B')$, where $B' \in \mathbf{S}_{D-2}$. Let i be the smallest number in $\bigcup_{I \in B^1} I$. We have $i \leq j$. We show:

(a) *For any* $h \in [i,j]$, *we have* $[h, \tilde{h}] \in B^1$ *for a unique* $\tilde{h} \in [h, D]$; *moreover we have* $j \leq \tilde{h}$.

We argue by induction on D. When $D \leq 1$ the result is obvious. We now assume that $D \geq 2$. Assume first that $i = j$. By (P_0), $\{j\} \in B^1$ implies that we cannot have $[j,b] \in B^1$ with $j < b$; thus (a) holds in this case. We can assume that $i < j$. We have $[i,b] \in B^1$ for some $b > i$, hence $|B^1| \geq 2$ so that $|B'^1| \geq 1$ and $B' \notin \mathbf{S}_{D-2}^{prim}$. Then i', j' are defined in terms of B' in the same way as i, j are defined in terms of B. From (P_1) we see that there exists j_1 such that $i < j_1 < b$ and such that $\{j_1\} \in B$. By the minimality of j we must have $j \leq j_1$. Thus we have $i < j < b$. We have $[i,b] = \xi_j[i, b-2]]$, hence $[i, b-2] \in B'^1$. This implies that $i' \leq i$. We have $[i', c] \in B'^1$ for some $c \in [i', D-2]$, $c =_2 i'$; hence $[i', c'] \in B^1$ for some $c' \geq i'$ so that $i' \geq i$. Thus we have $i' = i$. By the induction hypothesis, the following holds:

(b) *For any* $r \in [i, j']$, *we have* $[r, r_1] \in B'^1$ *for a unique* r_1; *moreover* $j' \leq r_1$.

If $j' \leq j-2$, then $\{j'\} = \xi_j(\{j'\}) \in B$. Hence $j' \geq j$ by the minimality of j; this is a contradiction. Thus we have $j' \geq j-1$.

Let $r \in [i, j-1]$. Then we have also $r \in [i, j']$, hence r_1 is defined as in (b). We have $[r, r_1] \in B'^1$, hence $[r, r_1+2] \in B^1$ (we use that $r < j \leq j'+1 \leq r_1+1 < r_1+2$); we have $j < r_1+2$. Assume now that $[r, r_2] \in B^1$ with $r \leq r_2$. Then $r < r_2$ (by the minimality of j). If $j = r_2$ or $j = r_2+1$, then applying (P_0) to $\{j\}, [r, r_2]$ gives a contradiction. Thus we must have either $r < j < r_2$ or $j > r_2+1$. If $j > r_2+1$, then $[r, r_2] \in B'^1$; hence by (b), $r_2 = r_1$, hence $j > r_1+1$ contradicting $j < r_1+2$. Thus we have $r < j < r_2$, so that $[r, r_2-2] \in B'^1$; hence by (b), $r_2 - 2 = r_1$. Thus we have $r < j < r_2$ so that $[r, r_2-2] \in B'^1$; hence by (b), $r_2 - 2 = r_1$.

Next we assume that $r = j$. In this case we have $\{r\} \in B^1$. Moreover, if $[r, r'] \in B^1$ with $r \leq r' \leq D$, then we cannot have $r < r'$ (if $r < r'$, then applying (P_0) to $\{r\}, [r, r']$ gives a contradiction). This proves (a).

We show:

(c) *Assume that $j < D$ and that $i \le h < j$. Then $\tilde h$ in (a) satisfies $\tilde h > j$.*

Assume that $\tilde h = j$, so that $[h, j] \in B^1$. Since $h < j$, applying (P_0) to $\{j\}, [h, j]$ gives a contradiction. This proves (c).

We show:

(d) *Assume that $j < D$ and that $r \in [j + 1, D]$. We have $[j + 1, r] \notin B^1$.*

Assume that $[j+1, r] \in B^1$. Applying (P_0) to $\{j\}, [j+1, r]$ gives a contradiction. This proves (d).

We show:

(e) *For $h \in [i, j]$ we have $|B^1_h| = h - i + 1$. If $j < D$ we have $|B^1_{j+1}| = j - i$.*

Let $h \in [i, j]$. Then for any $h' \in [i, h]$, B^1_h contains $[h', \tilde h']$ (since $h \le \tilde h'$); see (a). Conversely, assume that $[a, b] \in B^1_h$. We have $a \le h$. By the definition of i we have $i \le a$. By the uniqueness statement in (a) we have $b = \tilde a$ so that $[a, b]$ is one of the $h - i + 1$ intervals $[h', \tilde h']$ above. This proves the first assertion of (e). Assume now that $j < D$. If $h' \in [i, j]$, $h' < j$, then $[h', \tilde h'] \in B^1_{j+1}$, by (c). Conversely, assume that $[a, b] \in B^1_{j+1}$. We have $a \le j+1$ and by (d) we have $a \ne j+1$ so that $a \le j$. If $a = j$, then by the uniqueness in (a) we have $b = j$ which contradicts $j + 1 \in [a, b]$. Thus we have $a \le j - 1$. We see that $[a, b]$ is one of the $j - i$ intervals $[h', \tilde h']$ with $h' \in [i, j]$, $h' < j$. This proves (e).

We show:

(f) *Let $e = |B^0_i|$. For $h \in [i, j]$ we have $|B^0_h| = e$. If $j < D$ we have $|B^0_{j+1}| = e$.*

Let $I \in B^0_i$. Since I and $[i, \tilde i]$ are not disjoint and not equal, we must have $[i, \tilde i] \prec I$ or $I \prec [i, \tilde i]$ (this last case cannot occur since $i \in [i, \tilde i]$). Thus we have $[i, \tilde i] \prec I$. We have $[i, j] \subset [i, \tilde i]$; hence $[i, j] \prec I$ so that $I \in B^0_h$ for any $h \in [i, j]$. If in addition $j < D$, then from $[i, j] \prec I$ we deduce $[i, j + 1] \subset I$ so that $I \in B^0_{j+1}$. Conversely, assume that $h \in [i, j]$ and $I' \in B^0_h$. Since I' and $[h, \tilde h]$ are not disjoint and not equal, we must have $[h, \tilde h] \prec I'$ or $I' \prec [h, \tilde h]$ (this last case cannot occur since $h \in I'$). Thus we have $[h, \tilde h] \prec I'$. If $i < h$, it follows that $h - 1 \in I'$ so that $[h - 1, \widetilde{h - 1}] \prec I'$. Repeating this argument we see that $[h'', \tilde h''] \prec I'$ for any $h'' \in [i, h]$, so that in particular we have $i \in I'$ and $I' \in B^0_i$. If in addition $j < D$ and $I' \in B^0_{j+1}$, then $I', [j, \tilde j]$ are not non-touching and are not equal hence we must have $[j, \tilde j] \prec I'$ or $I' \prec [j, \tilde j]$ (this last case cannot occur since it contradicts 1.3(d)). Thus we have $[j, \tilde j] \prec I'$ which by the earlier part of the proof implies $I' \in B^0_i$. This proves (f).

1.6. For any $n \in \mathbf{N}$ we define $\underline{n} \in \{0, 1\}$ by $n =_2 \underline{n}$. For $B \in \mathbf{S}_D$, $j \in [1, D]$, we set $\kappa = \underline{|B^0|}$ and

$$f_j(B) = |B^1_j| - |B^0_j| - \kappa \in \mathbf{Z},$$
$$\epsilon_j(B) = f_j(B)(f_j(B) + 1)/2 \in \mathbf{F}_2.$$

This extends a definition in [L5, 1.6]. We have

$\epsilon_j(B) = 1$ if $f_j(B) \in (4\mathbf{Z} + 1) \cup (4\mathbf{Z} + 2)$, $\epsilon_j(B) = 0$ if $f_j(B) \in (4\mathbf{Z} + 3) \cup (4\mathbf{Z})$.

Assume now that $B \notin S^{prim}_D$. Let $i \le j$ in $[1, D]$ be as in 1.5. Let $e = |B^0_i| + \kappa$. From 1.5(e),(f) we deduce:

(a) *We have*

$$(f_i(B), f_{i+1}(B), \dots, f_j(B)) = (1 - e, 2 - e, 3 - e, \dots, j - i - e, j - i + 1 - e).$$

If $j < D$, we have $f_{j+1}(B) = j - i - e$.

From (a) we deduce:
(b)

$$(\epsilon_i(B),\epsilon_{i+1}(B),\dots,\epsilon_j(B))=((1-e)(2-e)/2,(2-e)(3-e)/2,$$
$$(3-e)(4-e)/2,\dots,(j-i-e)(j-i-e+1)/2,(j-i-e+1)(j-i-e+2));$$

(c) *if* $j<D$, *then* $\epsilon_{j+1}(B)=(j-i-e)(j-i-e+1)/2)$.
This extends [L5, 1.6(b),(c)].
For future reference we note:
(d) *If* $c\in\mathbf{Z}$, *then* $c(c+1)/2\not=_2(c+2)(c+3)/2$.
(e) *If* $c\in 2\mathbf{Z}$, *then* $c(c+1)/2\not=_2(c+1)(c+2)/2$.

1.7. Let $B\in\mathbf{S}_D$, $\tilde B\in\mathbf{S}_D$ be such that $B,\tilde B$ are not primitive and $\epsilon_h(B)=\epsilon_h(\tilde B)$ for any $h\in[1,D]$ and $|B^0|=|\tilde B^0|$. We show (extending [L5, 1.7(a)]):
(a) *We can find* $z\in[1,D]$ *such that* $\{z\}\in B$, $\{z\}\in\tilde B$.
Let $h_0<h_1<h_2<\dots<h_{2k}<h_{2k+1}$ be the sequence attached to B in (P_2); let $\tilde h_0<\tilde h_1<\tilde h_2<\dots<\tilde h_{2\tilde k}<\tilde h_{2\tilde k+1}$ be the analogous sequence attached to $\tilde B$. Here $k=\tilde k=|B^0|=|\tilde B^0|$. We shall need the following preparatory result.
(b) *Assume that* $s\in[0,k-1]$ *is such that*

$$(h_0,h_1,\dots,h_s)=(\tilde h_0,\tilde h_1,\dots,\tilde h_s)=(0,1,\dots,s).$$

Then either $h_{s+1}=\tilde h_{s+1}=s+1$, *or the conclusion of* (a) *holds.*
Let $i\le j$ be attached to B as in 1.5. Let $\tilde i\le\tilde j$ be similarly attached to $\tilde B$. Assume first that $h_{s+1}>s+1$, $\tilde h_{s+1}=s+1$. We have $|B^0_{s+1}|=s$, $|\tilde B^0_{s+1}|=s+1$, $|\tilde B^1_{s+1}|=0$ (we use 1.3(e)) and by (P_2) we have $|B^1_{s+1}|\ge 1$. We see that $i=s+1$ and from 1.5(e) we have $|B^1_{s+1}|=1$. Thus, $f_{s+1}(B)=1-s-\kappa$, $f_{s+1}(\tilde B)=-1-s-\kappa$ (where $\kappa=\underline{k}$), so that $\epsilon_{s+1}(B)=(1-s-\kappa)(2-s-\kappa)/2$, $\epsilon_{s+1}(\tilde B)=(-1-s-\kappa)(-s-\kappa)/2$. It follows that $(1-s-\kappa)(2-s-\kappa)/2=_2(-1-s-\kappa)(-s-\kappa)/2$, contradicting 1.6(d). Thus, if $\tilde h_{s+1}=s+1$, then $h_{s+1}=s+1$. Similarly, if $h_{s+1}=s+1$, then $\tilde h_{s+1}=s+1$. Assume now that $h_{s+1}>s+1$ and $\tilde h_{s+1}>s+1$. By (P_2) we have $i=\tilde i=s+1$. If $j<\tilde j$, then $j<D$ and from 1.6(b),(c), we see that

$$\epsilon_{j+1}(B)=(j-i-s-\kappa)(j-i-s-\kappa+1)/2,\epsilon_{j+1}(\tilde B)=(j-i-s-\kappa+2)(j-i-s-\kappa+3)/2$$

so that

$$(j-i-s-\kappa)(j-i-s-\kappa+1)/2=_2(j-i-s-\kappa+2)(j-i-s-\kappa+3)/2,$$

contradicting 1.6(d). Thus we have $j\ge\tilde j$. Similarly, we have $\tilde j\ge j$. Hence $\tilde j=j$, so that (a) holds with $z=j=\tilde j$. The only remaining case is that where $h_{s+1}=s+1$ and $\tilde h_{s+1}=s+1$. This proves (b).
We shall need a second preparatory result.
(c) *Assume that* $s\in[0,k-1]$ *is such that*

$$(h_{2k-s+1},\dots,h_{2k},h_{2k+1})=(\tilde h_{2k-s+1},\dots,\tilde h_{2k},\tilde h_{2k+1})=(D-s+1,\dots,D,D+1).$$

Then either $h_{2k-s}=\tilde h_{2k-s}=D-s$ *or the conclusion of* (a) *holds.*
We note that the assumptions of (b) are satisfied when $B,\tilde B$ are replaced by $\tau_D(B),\tau_D(\tilde B)$ (see 1.2). Hence from (b) we deduce that either $h_{2k-s}=\tilde h_{2k-s}=D-s$ or there exists $u\in[1,D]$ such that $\{u\}\in\tau_D(B)$, $\{u\}\in\tau_D(\tilde B)$ (which implies that $\{\tau_D(u)\}\in B$, $\{\tau_D(u)\}\in\tilde B$. This proves (c).

Next we note that the assumption of (b) (and that of (c)) is satisfied when $s=0$. Hence from (b),(c) we obtain by induction on s the following result.

(d) *We have either*

$$(h_0,h_1,\dots,h_k,h_{k+1},\dots,h_{2k+1}) = (\tilde h_0,\tilde h_1,\dots,\tilde h_k,\tilde h_{k+1},\dots,\tilde h_{2k+1})$$
$$= (0,1,\dots,k,D-k+1,\dots,D,D+1)$$

or the conclusion of (a) *holds.*

Thus, to prove (a) we can assume that $B,\tilde B$ are as in the first alternative of (d). We have $k<i\le j<D-k+1$, $k<\tilde i\le\tilde j<D-k+1$ (we use 1.3(e) and (P_1)). Assume first that $j<\tilde j$ (so that $j<D$) and $i<\tilde i$.

From 1.6 for B we have $\epsilon_i(B)=(1-k-\underline{k})(2-k-\underline{k})/2$. From $i<\tilde i$ we have $\epsilon_i(\tilde B)=(-k-\underline{k})(1-k-\underline{k})/2$. Thus $(1-k-\underline{k})(2-k-\underline{k})/2 =_2 (-k-\underline{k})(1-k-\underline{k})/2$. This contradicts 1.6(e) since $k+\underline{k}$ is even. Thus we must have $i\ge\tilde i$. Next we asssume that $j<\tilde j$ (so that $j<D$) and $\tilde i<i$. From 1.6 for $\tilde B$ we have $\epsilon_{\tilde i}(\tilde B)=(1-k-\underline{k})(2-k-\underline{k})/2$. From $\tilde i<i$ we have $\epsilon_{\tilde i}(B)=(-k-\underline{k})(-k-\underline{k}+1)/2$. $(1-k-\underline{k})(2-k-\underline{k})/2 =_2 (-k-\underline{k})(1-k-\underline{k})/2$. This contradicts 1.6(e) since $k+\underline{k}$ is even. Thus, when $j<\tilde j$ we must have $i=\tilde i$. From 1.6(c) for B we have $e_{j+1}(B)=(j-i-k-\underline{k})(j-i-k-\underline{k}+1)/2$ and from 1.6(b) for $\tilde B$ we have $e_{j+1}(\tilde B)=(j-i-k-\underline{k}+2)(j-i-k-\underline{k}+3)/2$. It follows that

$$(j-i-k-\underline{k})(j-i-k-\underline{k}+1)/2 =_2 (j-i-k-\underline{k}+2)(j-i-k-\underline{k}+3)/2,$$

contradicting 1.6(d). We see that $j<\tilde j$ leads to a contradiction. Similarly, $\tilde j<j$ leads to a contradiction. Thus we must have $j=\tilde j$, so that (a) holds with $z=j=\tilde j$. This completes the proof of (a).

1.8. Let $B\in\mathbf{S}_D$, $\tilde B\in\mathbf{S}_D$.

(a) *Assume that* $\tilde B\in\mathbf{S}_D^{prim}$, *that* $\epsilon_h(B)=\epsilon_h(\tilde B)$ *for any* $h\in[1,D]$, *and that* $|B^0|=|\tilde B^0|$. *Then* $\tilde B=B$.

The proof is similar to that of 1.7(a). Assume that $B\notin\mathbf{S}_D^{prim}$. Let $i\le j$ be attached to B as in 1.5. Let $h_0<h_1<h_2<\dots<h_{2k}<h_{2k+1}$ be the sequence attached to B in (P_2); let $\tilde h_0<\tilde h_1<\tilde h_2<\dots<\tilde h_{2\tilde k}<\tilde h_{2\tilde k+1}$ (that is, $0<1<\dots<k<D+1-k<\dots<D<D+1$) be the analogous sequence attached to $\tilde B$. We have $k=\tilde k=|B^0|=|\tilde B^0|$.

We show the following variant of 1.7(b).

(b) *Assume that* $s\in[0,k-1]$ *is such that* $(h_0,h_1,\dots,h_s)=(0,1,\dots,s)$. *Then* $h_{s+1}=s+1$.

Assume first that $h_{s+1}>s+1$. We have $|B^0_{s+1}|=s$, $|\tilde B^0_{s+1}|=s+1$, $|\tilde B^1_{s+1}|=0$ (we use 1.3(e)) and by (P_2) we have $|B^1_{s+1}|\ge1$. We see that $i=s+1$ and from 1.5(e) we have $|B^1_{s+1}|=1$. Thus, $f_{s+1}(B)=1-s-\underline{k}$, $f_{s+1}(\tilde B)=-1-s-\underline{k}$, so that

$$\epsilon_{s+1}(B)=(1-s-\underline{k})(2-s-\underline{k})/2, \epsilon_{s+1}(\tilde B)=(-1-s-\underline{k})(-s-\underline{k})/2.$$

It follows that $(1-s-\underline{k})(2-s-\underline{k})/2 =_2 (-1-s-\underline{k})(-s-\underline{k})/2$, contradicting 1.6(d). Thus, we must have $h_{s+1}=s+1$. This proves (b).

Next we show the following variant of 1.7(c).

(c) *Assume that* $s\in[0,k-1]$ *is such that* $(h_{2k-s+1},\dots,h_{2k},h_{2k+1})=(D-s+1,\dots,D,D+1)$. *Then* $h_{2k-s}=D-s$.

We note that the assumptions of (b) are satisfied when $B, \tilde{B}$ are replaced by $\tau_D(B), \tau_D(\tilde{B})$. Hence from (b) we deduce that $h_{2k-s} = D - s$. This proves (c).

Now we note that the assumption of (b) (and that of (c)) is satisfied when $s = 0$. Hence from (b),(c) we obtain by induction on s the following result.

(d) *We have*

$$(h_0, h_1, \dots, h_k, h_{k+1}, \dots, h_{2k+1}) = (\tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_k, \tilde{h}_{k+1}, \dots, \tilde{h}_{2k+1})$$
$$= (0, 1, \dots, k, D - k + 1, \dots, D, D + 1).$$

Using (d) and 1.6 we see that $e_i(B) = (1 - k - \underline{k})(2 - k - \underline{k})/2$. On the other hand we have $e_i(\tilde{B}) = (-k - \underline{k})(-k - \underline{k} + 1)/2$. We get $(1 - k - \underline{k})(2 - k - \underline{k})/2 =_2 (-k - \underline{k})(-k - \underline{k} + 1)/2$, contradicting 1.6(e) since $k + \underline{k}$ is even. Thus $B \notin \mathbf{S}_D^{prim}$ leads to a contradiction. Thus both $B, \tilde{B}$ are primitive. Since $B, \tilde{B}$ are primitive and $|B^0| = |\tilde{B}^0|$, we see that $B = \tilde{B}$. This proves (a).

1.9. We no longer assume that D is even. Let V be the $\mathbf{F}_2$-vector space with basis $\{e_i; i \in [1, D]\}$. For any subset I of $[1, D]$ let $e_I = \sum_{i \in I} e_i \in V$. We define a symplectic form $(,) : V \times V \to \mathbf{F}_2$ by $(e_i, e_j) = 1$ if $i - j = \pm 1$, $(e_i, e_j) = 0$ if $i - j \neq \pm 1$. This symplectic form is non-degenerate if D is even while if D is odd it has a one-dimensional radical spanned by $e_1 + e_3 + e_5 + \cdots + e_D$.

For any subset Z of V we set $Z^\perp = \{x \in V; (x, z) = 0 \quad \forall z \in Z\}$.

When $D \geq 2$ we denote by V' the $\mathbf{F}_2$-vector space with basis $\{e'_i; i \in [1, D-2]\}$. For any $I' \subset [1, D - 2]$ let $e'_{I'} = \sum_{i \in I'} e'_i \in V'$. We define a symplectic form $(,)' : V' \times V' \to \mathbf{F}_2$ by $(e'_i, e'_j) = 1$ if $i - j = \pm 1$, $(e'_i, e'_j) = 0$ if $i - j \neq \pm 1$.

When $D \geq 2$, for any $i \in [1, D]$ there is a unique linear map $T_i : V' \to V$ such that the sequence $T_i(e'_1), T_i(e'_2), \dots, T_i(e'_{D-2})$ is:

$e_1, e_2, \dots, e_{i-2}, e_{i-1} + e_i + e_{i+1}, e_{i+2}, e_{i+3}, \dots, e_D$ (if $1 < i < D$),

$e_3, e_4, \dots, e_D$ (if $i = 1$),

$e_1, e_2, \dots, e_{D-2}$ (if $i = D$).

Note that T_i is injective and $(x, y)' = (T_i(x), T_i(y))$ for any x, y in V'. For any $I' \in \mathcal{I}_{D-2}$ we have $T_i(e'_{I'}) = e_{\xi_i(I')}$. Let V_i be the image of $T_i : V' \to V$. From the definitions we deduce:

(a) $e_i^\perp = V_i \oplus \mathbf{F}_2 e_i$.

In the remainder of this section we assume that D is even.

If $D \geq 2$, for $j \in [1, D - 2]$ let $f'_j : \mathbf{S}_{D-2} \to \mathbf{Z}$, $\epsilon'_j : \mathbf{S}_{D-2} \to \mathbf{F}_2$ be the analogues of $f_i : \mathbf{S}_D \to \mathbf{Z}$, $\epsilon_i : \mathbf{S}_D \to \mathbf{F}_2$ when D is replaced by $D - 2$.

For $B \in \mathbf{S}_D$, we define $\epsilon(B) \in V$ by $\epsilon(B) = \sum_{i \in [1,D]} \epsilon_i(B) e_i$. If $D \geq 2$, for $B' \in \mathbf{S}_{D-2}$ we define $\epsilon'(B') \in V'$ by $\epsilon'(B') = \sum_{j \in [1,D-2]} \epsilon'_j(B') e'_j$. We show (extending [L5, 1.9(b)]):

(b) *Assume that $D \geq 2$, $i \in [1, D]$. Let $B' \in \mathbf{S}_{D-2}, B = t_i(B') \in \mathbf{S}_D$. Then $\epsilon(B) = T_i(\epsilon'(B')) + ce_i$ for some $c \in \mathbf{F}_2$.*

An equivalent statement is: for any $j \in [1, D] - \{i\}$ we have $\epsilon_j(B) = \epsilon'_{j'}(B')$ if $j' \in [1, D - 2]$ is such that $j \in \xi_i(\{j'\})$; and $\epsilon_j(B) = 0$ if no such j' exists. It is enough to show:

$f'_h(B') = f_h(B)$ if $1 \leq h \leq i - 2$,

$f'_{h-2}(B') = f_h(B)$ if $i + 2 \leq h \leq D$,

$f_{i-1}(B) = f_{i+1}(B) = f'_{i-1}(B')$ if $1 < i < D$,

$f_{i-1}(B) \in \{0, -1\}$ (hence $\epsilon_{i-1}(B) = 0$) if $i = D$,

$f_{i+1}(B) \in \{0, -1\}$ (hence $\epsilon_{i+1}(B) = 0$) if $i = 1$.

This follows from 1.4(a).

For $B \in \mathbf{S}_D$ let $\langle B\rangle$ be the subspace of V generated by $\{e_I; I \in B\}$. For $B' \in \mathbf{S}_{D-2}$ let $\langle B'\rangle$ be the subspace of V' generated by $\{e'_{I'}; I' \in B'\}$. We show (extending [L5, 1.9(c)]):

(c) *Let* $B \in \mathbf{S}_D$. *We have* $\epsilon(B) \in \langle B\rangle$. *If* $D \ge 2, i \in [1, D]$, $B' \in \mathbf{S}_{D-2}, B = t_i(B') \in \mathbf{S}_D$, *then* $\langle B\rangle = T_i(\langle B'\rangle) \oplus \mathbf{F}_2 e_i$.

To prove the first assertion of (c) we argue by induction on D. For $d = 0$ there is nothing to prove. Assume that $d \ge 1$. Let i, B' be as in (b). By the induction hypothesis we have $\epsilon'(B') \in \langle B'\rangle \subset V'$. Using (b) we see that it is enough to show that $T_i(\langle B'\rangle) \subset \langle B\rangle$. (Since $\{i\} \in B$, we have $e_i \in \langle B\rangle$.) Using the equality $T_i(e'_{I'}) = e_{\xi_i(I')}$ for any $I' \in B'$ it remains to note that $\xi_i(I') \in B$ for $I' \in B'$. This proves the first assertion of (c). The same proof shows the second assertion of (c).

For $s \in [0, D/2]$ we set $t = s/2$ if s is even and $t = (s + 1)/2$ if s is odd; we denote by $V(s)$ the set of vectors $x \in V$ such that $x = e_{[a_1,b_1]} + e_{[a_2,b_2]} + \dots + e_{[a_t,b_t]}$ with $[a_r, b_r] \in \mathcal{I}_D$ with any two of them non-touching and with $a_1 =_2 a_2 =_2 \dots =_2 a_t =_2 s$, $b_1 =_2 b_2 =_2 \dots =_2 b_t =_2 s + 1$. For such x we set $n(x) = a_1 + b_1 + a_2 + b_2 + \dots + a_t + b_t \in \mathbf{N}$.

Assume for example that

$$B = \{[1, D], [2, D - 1], \dots, [D/2, (D/2) + 1]\}.$$

We have $|B^0_i| = i$ for $i \in [1, D/2]$, $|B^0_i| = D - i + 1$ for $i \in [(D/2) + 1, D]$, $|B^1_i| = 0$ for all i. It follows that

(d) $\quad \epsilon(B) = e_{[2,3]} + e_{[6,7]} + e_{[10,11]} + \dots + e_{[D-2,D-1]} \in V(D/2)$ if $D/2$ is even,

(e) $\quad \epsilon(B) = e_{[1,2]} + e_{[5,6]} + e_{[9,10]} + \dots + e_{[D-1,D]} \in V(D/2)$ if $D/2$ is odd.

More generally, assume that

(f) $\quad B = \{[1, D], [2, D - 1], \dots, [s, D + 1 - s]\}$, where $s \in [0, D/2]$.

We have $|B^0_i| = i$ for $i \in [1, s]$, $|B^0_i| = D - i + 1$ for $i \in [D - s + 1, D]$, $|B^1_i| = 0$ for all i. It follows that:

if $s = 0$, then $\epsilon(B) = 0$;
if $s = 1$, then $\epsilon(B) = e_{[1,D]}$;
if $s = 2$, then $\epsilon(B) = e_{[2,D-1]}$;
if $s = 3$, then $\epsilon(B) = e_{[1,2]} + e_{[D-1,D]}$;
if $s = 4$, then $\epsilon(B) = e_{[2,3]} + e_{[D-2,D-1]}$;
if $s = 5$, then $\epsilon(B) = e_{[1,2]} + e_{[5,D-4]} + e_{[D-1,D]}$;
if $s = 6$, then $\epsilon(B) = e_{[2,3]} + e_{[6,D-5]} + e_{[D-2,D-1]}$, etc.

Thus,

(g) $\epsilon(B) \in V(s)$.

Let $B \in \mathbf{S}_D$. Using (P_0) we deduce:

(h) $\langle B\rangle$ *is an isotropic subspace of* V.

We show (extending [L5, 2.1(b)]):

(i) $\{e_I; I \in B\}$ *is an* $\mathbf{F}_2$*-basis of* $\langle B\rangle$.

We can assume that $D \ge 2$. Assume that $\sum_{I\in B} c_I e_I = 0$ with $c_I \in \mathbf{F}_2$ not all zero. We can find $I_1 = [a, b] \in B$ with $c_{I_1} \ne 0$ and $|I_1|$ maximal. If $a \in I'$ with $I' \in B$, $I' \ne I_1$, $c_{I'} \ne 0$, then by (P_0) we have $I_1 \prec I'$ (contradicting the maximality of $|I_1|$) or $I' \prec I_1$ (contradicting $a \in I'$). Thus no I' as above exists.

Thus when $\sum_{I\in B} c_I e_I$ is written in the basis $\{e_j; j \in [1, 2d]\}$, the coefficient of e_a is c_{I_1} and hence $c_{I_1} = 0$, contradicting $c_{I_1} \neq 0$. Thus (i) holds for B.

1.10. Let $B \in \mathbf{S}_D, \tilde{B} \in \mathbf{S}_D$. We show:

(a) *If* $\epsilon(B) = \epsilon(\tilde{B})$ *and* $|B^0| = |\tilde{B}^0|$, *then* $B = \tilde{B}$.

We argue by induction on D. If $D = 0$, there is nothing to prove. Assume that $D \geq 2$. If $\tilde{B} \in \mathbf{S}_D^{prim}$, then (a) follows from 1.8(a). Similarly, (a) holds if $B \in \mathbf{S}_D^{prim}$. Thus, we can assume that B and $\tilde{B}$ are not primitive. By 1.7(a) we can find $i \in [1, D]$ such that $\{i\} \in B^1$, $\{i\} \in \tilde{B}^1$. By 1.3(f) we then have $B = t_i(B')$, $\tilde{B} = t_i(\tilde{B}')$ with $B' \in \mathbf{S}_{D-2}$, $\tilde{B}' \in \mathbf{S}_{D-2}$. Using our assumption and 1.9(b) we see that $T_i(\epsilon'(B')) = T_i(\epsilon'(\tilde{B}')) + ce_i$ for some $c \in \mathbf{F}_2$. Using 1.9(a) we see that $c = 0$ so that $T_i(\epsilon'(B')) = T_i(\epsilon'(\tilde{B}'))$. Since T_i is injective, we deduce $\epsilon'(B') = \epsilon'(\tilde{B}')$. We have also $|B'^0| = |\tilde{B}'^0|$. By the induction hypothesis we have $B' = \tilde{B}'$ and hence $B = \tilde{B}$. This proves (a).

1.11. Any $x \in V$ can be written uniquely in the form

$$x = e_{[a_1,b_1]} + e_{[a_2,b_2]} + \cdots + e_{[a_r,b_r]},$$

where $[a_r, b_r] \in \mathcal{I}_D$ are such that any two of them are non-touching and $r \geq 0$, $1 \leq a_1 \leq b_1 < a_1 \leq b_2 < \cdots < a_r \leq b_r \leq D$. Following [L2, 3.3] and [L5, 1.11(a)] we set

(a) $\quad u(v) = |\{s \in [1, r]; a_s =_2 0, b_s =_2 1\}| - |\{s \in [1, r]; a_s =_2 1, b_s =_2 0\}| \in \mathbf{Z}.$

This defines a function $u : V \to \mathbf{Z}$. When $D \geq 2$ we denote by $u' : V' \to \mathbf{Z}$ the analogous function with D replaced by $D - 2$. The following result appears also in [L5, 1.11(b)].

(b) *Assume that* $D \geq 2, i \in [1, D]$. *Let* $v' \in V'$, *and let* $v = T_i(v') + ce_i \in V$, *where* $c \in \mathbf{F}_2$. *We have* $u(v) = u'(v')$.

We write $v' = e'_{[a'_1,b'_1]} + e'_{[a'_2,b'_2]} + \cdots + e'_{[a'_r,b'_r]}$, where $r \geq 0$, $[a'_s, b'_s] \in \mathcal{I}_{D-2}$ for all s and any two of $[a'_s, b'_s]$ are non-touching. For each s, we have $T_i(e'_{[a'_s,b'_s]}) = e_{[a_s,b_s]}$, where $[a_s, b_s] = \xi_i[a'_s, b'_s]$ so that $a_s =_2 a'_s$, $b_s =_2 b'_s$ and the various $[a_s, b_s]$ which appear are still non-touching with each other. Hence $u(T_i(v')) = u'(v')$. We have $v = T_i(v')$ or $v = T_i(v') + e_i$. If $v = T_i(v')$, we have $u(v) = u'(v')$, as desired. Assume now that $v = T_i(v') + e_i$. From the definition of ξ_i we see that either

(i) $[i, i]$ is non-touching with any $[a_s, b_s]$, or

(ii) $[i, i]$ is not non-touching with some $[a, b] = [a_s, b_s]$ which is uniquely determined and we have $a < i < b$.

If (i) holds, then e_i does not contribute to $u(v)$ and $u(v) = u(T_i(v')) = u'(v')$. We now assume that (ii) holds. Then $e_{[a,b]} + e_i = e_{[a,i-1]} + e_{[i+1,b]}$. We consider six cases.

(1) a is even b is odd, i is even; then $|[i + 1, b]|$ is odd so that the contribution of $e_{[a,i-1]} + e_{[i+1,b]}$ to $u(v)$ is $1 + 0$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is 1.

(2) a is even, b is odd, i is odd; then $|[a, i - 1]|$ is odd so that the contribution of $e_{[a,i-1]} + e_{[i+1,b]}$ to $u(v)$ is $0 + 1$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is 1.

(3) a is odd, b is even, i is even; then $|[i + 1, b]|$ is odd so that the contribution of $e_{[a,i-1]} + e_{[i+1,b]}$ to $u(v)$ is $0 - 1$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is -1.

(4) a is odd, b is even, i is odd; then $|[a,i-1]|$ is odd so that the contribution of $e_{[a,i-1]}+e_{[i+1,b]}$ to $u(v)$ is $-1+0$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is -1.

(5) $a=_2 b=_2 i+1$; then $|[a,i-1]|$ is odd, $|[i+1,b]|$ is odd so that the contribution of $e_{[a,i-1]}+e_{[i+1,b]}$ to $u(v)$ is $0+0$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is 0.

(6) $a=_2 b=_2 i$; then the contribution of $e_{[a,i-1]}+e_{[i+1,b]}$ to $u(v)$ is $1-1$ or $-1+1$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is 0.

This proves (b).

Let $s\in[0,D/2]$, and let $x\in V(s)$; see 1.9. From the definition, the following holds:

(c) *If s is even, then $u(x)=s/2$; if s is odd, then $u(x)=-(s+1)/2$.*

We now define $\tilde{u}:V\to\mathbf{N}$ by $\tilde{u}(x)=2u(x)$ if $u(x)\ge 0$, $\tilde{u}(x)=-2u(x)-1$ if $u(x)<0$. From (c) we deduce:

(d) *If $s\in[0,D/2]$ and $x\in V(s)$, then $\tilde{u}(x)=s$.*

1.12. As in [L5, 1.12], we view V as the set of vertices of a graph in which x,x' in V are joined whenever there exists $i\in[1,D]$ such that $x+x'=e_i$, $(x,e_i)=(x',e_i)=0$. (We then write $x\diamond x'$.) We show:

(a) *Let $s\in[0,D/2]$, and let x,x' be in $V(s)$; see* 1.9*. Then x,x' are in the same connected component of the graph V.*

As in 1.9 we set $t=s/2$ if s is even and $t=(s+1)/2$ if s is odd. There is a unique element $x_s\in V(s)$ such that $n(x_s)\le n(y)$ for any $y\in V(s)$ (see 1.9 for the definition of $n(y)$). This element is of the form $x_s=e_{[a_1^0,b_1^0]}+e_{[a_2^0,b_2^0]}+\dots+e_{[a_t^0,b_t^0]}$, where $a_1^0,b_1^0,a_2^0,b_2^0,a_3^0,b_3^0,\dots$ is $2,3,6,7,10,11,\dots$ if s is even and is $1,2,5,6,9,10,\dots$ if s is odd. Let Γ be the connected component of the graph V that contains x_s. Let $x=e_{[a_1,b_1]}+e_{[a_2,b_2]}+\dots+e_{[a_t,b_t]}\in V(s)$ be as in the definition of $V(s)$; see 1.9. We show that $x\in\Gamma$ by induction on $n(x)$. If $n(x)=n(x_s)$, then $x=x_s$ and there is nothing to prove. Assume now that $n(x)>n(x_s)$. Then one of (i), (ii) below holds:

(i) for some $z\ge 1$ we have $a_j=a_j^0$, $b_j=b_j^0$ for $j\in[1,z-1]$, $a_z>a_z^0$;

(ii) for some $z\ge 1$ we have $a_j=a_j^0$, $b_j=b_j^0$ for $j\in[1,z-1]$, $a_z=a_z^0$, $b_z>b_z^0$.

In case (i) we have $a_z-2\in[1,2d]$, $(e_{a_z-2},x)=0$ hence $x+e_{a_z-2}\diamond x$. We have $(e_{a_z-1},x+e_{a_z-2})=0$ hence $x':=x+e_{a_z-2}+e_{a_z-1}\diamond x+e_{a_z-2}$. We have $n(x')=n(x)-2$. By the induction hypothesis we have $x'\in\Gamma$ hence $x\in\Gamma$.

In case (ii) we have $(e_{b_z-1},x)=0$ hence $x+e_{b_z-1}\diamond x$. We have $(e_{b_z},x+e_{b_z-1})=0$ hence $x':=x+e_{b_z-1}+e_{b_z}\diamond x+e_{b_z-1}$. We have $n(x')=n(x)-2$. By the induction hypothesis we have $x'\in\Gamma$ hence $x\in\Gamma$. This proves (a).

1.13. For $x\in V$ we show:

(a) *There exist $s\in[0,D/2]$ and $\tilde{x}\in V(s)$ (see* 1.9*) such that $x,\tilde{x}$ are in the same component of the graph V.*

We argue by induction on D. If $D=0$ there is nothing to prove. Assume now that $D\ge 1$. Assume first that x is the element described in 1.9(d) or (e). Then $x\in V(D/2)$ so that there is nothing to prove. Next we assume that x is not the element described in 1.9(d) or (e). Then $(x,e_i)=0$ for some $i\in[1,D]$. By 1.9(a) we have $x=T_i(x')+ce_i$ for some $x'\in V'$ and some $c\in\mathbf{F}_2$. We first show the following result which appears also in [L5, 1.12(a)].

(b) *If y, y' in V' are joined in the graph V' (analogue of the graph V), then $T_i(y), T_i(y')$ are in the same connected component of the graph V.*

We can find $j \in [1, D-2]$ such that $(y, e'_j)' = (y', e'_j)' = 0$, $y + y' = e'_j$. Hence $(\tilde{y}, T_i(e'_j)) = (\tilde{y}', T_i(e'_j)) = 0$, $\tilde{y} + \tilde{y}' = T_i(e'_j)$, where $\tilde{y} = T_i(y), \tilde{y}' = T_i(y')$. If $T_i(e'_j) = e_h$ for some $h \in [1, D]$, then $\tilde{y}, \tilde{y}'$ are joined in V, as required. If this condition is not satisfied, then $1 < i < D$, $j = i - 1$ and $T_i(e'_j) = e_j + e_{j+1} + e_{j+2}$. We have $(\tilde{y}, e_j + e_{j+1} + e_{j+2}) = 0$, $\tilde{y} + \tilde{y}' = e_j + e_{j+1} + e_{j+2}$. Since $\tilde{y} \in V_i$ we have $(\tilde{y}, e_i) = 0$ hence $(\tilde{y}, e_{j+1}) = 0$ so that $(\tilde{y}, e_j) = (\tilde{y}, e_{j+2})$. We are in one of the two cases below.

(1) We have $(\tilde{y}, e_j) = (\tilde{y}, e_{j+2}) = 0$.

(2) We have $(\tilde{y}, e_j) = (\tilde{y}, e_{j+2}) = 1$.

In case (1) we consider the four-term sequence $\tilde{y}, \tilde{y} + e_j, \tilde{y} + e_j + e_{j+2}, \tilde{y} + e_j + e_{j+1} + e_{j+2} = \tilde{y}'$; any two consecutive terms of this sequence are joined in the graph V. In case (2) we consider the four-term sequence $\tilde{y}, \tilde{y} + e_{j+1}, \tilde{y} + e_j + e_{j+1}, \tilde{y} + e_j + e_{j+1} + e_{j+2} = \tilde{y}'$; any two consecutive terms of this sequence are joined in the graph V. We see that in both cases $\tilde{y}, \tilde{y}'$ are in the same connected component of V and (b) is proved.

We now continue the proof of (a). By the induction hypothesis there exists $s \in [0, (D/2) - 1]$ and $x'' \in V'(s)$ such that x', x'' are in the same connected component of V'. Here $V'(s)$ is defined like $V(s)$ (replacing V by V'). By (b), $T_i(x'), T_i(x'')$ are in the same connected component of V. From the definitions we see that $T_i(V'(s)) \subset V(s)$. Thus $T_i(x'') \in V(s)$. Clearly $x, T_i(x')$ are joined in the graph V. Hence $x, T_i(x'')$ are joined in the graph V. We see that (a) holds.

1.14. The following result follows by repeated application of 1.11(b).

(a) *If x, x' in V are in the same connected component of the graph V, then $u(x) = u(x')$.*

We can assume that x, x' are joined in the graph V. Then for some $i \in [1, D]$ we have $x = T_i(y) + ce_i$, $x' = T_i(y) + c'e_i$, where $y \in V'$, $c \in \mathbf{F}_2, c' \in \mathbf{F}_2$. By 1.11(b) we have $u(x) = u'(y)$, $u(x') = u'(y)$, hence $u(x) = u(x')$. This proves (a).

We now show the converse.

(b) *If x, x' in V satisfy $u(x) = u(x')$, then x, x' are in the same connected component of the graph V.*

By 1.13(a) we can find s, s' in $[0, D/2]$ and $x_1 \in V(s)$, $x'_1 \in V(s')$ such that x, x_1 are in the same connected component of the graph V and x', x'_1 are in the same connected component of the graph V. Thus, it is enough to prove that x_1, x'_1 are in the same connected component of the graph V. By (a), we have $u(x_1) = u(x'_1)$ hence $\tilde{u}(x_1) = \tilde{u}(x'_1)$. From 1.11(d) we have $\tilde{u}(x_1) = s$, $\tilde{u}(x'_1) = s'$. Using $\tilde{u}(x_1) = \tilde{u}(x'_1)$ we deduce that $s = s'$. Since $x_1 \in V(s), x'_1 \in V(s)$, they are in the same connected component of the graph V, by 1.12(a). This proves (b).

We show:

(c) *Let $B \in \mathbf{S}_D$. Let $k = |B^0|$, $k' = \tilde{u}(\epsilon(B)) \in \mathbf{Z}$. Then $k' = k$.*

We argue by induction on D. If $D = 0$ there is nothing to prove. Assume now that $D \geq 2$. If $B \in \mathbf{S}_D^{prim}$, then $\epsilon(B) \in V(k)$ (see 1.9(g)), and the result follows from 1.11(d). We now assume that $B \notin \mathbf{S}_D^{prim}$. We can find $i \in [1, D]$ and $B' \in \mathbf{S}_{D-2}$ such that $B = t_i(B')$. By 1.9(b) we have $\epsilon(B) = T_i(\epsilon'(B')) + ce_i$, where $c \in \mathbf{F}_2$. Using 1.11(b) we deduce $u(\epsilon(B)) = u'(\epsilon'(B'))$. Hence $\tilde{u}(\epsilon(B)) = \tilde{u}'(\epsilon'(BB'))$, where $\tilde{u}' : V' \to \mathbf{Z}$ is defined in terms of u' in the same way as $\tilde{u}$ is defined in terms of u. By the induction hypothesis we have $\tilde{u}'(\epsilon'(B')) = k$. This proves (c).

(d) *The map* $\epsilon : \mathbf{S}_D \to V$ *(see* 1.19*) is injective.*

Assume that $B, \tilde{B}$ in $\mathbf{S}_D$ are such that $\epsilon(B) = \epsilon(\tilde{B})$. Let $k = |B^0|$, $\tilde{k} = |\tilde{B}^0|$. Let $k' = \tilde{u}(\epsilon(B)) = \tilde{u}(\epsilon(\tilde{B}))$. By (c) we have $k' = k$, $k' = \tilde{k}$. It follows that $k = \tilde{k}$. Using now 1.10(a), we see that $B = \tilde{B}$. This proves (d).

1.15. Let $s \in [0, D/2]$, and let $B \in \mathbf{S}_D$ be as in 1.9(f), so that $\epsilon(B) \in V(s)$. We show:

(a) *For any* $x \in \langle B \rangle$ *we have* $\tilde{u}(x) \le s$*; moreover, we have* $\tilde{u}(x) = s$ *for a unique* $x \in \langle B \rangle$*.*

We argue by induction on D. If $D = 0$ the result is obvious. We now assume that $D \ge 2$. Assume first that $s = D/2$. If $x_1 = \epsilon(B)$, then $x_1 \in \langle B \rangle$ (see 1.9(c)), and $\tilde{u}(x_1) = D/2$ (see 1.14(c)). Conversely, assume that $x' \in V$, $\tilde{u}(x') = D/2$. Using 1.14(b), we see that x', x_1 are in the same connected component of V. From 1.9(d),(e), we see that $(x_1, e_i) = 1$ for any $i \in [1, D]$. Thus, x_1 is a connected component of V by itself, so that $x' = x_1$. Hence in this case (a) holds. Next we assume that $s < D/2$. Then $B' = \{[1, D-2], [2, D-3], \dots, [s, D-1-s]\} \in \mathbf{S}_{D-2}$ satisfies $\epsilon'(B') = s$ (by 1.9(f)). We have $B = \{\xi_i(I'); I' \in B'\}$. Let $i = s+1$. Let $\tilde{B} = t_i(B') = B \sqcup \{i\}$. Using the induction hypothesis for B' and 1.11(b) we see that for any

$$x \in T_i(\langle B' \rangle) \oplus \mathbf{F}_2 e_i = \langle B \rangle \oplus \mathbf{F}_2 e_i = \langle \tilde{B} \rangle$$

(see 1.9(c)) we have $\tilde{u}(x) \le s$; moreover, we have $\tilde{u}(x) = s$ for exactly two values of $x \in \langle \tilde{B} \rangle$ (whose sum is e_i). One of these values is in $\langle B \rangle$ and the other is not in $\langle B \rangle$. This proves (a).

1.16. Let F be the $\mathbf{C}$-vector space consisting of functions $V \to \mathbf{C}$. For $x \in V$ let $\psi_x \in F$ be the characteristic function of x. For $B \in \mathbf{S}_D$ let $\Psi_B \in F$ be the characteristic function of $\langle B \rangle$. Let $\tilde{F}$ be the $\mathbf{C}$-subspace of F generated by $\{\Psi_B; B \in \mathbf{S}_D\}$. When $D \ge 2$ we define $\psi'_{x'}$ for $x' \in V'$ and $\Psi'_{B'}$ for $B' \in \mathbf{S}_{D-2}$, $F', \tilde{F}'$, in terms of $\mathbf{S}_{D-2}$ in the same way as $\psi_x, \Psi_B, F, \tilde{F}$ were defined in terms of $\mathbf{S}_D$. For any $i \in [1, D]$ we define a linear map $\theta_i : F' \to F$ by $f' \mapsto f$, where $f(T_i(x') + ce_i) = f'(x')$ for $x' \in V', c \in \mathbf{F}_2$, $f(x) = 0$ for $x \in V - e_i^{\perp}$. We have

$\theta_i(\psi'_{x'}) = \psi_{T_i(x')} + \psi_{T_i(x')+e_i}$ for any $x' \in V'$,

$\theta_i(\Psi'_{B'}) = \Psi_{t_i(B')}$ for any $B' \in \mathbf{S}_{D-2}$.

We show:

(a) *For any* $x \in V$*, we have* $\psi_x \in \tilde{F}$*.*

We argue by induction on D. If $D = 0$ the result is obvious. We now assume that $D \ge 2$. We first show:

(b) *If* $x, \tilde{x}$ *in* V *are joined in the graph* V *and if* (a) *holds for* x*, then* (a) *holds for* $\tilde{x}$*.*

We can find $j \in [1, D]$ such that $x + \tilde{x} = e_j$, $(x, e_j) = 0$. We have $x = T_j(x') + ce_j$, $\tilde{x} = T_j(x') + c'e_j$, where $x' \in V'$ and $c \in \mathbf{F}_2, c' \in \mathbf{F}_2$, $c + c' = 1$. By the induction hypothesis we have $\psi'_{x'} = \sum_{B' \in \mathbf{S}_{D-2}} a_{B'} \Psi'_{B'}$, where $a_{B'} \in \mathbf{C}$. Applying θ_j we obtain

$$\psi_x + \psi_{\tilde{x}} = \sum_{B' \in \mathbf{S}_{D-2}} a_{B'} \Psi_{t_j(B')}.$$

We see that $\psi_x + \psi_{\tilde{x}} \in \tilde{F}$. Since $\psi_x \in \tilde{F}$, by assumption, we see that $\psi_{\tilde{x}} \in \tilde{F}$. This proves (b).

For any $s \in [0, D/2]$ we show:

(c) *If $x \in V$ is such that $\tilde{u}(x) = s$, then $\psi_x \in \tilde{F}$.*

We argue by induction on s. Let $\tilde{x} = \epsilon(B)$, where B is as in 1.9(f) so that $\tilde{x} \in V(s)$ (see 1.9(g)) and $\tilde{u}(\tilde{x}) = s$ (see 1.11(d)). Using 1.14(b) we see that $x, \tilde{x}$ are in the same connected component of the graph V and using (b) we see that it is enough to show that $\psi_{\tilde{x}} \in \tilde{F}$. Let x_0 be the unique element of $\langle B \rangle$ such that $\tilde{u}(x_0) = s$ (see 1.15(a)). By the uniqueness of x_0 we must have $x_0 = \tilde{x}$. From 1.15 we see that for $x_1 \in \langle B \rangle - \{\tilde{x}\}$ we have $\tilde{u}(x_1) < s$; for such x_1 we have $\psi_{x_1} \in \tilde{F}$ by the induction hypothesis. We have $\Psi_B = \psi_{\tilde{x}} + \sum_{x_1 \in \langle B \rangle - \{\tilde{x}\}} \psi_{x_1} = \psi_{\tilde{x}} \mod \tilde{F}$. Since $\Psi_B \in \tilde{F}$, we see that $\psi_{\tilde{x}} \in \tilde{F}$. This proves (c) hence also (a).

Since $\tilde{F} \subset F$, we see that (a) implies:

(d) $F = \tilde{F}$.

This extends [L5, 1.15(c)]. We have the following result which extends [L5, 1.16].

Theorem 1.17. *(a)$\{\Psi_B; B \in \mathbf{S}_D\}$ is a $\mathbf{C}$-basis of F.*

(b)$\epsilon : \mathbf{S}_D \to V$ is a bijection.

From the definition of $\tilde{F}$ we have $\dim \tilde{F} \le |\mathbf{S}_D|$. By 1.14(d) we have $|\mathbf{S}_D| \le |V| = \dim F$. Since $F = \tilde{F}$ (see 1.16(d)), it follows that $\dim \tilde{F} = |\mathbf{S}_D| = |V| = \dim F$. Using again the definition of $\tilde{F}$ and the equality $F = \tilde{F}$ we see that (a) holds. Since the map in (b) is injective (see 1.14(d)) and $|\mathbf{S}_D| = |V|$ we see that it is a bijection so that (b) holds.

Let $\mathcal{F}(V)$ be the set of (isotropic) subspaces of V of the form $\langle B \rangle$ for some $B \in \mathbf{S}_D$. By definition, the map $\mathbf{S}_D \to \mathcal{F}(V)$, $B \mapsto \langle B \rangle$ is surjective. In fact,

(c) *this map is a bijection.*

Indeed, if $B, \tilde{B}$ in $\mathbf{S}_D$ satisfy $\langle B \rangle = \langle \tilde{B} \rangle$, then the functions $\Psi_B, \Psi_{\tilde{B}}$ in F coincide and (d) follows from (a).

Note that $\mathcal{F}(V)$ admits an inductive definition similar to that of $\mathbf{S}_D$. If $D = 0$, $\mathcal{F}(V)$ consists of the subspace $\{0\}$. If $D \ge 2$, a subspace E of V is in $\mathcal{F}(V)$ if and ony if it is either of the form $\langle B \rangle$ for some $B \in \mathbf{S}_D^{prim}$ or if there exists $i \in [1, D]$ and $E' \in \mathcal{F}(V')$ such that $E = T_i(E') \oplus \mathbf{F}_2 e_i$.

1.18. Assume that $D \ge 2$. Let $B \in \mathbf{S}_D$, and let $i \in [1, D]$ be such that $\{i\} \in B$. Let Z_i be the set of all $[a, b] \in B^1$ such that $a < i < b$. If $I \in Z_i, I' = [a', b'] \in Z_i$, then $I \cap I' \ne \emptyset$ and hence we have either $I \subset I'$ or $I' \subset I$. It follows that if $Z_i \ne \emptyset$, then Z_i contains a unique interval $[a, b]$ such that $b - a$ is minimum; we set $Z_i^{min} = \{[a, b]\}$. We show:

(a) *If $Z_i \ne \emptyset$ and $Z_i^{min} = \{[a, b]\}$, then $a =_2 b =_2 i + 1$.*

If this is not so, then $a =_2 b =_2 i$. By (P_1) there exists $[a_1, b_1] \in B^1$ such that $a < a_1 \le i - 1 \le b_1 < b$. If $b_1 = i - 1$, then applying (P_0) to $[a_1, b_1], \{i\}$ gives a contradiction. Thus $b_1 \ge i$ and $i \in [a_1, b_1]$. By the minimality of $b - a$, we have $[a_1, b_1] = \{i\}$. This contradicts $i - 1 \in [a_1, b_1]$ and proves (a).

Let $h_0 < h_1 < \cdots < h_{2k+1}$ be the sequence attached to B in (P_2). We show:

(b) *Assume that $h_s < i < h_{s+1}$. If $s \in [0, k-1]$ and $i =_2 s$, then $Z_i \ne \emptyset$. If $s \in [k+1, 2k]$ and $i =_2 s + 1$, then $Z_i \ne \emptyset$.*

We prove the first assertion of (b). We have $h_s < i - 1 < h_{s+1}$ (since $h_s \ne_2 i - 1$). By (P_2) we can find $[a, b] \in B^1$ such that $h_s < a \le i - 1 \le b < h_{s+1}$. If $b = i - 1$, then applying (P_0) to $[a, b], \{i\}$ gives a contradiction. Thus, $b \ge i$ and $i \in [a, b]$. Since $a < i$ we have $\{i\} \prec [a, b]$ so that $[a, b] \in Z_i$. This proves the first assertion of (b). The second assertion of (b) can be deduced from the first assertion using the involution $\tau_D : \mathbf{S}_D \to \mathbf{S}_D$ in 1.2.

We show:

(c) *If $h_s < i < h_{s+1}$, $s \in [0, k-1]$, $i =_2 s+1$, then either $Z_i = \emptyset$ or $Z_i \neq \emptyset$ and $Z_i - Z_i^{min} \neq \emptyset$.*

Assume that $Z_i \neq \emptyset$. Let $[a,b] \in Z_i^{min}$, so that $a < i < b$. Using 1.3(e) we see that $h_s < a < b < h_{s+1}$. By (a) we have $a =_2 h_s$. Since $h_s < a$, we must have $h_s < a-1 < h_{s+1}$. By (P_2) we can find $[a',b'] \in B^1$ such that $h_s < a' \le a-1 \le b' < h_{s+1}$. If $b' = a-1$, then applying (P_0) to $[a,b],[a',b']$ gives a contradiction. Thus, $b' \ge a$, so that $[a',b'] \cap [a,b] \neq \emptyset$. This implies that either $[a',b'] \subset [a,b]$ or $[a,b] \prec [a',b']$. The first alternative does not hold since $a-1 \in [a',b'], a-1 \notin [a,b]$. Thus we have $[a,b] \prec [a',b']$ so that $[a',b'] \in Z_i - Z_i^{min}$. This proves (c).

We define a collection C of subsets of $\mathcal{I}_D$ as follows:

(i) If $h_k < i < h_{k+1}$ and $Z_i = \emptyset$, then $C = B - \{i\}$.

(ii) If $h_k < i < h_{k+1}$ and $Z_i \neq \emptyset$, then $C = (B - \{[a,b],\{i\}\}) \sqcup \{[a,i-1],[i+1,b]\}$, where $Z_i^{min} = \{[a,b]\}$.

(iii) If $h_s < i < h_{s+1}$, $s \in [0,k-1]$, $i =_2 s$, so that $Z_i \neq \emptyset$ (see (b)), then $C = (B - \{[a,b],\{i\}\}) \sqcup \{[a,i-1],[i+1,b]\}$, where $Z_i^{min} = \{[a,b]\}$.

(iv) If $h_s < i < h_{s+1}$, $s \in [0,k-1]$, $i =_2 s+1$ and $Z_i \neq \emptyset$, then $C = (B - \{[a,b],\{i\}\}) \sqcup \{[a,i-1],[i+1,b]\}$, where $Z_i^{min} = \{[a,b]\}$.

(v) If $h_s < i < h_{s+1}$, $s \in [k+1,2k]$, $i =_2 s+1$ so that $Z_i \neq \emptyset$ (see (b)), then $C = (B - \{[a,b],\{i\}\}) \sqcup \{[a,i-1],[i+1,b]\}$, where $Z_i^{min} = \{[a,b]\}$.

(vi) If $h_s < i < h_{s+1}$, $s \in [k+1,2k]$, $i =_2 s$ and $Z_i \neq \emptyset$, then $C = (B - \{[a,b],\{i\}\}) \sqcup \{[a,i-1],[i+1,b]\}$, where $Z_i^{min} = \{[a,b]\}$.

(vii) If $h_s < i < h_{s+1}$, $s \in [0,k-1]$, $i =_2 s+1$ and $Z_i = \emptyset$, then $C = (B - \{[h_{s+1},h_{2k-s}],\{i\}\}) \sqcup \{[i,h_{2k-s}],[i+1,h_{s+1}-1]\}$.

(viii) If $h_s < i < h_{s+1}$, $s \in [k+1,2k]$, $i =_2 s$ and $Z_i = \emptyset$, then $C = (B - \{[h_{2k-s+1},h_s],\{i\}\}) \sqcup \{[h_{2k-s},i],[h_s+1,i-1]\}$.

For $h \in \{0,1\}$ let C^h be the set of all $[a',b'] \in C$ such that $b-a =_2 h+1$. We show:

(d) *C satisfies properties $(P_0),(P_1),(P_2)$.*

We refer to properties $(P_0),(P_1),(P_2)$ for C as $(P_0'),(P_1'),(P_2')$. The verification of (P_0') is immediate. We check (P_2'). The sequence $h_0' < h_1' < \cdots < h_{2k+1}'$ in (P_2') is:

$h_0 < h_1 < \cdots < h_{2k+1}$ (of (P_2) for B) in cases (i)–(vi) (in these cases we use that $a =_2 i+1, b =_2 i+1$; see (a));

$h_0 < h_1 < \cdots < h_s < i < h_{s+2} < \cdots < h_{2k+1}$ in case (vii);

$h_0 < h_1 < \cdots < h_{s-1} < i < h_{s+1} < \cdots < h_{2k+1}$ in case (viii).

We check (P_1'). In case (i), (P_1') is immediate. In cases (ii)–(vi) let c be such that $a < c < i-1$ or $i+1 < c < b$, $c =_2 a+1$. By (P_1) for B we can find $[a_1,b_1] \in B^1$ such that $a < a_1 \le c \le b_1 < b$. If $c < i-1$, $b_1 \ge i$, then $[a_1,b_1] \in Z_i$, contradicting $Z_i = \emptyset$; if $i+1 < c$, $a_1 \le i$, then $[a_1,b_1] \in Z_i$, contradicting $Z_i = \emptyset$. Thus, we have $a < a_1 \le c \le b_1 \le i-1$ or $i+1 \le a_1 \le c \le b_1 < b$. If $b_1 = i-1$ or $a_1 = i+1$, then applying (P_0) for B to $[a_1,b_1],\{i\}$ gives a contradiction; thus we have $a < a_1 \le c \le b_1 < i-1$ or $i+1 < a_1 \le c \le b_1 < b$. Moreover, since $[a_1,b_1] \in B^1$ we have $[a_1,b_1] \in C^1$ so that (P_1') holds. In case (vii) let c be such that $i+1 < c < h_{s+1}-1$, $c =_2 i$. By (P_2) for B we can find $[a,b] \in B^1$ such that $h_s < a \le c \le b < h_{s+1}$. We have $b \le h_{s+1}-1$. If $a \le i$, then $[a,b] \in Z_i$, contradicting $Z_i = \emptyset$. Thus, $a > i$, so that $i+1 \le a \le c \le b \le h_{s+1} - 1$. If $b = h_{s+1} - 1$, then applying (P_0) for B to $[a,b], [h_{s+1}, h_{2k-s}]$ gives a contradiction. Thus, $b < h_{s+1} - 1$. If $a = i+1$, then applying (P_0) for B to $[a,b], \{i\}$ gives a contradiction. Thus $i+1 < a$. Moreover, since $[a,b] \in B^1$ we have $[a,b] \in C^1$ so that (P_1') holds. In case (viii), (P_1') is proved by an argument similar (and symmetric under τ_D) to that in case (vii).

We check (P_2') with $j \in [0, k-1]$. In case (i), (P_2') is immediate. Let c be such that $h_j' < c < h_{j+1}'$, $c =_2 j+1$. In cases (ii)–(vi), by (P_2) for B we can find $[a', b'] \in B^1$ such that $h_j < a' \le c \le b' < h_{j+1}$. If we are in case (ii),(v) or (vi), or (iii),(iv) with $s \ne j$, we have $[a', b'] \in C^1$ and (P_2') holds. Assume that we are in case (iii) or (iv) with $s = j$. Let $[a,b] \in B^1$ be the unique interval in Z_i^{min}. If $[a', b'] \ne [a,b]$, then $[a', b'] \in C^1$ and (P_2') holds. Thus we can assume that $[a', b'] = [a,b]$ so that $a \le c \le b$. If $i \notin [a', b']$, then $[a', b'] \in C^1$ and (P_2') holds. Thus we can assume that $i \in [a,b] = [a', b']$. In case (iii) (with $s = j$) we have $c \ne i$ (since $c =_2 j+1, i =_2 s, s =_2 j$) hence $c < i$ or $c > i$. Thus we have $c \in [a, i-1]$ or $c \in [i+1, b]$ and $[a, i-1] \in C^1$, $[i+1, b] \in C^1$ and (P_2') holds. In case (iv) with $s = j$, by (c) we can find $[a'', b''] \in Z_i$ such that $[a,b] \prec [a'', b'']$. We have $[a'', b''] \in C^1$ and $h_j < a'' \le c \le b'' < h_{j+1}$. Thus, (P_2') holds. Assume now that we are in case (vii). If $j \ne s+1$, then by (P_2) for B we can find $[a,b] \in B^1$ such that $h_j < a \le c \le b < h_{j+1}$. If in addition we have $j \ne s$, then $h_j' < a \le c \le b < h_{j+1}'$, $[a,b] \in C^1$ and (P_2') holds. If $j = s$, we have $c < i$ hence $a < i$. We show that $h_s < a \le c \le b < i$ (in particular, $[a,b] \in C^1$). Now $h_s < a$ holds since $h_s = h_j'$. To prove that $b < i$, we assume that $i \le b$ so that $i \in [a,b]$. Since $Z_i = \emptyset$ we deduce that $a = b = i$ hence $c = i$. This contradicts $c < h_{s+1}' = i$ and proves (P_2') in this case. If $j = s+1$, then taking $[a,b] = [i+1, h_{s+1} - 1] \in C^1$, we have $h_{s+1}' < i+1 \le c \le h_{s+1} - 1 < h_{s+2}'$ so that (P_2') holds.

Assume now that we are in case (viii). By (P_2) for B we can find $[a,b] \in B^1$ such that $h_j < a \le c \le b < h_{j+1}$ hence $h_j' < a \le c \le b < h_{j+1}'$. We have $[a,b] \in C^1$ so that (P_2') holds.

The proof of (P_2') with $j \in [k+1, 2k]$ is similar (and symmetric under τ_D) to the proof of (P_2') with $j \in [0, k-1]$. This completes the proof of (d).

From (d) and 1.3(c) we deduce:

(e) *We have* $C \in \mathbf{S}_D$.

From the definitions we deduce:

(f) *For* $j \in [1, D] - \{i\}$ *we have* $f_j(C) = f_j(B)$. *In case* (i) *we have* $f_i(C) = f_i(B) - 1$. *In cases* (ii)–(viii) *we have* $f_i(C) = f_i(B) - 2$.

From (f) we deduce:

(g) *For* $j \in [1, D] - \{i\}$ *we have* $\epsilon_j(C) = \epsilon_j(B)$. *We have* $\epsilon_i(C) = \epsilon_i(B) + 1$.

(For the second assertion of (g) in cases (ii)–(viii) we use 1.6(d); in case (i) we have $f_i(C) = -k - \underline{k}, f_i(B) = -k - \underline{k} + 1$ and $k + \underline{k} \in 2\mathbf{Z}$, so that the second assertion of (g) holds by 1.6(e).)

We show:

(h) *We have* $\epsilon(C) = \epsilon(B) + e_i$, $\epsilon(B) \in e_i^\perp$. *In other words,* $\epsilon(C), \epsilon(B)$ *are joined in the graph* V.

The first assertion of (h) is a restatement of (g). For the second assertion we note that by 1.3(f) we have $B = t_i(B')$ for some $B' \in \mathbf{S}_{D-2}$, so that $\langle B \rangle \subset V_i \oplus \mathbf{F}_2 e_i$ and it remains to use 1.9(a) and 1.9(c).

(i) *We shall also use the notation $C = B[i]$ when C is obtained from B, i as above.*

1.19. We view $\mathbf{S}_D$ as the set of vertices of a graph in which B_1, B_2 in $\mathbf{S}_D$ are joined whenever $\epsilon(B_1) \diamond \epsilon(B_2)$; see 1.12. (We then write $B_1 \diamond B_2$.) Thus the bijection $\epsilon : \mathbf{S}_D \xrightarrow{\sim} V$ is a graph isomorphism. We show:

(a) *Let B_1, B_2 in $\mathbf{S}_D$ be such that $B_1 \diamond B_2$. Define $i \in [1, D]$ by $e_i = \epsilon(B_1) + \epsilon(B_2)$. Then $\{i\}$ belongs to exactly one of B_1, B_2, say B_1, and we have $B_2 = B_1[i]$; see 1.18(i). Moreover, we have $B_1 = t_i(B')$ for a well-defined $B' \in \mathbf{S}_{D-2}$ and $\epsilon(B_1) = T_i(\epsilon'(B')) \mod \mathbf{F}_2 e_i$, $\epsilon(B_2) = T_i(\epsilon'(B')) \mod \mathbf{F}_2 e_i$.*

We have $\epsilon(B_1) = T_i(x') + c_1 e_i$, $\epsilon(B_2) = T_i(x') + c_2 e_2$ for a well-defined $x' \in V'$, $c_1 \in \mathbf{F}_2$, $c_2 \in \mathbf{F}_2$ such that $c_1 + c_2 = 1$. Define $B' \in \mathbf{S}_{D-2}$ by $\epsilon'(B') = x'$. By 1.9(b) we have $\epsilon(t_i(B')) = T_i(x') + ce_i$ with $c \in \mathbf{F}_2$. Since $c_1 + c_2 = 1$ we have $c = c_1$ or $c = c_2$. Assume for example that $c = c_1$. Then $\epsilon(t_i(B')) = \epsilon(B_1)$. Since ϵ is a bijection we deduce that $B_1 = t_i(B')$, so that $\{i\} \in B_1$. Let $C_1 = B_1[i] \in \mathbf{S}_D$; see 1.18(i). By 1.18(h) we have $\epsilon(C_1) = \epsilon(B_1) + e_i$ so that $\epsilon(C_1) = T_i(x') + c_1 e_i + e_i = T_i(x') + c_2 e_2 = \epsilon(B_2)$. Since ϵ is a bijection we deduce that $C_1 = B_2$. Note that $\{i\} \notin C_1$ so that $\{i\} \notin B_2$. This proves (a).

1.20. For $B, \tilde{B}$ in $\mathbf{S}_D$ we say that $B \le \tilde{B}$ if either

(i) $|B^0| < |\tilde{B}^0|$ or

(ii) $|B^0| = |\tilde{B}^0|$ and for any $i \in [1, D]$ we have $f_i(B) \le f_i(\tilde{B})|$.

We show:

(a) *This is a partial order on $\mathbf{S}_D$.*

It is enough to prove that for $B, \tilde{B}$ in $\mathbf{S}_D$ such that $B \le \tilde{B}$ and $\tilde{B} \le B$ we have $B = B'$. We have $|B^0| \le |\tilde{B}^0| \le |B^0|$ hence $|B^0| = |\tilde{B}^0|$ and $f_i(B) = f_i(\tilde{B})$ for all i hence $\epsilon_i(B) = \epsilon_i(\tilde{B})$ and $\epsilon(B) = \epsilon(\tilde{B})$. Since ϵ is a bijection (1.17(b)), we deduce that $B = B'$. This proves (a).

For $x, \tilde{x}$ in V we say that $x \le \tilde{x}$ if $\epsilon^{-1}(x) \le \epsilon^{-1}(\tilde{x})$, where $\epsilon^{-1} : V \to \mathbf{S}_D$ is the bijection inverse to $\epsilon : \mathbf{S}_D \to V$. This is a partial order on V. We shall write $x < \tilde{x}$ whenever $x \le \tilde{x}$ and $x \ne \tilde{x}$. Using the definitions and 1.4(a) we deduce:

(b) *Assume that $D \ge 2$, $i \in [1, D]$, $B' \in \mathbf{S}_{D-2}$, $\tilde{B}' \in \mathbf{S}_{D-2}$. If $B' \le \tilde{B}'$, then $t_i(B) \le t_i(\tilde{B})$. Hence if $x' \in V'$, $\tilde{x}' \in V'$, $x' \le \tilde{x}'$, then $t_i(\epsilon'^{-1}(x')) \le t_i(\epsilon'^{-1}(\tilde{x}'))$.*

Clearly, for any $x \in V$ we have $0 \le x$. We denote by $\nu(x)$ the largest number $r \ge 0$ such that there exists a sequence $0 = x_0 < x_1 < \cdots < x_r = x$ in V. We have $\nu(0) = 0$ and $\nu(x) > 0$ if $x \ne 0$.

We show:

(c) *Assume that $B \in \mathbf{S}_D^{prim}$. Recall that $z := \epsilon(B) \in \langle B \rangle$ (see 1.9(c)). If $y \in \langle B \rangle$ and $y \ne z$, then $y < z$.*

We set $k = |B^0| \in [0, D/2]$. By 1.9(g) we have $z \in V(k)$ and by 1.15(a) we have $\tilde{u}(x_0) = k$ for a unique $x_0 \in \langle B \rangle$, $\tilde{u}(x) < k$ for any $x \in \langle B \rangle$ such that $x \ne x_0$. By 1.14(c) we have $\tilde{u}(z) = |B^0| = k$ so that $x_0 = z$. Thus for y as in (c) we have $y \ne x_0$ so that $\tilde{u}(y) < k$, that is, $\tilde{u}(\epsilon(B')) < k$, where $B' = \epsilon^{-1}(y)$. By 1.14(c) this implies $|B'^0| < k$, that is, $|B'^0| < |B^0|$ so that $B' < B$ and $y < z$. This proves (c).

Let $x \in V$. By 1.16(d) we have $\psi_x = \sum_{\tilde{x} \in V} c_{x,\tilde{x}} \Psi_{\epsilon^{-1}(\tilde{x})}$, where $c_{x,\tilde{x}} \in \mathbf{C}$. Moreover, by 1.17, the coefficients $c_{x,\tilde{x}}$ are uniquely determined. We state:

Theorem 1.21. *If $x \in V, \tilde{x} \in V$, $c_{x,\tilde{x}} \ne 0$, then $\tilde{x} \le x$. Moreover, $c_{x,x} = 1$.*

We argue by induction on D; for fixed D we argue by (a second) induction on $\nu(x)$. If $D = 0$ the result is obvious. Now assume that $D \geq 2$. Assume first that $\epsilon^{-1}(x) \in \mathbf{S}_D^{prim}$. Since $x \in \langle \epsilon^{-1}(x) \rangle$, we have $\Psi_{\epsilon^{-1}(x)} = \psi_x + \sum_{x_1 \in Z} \psi_{x_1}$, where $Z = \langle \epsilon^{-1}(x) \rangle - \{x\}$. By 1.20(c), for any $x_1 \in Z$ we have $x_1 < x$ so that $\nu(x_1) < \nu(x)$. By the (second) induction hypothesis, for any $x_1 \in Z$, ψ_{x_1} is a linear combination of elements $\Psi_{\epsilon^{-1}(x_2)}$ with $x_2 \in V$, $x_2 \leq x_1$ (hence $x_2 < x$). It follows that the statement of the theorem holds for our x.

Next we assume that $B = \epsilon^{-1}(x) \notin \mathbf{S}_D^{prim}$. We can find $i \in [1, D]$ such that $\{i\} \in \epsilon^{-1}(x)$. We have $\epsilon^{-1}(x) = t_i(B')$, where $B' \in \mathbf{S}_{D-2}$. Let $x' = \epsilon'(B') \in V'$. We have $t_i(\epsilon'^{-1}(x')) = B$. From the first induction hypothesis we have

$$\text{(a)} \qquad \psi'_{x'} = \sum_{\tilde{x}' \in V'; \tilde{x}' \leq x'} c'_{x', \tilde{x}'} \Psi'_{\epsilon'^{-1}(\tilde{x}')},$$

where $c'_{x',\tilde{x}'} \in \mathbf{C}$ and $c'_{x',x'} = 1$. Let $C = B[i]$; see 1.18(i). We have $|C^0| = |B^0|$ and from 1.18(f) we see that $C < B$ hence $y < x$, where $y = \epsilon(C) \in V$. Applying to (a) θ_i (as in the proof of 1.16(b)) we obtain

$$\psi_x + \psi_y = \sum_{\tilde{x}' \in V'; \tilde{x}' \leq x'} c'_{x', \tilde{x}'} \Psi_{t_i(\epsilon'^{-1}(\tilde{x}'))}$$

(we have used 1.19(a)). By 1.20(b) the inequality $\tilde{x}' \leq x'$ implies $t_i(\epsilon'^{-1}(\tilde{x}')) \leq t_i(\epsilon'^{-1}(x')) = B$; moreover if $\tilde{x}' \neq x'$, then $t_i(\epsilon'^{-1}(\tilde{x}')) \neq t_i(\epsilon'^{-1}(x')) = B$. We see that $\psi_x + \psi_y$ is a linear combination of terms $\Psi_{\epsilon^{-1}(z)}$ with $z \in V$, $z \leq x$, and the coefficient of $\Psi_{\epsilon^{-1}(x)}$ is 1.

Since $y < x$ we have $\nu(y) < \nu(x)$. By the (second) induction hypothesis ψ_y is a linear combination of terms $\Psi_{\epsilon^{-1}(z)}$ with $z \in V$, $z \leq y$ hence $z < x$. We see that ψ_x is a linear combination of terms $\Psi_{\epsilon^{-1}(z)}$ with $z \in V$, $z \leq x$ and the coefficient of $\Psi_{\epsilon^{-1}(x)}$ is 1. This proves the theorem.

1.22. For $x \in V$ we have $\Psi_{\epsilon^{-1}(x)} = \sum_{\tilde{x} \in V} d_{x,\tilde{x}} \psi_{\tilde{x}}$, where $d_{x,\tilde{x}} = 1$ if $\tilde{x} \in \langle \epsilon^{-1}(x) \rangle$ and $d_{x,\tilde{x}} = 0$ if $\tilde{x} \notin \langle \epsilon^{-1}(x) \rangle$. Recall that $d_{x,x} = 1$. We show:

(a) *If $d_{x,\tilde{x}} \neq 0$, then $\tilde{x} \leq x$.*

From the definitions for x, x' in V we have $\sum_{\tilde{x} \in V} c_{x,\tilde{x}} d_{\tilde{x},x'} = \delta_{x,x'}$ (Kronecker δ). Using 1.21 we deduce $d_{x,x'} + \sum_{\tilde{x} \in S_D; \tilde{x} < x} c_{x,\tilde{x}} d_{\tilde{x},x'} = \delta_{x,x'}$. From this the desired result follows by induction on $\nu(x)$.

We show:

(b) *There is a unique bijection $e : V \xrightarrow{\sim} \mathcal{F}(V)$ (see* 1.17*) such that $x \in e(x)$ for any $x \in V$.*

The map $e : x \mapsto \langle \epsilon^{-1}(x) \rangle$, $V \to \mathcal{F}(V)$ is a well-defined bijection; see 1.17(b),(c). For $x \in V$ we have $x \in e(x)$ by 1.9(c). This proves the existence of e. We prove uniqueness. Let $e' : V \to \mathcal{F}(V)$ be a bijection such that $x \in e'(x)$ for any $x \in V$. We define a bijection $\sigma : V \xrightarrow{\sim} V$ by $\sigma = e'^{-1}e$. Then for any $X \in \mathcal{F}(V)$ we have $\sigma(e^{-1}(X)) = e'^{-1}(X)$. Setting $x = e^{-1}(X)$ we have $\sigma(x) = e'^{-1}(X) \in X = e(x)$. Thus $\sigma(x) \in e(x)$ for any $x \in V$. From (a) we have $x' \leq x$ for any $x' \in e(x)$. Hence $\sigma(x) \leq x$ for any $x \in V$. In a finite partially ordered set Z any bijection $a : Z \to Z$ such that $a(z) \leq z$ for all z must be the identity map. It follows that $\sigma = 1$ so that $e = e'$. This proves (b).

1.23. In 1.24–1.26 we describe the bijection in 1.17(c) assuming that D is 2, 4 or 6. In each case we give a table in which there is one row for each $B \in \mathbf{S}_D$; the row corresponding to B is of the form $< B >$: $(\dots)$, where B is represented by the list of intervals of B (we write an interval such as $[4,6]$ as 456) and $(\dots)$ is a list of the vectors in $\langle B \rangle$ (we write 1235 instead of $e_1+e_2+e_3+e_5$, etc). In each list $(\dots)$ we single out the vector $\epsilon(B)$ in 1.17(b) by putting it in a box. Any non-boxed entry in $(\dots)$ appears as a boxed entry in some previous row. These tables extend the tables in [L5, 1.17].

1.24. **The table for $D = 2$.**

$\emptyset : (\boxed{0})$

$< 1 >: (0, \boxed{1})$

$< 2 >: (0, \boxed{2})$

$< 12 >: (0, \boxed{12})$.

1.25. **The table for $D = 4$.**

$\emptyset : (\boxed{0})$

$< 1 >: (0, \boxed{1})$

$< 2 >: (0, \boxed{2})$

$< 3 >: (0, \boxed{3})$

$< 4 >: (0, \boxed{4})$

$< 1, 3 >: (0, 1, 3, \boxed{13})$

$< 1, 4 >: (0, 1, 4, \boxed{14})$

$< 2, 4 >: (0, 2, 4, \boxed{24})$

$< 2, 123 >: (0, 2, 13, \boxed{123})$

$< 3, 234 >: (0, 3, 24, \boxed{234})$

$< 1234 >: (0, \boxed{1234})$

$< 3, 1234 >: (0, 3, 1234, \boxed{124})$

$< 2, 1234 >: (0, 2, 1234, \boxed{134})$

$< 4, 12 >: (0, 4, 124, \boxed{12})$

$< 1, 34 >: (0, 1, 134, \boxed{34})$

$< 1234, 23 >: (0, 1234, 14, \boxed{23})$.

1.26. **The table for $D = 6$.**

$\emptyset : (\boxed{0})$

$< 1 >: (0, \boxed{1})$

$< 2 >: (0, \boxed{2})$

$< 3 >: (0, \boxed{3})$

$< 4 >: (0, \boxed{4})$

$< 5 >: (0, \boxed{5})$

$< 6 >: (0, \boxed{6})$

$< 1, 4 >: (0, 1, 4, \boxed{14})$

$< 1, 6 >: (0, 1, 6, \boxed{16})$

$< 2, 4 >: (0, 2, 4, \boxed{24})$

$< 2, 5 >: (0, 2, 5, \boxed{25})$

$< 2, 6 >: (0, 2, 6, \boxed{26})$

$<3,6>$: $(0,3,6,\boxed{36})$
$<4,6>$: $(0,4,6,\boxed{46})$
$<1,3>$: $(0,1,3,\boxed{13})$
$<1,5>$: $(0,1,5,\boxed{15})$
$<3,5>$: $(0,3,5,\boxed{35})$
$<2,123>$: $(0,2,13,\boxed{123})$
$<3,234>$: $(0,3,24,\boxed{234})$
$<4,345>$: $(0,4,35,\boxed{345})$
$<5,456>$: $(0,5,46,\boxed{456})$
$<1,3,5>$: $(0,1,3,5,13,15,35,\boxed{135})$
$<1,3,6>$: $(0,1,3,6,13,16,36,\boxed{136})$
$<1,4,345>$: $(0,1,4,345,14,35,135,\boxed{1345})$
$<1,4,6>$: $(0,1,4,6,14,16,46,\boxed{146})$
$<2,4,6>$: $(0,2,4,6,24,26,46,\boxed{246})$
$<1,5,456>$: $(0,1,5,456,15,46,146,\boxed{1456})$
$<2,5,456>$: $(0,2,5,456,25,46,246,\boxed{2456})$
$<2,5,123>$: $(0,2,5,123,25,13,135,\boxed{1235})$
$<2,6,123>$: $(0,2,6,123,26,13,136,\boxed{1236})$
$<2,4,12345>$: $(0,2,4,24,1345,1235,135,\boxed{12345})$
$<3,234,12345>$: $(0,3,234,12345,24,15,135,\boxed{1245})$
$<3,6,234>$: $(0,3,6,234,24,36,246,\boxed{2346})$
$<3,5,23456>$: $(0,3,5,2456,35,2346,246,\boxed{23456})$
$<4,345,23456>$: $(0,4,345,23456,35,26,246,\boxed{2356})$.
$<123456>$: $(0,\boxed{123456})$
$<5,123456>$: $(0,5,123456,\boxed{12346})$
$<4,123456>$: $(0,4,123456,\boxed{12356})$
$<3,123456>$: $(0,3,123456,\boxed{12456})$
$<2,123456>$: $(0,2,123456,\boxed{13456})$
$<6,1234>$: $(0,6,12346,\boxed{1234})$
$<1,3456>$: $(0,1,13456,\boxed{3456})$
$<2,5,123456>$: $(0,2,5,25,123456,13456,12346,\boxed{1346})$
$<3,5,123456>$: $(0,3,5,35,123456,12456,12346,\boxed{1246})$
$<2,4,123456>$: $(0,2,4,24,123456,13456,12356,\boxed{1356})$
$<3,6,1234>$: $(0,3,6,36,1234,12346,1246,\boxed{124})$
$<1,4,3456>$: $(0,1,4,14,3456,13456,1356,\boxed{356})$
$<2,6,1234>$: $(0,2,6,26,1234,12346,1346,\boxed{134})$
$<1,5,3456>$: $(0,1,5,14,3456,13456,1346,\boxed{346})$
$<3,234,123456>$: $(0,3,234,24,123456,12456,1356,\boxed{156})$
$<4,345,123456>$: $(0,4,345,35,123456,12356,1246,\boxed{126})$
$<4,6,12>$: $(0,4,6,46,124,126,1246,\boxed{12})$
$<1,3,56>$: $(0,1,3,13,156,356,1356,\boxed{56})$

$< 1,6,34 >$: $(0,1,6,16,134,346,1346,\boxed{34})$
$< 5,12,456 >$: $(0,5,12,456,12456,46,1246,\boxed{125})$
$< 2,56,123 >$: $(0,2,56,123,12356,13,1356,\boxed{256})$
$< 123456,2345 >$: $(0,123456,16,\boxed{2345})$
$< 123456,3,2345 >$: $(0,123456,3,2345,12456,16,136,\boxed{245})$
$< 123456,4,2345 >$: $(0,123456,4,2345,12356,16,146,\boxed{235})$
$< 123456,2,45 >$: $(0,2,123456,13456,1236,245,136,\boxed{45})$
$< 123456,5,23 >$: $(0,5,123456,12346,1456,235,146,\boxed{23})$
$< 3456,1,45 >$: $(0,1,45,3456,13456,36,136,\boxed{145})$
$< 1234,6,23 >$: $(0,6,23,1234,12346,14,146,\boxed{236})$
$< 123456,2345,34 >$: $(0,123456,2345,34,16,25,1346,\boxed{1256})$.

1.27. For $m \in \mathbf{N}$ such that $m \le D/2$ let $\mathbf{S}_D^m = \{B \in \mathbf{S}_D; |B^0| = m\}$. One can show:

(a) $|\mathbf{S}_D^m| = \binom{D+1}{(D/2)-m}$.

Indeed $\mathbf{S}_D^m$ can be identified with a fiber of the map $\tilde{u} : V \to \mathbf{N}$ in 1.11 and that fiber is in bijection with a set of symbols with fixed defect as in [L2]. These symbols can be counted and we find (a).

If $B \in \mathbf{S}_D$, then $B^1 \in \mathbf{S}_D^0$. This is seen by induction on D. Alternatively, B^1 satisfies $(P_0), (P_1), (P_2)$ hence is in $\mathbf{S}_D$, by 1.3(c). Thus $B \mapsto B^1$ is a well-defined (surjective) map $\mathbf{S}_D \to \mathbf{S}_D^0$. One can show:

(b) *This map induces a bijection* $\{B \in \mathbf{S}_D; |B| = D/2\} \xrightarrow{\sim} \mathbf{S}_D^0$.

1.28. We now assume that G in 0.1 is of type B_n or C_n, $n \ge 2$, or D_n, $n \ge 4$. We define the set $\tilde{\mathbf{B}}_c$ in 0.1. If $|c| = 1$, $\tilde{\mathbf{B}}_c$ consists of $(1,1)$. Assume now that $|c| \ge 2$. We associate to c a number $D \in 2\mathbf{N}$, and an $\mathbf{F}_2$-vector space V with basis $\{e_i; i \in [1,D]\}$ as in 1.9 so that Irr_c is identified with $M(\mathcal{G}_c) = V$ as in [L3]. Then $\mathbf{C}[M(\mathcal{G}_c)]$ becomes the vector space of functions $V \to \mathbf{C}$. The elements of $\tilde{\mathbf{B}}_c$ are the characteristic functions of the subsets $\langle B \rangle$ of V for various $B \in \mathbf{S}_D$. This has the properties (I)–(V) in 0.1. (The bipositivity property (I) in 0.1 follows from the fact that $\langle B \rangle$ is an isotropic subspace of V for any $B \in \mathbf{S}_D$.)

2. The case where D is odd

2.1. In this section we will sketch without proof a variant of the definitions and results in §1 in which $D \in \mathbf{N}$ is taken to be odd.

We say that $B \in R_D$ is primitive if either $B = \emptyset$ or B is of the form

(a) $B = \{[1, D-1], [2, D-2], [k, D-k]\}$ for some odd $k \in \mathbf{N}$ such that $k \le (D-1)/2$.

We define a subset $\mathbf{S}_D$ of R_D by induction on D as follows. If $D = 1$, $\mathbf{S}_D$ consists of a single element, namely $\emptyset \in R_D$. If $D \ge 3$ we say that $B \in R_D$ is in $\mathbf{S}_D$ if either B is primitive, or

(b) $|B^0| \ne 0$ and there exist $i \in [1,D]$ and $B' \in \mathbf{S}_{D-2}$ such that $B = t_i(B')$, or

(c) $|B^0| = 0$ and there exist $i \in [1,D-1]$ and $B' \in \mathbf{S}_{D-2}$ such that $B = t_i(B')$.

Here t_i is as in 1.1.

2.2. We shall use the notation of 1.9 (with D odd). Let $\underline{V} = V/\mathbf{F}_2\zeta$, where $\zeta = e_1 + e_3 + e_5 + \cdots + e_D$. Now $(,) : V \times V \to \mathbf{F}_2$ induces a non-degenerate symplectic form $\underline{V} \times \underline{V} \to \mathbf{F}_2$. Let $\pi : V \to \underline{V}$ be the obvious map. Now $\underline{V}$ with its basis $\{\pi(e_i); i \in [0, D-1]\}$ is like V in 1.9 (of even dimension). Hence $\mathcal{F}(\underline{V})$ is defined and we have canonical bijections $\underline{\alpha} : \mathbf{S}_{D-1} \xrightarrow{\sim} \mathcal{F}(\underline{V})$ (as in 1.17(c)) and $\underline{e} : \underline{V} \xrightarrow{\sim} \mathcal{F}(\underline{V})$ (as in 1.22(b)).

For $B \in \mathbf{S}_D$ let $\langle B \rangle$ be the subspace of V generated by $\{e_I; I \in B\}$; this is in fact a basis of $\langle B \rangle$ and $\{\pi(e_I); I \in B\}$ is a basis of $\pi(\langle B \rangle)$. Let $\mathcal{F}(V)$ be the set of (isotropic) subspaces of $\underline{V}$ of the form $\pi(\langle B \rangle)$ for some $B \in \mathbf{S}_D$. Now $\mathcal{F}(V)$ does not in general coincide with $\mathcal{F}(\underline{V})$.

One can show that the map $\alpha : \mathbf{S}_D \to \mathcal{F}(V)$, $B \mapsto \pi(\langle B \rangle)$ is a bijection and that there is a unique bijection $e : \underline{V} \xrightarrow{\sim} \mathcal{F}(V)$ such that for any $x \in \underline{V}$ we have $x \in e(x)$. Consider the matrix indexed by $\underline{V} \times \underline{V}$ whose entry at $(x, x') \in \underline{V} \times \underline{V}$ is 1 if $x' \in e(x)$ and is 0 if $x' \notin e(x)$. One can show that this matrix is upper triangular with 1 on the diagonal for a suitable partial order on $\underline{V}$.

2.3. For $m \in \mathbf{N}$ such that $m \le (D-1)/2$ let $\mathbf{S}_D^m = \{B \in \mathbf{S}_D; |B^0| = m\}$. For $m > 0$, even, we have $\mathbf{S}_D^m = \emptyset$. One can show that the bijection $\alpha^{-1} e \underline{e}^{-1} \underline{\alpha} : \mathbf{S}_{D-1} \xrightarrow{\sim} \mathbf{S}_D$ (see 2.2) restricts to the identity map $\mathbf{S}_D^0 \to \mathbf{S}_{D-1}^0$ and to a bijection $\mathbf{S}_{D-1}^m \cup \mathbf{S}_{D-1}^{m+1} \xrightarrow{\sim} \mathbf{S}_D^m$ for m odd.

2.4. In 2.5–2.7 we describe the bijection $\mathbf{S}_D \xrightarrow{\sim} \mathcal{F}(V)$, $B \mapsto \pi(\langle B \rangle)$ in 2.2 assuming that D is $3, 5$ or 7. In each case we give a table in which there is one row for each $B \in \mathbf{S}_D$; the row corresponding to B is of the form $< B >$: $(\dots)$, where B is represented by the list of intervals of B. We use conventions similar to those in 1.23, except that now $(\dots)$ is a list of vectors in $\underline{V}$ (we write 1235 instead of $\pi(e_1) + \pi(e_2) + \pi(e_3) + \pi(e_5)$, etc.). In each list $(\dots)$ we single out (by putting it in a box) the vector $x \in \underline{V}$ such that $e(x) = \pi(\langle B \rangle)$ with e as in 2.2. Any non-boxed entry in $(\dots)$ appears as a boxed entry in some previous row.

2.5. **The table for $D = 3$.**

$\emptyset : (\boxed{0})$

$< 1 >: (0, \boxed{1})$

$< 2 >: (0, \boxed{2})$.

$< 12 >: (0, \boxed{12})$.

2.6. **The table for $D = 5$.**

$\emptyset : (\boxed{0})$

$< 1 >: (0, \boxed{1})$

$< 2 >: (0, \boxed{2})$

$< 3 >: (0, \boxed{3})$

$< 4 >: (0, \boxed{4})$

$< 1, 3 >: (0, 1, 3, \boxed{13})$

$< 1, 4 >: (0, 1, 4, \boxed{14})$

$< 2, 4 >: (0, 2, 4, \boxed{24})$

$< 2, 123 >: (0, 2, 13, \boxed{123})$

$< 3, 234 >: (0, 3, 24, \boxed{234})$

$< 1234 >$: $(0, \boxed{1234})$

$< 3, 1234 >$: $(0, 3, 1234, \boxed{124})$

$< 2, 1234 >$: $(0, 2, 1234, \boxed{134})$

$< 4, 12 >$: $(0, 4, 124, \boxed{12})$

$< 1, 34 >$: $(0, 1, 134, \boxed{34})$

$< 5, 12 >$: $(0, 12, 13, \boxed{23})$.

2.7. **The table for $D = 7$.**

$\emptyset$: $(\boxed{0})$

$< 1 >$: $(0, \boxed{1})$

$< 2 >$: $(0, \boxed{2})$

$< 3 >$: $(0, \boxed{3})$

$< 4 >$: $(0, \boxed{4})$

$< 5 >$: $(0, \boxed{5})$

$< 6 >$: $(0, \boxed{6})$

$< 1, 4 >$: $(0, 1, 4, \boxed{14})$

$< 1, 6 >$: $(0, 1, 6, \boxed{16})$

$< 2, 4 >$: $(0, 2, 4, \boxed{24})$

$< 2, 5 >$: $(0, 2, 5, \boxed{25})$

$< 2, 6 >$: $(0, 2, 6, \boxed{26})$

$< 3, 6 >$: $(0, 3, 6, \boxed{36})$

$< 4, 6 >$: $(0, 4, 6, \boxed{46})$

$< 1, 3 >$: $(0, 1, 3, \boxed{13})$

$< 1, 5 >$: $(0, 1, 5, \boxed{15})$

$< 3, 5 >$: $(0, 3, 5, \boxed{35})$

$< 2, 123 >$: $(0, 2, 13, \boxed{123})$

$< 3, 234 >$: $(0, 3, 24, \boxed{234})$

$< 4, 345 >$: $(0, 4, 35, \boxed{345})$

$< 5, 456 >$: $(0, 5, 46, \boxed{456})$

$< 1, 3, 5 >$: $(0, 1, 3, 5, 13, 15, 35, \boxed{135})$

$< 1, 3, 6 >$: $(0, 1, 3, 6, 13, 16, 36, \boxed{136})$

$< 1, 4, 345 >$: $(0, 1, 4, 345, 14, 35, 135, \boxed{1345})$

$< 1, 4, 6 >$: $(0, 1, 4, 6, 14, 16, 46, \boxed{146})$

$< 2, 4, 6 >$: $(0, 2, 4, 6, 24, 26, 46, \boxed{246})$

$< 1, 5, 456 >$: $(0, 1, 5, 456, 15, 46, 146, \boxed{1456})$

$< 2, 5, 456 >$: $(0, 2, 5, 456, 25, 46, 246, \boxed{2456})$

$< 2, 5, 123 >$: $(0, 2, 5, 123, 25, 13, 135, \boxed{1235})$

$< 2, 6, 123 >$: $(0, 2, 6, 123, 26, 13, 136, \boxed{1236})$

$< 2, 4, 12345 >$: $(0, 2, 4, 24, 1345, 1235, 135, \boxed{12345})$

$< 3, 234, 12345 >$: $(0, 3, 234, 12345, 24, 15, 135, \boxed{1245})$

$< 3, 6, 234 >$: $(0, 3, 6, 234, 24, 36, 246, \boxed{2346})$

$< 3, 5, 23456 >$: $(0, 3, 5, 2456, 35, 2346, 246, \boxed{23456})$

$< 4, 345, 23456 >$: $(0, 4, 345, 23456, 35, 26, 246, \boxed{2356})$
$< 123456 >$: $(0, \boxed{123456})$
$< 5, 123456 >$: $(0, 5, 123456, \boxed{12346})$
$< 4, 123456 >$: $(0, 4, 123456, \boxed{12356})$
$< 3, 123456 >$: $(0, 3, 123456, \boxed{12456})$
$< 2, 123456 >$: $(0, 2, 123456, \boxed{13456})$
$< 6, 1234 >$: $(0, 6, 12346, \boxed{1234})$
$< 1, 3456 >$: $(0, 1, 13456, \boxed{3456})$
$< 2, 5, 123456 >$: $(0, 2, 5, 25, 123456, 13456, 12346, \boxed{1346})$
$< 3, 5, 123456 >$: $(0, 3, 5, 35, 123456, 12456, 12346, \boxed{1246})$
$< 2, 4, 123456 >$: $(0, 2, 4, 24, 123456, 13456, 12356, \boxed{1356})$
$< 3, 6, 1234 >$: $(0, 3, 6, 36, 1234, 12346, 1246, \boxed{124})$
$< 1, 4, 3456 >$: $(0, 1, 4, 14, 3456, 13456, 1356, \boxed{356})$
$< 2, 6, 1234 >$: $(0, 2, 6, 26, 1234, 12346, 1346, \boxed{134})$
$< 1, 5, 3456 >$: $(0, 1, 5, 14, 3456, 13456, 1346, \boxed{346})$
$< 3, 234, 123456 >$: $(0, 3, 234, 24, 123456, 12456, 1356, \boxed{156})$
$< 4, 345, 123456 >$: $(0, 4, 345, 35, 123456, 12356, 1246, \boxed{126})$
$< 4, 6, 12 >$: $(0, 4, 6, 46, 124, 126, 1246, \boxed{12})$
$< 1, 3, 56 >$: $(0, 1, 3, 13, 156, 356, 1356, \boxed{56})$
$< 1, 6, 34 >$: $(0, 1, 6, 16, 134, 346, 1346, \boxed{34})$
$< 5, 12, 456 >$: $(0, 5, 12, 456, 12456, 46, 1246, \boxed{125})$
$< 2, 56, 123 >$: $(0, 2, 56, 123, 12356, 13, 1356, \boxed{256})$
$< 7, 1234 >$: $(0, 135, 1234, \boxed{245})$
$< 3, 7, 1234 >$: $(0, 3, 15, 1234, 135, 124, 245, \boxed{2345})$
$< 4, 7, 12 >$: $(0, 4, 135, 12, 124, 1345, 2345, \boxed{235})$
$< 2, 7, 1234 >$: $(0, 2, 135, 1234, 1235, 134, 245, \boxed{45})$
$< 5, 7, 12 >$: $(0, 5, 13, 12, 135, 125, 235, \boxed{23})$
$< 1, 7, 34 >$: $(0, 1, 34, 135, 134, 35, 45, \boxed{145})$
$< 6, 12, 567 >$: $(0, 6, 12, 136, 13, 126, 23, \boxed{236})$
$< 123456, 2345, 34 >$: $(0, 123456, 2345, 34, 16, 25, 1346, \boxed{1256})$.

3. Exceptional groups

3.1. Let Γ be a finite group. Let $x \in \Gamma$, and let ρ be a not necessarily irreducible representation over $\mathbf{C}$ of the centralizer $Z_\Gamma(x)$ of x in Γ. We define $(x, \rho) \in M(\Gamma)$ to be $\sum_\sigma (\sigma : \rho)(x, \sigma)$, where σ runs over the irreducible representations of $Z_\Gamma(x)$ up to isomorphism and : denotes multiplicity. Let H be a subgroup of Γ. Following [L3, p. 312] we define a linear map $i_{H,\Gamma} : \mathbf{C}[M(H)] \to \mathbf{C}[M(\Gamma)]$ by

(a) $$(x, \sigma) \mapsto (x, \mathrm{Ind}_{Z_H(x)}^{Z_\Gamma(x)}(\sigma)).$$

As stated in loc. cit. we have

(b) $i_{H,\Gamma}(A_H(f)) = A_\Gamma(i_{H,\Gamma}(f))$ *for any* $f \in \mathbf{C}[M(H)]$.

If $f \in \mathbf{C}[M(H)]$ is ≥ 0, then clearly $i_{H,\Gamma}(f)$ is ≥ 0. Using this and (b) we see that

(c) *If $f \in \mathbf{C}[M(H)]$ is bipositive, then $i_{H,\Gamma}(f) \in \mathbf{C}[M(\Gamma)]$ is bipositive.*

Assume now that H is a normal subgroup of Γ and let $\pi : \Gamma \to \Gamma/H$ be the canonical map. Following loc.cit. we define a linear map $\pi_{H,\Gamma} : \mathbf{C}[M(\Gamma/H)] \to \mathbf{C}[M(\Gamma)]$ by

$$\text{(d)} \qquad (x,\sigma) \mapsto \sum_{y \in \pi^{-1}(x)} \sum_{\tau \in \mathrm{Irr}(Z_\Gamma(y))} |Z_\Gamma(y)||Z_{\Gamma/H}(x)|^{-1}|\Gamma|^{-1}|\Gamma/H|(\tau:\sigma)(y,\tau),$$

where τ runs over the irreducible representations of $Z_\Gamma(y)$ up to isomorphism and $\tau : \sigma$ denotes the multiplicity of τ in σ viewed as a representation of $Z_\Gamma(y)$ via the obvious homomorphism $Z_\Gamma(y) \to Z_{\Gamma/H}(x)$. As stated in loc.cit. we have

(e) $\pi_{H,\Gamma}(A_{\Gamma/H}(f)) = A_\Gamma(\pi_{H,\Gamma}(f))$ *for any* $f \in \mathbf{C}[M(\Gamma/H)]$.

If $f \in \mathbf{C}[M(\Gamma/H)]$ is ≥ 0, then clearly $\pi_{H,\Gamma}(f)$ is ≥ 0. Using this and (e) we see that

(f) *If $f \in \mathbf{C}[M(\Gamma/H)]$ is bipositive, then $\pi_{H,\Gamma}(f) \in \mathbf{C}[M(\Gamma)]$ is bipositive.*

Now let $H \subset H'$ be two subgroups of Γ such that H is normal in H'. We define a linear map $\mathbf{s}_{H,H'} : \mathbf{C}[M(H'/H)] \to \mathbf{C}[M(\Gamma)]$ by $f \mapsto i_{H',\Gamma}(\pi_{H,H'}(f))$. From (c),(f) we deduce:

(g) *If $f \in \mathbf{C}[M(H'/H)]$ is bipositive, then $\mathbf{s}_{H,H'}(f) \in \mathbf{C}[M(\Gamma)]$ is bipositive.*

Note that $\mathbf{s}_{H,H'}(1,1)$ is the same as $S_{H,H'}$ defined in [L5]; in this special case (g) can be also deduced from [L5, 0.7].

3.2. For $N \geq 1$ let S_N be the group of all permutations of $[1,N]$. We shall use the notation of [L3, 4.3] for the elements of $M(S_N)$ with $N = 2,3,4$ or 5 (but we replace $\bar{Q}_l$ by $\mathbf{C}$). We now give some examples of bipositive elements. Note that $(1,1) \in M(\Gamma)$ is bipositive for any finite group Γ. Indeed, we have

$$A_\Gamma(1,1) = \sum_{(x,\sigma)\in M(\Gamma)} \dim\sigma |Z_\Gamma(z)|^{-1}(x,\sigma).$$

Let

$\Lambda_{-1} = (g_2,\epsilon) + (1,1) \in M(S_2)$,

$\Lambda'_{\theta^j} = (g_3,\theta^j) + (g_2,1) + (1,1) \in M(S_3) (j=1,2)$,

$\Lambda_{\theta^j} = (g_3,\theta^j) + (g_2,\epsilon) + (1,1) \in M(S_3) (j=1,2)$,

$\Lambda_{i^k} = (g_4,i^k) + (g_4,-1) + (g_3,1) + (1,\lambda^2) + (1,1) \in M(S_4) (k=1,-1)$,

$\Lambda_{\zeta^j} = (g_5,\zeta^j) + (1,\lambda^4) + 2(1,\lambda^2) + (1,\nu) + (1,\nu') + (1,1) \in M(S_5) (j=1,2,3,4)$.

$$\begin{aligned}\Lambda'_{\zeta^l,\zeta^{2l}} &= (g_5,\zeta^l) + (g_5,\zeta^{2l}) + (g_2',1) + (g_2',\epsilon') \\ &+ (g_2',\epsilon'') + (g_2',\epsilon) + (1,\lambda^2) + (1,\nu) + (1,1) \in M(S_5), \quad l = 1,2,3,4.\end{aligned}$$

Here $\theta = \exp(2\pi i/3), \zeta = \exp(2\pi i/5)$.

One can verify by computation that each of the elements above (except for Λ_{ζ^j}) is fixed by the non-abelian Fourier transform hence is bipositive. In 3.3 we will show that Λ_{ζ^j} is also bipositive. We say that

$(1,1)$ is the primitive element of $M(S_1)$;

$\Lambda_{-1}, (1,1)$ are the primitive elements of $M(S_2)$;

$\Lambda'_{\theta^j}, (1,1)$ are the primitive elements of $M(S_3)$ (when G is not simply laced);

$\Lambda_{\theta^j}, (1,1)$ are the primitive elements of $M(S_3)$ (when G is simply laced);

$\Lambda_{i^k}, (1,1)$ are the primitive elements of $M(S_4)$;
$\Lambda_\zeta, \Lambda'_{\zeta,\zeta^2}, \Lambda'_{\zeta^2,\zeta^4}, \Lambda'_{\zeta^3,\zeta}, (1,1)$ are the primitive elements of $M(S_5)$.
It follows that the following elements are bipositive:
$\Lambda_{-1,-1} = \Lambda_{-1} \boxtimes \Lambda_{-1} \in M(S_2) \otimes M(S_2) = M(S_2 \times S_2)$;
$\Lambda_{-1,1} = \Lambda_{-1} \boxtimes (1,1) \in M(S_2) \otimes M(S_2) = M(S_2 \times S_2)$;
$\Lambda_{1,-1} = (1,1) \boxtimes \Lambda_{-1} \in M(S_3) \otimes M(S_2) = M(S_3 \times S_2)$;
$\Lambda_{\theta^j,-1} = \Lambda_{\theta^j} \boxtimes \Lambda_{-1} \in M(S_3) \otimes M(S_2) = M(S_3 \times S_2) (j = 1, 2)$;
$\Lambda_{\theta^j,1} = \Lambda_{\theta^j} \boxtimes (1,1) \in M(S_3) \otimes M(S_2) = M(S_3 \times S_2) (j = 1, 2)$.
Note that both $\Lambda_{-1,-1}, \Lambda_{\theta^j,-1}$ are fixed by the non-abelian Fourier transform. We say that
$\Lambda_{-1,-1}, \Lambda_{-1,1}, (1,1)$ are the primitive elements of $M(S_2 \times S_2)$;
$\Lambda_{\theta^j,-1}, \Lambda_{\theta^j,1}, \Lambda_{1,-1}, (1,1)$ are the primitive elements of $M(S_3 \times S_2)$.

3.3. Let H be a dihedral group of order 10. We denote by g_5 an element of order 5 of H and by g_2 an element of order 2 such that $g_2 g_5 g_2^{-1} = g_5^{-1}$. Now H has four conjugacy classes; they have representatives $1, g_2, g_5, g_5^2$ with centralizers of order $10, 2, 5, 5$. The irreducible representations of H are $1, r, r', \epsilon$, where r, r' are 2-dimensional and ϵ is the sign. We can assume that $\mathrm{tr}(g_5, r) = \mathrm{tr}(g_5, r') = \zeta + \zeta^{-1}$, $\mathrm{tr}(g_5^2, r) = \mathrm{tr}(g_5^2, r') = \zeta^2 + \zeta^{-2}$, $\mathrm{tr}(g_2, r) = \mathrm{tr}(g_2, r') = 0$, $\mathrm{tr}(g_5, \epsilon) = \mathrm{tr}(g_5^2, \epsilon) = 1$, $\mathrm{tr}(g_2, \epsilon) = -1$. The elements of $M(H)$ are $(1,1), (1,r), (1,r'), (1,\epsilon), (g_2, 1), (g_2, \epsilon)$, (g_5^k, ζ^l) with $k = 1, 2$, $l = 0, 1, \dots, 4$. Here ζ^l is the character of the cyclic group generated by g_5 which takes the value ζ^l at g_5. For $C \in \mathbf{Z}$ we set $[C] = \zeta^C + \zeta^{-C}$. Note that $[C]$ depends only on the residue class of C modulo 5. We write A instead of A_H. We have

$$A(1,1) = (1/10)(1,1) + (1/5)(1,r) + (1/5)(1,r') + (1/10)(1,\epsilon) + (1/2)(g_2,1) + (1/2)(g_2,\epsilon) + \sum_{k' \in \{1,2\}, l' \in \{0,4\}} (1/5)(g^{k'}, \zeta^{l'}),$$

$$A(1,\epsilon) = (1/10)(1,1) + (1/5)(1,r) + (1/5)(1,r') + (1/10)(1,\epsilon) - (1/2)(g_2,1) - (1/2)(g_2,\epsilon) + \sum_{k' \in \{1,2\}, l' \in \{0,4\}} (1/5)(g^{k'}, \zeta^{l'}),$$

$$A(g_2, 1) = (1/2)(1,1) - (1/2)(1,\epsilon) + (1/2)(g_2,1) - (1/2)(g_2,\epsilon),$$

$$A(g_5^k, \zeta^l) = (1/5)(1,1) + (1/5)[k](1,r) + (1/5)[2k](1,r') + (1/5)(1,\epsilon) + \sum_{k' \in \{1,2\}, l' \in \{0,4\}} (1/5)[kl' - k'l](g^{k'}, \zeta^{l'}).$$

Assume that $k = 1$ and $l \in [1,4]$. Using $[1] + [2] = -1$, $[2] + [4] = -1$, we have

$$A(g_5, \zeta^l) + A(g_5^2, \zeta^{2l}) = (2/5)(1,1) - (1/5)(1,r) - (1/5)(1,r') + (2/5)(1,\epsilon) + \sum_{k' \in \{1,2\}, l' \in \{0,4\}} (1/5)([l' - k'l] + [2l' - 2k'l])(g^{k'}, \zeta^{l'}).$$

Let $N_1 = l' - k'l$, $N_2 = 2N_1$. If $N_1 = 0 \mod 5$ we have $[N_1] + [N_2] = [0] + [0] = 4$. Assume now that $N_1 \neq 0 \mod 5$. If $N_1 + N_2 = 0 \mod 5$, then $3N_1 = 0 \mod 5$ so that $N_1 = 0 \mod 5$, contradicting our assumption. Thus N_1, N_2 are $\neq 0$ in $\mathbf{Z}/5$

and their sum is $\ne 0$ in $\mathbf{Z}/5$. This implies that $[N_1]+[N_2]=[1]+[2]=-1$. We see that

$$A(g_5,\zeta^l)+A(g_5^2,\zeta^{2l})=(2/5)(1,1)-(1/5)(1,r)-(1/5)(1,r')+(2/5)(1,\epsilon)$$
$$+(4/5)(g_5,\zeta^l)+(4/5)(g_5^2,\zeta^{2l})+\sum_{k'\in\{1,2\},l'\in\{0,4\};l'-k'l\ne 0 \mod 5}(-1/5)(g^{k'},\zeta^{l'}).$$

Hence

$$A(g_5,\zeta^l)+A(g_5^2,\zeta^{2l})+A(g_2,1)+A(1,1)$$
$$=(2/5)(1,1)-(1/5)(1,r)-(1/5)(1,r')+(2/5)(1,\epsilon)$$
$$+(4/5)(g_5,\zeta^l)+(4/5)(g_5^2,\zeta^{2l})+\sum_{k'\in\{1,2\},l'\in\{0,4\};l'-k'l\ne 0 \mod 5}(-1/5)(g^{k'},\zeta^{l'})$$
$$+(1/2)(1,1)-(1/2)(1,\epsilon)+(1/2)(g_2,1)-(1/2)(g_2,\epsilon)$$
$$+(1/10)(1,1)+(1/5)(1,r)+(1/5)(1,r')$$
$$+(1/10)(1,\epsilon)+(1/2)(g_2,1)+(1/2)(g_2,\epsilon)$$
$$+\sum_{k'\in\{1,2\},l'\in\{0,4\}}(1/5)(g^{k'},\zeta^{l'})=(g_5,\zeta^l)+(g_5^2,\zeta^{2l})+(g_2,1)+(1,1),$$

that is,

$$(g_5,\zeta^l)+(g_5^2,\zeta^{2l})+(g_2,1)+(1,1) \text{ is fixed by } A.$$

Next we show that the coefficient of any basis element (x,σ) in

$$A(g_5^k,\zeta^l)+A(1,\epsilon)+A(1,1)=(1/5)(1,1)+(1/5)[k](1,r)+(1/5)[2k](1,r')$$
$$+(1/5)(1,\epsilon)+\sum_{k'\in\{1,2\},l'\in\{0,4\}}(1/5)[kl'-k'l](g^{k'},\zeta^{l'})$$
$$+(1/10)(1,1)+(1/5)(1,r)+(1/5)(1,r')$$
$$+(1/10)(1,\epsilon)+(1/2)(g_2,1)+(1/2)(g_2,\epsilon)$$
$$+\sum_{k'\in\{1,2\},l'\in\{0,4\}}(1/5)(g^{k'},\zeta^{l'})+(1/10)(1,1)+(1/5)(1,r)+(1/5)(1,r')$$
$$+(1/10)(1,\epsilon)-(1/2)(g_2,1)-(1/2)(g_2,\epsilon)+\sum_{k'\in\{1,2\},l'\in\{0,4\}}(1/5)(g^{k'},\zeta^{l'})$$

is ≥ 0. It is enough to show that if $k'\in\{1,2\}, l'\in\{0,4\}$, then $[kl'-k'l]+2\ge 0$ and that $[k]+2\ge 0$, $[2k]+2\ge 0$. More generally, for any $C\in\mathbf{Z}$ we have $[C]+2\ge 0$.

We can regard H as a subgroup of S_5 so that $g_5\in H$ becomes a 5-cycle $g_5\in S_5$. Then $\mathbf{s}_{1,H}: M(H)\to M(S_5)$ is defined and for $l\in[1,4]$ we have

(a) $$\mathbf{s}_{1,H}((g_5,\zeta^l)+(g_5^2,\zeta^{2l})+(g_2,1)+(1,1))=\Lambda'_{\zeta^l,\zeta^{2l}}\in M(S_5),$$

(b) $$\mathbf{s}_{1,H}((g_5,\zeta^l)+(1,\epsilon)+(1,1))=\Lambda_{\zeta^l}\in M(S_5).$$

It follows that the elements (a),(b) are bipositive. (The element (a) is fixed by A_{S_5}.)

3.4. In the remainder of this section we assume that G in 0.1 is of exceptional type. We are in one of the following cases:

(i) $|c| = 1$, $\mathcal{G}_c = S_1$.
(ii) $|c| = 2$ (with W of type E_7 or E_8), $\mathcal{G}_c = S_2$.
(iii) $|c| = 3$, $\mathcal{G}_c = S_2$.
(iv) $|c| = 4$ (with W of type G_2), $\mathcal{G}_c = S_3$.
(v) $|c| = 5$ (with W of type E_6, E_7, or E_8), $\mathcal{G}_c = S_3$.
(vi) $|c| = 11$ (with W of type F_4), $\mathcal{G}_c = S_4$.
(vii) $|c| = 17$ (with W of type E_8), $\mathcal{G}_c = S_5$.

3.5. In the case 3.4(i) we define $\tilde{\mathbf{B}}_c$ as the set consisting of $(1,1) \in M(S_1)$.

In the cases 3.4(ii),3.4(iii) we define $\tilde{\mathbf{B}}_c$ as the subset of $\mathbf{C}[M(S_2)]$ consisting of
$\widehat{(1,1)} = \mathbf{s}_{1,S_2}(1,1) = (1,1)$,
$\widehat{(g_2,1)} = \mathbf{s}_{S_2,S_2}(1,1) = (g_2,1) + (1,1)$,
$\widehat{(1,\epsilon)} = \mathbf{s}_{1,1}(1,1) = (1,\epsilon) + (1,1)$,
$\widehat{(g_2,\epsilon)} = \mathbf{s}_{1,S_2}(\Lambda_{-1}) = \Lambda_{-1} = (g_2,\epsilon) + (1,1)$.

3.6. In cases 3.4(iv),(v) we define $\tilde{\mathbf{B}}_c$ as the subset of $\mathbf{C}[M(S_3)]$ consisting of
$\widehat{(1,1)} = \mathbf{s}_{1,S_3}(1,1) = (1,1)$,
$\widehat{(1,r)} = \mathbf{s}_{1,H_{21}}(1,1) = (1,r) + (1,1)$,
$\widehat{(g_2,1)} = \mathbf{s}_{H_{21},H_{21}}(1,1) = (g_2,1) + (1,r) + (1,1)$,
$\widehat{(g_3,1)} = \mathbf{s}_{S_3,S_3}(1,1) = (g_3,1) + (g_2,1) + (1,1)$,
$\widehat{(1,\epsilon)} = \mathbf{s}_{1,1}(1,1) = (1,\epsilon) + 2(1,r) + (1,1)$,
$\widehat{(g_2,\epsilon)} = \mathbf{s}_{1,H_{21}}\Lambda_{-1} = (g_2,\epsilon) + (1,r) + (1,1)$,
and of
$\widehat{(g_3,\theta^j)} = \mathbf{s}_{1,S_3}\Lambda'_{\theta^j} = (g_3,\theta^j) + (g_2,1) + (1,1)$ $(j = 1,2)$ (in case 3.4(iv)),
$\widehat{(g_3,\theta^j)} = \mathbf{s}_{1,S_3}\Lambda_{\theta^j} = (g_3,\theta^j) + (g_2,\epsilon) + (1,1)$ $(j = 1,2)$ (in case 3.4(v)).
Here the index H, H' in $\mathbf{s}_{H,H'}$ is a pair of subgroups of S_3 as in [L5, 3.10].

3.7. In the case 3.4(vi) we define $\tilde{\mathbf{B}}_c$ as the subset of $\mathbf{C}[M(S_4)]$ consisting of
$\widehat{(1,1)} = \mathbf{s}_{1,S_4}(1,1)$,
$\widehat{(1,\lambda^1)} = \mathbf{s}_{1,H_{31}}(1,1)$,
$\widehat{(1,\sigma)} = \mathbf{s}_{1,H_{22}}(1,1)$,
$\widehat{(1,\lambda^2)} = \mathbf{s}_{1,H_{211}}(1,1)$,
$\widehat{(g_2,1)} = \mathbf{s}_{\tilde{H}_{211},H_{22}}(1,1)$,
$\widehat{(g'_2,1)} = \mathbf{s}_{\tilde{H}_{22},\tilde{H}}(1,1)$,
$\widehat{(g_2,\epsilon'')} = \mathbf{s}_{H_{211},H_{221}}(1,1)$,
$\widehat{(g_3,1)} = \mathbf{s}_{H_{31},H_{31}}(1,1)$,
$\widehat{(g_4,1)} = \mathbf{s}_{S_4,S_4}(1,1)$,
$\widehat{(g'_2,\epsilon'')} = \mathbf{s}_{H_{22},H_{22}}(1,1)$,
$\widehat{(g'_2,\epsilon')} = \mathbf{s}_{\tilde{H},\tilde{H}}(1,1)$,
$\widehat{(g_2,\epsilon')} = \mathbf{s}_{1,H_{22}}\Lambda_{-1,1} = (g_2,\epsilon') + (1,\sigma) + (1,\lambda^1) + (1,1)$,
$\widehat{(g'_2,r)} = \mathbf{s}_{\tilde{H}_{211},H_{22}}\Lambda_{-1} = (g_2,\epsilon') + (g'_2,r) + (g_2,1) + (1,\lambda^1) + (1,\sigma) + (1,1)$,

$\widehat{(g_4,-1)} = \mathbf{s}_{H_{22},\tilde{H}}\Lambda_{-1} = (g_4,-1) + (g_2',r) + (g_2',1) + (g_2,1) + (1,\sigma) + (1,1)$

$\widehat{(1,\lambda^3)} = \mathbf{s}_{1,1}(1,1) = (1,\lambda^3) + 3(1,\lambda^2) + 3(1,\lambda^1) + 2(1,\sigma) + (1,1),$

$\widehat{(g_2,\epsilon)} = \mathbf{s}_{1,H_{211}}\Lambda_{-1} = (g_2,\epsilon) + (g_2,\epsilon') + 2(1,\lambda^1) + (1,\lambda^2) + (1,\sigma) + (1,1),$

$\widehat{(g_2',\epsilon)} = \mathbf{s}_{1,H_{22}}\Lambda_{-1,-1} = (g_2',\epsilon)+(g_2',1)+(g_2,\epsilon')+(g_2,\epsilon'')+(1,\lambda^1)+(1,\sigma)+(1,1),$

$\widehat{(g_3,\theta^j)} = \mathbf{s}_{1,H_{31}}\Lambda'_{\theta^j} = (g_3,\theta^j) + (g_2,1) + (g_2,\epsilon') + (1,\lambda^1) + (1,1)(j=1,2),$

$\widehat{(g_4,i^k)} = \mathbf{s}_{1,S_4}\Lambda_{i^k} = (g_4,i) + (g_4,-1) + (g_3,1) + (1,\lambda^2) + (1,1),\ (k=1,-1).$

Here the index H, H' in $\mathbf{s}_{H,H'}$ is a pair of subgroups of S_4 as in [L5, 3.10] except that $\mathbf{s}_{1,1}$ does not appear there. In each case H/H' is a product of symmetric groups.

Consider the matrix (from [L5]):

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0\\
1&2&1&1&0&0&0&0&0&0&0\\
1&1&1&0&1&0&0&0&0&0&0\\
1&0&1&0&1&1&0&0&0&0&0\\
1&2&1&1&1&0&1&0&0&0&0\\
1&1&0&0&1&0&1&1&0&0&0\\
1&0&0&0&1&1&0&1&1&0&0\\
1&1&1&0&2&1&0&0&0&1&0\\
1&0&1&0&1&2&0&0&1&0&1
\end{pmatrix}$$

with rows indexed from left to right and columns indexed from top to bottom by the elements of $M_0(S_4)$ in the order

$$(1,1),(1,\lambda^1),(1,\sigma),(1,\lambda^2),(g_2,1),(g_2',1),(g_2,\epsilon''),(g_3,1),(g_4,1),(g_2',\epsilon''),(g_2',\epsilon').$$

For $(x,\sigma) \in M_0(S_4)$, the coefficient of $(x',\sigma') \in M_0(S_4)$ in $\widehat{(x,\sigma)} \in \mathbf{C}[M(S_4)]$ is the entry of the matrix above in row (x,σ) and column (x',σ'); the coefficient of any $(x',\sigma') \in M(S_4) - M_0(S_4)$ is 0.

3.8. In the case 3.4(vii) we define $\tilde{\mathbf{B}}_c$ as the subset of $\mathbf{C}[M(S_5)]$ consisting of

$\widehat{(1,1)} = \mathbf{s}_{1,S_5}(1,1),$

$\widehat{(1,\lambda^1)} = \mathbf{s}_{1,H_{41}}(1,1),$

$\widehat{(1,\nu)} = \mathbf{s}_{1,H_{32}}(1,1),$

$\widehat{(1,\lambda^2)} = \mathbf{s}_{1,H_{311}}(1,1),$

$\widehat{(1,\nu')} = \mathbf{s}_{1,H_{221}}(1,1),$

$\widehat{(1,\lambda^3)} = \mathbf{s}_{1,H_{2111}}(1,1),$

$\widehat{(g_2,1)} = \mathbf{s}_{\tilde{H}_{2111},H_{32}}(1,1),$

$\widehat{(g_2,r)} = \mathbf{s}_{\tilde{H}_{2111},H_{221}}(1,1),$

$\widehat{(g_3,1)} = \mathbf{s}_{\tilde{H}_{311},H_{32}}(1,1),$

$\widehat{(g_2',1)} = \mathbf{s}_{\tilde{H}_{221},\tilde{H}}(1,1),$

$\widehat{(g_2',\epsilon'')} = \mathbf{s}_{H_{221},H_{221}}(1,1),$

$\widehat{(g_6,1)} = \mathbf{s}_{H_{32},H_{32}}(1,1),$

$\widehat{(g_2,\epsilon)} = \mathbf{s}_{H_{2111},H_{2111}}(1,1),$

$\widehat{(g_3,\epsilon)}=\mathbf{s}_{H_{311},H_{311}}(1,1)$,
$\widehat{(g_4,1)}=\mathbf{s}_{H_{41},H_{41}}(1,1)$,
$\widehat{(g_5,1)}=\mathbf{s}_{S_5,S_5}(1,1)$,
$\widehat{(g'_2,\epsilon')}=\mathbf{s}_{\tilde H,\tilde H}(1,1)$,
$\widehat{(g_2,-1)}=\mathbf{s}_{1,H_{32}}\Lambda_{1,-1}=(g_2,-1)+(1,\lambda^1)+(1,\nu)+(1,1)$,
$\widehat{(g_2,-r)}=\mathbf{s}_{1,H_{221}}\Lambda_{-1,1}=(g_2,-r)+(g_2,-1)+(1,\lambda^2)+(1,\nu')+2(1,\nu)+2(1,\lambda^1)+(1,1)$,
$\widehat{(g'_2,r)}=\mathbf{s}_{\tilde H_{2111},H_{221}}\Lambda_{-1}=(g'_2,r)+(g_2,-r)+(g_2,-1)+(g_2,1)+(g_2,r)+(1,\lambda^2)+(1,\nu')+2(1,\nu)+2(1,\lambda^1)+(1,1)$,
$\widehat{(g_4,-1)}=\mathbf{s}_{\tilde H_{221},\tilde H}\Lambda_{-1}=(g_4,-1)+(g'_2,r)+(g'_2,1)+(g_2,r)+(g_2,1)+(1,\lambda^1)+(1,\nu)+(1,\nu')+(1,1)$,

$$\widehat{(g_6,-1)}=\mathbf{s}_{\tilde H_{311},H_{32}}\Lambda_{-1}$$
$$=(g_6,-1)+(g'_2,r)+(g_2,-1)+(g_3,1)+(g_2,1)+(g_2,r)+(1,\lambda^1)+(1,\nu)+(1,1),$$

$$\widehat{(g_3,\theta^j)}=\mathbf{s}_{1,H_{32}}\Lambda_{1,\theta^j}=(g_3,\theta^j)+(g_2,r)+(g_2,\epsilon)+(1,\lambda^1)+(1,\nu)+(1,1)(j=1,2),$$

$$\widehat{(g_6,\theta^j)}=\mathbf{s}_{\tilde H_{2111},H_{32}}\Lambda_{\theta^j}=(g_6,\theta^j)$$
$$+(g_3,\theta)+(g'_2,r)+(g_2,r)+(g_2,\epsilon)+(g_2,1)+(1,\lambda^1)+(1,\nu)+(1,1)(j=1,2),$$

$\widehat{(1,\lambda^4)}=\mathbf{s}_{1,1}(1,1)=4(1,\lambda^1)+6(1,\lambda^2)+4(1,\lambda^3)+(1,\lambda^4)+5(1,\nu)+5(1,\nu')+(1,1)$,
$\widehat{(g_2,-\epsilon)}=\mathbf{s}_{1,H_{2111}}\Lambda_{-1}=(g_2,-\epsilon)+2(g_2,-r)+(g_2,-1)+3(1,\lambda^1)+3(1,\lambda^2)+(1,\lambda^3)+3(1,\nu)+2(1,\nu')+(1,1)$,
$\widehat{(g_3,\epsilon\theta^j)}=\mathbf{s}_{1,H_{311}}\Lambda_{\theta^j}=(g_3,\epsilon\theta^j)+(g_3,\theta)+(g_2,1)+2(g_2,r)+(g_2,\epsilon)+2(1,\lambda^1)+(1,\lambda^2)+(1,\nu)+(1,1)(j=1,2)$,
$\widehat{(g'_2,\epsilon)}=\mathbf{s}_{1,H_{221}}\Lambda_{-1,-1}=(g'_2,\epsilon)+(g'_2,1)+2(g_2,-1)+2(g_2,-r)+(1,\lambda^2)+(1,\nu')+2(1,\nu)+2(1,\lambda^1)+(1,1)$,
$\widehat{(g_6,-\theta^j)}=\mathbf{s}_{1,H_{32}}\Lambda_{\theta^j,-1}=(g_6,-\theta^j)+(g_3,\theta)+(g'_2,r)+(g_2,1)+(g_2,r)+(g_2,-1)+(1,\lambda^1)+(1,\nu)+(1,1)(j=1,2)$,
$\widehat{(g_4,i^k)}=\mathbf{s}_{1,H_{41}}\Lambda_{i^k}=(g_4,i^k)+(g_4,-1)+(g_3,1)+(g_3,\epsilon)+(1,\lambda^2)+(1,\lambda^3)+(1,\lambda^1)+(1,\nu)+(1,1)(k=1,-1)$,
$\widehat{(g_5,\zeta)}=\Lambda_\zeta$,
$\widehat{(g_5,\zeta^2)}=\Lambda'_{\zeta,\zeta^2}$,
$\widehat{(g_5,\zeta^3)}=\Lambda'_{\zeta^3,\zeta}$,
$\widehat{(g_5,\zeta^4)}=\Lambda'_{\zeta^2,\zeta^4}$.

Here the index H,H' in $\mathbf{s}_{H,H'}$ is a pair of subgroups of S_5 as in [L5, 3.10] except that $\mathbf{s}_{1,1}$ does not appear there. In each case H/H' is a product of symmetric groups.

Consider the matrix (from [L5]):

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&2&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&3&3&3&2&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&2&2&1&1&0&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&0&1&0&1&1&0&1&0&0&0&0&0&0&0\\
1&2&2&1&1&0&2&2&0&1&1&0&0&0&0&0&0\\
1&1&1&0&0&0&2&1&1&1&1&1&0&0&0&0&0\\
1&3&3&3&2&1&1&2&0&0&0&0&1&0&0&0&0\\
1&2&1&1&0&0&1&2&1&0&0&0&1&1&0&0&0\\
1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&0&0\\
1&0&0&0&0&0&1&0&1&1&0&1&0&0&1&1&0\\
1&1&1&0&1&0&1&1&0&2&0&0&0&0&1&0&1
\end{pmatrix}$$

with rows indexed from left to right and columns indexed from top to bottom by the elements of $M_0(S_5)$ in the order

$$(1,1),(1,\lambda^1),(1,\nu),(1,\lambda^2),(1,\nu'),(1,\lambda^3),(g_2,1),(g_2,r),(g_3,1),(g_2',1),(g_2',\epsilon''),$$
$$(g_6,1),(g_2,\epsilon),(g_3,\epsilon),(g_4,1),(g_5,1),(g_2',\epsilon').$$

For $(x,\sigma)\in M_0(S_5)$, the coefficient of $(x',\sigma')\in M_0(S_5)$ in $\widehat{(x,\sigma)}\in \mathbf{C}[M(S_5)]$ is the entry of the matrix above in the row (x,σ) and column (x',σ'); the coefficient of any $(x',\sigma')\in M(S_5)-M_0(S_5)$ is 0.

3.9. The basis $\tilde{\mathbf{B}}_c$ defined above satisfies properties (I)–(V) in 0.1. (For (I) we use 3.1(g) and the results in 3.2.) It also satisfies the property stated in 0.2 (with the notion of primitive elements as in 3.2).

References

[L1] George Lusztig, *Unipotent representations of a finite Chevalley group of type E_8*, Quart. J. Math. Oxford Ser. (2) **30** (1979), no. 119, 315–338, DOI 10.1093/qmath/30.3.315. MR545068

[L2] George Lusztig, *Unipotent characters of the symplectic and odd orthogonal groups over a finite field*, Invent. Math. **64** (1981), no. 2, 263–296, DOI 10.1007/BF01389170. MR629472

[L3] George Lusztig, *Characters of reductive groups over a finite field*, Annals of Mathematics Studies, vol. 107, Princeton University Press, Princeton, NJ, 1984. MR742472

[L4] G. Lusztig, *Leading coefficients of character values of Hecke algebras*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), Proc. Sympos. Pure Math., vol. 47, Amer. Math. Soc., Providence, RI, 1987, pp. 235–262. MR933415

[L5] G. Lusztig, *A new basis for the representation ring of a Weyl group*, Represent. Theory **23** (2019), 439–461, DOI 10.1090/ert/534. MR4021825

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Email address: gyuri@mit.edu