REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 24, Pages 292–322 (July 21, 2020)
https://doi.org/10.1090/ert/545

THE LANGLANDS DUAL AND UNITARY DUAL OF QUASI-SPLIT $PGSO_8^E$

CAIHUA LUO

Abstract. This paper serves two purposes, by adopting the classical Casselman–Tadić's Jacquet module machine and the profound Langlands–Shahidi theory, we first determine the explicit Langlands classification for quasi-split groups $PGSO_8^E$ which provides a concrete example to guess the internal structures of parabolic inductions. Based on the classification, we further sort out the unitary dual of $PGSO_8^E$ and compute the Aubert duality which could shed light on the final answer of Arthur's conjecture for $PGSO_8^E$. As an essential input to obtain a complete unitary dual, we also need to determine the local poles of triple product L-functions which is done in the appendix. As a byproduct of the explicit unitary dual, we verified Clozel's finiteness conjecture of special exponents and Bernstein's unitarity conjecture concerning AZSS duality for $PGSO_8^E$.

Introduction

Let $PGSO_8^E$ be an adjoint quasi-split group of type D_4 over a non-archimedean field F of characteristic zero, where E is a cubic field extension of F. As part of the Langlands program, it is pivotal to understand the decomposition of induced representations and classify the unitary dual. Following Harish-Chandra, Knapp–Stein and others developed the R-group theory to determine the structure of tempered induced representations (cf. [23, 44]), and based on the R-group theory Winarsky [52], Keys [20], and others have completely determined the structure of tempered principal series for split p-adic Chevalley groups. As for generalized principal series (tempered or not), Shahidi [42] has built up the profound Langlands–Shahidi theory to tackle this problem and produced quite fruitful results [11, 43]. Along another direction, Casselman [6], Rodier [34], Tadić–Sally [45, 47], Janzten [19], and others have developed the Jacquet module machine to analyze the constituents of non-tempered principal series representations. But it is still far from its completeness (to the author's knowledge). Motivated by the work of Rodier on regular characters, it should be reasonable to believe the existence of an internal structure for the non-tempered principal series. On the other hand, in light of unitary dual, Vogan and his collaborators have produced many influential works and created a unitary kingdom (cf. [24, 49–51]). As a test, some low rank groups have been computed (cf. [12–14, 25, 29, 31, 35, 37]). From the perspective of global Langlands conjectures, AZSS (Aubert, Zelevinsky, Schneider–Stuhler) duality also plays an

---

Received by the editors May 5, 2019, and, in revised form, December 25, 2019, and February 21, 2020.

2010 *Mathematics Subject Classification.* Primary 22E35.

Key words and phrases. Langlands classification (dual), unitary dual, Jacquet module, principal series, adjoint $^3D_4$.

important role in formulating Arthur's conjecture [1] (as always cited as Aubert duality). It is conjectured that the AZSS duality preserves unitarity (cf. [3, 36]) and corresponds to the switch of SL_2-components of the A-parameter on the Galois side (see [17]). In this paper, we will carry out the project for $PGSO_8^E$, whilst a similar result of the unitary dual of Spin_8^E will be discussed somewhere else, and we hope to finish the Langlands dual for Sp_6 in the near future to get a glimpse of possible internal structures of the decomposition of principal series. Even though the Jacquet module method applied here is not completely new, we want to emphasize that our ϵ-revised method originating from Muić's work on G_2 (cf. [31]) should work for all (relative) rank 2 groups and it is much more intuitive. On the other hand, as $PGSO_8$ is closely related to G_2, one may expect a similar result as G_2 concerning the unitary dual of $PGSO_8^E$. But new phenomenon appears, there are two isolated families of unitary representations of $PGSO_8^E$ instead of one isolated family of unitary dual for G_2, i.e.,

$$I_\alpha(1, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1) \text{ with } \chi_{1_{|_{F^\times}}=1}, \chi_1 \neq 1$$

and

$$I_\beta(3, 1 \otimes I^\beta(\chi_2, \chi_2^{-1})) \text{ with } \chi_2 \circ N_{E/F} = 1 \ , \chi_2 \neq 1.$$

We also want to point out that the determination of local poles of the triple product L-function is completely new, even the global problem is known by Ikeda (cf. [18]). In the meantime, we expect that such a detailed study could play a role on the understanding of Arthur's conjecture for G_2 as G_2 is a triality-twisted elliptic endoscopic group of $PGSO_8$. Finally, we would like to mention that recently Tadić has made major progress on the unitary dual of relative rank at most 3 parabolic inductions of classical groups (cf. [48]). We also would like to mention that recently, via Casselman–Tadić's Jacquet module machine, we have generalized Rodier's structure theorem for regular principal series (cf. [34]) and Muller's irreducibility theorem for principal series (cf. [32]) to their counterparts for generalized principal series (cf. [26, 27]). All of those should be regarded as preliminary steps to understand the internal structures of parabolic inductions.

Here is an outline of the paper. In the first section, we establish notation and recall some basic structure results for $PGSO_8^E$ with E/F a cyclic extension and some basic representation theory facts. As the non-Galois case is almost the same, we will treat it as a remark accordingly throughout the paper. Finally, we will do some basic computations for later use. In the second section, we compute the explicit Langlands classification for $PGSO_8^E$, while the last section is devoted to sorting out the unitary dual and showing the unitarizability of two isolated families $I_\alpha(1, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1)$ with $\chi_{1_{|_{F^\times}}=1}, \chi_1 \neq 1$, and $I_\beta(3, 1 \otimes I^\beta(\chi_2, \chi_2^{-1}))$ with $\chi_2 \circ N_{E/F} = 1$, $\chi_2 \neq 1$.

1. Preliminaries

Let F be a non-archimedean field of characteristic zero, let $\bar{F}$ be the algebraic closure of F, and let E be a cubic Galois field extension of F with $\mathrm{Gal}(E/F) = \langle \sigma \rangle$. Write $F^\times$ (resp., $\bar{F}^\times$, $E^\times$) to be the group of invertible elements in F (resp., $\bar{F}$, E). Denote by $|\cdot|$ the absolute value of F and by $|\cdot| \circ N_{E/F}$ the absolute value of E with $N_{E/F}$ the normal map from E to F, and write $\nu_F = |\cdot| \circ \det$ and $\nu_E = |\cdot| \circ N_{E/F} \circ \det$. Given such an E, we know there is an associated adjoint quasi-split group $G = PGSO_8^E$ of type D_4. For simplicity, we also write G to be $G(F)$ if no confusion arises with similar notions for other groups.

Denote by T a maximal torus and by $B = TU$ a Borel subgroup of $PGSO_8^E$. We know that the absolute root lattice

$$X^*(T) = \mathbb{Z}\left\langle \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_3 + e_4 \right\rangle,$$

and the absolute coroot lattice

$$X_*(T) = \mathbb{Z}\left\langle e_1^*, e_1^* + e_2^*, \frac{1}{2}(e_1^* + e_2^* + e_3^* - e_4^*), \frac{1}{2}(e_1^* + e_2^* + e_3^* + e_4^*) \right\rangle,$$

where $\{e_i^*\}$ is the basis dual to $\{e_i\}$.

As G is an adjoint group and T splits over E, thus we may parameterize $T(F)$ in the following way:

$$t \in T(F) \leftrightarrow t = H_{e_1^*}(t_1) H_{\frac{1}{2}(e_1^*+e_2^*+e_3^*-e_4^*)}(t_1^{\sigma}) H_{\frac{1}{2}(e_1^*+e_2^*+e_3^*+e_4^*)}(t_1^{\sigma^2}) H_{e_1^*+e_2^*}(t_2),$$

where $t_1 \in E^\times$, $t_2 \in F^\times$, and $H_{\gamma^\vee}(t) := \gamma^\vee \otimes t \in X_\star(T) \otimes \bar{F}^\times$ for a coroot $\gamma^\vee \in X_*(T)$. Under the natural restriction map from absolute roots to relative roots as in [8, 3.2], we denote by α the relative short root given by the restriction of any α_i with $i = 1, 3, 4$, by β the relative long root of the restriction of α_2, and by $\Phi_E^+ := \{\alpha,\ \beta,\ \alpha + \beta,\ 2\alpha + \beta,\ 3\alpha + \beta,\ 3\alpha + 2\beta\}$ the set of relative positive roots which is of type G_2. As α is a character of $F^\times \times F^\times$ mapping (t, t_2) to t for any $t,\ t_2 \in F^\times$, we naturally extend it to be a character of $E^\times \times F^\times$ defined by $(t_1, t_2) \mapsto t_1$ for any $t_1 \in E^\times$ and $t_2 \in F^\times$. Accordingly, the coroot $\alpha^\vee$ naturally extended to $E^\times$ is defined by $t_1 \mapsto (t_1^2, N_{E/F}(t_1)^{-1})$ for any $t_1 \in E^\times$. Then we have $\mathfrak{a}_T^* := X^*(T)_F \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}\langle \alpha, \beta \rangle$ and the positive Weyl chamber

$$C^+ = (\mathfrak{a}_T^*)_+ := \{x \in \mathfrak{a}_T^* : (x, \alpha^\vee) > 0, (x, \beta^\vee) > 0\} = \left\{s_1\alpha + s_2\beta : \frac{3}{2}s_2 < s_1 < 2s_2\right\},$$

where $\alpha^\vee$ and $\beta^\vee$ are coroots of α and β, respectively, and

$$(\alpha, \alpha^\vee) = 2, \quad (\beta, \beta^\vee) = 2, \quad (\alpha, \beta^\vee) = -1 \text{ and } (\beta, \alpha^\vee) = -3.$$

For any root $\gamma \in \Phi_E^+$, we denote w_γ to be the corresponding reflection in the Weyl group $W = \langle w_\alpha, w_\beta \rangle$ of G which satisfies that

$$w_\alpha = w_{\alpha_1} w_{\alpha_3} w_{\alpha_4} \text{ and } w_\beta = w_{\alpha_2};$$

here w_{α_i}, $i = 1, 2, 3, 4$, is the simple reflection corresponding to α_i in the absolute Weyl group W_{D_4} of type D_4. Thus an easy calculation shows that

$$\begin{aligned}
&w_\alpha.e_1^* = -e_1^* + (e_1^* + e_2^*),\\
&w_\beta.e_1^* = e_1^*,\\
&w_\alpha.\frac{1}{2}(e_1^* + e_2^* + e_3^* - e_4^*) = -\frac{1}{2}(e_1^* + e_2^* + e_3^* - e_4^*) + (e_1^* + e_2^*),\\
&w_\beta.\frac{1}{2}(e_1^* + e_2^* + e_3^* - e_4^*) = \frac{1}{2}(e_1^* + e_2^* + e_3^* - e_4^*),\\
&w_\alpha.\frac{1}{2}(e_1^* + e_2^* + e_3^* + e_4^*) = -\frac{1}{2}(e_1^* + e_2^* + e_3^* + e_4^*) + (e_1^* + e_2^*),\\
&w_\beta.\frac{1}{2}(e_1^* + e_2^* + e_3^* + e_4^*) = \frac{1}{2}(e_1^* + e_2^* + e_3^* + e_4^*),\\
&w_\alpha.(e_1^* + e_2^*) = (e_1^* + e_2^*),\\
&w_\beta.(e_1^* + e_2^*) = -(e_1^* + e_2^*) + e_1^* + \frac{1}{2}(e_1^* + e_2^* + e_3^* - e_4^*) + \frac{1}{2}(e_1^* + e_2^* + e_3^* + e_4^*).
\end{aligned}$$

For Levi subgroups of $PGSO_8^E$, we have the following isomorphisms (please refer to [9, Formula (2.28)] for P_β and [8, Formula (3.2)] or [38, Formula (1.1)] for the dual of P_α):

$$\begin{tikzcd}
B = TU : & T \arrow[r, "\sim"] & E^\times \times F^\times \\
& t \arrow[r, mapsto] & (\alpha(t), \beta(t)), \\
P_\alpha = M_\alpha N_\alpha : & M_\alpha \arrow[r, "\sim"] & GL_2(E) \times F^\times / \Delta E^\times \\
& t = (t_1, t_2) \arrow[r, mapsto] & (\mathrm{diag}(t_1, 1), t_2^{-1}), \\
P_\beta = M_\beta N_\beta : & M_\beta \arrow[r, "\sim"] & E^\times \times GL_2(F) / \Delta F^\times \\
& t = (t_1, t_2) \arrow[r, mapsto] & (t_1^{-1}, \mathrm{diag}(t_2, 1)),
\end{tikzcd}$$

where $\Delta : E^\times \hookrightarrow GL_2(E) \times F^\times$ is given by $x \mapsto (x \cdot \mathrm{diag}(1,1), N_{E/F}(x))$, and $\Delta : F^\times \hookrightarrow E^\times \times GL_2(F)$ is given by $y \mapsto (y, y \cdot \mathrm{diag}(1,1))$.

Under the above realization, we have an explicit description of the Weyl group action on $(t_1, t_2) \in T(F)$ given by $w.t := wtw^{-1}$ for any $w \in W$ and $t \in T(F)$ which is listed as follows under the above isomorphism $T(F) \simeq E^\times \times F^\times : \ t \mapsto (t_1, t_2)$:

$$w_\alpha(t_1, t_2) = (t_1^{-1}, N_{E/F}(t_1)t_2), \ w_\beta(t_1, t_2) = (t_1 t_2, t_2^{-1}).$$

See Figure 1.

$$\begin{tikzcd}
W.T : & (t_1, t_2) \arrow[r, "w_\alpha"] & (t_1^{-1}, N_{E/F}(t_1)t_2) \arrow[r, "w_\beta"] & (t_1^{-1} N_{E/F}(t_1)t_2, N_{E/F}(t_1)^{-1} t_2^{-1}) \arrow[d, "w_\alpha"] \\
.(t_1^{-1}, t_2^{-1}) & (t_1^{-1} t_2^{-1}, t_2) \arrow[l, "w_\beta"'] & (t_1 t_2, N_{E/F}(t_1)^{-1} t_2^{-2}) \arrow[l, "w_\alpha"'] & (t_1 N_{E/F}(t_1)^{-1} t_2^{-1}, N_{E/F}(t_1) t_2^2) \arrow[l, "w_\beta"']
\end{tikzcd}$$

FIGURE 1. W-action on torus

Similarly, an explicit description of the Weyl group action on characters (χ_1, χ_2) of $T(F) \simeq E^\times \times F^\times$ defined by $\chi^w := \chi \circ \mathrm{Ad}(w)$ and $w.\chi := \chi \circ \mathrm{Ad}(w^{-1})$ for any $w \in W$ and $\chi : \ E^\times \times F^\times \to \mathbb{C}^\times$ is listed as shown in Figure 2..

$$\begin{tikzcd}
\chi^W : & (\chi_1,\chi_2) \arrow[r, "w_\alpha"] & (\chi_1^{-1}\chi_2\circ N_{E/F},\chi_2), \arrow[r, "w_\beta w_\alpha"] & (\chi_1^{-1}\chi_1\circ N_{E/F}\chi_2\circ N_{E/F}^{-1},\chi_1\chi_2^{-1}), \arrow[d, "w_\alpha w_\beta w_\alpha"] \\
& (\chi_1^{-1},\chi_1^{-1}\chi_2) \arrow[d, "w_\beta w_\alpha w_\beta w_\alpha w_\beta w_\alpha"] & ,(\chi_1\chi_2\circ N_{E/F}^{-1},\chi_1\chi_2^{-2}) \arrow[l, "w_\alpha w_\beta w_\alpha w_\beta w_\alpha"'] & ,(\chi_1\chi_1\circ N_{E/F}^{-1}\chi_2\circ N_{E/F},\chi_1^{-1}\chi_2^2) \arrow[l, "w_\beta w_\alpha w_\beta w_\alpha"'] \\
& (\chi_1^{-1},\chi_2^{-1}). & &
\end{tikzcd}$$

Figure 2. W-action on character

Let $R(G)$ be the Grothendieck group of admissible smooth representations of finite length of G with similar notions for other groups. We denote by r_γ the normalized Jacquet functor w.r.t. P_γ, and by $r_\varnothing$ the normalized Jacquet functor w.r.t. B. Now we recall that (cf. [4,6])

$$r_\alpha(I_\alpha(\sigma)) = \sigma + w_{3\alpha+2\beta}.\sigma + I^\alpha(w_{\alpha+\beta}.r_\varnothing(\sigma)) + I^\alpha(w_\beta.r_\varnothing(\sigma)),$$
$$r_\beta(I_\alpha(\sigma)) = I^\beta(r_\varnothing(\sigma)) + I^\beta(w_{2\alpha+\beta}w_\alpha.r_\varnothing(\sigma) + I^\beta(w_{3\alpha+\beta}w_\alpha.r_\varnothing(\sigma))$$

if $\sigma \in R(M_\alpha)$, and

$$r_\beta(I_\beta(\sigma)) = \sigma + w_{2\alpha+\beta}.\sigma + I^\beta(w_\alpha.r_\varnothing(\sigma)) + I^\beta(w_{3\alpha+\beta}.r_\varnothing(\sigma)),$$
$$r_\alpha(I_\beta(\sigma)) = I^\alpha(r_\varnothing(\sigma)) + I^\alpha(w_{\alpha+\beta}w_\alpha.r_\varnothing(\sigma)) + I^\alpha(w_\beta w_\alpha.r_\varnothing(\sigma))$$

if $\sigma \in R(M_\beta)$. Here the action of Weyl group W on the representation σ of M_γ with $\gamma \in \{\alpha,\beta\}$ is defined by $w.\sigma := \sigma \circ \mathrm{Ad}(w^{-1})$ for $w \in W$.

We have the Aubert involution endomorphism of $R(G)$

$$D_G(\pi) = I^G \circ r_\varnothing(\pi) - I_\alpha \circ r_\alpha(\pi) - I_\beta \circ r_\beta(\pi) + \pi.$$

It follows from [3,36] that $\pm D_G(\pi)$ preserves irreducibility and we have [3, Theorem 1.7(3)],

$$D_G \circ I_\gamma = I_\gamma \circ D_{M_\gamma},$$
$$r_\gamma \circ D_G = \tilde{w}_\gamma \circ D_{M_\gamma} \circ r_\gamma.$$

Here $\tilde{w}_\alpha = w_{3\alpha+2\beta}$ and $\tilde{w}_\beta = w_{2\alpha+\beta}$.

For

$$(s_1,s_2) := 3s_1\alpha + s_2\beta \in X^*(T)\otimes_{\mathbb{Z}}\mathbb{C},$$

we define the associated unramified character of $T(F)$ as

$$(t_1,t_2) \mapsto |N_{E/F}(t_1)|^{s_1}|t_2|^{s_2}.$$

For $\gamma \in \{\alpha,\beta\}$, $s_1,s_2 \in \mathbb{R}$, and $\chi_1\times\chi_2$ a unitary character of $E^\times \times F^\times \simeq T(F)$, set

$$I^\gamma(s_1,s_2,\chi_1,\chi_2) = I^\gamma(|N_{E/F}(\cdot)|^{s_1}\chi_1 \otimes |\cdot|^{s_2}\chi_2) = \mathrm{Ind}_T^{M_\gamma}(|N_{E/F}(\cdot)|^{s_1}\chi_1 \otimes |\cdot|^{s_2}\chi_2).$$

Similarly, we write $I(s_1,s_2,\chi_1,\chi_2) = I^G(s_1,s_2,\chi_1,\chi_2) = \mathrm{Ind}_T^G(|N_{E/F}(\cdot)|^{s_1}\chi_1 \otimes |\cdot|^{s_2}\chi_2)$ for the normalized induced representation from B to G, and denote by $I_\gamma(-)$ the normalized parabolic induction inducing from P_γ to G for $\gamma \in \{\alpha,\beta\}$. Throughout the paper, for simplicity, we would like to use the same symbol χ_1 to be $\chi_1|_{F^\times}$ when restricting a character χ_1 of $E^\times$ to $F^\times$ if there is no confusion.

Now we recall the Langlands quotient theorem and Casselman's temperedness criterion in the $PGSO_8^E$-setting (cf. [5, XI Proposition 2.6 and Corollary 2.7]) for later use as follows.

Langlands quotient theorem. *Denote by σ an irreducible tempered representations of GL_2.*

When χ_2 is unitary and $s > 0$, the induced representation $\mathrm{Ind}_{P_\alpha}^G(\nu_E^s\sigma \otimes \chi_2\nu_F^{-2s})$ has a unique irreducible quotient, i.e., the Langlands quotient $J_\alpha(s, \sigma \otimes \chi_2)$.

When χ_1 is unitary and $s > 0$, the induced representation $\mathrm{Ind}_{P_\beta}^G(\chi_1\nu_E^{\frac{-2s}{3}} \otimes \nu_F^s\sigma)$ has a unique irreducible quotient, i.e., the Langlands quotient $J_\beta(s, \chi_1 \otimes \sigma)$.

When χ_1, χ_2 are unitary and $\frac{3}{2}s_2 < 3s_1 < 2s_2$, the induced representation $I(s_1, s_2, \chi_1, \chi_2)$ has a unique irreducible quotient, i.e., the Langlands quotient $J(s_1, s_2, \chi_1, \chi_2)$.

Casselman's temperedness criterion. Suppose π is an irreducible representation of G supported on a minimal parabolic subgroup; then π is square-integrable (resp., tempered) if and only if for any irreducible subquotient $(s_1, s_2, \chi_1, \chi_2)$ of $r_\varnothing(\pi)$ ($s_i \in \mathbb{R}$, χ_i unitary), we have

$$(s_1, s_2) \in {}^+\mathfrak{a}_T^* = \{a\alpha + b\beta : a > 0,\ b > 0\}\ (\textit{resp., } {}^+\bar{\mathfrak{a}}_T^*).$$

Notice that for $(s_1, s_2) = 3s_1\alpha + s_2\beta$, there exists $w \in W$ such that $(s_1, s_2)^w \in \bar{C}^+$ the closure of C^+, and we have $I(s_1, s_2, \chi_1, \chi_2) = I((s_1, s_2, \chi_1, \chi_2)^w)$ in $\bar{R}(G)$. Thus we may only need to analyze those $I(s_1, s_2, \chi_1, \chi_2)$ where $(s_1, s_2) \in \bar{C}^+$, i.e., $0 \leq \frac{3}{2}s_2 \leq 3s_1 \leq 2s_2$. To do so, we need to classify two pivotal data as follows.

Singular character. As the composition series of $I(s_1, s_2, \chi_1, \chi_2)$ have been determined completely by Rodier for regular characters $(s_1, s_2, \chi_1, \chi_2)$ and by Keys for unitary characters, it will be helpful to first sort out the singular characters $\chi = (s_1, s_2, \chi_1, \chi_2)$, i.e.,

$$W_\chi := \{w \in W : \ \chi^w = \chi\} \neq \{1\}.$$

We call the cardinality of the set W_χ the multiplicity of χ which measures the extent of singularity. To keep track of the stabilizer group and constraint conditions of χ (if any), we encode this information into χ as, for simplicity,

$$\big(s_1, s_2, \chi_1, \chi_2; W_\chi;\ \text{constraint conditions}\big).$$

If χ is unitary (resp., $\chi \in \mathfrak{a}_T^*$), we write (χ_1, χ_2) (resp., (s_1, s_2)) to be $(0, 0, \chi_1, \chi_2)$ (resp., $(s_1, s_2, 1, 1)$) for simplicity.

For the convenience of the reader, we recall the action of W on $(t_1, t_2) \in T(F)$ in Table 1.

Table 1. Relation 1

$w_\alpha(t_1, t_2) = (t_1^{-1}, N_{E/F}(t_1)t_2)$	$(w_\beta w_\alpha)(t_1, t_2) = (t_1^{-1}N_{E/F}(t_1)t_2, N_{E/F}(t_1)^{-1}t_2^{-1})$
$w_\alpha(w_\beta w_\alpha)(t_1, t_2) = (t_1 N_{E/F}(t_1)^{-1}t_2^{-1}, N_{E/F}(t_1)t_2^2)$	$(w_\beta w_\alpha)^2(t_1, t_2) = (t_1 t_2, N_{E/F}(t_1)^{-1}t_2^{-2})$
$w_\alpha(w_\beta w_\alpha)^2(t_1, t_2) = (t_1^{-1}t_2^{-1}, t_2)$	$(w_\beta w_\alpha)^3(t_1, t_2) = (t_1^{-1}, t_2^{-1})$

Thus it gives rise to the action of W on unramified characters (s_1, s_2) of $T(F)$ in Table 2

TABLE 2. Relation 2

$(s_1,s_2)^{w_\alpha} = (-s_1+s_2,s_2)$	$(s_1,s_2)^{(w_\beta w_\alpha)} = (2s_1-s_2,3s_1-s_2)$
$(s_1,s_2)^{w_\alpha(w_\beta w_\alpha)} = (-2s_1+s_2,-3s_1+2s_2)$	$(s_1,s_2)^{(w_\beta w_\alpha)^2} = (s_1-s_2,3s_1-2s_2)$
$(s_1,s_2)^{w_\alpha(w_\beta w_\alpha)^2} = (-s_1,-3s_1+s_2)$	$(s_1,s_2)^{(w_\beta w_\alpha)^3} = (-s_1,-s_2)$

Thus we have the following action of the Wey group W on unramified characters:

$$\begin{aligned}&\{(s_1,s_2)^w : w\in W\}\\ &\quad=\Big\{\pm(s_1,s_2),\pm(s_1-s_2,-s_2),\pm(2s_1-s_2,3s_1-s_2),\pm(2s_1-s_2,3s_1-2s_2),\\ &\qquad\qquad\pm(s_1-s_2,3s_1-2s_2),\pm(s_1,3s_1-s_2)\Big\}.\end{aligned}$$

In conjunction with Figure 2, we know that for those s_1 and s_2 satisfying the condition that $0\le\frac{3}{2}s_2\le 3s_1\le 2s_2$, the set S of singular characters consists of those unitary χ with multiplicity $m>2$:

$$\begin{aligned}&(1,1;D_6),(\chi_1,1;S_3;\chi_1\neq1,\ \chi_1|_{F^\times}=1), && (\chi_1,1;\langle w_\alpha,w_{3\alpha+2\beta}\rangle;\chi_1^2=1,\ \chi_1|_{F^\times}\neq1),\\ &(1,\chi_2;\langle w_\beta,w_{2\alpha+\beta}\rangle;\chi_2^2=1,\ \chi_2\neq1), && (\chi_1,\chi_1;\langle w_{3\alpha+\beta},w_{\alpha+\beta}\rangle;\chi_1^2=1,\ \chi_1|_{F^\times}\neq1).\end{aligned}$$

Here D_6 stands for the Dihedral group of order 12, and S_3 is the permutation group of order 6, and those unitary χ with multiplicity $m=2$:

$$\begin{aligned}&(\chi_1,\chi_2;\langle w_\alpha w_{3\alpha+2\beta}\rangle;\chi_1^2=1,\ \chi_2^2=1), && (\chi_1,\chi_2;\langle w_\alpha\rangle;\chi_1^2=\chi_2\circ N_{E/F}),\\ &(\chi_1,\chi_2;\langle w_{\alpha+\beta}\rangle;\chi_1=\chi_2\circ N_{E/F}), && (1,\chi_2;\langle w_{2\alpha+\beta}\rangle),(\chi_1,1;\langle w_{3\alpha+2\beta}\rangle),\\ &(\chi_1,\chi_2;\langle w_\beta\rangle;\chi_1|_{F^\times}=\chi_2^2), && (\chi_1,\chi_2;\langle w_{3\alpha+\beta}\rangle;\chi_1|_{F^\times}=\chi_2),\end{aligned}$$

and those non-unitary χ with multiplicity $m=2$:

$$\begin{aligned}&(s_1,2s_1,\chi_1,\chi_2;\langle w_\alpha\rangle;s_1>0,\ \chi_1^2\chi_2\circ N_{E/F}),\\ &\qquad\qquad(s_1,\frac{3}{2}s_1,\chi_1,\chi_2;\langle w_\beta\rangle;s_1>0,\ \chi_1|_{F^\times}=\chi_2^2).\end{aligned}$$

Reducibility point. In what follows, we will describe the rank 1 reducibility points, i.e., the reducibility points arising from those rank 1 parabolic inductions $I^\gamma((s_1,s_2,\chi_1,\chi_2)^w)$ with $\gamma\in\{\alpha,\beta\}$ and $w\in W$, as we believe that, in most cases, rank 1 irreducibility should determine the irreducibility of the full induced representation. Note that the derived groups of those Levi subgroups M_γ with $\gamma\in\{\alpha,\beta\}$ are isogenous to $SL_2(E)$ or $SL_2(F)$, thus those rank 1 reducibility points are determined by $\chi_{\gamma^\vee}:=\chi\circ\gamma^\vee=|\cdot|\circ N_{E/F}$ if $\gamma=\alpha$ up to conjugation by some $w\in W$ or $\chi_{\gamma^\vee}=|\cdot|$ if $\gamma=\beta$ up to conjugation by some $w\in W$. Note that $(\alpha,\beta^\vee)=-1$ and $(\beta,\alpha^\vee)=-3$. In conjugation with Figure 2, an easy calculation shows that the set R of rank 1 reducibility points is listed as shown in Tables 3 and 4.

TABLE 3. Reducibility point

	Rank 1 reducibility $(s_1, s_2, \chi_1, \chi_2)$
$\alpha^\vee$	$2s_1 - s_2 = 1,\ \chi_1^2 = \chi_2 \circ N_{E/F}$
$\beta^\vee$	$-3s_1 + 2s_2 = 1,\ \chi_1 = \chi_2^2$
$(\alpha+\beta)^\vee$	$-s_1 + s_2 = 1,\ \chi_1 = \chi_2 \circ N_{E/F}$
$(2\alpha+\beta)^\vee$	$s_1 = 1,\ \chi_1 = 1$
$(3\alpha+\beta)^\vee$	$3s_1 - s_2 = 1,\ \chi_1 = \chi_2$
$(3\alpha+2\beta)^\vee$	$s_2 = 1,\ \chi_2 = 1$

TABLE 4. $\#R > 1$

	Reducible coroot $(\cdot)^\vee$	relation for $(\cdot)^\vee$	$(s_1, s_2, \chi_1, \chi_2)$
$\#R = 4$	$(\alpha+\beta),\ \beta,\ (2\alpha+\beta),\ (3\alpha+\beta)$	w_α	(1,2,1,1)
$\#R = 2$	$(3\alpha+\beta),\ (3\alpha+2\beta)$	w_β	$(\frac{2}{3}, 1, \chi_1, 1; \chi_1\|_{F^\times} = 1)$
	$\alpha,\ (\alpha+\beta)$	w_β	$(2, 3, 1, \chi_2; \chi_2 \circ N_{E/F} = 1)$
	$\alpha,\ \beta$		$(3, 5, 1, 1)$
	$(\alpha+\beta),\ (2\alpha+\beta)$	w_α	$(1, 2, 1, \chi_2; \chi_2 \neq 1\ \&\ \chi_2 \circ N_{E/F} = 1)$
	$\beta,\ (2\alpha+\beta)$		$(1, 2, 1, \chi_2; \chi_2 \neq 1\ \&\ \chi_2^2 = 1)$
	$(\alpha+\beta),\ (3\alpha+\beta)$		$(1, 2, \chi_1, \chi_2; \chi_2 \neq 1\ \&\ \chi_1^2 = 1,\ \chi_1 = \chi_2)$
	$\beta,\ (3\alpha+\beta)$	w_α	$(1, 2, \chi_1, 1; \chi_1 \neq 1\ \&\ \chi_1\|_{F^\times} = 1)$

Before moving to the next computation section, we recall Shahidi's local coefficient formula in the $PGSO_8^E$-setting based on its multiplicative property as follows (please refer to [42] for the notions), up to a monomial in q^{-s}:

$$\begin{aligned} C_\psi(s, \delta(\chi_1) \otimes \chi_2, w_{3\alpha+2\beta}) &= \frac{L_F(\frac{5}{2} - s, \chi_2^2\chi_1) L_F(1 - 2s, \chi_2) L_F(-\frac{1}{2} - s, \chi_1^{-1}\chi_2^{-1})}{L_F(s - \frac{3}{2}, \chi_2^{-2}\chi_1^{-1}) L_F(2s, \chi_2^{-1}) L_F(\frac{3}{2} + s, \chi_1\chi_2)} \\ &\times \frac{L_E(\frac{3}{2} - s, \chi_1^{-2}(\chi_1\chi_2) \circ N_{E/F}) L_E(\frac{1}{2} - s, \chi_1^{-1} \circ N_{E/F}\chi_1^2)}{L_E(s - \frac{1}{2}, \chi_1^2(\chi_1\chi_2)^{-1} \circ N_{E/F}) L_E(s + \frac{1}{2}, \chi_1 \circ N_{E/F}\chi_1^{-2})} \end{aligned}$$

$$\begin{aligned} &C_\psi(s, \chi_1 \otimes \delta(\chi_2), w_{2\alpha+\beta}) \\ &= \frac{L_E(\frac{3}{2} - \frac{s}{3}, \chi_1^2\chi_2 \circ N_{E/F}) L_E(1 - \frac{2s}{3}, \chi_2^{-2}\chi_1 \circ N_{E/F}) L_E(\frac{1}{2} - \frac{s}{3}, \chi_1^2(\chi_1\chi_2)^{-1} \circ N_{E/F})}{L_E(-\frac{1}{2} + \frac{s}{3}, \chi_1^{-2}\chi_2^{-1} \circ N_{E/F}) L_E(\frac{2s}{3}, \chi_2\chi_1^{-1} \circ N_{E/F}) L_E(\frac{1}{2} + \frac{s}{3}, \chi_1^{-2}(\chi_1\chi_2) \circ N_{E/F})} \\ &\times \frac{L_F(\frac{3}{2} - s, \chi_1\chi_2) L_F(\frac{1}{2} - s, \chi_2^{-1})}{L_F(s - \frac{1}{2}, (\chi_1\chi_2)^{-1}) L_F(\frac{1}{2} + s, \chi_2)}. \end{aligned}$$

In view of the above formulas, we have the following lemma which results from [39, Proposition 3.3.1].

Lemma 1.1. *We have the genericity of those representations which will be used in the next section.*

$$J_\alpha(\nu_E^{\frac{1}{2}}\delta(1) \otimes \chi_2\nu_F^{-1})|_{\chi_2 \neq 1, \chi_2 \circ N_{E/F} = 1}, \quad J_\alpha(\nu_E^{\frac{3}{2}}\delta(1) \otimes \nu_F^{-3}), \quad J_\beta(\nu_E^{\frac{-1}{3}}\chi_1 \otimes \nu_F^{\frac{1}{2}}\delta(1))|_{\chi_1|_{F^\times} = 1}.$$

2. Langlands classification

In this section, we will carry out the computation of the constituents of principal series in detail following Casselman–Tadić's Jacquet module machine. Recall that given a character $\chi := (s_1, s_2, \chi_1, \chi_2)$ of T, under the previous realization of Levi subgroups, we have

$$\begin{aligned}
&I^{\alpha}(\chi) = \mathrm{Ind}^{GL_2}(|N_{E/F}(\cdot)|^{s_1}\chi_1 \otimes |N_{E/F}(\cdot)|^{-s_1+s_2}\chi_1^{-1}\chi_2 \circ N_{E/F}) \otimes |\cdot|^{-s_2}\chi_2^{-1},\\
&I^{\beta}(\chi) = \chi_1^{-1}|N_{E/F}(\cdot)|^{-s_1} \otimes \mathrm{Ind}^{GL_2}(|\cdot|^{s_2}\chi_2 \otimes |\cdot|^{3s_1-s_2}\chi_1\chi_2^{-1}).
\end{aligned}$$

It is well known that they are reducible if and only if

$$(2s_1 - s_2, \chi_1^2\chi_2^{-1} \circ N_{E/F}) = (\pm 1, 1) \text{ and } (2s_2 - 3s_1, \chi_2^2\chi_1^{-1}) = (\pm 1, 1), \text{ respectively.}$$

Also, their Jacquet modules $r_{\varnothing}$ have the form

$$\begin{aligned}
&r_{\varnothing}^{M_\alpha}(I^{\alpha}(\chi)) = \{(s_1, s_2, \chi_1, \chi_2), (-s_1 + s_2, s_2, \chi_1^{-1}\chi_2 \circ N_{E/F}, \chi_2)\},\\
&r_{\varnothing}^{M_\beta}(I^{\beta}(\chi)) = \{(s_1, s_2, \chi_1, \chi_2), (s_1, 3s_1 - s_2, \chi_1, \chi_1\chi_2^{-1})\},\\
&r_{\varnothing}^{G}(I^{G}(\chi))\\
&\quad \overset{M_\alpha}{=} \{\pm(s_1, s_2, \chi_1, \chi_2), \pm(-s_1 + s_2, s_2, \chi_1^{-1}\chi_2 \circ N_{E/F}, \chi_2)\}\\
&\quad \cup \{\pm(s_1, 3s_1 - s_2, \chi_1, \chi_1\chi_2^{-1}), \pm(2s_1 - s_2, 3s_1 - s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-1})\}\\
&\quad \cup \{\pm(s_1 - s_2, 3s_1 - 2s_2, \chi_1\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2}),\\
&\qquad\quad \pm(2s_1 - s_2, 3s_1 - 2s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2})\},\\
&r_{\varnothing}^{G}(I^{G}(\chi))\\
&\quad \overset{M_\beta}{=} \{\pm(s_1, s_2, \chi_1, \chi_2), \pm(s_1, 3s_1 - s_2, \chi_1, \chi_1\chi_2^{-1})\}\\
&\quad \cup \{\pm(s_1 - s_2, 3s_1 - 2s_2, \chi_1\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2}), \mp(-s_1 + s_2, s_2, \chi_1^{-1}\chi_2 \circ N_{E/F}, \chi_2)\}\\
&\quad \cup \{\pm(2s_1 - s_2, 3s_1 - 2s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2}),\\
&\qquad\quad \pm(2s_1 - s_2, 3s_1 - s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-1})\}.
\end{aligned}$$

Now suppose that s_1 and s_2 satisfy the condition that $2s_2 \geq 3s_1 \geq \frac{3}{2}s_2 \geq 0$. We are ready to carry out the calculation case-by-case as follows, as it may show some hidden structures.

$\#R = 0$, $(s_1, s_2, \chi_1, \chi_2)$ **non-unitary.**

Claim. $I(\chi)$ is irreducible.

If χ is regular, i.e., $\mathrm{Stab}_W(\chi) := W_\chi = \{1\}$, the diagram chasing looks pretty easy. We write down the diagram as a template for other cases.

$$\xRightarrow{\alpha} \{(s_1, s_2, \chi_1, \chi_2), (-s_1 + s_2, s_2, \chi_1^{-1}\chi_2 \circ N_{E/F}, \chi_2)\}$$

$$\xRightarrow{\beta} \begin{array}{l} \{(s_1, s_2, \chi_1, \chi_2), (s_1, 3s_1 - s_2, \chi_1, \chi_1\chi_2^{-1})\}; \\ \{-(s_1 - s_2, 3s_1 - 2s_2, \chi_1\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2}), (-s_1 + s_2, s_2, \chi_1^{-1}\chi_2 \circ N_{E/F}, \chi_2)\} \end{array}$$

$$\xRightarrow{\alpha} \begin{array}{l} \{(s_1, 3s_1 - s_2, \chi_1, \chi_1\chi_2^{-1}), (2s_1 - s_2, 3s_1 - s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-1})\}; \\ \{-(s_1 - s_2, 3s_1 - 2s_2, \chi_1\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2}), -(2s_1 - s_2, 3s_1 - 2s_2, \chi_1^{-1}\chi_1 \\ \hfill \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2})\} \end{array}$$

$$\xRightarrow{\beta} \begin{array}{l} \{(2s_1 - s_2, 3s_1 - 2s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2}), \\ \hfill (2s_1 - s_2, 3s_1 - s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-1})\}; \\ \{-(2s_1 - s_2, 3s_1 - 2s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2}), \\ \hfill -(2s_1 - s_2, 3s_1 - s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-1})\} \end{array}$$

$$\xRightarrow{\alpha} \begin{array}{l} \{(s_1 - s_2, 3s_1 - 2s_2, \chi_1\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2}), (2s_1 - s_2, 3s_1 - 2s_2, \chi_1^{-1}\chi_1 \\ \hfill \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2})\}; \\ \{-(s_1, 3s_1 - s_2, \chi_1, \chi_1\chi_2^{-1}), -(2s_1 - s_2, 3s_1 - s_2, \chi_1^{-1}\chi_1 \circ N_{E/F}\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-1})\} \end{array}$$

$$\xRightarrow{\beta} \begin{array}{l} \{(s_1 - s_2, 3s_1 - 2s_2, \chi_1\chi_2^{-1} \circ N_{E/F}, \chi_1\chi_2^{-2}), -(-s_1 + s_2, s_2, \chi_1^{-1}\chi_2 \circ N_{E/F}, \chi_2)\}; \\ \{-(s_1, s_2, \chi_1, \chi_2), -(s_1, 3s_1 - s_2, \chi_1, \chi_1\chi_2^{-1})\} \end{array}$$

$$\xRightarrow{\alpha} \{-(s_1, s_2, \chi_1, \chi_2), -(-s_1 + s_2, s_2, \chi_1^{-1}\chi_2 \circ N_{E/F}, \chi_2)\}.$$

Whence $I(\chi)$ is irreducible. If χ is singular, as the singularity is given by $\langle w_\alpha \rangle$ or $\langle w_\beta \rangle$, we may obtain $I(\chi)$ is irreducible as well by the same argument.

$\underline{(1,2,1,1;\langle w_\alpha\rangle),\ (\#R = 4, w_\alpha)}$.

Claim. $I(\chi)$ is of length $2^{\#R/2} + 2$ and multiplicity at most 2, and the two subrepresentations are square-integrable.

Comparing Tables 3 and 4, we find that χ is singular. The Jacquet modules $r_\varnothing$ of the constituents are listed as follows. We write (s_1, s_2) for $(s_1, s_2, 1, 1)$ for simplicity.

The subrepresentation $\pi(1)$:

$$2(1,2),\ (1,1);$$

subrepresentation $\pi(1)'$:

$$(1,1);$$

subquotient $J_\alpha(\nu_E^{1/2}\delta(1) \otimes \nu_F^{-1})$ (multiplicity 2):

$$(0,1),\ (0,-1);$$

quotient $J_\beta(\nu_E^{-1} \otimes \nu_F^{3/2}\delta(1))$:

$$(-1,-1);$$

the Langlands quotient $J_\alpha(\nu_E I^\alpha(1 \otimes 1) \otimes \nu_F^{-2})$:

$$2(-1,-2),\ (-1,-1).$$

Proof. In $R(G)$,

$$\begin{aligned} I(1,2) = I(0,1) = I(1,1) &= I_\alpha(\nu_E^{1/2}\delta(1) \otimes \nu_F^{-1}) + I_\alpha(\nu_E^{1/2} 1_{GL_2} \otimes \nu_F^{-1}) \\ &= I_\beta(\nu_E^{-1} \otimes \nu_F^{3/2} 1_{GL_2}) + I_\beta(\nu_E^{-1} \otimes \nu_F^{3/2}\delta(1)). \end{aligned}$$

We write the semisimplification of Jacquet modules as follows:

$$r_\beta(I_\alpha(\nu_E^{1/2}\delta(1)\otimes\nu_F^{-1})) = 2\{(1,1)\} + 2\{(1,2)\} + \{(0,1),(0,-1)\},$$
$$r_\beta(I_\beta(\nu_E^{-1}\otimes\nu_F^{3/2}1_{GL_2})) = \{(1,1)\} + 2\{(-1,-2)\} + \{(-1,-1)\} + \{(0,1),(0,-1)\}.$$

It is easy to see

$$\pi(1)' := I_\alpha(\nu_E^{1/2}\delta(1)\otimes\nu_F^{-1}) \cap I_\beta(\nu_E^{-1}\otimes\nu_F^{3/2}1_{GL_2}) \neq \varnothing.$$

Notice that

$$J_\alpha(\nu_E^{1/2}\delta(1)\otimes\nu_F^{-1}) \hookrightarrow I_\alpha(\nu_E^{-1/2}\delta(1)\otimes\nu_F) \hookrightarrow I(0,-1);$$

it implies $r_\varnothing(\pi(1)') = (1,1)$.

Now consider

$$I(0,-1) \simeq I(0,1) = I_\alpha(\nu_E^{1/2}1_{GL_2}\otimes\nu_F^{-1}) + I_\alpha(\nu_E^{1/2}\delta(1)\otimes\nu_F^{-1}).$$

We write the semisimplification of Jacquet modules as follows:

$$r_\beta(I_\alpha(\nu_E^{-1/2}\delta(1)\otimes\nu_F)) = 2\{(1,1)\} + 2\{(1,2)\} + \{(0,1),(0,-1)\},$$
$$r_\beta(I_\alpha(\nu_E^{1/2}1_{GL_2}\otimes\nu_F^{-1})) = 2\{(-1,-2)\} + 2\{(-1,-1)\} + \{(0,1),(0,-1)\}.$$

It is easy to see

$$I_\alpha(\nu_E^{-1/2}\delta(1)\otimes\nu_F) \cap I_\alpha(\nu_E^{1/2}1_{GL_2}\otimes\nu_F^{-1}) \neq \varnothing$$

with the Jacquet module $\{(0,1),(0,-1)\}$.

Note also that under the Aubert duality,

$$r_\varnothing \circ D_G(\pi(1)') = (-1,-1).$$

Observe that

$$r_\beta \circ I_\beta(\nu_E^{-1}\otimes\nu_F^{3/2}\delta(1)) = 2\{(1,2)\} + \{(0,1),(0,-1)\} + \{(1,1)\} + \{(-1,-1)\}$$

and the possible Langlands quotients associated to $I(1,2)$ are

$$J_\beta(\nu_E^{-1}\otimes\nu_F^{3/2}\delta(1)),\ J_\alpha(\nu_E^{1/2}\delta(1)\otimes\nu_F^{-1}) \text{ and } J_\alpha(\nu_E I^\alpha(1\otimes 1)\otimes\nu_F^{-2}).$$

We may conclude that $D_G(\pi(1)')$ is of multiplicity 1 in $I(1,2)$. □

$(1,2,1,\chi_2;\langle w_\alpha\rangle;\chi_2\neq 1\ \&\ \chi_2\circ N_{E/F}=1)$, $(\#R=2, w_\alpha)$.

Claim. $I(\chi)$ is of length $2^{\#R/2}$ and multiplicity 1, and the subrepresentation is square-integrable and maps to the Langlands quotient under the Aubert duality.

Comparing Tables 3 and 4, we find that χ is singular. Based on the fact that $J_\alpha(\nu_E^{1/2}\delta(1)\otimes\chi_2\nu_F^{-1})$ is generic (cf. Lemma 1.1) and Rodier's heredity theorem [33, Theorem 2], one can readily reach the claim and the Jacquet modules $r_\varnothing$ of the constituents are listed as follows:
the subrepresentation $I_\alpha(\nu_E^{1/2}\delta(1)\otimes\chi_2\nu_F^{-1})$:

$$2(1,2,1,\chi_2), 2(1,1,1,\chi_2^{-1}), (0,1,1,\chi_2^2), (0,-1,1,\chi_2);$$

the Langlands quotient $J_\alpha(\nu_E I^\alpha(1\otimes 1)\otimes\chi_2^{-1}\nu_F^{-2})$:

$$-2(1,2,1,\chi_2), -2(1,1,1,\chi_2^{-1}), (0,1,1,\chi_2^2), (0,-1,1,\chi_2).$$

$(1,2,1,\chi_2;\langle w_\alpha\rangle;\chi_2\neq 1\ \&\ \chi_2^2=1),\ (\#R=2,\varnothing)$.

Claim. $I(\chi)$ is of length $2^{\#R}$ and multiplicity 1, and the subrepresentation is square-integrable and maps to the Langlands quotient under the Aubert duality.

Comparing Tables 3 and 4, we find that χ is regular. The Jacquet modules $r_\varnothing$ of the constituents are listed as follows:
the subrepresentation $\pi(\chi_2)$:

$$(1,2,1,\chi_2),\ (1,2,\chi_2\circ N_{E/F},\chi_2),\ (1,1,\chi_2\circ N_{E/F},1);$$

subquotient $J_\beta(\nu_E^{-1}\otimes\nu_F^{3/2}\delta(\chi_2))$:

$$(0,-1,\chi_2\circ N_{E/F},\chi_2^2),\ (0,1,\chi_2\circ N_{E/F},\chi_2),\ (-1,-1,1,\chi_2);$$

subquotient $J_\alpha(\nu_E^{1/2}\delta(\chi_2\circ N_{E/F})\otimes\nu_F^{-1})$:

$$-(0,-1,\chi_2\circ N_{E/F},\chi_2^2),\ -(0,1,\chi_2\circ N_{E/F},\chi_2),\ -(-1,-1,1,\chi_2);$$

the Langlands quotient $J_\alpha(\nu_E I^\alpha(1\otimes\chi_2\circ N_{E/F})\otimes\chi_2^{-1}\nu_F^{-2})$:

$$-(1,2,1,\chi_2),\ -(1,2,\chi_2\circ N_{E/F},\chi_2),\ -(1,1,\chi_2\circ N_{E/F},1).$$

$(1,2,\chi_1,\chi_2;\langle w_\alpha\rangle;\chi_2\neq 1\ \&\ \chi_1^2=1,\ \chi_1=\chi_2),\ (\#R=2,\varnothing)$.

Claim. $I(\chi)\simeq I(1,2,1,\chi_2;\chi_2^2=1)$.

As $I(1,2,1,\chi_2)$

$$\begin{aligned}&=I_\alpha(I^\alpha(\nu_E\otimes\nu_E\chi_2\circ N_{E/F})\otimes\chi_2\nu_F^{-2})\simeq I_\alpha(I^\alpha(\nu_E\chi_2\circ N_{E/F}\otimes\nu_E)\otimes\chi_2\nu_F^{-2})\\&=I(1,2,\chi_2\circ N_{E/F},\chi_2).\end{aligned}$$

$(1,2,\chi_1,1;\chi_1\neq 1\ \&\ \chi_1|_{F^\times}=1),\ (\#R=2,w_\alpha)$.

Claim. $I(\chi)\simeq I(1,2,\chi_1^{-1},1)$ is of length $2^{\#R}$ and multiplicity 1, and the subrepresentation is square-integrable and maps to the Langlands quotient under the Aubert duality.

Comparing Tables 3 and 4, we find that χ is regular. The Jacquet modules $r_\varnothing$ of the constituents are listed as follows. We write (s_1,s_2,μ) for $(s_1,s_2,\mu,1)$ for simplicity.

The subrepresentation $\pi(\chi)$:

$$(1,2,\chi_1),\ (1,2,\chi_1^{-1});$$

subquotient $J_\beta(\nu_E^{-1}\chi_1\otimes\nu_F^{3/2}\delta(1))$:

$$(1,1,\chi_1),\ (0,1,\chi_1^{-1}),\ (0,-1,\chi_1^{-1}),\ (-1,-1,\chi_1);$$

subquotient $J_\beta(\nu_E^{-1}\chi_1^{-1}\otimes\nu_F^{3/2}\delta(1))$:

$$-(1,1,\chi_1),\ -(0,1,\chi_1^{-1}),\ -(0,-1,\chi_1^{-1}),\ -(-1,-1,\chi_1);$$

the Langlands quotient $J_\alpha(\nu_E I^\alpha(\chi_1,\chi_1^{-1})\otimes\nu_F^{-2})$:

$$(-1,-2,\chi_1^{-1}),\ (-1,-2,\chi_1).$$

$(2/3,1,\chi_1,1;\langle w_\beta\rangle;\chi_1|_{F^\times}=1)$, $(\#R=2,w_\beta)$.

Claim. $I(\chi)$ is of length $2^{\#R/2}$ and multiplicity 1, and the subrepresentation maps to the Langlands quotient under the Aubert duality.

The above claim follows readily from the fact that $J_\beta(\nu_E^{-1/3}\chi_1\otimes\nu_F^{1/2}\delta(1))$ is generic (see Lemma 1.1) and Rodier's heredity theorem [33, Theorem 2]. The Jacquet modules $r_\varnothing$ of the constituents are listed as follows. We write (s_1,s_2) for $(s_1,s_2,1,1)$ for simplicity.

The subrepresentation $I_\beta(\nu_E^{-1/3}\chi_1\otimes\nu_F^{1/2}\delta(1))$:

$$2(2/3,1),\ 2(1/3,1),\ (1/3,0),\ (-1/3,0);$$

the Langlands quotient $J_\beta(\nu_E^{-2/3}\chi_1^{-1}\otimes\nu_F I^\beta(1\otimes 1))$:

$$2(-2/3,-1),\ 2(-1/3,-1),\ (1/3,0),\ (-1/3,0).$$

$(2,3,1,\chi_2;\langle w_\beta\rangle;\chi_2\circ N_{E/F}=1,\chi_2\neq 1)$, $(\#R=2,w_\beta)$.

Claim. $I(\chi)\simeq I(2,3,1,\chi_2^{-1})$ is of length $2^{\#R}$ and multiplicity 1, and the subrepresentation is square-integrable and maps to the Langlands quotient under the Aubert duality.

It is easy to see that such a χ is regular. The Jacquet modules $r_\varnothing$ of the constituents are listed as follows:
the subrepresentation $\pi(\chi_2)$:

$$(2,3,1,\chi_2),\ (2,3,1,\chi_2^{-1});$$

subquotient $J_\alpha(\nu_E^{3/2}\delta(1)\otimes\chi_2\nu_F^{-3})$:

$$(1,3,1,\chi_2),\ (1,0,1,\chi_2^{-1}),\ (-1,0,1,\chi_2^{-1}),\ (-1,-3,1,\chi_2);$$

subquotient $J_\alpha(\nu_E^{3/2}\delta(1)\otimes\chi_2^{-1}\nu_F^{-3})$:

$$-(1,3,1,\chi_2),\ -(1,0,1,\chi_2^{-1}),\ -(-1,0,1,\chi_2^{-1}),\ -(-1,-3,1,\chi_2);$$

the Langlands quotient $J_\beta(\nu_E^{-2}\otimes\nu_F^{3}I^\beta(\chi_2\otimes\chi_2^{-1}))$:

$$-(2,3,1,\chi_2),\ -(2,3,1,\chi_2^{-1}).$$

$(2,3,1,1;\langle w_\beta\rangle)$, $(\#R=2,w_\beta)$.

Claim. $I(\chi)$ is of length $2^{\#R/2}$ and multiplicity 1, and the subrepresentation maps to the Langlands quotient under the Aubert duality.

It is easy to see that such a χ is singular. In view of the fact that $J_\alpha(\nu_E^{3/2}\delta(1)\otimes\nu_F^{-3})$ is generic (cf. Lemma 1.1) and Rodier's heredity theorem [33, Theorem 2], it is easy to see the above claim holds, and the Jacquet modules $r_\varnothing$ of the constituents are listed as follows. We write (s_1,s_2) for $(s_1,s_2,1,1)$ for simplicity.

The subrepresentation $I_\alpha(\nu_E^{3/2}\delta(1)\otimes\chi_2^{-1}\nu_F^{-3})$:

$$2(2,3),\ (1,0),\ (1,3),\ (-1,0),\ (-1,-3);$$

the Langlands quotient $J_\beta(\nu_E^{-2}\otimes\nu_F^{3}I^\beta(\chi_2\otimes\chi_2^{-1}))$:

$$2(-2,-3),\ (1,0),\ (1,3),\ (-1,0),\ (-1,-3).$$

$(3,5,1,1), (\#R = 2)$.

Claim. $I(\chi)$ is of length $2^{\#R}$ and multiplicity 1, and the subrepresentation is square-integrable and maps to the Langlands quotient under the Aubert duality.

The Jacquet modules $r_{\varnothing}$ of the constituents of $I(\chi)$ are listed as follows. We write (s_1, s_2) for $(s_1, s_2, 1, 1)$ for simplicity.

The subrepresentation St_G:

$$(3,5);$$

subquotient $J_\beta(\nu_E^{-3} \otimes \nu_F^{9/2}\delta(1))$:

$$(2,5),\ (2,1),\ (1,-1),\ (-1,-4),\ (-3,-4);$$

subquotient $J_\alpha(\nu_E^{5/2}\delta(1) \otimes \nu_F^{-5})$:

$$(3,4),\ (1,4),\ (1,-1),\ (-2,-1),\ (-2,-5);$$

the Langlands quotient 1_G:

$$(-3,-5).$$

$\#R = 1$.

Claim. $I(\chi)$ is of length 2 and multiplicity 1, and the subrepresentation maps to the quotient under the Aubert duality.

Comparing Tables 3 and 4, we find that only

$$(1, 3/2, 1, \chi_2; \langle w_\beta \rangle; \chi_2^2 = 1) \textit{ and } (1/2, 1, \chi_1, 1; \langle w_\alpha \rangle; \chi_1^2 = 1)$$

are singular characters. The claim that $I(\chi)$ is of length 2 can be checked easily by diagram chasing. As for multiplicity 1, notice that the singularity given by $\langle w_\alpha \rangle$ or $\langle w_\beta \rangle$ is not the one giving rise to the rank 1 reducibility, so it is of multiplicity 1.

$\#R = 0,\ (\chi_1, \chi_2; \langle ? \rangle)|_{?^2=1}$.

Claim. $I(\chi)$ is irreducible except the case $? = w_\alpha w_{3\alpha+2\beta}$ which is reducible (see the paragraph below) and its constituents are invariant under the Aubert duality.

For $\langle w_\alpha w_{3\alpha+2\beta} \rangle$, if $I(\chi)$ is reducible, then it is of multiplicity 1, otherwise

$$\dim\ \mathrm{Hom}_G(I(\chi), I(\chi)) \le 2,$$

contradiction.

As for other cases, it is easy to verify that $I(\chi)$ is irreducible.

Other (χ_1, χ_2).

Claim. They are irreducible.

Note that the rank 1 groups are $GL_2(F) \times E^\times/\Delta F^\times$ and $GL_2(E) \times F^\times/\Delta E^\times$, so the associated Plancherel measures of unitary induced representations are the same as in $GL_2(F)$ and $GL_2(E)$, respectively. So by Keys' theorem [20, Theorem 1], the R-group can be described as

$$R = \{w \in W_\chi = \mathrm{Stab}_W(\chi) : \ \gamma > 0 \text{ and } \chi_\gamma := \chi \circ \gamma^\vee = 1 \text{ imply that } w.\gamma > 0\}.$$

Whence they are reducible unless χ_1 and χ_2 are different characters of order 2 which results from the same reason as in [20, Theorem G_2].

Remark 1. From the above computation for the case $(\#R = 2, m = 2)$, we know that $I(\chi)$ is of length 2 and multiplicity 1. Heuristically, this may be a general result for reductive groups based on the following strategy by a case-by-case check:

(i) Possible Jacquet module decomposition of $r_{\varnothing}(I(\chi))$:

$$\{\chi^w : w \in W_1 W_\chi\}, \quad \{\chi^w : w \in W_2\}, \quad \{\chi^w : w \in W_2\}, \quad \{\chi^w : w \in W_3 W_\chi\},$$

where W_i, $i = 1, 2, 3$, are subsets of the Weyl group W.

(ii) Genericity of the quotient π of the subrepresentation of $I(\chi)$ associated to $\{\chi^w : w \in W_2\}$. This may be checked using the Langlands–Shahidi theory.

(iii) Rodier's heredity theorem which implies that the generic subquotient of $I(\chi)$ is of multiplicity 1.

Note that once we know π is generic, the above assertion also follows from the standard module conjecture proved by Heiermann and Muić (cf. [15]).

Remark 2. The same decomposition pattern of the case $(\#R = 4, m = 2)$ also appears in Zelevinsky's work on GL_n (cf. [54, Example 11.4]). Idealistically, one should be able to guess and prove a formula for the case $m = 2$ for connected reductive groups.

Corollary 2.1. (i) *$I_\alpha(s, \delta(\chi_1) \otimes \chi_2)$ reduces if and only if*

$$s = \pm 1/2, \chi_1 = \chi_1 \circ N_{E/F}, \chi_2 = 1 \text{ or } s = \pm 3/2, \chi_1 = 1, \chi_2 \neq 1, \chi_2 \circ N_{E/F} = 1$$

or

$$s = \pm 5/2, \chi_1 = 1, \chi_2 = 1;$$

(ii) *$I_\beta(s, \chi_1 \otimes \delta(\chi_2))$ reduces if and only if*

$$s = \pm 3/2, \chi_1|_{F^\times} = 1, \chi_2 = 1 \quad \text{or} \quad s = \pm 3/2, \chi_1 = 1, \chi_2^2 = 1 \quad \text{or} \quad s = \pm 9/2, \chi_1 = 1, \chi_2 = 1.$$

Conclusion. In Tables 5, 6, and 7 we summarize our previous computation for later use.

Table 5. Regular $\#R = 1$

Regular $\#R = 1$	$I(s_1, s_2, \chi_1, \chi_2)$	
	subrepresentation	Langlands quotient
$2s_1 - s_2 = 1$ and $\chi_1^2 = \chi_2 \circ N_{E/F}$	$I_\alpha(s_1 - 1/2, \delta(\chi_1) \otimes \chi_2^{-1})$	$I_\alpha(s_1 - 1/2, \chi_1 \circ \det \otimes \chi_2^{-1})$
$2s_2 - 3s_1 = 1$ and $\chi_2^2 = \chi_1$	$I_\beta(s_2 - 1/2, \chi_1^{-1} \otimes \delta(\chi_2))$	$I_\beta(s_2 - 1/2, \chi_1^{-1} \otimes \chi_2 \circ \det)$
$s_2 - s_1 = 1$ and $\chi_1 = \chi_2 \circ N_{E/F}$	$I_\alpha(s_1 - 1/2, \delta(\chi_1) \otimes \chi_2^{-2})$	$I_\alpha(s_1 - 1/2, \chi_1 \circ \det \otimes \chi_2^{-2})$
$s_1 = 1$ and $\chi_1 = 1$	$I_\alpha(s_2 - 3/2, \delta(\chi_2 \circ N_{E/F}) \otimes \chi_2^{-2})$	$I_\alpha(s_2 - 3/2, (\chi_2 \circ N_{E/F}) \circ \det \otimes \chi_2^{-2})$
$3s_1 - s_2 = 1$ and $\chi_1 = \chi_2$	$I_\beta(s_2 - 1/2, \chi_1 \chi_2^{-1} \circ N_{E/F} \otimes \delta(\chi_2))$	$I_\beta(s_2 - 1/2, \chi_1 \chi_2^{-1} \circ N_{E/F} \otimes \chi_2 \circ \det)$
$s_2 = 1$ and $\chi_2 = 1$	$I_\beta(3s_1 - 3/2, \chi_1 \chi_1^{-1} \circ N_{E/F} \otimes \delta(\chi_1))$	$I_\beta(3s_1 - 3/2, \chi_1 \chi_1^{-1} \circ N_{E/F} \otimes \chi_1 \circ \det)$

Table 6. Regular $\#R = 2$

Regular $\#R = 2$	$I(s_1, s_2, \chi_1, \chi_2)$			
	subrepresentation		quotient	
	subrepresentation	quotient	subrepresentation	Langlands quotient
$(1, 2, \chi_1, 1; \chi_1\|_{F^\times} = 1)$	$\pi(\chi_1) \simeq \pi(\chi_1^{-1})$	$J_\beta(3/2, \chi_1 \otimes \delta(1))$	$J_\beta(3/2, \chi_1^{-1} \otimes \delta(1))$	$J_\alpha(1, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1)$
$(1, 2, 1, \chi_2; \chi_2^2 = 1)$	$\pi(\chi_2)$	$J_\alpha(1/2, \delta(\chi_2 \circ N_{E/F}) \otimes 1)$	$J_\beta(3/2, 1 \otimes \delta(\chi_2))$	$J_\alpha(1, I^\alpha(1, \chi_2 \circ N_{E/F}) \otimes \chi_2)$
$(1, 2, \chi_1, \chi_2; \chi_1^2 = 1, \chi_1 = \chi_2)$	$\simeq (1, 2, 1, \chi_2; \chi_2^2 = 1)$			
$(2, 3, 1, \chi_2; \chi_2 \circ N_{E/F} = 1)$	$\pi(\chi_2) \simeq \pi(\chi_2^{-1})$	$J_\alpha(3/2, \delta(1) \otimes \chi_2^{-1})$	$J_\alpha(3/2, \delta(1) \otimes \chi_2)$	$J_\beta(3, 1 \otimes I^\beta(\chi_2 \otimes \chi_2^{-1}))$
$(3, 5, 1, 1)$	St_{G_2}	$J_\beta(9/2, 1 \otimes \delta(1))$	$J_\alpha(5/2, \delta(1) \otimes 1)$	1_{G_2}

TABLE 7. Singular length 2

Singular $1 \leq \#R \leq 2$	$I(s_1, s_2, \chi_1, \chi_2)$	
	subrepresentation	Langlands quotient
$(1, 2, 1, \chi_2; \chi_2 \circ N_{E/F} = 1)$	$I_\alpha(1/2, \delta(1) \otimes \chi_2)$	$J_\alpha(1, I^\alpha(1 \otimes 1) \otimes \chi_2^{-1})$
$(2/3, 1, \chi_1, 1; \chi_1\|_{F^\times} = 1)$	$I_\beta(1/2, \chi_1 \otimes \delta(1))$	$J_\beta(1, \chi_1^{-1} \otimes I^\beta(1 \otimes 1))$
$(2, 3, 1, 1)$	$I_\alpha(3/2, \delta(1) \otimes 1)$	$J_\beta(3, , 1 \otimes I^\beta(1 \otimes 1))$
$(1, 3/2, 1, \chi_2; \chi_2^2 = 1)$	$I_\alpha(0, \delta(\chi_2 \circ N_{E/F}) \otimes 1)$	$J_\beta(3/2, 1 \otimes I^\beta(\chi_2 \otimes \chi_2))$
$(1/2, 1, \chi_1, 1; \chi_1^2 = 1)$	$I_\beta(0, 1 \otimes \delta(\chi_1))$	$J_\alpha(1/2, I^\alpha(\chi_1 \otimes \chi_1) \otimes 1)$

Remark 3. If E/F is a non-Galois cubic field extension, the previous Langlands classification almost holds. The only difference is that $N_{E/F}(E^\times) = F^\times$ (cf. Norm Limitation Theorem [30, Theorem 3.16]). That is to say $(2, 3, 1, \chi_2; \chi_2 \circ N_{E/F} = 1)$ and $(1, 2, 1, \chi_2; \chi_2 \circ N_{E/F} = 1)$ will not appear in Tables 6 and 7, respectively.

3. UNITARY DUAL

In this section, we would like to sort out the unitary dual from our previous Langlands classification for $PGSO_8^E$. To do so, we first classify the Hermitian dual which states: denote by Δ^G(resp., Δ^M) the set of simple positive roots in G (resp., M) and by A_M the split component of the center of M.

For $\nu \in (\mathfrak{a}_M^*)_+ := \{x \in X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} : (x, \alpha) > 0 \ \forall \alpha \in \Delta^G \backslash \Delta^M\}$ *and* σ *tempered, the Langlands quotient* $J_P(\sigma \otimes \nu)$ *is Hermitian if and only if there exists* $w \in W(G, A_M) := N_G(A_M)/C_G(A_M)$ *such that* $\sigma \simeq w.\sigma$ *and* $-\nu = w.\nu$.

Applying the above criterion of Hermitian dual to our group $PGSO_8^E$, we have: denote by σ an irreducible tempered representation of GL_2.

When χ_2 is unitary and $s > 0$, the Langlands quotient $J_\alpha(s, \sigma \otimes \chi_2)$ is Hermitian if and only if $w_{3\alpha+2\beta}.(\sigma \otimes \chi_2) \simeq \sigma \otimes \chi_2$, i.e.,

$$\sigma = \delta(\chi_1) \otimes 1|_{\chi_1^2=1} \text{ or } I^\alpha(\chi_1, \chi_1^{-1}\chi_2 \circ N_{E/F}) \otimes \chi_2|_{\chi_1^2=1, \chi_2^2=1} \text{ or } I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1.$$

When χ_1 is unitary and $s > 0$, the Langlands quotient $J_\beta(s, \chi_1 \otimes \sigma)$ is Hermitian if and only if $w_{2\alpha+\beta}.(\chi_1 \otimes \sigma) \simeq \chi_1 \otimes \sigma$, i.e.,

$$\sigma = 1 \otimes \delta(\chi_2)|_{\chi_2^2=1} \text{ or } \chi_1^{-1} \otimes I^\beta(\chi_2, \chi_2^{-1}\chi_1)|_{\chi_1^2=1, \chi_2^2=1} \text{ or } 1 \otimes I^\beta(\chi_2, \chi_2^{-1}).$$

When χ_1, χ_2 are unitary and $\frac{3}{2}s_2 < 3s_1 < 2s_2$, the Langlands quotient $J(s_1, s_2, \chi_1, \chi_2)$ is Hermitian if and only if $\chi_1^2 = 1$ and $\chi_2^2 = 1$.

For those Hermitian representations, we have the following associated reducibility conditions based on the classification result in Section 2. As the discrete case has been discussed in Corollary 2.1, here we only consider the tempered non-discrete case.

Lemma 3.1. *Keep the notions as before. For unitary characters* χ_1, χ_2, *and* $s > 0$, *we have*

(i) *For* $\chi_1^2 = 1$ *and* $\chi_2^2 = 1$, $I_\alpha(s, I^\alpha(\chi_1, \chi_1^{-1}\chi_2 \circ N_{E/F}) \otimes \chi_2)$ *reduces if and only if*

$$s = 1/2, \chi_2 = 1 \quad \textit{or} \quad s = 1, \chi_1|_{F^\times} = 1 \quad \textit{or} \quad s = 1, \chi_1 = \chi_2.$$

(ii) $I_\alpha(s, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1)$ *reduces if and only if*

$$s = 1/2 \quad \textit{or} \quad s = 1, \chi_1|_{F^\times} = 1.$$

(iii) *For* $\chi_1^2 = 1$ *and* $\chi_2^2 = 1$, $I_\beta(s, \chi_1^{-1} \otimes I^\beta(\chi_2, \chi_1\chi_2^{-1}))$ *reduces if and only if*

$$s = 3/2, \chi_1 = 1 \quad \text{or} \quad s = 1, \chi_1 = \chi_2 \text{ or } \chi_2 = 1 \quad \text{or} \quad s = 3, \chi_1 = \chi_2 \text{ or } \chi_2 = 1.$$

(iv) $I_\beta(s, 1 \otimes I^\beta(\chi_2, \chi_2^{-1}))$ $(\chi_2 \neq 1)$ *reduces if and only if*

$$s = \frac{3}{2} \quad \text{or} \quad s = 3, \chi_2 \circ N_{E/F} = 1.$$

In order to detect the unitarizability, we need to introduce another key input developed by Tadić and Speh, and summarized by Muić [31, Lemma 5.1]. For an F-parabolic subgroup $P = MN$ of G, we denote by the $\mathrm{Unr}(M)$ the group of unramified characters. For any irreducible representation σ of M and $\chi \in \mathrm{Unr}(M)$, denote $I(\chi, \sigma) = \mathrm{Ind}_P^G(\chi \otimes \sigma)$.

Lemma 3.2 ([31, Lemma 5.1])**.** *Under the above assumptions, we have*

(i) *The set of those* $\chi \in Unr(M)$*, such that* $I(\chi, \sigma)$ *has a unitarizable irreducible subquotient, is compact.*

(ii) *Let* $S \subset Unr(M)$ *be a connected set. Suppose that for all* $\chi \in S$*, the representation* $I(\chi, \sigma)$ *is an irreducible unitarizable representation. Then for* $\chi \in \bar{S}$ *the closure of* S*, any irreducible subquotient of* $I(\chi, \sigma)$ *is unitarizable.*

(iii) *Suppose that* σ *is Hermitian, and* $I(1, \sigma)$ *is irreducible and unitarizable. Then* σ *is unitarizable.*

Before proceeding to sort out the whole unitary dual, we first verify some special cases as follows.

Lemma 3.3. *Suppose that* χ_1, χ_2 *are quadratic unitary characters and* $s > 0$. *Then*

$I_\alpha(s, \chi_1 \circ \det \otimes 1)$ *is unitarizable (away from points of reducibility) if and only if* $s < 1/2$,

and

$I_\beta(s, 1 \otimes \chi_2 \circ \det)$ *is unitarizable (away from points of reducibility) if and only if* $s < 3/2$.

Proof. This follows from the same argument as in [31, Lemma 5.2]. □

Now let us turn to determining the unitary dual of $PGSO_8^E$. By Corollary 2.1 and Lemmas 3.1 and 3.2, we have the following.

Theorem 3.4 (**Unitary dual supported on** B**. I**)**.** *Keep the notation as before. For* χ_1, χ_2 *unitary characters of* $F^\times$ *and* $s > 0$*, we have*

(i) *For* $\chi_1^2 = 1$, $J_\alpha(s, \delta(\chi_1) \otimes 1)$ *is unitarizable if and only if* $s \leq 1/2$.

(ii) *For* $\chi_2^2 = 1$, $J_\beta(s, 1 \otimes \delta(\chi_2))$ *is unitarizable if and only if* $s \leq 3/2$.

(iii) $J_\alpha(s, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1)$ *is unitarizable if and only if* $s \leq 1/2$*, or* $\chi_1|_{F^\times} = 1$ *and* $s = 1$.

(iv) *For* $\chi_1^2 = 1$ *and* $\chi_2^2 = 1$ $(\chi_2 \neq 1)$, $J_\alpha(s, I^\alpha(\chi_1, -) \otimes \chi_2)$ *is unitarizable if and only if* $\chi_1 = 1$ *and* $s \leq 1$*, or* $\chi_1 = \chi_2$ *and* $s \leq 1$.

(v) *For* $\chi_1^2 = 1$ *and* $\chi_2^2 = 1$ $(\chi_1 \neq 1)$, $J_\beta(s, \chi_1 \otimes I^\beta(\chi_2, -))$ *is unitarizable if and only if* $\chi_2 = 1$ *and* $s \leq 1$*, or* $\chi_1 = \chi_2$ *and* $s \leq 1$.

(vi) $J_\beta(s, 1 \otimes I^\beta(\chi_2, \chi_2^{-1}))$ *is unitarizable if and only if* $s \leq \frac{3}{2}$*, or* $s = 3$ *provided* $\chi_2 \circ N_{E/F} = 1$ *and* $\chi_2 \neq 1$.

Proof. Following the standard procedure to construct families of positive definite Hermitian forms as in [31, Theorem 5.1], we have the following.

Proof of (i)(ii). It suffices to show $J_\alpha(5/2, \delta(\chi_1) \otimes 1)$ and $J_\beta(9/2, 1 \otimes \delta(\chi_2))$ are non-unitarizable which is well known (see [5, Chapter XI Theorem 4.5]).

Proof of (iii)(iv). It suffices to show

$$I_\alpha(s, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1) \text{ is non-unitarizable for some } s \in (1/2, 1)$$

which follows from the fact that $I_\alpha(s, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1) = I_\beta(\chi_1 \otimes I^\beta(\nu_F^s \otimes \nu_F^{-s}))$ and Lemma 3.2(iii), and

$$(\star) \qquad J_\alpha(1, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1) \text{ is unitarizable.}$$

If $\chi_1 = 1$, $(\star)$ follows from Lemma 3.3 and the fact that

$$I_\alpha(1/2, 1_{GL_2} \otimes 1) \twoheadrightarrow J_\alpha(1, I^\alpha(1,1) \otimes 1).$$

If $\chi_1 \neq 1$, $(\star)$ will be proved later on.

Proof of (v). It suffices to show

$$I_\beta(s, \chi_1^{-1} \otimes I^\beta(1,1)) \text{ and } I_\beta(s, \chi_1^{-1} \otimes I^\beta(\chi_2, 1))|_{\chi_1 = \chi_2} \text{ are non-unitarizable for some } s \in (1,3)$$

which results from the fact that they are isomorphic to $I_\alpha(I^\alpha(\nu_E^{\frac{1}{3}s}\chi_1 \otimes \nu_E^{-\frac{1}{3}s}) \otimes \chi_1^{-1})$ and Lemma 3.2(iii), and

$$J_\beta(2,3,\chi_1,1) \simeq I_\alpha(3/2, \chi_1 \circ \det \otimes 1) \text{ and } J_\beta(2,3,\chi_1,\chi_1) \simeq I_\alpha(3/2, \chi_1 \circ \det \otimes 1) \text{ are non-unitarizable}$$

which is considered in Lemma 3.3.

Proof of (vi). It suffices to show

$$I_\beta(\frac{2}{3}s, s, 1, \chi_2) \simeq I_\alpha(I^\alpha(\nu_E^{s/3} \otimes \nu_E^{-s/3}) \otimes \chi_2^{-1}) \text{ is non-unitarizable for some } s \in (3/2, 3)$$

which is known by Lemma 3.2(iii), and

$$I_\beta(\frac{2}{3}s, s, 1, 1) \simeq I_\alpha(I^\alpha(\nu_E^{\frac{1}{3}s} \otimes \nu_E^{-\frac{1}{3}s}) \otimes 1) \text{ is unitarizable for } s \in (1, 3/2)$$

which is also known by Lemma 3.2(iii), and

$$J_\beta(3, 1 \otimes I^\beta(1 \otimes 1)) \simeq I_\alpha(3/2, 1_{GL_2} \otimes 1) \text{ is non-unitarizable}$$

which is known by Lemma 3.3, and

$$J_\beta(3, 1 \otimes I^\beta(\chi_2, \chi_2^{-1})) \text{ is unitarizable provided } \chi_2 \circ N_{E/F} = 1 \text{ and } \chi_2 \neq 1$$

which will be proved later on.

□

Before heading to the last case of unitarizable non-tempered Langlands quotients supported on the minimal parabolic subgroup, we recall the associated reducibility conditions as usual in the following which results from the classification result in Section 2.

Lemma 3.5. *For quadratic unitary characters* χ_1, χ_2, *and* $(s_1, s_2) \in C^+$ *the positive Weyl chamber, i.e.,* $\frac{1}{2}s_2 < s_1 < \frac{2}{3}s_2$. *We know that* $I(s_1, s_2, \chi_1, \chi_2)$ *reduces if and only if* $(s_1, s_2, \chi_1, \chi_2)$ *is one of the following:*

$$(s_1, 1, \chi_1, 1; 1/2 < s_1 < 2/3),$$
$$(s_1, 2s_1 - 1, \chi_1, 1; s_1 > 2),$$
$$(1, s_2, 1, \chi_2; 3/2 < s_2 < 2),$$
$$(\frac{2s_2 - 1}{3}, s_2, 1, \chi_2; s_2 > 2),$$
$$(s_1, 3s_1 - 1, \chi_1, \chi_1; 2/3 < s_1 < 1),$$
$$(s_1, s_1 + 1, \chi_1, \chi_1; 1 < s_1 < 2).$$

Theorem 3.6 (**Unitary dual supported on** B**. II**)**.** *Suppose that* χ_1 *and* χ_2 *are unitary characters, and* s_1 *and* s_2 *satisfy the condition that* $\frac{1}{2}s_2 < s_1 < \frac{2}{3}s_2$. *Then* $J(s_1, s_2, \chi_1, \chi_2)$ *is unitarizable if and only if one of the following conditions holds:*

(i) $\chi_1 = 1$, $\chi_2 = 1$, *and* $s_2 \le 1$ *or* $3s_1 - s_2 \ge 1, s_2 - s_1 \le 1$, *or* $s_1 = 3, s_2 = 5$.
(ii) $\chi_1 = 1, \chi_2$ *is of order* 2, *and* $s_1 \le 1$.
(iii) $\chi_2 = 1, \chi_1$ *is of order* 2, *and* $s_2 \le 1$.
(iv) $\chi_1 = \chi_2$, χ_1 *is of order* 2, *and* $3s_1 - s_2 \le 1$.

Proof. By Lemma 3.5, we only have to discuss four cases as follows:

(i) $\underline{\chi_1 = 1,\ \chi_2 = 1}$: This results from the analysis in Figure 3 on the bounded domains partitioned by the reducibility lines case by case.

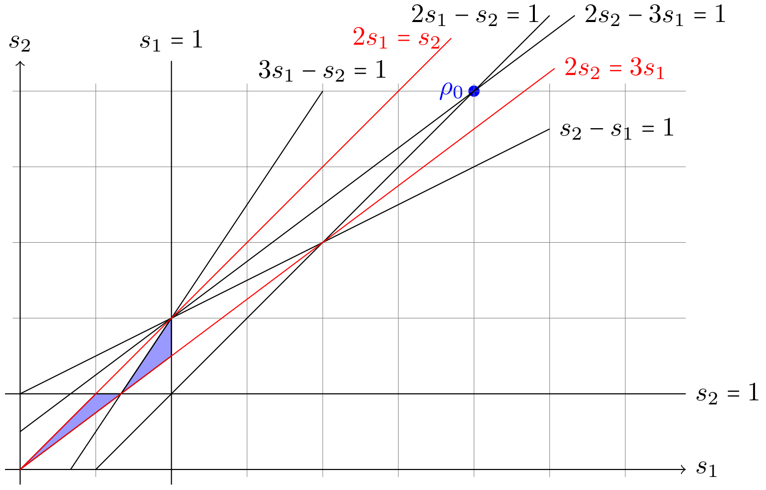

FIGURE 3. Spherical unitary dual with real infinitesimal character

(i1) $\underline{s_2 \le 1}$: This is because $I(1 \otimes 1)$ is unitarizable.
(i2) $\underline{s_2 > 1, 3s_1 - s_2 < 1:}$ As $I(s_1, 2s_1, 1, 1) \simeq I_\alpha(s_1, I^\alpha(1 \otimes 1) \otimes 1)$ is non-unitarizable for $s_1 \in (1/2, 1)$ by Theorem 3.4(iii).
(i3) $\underline{3s_1 - s_2 \ge 1, s_1 \le 1:}$ It suffices to prove the unitarizability of one of the representations $J(s_1, s_2, 1, 1)$ under the condition $3s_1 - s_2 > 1, s_1 < 1$

by Lemma 3.2(ii). The argument is the same as in the proof part (i3) of [31, Theorem 5.2] by replacing β by α. But for completeness, we write down the argument as follows.
Consider

$$X_{t,s} := I_\alpha(t, I^\alpha(\nu_E^s \otimes \nu_E^{-s}) \otimes 1) = I(t+s, 2t, 1, 1).$$

The idea is to show the existence of an irreducible unitarizable domain U of $X_{t,s}$ such that

$$(\star) \quad \{(t+s, 2t)^w : w \in W,\ (t,s) \in U\} \cap \{(s_1, s_2) \in C^+ : 3s_1 - s_2 > 1,\ s_1 < 1\} \neq \varnothing.$$

We first classify the reducibility lines of $X_{t,s}$ as follows:

$$s = \frac{1}{2}, \quad t \pm 3s = 1, \quad t \pm s = 1, \quad t = \frac{1}{2}.$$

Then we sort out an irreducible unitarizable domain U of $X_{t,s}$

$$U =: \{(t,s) : s \in (\frac{1}{3}, \frac{1}{2}),\ t \in (0, 1-s)\}.$$

It is quite easy to check that such a U satisfies the above requirement $(\star)$.

(i4) $\underline{s_1 > 1, s_2 - s_1 < 1}$: As $I(s_1, \frac{3}{2}s_1, 1, 1) \simeq I_\beta(\frac{3}{2}s_1, 1 \otimes I^\beta(1 \otimes 1))$ is non-unitarizable for $1 < s_1 < 2$ by Theorem 3.4(vi).

(i5) $\underline{s_2 - s_1 \geq 1, 2s_2 - 3s_1 < 1, 2s_1 - s_2 > 1}$: On the boundary $s_2 - s_1 = 1$ with $1 < s_1 < 2$, we know the non-unitarizability of $J(s_1, s_1+1, 1, 1)$ which follows from the fact that

$$J(s_1, s_1+1, 1, 1) \simeq I_\alpha(s_1 - 1/2, 1_{GL_2} \otimes 1)$$

is non-unitarizable for $1 < s_1 < 2$ by Lemma 3.3.

(i6) $\underline{2s_2 - 3s_1 = 1, 1 < s_1 < 3}$: As was known,

$$J\left(\frac{2s_2 - 1}{3}, s_2, 1, 1\right) \simeq I_\beta(s_2 - 1/2, 1 \otimes 1_{GL_2})$$

is non-unitarizable for $s_2 \in (2, 5)$ by Lemma 3.3.

(i7) $\underline{2s_1 - s_2 = 1, 2 < s_1 < 3}$: Similarly, this follows from the fact that

$$J(s_1, 2s_1 - 1, 1, 1) \simeq I_\alpha(s_1 - 1/2, 1_{GL_2} \otimes 1)$$

is non-unitarizable for $s_1 \in (2, 3)$ by Lemma 3.3.

(i8) $\underline{s_1 = 3, s_2 = 5}$: $J(3, 5, 1, 1) \simeq 1_G$ is a unitarizable representation.

(ii) $\underline{\chi_1 = 1, \chi_2 \text{ order } 2}$: This follows from the fact that there is only one connected bounded domain determined by the reducibility lines.

(iii) $\underline{\chi_2 = 1, \chi_1 \text{ order } 2}$: This follows from the fact that

$$J(s_1, 2s_1 - 1, \chi_1, 1) \simeq I_\alpha(s_1 - 1/2, \chi_1 \circ \det \otimes 1)$$

is non-unitarizable for $s_1 > 2$ by Lemma 3.3.

(iv) $\underline{\chi_1 = \chi_2,\ \chi_1 \text{ order } 2}$: This follows from the fact that

$$J(s_1, s_1+1, \chi_1, \chi_1) \simeq I_\alpha(s_1 - 1/2, \chi_1 \circ \det \otimes 1)$$

is non-unitarizable for $s_1 > 1$ by Lemma 3.3. ☐

Unitary dual supported on P_γ. Let K be a non-archimedean field of characteristic zero, and denote by W_K the associated Weil group of K. Let $\rho = \pi(\tau)$ be any supercuspidal representation of $GL_2(K)$, where

$$\tau : W_K \longrightarrow GL_2(\mathbb{C})$$

is an attached irreducible admissible homomorphism. Then det $\tau = \omega_\rho$ (the central character of ρ) via class field theory (see [41, Section 1] for the details).

Theorem 3.7 (**Unitary dual supported on P_γ**). *Suppose that ρ is a unitary supercuspidal representation of $GL_2(K)$ for $K = F$ or E. We have*

(i) *The Langlands quotient $J_\alpha(s, \rho \otimes \chi_2)$ provided $\omega_\rho \chi_2 \circ N_{E/F} = 1$ is unitarizable if and only if $\rho \simeq \tilde{\rho}$ (the contragredient) and one of the following conditions holds:*
  - $\chi_2 = 1$ *and* $0 < s \leq 1/2$.
  - $0 < s \leq 1$ *and* $\rho = \mathrm{Ind}_{W_{E^c}}^{W_E}(\chi_0)$ *provided* $\chi_0|_S = 1$ *and* $\chi_2 \circ N_{S/F} = 1$, *where E^c/F is a Galois extension of degree 6 and $S \subset E^c$ is the unique quadratic extension over F.*

(ii) *The Langlands quotient $J_\beta(s, \chi_1 \otimes \rho)$ provided $\omega_\rho \chi_1 = 1$ is unitarizable if and only if $\rho \simeq \tilde{\rho}$ and one of the following conditions is satisfied:*
  - $\chi_1 = 1$ *and* $0 < s \leq 1/2$.
  - $\mathrm{Im}(\tau) \simeq S_3$ *(the symmetric group) given by the non-Galois extension E over F, and* $0 < s \leq 1$.

If $I_\alpha(s_0, \rho \otimes \chi_2)$ (resp., $I_\beta(s_0, \chi_1 \otimes \rho)$), $s_0 > 0$, reduces, then it has a unique irreducible subrepresentation $\pi_\alpha(s_0, \rho \otimes \chi_1)$ (resp., $\pi_\beta(s_0, \chi_1 \otimes \rho)$). Those subrepresentations are square-integrable and different (s_0 is uniquely determined by the pair (ρ, χ_i)). If $I_\alpha(0, \rho \otimes \chi_2)$ (or $I_\beta(0, \chi_1 \otimes \rho)$) reduces, then it is of length 2 and multiplicity 1.

Proof. This follows from the L-factor computation in [22, 40] and the recent result of Henniart and Lomelí [16]. For $M_\alpha \simeq GL_2(E) \times F^\times/\Delta E^\times$, we have, using the standard notation as in [42],

$$L(s, \rho \otimes \chi_2, r_2) = L_F(s, \chi_2^{-1})$$

and

$$\begin{aligned} &L(s, \rho \otimes \chi_2, r_1) \\ &= L_F(s, \bigotimes\text{-}\mathrm{Ind}_{W_E}^{W_F}(\tau) \otimes \overset{-1}{\det}(\tau))(\bigotimes\text{-}\mathrm{Ind} \text{ twisted tensor induction [10, §6.1]}). \end{aligned}$$

In view of those and the poles of twisted local triple product L-function which is proved in the appendix (see also [18, Theorem 2.6]), part (i) holds. As for $M_\beta \simeq E^\times \times GL_2(F)/\Delta F^\times$, we have

$$L(s, \chi_1 \otimes \rho, r_2) = L_E(s, \chi_1^{-1}), \quad L(s, \chi_1 \otimes \rho, r_3) = L_F(s, \tau),$$

and

$$L(s, \chi_1 \otimes \rho, r_1) = L_E(s, \chi_1 \cdot \rho_E),$$

where ρ_E is the base change of ρ. In view of those, part (ii) holds. □

Remark 4. For the non-Galois cubic extension E/F case, there is a new family of unitary representations concerning part (i) of Theorem 3.7 under the conditions that

$0 < s \leq 1$ and $\tau|_{W_{E^c}} = \mathrm{Ind}_{W_L}^{W_{E^c}}(\chi_0)$ is irreducible, where L/F is a Galois extension with $\mathrm{Gal}(L/F) = D_{12}$ and E^c/F is the Galois closure of E/F, such that

- $\chi_0|_{S^\times} \cdot \chi_2 \circ N_{S/F} = 1$, where $S \subset L$ is the degree 4 extension over F.
- $\omega_\rho \cdot \chi_2 \circ N_{E/F} = 1$.
- $\omega_\rho \circ N_{E^c/E} = \chi_0|_{(E^c)^\times} \cdot \omega_{L/E^c}$, where ω_{L/E^c} is the quadratic character associated to L/E^c.

Note that J. Bernstein's unitarity conjecture says that the Aubert duality preserves unitarity. Back to our $PGSO_8^E$-setting, based on our computation, we have the following.

Corollary 3.8. *Keep the notation as before. The unitary dual is preserved under the Aubert duality.*

Note also that L. Clozel's finiteness conjecture (see [7] for the details) says that the set of exponents of discrete series is finite. Put in our setting, we have the following.

Corollary 3.9. *Keep the notions as before. Clozel's finiteness conjecture of special exponents holds for $PGSO_8^E$.*

Remark 5. Recently, under the assumption of the finiteness of special exponents for relative rank 1 cases, we have found a proof of Clozel's finiteness conjecture for general cases via Casselman–Tadić's Jacquet module machine (cf. [28]).

Unitarizability of $J_\alpha(1, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1)$ and $J_\beta(3, 1 \otimes I^\beta(\chi_2, \chi_2^{-1}))$. In what follows, we prove that $J_\alpha(1, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1)$ (resp., $J_\beta(3, 1 \otimes I^\beta(\chi_2, \chi_2^{-1}))$), where $\chi_1|_{F^\times} = 1$ and $\chi_1 \neq 1$ (resp., $\chi_2 \neq 1$ and $\chi_2 \circ N_{E/F} = 1$), is a unitary representation. Then it is an isolated point in the unitary dual of $PGSO_8^E$ by [46, Theorem 2.2]. The main idea is to show that they appear as components of some specific residual spectrum of G as in [21, 31, 53]. Let us start with some notation. For a global field $\dot{K}$, let $\mathbb{A}_{\dot{K}}$ be the ring of Adeles of $\dot{K}$. As in the local field case, given $\dot{E}$ a cubic field extension of a global field $\dot{F}$, we have an associated quasi-split adjoint group $G = PGSO_8^{\dot{E}}$ of type D_4. For grössencharacters μ_1 and μ_2 of $\dot{E}$ and $\dot{F}$, respectively, we define a unitary character $\chi = (\mu_1, \mu_2)$ of $T(\mathbb{A}_{\dot{F}})$ by $\chi(t(a,b)) = \mu_1(a)\mu_2(b)$. We take the coordinates in $\mathfrak{a}_{\mathbb{C}}^* = X^*(T) \otimes \mathbb{C}$ with respect to the basis α, β; the ordered pair $(s_1, s_2) \in \mathbb{C}^\times$ corresponds to the character $\lambda = 3s_1\alpha + s_2\beta$. For λ and χ as above, let $I_B(\lambda, \chi) = I_{T(\mathbb{A}_{\dot{F}})}^{G(\mathbb{A}_{\dot{F}})}(\lambda, \chi)$ be the space for the standard normalized induction (sometimes written as $I_B(\nu_{\dot{E}}^{s_1}\mu_1 \otimes \nu_{\dot{F}}^{s_2}\mu_2)$). Finally, let ρ_B be the half sum of positive roots, i.e., $\rho_B = 5\alpha + 3\beta$, and let C^+ be the positive Weyl chamber in $\mathfrak{a}_{\mathbb{C}}^*$:

$$C^+ = \{s_1\alpha + s_2\beta : \frac{3}{2}\mathrm{Re}(s_2) < \mathrm{Re}(s_1) < 2\mathrm{Re}(s_2)\}.$$

Following the standard procedure of investigating $L_d^2(B)$,

- (Eisenstein series) For $f \in I_B(\lambda, \chi)$, one forms Eisenstein series
$$E(g, f, \lambda) = \sum_{\gamma \in B(\dot{F}) \backslash G(\dot{F})} f(\gamma g)$$
which converges absolutely for $\mathrm{Re}\lambda \in C^+ + \rho_B$ and extends to a meromorphic function of λ. It is an automorphic form and its singularities coincide with

those of its constant term along B, i.e.,

$$E_0(g,f,\lambda)=\sum_{w\in W}(M(w,\lambda,\chi)f)(g), \tag{C}$$

where $M(w,\lambda,\chi)=\otimes_\nu M(w,\lambda,\chi_\nu)$ are the so-called non-normalized intertwining operators from $I_B(\lambda,\chi)$ to $I_B(w\lambda,w\chi)$.

- (Normalization) Let $\psi=\otimes_\nu\psi_\nu$ be a fixed non-trivial additive character of $\dot{F}\backslash\mathbb{A}_{\dot{F}}$. The standard normalization of the intertwining operators $M(w,\lambda,\chi)$ for all ν by factors (assume $\dot{E}/\dot{F}$ is Galois for simplicity),

$$r(w,\lambda,\chi_\nu)=\prod_{\substack{\{\gamma>0,w.\gamma<0\}\\ \gamma\text{ long}}}\frac{L(\langle\lambda,\gamma^\vee\rangle,\chi_\nu\circ\gamma^\vee)}{L(\langle\lambda,\gamma^\vee\rangle+1,\chi_\nu\circ\gamma^\vee)\epsilon(\langle\lambda,\gamma^\vee\rangle,\chi_\nu\circ\gamma^\vee,\psi_\nu)}$$
$$\times\prod_{\substack{\{\gamma>0,w.\gamma<0\}\\ \gamma\text{ short}}}\frac{L(\langle\lambda,\gamma^\vee\rangle/3,\chi_\nu\circ\gamma^\vee)}{L(\langle\lambda,\gamma^\vee\rangle/3+1,\chi_\nu\circ\gamma^\vee)\epsilon(\langle\lambda,\gamma^\vee\rangle/3,\chi_\nu\circ\gamma^\vee,\psi_\nu)}$$

are as follows:

$$N(w,\lambda,\chi_\nu)=r(w,\lambda,\chi_\nu)^{-1}M(w,\lambda,\chi_\nu)\text{ which are multiplicative.}$$

Let $N(w,\lambda,\chi)=\otimes_\nu N(w,\lambda,\chi_\nu)$. It is well known that

$$M(w,\lambda,\chi_\nu)\prod_{\substack{\{\gamma>0,w.\gamma<0\}\\ \gamma\text{ long}}}L(\langle\lambda,\gamma^\vee\rangle,\chi_\nu\circ\gamma^\vee)^{-1}\prod_{\substack{\{\gamma>0,w.\gamma<0\}\\ \gamma\text{ short}}}L(\langle\lambda,\gamma^\vee\rangle/3,\chi_\nu\circ\gamma^\vee)^{-1}$$

is holomorphic for all ν.

- (Singularities) The possible singularities of $M(w,\lambda,\chi)$ are rank 1 reducibility points as in Table 3, zeros of the denominator of $r(w,\lambda,\chi)=\prod_\nu r(w,\lambda,\chi_\nu)$, and poles of $N(w,\lambda,\chi)$. It is easy to see that only the point $3\alpha+\beta$ could provide a pole of $N(w,\lambda,\chi)$ as in [53].
- (Langlands square-integrable criterion) $\mathrm{Res}_{\lambda_0}\mathrm{Res}_{\langle\lambda,\gamma^\vee\rangle=1}E(g,f,\lambda)$ is square-integrable if and only if

$$\mathrm{Re}(w\lambda_0)\in\{-u\alpha-v\beta:u,v>0\}\text{ for all }w\in W_0,$$

where $W_0\subset W$ consists of those elements that give non-zero residue on the right-hand side of (C) which is non-canceled by residue of any other term.

$\underline{J_\beta(3,1\otimes I^\beta(\chi_2,\chi_2^{-1}))\text{ unitary}}$: This results from the following two lemmas as in [31, Theorem 6.2].

Lemma 3.10. *Let χ be a grössencharacter of $\dot{F}$ of order 3. Then the representation*

$$J_\beta(3,1\otimes I^\beta(\chi,\chi^{-1}))=J_\beta(3,1\otimes I^\beta(\chi^{-1},\chi))=\bigotimes_\nu J_\beta(3,1\otimes I^\beta(\chi_\nu,\chi_\nu^{-1}))$$

occurs in the residual spectrum of G.

Proof. This is to take residue at $\Lambda=6\alpha+3\beta$. It is easy to see that the point $\Lambda=6\alpha+3\beta$ only gives rise to simple poles arising from $r(w,\Lambda,\chi)$. So $W_0\subset W_{1,5}:=\{w\in W:w.\alpha<0,w.(\alpha+\beta)<0\}=\{w_{2\alpha+\beta},w_\beta w_{2\alpha+\beta}\}$. By the same argument as in [53, Case a) Residue at $\Lambda=2\alpha+\beta$], we know that $W_0=W_{1,5}$ and the residue of the constant term (C) produces $J_\beta(3,1\otimes I^\beta(\chi,\chi^{-1}))=\otimes_\nu J_\beta(3,1\otimes I^\beta(\chi_\nu,\chi_\nu^{-1}))$, whence the lemma holds. □

Lemma 3.11 ([2, Theorem 5]). *Let $\dot{K}$ be a global field, and let S be a finite set of places of $\dot{K}$. For $\nu \in S$, let χ_ν be a character of $\dot{K}_\nu^\times$ of order dividing $n \in \mathbb{N}$. Then there exists a character μ of $\dot{K}\backslash\mathbb{A}_{\dot{K}}^\times$ of order dividing $2n$, such that $\mu_\nu = \chi_\nu$ for $\nu \in S$.*

$\underline{J_\alpha(1, I^\alpha(\chi_1, \chi_1^{-1}) \otimes 1) \text{ unitary}}$: This results from the same argument as above.

Lemma 3.12. *Let χ be a grössencharacter of $\dot{E}$ such that $\chi|_{\mathbb{A}_{\dot{F}}^\times} = 1$. Then the representation*

$$J_\alpha(1, I^\alpha(\chi, \chi^{-1}) \otimes 1) = \bigotimes_\nu J_\alpha(1, I^\alpha(\chi_\nu, \chi_\nu^{-1}) \otimes 1)$$

occurs in the residual spectrum of G.

Proof. This is about taking residue at $3\alpha + 2\beta$. It is easy to see that the point $\Lambda = 3\alpha + 2\beta$ only gives rise to simple poles arising from $r(w, \Lambda, \chi)$. So $W_0 \subset W_{2,6} := \{w \in W : w.\beta < 0, w.(3\alpha + \beta) < 0\} = \{w_{3\alpha+2\beta}, w_\alpha w_{3\alpha+2\beta}\}$. By the same argument as in [53, Case a) Residue at $\Lambda = 2\alpha + \beta$], we know that $W_0 = W_{2,6}$ and the residue of the constant term (C) produces $J_\alpha(1, I^\alpha(\chi, \chi^{-1}) \otimes 1) = \otimes_\nu J_\alpha(1, I^\alpha(\chi_\nu, \chi_\nu^{-1}) \otimes 1)$, whence the lemma holds. □

Lemma 3.13. *Let $\dot{E}$ be a cubic extension of a global field $\dot{F}$, and let S be a finite set of places of $\dot{F}$. For $\nu \in S$, let χ_ν be a character of $\dot{E}_\nu^\times$ such that $\chi_\nu|_{\dot{F}_\nu^\times} = 1$. Then there exists a grössencharacter μ of $\dot{E}$, such that $\mu|_{\mathbb{A}_{\dot{F}}^\times} = 1$ and $\mu_\nu = \chi_\nu$ for $\nu \in S$.*

Proof. This follows from the Pontryagin duality and the fact that:

$$\prod_{\nu \in S} \dot{E}_\nu^\times / \dot{F}_\nu^\times \longrightarrow \mathbb{A}_{\dot{E}}^\times / \dot{E}^\times \mathbb{A}_{\dot{F}}^\times \text{ is continuous and injective.}$$

□

APPENDIX: POLES OF LOCAL TRIPLE PRODUCT L-FUNCTIONS

In this appendix, we will determine the poles of local triple product L-functions, which turns out to be the same as in the global case (treated by Ikeda in [18]), but the proof is of course completely different, since the local proof proceeds on the Galois side based on the recent work of Henniart and Lomelí [16].

Let us first consider the case when $E = F \times F \times F$. Hence, let ϕ_1, ϕ_2, $\phi_3 : W_F \to GL_2(\mathbb{C})$ be three irreducible representations (corresponding to supercuspidal representations of $GL_2(F)$). We are interested in determining if $(\phi_1 \otimes \phi_2 \otimes \phi_3)^{W_F} \neq 0$ and, equivalently, whether $\phi_1 \otimes \phi_2$ can contain an irreducible 2-dimensional summand.

Suppose that $\phi_1 \otimes \phi_2 = \rho_1 \oplus \rho_2$ with $dim(\rho_i) = 2$.

Claim. ϕ_1 and ϕ_2 must have the form $\phi_i = \mathrm{Ind}_{W_K}^{W_F}(\chi_i)$ for some quadratic field extension K/F (independent of i), i.e., ϕ_1 and ϕ_2 are dihedral w.r.t. K/F.

Before justifying the claim, we first recall the following possibilities for $\phi := \phi_i$:

(a) ϕ is not dihedral

$\Leftrightarrow \phi \bigotimes \chi \neq \phi$ for any quadratic character $\chi \neq 1$.

$\Leftrightarrow \phi|_{W_K}$ is irreducible for any quadratic extension K/F.

$\Leftrightarrow \mathrm{Sym}^2\phi = \wedge^2\phi \bigotimes \mathrm{Ad}(\phi)$ is irreducible.

(b) ϕ is dihedral w.r.t. a unique quadratic extension K/F

$\Leftrightarrow \phi \bigotimes \omega_{K/F} = \phi$, but $\phi \bigotimes \chi \neq \phi$ for any quadratic character $\chi \neq \omega_{K/F}$ or 1.

$\Leftrightarrow$ Ad(ϕ) contains $\omega_{K/F}$, but not other quadratic characters.

In this case, we may write $\phi = \mathrm{Ind}_{W_K}^{W_F}(\chi)$ for some character χ of W_K.

(c) ϕ is dihedral w.r.t. three quadratic extensions K_i of F, $i = 1, 2, 3$.

$\Leftrightarrow$ Ad(ϕ) is the sum of three quadratic characters χ_1, χ_2, χ_3, such that $\chi_1\chi_2\chi_3 = 1$.

In this case, we may write $\phi = \mathrm{Ind}_{W_{K_i}}^{W_F}(\chi_i)$ for each i.

Now to justify the claim, we consider $\wedge^2$ on both sides of the equation $\phi_1 \otimes \phi_2 = \rho_1 \oplus \rho_2$. This gives:

$$(\star\star) \qquad (\wedge^2\phi_1 \bigotimes \mathrm{Sym}^2\phi_2) \bigoplus (\mathrm{Sym}^2\phi_1 \bigotimes \wedge^2\phi_2) = \wedge^2\rho_1 \bigoplus \wedge^2\rho_2 \bigoplus \rho_1 \bigotimes \rho_2.$$

We now argue:

- At least one of ϕ_1, ϕ_2 is dihedral. If not, then the left-hand side of $(\star\star)$ is the sum of two 3-dimensional irreducible summands, whereas the right-hand side is not.
- If ϕ_1 is dihedral, say $\phi_1 = \mathrm{Ind}_{W_K}^{W_F}(\chi)$, but ϕ_2 is not dihedral. Then
$$\phi_1 \bigotimes \phi_2 = \mathrm{Ind}_{W_K}^{W_F}(\chi \cdot \phi_2|_{W_K}).$$
Since $\phi_2|_{W_K}$ is irreducible, $\phi_1 \otimes \phi_2$ is either irreducible, or a sum $\rho_1 \oplus \rho_2 = \rho \oplus \rho \cdot \omega_{K/F}$. Looking at $(\star\star)$, one sees that the left-hand side contains either one or three distinct 1-dimensional characters, whereas the right-hand side contains $\wedge^2\rho_1 = \wedge^2\rho_2$ with multiplicity ≥ 2.
- Thus both ϕ_1 and ϕ_2 are dihedral. If they are not dihedral w.r.t. the same K, then ϕ_1, ϕ_2 are as in case (b) above. Let $\phi_i = \mathrm{Ind}_{W_{K_i}}^{W_F}(\chi_i)$. Then $\phi_1 \otimes \phi_2 = \mathrm{Ind}_{W_{K_1}}^{W_F}(\chi_1 \cdot \phi_2|_{W_{K_1}})$ is either irreducible or the sum $\rho \oplus \rho \cdot \omega_{K/F}$. Looking at $(\star\star)$, we see that the left-hand side contains two distinct 1-dimensional characters, whereas the right-hand side contains $\wedge^2\rho_1 = \wedge^2\rho_2$ with multiplicity ≥ 2.
- We have thus shown that there exists a quadratic extension K/F such that $\phi_i = \mathrm{Ind}_{W_K}^{W_F}(\chi_i)$. Then
$$\phi_1 \bigotimes \phi_2 = \mathrm{Ind}_{W_K}^{W_F}(\chi_1\chi_2) \bigoplus \mathrm{Ind}_{W_K}^{W_F}(\chi_1\chi_2^\tau),$$
where $\mathrm{Gal}(K/F) = \langle \tau \rangle$. Hence if $\tilde{\phi}_2$ (the contragredient) is a summand of $\phi_1 \otimes \phi_2$, then $\tilde{\phi}_3$ is one of the two summands above, i.e., $\phi_3 = \mathrm{Ind}_{W_K}^{W_F}(\chi_1\chi_2)^{-1}$ (replacing χ_2 by χ_2^τ if necessary).

We have shown the following.

Proposition 3.14. *Let* ϕ_1, ϕ_2, ϕ_3 : $W_F \to GL_2(\mathbb{C})$ *be irreducible. Then* $(\phi_1 \otimes \phi_2 \otimes \phi_3)^{W_F} \neq 0$

$\Leftrightarrow$ *there exists quadratic extension* K/F *s.t.* $\phi_i = \mathrm{Ind}_{W_K}^{W_F}(\chi_i)$, *with* $\chi_i^\tau \neq \chi_i$ *and* $\chi_1\chi_2\chi_3 = 1$, *in which case, the quadratic extension* K/F *is uniquely determined by* ϕ_1, ϕ_2, ϕ_3 *via:*

$$\det(\phi_1) \cdot \det(\phi_2) \cdot \det(\phi_3) = \omega_{K/F}.$$

Proof. We have already shown the ($\Leftrightarrow$). It remains to prove the last assertion.

With $\phi_i = \mathrm{Ind}_{W_K}^{W_F}(\chi_i)$, $\chi_1\chi_2\chi_3 = 1$, one has $\det(\phi_i) = \chi_i|_{F^\times} \cdot \omega_{K/F}$. So

$$\det(\phi_1)\cdot\det(\phi_2)\cdot\det(\phi_3) = \chi_1\chi_2\chi_3|_{F^\times}\omega_{K/F}^3 = \omega_{K/F}.$$

□

Remark 6. As a consequence, we see that one cannot have ϕ_1 and ϕ_2 to be both dihedral w.r.t. the same three quadratic extensions K_1, K_2, K_3. This will contradict the uniqueness part of the proposition.

Now we consider the main case of interest where E/F is a cubic field extension.

$\underline{E/F}$ **Galois.** We first consider the case that E/F is a Galois extension. Suppose $\mathrm{Gal}(E/F) = \langle\sigma\rangle$ and let $\tilde\sigma \in W_F$ be an element which projects to σ under $W_F \twoheadrightarrow \mathrm{Gal}(E/F)$. Let $\phi\colon W_E \to GL_2(\mathbb{C})$ be an irreducible representation and set

$$\rho = \bigotimes\text{–}\mathrm{Ind}_{W_E}^{W_F}(\phi)$$

to be the tensor induction of ϕ from W_E to W_F (see [16, §2.1] for the notion of tensor induction), so that $dim(\rho) = 8$. We are interested in determining when $\rho^{W_F} \neq 0$.

Now $\rho^{W_F} \neq 0 \Rightarrow \rho^{W_E} \neq 0$. Since $\rho|_{W_E} = \phi \otimes \phi^\sigma \otimes \phi^{\sigma^2}$, our proposition shows that there exists a unique quadratic extension L/E such that

$$\phi = \mathrm{Ind}_{W_L}^{W_E}(\chi), \quad \phi^\sigma = \mathrm{Ind}_{W_L}^{W_E}(\chi'), \quad \phi^{\sigma^2} = \mathrm{Ind}_{W_L}^{W_E}(\chi'') \textit{ with } \chi\chi'\chi'' = 1.$$

Claim. L/F is a Galois extension.

Proof. It suffices to show that $\tilde\sigma(L) = L$. If not, then $L, \tilde\sigma(L), \tilde\sigma^2(L)$ are three distinct quadratic extensions of E. Moreover,

$$\phi^\sigma = \mathrm{Ind}_{W_L}^{W_E}(\chi') \Rightarrow \phi = \mathrm{Ind}_{W_{\tilde\sigma^2(L)}}^{W_E}(\tilde\sigma^2(\chi')),$$
$$\phi^{\sigma^2} = \mathrm{Ind}_{W_L}^{W_E}(\chi'') \Rightarrow \phi = \mathrm{Ind}_{W_{\tilde\sigma(L)}}^{W_E}(\tilde\sigma(\chi'')).$$

So ϕ is dihedral w.r.t. L, $\tilde\sigma(L)$, and $\tilde\sigma^2(L)$. A similar argument shows the same for ϕ^σ and ϕ^{σ^2}. This contradicts our earlier proposition, or rather the remark following it. So we must have $\tilde\sigma(L) = L$. □

As a consequence of the claim, $\mathrm{Gal}(L/F) = \langle c\rangle$ is a cyclic group of order 6, and we have:

$$\begin{tikzcd}
& L = K\cdot E \arrow[dl] \arrow[dr] & \\
K \arrow[dr, "2"'] & & E \arrow[dl, "3"] \\
& F &
\end{tikzcd}$$

with $\mathrm{Gal}(L/K) = \langle\tilde\sigma|_L\rangle = \langle c^2\rangle$ and $\mathrm{Gal}(L/E) = \langle\tau\rangle = \langle c^3\rangle$.

Now a short computation shows (see [18, Theorem 2.6]) the following.

Lemma 3.15.

$$\begin{aligned}\rho := & \bigotimes\text{-}\mathrm{Ind}_{W_E}^{W_F}(\mathrm{Ind}_{W_L}^{W_E}(\chi))\\ \simeq & \mathrm{Ind}_{W_K}^{W_F}(\chi|_{K^\times})\bigoplus \mathrm{Ind}_{W_L}^{W_F}(\chi^\tau\cdot\chi^{\tilde{\sigma}}\cdot\chi^{\tilde{\sigma}^2}),\end{aligned}$$

where we have regarded χ *as a character of* $L^\times$.

For $\rho^{W_F}\neq 0$, we need

$$\text{either}\quad \chi|_{K^\times}=1 \qquad \text{or}\quad \chi^\tau\chi^{\tilde{\sigma}}\chi^{\tilde{\sigma}^2}=1.$$

Let us show that the latter case is not possible. Indeed, if

$$1=\chi^\tau\chi^{\tilde{\sigma}}\chi^{\tilde{\sigma}^2}=\chi^{c^3}\chi^{c^2}\chi^{c^4},$$

then applying c gives:

$$1=\chi^{c^4}\chi^{c^3}\chi^{c^5}.$$

Comparing the two equations gives:

$$\chi^{c^5}=\chi^{c^2},\ \text{i.e.,}\ \chi^{c^3}=\chi,\ \text{i.e.,}\ \chi^\tau=\chi.$$

But $\chi^\tau\neq\chi$ since ϕ is irreducible.

Hence we have shown the following.

Theorem 3.16 (E/F Galois)**.** *Let* $\phi:\ W_E\to GL_2(\mathbb{C})$ *be irreducible. Then* $\rho:=\otimes\text{-}\mathrm{Ind}_{W_E}^{W_F}(\phi)$ *contains the trivial character*

$\Leftrightarrow$ *there exists a quadratic extension* K/F *and a character* χ *of* $L^\times=(K\cdot E)^\times$ *s.t.* $\phi=\mathrm{Ind}_{W_L}^{W_E}(\chi)$ *and* $\chi|_{K^\times}=1$,

in which case,

$$\rho\simeq\mathrm{Ind}_{W_K}^{W_F}(1)\bigoplus\mathrm{Ind}_{W_L}^{W_F}(\chi^\tau\chi^{-1})$$

and K *is uniquely determined by*

$$\omega_{K/F}=\omega_{L/E}|_{F^\times}=\det(\phi)|_{F^\times}.$$

E/F non-Galois. Now we turn to the non-Galois case. Let E^c/F be the Galois closure of E/F with

$$\mathrm{Gal}(E^c/F)=S_3:=\langle\tau,\sigma|\tau^2=\sigma^3=1,\tau\sigma\tau=\sigma^{-1}\rangle.$$

We have two cases:

(i) $\phi|_{W_{E^c}}$ reducible: Similar argument as above shows that this gives rise to the same condition as in Theorem 3.16 for $\rho^{W_F}\neq 0$.

(ii) $\phi|_{W_{E^c}}$ irreducible: Similar argument as in the Galois case, $\rho^{W_{E^c}}\neq 0$ implies that there exists a unique quadratic extension L/E^c such that

$$\phi|_{W_{E^c}}=\mathrm{Ind}_{W_L}^{W_{E^c}}(\chi),\quad \phi^\sigma|_{W_{E^c}}=\mathrm{Ind}_{W_L}^{W_{E^c}}(\chi'),\quad \phi^{\sigma^2}|_{W_{E^c}}=\mathrm{Ind}_{W_L}^{W_{E^c}}(\chi'')\ \text{with}\ \chi\chi'\chi''=1.$$

Suppose $\mathrm{Gal}(L/E^c)=\langle\tau'\rangle$. As $\phi|_{W_{E^c}}$ is irreducible, so

$$\mathrm{Ind}_{W_L}^{W_{E^c}}(\chi)\simeq\mathrm{Ind}_{W_{L^\tau}}^{W_{E^c}}(\chi^\tau),$$

which in turn implies that

$$L^\tau=L,\quad\text{and}\quad \chi=\chi^\tau\ \text{or}\ \chi^{\tau'}=\chi^\tau.$$

This is to say L/E is a Galois extension and

$$\mathrm{Gal}(L/E)\simeq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}.$$

Further applying Proposition 3.14, we know that $L^\sigma = L$, which in turn implies that L/F is a Galois extension with

$$\mathrm{Gal}(L/F) \simeq D_{12} := \langle \tau, \sigma_0 | \tau^2 = \sigma_0^6 = 1, \tau\sigma_0\tau = \sigma_0^{-1} \rangle,$$

and we have:

$$\begin{tikzcd}
& L \arrow[dl, "3"'] \arrow[d, "2"] \\
K \arrow[dr, "4"'] & E^c \arrow[d, "6"] \\
& F
\end{tikzcd}$$

with $\mathrm{Gal}(L/K) = \langle \sigma_0^2 \rangle$ and $\mathrm{Gal}(K/F) = \langle \tau, \sigma_0^3 \rangle$.

Now a short computation shows that

$$\rho|_{W_K} = \chi\chi^{\sigma_0^2}\chi^{\sigma_0^4} + \chi^{\sigma_0}\chi^{\sigma_0^3}\chi^{\sigma_0^5} + \mathrm{Ind}_{W_L}^{W_K}(\chi\chi^{\sigma_0}\chi^{\sigma_0^2}) + \mathrm{Ind}_{W_L}^{W_K}(\chi\chi^{\sigma_0}\chi^{\sigma_0^5}).$$

So $\rho^{W_K} \neq 0$ implies that, applying the same argument as in Lemma 3.15,

$$\chi\chi^{\sigma_0^2}\chi^{\sigma_0^4}|_{W_K} = 1, \text{ i.e., } \chi|_{K^\times} = 1.$$

On the other hand, given $\chi|_{K^\times} = 1$, an easy calculation shows that $\rho^{W_F} \neq 0$. Thus we obtain the following.

Theorem 3.17 (E/F non-Galois)**.** *Let $\phi: W_E \to GL_2(\mathbb{C})$ be irreducible. Denote by E^c/F the Galois closure of E/F. Then $\rho := \bigotimes\text{-}\mathrm{Ind}_{W_E}^{W_F}(\phi)$ contains the trivial character if and only if one of the following conditions holds:*

(i) *There exists a character χ of $(E^c)^\times$, such that $\phi = \mathrm{Ind}_{W_{E^c}}^{W_E}(\chi)$ and $\chi|_{K^\times} = 1$. Here K/F is the unique intermediate quadratic extension, in which case,*

$$\rho \simeq \mathrm{Ind}_{W_K}^{W_F}(1) \bigoplus \mathrm{Ind}_{W_L}^{W_F}(\chi^\tau\chi^{-1}).$$

(ii) *There exists a quadratic extension L/E^c and a character χ of $L^\times$, such that Gal$(L/F) = D_{12}$, $\phi|_{W_{E^c}} = \mathrm{Ind}_{W_L}^{W_{E^c}}(\chi)$ is irreducible and $\chi|_{K^\times} = 1$. Here K/F is the unique quartic intermediate extension.*

Remark 7. As pointed out by Professor T. Ikeda, part (ii) of Theorem 3.17 is indeed a dihedral case given as follows.

As $\mathrm{Gal}(L/E) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we have the following diagram:

$$\begin{tikzcd}
& L \arrow[dl, "\langle\sigma_1\sigma_2\rangle"'] \arrow[d, "\langle\sigma_1\rangle"] \arrow[dr, "\langle\sigma_2\rangle"] & \\
M_2 \arrow[dr] & E^c \arrow[d] & M_1 \arrow[dl] \\
& E &
\end{tikzcd}$$

The point is to show that

$$\text{one of } \mathrm{Ind}_{W_L}^{W_{M_i}}(\chi),\ i = 1, 2, \text{ is irreducible.}$$

Otherwise,

$$\text{(A)} \qquad \mathrm{Ind}_{W_L}^{W_{M_i}}(\chi) \text{ is irreducible for } i = 1, 2.$$

That is to say

$$\chi \neq \chi^{\sigma_2} \text{ and } \chi \neq \chi^{\sigma_1\sigma_2}.$$

Note that

$$\text{(B)} \qquad \phi|_{W_L} = \chi + \chi^{\sigma_1} \text{ with } \chi \neq \chi^{\sigma_1} \text{ (as } \phi|_{W_{E^c}} = \mathrm{Ind}_{W_L}^{W_{E^c}}(\chi) \text{ is irreducible).}$$

Therefore

$$\begin{aligned} 0 \neq \mathrm{Hom}_{W_L}(\chi, \phi) =& \mathrm{Hom}_{W_E}(\mathrm{Ind}_{W_L}^{W_E}(\chi), \phi) \\ =& \mathrm{Hom}_{W_{M_i}}(\mathrm{Ind}_{W_L}^{W_{M_i}}(\chi), \phi). \end{aligned}$$

Thus (A) implies

$$\phi|_{W_{M_i}} = \mathrm{Ind}_{W_L}^{W_{M_i}}(\chi) \text{ for } i = 1, 2.$$

This in turn says that

$$\phi|_{W_L} = \chi + \chi^{\sigma_2} = \chi + \chi^{\sigma_1\sigma_2} \overset{(B)}{=} \chi + \chi^{\sigma_1}.$$

Contradiction.

Acknowledgments

The author is much indebted to Professor Wee Teck Gan for his constant help and support, and useful discussions on various topics. The author would also like to thank Professor Tamotsu Ikeda for discussions on the poles of local triple product L-functions during a conference at IMUS, Seville, Spain. Thanks are also due to the referee for detailed comments.

References

[1] James Arthur, *The endoscopic classification of representations: Orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. MR3135650

[2] E. Artin and J. Tate, *Class field theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968. MR0223335

[3] Anne-Marie Aubert, *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p-adique* (French, with English summary), Trans. Amer. Math. Soc. **347** (1995), no. 6, 2179–2189, DOI 10.2307/2154931. MR1285969

[4] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive $\mathfrak{p}$-adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472. MR579172

[5] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, 2nd ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000. MR1721403

[6] W. Casselman, *Introduction to the theory of admissible representations of reductive p-adic groups*, Preprint, 1995.

[7] Laurent Clozel, *Sur une conjecture de Howe. I* (English, with French summary), Compositio Math. **56** (1985), no. 1, 87–110. MR806844

[8] Wee Teck Gan, Nadya Gurevich, and Dihua Jiang, *Cubic unipotent Arthur parameters and multiplicities of square integrable automorphic forms*, Invent. Math. **149** (2002), no. 2, 225–265, DOI 10.1007/s002220200210. MR1918673

[9] Wee Teck Gan and Joseph Hundley, *The spin L-function of quasi-split D_4*, IMRP Int. Math. Res. Pap. (2006), Art. ID 68213, 74. MR2268487

[10] Wee Teck Gan and Gordan Savin, *Twisted Bhargava cubes*, Algebra Number Theory **8** (2014), no. 8, 1913–1957, DOI 10.2140/ant.2014.8.1913. MR3285619

[11] David Goldberg, *Reducibility of induced representations for* Sp(2n) *and* SO(n), Amer. J. Math. **116** (1994), no. 5, 1101–1151, DOI 10.2307/2374942. MR1296726

[12] Marcela Hanzer, *The unitary dual of the Hermitian quaternionic group of split rank 2*, Pacific J. Math. **226** (2006), no. 2, 353–388, DOI 10.2140/pjm.2006.226.353. MR2247868

[13] Marcela Hanzer, *Unitary dual of the non-split inner form of* $\mathrm{Sp}(8, F)$, Trans. Amer. Math. Soc. **360** (2008), no. 2, 1005–1034, DOI 10.1090/S0002-9947-07-04286-9. MR2346481

[14] Marcela Hanzer and Ivan Matić, *The unitary dual of p-adic* $\widetilde{\mathrm{Sp}(2)}$, Pacific J. Math. **248** (2010), no. 1, 107–137, DOI 10.2140/pjm.2010.248.107. MR2734167

[15] V. Heiermann and G. Muić, *On the standard modules conjecture*, Math. Z. **255** (2007), no. 4, 847–853, DOI 10.1007/s00209-006-0052-9. MR2274538

[16] G. Henniart and L. Lomelí, *Asai cube L-functions and the local Langlands conjecture*, arXiv preprint `arXiv:1701.01516`, (2017).

[17] Kaoru Hiraga, *On functoriality of Zelevinski involutions*, Compos. Math. **140** (2004), no. 6, 1625–1656, DOI 10.1112/S0010437X04000892. MR2098406

[18] Tamotsu Ikeda, *On the location of poles of the triple L-functions*, Compositio Math. **83** (1992), no. 2, 187–237. MR1174424

[19] Chris Jantzen, *Degenerate principal series for symplectic and odd-orthogonal groups*, Mem. Amer. Math. Soc. **124** (1996), no. 590, viii+100, DOI 10.1090/memo/0590. MR1346929

[20] Charles David Keys, *On the decomposition of reducible principal series representations of p-adic Chevalley groups*, Pacific J. Math. **101** (1982), no. 2, 351–388. MR675406

[21] Henry H. Kim, *The residual spectrum of* G_2, Canad. J. Math. **48** (1996), no. 6, 1245–1272, DOI 10.4153/CJM-1996-066-3. MR1426903

[22] Henry H. Kim and Wook Kim, *On local L-functions and normalized intertwining operators II; quasi-split groups*, On certain L-functions, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 265–295. MR2767519

[23] A. W. Knapp and E. M. Stein, *Intertwining operators for semisimple groups*, Ann. of Math. (2) **93** (1971), 489–578, DOI 10.2307/1970887. MR460543

[24] Anthony W. Knapp and David A. Vogan Jr., *Cohomological induction and unitary representations*, Princeton Mathematical Series, vol. 45, Princeton University Press, Princeton, NJ, 1995. MR1330919

[25] Kazuko Konno, *Induced representations of* $\mathrm{U}(2, 2)$ *over a p-adic field*, J. Reine Angew. Math. **540** (2001), 167–204, DOI 10.1515/crll.2001.083. MR1868602

[26] C. Luo, *Muller type irreducibility criterion for generalized principal series*, `arXiv:1903.07278`, (2019).

[27] C. Luo. *Rodier type theorem for regular generalized principal series*, `arXiv:1903.06887`, (2019).

[28] C. Luo, *Universal hierarchical structure of reducibility of Harish-Chandra parabolic induction*, `arXiv:1903.09774`, (2019).

[29] Ivan Matić, *The unitary dual of p-adic* $\mathrm{SO}(5)$, Proc. Amer. Math. Soc. **138** (2010), no. 2, 759–767, DOI 10.1090/S0002-9939-09-10065-5. MR2557193

[30] J. S. Milne, *Class field theory*, lecture notes available at `http://www.math.lsa.umich.edu/jmilne`, 1997.

[31] Goran Muić, *The unitary dual of p-adic* G_2, Duke Math. J. **90** (1997), no. 3, 465–493, DOI 10.1215/S0012-7094-97-09012-8. MR1480543

[32] I. Muller, *Intégrales d'entrelacement pour un groupe de Chevalley sur un corps p-adique* (French), Analyse harmonique sur les groupes de Lie (Sém., Nancy-Strasbourg 1976–1978), II, Lecture Notes in Math., vol. 739, Springer, Berlin, 1979, pp. 367–403. MR560847

[33] François Rodier, *Whittaker models for admissible representations of reductive p-adic split groups*, Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), Amer. Math. Soc., Providence, R.I., 1973, pp. 425–430. MR0354942

[34] François Rodier, *Décomposition de la série principale des groupes réductifs p-adiques* (French), Noncommutative harmonic analysis and Lie groups (Marseille, 1980), Lecture Notes in Math., vol. 880, Springer, Berlin-New York, 1981, pp. 408–424. MR644842

[35] Paul J. Sally Jr. and Marko Tadić, *Induced representations and classifications for* $\mathrm{GSp}(2, F)$ *and* $\mathrm{Sp}(2, F)$ (English, with English and French summaries), Mém. Soc. Math. France (N.S.) **52** (1993), 75–133. MR1212952

[36] Peter Schneider and Ulrich Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. **85** (1997), 97–191. MR1471867

[37] C. Schoemann, *Unitary Dual of p-adic U(5)*, arXiv preprint `arXiv:1412.2924`, (2014).

[38] Avner Segal, *The degenerate Eisenstein series attached to the Heisenberg parabolic subgroups of quasi-split forms of* $Spin_8$, Trans. Amer. Math. Soc. **370** (2018), no. 8, 5983–6039, DOI 10.1090/tran/7293. MR3803152

[39] Freydoon Shahidi, *On certain L-functions*, Amer. J. Math. **103** (1981), no. 2, 297–355, DOI 10.2307/2374219. MR610479

[40] Freydoon Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Ann. of Math. (2) **127** (1988), no. 3, 547–584, DOI 10.2307/2007005. MR942520

[41] Freydoon Shahidi, *Third symmetric power L-functions for* GL(2), Compositio Math. **70** (1989), no. 3, 245–273. MR1002045

[42] Freydoon Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2) **132** (1990), no. 2, 273–330, DOI 10.2307/1971524. MR1070599

[43] Freydoon Shahidi, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J. **66** (1992), no. 1, 1–41, DOI 10.1215/S0012-7094-92-06601-4. MR1159430

[44] Allan J. Silberger, *Introduction to harmonic analysis on reductive p-adic groups*, Mathematical Notes, vol. 23, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979. Based on lectures by Harish-Chandra at the Institute for Advanced Study, 1971–1973. MR544991

[45] M. Tadić, *Harmonic analysis of spherical functions on reductive groups over* $\mathfrak{p}$*-adic fields*, Pacific J. Math. **109** (1983), no. 1, 215–235. MR716298

[46] Marko Tadić, *Geometry of dual spaces of reductive groups (non-Archimedean case)*, J. Analyse Math. **51** (1988), 139–181, DOI 10.1007/BF02791122. MR963153

[47] Marko Tadić, *Representations of p-adic symplectic groups*, Compositio Math. **90** (1994), no. 2, 123–181. MR1266251

[48] M. Tadić, *Unitarizability in generalized rank three for classical p-adic groups*, arXiv preprint `arXiv:1709.00630`, (2017).

[49] David A. Vogan Jr., *The unitary dual of* GL(n) *over an Archimedean field*, Invent. Math. **83** (1986), no. 3, 449–505, DOI 10.1007/BF01394418. MR827363

[50] David A. Vogan Jr., *The unitary dual of* G_2, Invent. Math. **116** (1994), no. 1-3, 677–791, DOI 10.1007/BF01231578. MR1253210

[51] David A. Vogan Jr., *A Langlands classification for unitary representations*, Analysis on homogeneous spaces and representation theory of Lie groups, Okayama–Kyoto (1997), Adv. Stud. Pure Math., vol. 26, Math. Soc. Japan, Tokyo, 2000, pp. 299–324, DOI 10.2969/aspm/02610299. MR1770725

[52] Norman Winarsky, *Reducibility of principal series representations of p-adic Chevalley groups*, Amer. J. Math. **100** (1978), no. 5, 941–956, DOI 10.2307/2373955. MR517138

[53] Siniša Žampera, *The residual spectrum of the group of type* G_2 (English, with English and French summaries), J. Math. Pures Appl. (9) **76** (1997), no. 9, 805–835, DOI 10.1016/S0021-7824(97)89971-0. MR1485422

[54] A. V. Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*-adic groups. II. On irreducible representations of* GL(n), Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210. MR584084

Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, Chalmers Tvärgata 3, SE-412 96 Göteborg, Sweden

Email address: `caihua@chalmers.se`