REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 24, Pages 229–234 (June 25, 2020)
https://doi.org/10.1090/ert/540

RESOLVING IRREDUCIBLE $\mathbb{C}S_n$-MODULES BY MODULES RESTRICTED FROM $GL_n(\mathbb{C})$

CHRISTOPHER RYBA

Abstract. We construct a resolution of irreducible complex representations of the symmetric group S_n by restrictions of representations of $GL_n(\mathbb{C})$ (where S_n is the subgroup of permutation matrices). This categorifies a recent result of Assaf and Speyer. Our construction also gives projective resolutions of simple $\mathcal{F}$-modules (here $\mathcal{F}$ is the category of finite sets).

1. Introduction

The symmetric group S_n may be viewed as the subgroup of the general linear group $GL_n(\mathbb{C})$ consisting of permutation matrices. We may therefore consider the restriction to S_n of irreducible $GL_n(\mathbb{C})$ representations. Let S^λ denote the irreducible representation of $\mathbb{C}S_n$ indexed by the partition λ (so necessarily n is the size of λ). Let $\mathbb{S}^\lambda$ denote the Schur functor associated to a partition λ, so that $\mathbb{S}^\lambda(\mathbb{C}^n)$ is an irreducible representation of $GL_n(\mathbb{C})$, provided that $l(\lambda) \leq n$. Let us write $[M]$ for the image of a module in the Grothendieck ring of $\mathbb{C}S_n$-modules. Thus, the restriction multiplicities a^λ_μ are defined via

$$[\mathrm{Res}^{GL_n}_{S_n}(\mathbb{S}^\lambda(\mathbb{C}^n))] = \sum_{\mu \vdash n} a^\lambda_\mu [S^\mu].$$

Although a positive combinatorial formula for the restriction multiplicities is not currently known, there is an expression using plethysm of symmetric functions (see [Mac95], Chapter 1, Section 8, for background about plethysm). Let us write s_λ for the Schur functions (indexed by partitions λ). The complete symmetric functions, h_n, are the Schur functions indexed by the one-part partitions (n). We recall the Schur functions s_λ form an orthonormal basis of the ring of symmetric functions with respect to the usual inner product, denoted $\langle -, - \rangle$ (see [Mac95], Chapter 1, Section 4). Let $f[g]$ denote the plethysm of a symmetric function f with another symmetric function g. Then,

$$a^\lambda_\mu = \langle s_\lambda, s_\mu[1 + h_1 + h_2 + \cdots] \rangle;$$

see [Gay76] or Exercise 7.74 of [SF99]. We will need to consider the Lyndon symmetric function,

$$L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d},$$

where $\mu(d)$ is the Möbius function and p_d is the dth power-sum symmetric function. It is important for us that L_n is the $GL(V)$ character of the degree n component of the free Lie algebra on V (see the first proof of Theorem 8.1 of [Reu93], which

Received by the editors January 4, 2019, and, in revised form, September 24, 2019.
2010 *Mathematics Subject Classification.* Primary 05E10, 20C30.

proves this to deduce a related result). For convenience we define the total Lyndon symmetric function $L = L_1 + L_2 + \cdots$; this is the character of the (whole) free Lie algebra on V.

Instead of asking for the restriction coefficients a^λ_μ, we may ask the inverse question: how can one express $[S^\mu]$ in terms of $[\mathrm{Res}^{GL_n}_{S_n}(\mathbb{S}^\lambda(\mathbb{C}^n))]$? This question was recently answered by Assaf and Speyer in [AS20]. For a partition $\mu = (\mu_1, \mu_2, \ldots)$ of any size, let $\mu[n]$ denote $(n - |\mu|, \mu_1, \mu_2, \ldots)$ (a partition of n provided that $n \geq |\mu| + \mu_1$). Assaf and Speyer showed

$$[S^{\mu[n]}] = \sum_\lambda b^\mu_\lambda [\mathbb{S}^\lambda(\mathbb{C}^n)],$$

where

$$b^\mu_\lambda = (-1)^{|\mu|-|\lambda|} \sum_{\mu/\nu \text{ vert. strip}} \langle s_{\nu'}, s_{\lambda'}[L]\rangle.$$

The notation μ/ν vert. strip means that the diagram of μ may be obtained from the diagram of ν by adding boxes, no two in the same row, and primes indicate dual partitions.

It is more convenient to work with

$$M^\mu_n = \mathrm{Ind}^{S_n}_{S_{|\mu|} \times S_{n-|\mu|}} (S^\mu \boxtimes \mathbf{1}),$$

which decompose into the irreducible $S^{\nu[n]}$ via the Pieri rule:

$$[M^\mu_n] = \sum_{\mu/\nu \text{ horiz. strip}} [S^{\nu[n]}].$$

Here, μ/ν horiz. strip means that the diagram of μ may be obtained from the diagram of ν by adding boxes, no two in the same column. The formula for b^μ_λ is equivalent to the following statement (see Theorem 3 and Proposition 5 of [AS20]):

$$[M^\mu_n] = \sum_\lambda (-1)^{|\mu|-|\lambda|} \langle s_{\mu'}, s_{\lambda'}[L]\rangle [\mathbb{S}^\lambda(\mathbb{C}^n)].$$

The purpose of this note is to give a categorification of this answer, namely a (minimal) resolution of M^μ_n by restrictions of $\mathbb{S}^\lambda(\mathbb{C}^n)$; this is accomplished in Theorem 3.2. Along the way, this explains the presence of the character of the free Lie algebra in the formula, and constructs projective resolutions in the category of $\mathcal{F}$-modules (over $\mathbb{Q}$) introduced by Wiltshire-Gordon in [WG14].

2. The resolution

We begin by calculating the cohomology of the free Lie algebra on a fixed vector space. Although this result is very well known (for example, by direct application of Koszul duality), it is instrumental in what follows, so we include it for completeness.

Let $\mathfrak{l}$ be the free Lie algebra on $V = \mathbb{C}^m$. Then $\mathfrak{g} = \mathfrak{l}^{\oplus n} = \mathfrak{l} \otimes \mathbb{C}^n$ is again a Lie algebra. It has an action of S_n by permuting the n summands, coming from an action of $GL_n(\mathbb{C})$ that does not respect the Lie algebra structure. We consider the Lie algebra cohomology of $\mathfrak{g}$ (with coefficients in the trivial module).

Recall that the Lie algebra cohomology is $\mathrm{Ext}^i_{U(\mathfrak{g})}(\mathbb{C}, \mathbb{C})$. We first consider the case for $n = 1$, so $\mathfrak{g} = \mathfrak{l}$. Now $U(\mathfrak{l})$ is just the tensor algebra of V, which we denote $T(V)$. We therefore have a (graded) free resolution

$$0 \to T(V) \otimes V \xrightarrow{d_1} T(V) \xrightarrow{d_0} \mathbb{C} \to 0.$$

Here, $d_1(x \otimes v) = xv$ (product in $T(V)$), while d_0 simply projects onto the degree zero component. Crucially, $GL(V)$ acts by automorphisms on $\mathfrak{l}$ (which was the free Lie algebra on V), and the above complex is equivariant for this action. The Lie algebra cohomology is given by the cohomology of the complex

$$0 \leftarrow \hom_{T(V)}(T(V) \otimes V, \mathbb{C}) \xleftarrow{d_1^*} \hom_{T(V)}(T(V), \mathbb{C}) \leftarrow 0.$$

We easily see the differential d_1^* is zero because any element of $\hom_{T(V)}(T(V), \mathbb{C})$ is zero on a positive degree element of $T(V)$, but the image of d_1 is contained in degrees greater than or equal to 1. We thus conclude that $H^0(\mathfrak{l}, \mathbb{C}) = \mathbb{C}$, and $H^1(\mathfrak{l}, \mathbb{C}) = V^*$, with all higher cohomology vanishing. Next, we obtain the Lie algebra cohomology of $\mathfrak{g} = \mathfrak{l}^{\oplus n}$.

Proposition 2.1. *For $0 \leq i \leq n$:*

$$H^i(\mathfrak{g}, \mathbb{C}) = \mathrm{Ind}_{S_i \times S_{n-i}}^{S_n}((V^*)^{\otimes i} \otimes \varepsilon_i \boxtimes \mathbb{C}^{\otimes(n-i)}),$$

where ε_i is the sign representation of S_i, and $\mathbb{C}^{\otimes(n-i)}$ is the trivial representation of S_{n-i}. Further, for $i > n$, the cohomology $H^i(\mathfrak{g}, \mathbb{C})$ vanishes.

Proof. We apply the Künneth theorem in an S_n-equivariant way. The sign representation ε_i arises because of the Koszul sign rule (cohomology is only graded commutative). □

Now let us compute the Lie algebra cohomology of $\mathfrak{g}$ using the Chevalley-Eilenberg complex [Wei94]. Recall that the ith cochain group is

$$\hom_{\mathbb{C}}(\bigwedge^i(\mathfrak{g}), \mathbb{C})$$

and the differential d is given by the formula

$$d(f)(x_1 \wedge \cdots \wedge x_{k+1}) = \sum_{i<j} (-1)^{j-i} f([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{k+1}),$$

where hats indicate omitted terms. This differential is $GL(V) \times S_n$-equivariant and homogeneous in terms of the grading on $\mathfrak{g}$ (the grading corresponds to the degree of the action of $\mathbb{C}^\times = Z(GL(V))$).

Note that $\mathfrak{g}$ is graded in strictly positive degrees. As an algebraic representation of $GL(V)$, the ith cochain group

$$\hom_{\mathbb{C}}(\bigwedge^i(\mathfrak{g}), \mathbb{C})$$

is contained in degrees $\leq -i$. This means that if we are interested only in the degree $-i$ component of the cohomology, we may truncate the Chevalley-Eilenberg complex after i steps. Thus, if we write a subscript $-i$ to indicate the degree $-i$ component of a $GL(V)$ representation, we obtain the following.

Proposition 2.2. *The complex (with differential inherited from the Chevalley-Eilenberg complex)*

$$0 \leftarrow \hom_{\mathbb{C}}(\bigwedge^i(\mathfrak{g}), \mathbb{C})_{-i} \leftarrow \hom_{\mathbb{C}}(\bigwedge^{i-1}(\mathfrak{g}), \mathbb{C})_{-i} \leftarrow \cdots \leftarrow \hom_{\mathbb{C}}(\bigwedge^1(\mathfrak{g}), \mathbb{C})_{-i} \leftarrow \hom_{\mathbb{C}}(\bigwedge^0(\mathfrak{g}), \mathbb{C})_{-i} \leftarrow 0$$

has cohomology $H^i(\mathfrak{g}, \mathbb{C})$ on the far left, and zero elsewhere.

3. Resolving the $\mathbb{C}S_n$-modules M_n^μ

Let us take the multiplicity space of the $GL(V)$-irreducible $\mathbb{S}^{\mu'}(V^*)$.

Proposition 3.1. *The $\mathbb{S}^{\mu'}(V^*)$ multiplicity space in the cohomology $H^i(\mathfrak{g},\mathbb{C})$ is*

$$M_n^\mu = \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}(S^\mu \boxtimes \mathbf{1}).$$

Proof. We apply Schur-Weyl duality to Proposition 2.1, noting that $S^\lambda \otimes \varepsilon_i = S^{\lambda'}$:

$$H^i(\mathfrak{g},\mathbb{C}) = \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}((V^*)^{\otimes i}\otimes \varepsilon_i \boxtimes \mathbf{1}) = \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}(\bigoplus_{\lambda\vdash i}\mathbb{S}^\lambda(V^*)\otimes S^{\lambda'}\boxtimes \mathbf{1}).$$

Hence, the $\mathbb{S}^{\mu'}(V^*)$ multiplicity space is $\mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}(S^\mu \boxtimes \mathbf{1})$. □

Because the complex we constructed in Proposition 2.2 is $GL(V)$-equivariant, taking cohomology commutes with taking the $\mathbb{S}^{\mu'}(V^*)$ multiplicity space. We immediately obtain the following.

Theorem 3.2. *Consider the complex of S_n representations*

$$\hom_{GL(V)}\left(\mathbb{S}^{\mu'}(V^*), \hom_{\mathbb{C}}(\bigwedge^i(\mathfrak{g}),\mathbb{C})\right)$$

for $|\mu| \geq i \geq 0$ with maps induced by the differential of the Chevalley-Eilenberg complex. This is a resolution of M_n^μ by representations restricted from $GL_n(\mathbb{C})$.

Proof. This is immediate from Proposition 3.1 and Proposition 2.2. □

Should we wish to resolve the irreducible S^μ, rather than M_n^μ, we simply take $n = |\mu|$ so that $M_n^\mu = S^\mu$.

We now take the Euler characteristic of our complex, viewed as an element of the Grothendieck ring of $\mathbb{C}S_n$-modules tensored with the Grothendeick ring of $GL(V)$-modules; we view the latter as the ring of symmetric functions. In the language of symmetric functions, the Schur function s_λ corresponds to the irreducible representation $\mathbb{S}^\lambda(V)$ (strictly speaking, we must quotient out s_λ for λ with more parts than $m = \dim(V)$, but this will never be an issue). We express the cohomology groups in terms of symmetric functions; as in the proof of Proposition 3.1, Schur-Weyl duality gives

$$H^i(\mathfrak{g},\mathbb{C}) = \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}(\bigoplus_{\lambda\vdash i}\mathbb{S}^\lambda(V^*)\otimes S^{\lambda'}\boxtimes \mathbf{1}).$$

Letting $\lambda = \mu'$ and passing to Grothendieck rings, this becomes $\sum_{\mu\vdash i} s_{\mu'}(x^{-1})[M_n^\mu]$, where a Schur function indicates a representation of $GL(V)$ (the inverted variables account for the dualised space V^*). Calculating the Euler characteristic directly from the cochain groups, we consider the ith exterior power of $\mathfrak{g} = \mathfrak{l}\otimes\mathbb{C}^n$,

$$\bigwedge^i(\mathfrak{g}) = \bigoplus_{\lambda\vdash i}\mathbb{S}^{\lambda'}(\mathfrak{l})\otimes\mathbb{S}^\lambda(\mathbb{C}^n), \tag{1}$$

which gives $\sum_{\lambda\vdash i}[S^\lambda(\mathbb{C}^n)]s_{\lambda'}[L](x)$. The actual chain groups are the duals of these exterior powers, so we replace the symmetric function variables x with their inverses x^{-1}. When we introduce a factor of $(-1)^{|\lambda|-|\mu|}$ from the signs in the Euler characteristic, we obtain

$$\sum_\lambda (-1)^{|\lambda|-|\mu|}[\mathbb{S}^\lambda(\mathbb{C}^n)]s_{\lambda'}[L](x^{-1}).$$

Thus the coefficient of $[\mathbb{S}^\lambda(\mathbb{C}^n)]$ in $[M_\mu^n]$ is the coefficient of $s_{\mu'}(x^{-1})$ in $(-1)^{|\mu|-|\lambda|}s_{\lambda'}[L](x^{-1})$, which gives us

$$[M_\mu^n] = \sum_\lambda (-1)^{|\mu|-|\lambda|}[\mathbb{S}^\lambda(\mathbb{C}^n)]\langle s_{\lambda'}[L], s_{\mu'}\rangle.$$

This provides an alternative proof of the formula from [AS20] for expressing the irreducible representation $S^{\mu[n]}$ of S_n in terms of restrictions $\mathrm{Res}_{S_n}^{GL_n(\mathbb{C})}(\mathbb{S}^\lambda(\mathbb{C}^n))$. This construction addresses a remark of Assaf and Speyer by explaining the presence of the character of the free Lie algebra (namely, L) in the formula.

4. Application to $\mathcal{F}$-modules

Let $\mathcal{F}$ denote the category of finite sets. An $\mathcal{F}$-module is a functor from $\mathcal{F}$ to vector spaces over a fixed field. These were introduced in [WG14], and their homological algebra was studied over $\mathbb{Q}$. An $\mathcal{F}$-module consists of an S_n-module for each n together with suitably compatible maps between them. (This is because the image of an n-element set carries an action of $Aut(\{1, 2, \dots, n\}) = S_n$.) When μ is a partition different from (1^k) (i.e., not a single column), M_n^μ (considered for fixed μ but varying n) defines an irreducible $\mathcal{F}$-module, by demanding that an n-element set in $\mathcal{F}$ map to M_n^μ (see Theorem 5.5 of [WG14]). Furthermore, in this category, objects obtained by restricting $\mathbb{S}^\lambda(\mathbb{Q}^n)$ to S_n are projective (see Definition 4.8 and Proposition 4.12 of [WG14]). Our resolution (provided we replace all instances of $\mathbb{C}$ with $\mathbb{Q}$) therefore gives a projective resolution of these simple $\mathcal{F}$-modules M_n^μ.

This resolution is in fact minimal as a resolution of M_n^μ by Schur modules (in particular, there are no maps $\mathbb{S}^\lambda(\mathbb{Q}^{\oplus n}) \to \mathbb{S}^\lambda(\mathbb{Q}^{\oplus n})$). This follows from the following two facts. Firstly, the rth term in the resolution of M_n^μ is a sum of $\mathrm{Res}_{S_n}^{GL_n(\mathbb{Q})}(\mathbb{S}^\lambda(\mathbb{Q}^n))$ with $|\lambda| = |\mu| - r$, which is a consequence of equation (1). In particular, such a module with fixed λ can only appear in one step of the resolution. Secondly, a theorem of Littlewood (Theorem XI of [Lit58]) states that the restriction multiplicity a_μ^λ is equal to $\delta_{\mu,\lambda}$ if $|\mu| \geq |\lambda|$. Thus, $[\mathrm{Res}_{S_n}^{GL_n(\mathbb{Q})}(\mathbb{S}^\lambda(\mathbb{Q}^n))]$ are linearly independent elements of the Grothendieck ring of S_n-modules, provided n is sufficiently large. Furthermore, the $[\mathrm{Res}_{S_n}^{GL_n(\mathbb{Q})}(\mathbb{S}^\lambda(\mathbb{Q}^n))]$ should only occur in the resolution in order of decreasing $|\lambda|$ (as in our resolution). Together with Observation 4.25 of [WG14], which provides a projective resolution of certain $\mathcal{F}$-modules D_k (which can be thought of as substitutes for M_n^μ when $\mu = (1^k)$), we obtain projective resolutions of all finitely generated $\mathcal{F}$-modules over $\mathbb{Q}$.

Remark 4.1. The $\mathcal{F}$-modules corresponding to $\mathbb{S}^\lambda(\mathbb{Q}^{\oplus n})$ are not in general indecomposable. As a result, the complex itself may be decomposable (and hence not minimal as a projective resolution in the category of $\mathcal{F}$-modules). For example, consider the case of $\lambda = (2)$:

$$0 \to \mathbb{Q}^{\oplus n} \xrightarrow{\pi} Sym^2(\mathbb{Q}^{\oplus n}) \to M_n^{(2)} \to 0.$$

If $\{e_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{Q}^{\oplus n}$, then $\pi(e_i) = e_i^2$. However, there is a splitting defined by $\psi(e_ie_j) = \frac{1}{2}(e_i + e_j)$. It follows that $Sym^2(\mathbb{Q}^{\oplus n}) = \mathbb{Q}^{\oplus n} \oplus M_n^{(2)}$, and hence that $M_n^{(2)}$ is already projective. The author is grateful to the referee for this remark.

Acknowledgments

The author would like to thank Gurbir Dhillon for helpful comments on this paper, and the referee for Remark 4.1.

References

[AS20] Sami H. Assaf and David E. Speyer, *Specht modules decompose as alternating sums of restrictions of Schur modules*, Proc. Amer. Math. Soc. **148** (2020), no. 3, 1015–1029, DOI 10.1090/proc/14815. MR4055931

[Gay76] David A. Gay, *Characters of the Weyl group of $SU(n)$ on zero weight spaces and centralizers of permutation representations*, Rocky Mountain J. Math. **6** (1976), no. 3, 449–455, DOI 10.1216/RMJ-1976-6-3-449. MR414794

[Lit58] D. E. Littlewood, *Products and plethysms of characters with orthogonal, symplectic and symmetric groups*, Canadian J. Math. **10** (1958), 17–32, DOI 10.4153/CJM-1958-002-7. MR95209

[Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky; Oxford Science Publications. MR1354144

[Reu93] Christophe Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series, vol. 7, The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications. MR1231799

[SF99] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR1676282

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324

[WG14] John D. Wiltshire-Gordon, *Uniformly presented vector spaces*, arXiv preprint `arXiv:1406.0786`, 2014.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Email address: `ryba@mit.edu`