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SPINORIALITY OF ORTHOGONAL REPRESENTATIONS OF REDUCTIVE GROUPS

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Abstract. Let G be a connected reductive group over a field F of characteristic 0, and $\varphi : G \to \mathrm{SO}(V)$ an orthogonal representation over F. We give criteria to determine when φ lifts to the double cover $\mathrm{Spin}(V)$.

Contents

1. Introduction

Let G be a connected reductive group over a field F of characteristic 0. Let (φ, V) be a representation of G, which in this paper always means a finite-dimensional F-representation of G. Suppose that V is orthogonal, i.e., carries a symmetric nondegenerate bilinear form preserved by φ. Thus φ is a morphism from G to $\mathrm{SO}(V)$. Write $\rho : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ for the usual isogeny ([SV00]). Following [Bou05], we say that φ is *spinorial* when it lifts to $\mathrm{Spin}(V)$, i.e., provided there

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exists a morphism $\hat{\varphi} : G \to \mathrm{Spin}(V)$ so that $\varphi = \rho \circ \hat{\varphi}$. We call φ *aspinorial* otherwise.

By an argument in Section 14, we may assume that F is algebraically closed, which we do for the rest of this introduction. Let T be a maximal torus of G. Write $\pi_1(G)$ for the fundamental group of G (the cocharacter group of T modulo the subgroup $Q(T)$ generated by coroots), and T_V for a maximal torus of $\mathrm{SO}(V)$ containing $\varphi(T)$. Then φ induces a homomorphism $\varphi_* : \pi_1(G) \to \pi_1(\mathrm{SO}(V)) \cong \mathbb{Z}/2\mathbb{Z}$, and φ is spinorial iff φ_* is trivial. If we take a set of cocharacters $\underline{\nu} = \{\nu_1, \ldots, \nu_r\}$ whose images generate $\pi_1(G)$, then φ is spinorial iff each cocharacter $\varphi_* \nu_i$ of T_V lifts to $\mathrm{Spin}(V)$. (See Section 3.)

Write $\mathfrak{g}$ for the Lie algebra of G, and $X^*(T)$ for the character group of T. Suppose (φ, V) is an orthogonal representation of G. Write C for the Casimir element associated to the Killing form. Given a cocharacter ν of T, put

$$|\nu|^2 = \sum_{\alpha \in R} \langle \alpha, \nu \rangle^2 \in 2\mathbb{Z}.$$

We introduce the integer

$$p(\underline{\nu}) = \frac{1}{2} \gcd\left(|\nu_1|^2, \ldots, |\nu_r|^2\right).$$

Theorem 1. *Suppose that* $\mathfrak{g}$ *is simple and let* φ *be an orthogonal representation of* G*. Then* φ *is spinorial iff the integer*

$$p(\underline{\nu}) \cdot \frac{\mathrm{tr}(C, V)}{\dim \mathfrak{g}} \tag{1}$$

is even.

Alternatively, this can be reformulated in terms of the Dynkin index "$\mathrm{dyn}(\varphi)$" of φ and the dual Coxeter number $\check{h}$ of $\mathfrak{g}$. (We recall these integers in Section 7.)

Corollary 1. *Suppose* $\mathfrak{g}$ *is simple and let* φ *be an orthogonal representation of* G*. Then* φ *is spinorial iff the integer*

$$p(\underline{\nu}) \cdot \frac{\mathrm{dyn}(\varphi)}{2\check{h}}$$

is even.

If $\lambda \in X^*(T)$ is dominant, write φ_λ for the irreducible representation with highest weight λ. As λ varies, we may regard (1) as an integer-valued polynomial in λ. We show that the "spinorial weights" form a periodic subset of the highest weight lattice. To be more precise, let $X^+_{\mathrm{orth}} \subset X^*(T)$ be the set of highest weights of irreducible orthogonal representations.

Theorem 2. *There is a* $k \in \mathbb{N}$ *so that for all* $\lambda_0, \lambda \in X^+_{\mathrm{orth}}$*, the representation* φ_{λ_0} *is spinorial iff* $\varphi_{\lambda_0 + 2^k \lambda}$ *is spinorial.*

For any representation φ, one can form an orthogonal representation $S(\varphi) = \varphi \oplus \varphi^\vee$. When G is semisimple, $S(\varphi)$ is always spinorial. For the reductive case we have the following theorem.

Theorem 3. $S(\varphi_\lambda)$ *is spinorial iff the integers*

$$\langle \lambda, \nu^z \rangle \cdot \dim V_\lambda$$

are even for all $\nu \in \underline{\nu}$*.*

In this formula, ν^z is the $\mathfrak{z}$-component of ν corresponding to the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}'$ is the derived algebra of $\mathfrak{g}$.

This paper is organized as follows. Section 2 establishes general notation and Section 2.4 sets up preliminaries for the spin groups. In Section 3 we give a criterion for spinoriality in terms of the weights of φ. This approach is along the lines of [PR95] and [Bou05].

We advance the theory in Section 4 by employing an algebraic trick involving palindromic Laurent polynomials; this gives a lifting condition in terms of the integers

$$q_\varphi(\nu) = \frac{1}{2} \cdot \frac{d^2}{dt^2} \Theta_\varphi(\nu(t))|_{t=1}$$

for $\nu \in \underline{\nu}$. Here Θ_φ denotes the character of φ.

In Section 5 we compute $q_\varphi(\nu)$ for φ irreducible, essentially by taking two derivatives of Weyl's Character Formula. As a corollary we show that every nonabelian reductive group has a nontrivial spinorial irreducible representation. Section 6 works out the case of reducible orthogonal representations, in particular we prove Theorems 1 and 3. In Section 7 we explain the connection with the Dynkin index. Spinoriality for tensor products is understood in Section 8.

The next four sections apply our theory to groups G with $\mathfrak{g}$ simple. Our goal is to answer the question: For which such G is every orthogonal representation spinorial? Section 9 covers quotients of SL_n, Section 10 covers type C_n, Section 11 covers type D_n, and Section 12 presents the final answer to the question.

In Section 13 we prove Theorem 2, the periodicity of the spinorial weights. Finally, in Section 14 we reduce to the case of F algebraically closed.

2. Preliminaries

2.1. **Notation.** Throughout this paper G is a connected reductive algebraic group over F with Lie algebra $\mathfrak{g}$. Until the final section, F is algebraically closed. Write $\mathfrak{g}'$ for the derived algebra of $\mathfrak{g}$. We write T for a maximal torus of G, with Lie algebra $\mathfrak{t}$ and Weyl group W. Put $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{g}'$. Let $\mathrm{sgn} : W \to \{\pm 1\}$ be the usual sign character of W. As in [Spr98], let $(X^*, R, X_*, R^\vee)$ be the root datum associated to G.

The groups $X^* = X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_* = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ are the character and cocharacter lattices of T. One has injections $X^* \hookrightarrow \mathfrak{t}^*$ and $X_* \hookrightarrow \mathfrak{t}$ given by differentiation for the former, and $\nu \mapsto d\nu(1)$ for the latter. We will often identify X^*, R, X_*, and $R^\vee$ with their images under these injections. Let $Q(T) \subseteq X_*(T)$ be the group generated by the coroots of T in G. Write R^+ for a set of positive roots of T in G, and $\delta \in \mathfrak{t}^*$ for the half-sum of these positive roots. Let $w_0 \in W$ denote the longest Weyl group element.

For $\lambda, \lambda' \in X^*(T)$, we write $\lambda' \prec \lambda$ when $\lambda - \lambda'$ is a nonnegative combination of positive roots.

In this paper all representations V of G are finite-dimensional F-representations, equivalently morphisms $\varphi : G \to \mathrm{GL}(V)$ of algebraic groups. For $\mu \in X^*(T)$, write V^μ for the μ-eigenspace of V, and put $m_\varphi(\mu) = \dim V^\mu$, the multiplicity of μ as a weight of V.

If H is an algebraic group, write H° for the connected component of the identity. We frequently write $\mathrm{diag}(t_1, t_2, \ldots, t_n)$ for the $n \times n$ matrix with the given elements as entries.

2.2. **Pairings.** Write $\langle , \rangle_T : X^*(T) \times X_*(T) \to \mathbb{Z}$ for the pairing

$$\langle \mu, \nu \rangle_T = n \Leftrightarrow \mu(\nu(t)) = t^n$$

for $t \in F^\times$, and $\langle , \rangle_{\mathfrak{t}} : \mathfrak{t}^* \times \mathfrak{t}$ for the natural pairing. Note that for $\mu \in X^*(T)$ and $\nu \in X_*(T)$, we have

$$\langle d\mu, d\nu(1) \rangle_{\mathfrak{t}} = \langle \mu, \nu \rangle_T.$$

So we may drop the subscripts and simply write "$\langle \mu, \nu \rangle$".

Write $(,)$ for the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{t}$; it may be computed by

$$(x, y) = \sum_{\alpha \in R} \alpha(x)\alpha(y),$$

for $x, y \in \mathfrak{t}$. Also set $|x|^2 = (x, x)$. In particular, for $\nu \in X_*(T)$ we have $|\nu|^2 = \sum_{\alpha \in R} \langle \alpha, \nu \rangle^2$. The Killing form restricted further to $\mathfrak{t}'$ induces an isomorphism $\sigma : (\mathfrak{t}')^* \cong \mathfrak{t}'$. We use the same notation "$(,)$" to denote the inverse form on $(\mathfrak{t}')^*$ defined for $\mu_1, \mu_2 \in \mathfrak{t}'$ by

$$(\mu_1, \mu_2) = (\sigma(\mu_1), \sigma(\mu_2)).$$

In [Bou02] this form on $(\mathfrak{t}')^*$ is called the "canonical bilinear form" Φ_R. Write $|y|^2 = (y, y)$ for $y \in (\mathfrak{t}')^*$.

Let $\pi_1(G) = X_*(T)/Q(T)$. As in the introduction, fix a set $\underline{\nu} = \{\nu_1, \dots, \nu_r\}$ of cocharacters whose images generate $\pi_1(G)$, and put

$$p(\underline{\nu}) = \frac{1}{2} \gcd\left(|\nu_1|^2, \dots, |\nu_r|^2\right).$$

Often $\underline{\nu}$ will be a singleton $\{\nu_0\}$, in which case we may simply write

$$p(\nu_0) = p(\{\nu_0\}) = \frac{1}{2}|\nu_0|^2.$$

2.3. **Orthogonal representations.** Let $X^*(T)^+$ be the set of dominant characters, i.e., the $\lambda \in X^*(T)$ so that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in R^+$.

Put

$$X_{\text{sd}} = \{\lambda \in X^*(T) \mid w_0\lambda = -\lambda\}$$

and

$$X_{\text{orth}} = \{\lambda \in X_{\text{sd}} \mid \langle \lambda, 2\delta^\vee \rangle \text{ is even}\},$$

and use the superscript "+" to denote the dominant members of these sets. According to [Bou05], X_{sd}^+ is the set of highest weights of irreducible self-dual representations, and X_{orth}^+ is the set of highest weights of irreducible orthogonal representations.

For $\lambda \in X^*(T)^+$, the quantity

$$|\lambda + \delta|^2 - |\delta|^2 = (\lambda, \lambda + 2\delta)$$

is equal to $\chi_\lambda(C)$, the value of the central character of the irreducible representation φ_λ at the Casimir element C. (See [Jr.08].)

2.4. **Tori of spin groups.** In this section we recall material about the tori of spin groups. Our reference is Section 6.3 of [GW09].

For the even-dimensional case, let V be a vector space with basis $(e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1})$. For the odd-dimensional case, use the basis $(e_1, \ldots, e_n, e_0, e_{-n}, \ldots, e_{-1})$. In either case, give V the symmetric bilinear form $(,)$ so that $(e_i, e_{-i}) = 1$ and $(e_i, e_j) = 0$ for $j \neq -i$.

Let $C(V)$ be the corresponding Clifford algebra, i.e., the quotient of the tensor algebra of V by the relation

$$v \otimes w + w \otimes v = (v, w).$$

Let $\mathrm{Pin}(V)$ denote the subgroup of the invertible elements of $C(V)$, generated by the unit vectors in V. The morphism $\rho : \mathrm{Pin}(V) \to O(V)$ taking each unit vector to the corresponding reflection of V is a double cover. Then $\mathrm{Spin}(V) = \mathrm{Pin}(V)^\circ$ is the inverse image of $\mathrm{SO}(V)$ under ρ.

For $1 \leq j \leq n$, let $c_j(t) = te_je_{-j} + t^{-1}e_{-j}e_j \in \mathrm{Spin}(V)$. This gives a morphism $c_j : \mathbb{G}_m \to \mathrm{Spin}(V)$. Define $c : \mathbb{G}_m^n \to \mathrm{Spin}(V)$ by

$$c(t_1, \ldots, t_n) = c_1(t_1) \cdots c_n(t_n).$$

The kernel of c is

$$\{(t_1, \ldots, t_n) \mid t_i = \pm 1, t_1 \cdots t_n = 1\},$$

and the image of c is a maximal torus $\tilde{T}_V$ of $\mathrm{Spin}(V)$. The image of $\tilde{T}_V$ under ρ is the subgroup of diagonal matrices in $\mathrm{SO}(V)$, relative to the basis of V mentioned above. More precisely, the restriction of ρ to $\tilde{T}_V$ may be described by

$$\rho(c(t_1, \ldots, t_n)) = \begin{cases} \mathrm{diag}(t_1^2, \ldots, t_n^2, t_n^{-2}, \ldots, t_1^{-2}), \\ \mathrm{diag}(t_1^2, \ldots, t_n^2, 1, t_n^{-2}, \ldots, t_1^{-2}), \end{cases}$$

depending on whether $\dim V = 2n$ or $2n + 1$.

The kernel of ρ is generated by $z = c(-1, 1, \ldots, 1) = -1 \in C(V)$. Pick $\sqrt{-1} \in F$, and put $c^+ = c(\sqrt{-1}, \sqrt{-1}, \ldots, \sqrt{-1})$. Then $(c^+)^2 = z^n$.

We now describe the center Z of $\mathrm{Spin}(V)$.

(1) When $\dim V = 2n + 1$, Z is generated by z.
(2) When $\dim V = 2n$, with n odd, Z is cyclic of order 4, generated by c^+.
(3) When $\dim V = 2n$, with n even, Z is a Klein 4-group generated by z and c^+.

Define $\vartheta_i \in X^*(T_V)$ by

$$\vartheta_i : \mathrm{diag}(t_1, \ldots, t_n, \ldots) \mapsto t_i.$$

We identify $X^*(T_V)$ with $\mathbb{Z}^n$ through the bijection $\sum_i a_i\vartheta_i \leftrightarrow (a_1, \ldots, a_n)$, and $X_*(T_V)$ with $\mathbb{Z}^n$ by $\nu \leftrightarrow (b_1, \ldots, b_n)$ when $\nu(t) = \mathrm{diag}(t^{b_1}, \ldots, t^{b_n}, \ldots)$.

Let Σ be a set of weights formed by taking one representative from each pair $\{\vartheta_i, -\vartheta_i\}$. Then Σ is a $\mathbb{Z}$-basis of $X^*(T_V)$. Of course, one choice is $\Sigma_* = \{\vartheta_1, \ldots, \vartheta_n\}$. Put $\omega_\Sigma = \sum_{\omega \in \Sigma} \omega$.

Lemma 1. *Let d be a positive even integer, and $\zeta_d \in F^\times$ a primitive dth root of unity. Let $\nu \in X_*(T_V)$.*

(1) *ν lifts to a cocharacter $\tilde{\nu} \in X_*(\tilde{T}_V) \Leftrightarrow \langle \omega_\Sigma, \nu \rangle$ is even.*
(2) *$\nu(\zeta_d) = 1 \Leftrightarrow d \mid \langle \vartheta_i, \nu \rangle$ for all i.*
(3) *Assume the conditions in* (1) *and* (2) *above. Then $\tilde{\nu}(\zeta_d) = 1 \Leftrightarrow 2d \mid \langle \omega_\Sigma, \nu \rangle$.*

Proof. For the first statement, note that the image of $X_*(\tilde{T}_V)$ in $X_*(T_V)$ is exactly $Q(T_V)$. One checks that $\langle \omega_\Sigma, \nu\rangle$ is even iff $\nu \in Q(T_V)$.

For the second statement, just use that

$$\nu(t) = \mathrm{diag}(t^{b_1}, \dots, t^{b_n}, \dots), \tag{2}$$

with $b_i = \langle \vartheta_i, \nu\rangle$.

Now consider the third statement for $\Sigma = \Sigma_*$. By hypothesis each b_i in (2) is even, and $\langle \omega_{\Sigma_*}, \nu\rangle = b_1 + \cdots + b_n$ is even. Then

$$\tilde{\nu}(t) = c(t^{b_1/2}, \dots, t^{b_n/2}),$$

so

$$\tilde{\nu}(\zeta_d) = c\left(\zeta_d^{\frac{b_1}{2}}, \dots, \zeta_d^{\frac{b_n}{2}}\right).$$

Since each b_i is even, each $\zeta_d^{\frac{b_i}{2}} = \pm 1$. Therefore $\tilde{\nu}(\zeta_d) = 1$, i.e.,

$$\zeta_d^{\frac{b_1+\cdots+b_n}{2}} = 1,$$

equivalently $2d$ divides $\langle \omega_{\Sigma_*}, \nu\rangle$, as claimed. Finally, by hypothesis d divides each $\langle \vartheta_i, \nu\rangle$, so that $\langle \omega_{\Sigma_*}, \nu\rangle \equiv \langle \omega_\Sigma, \nu\rangle \mod 2d$. □

3. Lifting cocharacters

We reformulate the lifting problem for an orthogonal representation in terms of its weights. Throughout this section G is a connected reductive group over an algebraically closed field F, and T is a maximal torus of G.

Recall [Spa66] that for nice topological spaces such as manifolds, if $\rho : \tilde{Y} \to Y$ is a covering map, then a continuous function $\varphi : X \to Y$ lifts to $\hat{\varphi} : X \to \tilde{Y}$ iff $\varphi_*(\pi_1(X)) \le \rho_*(\pi_1(\tilde{Y}))$ (with compatibly chosen basepoints on $X, Y, \tilde{Y}$). The purpose of the next proposition is to extend this to the setting of algebraic groups.

Lemma 2. *Let G, H be connected reductive groups, with maximal tori $T \le G$ and $T_H \le H$. Let $\varphi : G \to H$ be a morphism with $\varphi(T) \le T_H$. The induced map $\varphi_* : X_*(T) \to X_*(T_H)$ takes $Q(T)$ to $Q(T_H)$.*

Proof. Suppose first that G, H are semisimple. Write $\rho_G : G_{\mathrm{sc}} \to G$ for the universal cover, with maximal torus T_{sc} above T. Similarly we have $\rho_H : H_{\mathrm{sc}} \to H$, with maximal torus $T_{H,\mathrm{sc}}$. Put $\Phi = \varphi \circ \rho_G : G \to H$.

Let $\tilde{G} = (G_{\mathrm{sc}} \times_H H_{\mathrm{sc}})^\circ$, with projection maps $\mathrm{pr}_1 : \tilde{G} \to G_{\mathrm{sc}}$ and $\mathrm{pr}_2 : \tilde{G} \to H_{\mathrm{sc}}$. It is easy to see that pr_1 is a central isogeny; since G_{sc} is simply connected, it is an isomorphism by 2.15 of [Spr79]. If we put $\widetilde{\varphi} = \mathrm{pr}_2 \circ (\mathrm{pr}_1)^{-1}$, then the following diagram commutes:

$$\begin{array}{ccc} G_{\mathrm{sc}} & \xrightarrow{\widetilde{\varphi}} & H_{\mathrm{sc}} \\ \rho_G \downarrow & & \downarrow \rho_H \\ G & \xrightarrow[\varphi]{} & H \end{array} \tag{3}$$

Note that φ restricts to a map from T_{sc} to $T_{H,\mathrm{sc}}$. Applying the covariant functor $T \mapsto X_*(T)$ gives the commutative square:

$$\begin{array}{ccc} X_*(T_{\mathrm{sc}}) & \xrightarrow{\widetilde{\varphi}_*} & X_*(T_{H,\mathrm{sc}}) \\ \downarrow & & \downarrow \\ X_*(T) & \xrightarrow{\varphi_*} & X_*(T_H) \end{array}$$

We have $X_*(T_{\mathrm{sc}}) = Q(T)$ and similarly for $T_{H,\mathrm{sc}}$ by simple connectedness. The downward maps, being isogenies, take coroots to coroots and we deduce that φ_* takes $Q(T)$ to $Q(T_H)$. This conclusion also holds for G and H connected reductive, for one applies the previous argument to the derived groups G_{der} and H_{der}, recalling that the coroots of G lie in $X_*(T_{\mathrm{der}})$. □

Proposition 1. *Let $\rho : \tilde{H} \to H$ be a central isogeny of connected reductive groups over F, and $\varphi : G \to H$ a morphism. Pick a maximal torus $T_H \leq H$ containing $\varphi(T)$, and write $\varphi_* : X_*(T) \to X_*(T_H)$ for the induced map. Let $\tilde{T}_H = \rho^{-1}(T_H) \leq \tilde{H}$, and write $\rho_* : X_*(\tilde{T}_H) \to X_*(T_H)$ for the induced map. Then there exists a morphism $\hat{\varphi} : G \to \tilde{H}$ such that $\rho \circ \hat{\varphi} = \varphi$, iff $\operatorname{im} \varphi_* \subseteq \operatorname{im} \rho_*$. Moreover when this morphism exists, it is unique.*

Proof. Let $\tilde{G} = (G \times_H \tilde{H})^\circ$, with projection maps $\rho_G : \tilde{G} \to G$ and $\widetilde{\varphi} : \tilde{G} \to \tilde{H}$. We have the diagram:

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\widetilde{\varphi}} & \tilde{H} \\ {\scriptstyle \rho_G}\downarrow & \overset{\hat{\varphi}}{\nearrow} & \downarrow{\scriptstyle \rho} \\ G & \xrightarrow[\varphi]{} & H \end{array} \tag{4}$$

Put $\tilde{T}_H = \rho^{-1}(T_H)$ and $\tilde{T} = \rho_G^{-1}(T)$. Let us see the equivalence of the following statements:

(1) φ lifts to $\hat{\varphi} : G \to \tilde{H}$.
(2) $\widetilde{\varphi}$ factors through ρ_G.
(3) $\ker \rho_G \leq \ker \widetilde{\varphi}$.
(4) $\ker \rho_G \leq \ker \widetilde{\varphi}|_{\tilde{T}}$.
(5) $\tilde{\varphi}|_{\tilde{T}}$ factors through T.
(6) $\varphi|_T$ lifts to $\tilde{T}_H$.
(7) φ_* lifts in the diagram:

$$\begin{array}{ccc} & & X_*(\tilde{T}_H) \\ & \nearrow & \downarrow{\scriptstyle \rho_*} \\ X_*(T) & \xrightarrow{\varphi_*} & X_*(T_H) \end{array}$$

(8) $\operatorname{im} \varphi_* \leq \operatorname{im} \rho_*$.

For (1) ⇒ (2), suppose φ lifts to $\hat{\varphi}$. Then

$$\rho(\widetilde{\varphi}(x)^{-1} \cdot \hat{\varphi}(\rho_G(x))) = 1 \in H,$$

so the algebraic map $m : \tilde{G} \to \tilde{H}$ defined by

$$m(x) = \widetilde{\varphi}(x)^{-1} \cdot \hat{\varphi}(\rho_G(x)),$$

takes values in $\ker \rho$. Since $\ker \rho$ is discrete, and $\tilde{G}$ is connected, it must be that m is constant. Thus $m(x) = 1 \in \tilde{H}$ for all x, i.e., $\widetilde{\varphi} = \hat{\varphi} \circ \rho_G$.

For (2) $\Rightarrow$ (1), suppose $\widetilde{\varphi} = \hat{\varphi} \circ \rho_G$ for some morphism $\hat{\varphi}$. From the identity $\varphi \circ \rho_G = \rho \circ \widetilde{\varphi}$ and the fact that ρ_G is surjective we deduce that $\varphi = \rho \circ \hat{\varphi}$.

(2) $\Rightarrow$ (3) is immediate.

The implication (3) $\Rightarrow$ (2) follows from the universal property of $\tilde{G}/\ker \rho_G$. (See Section 5.5, page 92 of [Spr98].)

Since $\ker \rho_G \leq Z(\tilde{G}) \leq \tilde{T}$, we have (3) $\Leftrightarrow$ (4).

The argument for (4) $\Leftrightarrow$ (5) is similar to the argument for (2) $\Leftrightarrow$ (3), and (5) $\Leftrightarrow$ (6) is similar to (1) $\Leftrightarrow$ (2).

To see (6) $\Leftrightarrow$ (7), note that the functors $T \mapsto X_*(T)$ and $L \mapsto L \otimes_{\mathbb{Z}} F^\times$ give an equivalence of categories between F-tori and free abelian groups of finite rank.

The equivalence (7) $\Leftrightarrow$ (8) is elementary. Thus (1)–(8) are equivalent.

Finally, suppose that $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are lifts of φ. Then $g \mapsto \hat{\varphi}_1(g)\hat{\varphi}_2(g)^{-1}$ is an algebraic map $G \to \ker \rho$ taking 1 to 1. Since G is connected it must be that $\hat{\varphi}_1 = \hat{\varphi}_2$. ☐

Remark 1. This proof did not use the property that F has characteristic zero. In the case of positive characteristic, it is sufficient for F to be separably closed, and then $\hat{\varphi}$ is defined over F. Suppose F is an arbitrary field, and the maps ρ and φ are defined over F. By uniqueness, $\hat{\varphi}$, when it exists, is fixed by the absolute Galois group of F and hence defined over F.

By Lemma 2, φ_* descends to

$$\varphi_* : \pi_1(G) \to \pi_1(H).$$

Again, since ρ is an isogeny, we have $\rho_*(Q(\tilde{T}_H)) = Q(T_H)$. Therefore a lift $\hat{\varphi}$ in the diagram (4) exists iff φ_* lifts in the diagram:

$$\begin{array}{ccc} & & \pi_1(\tilde{H}) \\ & \nearrow & \downarrow \rho_* \\ \pi_1(G) & \xrightarrow{\varphi_*} & \pi_1(H) \end{array}$$

Recall we have fixed a set $\underline{\nu}$ of cocharacters which generates $\pi_1(G)$.

Corollary 2. *A lift $\hat{\varphi}$ as in the above proposition exists iff $\varphi_*(\nu) \in \operatorname{im} \rho_*$ for each $\nu \in \underline{\nu}$.*

Definition 1. Let (φ, V) be a representation of G. For $\nu \in X_*(T)$, put

$$L_\varphi(\nu) = \sum_{\{\mu \in X^*(T) \mid \langle \mu, \nu \rangle > 0\}} m_\varphi(\mu) \langle \mu, \nu \rangle \in \mathbb{Z}.$$

Proposition 2. *Let $\varphi : G \to \mathrm{SO}(V)$ be an orthogonal representation. For $\nu \in X_*(T)$, the cocharacter $\varphi_*(\nu) \in \operatorname{im} \rho_*$ iff $L_\varphi(\nu)$ is even. Thus φ is spinorial iff the integers $L_\varphi(\nu)$ are even for all $\nu \in \underline{\nu}$.*

(Compare Exercise 7 in Section 8, Chapter IX of [Bou05] and Lemma 3 in [PR95].)

Proof. We may assume that $\varphi(T) \leq T_V$. By Corollary 2, φ is spinorial iff $\varphi_*(\nu) \in \operatorname{im} \rho_*$ for all $\nu \in \underline{\nu}$. By Lemma 1(1), we must check whether the integer $\langle \omega_\Sigma, \varphi_*(\nu) \rangle$ is even for a suitable Σ.

Write $P_V = \{\pm \vartheta_i \mid 1 \leq i \leq n\}$, the weights of V as a T_V-module. Let

$$P_V^1 = \{\omega \in P_V \mid \langle \varphi^* \omega, \nu \rangle \geq 0\}.$$

We may choose $\Sigma \subseteq P_V^1$ so that Σ contains one representative from each pair $\{\vartheta_i, -\vartheta_i\}$ as in Section 2.4.

Then

$$\begin{aligned} \langle \omega_\Sigma, \varphi_* \nu \rangle &= \sum_{\omega \in \Sigma} \langle \varphi^* \omega, \nu \rangle \\ &= \sum_{\{\mu \mid \langle \mu, \nu \rangle > 0\}} \langle \mu, \nu \rangle \cdot \dim V^\mu \\ &= L_\varphi(\nu). \end{aligned}$$

Thus φ lifts iff $L_\varphi(\nu)$ is even for all $\nu \in \underline{\nu}$. □

Since $\varphi(Q(T)) \subseteq Q(T_V)$ we note the following corollary.

Corollary 3. *If $\nu \in Q(T)$, then $L_\varphi(\nu)$ is even.*

For two representations φ_1, φ_2, we have

$$L_{\varphi_1 \oplus \varphi_2}(\nu) = L_{\varphi_1}(\nu) + L_{\varphi_2}(\nu), \tag{5}$$

since $m_{\varphi_1 \oplus \varphi_2}(\mu) = m_{\varphi_1}(\mu) + m_{\varphi_2}(\mu)$.

Corollary 4. *The adjoint representation of G on $\mathfrak{g}$ is spinorial iff $\delta \in X^*(T)$.*

Proof. If φ is the adjoint representation, then

$$\begin{aligned} L_\varphi(\nu) &= \sum_{\{\alpha \in R \mid \langle \alpha, \nu \rangle > 0\}} \langle \alpha, \nu \rangle \\ &\equiv \sum_{\alpha \in R^+} \langle \alpha, \nu \rangle \mod 2 \\ &= 2\langle \delta, \nu \rangle. \end{aligned}$$

The corollary follows since the pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ is perfect. □

Remark 2. This is well known; for G a compact connected Lie group, see Section 5.56 of [Ada69].

Example 1. Let $G = \mathrm{PGL}_2$, with diagonal maximal torus T. Then $\pi_1(G)$ is generated by $\nu_0(t) = \operatorname{diag}(t, 1)$ mod center. Let α be the positive root defined by $\alpha(\operatorname{diag}(a, b)) = ab^{-1}$, and let φ_j be the representation of PGL_2 with highest weight $j\alpha$. Then

$$\begin{aligned} L_{\varphi_j}(\nu_0) &= \langle \alpha, \nu_0 \rangle + \cdots + \langle j\alpha, \nu_0 \rangle \\ &= 1 + \cdots + j. \end{aligned}$$

Therefore φ_j is spinorial iff $j \equiv 0, 3 \mod 4$.

4. Palindromy

This section is the cornerstone of our paper. The difficulty with determining the parity of $L_\varphi(\nu)$ is in somehow getting ahold of "half" of the weights of V, one for each positive/negative pair. This amounts to knowledge of the polynomial part of a certain palindromic Laurent polynomial, and this we accomplish with a derivative trick.

Definition 2. For (φ, V) a representation of G and $\nu \in X_*(T)$, consider the function $Q_{(\varphi,\nu)} : F^\times \to F$ defined by

$$\begin{aligned} Q_{(\varphi,\nu)}(t) &= \Theta_\varphi(\nu(t)) \\ &= \mathrm{tr}(\varphi(\nu(t))). \end{aligned}$$

If φ is understood we may simply write "$Q_\nu(t)$". For $\gamma \in T$, we have

$$\Theta_\varphi(\gamma) = \sum_{\mu \in X^*} m_\varphi(\mu)\mu(\gamma),$$

so in particular

$$Q_\nu(t) = \sum_{\mu \in X^*} m_\varphi(\mu) t^{\langle \mu, \nu \rangle} \in \mathbb{Z}[t, t^{-1}]. \tag{6}$$

We note:

- $Q_\nu(1) = \dim V$,
- $Q'_\nu(1) = \sum_\mu m_\varphi(\mu)\langle \mu, \nu \rangle$,
- $Q''_\nu(1) = \sum_\mu \big(m_\varphi(\mu)\langle \mu, \nu\rangle^2 - m_\varphi(\mu)\langle \mu, \nu \rangle\big)$.

Definition 3. For (φ, V) a representation of G and $\nu \in X_*(T)$, we set

$$q_\varphi(\nu) = \frac{1}{2} Q''_\nu(1).$$

When φ is self-dual, $m_\varphi(-\mu) = m_\varphi(\mu)$ for all $\mu \in X^*$, so in this case:

- $Q_\nu(t) = Q_\nu(t^{-1})$, i.e., Q_ν is "palindromic",
- $Q'_\nu(1) = 0$,
- $Q''_\nu(1) = \sum_\mu m_\varphi(\mu)\langle \mu, \nu \rangle^2 \in 2\mathbb{Z}$.

In particular, $q_\varphi(\nu)$ is an integer for all $\nu \in X_*$.

Lemma 3. *For φ self-dual and $\nu_1, \nu_2 \in X_*$, we have*

$$q_\varphi(\nu_1 + \nu_2) \equiv q_\varphi(\nu_1) + q_\varphi(\nu_2) \mod 2.$$

Proof. Breaking the sum over μ into a sum over nonzero pairs $\{\mu, -\mu\}$ gives

$$\begin{aligned} q_\varphi &= \frac{1}{2} \sum_{\mu \in X^*} m_\varphi(\mu)\langle \mu, \nu \rangle^2 \\ &= \sum_{\{\mu, -\mu\}} m_\varphi(\mu)\langle \mu, \nu\rangle^2 \in \mathbb{Z}. \end{aligned}$$

Therefore

$$\begin{aligned} q_\varphi(\nu_1 + \nu_2) &= \sum_{\{\mu,-\mu\}} m_\varphi(\mu) \left(\langle \mu, \nu_1\rangle^2 + 2\langle \mu, \nu_1 \rangle \langle \mu, \nu_2 \rangle + \langle \mu, \nu_2 \rangle^2\right) \\ &\equiv q_\varphi(\nu_1) + q_\varphi(\nu_2) \mod 2. \end{aligned}$$

□

Thus when φ is self-dual, the function $q_\varphi : X_* \to \mathbb{Z}$ induces a group homomorphism $\overline{q}_\varphi : X_* \to \mathbb{Z}/2\mathbb{Z}$. Our goal in this section is to show that

$$\overline{q}_\varphi(\nu) = L_\varphi(\nu) \mod 2$$

when φ is orthogonal.

Since Q_ν is palindromic, it may be expressed in the form

$$Q_\nu(t) = H_\nu(t) + H_\nu(t^{-1})$$

for a unique polynomial $H_\nu \in \mathbb{Z}[t] + \frac{1}{2}\mathbb{Z}$. Thus H_ν has integer coefficients, except its constant term may be half-integral. More precisely,

$$H_\nu(t) = \sum_{\langle \mu,\nu\rangle>0} m_\varphi(\mu)t^{\langle \mu,\nu\rangle} + \frac{1}{2}\sum_{\langle \mu,\nu\rangle=0} m_\varphi(\mu).$$

What we want, at least mod 2, is the integer

$$H'_\nu(1) = \sum_{\langle \mu,\nu\rangle>0} m_\varphi(\mu)\langle \mu,\nu\rangle = L_\varphi(\nu).$$

By calculus we compute

$$Q''_\nu(1) = 2(H'_\nu(1) + H''_\nu(1)).$$

But $H''_\nu(1)$ is even! This gives the following crucial result.

Proposition 3. *If φ is self-dual, then*

$$L_\varphi(\nu) \equiv q_\varphi(\nu) \mod 2. \tag{7}$$

Corollary 5. *Let φ be an orthogonal representation of G. Then φ is spinorial iff $q_\varphi(\nu)$ is even for every $\nu \in \underline{\nu}$.*

Proof. This follows from Corollary 2, Proposition 2, and the above equation. □

5. Irreducible representations

In this section we compute $q_\varphi(\nu)$ when φ is irreducible (not necessarily self-dual). Our method follows the proof of Weyl's Character Formula in [GW09]. For $\lambda \in X^*(T)^+$, write $(\varphi_\lambda, V_\lambda)$ for the irreducible representation of G with highest weight λ. For simplicity, we use the notation q_λ, $m_\lambda(\mu)$, etc. for q_{φ_λ}, $m_{\varphi_\lambda}(\mu)$, etc.

5.1. **Two derivatives of Weyl's Character Formula.**

For $\nu \in \mathfrak{t}$, put

$$d_\nu = \prod_{\alpha\in R^+} \langle \alpha,\nu\rangle,$$

and for $\mu \in \mathfrak{t}^*$, put

$$d_\mu = \prod_{\alpha\in R^+} \langle \mu,\alpha^\vee\rangle.$$

Definition 4. Put

$$\mathfrak{t}_{\mathrm{reg}} = \{\nu \in \mathfrak{t} \mid d_\nu \neq 0\}.$$

Extend the function $q_\lambda : X_* \to \mathbb{Z}$ to the polynomial function $q_\lambda : \mathfrak{t} \to F$ defined by the formula

$$q_\lambda(\nu) = \frac{1}{2}\sum_{\mu\in X^*} \langle \mu,\nu\rangle^2 m_\lambda(\mu).$$

We let $\mathbb{Z}[\mathfrak{t}^*]$ denote the usual algebra of the monoid $\mathfrak{t}^*$ with basis e^μ for $\mu \in \mathfrak{t}^*$. It contains the elements

$$J(e^\mu) = \sum_{w \in W} \operatorname{sgn}(w) e^{w\mu} \text{ and } \operatorname{ch}(V_\lambda) = \sum_{\mu \in X^*} m_\lambda(\mu) e^\mu.$$

Recall the Weyl Character Formula (Prop. 5.10 in [Jan03]):

$$\operatorname{ch}(V_\lambda) J(e^\delta) = J(e^{\lambda+\delta}).$$

Write $\varepsilon : \mathbb{Z}[\mathfrak{t}^*] \to \mathbb{Z}$ for the $\mathbb{Z}$-linear map so that $\varepsilon(e^\mu) = 1$ for all $\mu \in \mathfrak{t}^*$ (i.e., the augmentation); it is a ring homomorphism. Given $\nu \in \mathfrak{t}$, write $\frac{\partial}{\partial \nu} : \mathbb{Z}[\mathfrak{t}^*] \to \mathbb{Z}[\mathfrak{t}^*]$ for the $\mathbb{Z}$-linear map so that $\frac{\partial}{\partial \nu}(e^\mu) = \langle \mu, \nu \rangle e^\mu$; it is a $\mathbb{Z}$-derivation. Note that $\varepsilon(\operatorname{ch}(V_\lambda)) = \dim V_\lambda$, and

$$\left(\varepsilon \circ \frac{\partial^2}{\partial \nu^2}\right) \operatorname{ch}(V_\lambda) = Q_\nu''(1). \tag{8}$$

Proposition 4. *For $\nu \in \mathfrak{t}_{\mathrm{reg}}$, we have*

$$q_\lambda(\nu) = \frac{\sum_{w \in W} \operatorname{sgn}(w) \langle w(\lambda+\delta), \nu \rangle^{N+2}}{(N+2)! d_\nu} - \frac{1}{48} \dim V_\lambda |\nu|^2,$$

where $N = |R^+|$.

Proof. We apply $\varepsilon \circ \frac{\partial^{N+2}}{\partial \nu^{N+2}}$ to both sides of $J(e^{\lambda+\delta}) = \operatorname{ch}(V_\lambda) J(e^\delta)$. On the left we have

$$\left(\varepsilon \circ \frac{\partial^{N+2}}{\partial \nu^{N+2}}\right) J(e^{\lambda+\delta}) = \sum_{w \in W} \operatorname{sgn}(w) \langle w(\lambda+\delta), \nu \rangle^{N+2}. \tag{9}$$

The right-hand side requires more preparation. For $\alpha \in R^+$, let $r_\alpha = e^{\alpha/2} - e^{-\alpha/2}$. Then:

- $\varepsilon(r_\alpha) = 0$,
- $\varepsilon \circ \frac{\partial}{\partial \nu}(r_\alpha) = \langle \alpha, \nu \rangle$,
- $\frac{\partial^2}{\partial \nu^2} r_\alpha = \frac{1}{4} \langle \alpha, \nu \rangle^2 r_\alpha$,
- $J(e^\delta) = \prod_{\alpha \in R^+} r_\alpha$.

The last equality is a familiar identity from [Bou02]. We may now apply the following lemma.

Lemma 4. *Let R be a commutative ring, $D : R \to R$ a derivation, and $\varepsilon : R \to R'$ a ring homomorphism. Suppose that $r_1, \ldots, r_N \in \ker \varepsilon$. Then:*

(1) $\varepsilon(D^n(r_1 \cdots r_N)) = 0$ *for* $0 \leq n < N$.
(2) $\varepsilon(D^N(r_1 \cdots r_N)) = N! \prod_{i=1}^N \varepsilon(D(r_i))$.
(3) *If also* $D^2(r_i) \in \ker \varepsilon$ *for all* i, *then* $\varepsilon(D^{N+1}(r_1 \cdots r_N)) = 0$.
(4) *Suppose further that there are* $c_i \in R$ *so that* $D^2(r_i) = c_i r_i$. *Then:*

$$\varepsilon(D^{N+2}(r_1 \cdots r_N)) = \frac{(N+2)!}{6} \left(\prod_i \varepsilon(D(r_i))\right) \left(\sum_i c_i\right).$$

Proof. This follows from the Leibniz rule for derivations:

$$D^n(r_1 \cdots r_k) = \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, \ldots, i_k} D^{i_1}(r_1) \cdots D^{i_k}(r_k).$$

∎

Thus in our case,

(1) $(\varepsilon \circ \frac{\partial^n}{\partial \nu^n})J(e^\delta) = 0$ for $0 \le n < N$,
(2) $(\varepsilon \circ \frac{\partial^N}{\partial \nu^N})J(e^\delta) = N! d_\nu$,
(3) $(\varepsilon \circ \frac{\partial^{N+1}}{\partial \nu^{N+1}})J(e^\delta) = 0$,
(4) $(\varepsilon \circ \frac{\partial^{N+2}}{\partial \nu^{N+2}})J(e^\delta) = \frac{(N+2)!}{24} d_\nu \sum_{\alpha>0} \langle \alpha, \nu \rangle^2$.

Now we are ready to consider

$$\left(\varepsilon \circ \frac{\partial^{N+2}}{\partial \nu^{N+2}} \right) (\mathrm{ch}(V_\lambda) J(e^\delta)).$$

Applying the Leibniz rule to the above gives

$$\binom{N+2}{2} Q_\nu''(1) N! d_\nu + \dim V_\lambda \frac{(N+2)!}{24} d_\nu \sum_{\alpha>0} \langle \alpha, \nu \rangle^2.$$

Equating this with (9) yields the identity

$$\sum_{w \in W} \mathrm{sgn}(w) \langle w(\lambda+\delta), \nu \rangle^{N+2} = (N+2)! d_\nu \left(q_\lambda(\nu) + \frac{\dim V_\lambda}{24} \sum_{\alpha>0} \langle \alpha, \nu \rangle^2 \right), \tag{10}$$

whence the proposition. □

5.2. **Anti-W-invariant polynomials.** The expression

$$\sum_{w \in W} \mathrm{sgn}(w) \langle w(\lambda+\delta), \nu \rangle^{N+2}$$

in our formula demands simplification. This can be done by applying the theory of anti-W-invariant polynomials.

Let $f : \mathfrak{t} \to F$ be a polynomial function. We say that f is *anti-W-invariant*, provided for all $w \in W$ and $\nu \in \mathfrak{t}$ we have

$$f(w(\nu)) = \mathrm{sgn}(w) f(\nu).$$

The polynomial $\nu \mapsto d_\nu$ is a homogeneous anti-W-invariant polynomial of degree N. According to [Bou02], page 118, if f is a homogeneous anti-W-invariant polynomial of degree d, then there exists a homogeneous W-invariant polynomial $p : \mathfrak{t} \to F$ so that $f(\nu) = p(\nu) d_\nu$. Necessarily $d = \deg f \ge N$ and p has degree $d - N$. Similarly, if $g : \mathfrak{t}^* \to F$ is a homogeneous anti-W-invariant polynomial, then $g(\mu) = p(\mu) d_\mu$ for a W-invariant polynomial p on $\mathfrak{t}^*$.

In this section we will make use of the famous Weyl dimension formula, which we recall is $\dim V_\lambda = d_{\lambda+\delta}/d_\delta$.

Definition 5. Let k be a nonnegative integer. Put

$$F_k(\mu, \nu) = \sum_{w \in W} \mathrm{sgn}(w) \langle w(\mu), \nu \rangle^k$$

for $\mu \in \mathfrak{t}^*$ and $\nu \in \mathfrak{t}$.

Proposition 5. *Let $\mathfrak{g}$ be simple. Then*

$$F_k(\mu, \nu) = \begin{cases} 0 & \text{if } 0 \le k < N \text{ or } k = N+1, \\ N! \cdot \dfrac{d_\mu d_\nu}{d_\delta} & \text{if } k = N, \\ \dfrac{(N+2)!}{48|\delta|^2} \cdot \dfrac{d_\mu d_\nu}{d_\delta} |\mu|^2 |\nu|^2 & \text{if } k = N+2. \end{cases}$$

Proof. Each F_k may be viewed as a polynomial in two ways: as a function of μ and as a function of ν. It is either identically 0, or homogeneous of degree k. Both the functions $\mu \mapsto F_k(\mu,\nu)$ and $\nu \mapsto F_k(\mu,\nu)$ are anti-W-invariant. Therefore $F_k(\mu,\nu)$ either vanishes, or is the product of $d_\mu d_\nu$ and a homogeneous W-invariant polynomial of degree $k-N$ in both ν and μ. By degree considerations, F_k must vanish for $0 \leq k < N$.

Case $k = N$. Here $F_N(\mu,\nu) = c d_\mu d_\nu$ for some constant $c \in F$, independent of μ and ν. To determine c, we apply $\varepsilon \circ \frac{\partial^N}{\partial \nu^N}$ to both sides of $J(e^{\lambda+\delta}) = \mathrm{ch}(V_\lambda) J(e^\delta)$. On the left we have

$$\left(\varepsilon \circ \frac{\partial^N}{\partial \nu^N}\right) J(e^{\lambda+\delta}) = F_N(\lambda+\delta,\nu). \tag{11}$$

On the right we proceed as in the proof of Proposition 4 to obtain $N! \cdot \dim V_\lambda \cdot d_\nu$. Therefore

$$c \cdot d_{\lambda+\delta} d_\nu = N! \cdot \dim V_\lambda \cdot d_\nu,$$

so that $c = \frac{N!}{d_\delta}$.

Case $k = N+1$. Since $\mathfrak{g}$ is simple, both $\mathfrak{t}$ and $\mathfrak{t}^*$ are irreducible representations of W. If $\dim \mathfrak{t} > 1$, there is no 1-dimensional invariant subspace. When $\dim \mathfrak{t} = 1$, W acts by a nontrivial reflection. Therefore there is no W-invariant vector, i.e., no W-invariant polynomial of degree 1. Thus in all cases F_{N+1} vanishes.

Case $k = N+2$. Let us write $F_{N+2}(\mu,\nu) = \mathcal{Q}_\mu(\nu) d_\nu$ with $\mathcal{Q}_\mu$ a W-invariant quadratic form on $\mathfrak{t}$. The corresponding bilinear form on $\mathfrak{t}$ is W-invariant; as $\mathfrak{t}$ is an irreducible W-representation, this bilinear form must be a scalar multiple of the Killing form. Thus we may write

$$F_{N+2}(\mu,\nu) = c_R d_\mu d_\nu |\mu|^2 |\nu|^2; \tag{12}$$

it remains to determine c_R.

Let σ be as in Section 2.2. Employing [Bou05], Ch. VIII, Section 9, Exercise 7, we obtain the value at $\nu = \sigma(\delta) \in \mathfrak{t}$:

$$\begin{aligned} Q''_{\sigma(\delta)}(1) &= \sum_\mu \langle \mu, \sigma(\delta)\rangle^2 m_\lambda(\mu) \\ &= \frac{\dim V_\lambda}{24} \cdot (\lambda, \lambda+2\delta). \end{aligned}$$

Substituting this into (10) gives

$$\begin{aligned} F_{N+2}(\lambda+\delta, \sigma(\delta)) &= \frac{1}{2} d_{\sigma(\delta)} (N+2)! \left(Q''_{\sigma(\delta)}(1) + \frac{\dim V_\lambda}{24} |\delta|^2 \right) \\ &= d_{\sigma(\delta)} (N+2)! \frac{\dim V_\lambda}{48} |\lambda+\delta|^2. \end{aligned}$$

On the other hand, from (12) we have

$$\begin{aligned} F_{N+2}(\lambda+\delta, \sigma(\delta)) &= c_R d_{\lambda+\delta} d_{\sigma(\delta)} |\lambda+\delta|^2 |\delta|^2 \\ &= c_R \dim V_\lambda d_\delta d_{\sigma(\delta)} |\lambda+\delta|^2 |\delta|^2. \end{aligned}$$

We deduce that

$$c_R = \frac{(N+2)!}{48 d_\delta |\delta|^2}.$$

The proposition follows from this. □

For the general case, say that $\mathfrak{g} = \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^\ell \oplus \mathfrak{z}$ with each $\mathfrak{g}^i$ simple, and $\mathfrak{z}$ abelian. A Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is the direct sum of the center $\mathfrak{z}$ and Cartan subalgebras $\mathfrak{t}^i \subset \mathfrak{g}^i$, and the Weyl group $W = W(\mathfrak{g}, \mathfrak{t})$ is the direct product of the Weyl groups $W^i = W^i(\mathfrak{g}^i, \mathfrak{t}^i)$. Any $\mu \in \mathfrak{t}^*$ is equal to $\mu^z + \sum_i \mu^i$ with $\mu^i \in (\mathfrak{t}^i)^*$ and $\mu^z \in \mathfrak{z}^*$; similarly for $\nu \in \mathfrak{t}$. Let N_i (resp., N) be the number of positive roots in $\mathfrak{g}^i$ (resp., $\mathfrak{g}$).

Proposition 6. *Let $\mu \in \mathfrak{t}^*$ and $\nu \in \mathfrak{t}$, with notation as above. Then*

$$F_k(\mu,\nu) = \begin{cases} 0 & \text{if } 0 \le k < N, \\ N! \cdot \dfrac{d_\mu d_\nu}{d_\delta} & \text{if } k = N, \\ (N+1)! \cdot \dfrac{d_\mu d_\nu}{d_\delta} \langle \mu^z, \nu^z \rangle & \text{if } k = N+1, \\ \dfrac{(N+2)!}{48} \cdot \dfrac{d_\mu d_\nu}{d_\delta} \sum_i \dfrac{|\mu^i|^2 |\nu^i|^2}{|\delta^i|^2} & \\ \quad + \dfrac{(N+2)!}{2} \cdot \dfrac{d_\mu d_\nu}{d_\delta} \langle \mu^z, \nu^z \rangle^2 & \text{if } k = N+2. \end{cases}$$

Proof. If $\mathfrak{z} = 0$, we have

$$\begin{aligned} F_k(\mu,\nu) &= \sum_{w \in W} \operatorname{sgn}(w) \langle w(\mu^1 + \cdots + \mu^\ell), \nu^1 + \cdots + \nu^\ell \rangle^k \\ &= \sum_{w=(w_1,\ldots,w_\ell) \in W} \operatorname{sgn}(w) \left(\sum_{i=1}^{\ell} \langle w_i(\mu^i), \nu^i \rangle \right)^k \\ &= \sum_w \operatorname{sgn}(w) \sum_{k_1+\cdots+k_\ell=k} \binom{k}{k_1,\ldots,k_\ell} \prod_i \langle w_i(\mu^i), \nu^i \rangle^{k_i} \\ &= \sum_{k_1+\cdots+k_\ell=k} \binom{k}{k_1,\ldots,k_\ell} \prod_i \sum_{w_i \in W^i} \operatorname{sgn}(w_i) \langle w_i(\mu^i), \nu^i \rangle^{k_i} \\ &= \sum_{k_1+\cdots+k_\ell=k} \binom{k}{k_1,\ldots,k_\ell} \prod_i F_{k_i}(\mu^i, \nu^i). \end{aligned}$$

The product $\prod_i F_{k_i}(\mu^i,\nu^i)$ vanishes unless $k_i \ge N_i$ for all i. So $F_k(\mu,\nu)$ vanishes for $k < N$.

Now put $k = N+2$. Since $k_1 + \cdots + k_\ell = N+2$, we see by Proposition 5 that this product is only nonzero when some $k_i = N_i + 2$ and the other k_i equal N_i. Therefore

$$\begin{aligned} &F_{N+2}(\mu,\nu) \\ &\quad = \sum_{i=1}^{\ell} \binom{N+2}{N_1,\ldots,N_i+2,\ldots,N_\ell} F_{N_1}(\mu^1,\nu^1) \cdots F_{N_i+2}(\mu^i,\nu^i) \cdots F_{N_\ell}(\mu^\ell,\nu^\ell) \\ &\quad = \frac{(N+2)!}{48} \cdot \frac{d_\mu d_\nu}{d_\delta} \sum_{i=1}^{\ell} \frac{|\mu^i|^2 |\nu^i|^2}{|\delta_i|^2}. \end{aligned}$$

If $\mathfrak{z} \neq 0$, there is an extra term $\frac{(N+2)!}{2} \cdot \frac{d_\mu d_\nu}{d_\delta} \langle \mu^z, \nu^z \rangle^2$. The other cases are similar. ◻

5.3. **Main theorem for φ irreducible.**

Proposition 7. *Let $\mathfrak{g}$ be simple and $\varphi = \varphi_\lambda$ irreducible. Then for all $\nu \in \mathfrak{t}$, we have*

$$q_\lambda(\nu) = \frac{\dim V_\lambda \cdot \chi_\lambda(C)}{\dim \mathfrak{g}} \cdot \frac{|\nu|^2}{2}.$$

Proof. Let $\nu \in \mathfrak{t}_{\mathrm{reg}}$. By Proposition 4,

$$\begin{aligned} q_\lambda(\nu) &= \frac{F_{N+2}(\lambda+\delta,\nu)}{(N+2)!d_\nu} - \frac{1}{48}\dim V_\lambda |\nu|^2 \\ &= \frac{1}{48|\delta|^2} \cdot \frac{d_{\lambda+\delta}}{d_\delta} |\lambda+\delta|^2|\nu|^2 - \frac{1}{48}\dim V_\lambda|\nu|^2 \\ &= \frac{1}{48|\delta|^2}\dim V_\lambda |\nu|^2 \left(|\lambda+\delta|^2 - |\delta|^2\right). \end{aligned}$$

Recall that $\chi_\lambda(C) = |\lambda+\delta|^2 - |\delta|^2$. Moreover, by [Bou05], Exercise 7, page 256, we have $|\delta|^2 = \dim \mathfrak{g}/24$. These substitutions give the proposition for the case $\nu \in \mathfrak{t}_{\mathrm{reg}}$; by continuity it holds for $\nu \in \mathfrak{t}$. □

Example 2. Revisiting PGL_2 from Example 1, one computes $|\nu_0|^2 = 2$, $\dim V_{j\alpha} = 2j+1$, and $\chi_{j\alpha} = \frac{1}{2}(j^2+j)$, so

$$q_{j\alpha}(\nu_0) = \frac{j(j+1)(2j+1)}{6}.$$

So as before $\varphi_{j\alpha}$ is spinorial iff $j \equiv 0, 3 \mod 4$.

The case of G reductive is similar:

Proposition 8. *With notation as before, and φ_λ irreducible, we have*

$$q_\lambda(\nu) = \frac{1}{2}\dim V_\lambda \cdot \sum_i \frac{|\nu^i|^2 \chi_{\lambda^i}(C^i)}{\dim \mathfrak{g}^i}.$$

Proof. For $\mathfrak{z} = 0$, we have

$$\begin{aligned} q_\lambda(\nu) &= \frac{F_{N+2}(\lambda+\delta,\nu)}{(N+2)!d_\nu} - \frac{1}{48}\dim V_\lambda|\nu|^2 \\ &= \frac{1}{48}\cdot \dim V_\lambda \sum_{i=1}^{\ell} \frac{|\lambda^i+\delta^i|^2|\nu^i|^2}{|\delta^i|^2} - \frac{1}{48}\dim V_\lambda \sum_i |\nu_i|^2 \\ &= \frac{1}{48}\dim V_\lambda \sum_{i=1}^{l} |\nu_i|^2 \left(\frac{|\lambda^i+\delta^i|^2 - |\delta^i|^2}{|\delta^i|^2}\right). \end{aligned}$$

The substitution $|\delta^i|^2 = \dim \mathfrak{g}^i/24$ gives the proposition in the semisimple case. If $\mathfrak{z} \neq 0$, one must add $\frac{1}{2}\langle \lambda, \nu^z\rangle^2 \cdot \dim V_\lambda$. However for φ_λ irreducible orthogonal, necessarily λ annihilates the center. □

Corollary 6. *An irreducible orthogonal representation φ_λ of G is spinorial iff*

$$\frac{1}{2}\dim V_\lambda \sum_i \frac{|\nu^i|^2\chi_{\lambda^i}(C^i)}{\dim \mathfrak{g}^i}$$

is even for all cocharacters $\nu \in \underline{\nu}$.

Proof. This follows from Proposition 8 and Corollary 5. □

Example 3. For $G = \mathrm{SO}_4$, the Lie algebra $\mathfrak{g}$ is not simple. Here, $X^*(T) = X_{\mathrm{sd}} = X_{\mathrm{orth}}$, where T is the diagonal torus of G.

We may identify $\mathrm{Spin}_4 \to \mathrm{SO}_4$ with the cover $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$ as in Exercise 7.16 of [FH91]. In particular, we may identify $\mathfrak{g}$ with the Lie algebra of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, and $\mathfrak{t}$ with pairs of diagonal matrices in $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. The irreducible representations of $\mathrm{SL}_2 \times \mathrm{SL}_2$ are the external tensor products $V_{a,b} = \mathrm{Sym}^a V_0 \boxtimes \mathrm{Sym}^b V_0$, where V_0 is the standard 2-dimensional representation of SL_2. Here a, b are nonnegative integers; the representation $V_{a,b}$ descends to a representation $\varphi_{a,b}$ of G when $a \equiv b \mod 2$.

Let $\nu_s = \mathrm{diag}(s, -s)$; then $\nu_{s,t} = (\nu_s, \nu_t) \in \mathfrak{t}$ corresponds to a cocharacter of T iff either $s, t \in \mathbb{Z}$, or $2s$ and $2t$ are both odd integers. Proposition 8 gives

$$\begin{aligned} q_{\varphi_{a,b}}(\nu_{s,t}) &= \frac{1}{2}(a+1)(b+1)\left(\frac{4s^2 \cdot \frac{1}{4}a(a+2)}{3} + \frac{4t^2 \cdot \frac{1}{4}b(b+2)}{3}\right) \\ &= s^2(b+1)\binom{a+2}{3} + t^2(a+1)\binom{b+2}{3}. \end{aligned}$$

Since $\pi_1(G)$ is generated by $\nu_{\frac{1}{2},\frac{1}{2}}$, we deduce that $V_{a,b}$ is spinorial iff

$$(b+1)\binom{a+2}{3} + (a+1)\binom{b+2}{3},$$

which is always a multiple of 4, is divisible by 8.

By the following, spinoriality for irreducible orthogonal representations of connected reductive groups reduces to the semisimple case.

Proposition 9. *Let G be a connected reductive group and $\varphi : G \to \mathrm{SO}(V)$ an irreducible orthogonal representation. Then φ factors through the quotient $p : G \to G/Z(G)^\circ$, so that $\varphi = \varphi' \circ p$ with $\varphi' : G/Z(G)^\circ \to \mathrm{SO}(V)$. Moreover φ is spinorial iff φ' is spinorial.*

Proof. By Schur's Lemma, $\varphi(Z(G))$ is a subgroup of the scalars in $\mathrm{SO}(V)$, namely $\{\pm \mathrm{id}_V\}$. Therefore $\varphi(Z(G)^\circ)$ is trivial. This gives the first part, and the second part is similar. ☐

5.4. **Existence of spinorial representations.** We continue with G connected reductive. Let a be a positive multiple of 4 and $\lambda = a\delta$. Consider the irreducible representation $(\varphi_\lambda, V_\lambda)$. It is easy to see that V_λ is orthogonal, and $\dim V_\lambda = (a+1)^N$. Therefore for $\nu \in X_*(T)$, we have by Corollary 6:

$$\begin{aligned} q_\lambda(\nu) &= \frac{1}{2}(a+1)^N \sum_i \frac{|\nu^i|^2(\lambda_i, \lambda_i + 2\delta_i)}{\dim \mathfrak{g}^i} \\ &= \frac{1}{24}(a+1)^N a(a+2)\frac{|\nu|^2}{2}, \end{aligned}$$

since $(\delta_i, \delta_i) = \frac{\dim \mathfrak{g}_i}{24}$ for each i.

Therefore φ_λ is spinorial iff the quantity

$$\frac{p(\underline{\nu})}{24}(a+1)^N \cdot a(a+2)$$

is even. From this we deduce:

(1) The representation V_λ is spinorial when $a \equiv 0 \mod 8$.
(2) If $p(\underline{\nu})$ is even, then V_λ is spinorial.

In particular, we have the following corollary.

Corollary 7. *A nonabelian connected reductive group has a nontrivial irreducible spinorial representation.*

Proof. By the above, one may take $\lambda = 8\delta$. □

6. Reducible representations

In this section we treat the case of φ orthogonal, but not necessarily irreducible.

6.1. **Spinoriality of $\varphi \oplus \varphi^\vee$.** For a representation (φ, V) of a connected reductive group G, consider the orthogonal representation $(S(\varphi), V \oplus V^\vee)$ defined as follows. We give $V \oplus V^\vee$ the quadratic form

$$\mathcal{Q}((v, v^*)) = \langle v^*, v \rangle,$$

and write $S(\varphi)$ for the representation of G on $V \oplus V^\vee$ given by

$$^g(v, v^*) = (\varphi(g)v, \varphi^\vee(g)v^*).$$

For $\nu \in X_*(T)$, $\mu \in X^*(T)$, and $t \in F$, $\nu(t)$ acts on V^μ by the scalar $t^{\langle \mu, \nu \rangle}$. Therefore we have

$$\det \varphi(\nu(t)) = t^{s_\varphi(\nu)}, \tag{13}$$

where

$$s_\varphi(\nu) = \sum_{\mu \in X^*(T)} m_\varphi(\mu) \langle \mu, \nu \rangle.$$

Proposition 10. $L_{S(\varphi)}(\nu) \equiv s_\varphi(\nu) \mod 2$. *Therefore $S(\varphi)$ is spinorial iff $s_\varphi(\nu)$ is even for all $\nu \in \underline{\nu}$. If G is semisimple, then $S(\varphi)$ is spinorial.*

Proof. Since $m_{\varphi^\vee}(\mu) = m_\varphi(-\mu)$, we have

$$\begin{aligned} L_{S(\varphi)}(\nu) &= \sum_{\{\mu \mid \langle \mu, \nu \rangle > 0\}} (m_\varphi(\mu) + m_\varphi(-\mu)) \langle \mu, \nu \rangle \\ &\equiv \sum_{\{\mu \mid \langle \mu, \nu \rangle > 0\}} (m_\varphi(\mu) - m_\varphi(-\mu)) \langle \mu, \nu \rangle \mod 2 \\ &= s_\varphi(\nu). \end{aligned}$$

When G is semisimple, the image of φ lies in $\mathrm{SL}(V)$, and so $s_\varphi(\nu) = 0$. Therefore $L_{S(\varphi)}(\nu)$ is even and so $S(\varphi)$ is spinorial in this case. □

Now, assume $\varphi = \varphi_\lambda$ is irreducible. Let $\nu = \nu^z + \nu'$ correspond to the decomposition $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}'$.

Theorem 4. *$S(\varphi_\lambda)$ is spinorial iff the integers*

$$\langle \lambda, \nu^z \rangle \cdot \dim V_\lambda$$

are even for all $\nu \in \underline{\nu}$.

Proof. Differentiating both sides of (13) at $t = 1$ gives

$$s_\varphi(\nu) = \operatorname{tr} d\varphi(d\nu(1)),$$

where $\operatorname{tr} : \mathfrak{t}_V \to F$ is the trace.

Write $\mathfrak{z}_V$ for the center of the Lie algebra of $\mathrm{GL}(V)$, and $\mathfrak{t}'_V$ for the Lie algebra of the maximal torus in $\mathrm{SL}(V)$. We have a direct sum decomposition $\mathfrak{t}_V = \mathfrak{t}'_V \oplus \mathfrak{z}_V$, and similarly for $\mathfrak{t}$. Let $\mathrm{pr}_V : \mathfrak{t}_V \to \mathfrak{z}_V$ and $\mathrm{pr} : \mathfrak{t} \to \mathfrak{z}$ be the projections.

Note that the diagram

$$\begin{array}{ccccccc} F & \xrightarrow{d\nu} & \mathfrak{t} & \xrightarrow{d\varphi} & \mathfrak{t}_V & \xrightarrow{\mathrm{tr}} & F \\ & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}_V} & \nearrow{\scriptstyle \mathrm{tr}} & \\ & & \mathfrak{z} & \xrightarrow{d\varphi} & \mathfrak{z}_V & & \end{array}$$

is commutative. Moreover $\mathrm{tr}(d\varphi(z)) = d\lambda(z) \cdot \dim V_\lambda$ for $z \in \mathfrak{z}$, by Schur's Lemma. It follows that

$$s_\varphi(\nu) = d\lambda(\nu^z) \cdot \dim V_\lambda,$$

so the theorem follows from the previous proposition. □

Example 4. Let $G = \mathrm{GL}_2$. We may parametrize $X^*(T)^+$ with integers (m, n) with $0 \le n \le m$ via:

$$\lambda_{m,n} \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = t_1^m t_2^n.$$

Let $\nu_0(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so that $(\nu_0)^z = \frac{1}{2}(1,1)$. Then $\dim V_{\lambda_{m,n}} = m - n + 1$ and $\langle \lambda, \nu_0^z \rangle = \frac{1}{2}(m+n)$, so $s_{\lambda_{m,n}}(\nu_0) = \frac{1}{2}(m+n)(m-n+1)$. From Theorem 4, we deduce that the representation $S(\varphi_{\lambda_{m,n}})$ of GL_2 is spinorial iff the integer $\frac{1}{2}(m+n)(m-n+1)$ is even.

6.2. **General lifting condition.** We begin this section by gathering our results to give a general lifting condition for reducible orthogonal representations.

Recall we have $\mathfrak{g} = \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^\ell \oplus \mathfrak{z}$ with each $\mathfrak{g}^i$ simple and $\mathfrak{z}$ abelian. Thus our $\nu \in \mathfrak{t}$ decomposes into $\nu^z + \sum_i \nu^i$ with $\nu^i \in \mathfrak{t}^i$ and $\nu^z \in \mathfrak{z}$.

Proposition 11. *If φ is an orthogonal representation of G, then φ is a direct sum of representations of the following type:*

- *Irreducible orthogonal representations.*
- *The representations $S(\sigma)$, with σ irreducible.*

Proof. This follows from Lemma C in Section 3.11 of [Sam90]. □

Theorem 5. *Let $\varphi = S(\sigma) \oplus \bigoplus_j \varphi_j$, with each φ_j irreducible orthogonal with highest weight λ_j, and $\sigma = \bigoplus_k \sigma_k$, with each σ_k irreducible with highest weight γ_k. Then φ is spinorial iff for all $\nu \in \underline{\nu}$, the integer*

$$q_\varphi(\nu) = \sum_k \langle \gamma_k, \nu^z \rangle \cdot \dim V_{\gamma_k} + \sum_i \frac{|\nu^i|^2}{2} \sum_j \frac{\dim V_{\lambda_j} \cdot \chi_{\lambda_j^i}(C^i)}{\dim \mathfrak{g}^i}$$

is even.

Proof. We have

$$\begin{aligned} L_\varphi(\nu) &= \sum_k L_{S(\sigma_k)}(\nu) + \sum_j L_{\varphi_j}(\nu) \\ &\equiv \sum_k s_{\sigma_k}(\nu) + \sum_j q_{\varphi_j}(\nu) \mod 2. \end{aligned}$$

The first equality is by (5), and the congruence is by Proposition 10 and Proposition 3. The conclusion then follows from Theorem 4 and Corollary 6. □

Note that when G is semisimple, the sum over k vanishes.

6.3. **Case of $\mathfrak{g}$ simple.** The situation is much nicer when $\mathfrak{g}$ is simple; let us deduce Theorem 1 as a Corollary of Theorem 5.

Proof of Theorem 1. By Theorem 5, we have

$$\begin{aligned} q_\varphi(\nu) &= \frac{|\nu|^2}{2}\sum_j \frac{\dim V_{\lambda_j}\cdot \chi_{\lambda_j}(C)}{\dim \mathfrak{g}} \\ &= \frac{|\nu|^2}{2}\frac{\operatorname{tr}(C,V)}{\dim \mathfrak{g}}. \end{aligned}$$

This must be even for all $\nu \in \underline{\nu}$; equivalently

$$p(\underline{\nu})\cdot \frac{\operatorname{tr}(C,V)}{\dim \mathfrak{g}}$$

must be even. □

Corollary 8. *Let $\mathfrak{g}$ be simple, and let $\varphi = \varphi_1 \oplus \varphi_2$ with φ_1, φ_2 orthogonal. Then φ is spinorial iff either both φ_1, φ_2 are spinorial, or both φ_1, φ_2 are aspinorial.*

The following corollary will be useful when varying the isogeny class of G.

Corollary 9. *Let $\rho : \tilde{G} \to G$ be a cover, with simple Lie algebra, and let $\underline{\tilde{\nu}}, \underline{\nu}$ be two sets of cocharacters, with $\underline{\tilde{\nu}}$ generating $\pi_1(\tilde{G})$ and $\underline{\nu}$ generating $\pi_1(G)$. Suppose that $\operatorname{ord}_2(p(\underline{\tilde{\nu}})) = \operatorname{ord}_2(p(\underline{\nu}))$. Then an orthogonal representation φ of G is spinorial iff $\overline{\varphi} = \varphi \circ \rho$ is spinorial.*

Proof. This follows since then

$$\begin{aligned} \operatorname{ord}_2(q_\varphi) &= \operatorname{ord}_2(p(\underline{\nu})\tau(\varphi)) \\ &= \operatorname{ord}_2(p(\underline{\tilde{\nu}})\tau(\overline{\varphi})) \\ &= \operatorname{ord}_2(q_{\overline{\varphi}}). \end{aligned}$$

□

6.4. **A counterexample.** The simplicity hypothesis for Corollary 8 is necessary, for example let G_1 and G_2 be connected semisimple groups, with orthogonal representations (φ_1, V_1) and (φ_2, V_2), respectively. Let $G = G_1 \times G_2$, and write $\Phi_i : G_1 \times G_2 \to \mathrm{SO}(V_i)$ for the inflations of φ_1, φ_2 to G via the two projections. Put $\Phi = \Phi_1 \oplus \Phi_2$. For ν_1, ν_2 cocharacters of tori of G_1, G_2, put $\nu = \nu_1 \times \nu_2$. It is easy to see that

$$L_\Phi(\nu) = L_{\varphi_1}(\nu_1) + L_{\varphi_2}(\nu_2).$$

Therefore in this situation,

$$\begin{aligned} \Phi \text{ is spinorial } &\Leftrightarrow \varphi_1 \text{ and } \varphi_2 \text{ are spinorial} \\ &\Leftrightarrow \Phi_1 \text{ and } \Phi_2 \text{ are spinorial.} \end{aligned}$$

For example, if $G = \mathrm{SO}(3) \times \mathrm{SO}(3)$, and φ_1, φ_2 are aspinorial (e.g., the defining representation of $\mathrm{SO}(3)$), then each of Φ_1, Φ_2, and $\Phi_1 \oplus \Phi_2$ is aspinorial.

7. Dynkin index

Let $\mathfrak{g}$ be a simple Lie algebra with a long root α. The quantity

$$\check{h} = \frac{1}{|\alpha|^2}$$

is called the *dual Coxeter number of G*. (See Section 2 of [Kos76].)

Following [Dyn52], we define a bilinear form on $\mathfrak{t}$ by

$$(x, y)_d = 2\check{h} \cdot (x, y)$$

for $x, y \in \mathfrak{g}$. In other words, we renormalize the Killing form so that $(\alpha, \alpha)_d = 2$.

Definition 6. Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism of simple Lie algebras. Then there exists an integer $\mathrm{dyn}(\phi)$, called the Dynkin index of ϕ, so that for $x, y \in \mathfrak{g}$, we have

$$(\phi(x), \phi(y))_d = \mathrm{dyn}(\phi) \cdot (x, y)_d.$$

If $\phi \neq 0$, then $\mathrm{dyn}(\phi) \neq 0$. Also, if $f' : \mathfrak{g}_2 \to \mathfrak{g}_3$ is another homomorphism of simple Lie algebras, then $\mathrm{dyn}(f' \circ f) = \mathrm{dyn}(f')\,\mathrm{dyn}(f)$. We refer the reader to [Dyn00], page 195, Theorem 2.2, and (2.4).

We assume for the rest of this section that $\mathfrak{so}_V$ is simple, equivalently $\dim V \neq 1, 2, 4$. Note that there are no nontrivial irreducible orthogonal representations of $\mathfrak{g}$ with those degrees. The following is an easy calculation.

Lemma 5. *If $\iota_V : \mathfrak{so}_V \hookrightarrow \mathfrak{sl}_V$ is the standard inclusion, then* $\mathrm{dyn}(\iota_V) = 2$.

Now let $\varphi : \mathfrak{g} \to \mathfrak{sl}_V$ be a nontrivial orthogonal Lie algebra representation. Then we may write $\varphi = \iota_V \circ \varphi'$, where $\varphi' : \mathfrak{g} \to \mathfrak{so}_V$. We define $\mathrm{dyn}^o(\varphi) = \mathrm{dyn}(\varphi') \in \mathbb{N}$; thus $\mathrm{dyn}(\varphi) = 2\,\mathrm{dyn}^o(\varphi)$.

Theorem 6. *For $\varphi : \mathfrak{g} \to \mathfrak{sl}_V$ a representation, we have*

$$\mathrm{dyn}(\varphi) = 2\check{h}\frac{\mathrm{tr}(C; V)}{\dim \mathfrak{g}}.$$

Proof. This is a reformulation of Theorem 2.5 of [Dyn00], page 197. ☐

Corollary 10. *Let G have simple Lie algebra $\mathfrak{g}$, and let φ be an orthogonal representation of G. For a cocharacter ν we have*

$$q_\varphi(\nu) = \frac{|\nu|^2}{2} \cdot \frac{\mathrm{dyn}^o(\varphi)}{\check{h}}.$$

Therefore φ is spinorial iff

$$p(\underline{\nu}) \cdot \frac{\mathrm{dyn}^o(\varphi)}{\check{h}}$$

is even.

This formula is convenient because for simple $\mathfrak{g}$, the dual Coxeter numbers are tabulated in Section 6 of [Kac90], and Dynkin indices for fundamental representations are found in Table 5 of [Dyn52]. In the forthcoming examples, we will use these tables without further comment.

8. Tensor products

In this section, we explain how the spinoriality of a tensor product of two orthogonal representations is related to the spinoriality of the factors.

8.1. **Internal products.** Let G be a connected reductive group and (φ_1, V_1), (φ_2, V_2) orthogonal representations of G. Write $(\varphi, V) = (\varphi_1 \otimes \varphi_2, V_1 \otimes V_2)$ for the (internal) tensor product representation of G.

Proposition 12. *For $\nu \in X_*(T)$, we have*

$$q_\varphi(\nu) = \dim V_1 \cdot q_{\varphi_2}(\nu) + \dim V_2 \cdot q_{\varphi_1}(\nu). \tag{14}$$

Proof. For $t \in F^\times$, we have

$$\Theta_\varphi(\nu(t)) = \Theta_{\varphi_1}(\nu(t))\Theta_{\varphi_2}(\nu(t)).$$

Therefore

$$Q''_{(\varphi,\nu)} = Q_{(\varphi_1,\nu)}Q''_{(\varphi_2,\nu)} + 2Q'_{(\varphi_1,\nu)}Q'_{(\varphi_2,\nu)} + Q''_{(\varphi_1,\nu)}Q_{(\varphi_2,\nu)},$$

and so

$$Q''_{(\varphi,\nu)}(1) = \dim V_1 \cdot Q''_{(\varphi_2,\nu)}(1) + \dim V_2 \cdot Q''_{(\varphi_1,\nu)}(1).$$

The proposition follows. □

Corollary 11. *If φ_1, φ_2 are spinorial, then so is $\varphi_1 \otimes \varphi_2$.*

8.2. **External tensor products.** Next, let $(\varphi_1, V_1), (\varphi_2, V_2)$ be orthogonal representations of connected reductive groups G_1, G_2, respectively. Write $(\varphi, V) = (\varphi_1 \boxtimes \varphi_2, V_1 \otimes V_2)$ for the external tensor product representation of $G = G_1 \times G_2$. If T_1, T_2 are maximal tori for G_1, G_2, then $T = T_1 \times T_2$ is a maximal torus of G.

As in the previous proposition, we have:

Proposition 13. *For $\nu = (\nu_1, \nu_2) \in X_*(T) = X_*(T_1) \oplus X_*(T_2)$, we have*

$$q_\varphi(\nu) = \dim V_1 \cdot q_{\varphi_2}(\nu_2) + \dim V_2 \cdot q_{\varphi_1}(\nu_1).$$

8.3. **Positive orthogonal spanning sets.** In the examples to come, it will be convenient to have a set of orthogonal dominant weights of G which play the role of fundamental weights, but in X^+_{orth}.

Definition 7. Let S_o be a set of dominant orthogonal weights. We say that S_o is a positive orthogonal spanning set (POSS) for G, provided every dominant orthogonal weight can be written as a nonnegative integral combination of S_o.

The strategy will be to deduce the spinoriality of an arbitrary φ_λ from the spinoriality of the representations φ_{λ_0} with $\lambda_0 \in S_0$.

Lemma 6. *Let G be semisimple and μ_0, ν_0 dominant weights. Put $\lambda_0 = \mu_0 + \nu_0$. Suppose that $\Phi = \varphi_{\mu_0} \otimes \varphi_{\nu_0}$ is spinorial, and that one of the following conditions holds:*

1. *φ_λ is spinorial for any dominant orthogonal $\lambda \neq \lambda_0$ with $\lambda \prec \lambda_0$.*
2. *φ_λ is spinorial for any dominant orthogonal λ with $|\lambda| < |\lambda_0|$.*

Then φ_{λ_0} is spinorial.

Proof. By Proposition 11, Φ decomposes into a sum of irreducible orthogonal representations φ_λ possibly together with an $S(\sigma)$ summand.

Let us see that each φ_λ is spinorial, for $\lambda \neq \lambda_0$. If the first condition holds, this is clear by ([Bou05], page 132, Proposition 9 i).

Suppose the second condition holds. Each weight λ of Φ decomposes into $\lambda = \mu_1 + \nu_1$, with μ_1 a weight of φ_{μ_0} and ν_1 a weight of φ_{ν_0}. Therefore $\mu_0 - \mu_1$ and $\nu_0 - \nu_1$ are positive. Moreover φ_{λ_0} itself occurs with multiplicity one.

The inner product of a dominant weight with a positive one is nonnegative, thus

$$\begin{aligned}|\lambda|^2 &\le (\lambda, \mu_0 + \nu_0)\\ &\le |\mu_0|^2 + |\nu_0|^2 + (\mu_0, \nu_1) + (\mu_1, \nu_0)\\ &\le |\lambda_0|^2.\end{aligned}$$

Moreover by [Bou05], page 129, Proposition 5(iii), equality holds iff $\mu_0 = \mu_1$ and $\nu_0 = \nu_1$, i.e., iff $\lambda = \lambda_0$. Thus by the second condition, each orthogonal weight λ of Φ, except a priori λ_0, has φ_λ spinorial. Recall that $S(\sigma)$ is spinorial by Proposition 10. So by Corollary 8, it must be that φ_{λ_0} is spinorial. □

Proposition 14. *Let $\mathfrak{g}$ be simple and suppose S_o is a POSS for G. If φ_λ is spinorial for each $\lambda \in S_o$, then all orthogonal representations of G are spinorial.*

Proof. By Proposition 10 and Corollary 8 we reduce to the case of irreducible orthogonal φ_{λ_0}. We prove the proposition by induction on $|\lambda_0|$.

If $\lambda_0 \in S_o$, then φ_{λ_0} is spinorial. Otherwise $\lambda_0 = \mu_0 + \nu_0$ with $\nu_0 \in S_o$ and μ_0 dominant orthogonal. Since

$$|\lambda_0|^2 = |\mu_0|^2 + 2(\mu_0, \nu_0) + |\nu_0|^2 > |\mu_0|^2,$$

we can say that φ_{μ_0} is spinorial. Put $\Phi = \varphi_{\mu_0} \otimes \varphi_{\nu_0}$.

By Corollary 11, Φ is a spinorial orthogonal representation of G. Therefore φ_{λ_0} is spinorial, by Lemma 6. □

9. Type A_{n-1}

For the next few sections, we will pursue the question: For which groups G, with $\mathfrak{g}$ simple, is every orthogonal representation spinorial? This section treats the quotients of SL_n.

9.1. **Preliminaries for type A_{n-1}.** Let n be an even positive integer. The center of SL_n is cyclic of order n, and can be identified with the group μ_n of nth roots of unity in $F^\times$. Let T_1 be the diagonal torus of SL_n. Let $\vartheta_i \in X^*(T_1)$ be the character of T_1 given by taking the ith diagonal entry. The roots of T_1 are of the form $\vartheta_i - \vartheta_j$ for $i \neq j$.

Let d be a divisor of n, and μ_d the subgroup of μ_n of order d. In this section we consider the spinoriality of orthogonal representations of $G_d = \mathrm{SL}_n / \mu_d$. The maximal torus $T_d < G_d$ is the image of T_1 under this quotient.

Recall that generally $X_*(T_d)$ injects into $\mathfrak{t}$ by $\nu \mapsto d\nu(1)$. When T_n is the diagonal torus of PGL_n, the injection $X_*(T_n) \hookrightarrow \mathfrak{t}$ can be identified with the natural injection

$$\frac{\mathbb{Z}^n}{\mathbb{Z}(1,1,\dots,1)} \hookrightarrow \frac{F^n}{F(1,1,\dots,1)}.$$

In these terms, each coroot lattice $Q(T_d) = X_*(T_1)$ is given by

$$\left\{ (x_1, \dots, x_n) \in \frac{\mathbb{Z}^n}{\mathbb{Z}(1,1,\dots,1)} \mid \sum_{i=1}^n x_i \equiv 0 \mod n \right\},$$

and the subgroup $X_*(T_d) \le X_*(T_n)$ is equal to

$$\left\{ (x_1, \dots, x_n) \in \frac{\mathbb{Z}^n}{\mathbb{Z}(1,1,\dots,1)} \mid \sum_{i=1}^n x_i \equiv 0 \mod \frac{n}{d} \right\}.$$

Let $\nu_d \in X_*(T_d)$ be the cocharacter parametrized by $(\frac{n}{d}, 0, \ldots, 0)$ in these terms. Then ν_d generates $\pi_1(G_d)$, which is therefore cyclic of order d. Of course, when d is odd, every orthogonal representation of G_d is spinorial. Let us henceforth take d even. We compute

$$p(\nu_d) = \left(\frac{n}{d}\right)^2 (n-1).$$

Thus by Theorem 1 we deduce the following proposition.

Proposition 15. *Let φ_λ be an irreducible orthogonal representation of G_d. Then φ_λ is spinorial iff*

$$\left(\frac{n}{d}\right)^2 \dim V_\lambda \cdot \chi_\lambda(C)$$

is even.

(We regard a rational number as *even* if, when written in lowest terms, its numerator is even.)

Example 5. Let φ_λ be an irreducible orthogonal representation of GL_n. By Proposition 9, it descends to an orthogonal representation $\overline{\varphi}_\lambda$ of $G_n = \mathrm{PGL}_n$, and φ_λ is spinorial iff $\overline{\varphi}_\lambda$ is. By the above, we deduce that φ_λ is spinorial iff $\dim V_\lambda \cdot \chi_\lambda(C)$ is even.

Proposition 15 does not by itself answer the question at the beginning of this section, and the groups G_d are somewhat awkward to compute with directly. So instead we ask, which morphisms from SL_n to $\mathrm{Spin}(V)$ descend to G_d?

9.2. **Descent method.** Consider the following approach to determining the spinoriality of an orthogonal $\varphi : G/C \to \mathrm{SO}(V)$, where G is a simply connected and C is central. Write $\hat{\varphi} : G \to \mathrm{Spin}(V)$ for the lift of φ. Then φ is spinorial iff $C \leq \ker \hat{\varphi}$. In this subsection we pursue this method for certain C; this approach will tremendously simplify the theory for the groups $G/C = G_d$ of type A_{n-1}.

Resetting notation, let $\hat{\varphi} : G \to \mathrm{Spin}(V)$ be a morphism, with G connected semisimple. Put $\varphi = \rho \circ \hat{\varphi} : G \to \mathrm{SO}(V)$, and suppose that φ is irreducible. Let $C \leq Z(G)$, and suppose that $C \leq \ker \varphi$. Then φ descends to an orthogonal representation $\overline{\varphi}$ of G/C, which is spinorial iff $C \leq \ker \hat{\varphi}$.

Let d be a positive even integer, and $\zeta_d \in F^\times$ a primitive dth root of unity. If $\nu \in X_*(T)$ with $\nu(\zeta_d) \in C$, then for all weights μ of φ, we have $d | \langle \mu, \nu \rangle$.

Proposition 16. *Let $\nu : \mathbb{G}_m \to T$ be a cocharacter, so that C is generated by $\nu(\zeta_d)$. The following are equivalent:*

(1) *$\overline{\varphi}$ is spinorial.*
(2) *$\hat{\varphi}(\nu(\zeta_d)) = 1$.*
(3) *$2d$ divides $L_\varphi(\nu)$.*

Proof. Choose Σ as in the proof of Proposition 2. Note that $\hat{\varphi}_* \nu \in X_*(\tilde{T}_V)$ is a lift of $\varphi_* \nu \in X_*(T_V)$. By Lemma 1, we have $\hat{\varphi}(\nu(\zeta_d)) = 1$ iff

$$2d \mid \langle \omega_\Sigma, \varphi_* \nu \rangle,$$

which, as in the proof of Proposition 2, is equal to $L_\varphi(\nu)$. ∎

9.3. **Application to SL_n.** We return to $G = \mathrm{SL}_n$, with n even. For $1 \leq i \leq n$, put $\varpi_i = \vartheta_1 + \cdots + \vartheta_i$. Let $\nu_0 \in X_*(T)$ be the cocharacter defined by

$$\nu_0(t) = \mathrm{diag}(t, t, \ldots, t, t^{1-n}).$$

Let d be an even divisor of n. Then $\nu_0(\zeta_d)$ generates $\mu_d < G$, so by Proposition 16, the representation $(\overline{\varphi}_\lambda, V)$ of G_d is spinorial iff $2d$ divides $L_\varphi(\nu_0)$.

Proposition 17. *The adjoint representation of G_d is spinorial iff $\frac{n}{d}$ is even.*

Proof. For $\varphi = \mathrm{ad}$, we have

$$\begin{aligned} L_{\mathrm{ad}}(\nu_0) &= \sum_{\alpha \in R : \langle \alpha, \nu_0 \rangle > 0} \langle \alpha, \nu_0 \rangle \\ &= \sum_{i=1}^{n-1} \langle \vartheta_i - \vartheta_n, \nu_0 \rangle \\ &= n(n-1). \end{aligned}$$

This is divisible by $2d$ iff $\frac{n}{d}$ is even. □

9.4. **Case where n/d is even.** For all $q \in Q(T)$, the quantity $\langle q, \nu_0 \rangle$ is divisible by n. Since all weights of V_λ are congruent mod $Q(T)$, we deduce that

$$L_{\varphi_\lambda}(\nu_0) \equiv \langle \lambda, \nu_0 \rangle \cdot \left(\sum_{\mu : \langle \mu, \nu_0 \rangle > 0} m_\mu \right) \mod n. \tag{15}$$

Proposition 18. *Suppose that $2d$ divides n, and an irreducible orthogonal representation φ_λ of SL_n descends to the orthogonal representation $\overline{\varphi_\lambda}$ of G_d.*

(1) *If V_λ is odd-dimensional, then $\overline{\varphi}_\lambda$ is spinorial.*
(2) *If V_λ is even-dimensional, then $\overline{\varphi}_\lambda$ is spinorial iff the product $\frac{1}{2} \dim V_\lambda \cdot \langle \lambda, \nu_0 \rangle$ is divisible by $2d$.*

Proof. If φ_λ is orthogonal with odd degree, then the trivial weight must occur in V_λ, which implies that $\lambda \in Q(T)$. From (15), we see that $L_\varphi(\nu_0)$ is divisible by n, and the first statement follows.

Now suppose φ_λ has even degree. If $\langle \lambda, \nu_0 \rangle$ is divisible by n, then $\overline{\varphi}_\lambda$ is spinorial and the second statement is clear. If $\langle \lambda, \nu_0 \rangle$ is not divisible by n, then for all μ occurring in V_λ, it must be that $\langle \mu, \nu_0 \rangle \neq 0$. It follows that

$$\sum_{\mu : \langle \mu, \nu_0 \rangle > 0} m_\mu = \frac{1}{2} \dim V_\lambda,$$

and the second statement follows from (15). □

Let $\varpi_i^o = \varpi_i + \varpi_{n-i}$ for $1 \leq i < \frac{n}{2}$ and

$$S_o = \left\{ \varpi_i^o \mid 1 \leq i < \frac{n}{2} \right\} \cup \left\{ \varpi_{n/2} \right\}.$$

It is easy to see that S_o is a POSS for G_d. Note that $\langle \varpi_i^o, \nu_0 \rangle = n$ for $1 \leq i < \frac{n}{2}$, and $\langle \varpi_{n/2}, \nu_0 \rangle = n/2$.

Proposition 19. *Suppose $2d$ divides n. For each $1 \leq i < \frac{n}{2}$, the representation $V_{\varpi_i^o}$ of G_d is spinorial.*

Proof. This follows from Proposition 18. □

The representation V_λ for $\lambda = \varpi_{n/2}$ is the exterior power $\bigwedge^{n/2} V_0$, where V_0 is the standard representation of SL_n.

Proposition 20. *Let $\lambda = \varpi_{n/2}$. Then $\overline{\varphi}_\lambda$ is aspinorial iff n is a power of 2 and $d = n/2$.*

Proof. From elementary number theory we know that $\dim V_\lambda = \binom{n}{n/2}$ is even, and divisible by 4 iff n is not a power of 2. We have

$$\frac{1}{2}\dim V_\lambda \cdot \langle \lambda, \nu_0 \rangle = \frac{n}{4}\binom{n}{n/2}. \tag{16}$$

This is divisible by d, since the binomial coefficient is always even, and d divides $\frac{n}{2}$. Thus (16) is divisible by $2d$ unless both $\frac{n}{2d}$ is odd, and n is a power of 2. In this case it must be that n is a power of 2 and $n = 2d$. ☐

Theorem 7. *Suppose that n/d is even. Unless $n = 2^{k+1}$ for some $k \geq 1$ and $d = 2^k$, every orthogonal representation of $G_d = \mathrm{SL}_n / \mu_d$ is spinorial.*

Proof. By Proposition 14, it is enough to check that $\overline{\varphi}_\lambda$ is spinorial for each $\lambda \in S_o$. But we have done this. ☐

Example 6. Although the adjoint representation of $G_2 = \mathrm{SL}_4 / \{\pm 1\}$ is spinorial, the representation $V_{\varpi_2} = \bigwedge^2 F^4$ of G_2 is aspinorial.

Remark 3. What makes the "descent method" work in the case of G_d with n/d even is the fortunate fact that $\langle q, \nu_0 \rangle$ is divisible by n for $q \in Q(T)$. In other contexts, it is unclear how to compute $L_\varphi(\nu) \mod 2d$.

9.5. **Summary for the groups G_d.** Let n be a positive integer, d a divisor of n, and $G_d = \mathrm{SL}_n / \mu_d$.

From the above we have:

- If d is odd, then every orthogonal representation of G_d is spinorial.
- If n is even and n/d is odd, then the adjoint representation of G_d is aspinorial.
- If n is a power of 2 and $d = n/2$, then $\bigwedge^d V_0$ is an aspinorial representation of G_d.
- If n/d is even, then every orthogonal representation of G_d is spinorial, unless n is a power of 2 and $d = n/2$.

In particular, every orthogonal representation of G_d is spinorial iff n is odd, or n/d is even with $(n, d) \neq (2^{k+1}, 2^k)$.

10. Type C_n

Let J be the $2n \times 2n$ matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I is the $n \times n$ identity matrix. We let

$$G_{\mathrm{sc}} = \mathrm{Sp}_{2n} = \{ g \in \mathrm{GL}_{2n} \mid g^t J g = J \}.$$

Write T_{sc} for the diagonal torus in G_{sc}. A typical element is

$$\mathrm{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}).$$

We identify $X_*(T_{\mathrm{sc}})$ with $\mathbb{Z}^n$ by $(b_1, \ldots, b_n) \mapsto \nu$, where

$$\nu(t) = \mathrm{diag}(t^{b_1}, \ldots, t^{b_n}, \ldots).$$

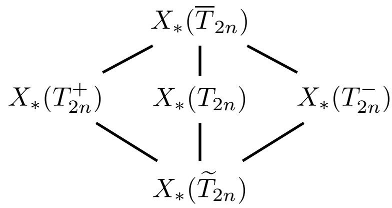

FIGURE 1. Cocharacter lattice for D_{2n}

Put $G = \mathrm{Sp}_{2n}/\{\pm 1\}$, and let T be the image of T_{sc} under the quotient. Then $X_*(T_{\mathrm{sc}})$ has index 2 in $X_*(T)$; more precisely we may write

$$X_*(T) = X_*(T_{\mathrm{sc}}) + \mathbb{Z}\cdot\nu_0,$$

where $\nu_0 = \frac{1}{2}(1,1,\ldots,1)$. In particular, $\pi_1(G)$ is cyclic of order 2, generated by ν_0.

Remark 4. One way to understand ν_0 is through the isomorphism $\mathrm{Sp}_{2n}/\{\pm 1\} \cong \mathrm{GSp}_{2n}/Z$, where GSp_{2n} is the general symplectic group defined with J, and Z is its center. The cocharacter

$$t \mapsto \mathrm{diag}(\underbrace{t,t,\ldots,t}_{n\ \text{times}},1,1,\ldots,1),$$

of the diagonal torus of GSp_{2n}, when projected to T, is ν_0.

We have $p(\nu_0) = \frac{1}{2}n(n+1)$. Every representation φ of G is orthogonal. Since $\check{h} = n+1$, we have by Corollary 10:

$$\begin{aligned} q_\varphi(\nu_0) &= \frac{\frac{1}{2}n(n+1)}{\check{h}}\,\mathrm{dyn}^o(\varphi) \\ &= \frac{1}{2}n\cdot\mathrm{dyn}^o(\varphi). \end{aligned} \tag{17}$$

Proposition 21. *Every representation of* $\mathrm{Sp}_{2n}/\{\pm 1\}$ *is spinorial iff* $4|n$.

Proof. If $n \equiv 0 \mod 4$, then every representation is spinorial by (17). If $n \equiv 1,2 \mod 4$, then the adjoint representation is aspinorial, and if $n \equiv 3 \mod 4$, then the second fundamental representation is aspinorial. □

11. TYPE D_n

The simply connected group of type D_n is $G_{\mathrm{sc}} = \mathrm{Spin}_{2n}$. The center Z of G_{sc} has order 4; in the notation of Section 2.4, it is generated by c^+ when n is odd, and generated by c^+ and z when n is even.

Thus the groups of type D_n for n odd are G_{sc} and its quotients SO_{2n} and PSO_{2n}. When n is odd, the adjoint representation of PSO_{2n} is aspinorial, which ends our investigation in this case. Henceforth in this section, we will assume that n is even, and to ensure $\mathfrak{g}$ is simple we take $n > 2$. (See Example 3 for SO_4.)

For n even, there are two more groups of type D_{2n}: the quotient G_{2n}^+ of G_{sc} by $\langle c^+\rangle$, and the quotient G_{2n}^- of G_{sc} by $\langle -c^+\rangle$. Write $T_{2n} = T_V$, where $V = F^{2n}$, write $\widetilde{T}_{2n} < G_{\mathrm{sc}}$ for its preimage, and write $T_{2n}^{\pm}$ and $\overline{T}_{2n}$ for the corresponding tori of $G_{2n}^{\pm}$ and PSO_{2n}. The lattice of cocharacters corresponding to these quotients is depicted in Figure 1.

Recall that we identified $X_*(T_{2n})$ with $\mathbb{Z}^n$ in Section 2.4. Let

$$Q = X_*(\widetilde{T}_{2n}) = \left\{(b_1, \dots, b_n) \in \mathbb{Z}^n \mid \sum b_i \text{ is even}\right\}.$$

Then

$$X_*(T_{2n}^{\pm}) = Q + \mathbb{Z} \cdot \frac{1}{2}(1, 1, \dots, \pm 1)$$

and

$$X_*(\overline{T}_{2n}) = \mathbb{Z}^n + \mathbb{Z} \cdot \frac{1}{2}(1, 1, \dots, 1).$$

All representations of groups of type D_n are orthogonal. The standard representation V_0 of SO_{2n} is evidently aspinorial. Thus, for the rest of this section we focus on the groups PSO_{2n} and $G_{2n}^{\pm}$ with n even.

Tables 1 and 2 record the quantities $p(\underline{\nu})$, $\dim V_\lambda$, and $\chi_\lambda(C)$ that we need for our formulas. Here $\varpi_k = (\underbrace{1, 1, \dots, 1}_{k \text{ times}}, 0, \dots, 0)$ and $\varpi_- = (1, 1, \dots, 1, -1)$. (We parametrize X_*, X^* by $\mathbb{Z}^n$ as in Section 2.4.)

TABLE 1. Computing $p(\underline{\nu})$

G	$\underline{\nu}$	$p(\underline{\nu})$
SO_{2n}	$(1, 0, \dots, 0)$	$2n-2$
PSO_{2n} $n \equiv 0 \mod 4$	$(1, 0, \dots, 0)$, $\frac{1}{2}(1, \dots, 1)$	$2n-2$
PSO_{2n} $n \equiv 2 \mod 4$	$(1, 0, \dots, 0)$, $\frac{1}{2}(1, \dots, 1)$	$n-1$
$G_{2n}^{\pm}$	$\frac{1}{2}(1, \dots, \pm 1)$	$\binom{n}{2}$

TABLE 2. $\dim V_\lambda$ and $\chi_\lambda(C)$ for type D_n, n even

λ	$\dim V_\lambda$	$\chi_\lambda(C)$
ϖ_k	$\binom{2n}{k}$	$\frac{k(2n-k)}{4n-4}$
$\frac{1}{2}\varpi_-$	2^{n-1}	$\frac{n(2n-1)}{16(n-1)}$
$\frac{1}{2}\varpi_n$	2^{n-1}	$\frac{n(2n-1)}{16(n-1)}$
ϖ_-	$\frac{(2n-1)!}{2^n}$	$\frac{n^2}{4n-4}$

Remark 5. Let n be odd. One has similarly $p(\nu_0) = n-2$ for SO_n, with

$$\nu_0(t) = \mathrm{diag}(t, 1, \dots, 1, t^{-1}).$$

11.1. **The case of** PSO_{2n}**.** If $n \equiv 2 \mod 4$, then the representation φ of PSO_{2n} on $\bigwedge^2 V_0$ is aspinorial by Corollary 10: here $\mathrm{dyn}^o(\varphi) = 2n-2$ and $\check{h} = 2n-2$, so

$$p(\underline{\nu}) \cdot \frac{\mathrm{dyn}^o(\varphi)}{\check{h}} = n - 1.$$

Let us assume for the rest of this section that n is a multiple of 4; we will prove every orthogonal representation is spinorial in this case. We first consider φ_λ, with λ in the set

$$S_0 = \{\varpi_k, \varpi_- \mid k \text{ even}, 1 \leq k \leq n\}.$$

Proposition 22. *Each representation φ_λ of SO_{2n} with $\lambda \in S_0$ is spinorial.*

Proof. Tables 1 and 2 give

$$\begin{aligned} q_{\varpi_k} &= \frac{1}{2}\frac{\binom{2n}{k}}{\binom{2n}{2}} \cdot k(2n-k) \\ &= \binom{2n-2}{k-1}. \end{aligned} \tag{18}$$

Since k is even, this is necessarily even, and we deduce that each φ_{ϖ_k} is spinorial. Similarly

$$q_{\varpi_-} = \frac{(2n-2)!n}{2^{n+1}}; \tag{19}$$

it is easy to see this is even for all n divisible by 4, thus φ_{ϖ_-} is spinorial. □

These representations descend to PSO_{2n}, which are also spinorial by Corollary 9. Moreover, formulas (18) and (19) remain the same when computed for PSO_{2n} (since the $p(\underline{\nu})$ are the same).

Let S_1 be the set

$$S_0 \cup \{\varpi_k + \varpi_\ell \mid k \equiv \ell \mod 2, 1 \leq k, \ell \leq n\} \cup \{\varpi_k + \varpi_- \mid k \text{ even}\} \cup \{2\varpi_-\}.$$

Note that S_1 has the following property. If $\lambda \in S_1$ and λ' is a dominant weight with $\lambda' \prec \lambda$, then $\lambda' \in S_1$.

Proposition 23. *Each φ_λ with $\lambda \in S_1$ is spinorial.*

Proof. Suppose, by way of contradiction, that there are aspinorial $\lambda \in S_1$. Let $\lambda \in S_1$ be a $\prec$-minimal with φ_λ aspinorial. We show below that for each of the three possibilities of $\lambda \in S_1$, the representation φ_λ is spinorial, a contradiction.

If $\lambda \in S_0$ it is spinorial by Proposition 22. Otherwise $\lambda = \lambda_1 + \lambda_2$ with $\lambda_1, \lambda_2 \in S_0$, or $\lambda = \varpi_k + \varpi_\ell$ with k, ℓ odd.

In the first case, let $\Phi = \varphi_{\lambda_1} \otimes \varphi_{\lambda_2}$, which is spinorial by Proposition 22. By the property of S_1 mentioned above, we may apply Lemma 6(1) to deduce that φ_λ is spinorial.

In the second case we have $\lambda = \varpi_k + \varpi_\ell$ with k, ℓ odd. Consider the representation $\Phi = \varphi_{\varpi_k} \otimes \varphi_{\varpi_\ell}$ of SO_{2n}. Applying equation (14) to the representations φ_{ϖ_k} and φ_{ϖ_ℓ} of SO_{2n} gives

$$q_\Phi = \binom{2n}{\ell}\binom{2n-2}{k-1} + \binom{2n}{k}\binom{2n-2}{\ell-1}.$$

Since this is even, Φ is a spinorial representation of SO_{2n}. By Corollary 9 it descends to a spinorial representation $\overline{\Phi}$ of PSO_{2n}.

Again, we may apply Lemma 6(1) to deduce that φ_λ is spinorial. Thus in all cases we have a contradiction. □

Theorem 8. *When n is divisible by 4, every representation of PSO_{2n} is spinorial.*

Proof. It is elementary to see that S_1 is a POSS. Thus the theorem follows by Propositions 14 and 23. □

11.2. **The groups $G_{2n}^{\pm}$.** Here $\check{h} = 2n-2$ and $p(\nu_0) = \binom{n}{2}$, so by Corollary 10:

$$q_\varphi(\nu_0) = \binom{n}{2} \cdot \frac{\mathrm{dyn}^o(\varphi)}{\check{h}} = \frac{n}{4} \cdot \mathrm{dyn}^o(\varphi). \tag{20}$$

If $n \equiv 2 \mod 4$, then the representation φ of $G_{2n}^{\pm}$ on $\bigwedge^2 V_0$ is aspinorial, since again $\mathrm{dyn}^o(\varphi) = 2n-2$. The half-spin representation $(\varphi_{\frac{1}{2}\varpi_4}, V_{\frac{1}{2}\varpi_4})$ of G_8^+, and the half-spin representation $(\varphi_{\frac{1}{2}\varpi_-}, V_{\frac{1}{2}\varpi_-})$ of G_8^- are also aspinorial, since here $\mathrm{dyn}^o(\varphi) = 1$.

Theorem 9. *Suppose $n > 4$ and a multiple of 4. Then every orthogonal representation of G_{2n}^+ and G_{2n}^- is spinorial.*

Proof. If n is a multiple of 8, then the conclusion follows from (20).

If $n \equiv 4 \mod 8$, then $\mathrm{ord}_2(p(\underline{\nu})) = 1 = \mathrm{ord}_2(p(\underline{\nu}'))$. Therefore we may apply Corollary 9 to see that a representation of $G_{2n}^{\pm}$ which descends to PSO_{2n} is spinorial iff it was originally spinorial. Thus by Theorem 8, all such representations of $G_{2n}^{\pm}$ are spinorial.

However there are representations of $G_{2n}^{\pm}$ which don't descend, so we must enlarge our POSS. Let $S^+ = S_1 \cup \{\frac{1}{2}\varpi_n\}$ and $S^- = S_1 \cup \{\frac{1}{2}\varpi_-\}$. Then $S^{\pm}$ is a POSS for $G^{\pm}$. By (20), we have

$$q_{\frac{1}{2}\varpi_n} = q_{\frac{1}{2}\varpi_-} = n2^{n-6},$$

which is certainly even. Thus for each $\lambda \in S^{\pm}$, the representation V_λ of $G_{2n}^{\pm}$ is spinorial. The conclusion then follows by Proposition 14. □

11.3. **Summary for groups of type D_n.** Let $n > 2$ be a positive integer. From the above we know:

- The standard representation of SO_{2n} is aspinorial.
- If n is a multiple of 4, then every representation of PSO_{2n} is spinorial.
- If n is odd, then the adjoint representation of PSO_{2n} is aspinorial.
- If $n \equiv 2 \mod 4$, then the representations of PSO_{2n} and $G_{2n}^{\pm}$ on $\bigwedge^2 V_0$ are aspinorial.
- The half-spin representation $\varphi_{\frac{1}{2}\varpi_4}$ of G_8^+ and the half-spin representation $\varphi_{\frac{1}{2}\varpi_-}$ of G_8^- are aspinorial.
- For $n > 4$ a multiple of 4, all representations of G_{2n}^+ and G_{2n}^- are spinorial.

12. Summary for simple $\mathfrak{g}$

Here is a list of all G with simple $\mathfrak{g}$, with the property that all orthogonal representations of G are spinorial:

- All G whose fundamental group has odd order.
- All SL_n / μ_d, when n/d is even, except when n is a power of 2 and $d = n/2$.
- $\mathrm{Sp}_{2n} / \pm 1$, when n is a multiple of 4.
- The groups PSO_n, when n is a multiple of 8.
- The groups $G_{2n}^{\pm}$, when $n > 4$ is a multiple of 4.

For the reader's convenience, we recall the G whose fundamental groups have odd order:

- Simply connected G.
- SL_n/μ_d with d odd.
- The adjoint group of type E_6.

Aspinorial representations for most groups not on this list have already been mentioned. To finish, we remark that the standard representation of an odd orthogonal group is aspinorial, and the adjoint representation of the adjoint group of type E_7 is aspinorial.

13. Periodicity

For the irreducible orthogonal representations φ_λ, our lifting criterion amounts to determining the parity of one or more $q_\lambda(\nu)$, each an integer-valued polynomial function of λ. As we explain in this section, this entails a certain periodicity of the spinorial highest weights in the character lattice.

13.1. **Polynomials with integer values.** Let V be a finite-dimensional rational vector space, V^* its dual, L a lattice in V, and $L^\vee$ the dual lattice in V^*. Recall that $L^\vee$ is the $\mathbb{Z}$-module of $\mathbb{Q}$-linear maps $f : V \to \mathbb{Q}$ so that $f(L) \subseteq \mathbb{Z}$. Denote by $\binom{L^\vee}{\mathbb{Z}}$ the $\mathbb{Z}$-algebra of polynomial functions on V which take integer values on L. Given $f \in L^\vee$, and $n \in \mathbb{N}$, define $\binom{f}{n} \in \binom{L^\vee}{\mathbb{Z}}$ by the prescription

$$\binom{f}{n} : x \mapsto \binom{f(x)}{n} = \frac{f(x)(f(x)-1)\cdots(f(x)-n+1)}{n!}$$

for $x \in L$.

Proposition 24. *The $\mathbb{Z}$-algebra $\binom{L^\vee}{\mathbb{Z}}$ is generated by the $\binom{f}{n}$ for $f \in L^\vee$ and $n \in \mathbb{N}$. If $\{f_1, \dots, f_r\}$ is a $\mathbb{Z}$-basis of $L^\vee$, then the products*

$$\binom{f_1}{n_1} \cdots \binom{f_r}{n_r},$$

where $n_1, \dots, n_r \in \mathbb{N}$, form a basis of the $\mathbb{Z}$-module $\binom{L^\vee}{\mathbb{Z}}$.

Proof. See Proposition 2 in [Bou05], Chapter 8, Section 12, no. 4. □

Given a basis of V, we can form the set C of its nonnegative linear combinations. Call C a "full polyhedral cone" if it arises in this way, and write $L^+ = L \cap C$.

Proposition 25. *Suppose f is a polynomial map from V to $\mathbb{Q}$ that takes integer values on L^+. Then $f \in \binom{L^\vee}{\mathbb{Z}}$.*

Proof. We omit the elementary proof (see [Jos18]) of the following lemma.

Lemma 7. *Suppose that V is a finite-dimensional rational vector space, that C is a full polyhedral cone in V, and that $L \subset V$ is a lattice. Let $p \in L$. Then:*

1. *$C \cap (p + C)$ is a translation of C.*
2. *The intersection $L \cap C \cap (p + C)$ is nonempty.*
3. *Suppose p' is in the above intersection, and write $v = p' - p$. Then $p + nv \in L \cap C \cap (p + C)$ for all positive integers n.*

Continuing with the proof of the proposition, let $\ell \in L$; we must show that $f(\ell) \in \mathbb{Z}$. By the lemma there is a $v \in L$ so that $\ell + nv \in L^+$ for all positive integers n. For $x \in \mathbb{Z}$, put $g(x) = f(\ell + xv)$. Then $g \in \mathbb{Q}[x]$, and by hypothesis it takes integer values on positive integers. It is elementary to see that such a polynomial takes integer values at *all* integers, and in particular $g(0) = f(\ell) \in \mathbb{Z}$. □

Lemma 8. *Fix an integer $n \geq 1$ and put $k = [\log_2 n] + 1$. Then $\binom{a+2^k}{n} \equiv \binom{a}{n}$ mod 2 for every integer $a \geq 1$.*

Proof. This follows from the Lucas congruence (see, e.g., [Sta12]). □

Proposition 26. *Let $f \in \binom{L^\vee}{\mathbb{Z}}$. Then there is a $k \in \mathbb{N}$ so that for all $x, y \in L$ we have*

$$f(x + 2^k y) \equiv f(x) \mod 2.$$

Proof. By Proposition 24, there are $f_1, \ldots, f_r \in L^\vee$, integers $n_1, \ldots, n_r$, and a polynomial $g \in \mathbb{Z}[x_1, \ldots, x_r]$ so that

$$f = g\left(\binom{f_1}{n_1}, \ldots, \binom{f_r}{n_r}\right).$$

Let $k_i = [\log_2 n_i] + 1$; by Lemma 8 we have

$$\binom{f_i}{n_i}(x + 2^{k_i} y) \equiv \binom{f_i}{n_i}(x) \mod 2$$

for all $x, y \in L$. If we put $k = \max(k_1, \ldots, k_r)$ we obtain the proposition. □

13.2. **Example: Parity of dimensions.** To illustrate the above, let G be connected reductive with notation as before. Take $L = X^*(T) \subset V = X^*(T) \otimes \mathbb{Q} \hookrightarrow \mathfrak{t}^*$. Define $f : \mathfrak{t}^* \to F$ by

$$f(\lambda) = \frac{d_{\lambda+\delta}}{d_\delta} = \dim V_\lambda.$$

From Propositions 25 and 26 we deduce the following corollary.

Corollary 12. *With notation as above:*

1. *$f(\lambda) \in \mathbb{Z}$ for all $\lambda \in X^*(T)$; equivalently $f \in \binom{X_*(T)}{\mathbb{Z}}$.*
2. *There is a $k \in \mathbb{N}$ so that $f(\lambda_0 + 2^k \lambda) \equiv f(\lambda_0)$ mod 2 for all $\lambda_0, \lambda \in X^*(T)$.*

13.3. **Proof of Theorem 2.** We continue with G connected reductive.

If $\mathfrak{g}$ is simple put

$$\eta_{\underline{\nu}}(\lambda) = p(\underline{\nu}) \cdot \frac{\dim V_\lambda \cdot \chi_\lambda(C)}{\dim \mathfrak{g}}.$$

Then:

1. $\eta_{\underline{\nu}}$ is a polynomial in λ,
2. $\eta_{\underline{\nu}}(\lambda) \in \mathbb{Z}$ for $\lambda \in X^+_{\mathrm{sd}}$, and
3. φ_λ is spinorial iff $\eta_{\underline{\nu}}(\lambda)$ is even.

If $\mathfrak{g}$ is not necessarily simple, we may instead put

$$\eta_{\underline{\nu}}(\lambda) = 1 + \prod_{\nu \in \underline{\nu}} (q_\lambda(\nu) - 1),$$

and the same three properties hold. From Propositions 25 and 26 we deduce the following corollary.

Corollary 13. *With notation as above:*

(1) $\eta_{\underline{\nu}}(\lambda) \in \mathbb{Z}$ *for all* $\lambda \in X_{\mathrm{orth}}$*; equivalently* $\eta_{\underline{\nu}} \in \binom{X^\vee_{\mathrm{orth}}}{\mathbb{Z}}$.

(2) *There is a* $k \in \mathbb{N}$ *so that* $\eta_{\underline{\nu}}(\lambda_0 + 2^k\lambda) \equiv \eta_{\underline{\nu}}(\lambda_0) \mod 2$ *for all* $\lambda_0, \lambda \in X_{\mathrm{orth}}$.

Theorem 2 in the introduction follows from this. If we put $L^+ = 2^k X^+_{\mathrm{orth}}$, then the theorem says that the set of spinorial highest weights is stable under addition from L^+. Since the index $[X_{\mathrm{orth}} : 2^k X_{\mathrm{orth}}]$ is finite, the determination of the full set of spinorial weights amounts to a finite computation.

The problem of finding the exact largest lattice $L \subseteq X_{\mathrm{orth}}$ so that the spinorialities of φ_{λ_0} and $\varphi_{\lambda_0+\ell}$ agree for all $\lambda \in X^+_{\mathrm{orth}}$ and $\ell \in L^+$ seems interesting, as does the problem of determining the proportion of spinorial irreducible representations. We do not settle these questions here, but see the next section for PGL_2 and SO_4, and [Jos18] for more examples.

13.4. **Examples.** Let us examine $G = \mathrm{PGL}_2$ more closely. We have $X^*(T) = X_{\mathrm{sd}} = X_{\mathrm{orth}}$. For integers $j \geq 0$ define $\lambda_j \in X^*(T)$ by

$$\lambda_j\left(\begin{pmatrix} a & \\ & b \end{pmatrix}\right) = (ab^{-1})^j.$$

Then $\dim V_{\lambda_j} = 2j+1$ and $\chi_{\lambda_j}(C) = \frac{1}{2}(j^2+j)$, so φ_{λ_j} is spinorial iff

$$\frac{j(j+1)(2j+1)}{2}$$

is even. Equivalently, $j \equiv 0, 3 \mod 4$. We may therefore take $k = 2$ in Theorem 2.

As a second example, recall the representations $V_{a,b}$ of SO_4 from Example 3. If we put

$$F(a,b) = \frac{1}{4}\left((b+1)\binom{a+2}{3} + (a+1)\binom{b+2}{3}\right),$$

then $V_{a,b}$ is spinorial iff $F(a,b)$ is even. It is elementary to see that $F(a+8i, b+8j) \equiv F(a,b) \mod 2$ for integers i, j. In particular we may take $k = 3$ in Theorem 2.

14. Reduction to algebraically closed fields

For this section, G is a connected reductive group defined over a field F of characteristic 0, not necessarily algebraically closed. Let V be a quadratic vector space over F, and $\varphi : G \to \mathrm{SO}(V)$ a morphism defined over F. The isogeny $\rho : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ is also defined over F. By extending scalars to the algebraic closure $\overline{F}$ of F, we may use the rest of this paper to determine whether there exists a lift $\hat{\varphi} : G \to \mathrm{Spin}(V)$ of φ defined over $\overline{F}$.

Lemma 9. *If* $\hat{\varphi} : G \to \mathrm{Spin}(V)$ *is a lift defined over* $\overline{F}$*, then it arises from a lift defined over* F.

Proof. The Galois group acts by Zariski-continuous automorphisms on the $\overline{F}$-points of G and $\mathrm{Spin}(V)$. We must show that for every $\sigma \in \mathrm{Gal}(F)$ and $x \in G(\overline{F})$, we have ${}^\sigma\hat{\varphi}(x) = \hat{\varphi}({}^\sigma x)$. Since ρ and φ are defined over F, the identity $\rho(\hat{\varphi}(x)) = \varphi(x)$ implies that

$$\rho(\hat{\varphi}(x)^{-1} \cdot {}^{\sigma^{-1}}\hat{\varphi}({}^\sigma x)) = 1.$$

Thus the argument of ρ above gives a Zariski-continuous map $G(\overline{F}) \to \ker \rho$. Since G is connected and $\ker \rho$ is discrete, it must be that $\hat{\varphi}(x) = {}^{\sigma^{-1}}\hat{\varphi}({}^\sigma x)$, and the lemma follows. ∎

Therefore: the F-representation φ is spinorial iff its extension to $\overline{F}$-points is spinorial.

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