REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 24, Pages 525–550 (October 27, 2020)
https://doi.org/10.1090/ert/550

SPEH REPRESENTATIONS ARE RELATIVELY DISCRETE

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Abstract. Let F be a p-adic field of characteristic zero and odd residual characteristic. Let $\mathbf{Sp}_{2n}(F)$ denote the symplectic group defined over F, where $n \geq 2$. We prove that the Speh representations $\mathcal{U}(\delta, 2)$, where δ is a discrete series representation of $\mathbf{GL}_n(F)$, lie in the discrete spectrum of the p-adic symmetric space $\mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F)$.

Contents

1. Introduction

Let F be a nonarchimedean local field of characteristic zero and odd residual characteristic p. Let $G = \mathbf{GL}_{2n}(F)$ be an even rank general linear group and let $H = \mathbf{Sp}_{2n}(F)$ be the symplectic group. This paper is concerned with the harmonic analysis on the p-adic symmetric space $X = H\backslash G$. We prove that the Speh representations $\mathcal{U}(\delta, 2)$ appear in the discrete spectrum of X, as predicted by the conjectures of Sakellaridis and Venkatesh [SV17]. Our main result, Theorem 6.1, is an unpublished result of H. Jacquet. We frame this result within the construction of (relative) discrete series representations for symmetric quotients of general linear groups carried out in [Smi18b, Smi18a]. The present work relies on the substantial contributions of Heumos and Rallis [HR90], and Offen and Sayag [OS07, OS08a, OS08b] in the study of symplectic periods for the general linear group.

All representations are assumed to be on complex vector spaces. In general, a smooth representation (π, V) of G is relevant to the harmonic analysis on $X = H\backslash G$ if and only if there exists a nonzero H-invariant linear form on the space V. If there exists a nonzero element λ of $\mathrm{Hom}_H(\pi, 1)$, then (π, V) is H-distinguished. Let (π, V) be an irreducible admissible representation of $\mathbf{GL}_{2n}(F)$. Heumos and Rallis proved that the dimension of the space of $\mathbf{Sp}_{2n}(F)$-invariant linear forms on

Received by the editors August 3, 2018, and, in revised form, July 9, 2020.
2010 *Mathematics Subject Classification.* Primary 22E50; Secondary 22E35.
Key words and phrases. Relative discrete series, distinguished representation, symplectic group, Speh representation.

V is at most one [HR90, Theorem 2.4.2]. In addition, Heumos and Rallis showed that any irreducible admissible representation of $\mathbf{GL}_{2n}(F)$ cannot be both generic and $\mathbf{Sp}_{2n}(F)$-distinguished. Recall that representation of $\mathbf{GL}_n(F)$ is generic if it admits a Whittaker model (see [Rod73] for more information on Whittaker models).

To see that an H-distinguished smooth representation (π, V) of G occurs in the space $C^\infty(X)$ of smooth (locally constant) functions on $X = H\backslash G$ one considers its relative matrix coefficients. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be nonzero. For any $v \in V$, define a function $\varphi_{\lambda,v}$ by declaring that $\varphi_{\lambda,v}(Hg) = \langle \lambda, \pi(g)v\rangle$. The functions $\varphi_{\lambda,v}$ are smooth, since π is smooth, and well-defined because λ is H-invariant. Moreover, the map that sends $v \in V$ to the λ-relative matrix coefficient $\varphi_{\lambda,v}$ intertwines (π, V) and the right regular representation of G on $C^\infty(X)$. It is a fundamental problem to determine which irreducible representations of G occur in the space $L^2(X)$ of square integrable functions on X. The discrete spectrum $L^2_{\mathrm{disc}}(X)$ of X is the direct sum of all irreducible G-subrepresentations of the space $L^2(X)$ of square integrable functions on X. We prove, in Theorem 6.1, that the Speh representations $\mathcal{U}(\delta, 2)$ appear in $L^2_{\mathrm{disc}}(\mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F))$. On the other hand, we do not prove that such representations are the only discrete series; we face the same obstacles discussed in [Smi18b, Remark 6.6].

Sakellaridis and Venkatesh have developed a framework ecnompassing the study of harmonic analysis on p-adic symmetric spaces and its deep connections with periods of automorphic forms and Langlands functoriality [SV17]. In addition to providing explicit Plancherel formulas, Sakellaridis and Venkatesh have made precise conjectures describing the Arthur parameters of representations in the discrete series of symmetric spaces (and, more generally, spherical varieties) [SV17, Conjectures 1.3.1 and 16.2.2]. In fact, their conjectures predict that the discrete series of $\mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F)$ consists precisely of the Speh representations.

We conclude the introduction with a summary of the contents of the paper. In Section 2 we establish notation regarding p-adic symmetric spaces and representations; in addition, we review the Relative Casselman Criterion established by Kato and Takano [KT10]. We review the construction of the Speh representation in Section 3. In Section 4, we review the conjectures of Sakellaridis and Venkatesh and we demonstrate that their work predicts that the Speh representations $\mathcal{U}(\delta, 2)$ should appear in the discrete spectrum of $\mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F)$ (see Proposition 4.5). We determine the fine structure of the symmetric space $\mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F)$ in Section 5; in particular, we realize the group $\mathbf{Sp}_{2n}(F)$ as the fixed points of an involution θ on $\mathbf{GL}_{2n}(F)$, and determine the restricted root system and maximal θ-split parabolic subgroups of $\mathbf{GL}_{2n}(F)$ relative to θ. In Section 6 we prove our main result, Theorem 6.1, by applying the Relative Casselman Criterion (see Theorem 2.7).

In Section 5.2, we make an effort to set the present work within the program started in [Smi18b, Smi18a], where relative discrete series representations have been systematically constructed via parabolic induction from distinguished discrete series representations of θ-elliptic Levi subgroups. In fact, we realize the Speh representations as quotients of representations induced from distinguished discrete series of certain maximal θ-elliptic Levi subgroups. The present setting is complicated by the fact that representations induced from discrete series are generic and therefore not distinguished by the symplectic group. In particular, although we expect that the construction of relative discrete series carried out in [Smi18b, Smi18a] should generalize, some care must be taken to handle the "disjointness-of-models" phenomena as in the case of the Whittaker and symplectic models [HR90, Theorem 3.2.2], and Klyachko models [OS08b].

2. Notation and terminology

Let F be a nonarchimedean local field of characteristic zero and odd residual characteristic p. Let $\mathcal{O}_F$ be the ring of integers of F. Fix a uniformizer ϖ of F. Let q be the cardinality of the residue field k_F of F. Let $|\cdot|_F$ denote the normalized absolute value on F such that $|\varpi|_F = q^{-1}$. We reserve the notation $|\cdot|$ for the usual absolute value on $\mathbb{C}$.

2.1. **Reductive groups and p-adic symmetric spaces.** Let $\mathbf{G}$ be a connected reductive group defined over F. Let θ be an F-involution of $\mathbf{G}$. Let $\mathbf{H} = \mathbf{G}^\theta$ be the subgroup of θ-fixed points in $\mathbf{G}$. Write $G = \mathbf{G}(F)$ for the group of F-points of $\mathbf{G}$, similarly $H = \mathbf{H}(F)$. The quotient $H\backslash G$ is a p-adic symmetric space. We will routinely abuse notation and identify an algebraic F-variety $\mathbf{X}$ with its F-points $X = \mathbf{X}(F)$. When the distinction is to be made, we will use boldface to denote the algebraic variety and regular typeface for the set of F-points.

For an F-torus $\mathbf{A} \subset \mathbf{G}$, let A^1 be the subgroup $\mathbf{A}(\mathcal{O}_F)$ of $\mathcal{O}_F$-points of $A = \mathbf{A}(F)$. We use Z_G to denote the centre of G and A_G to denote the F-split component of the centre of G. Let $X^*(G)$ denote the group of F-rational characters of the algebraic group $\mathbf{G}$. If Y is a subset of a group G, then let $N_G(Y)$ denote the normalizer of Y in G and let $C_G(Y)$ denote the centralizer of Y in G.

2.1.1. *Tori and root systems relative to involutions.* An element $g \in G$ is θ-split if $\theta(g) = g^{-1}$. An F-torus S contained in G is (θ, F)-split if S is F-split and every element of S is θ-split.

Let S_0 be a maximal (θ, F)-split torus of G. Let A_0 be a θ-stable maximal F-split torus of G that contains S_0 [HW93, Lemma 4.5(iii)]. Let $\Phi_0 = \Phi(G, A_0)$ be the root system of G with respect to A_0. Let $W_0 = W(G, A_0) = N_G(A_0)/C_G(A_0)$ be the Weyl group of G with respect to A_0.

The torus A_0 is θ-stable, so there is an action of θ on the F-rational characters $X^*(A_0)$; moreover, Φ_0 is a θ-stable subset of $X^*(A_0)$. Recall that a base of Φ_0 determines a choice of positive roots Φ_0^+.

Definition 2.1. A base Δ_0 of Φ_0 is called a θ-base if for every positive root $\alpha \in \Phi_0^+$ such that $\theta(\alpha) \neq \alpha$ we have that $\theta(\alpha) \in \Phi_0^- = -\Phi_0^+$.

Let Δ_0 be a θ-base of Φ_0. Let $r : X^*(A_0) \to X^*(S_0)$ be the surjective map defined by restriction of (F-rational) characters. Define $\overline{\Phi}_0 = r(\Phi_0) \setminus \{0\}$ and $\overline{\Delta}_0 = r(\Delta_0) \setminus \{0\}$. The set $\overline{\Phi}_0$ coincides with $\Phi_0(G, S_0)$ and is referred to as the restricted root system of $H\backslash G$ [HW93, Proposition 5.9]. The set $\overline{\Delta}_0$ is a base of the root system $\overline{\Phi}_0$. Note that $\overline{\Phi}_0$ is not necessarily reduced. Let Φ_0^θ and Δ_0^θ be the subsets of θ-fixed roots in Φ_0, respectively Δ_0. Observe that $\overline{\Phi}_0 = r(\Phi_0 \setminus \Phi_0^\theta)$ and $\overline{\Delta}_0 = r(\Delta_0 \setminus \Delta_0^\theta)$.

Let $\overline{\Theta}$ be a subset of $\overline{\Delta}_0$. Set $[\overline{\Theta}] = r^{-1}(\overline{\Theta}) \cup \Delta_0^\theta$. Subsets of Δ_0 of the form $[\overline{\Theta}]$ are called θ-split. Maximal θ-split subsets of Δ_0 are of the form $[\overline{\Delta}_0 \setminus \{\bar{\alpha}\}]$, where $\bar{\alpha} \in \overline{\Delta}_0$.

2.1.2. *Parabolic subgroups relative to involutions.* Let $\mathbf{P}$ be an F-parabolic subgroup of $\mathbf{G}$. We refer to an F-parabolic subgroup of $\mathbf{G}$ simply as a parabolic subgroup. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$. The reductive quotient $\mathbf{M} \cong \mathbf{P}/\mathbf{N}$ is called a Levi factor of $\mathbf{P}$. We denote by δ_P the modular character of $P = \mathbf{P}(F)$ given by $\delta_P(p) = |\det \mathrm{Ad}_{\mathfrak{n}}(p)|_F$, where $\mathfrak{n}$ is the Lie algebra of $\mathbf{N}$.

Let M be a Levi subgroup of G. Let A_M denote the F-split component of the centre of M. The (θ, F)-split component of M, denoted by S_M, is the largest (θ, F)-split torus of M that is contained in A_M. More precisely,

$$S_M = \left(\{a \in A_M : \theta(a) = a^{-1}\}\right)^{\circ},$$

where $(\cdot)^{\circ}$ denotes the Zariski-connected component of the identity.

Definition 2.2. A parabolic subgroup P of G is θ-split if $\theta(P)$ is opposite to P, in which case $M = P \cap \theta(P)$ is a θ-stable Levi subgroup of P.

If $\Theta \subset \Delta_0$ is θ-split, then the Δ_0-standard parabolic subgroup P_Θ is θ-split. Let Φ_Θ be the subsystem of Φ_0 generated by Θ. The standard parabolic subgroup P_Θ has unipotent radical N_Θ generated by the root subgroups N_α, where $\alpha \in \Phi_0^+ \setminus \Phi_\Theta^+$. The standard Levi subgroup of P_Θ is M_Θ, which is the centralizer in G of the F-split torus $A_\Theta = \left(\bigcap_{\alpha \in \Theta} \ker \alpha\right)^{\circ}$. Any Δ_0-standard θ-split parabolic subgroup of G arises from a θ-split subset of Δ_0 [KT08, Lemma 2.5(1)].

Let $\Theta \subset \Delta_0$ be θ-split. The (θ, F)-split component of M_Θ is equal to

$$S_\Theta = \left(\bigcap_{\bar{\alpha} \in r(\Theta)} \ker(\bar{\alpha} : S_0 \to F^{\times}) \right)^{\circ}.$$

For any $0 < \epsilon \leq 1$, define

$$S_\Theta^-(\epsilon) = \{s \in S_\Theta : |\alpha(s)|_F \leq \epsilon, \text{ for all } \alpha \in \Delta_0 \setminus \Theta\}. \tag{2.1}$$

We write S_Θ^- for $S_\Theta^-(1)$ and refer to S_Θ^- as the dominant part of S_Θ.

By [HH98, Theorem 2.9], the θ-split subset Δ_0^θ determines the standard minimal θ-split parabolic subgroup $P_0 = P_{\Delta_0^\theta}$. Let N_0 be the unipotent radical of P_0. The standard Levi subgroup M_0 of P_0 is the centralizer in G of the maximal (θ, F)-split torus S_0.

Lemma 2.3 ([KT08, Lemma 2.5]). *Let $S_0 \subset A_0$, Δ_0, and $P_0 = M_0 N_0$ be as above.*

1. *Any θ-split parabolic subgroup P of G is conjugate to a Δ_0-standard θ-split parabolic subgroup by an element $g \in (\mathbf{H}\mathbf{M}_0)(F)$.*
2. *If the group of F-points of the product $(\mathbf{H}\mathbf{M}_0)(F)$ is equal to HM_0, then any θ-split parabolic subgroup of G is H-conjugate to a Δ_0-standard θ-split parabolic subgroup.*

Let $P = MN$ be a θ-split parabolic subgroup. Pick $g \in (\mathbf{H}\mathbf{M}_0)(F)$ such that $P = gP_\Theta g^{-1}$ for some θ-split subset $\Theta \subset \Delta_0$. Since $g \in (\mathbf{H}\mathbf{M}_0)(F)$ we have that $g^{-1}\theta(g) \in \mathbf{M}_0(F)$, and we have $S_M = gS_\Theta g^{-1}$. For a given $\epsilon > 0$, one may extend the definition of S_Θ^- in (2.1) to the torus S_M. Set $S_M^-(\epsilon) = gS_\Theta^-(\epsilon)g^{-1}$ and define $S_M^- = S_M^-(1)$. Write S_M^1 to denote the group of $\mathcal{O}_F$-points $S_M(\mathcal{O}_F)$.

2.2. **Distinguished representations and relative matrix coefficients.** A representation (π, V) of G is smooth if for every $v \in V$ the stabilizer of v in G is an open subgroup. A smooth representation (π, V) of G is admissible if, for every compact open subgroup K of G, the subspace V^K of K-invariant vectors is finite dimensional. All of the representations that we consider are smooth and admissible. A quasi-character of G is a one-dimensional representation. Let (π, V) be a smooth representation of G. If ω is a quasi-character of Z_G, then (π, V) is called an ω-representation if π has central character ω.

Let P be a parabolic subgroup of G with Levi subgroup M and unipotent radical N. Given a smooth representation (ρ, V_ρ) of M we may inflate ρ to a representation of P, also denoted ρ, by declaring that N acts trivially. We define the representation $\iota_P^G \rho$ of G to be the (normalized) parabolically induced representation $\mathrm{Ind}_P^G(\delta_P^{1/2} \otimes \rho)$. We will also use the Bernstein–Zelevinsky [BZ77, Zel80] notation $\pi_1 \times \ldots \times \pi_k$ for the (normalized) parabolically induced representation $\iota_{P_{(m_1,\ldots,m_k)}}^{\mathbf{GL}_m(F)}(\pi_1 \otimes \ldots \otimes \pi_k)$ of $\mathbf{GL}_m(F)$ obtained from the standard (block-upper triangular) parabolic subgroup $P_{(m_1,\ldots,m_k)}$ and representations π_j of $\mathbf{GL}_{m_j}(F)$, where $\sum_{j=1}^k m_j = m$.

Let (π, V) be a smooth representation of G. Let (π_N, V_N) denote the normalized Jacquet module of π along P. Precisely, V_N is the quotient of V by the P-stable subspace $V(N) = \mathrm{span}\{\pi(n)v - v : n \in N, v \in V\}$, and the action of P on V_N is normalized by $\delta_P^{-1/2}$. The unipotent radical N of P acts trivially on (π_N, V_N), and we will regard (π_N, V_N) as a representation of the Levi factor $M \cong P/N$ of P.

We also let π denote its restriction to H. Let χ be a quasi-character of H.

Definition 2.4. The representation π is (H, χ)-distinguished if the space $\mathrm{Hom}_H(\pi, \chi)$ is nonzero. If π is $(H, 1)$-distinguished, where 1 is the trivial character of H, then we will simply call π H-distinguished.

Let (π, V) be a smooth H-distinguished ω-representation of G. Note that ω must be trivial on $Z_G \cap H$. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be a nonzero H-invariant linear functional on V. Let $v \in V$ be a nonzero vector. Define the λ-relative matrix coefficient associated to v to be the complex valued function $\varphi_{\lambda,v} : G \to \mathbb{C}$ given by $\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g)v \rangle$. When λ is understood, we drop it from the terminology and refer to the relative matrix coefficients of π. Since (π, V) is assumed to be smooth, for all $v \in V$, the function $\varphi_{\lambda,v}$ lies in the space $C^\infty(G)$ of smooth (that is, locally constant) $\mathbb{C}$-valued functions on G. Moreover, since π is an ω-representation, the functions $\varphi_{\lambda,v}$ lie in the subspace $C^\infty_\omega(G)$ consisting of smooth functions $f : G \to \mathbb{C}$ such that $f(zg) = \omega(z)f(g)$, for all $z \in Z_G$ and $g \in G$. Observe that, since λ is H-invariant, for all $g \in G, z \in Z_G$, and $h \in H$ we have

$$\begin{aligned}\varphi_{\lambda,v}(hzg) &= \langle \lambda, \pi(hzg)v \rangle \\ &= \omega(z)\langle \lambda, \pi(g)v \rangle \\ &= \omega(z)\varphi_{\lambda,v}(g).\end{aligned}$$

For any $v \in V$, the λ-relative matrix coefficient $\varphi_{\lambda,v}$ descends to well a defined function on $H\backslash G$ and satisfies $\varphi_{\lambda,v}(Hzg) = \omega(z)\varphi_{\lambda,v}(Hg)$, for $z \in Z_G$ and $Hg \in H\backslash G$. Further assume that the central character ω of (π, V) is unitary. In this case, the function $Z_G H \cdot g \mapsto |\varphi_{\lambda,v}(g)|$ is well defined on $Z_G H\backslash G$. The centre Z_G of G is unimodular since it is abelian. The fixed point subgroup H is also reductive and thus unimodular. By [Rob83, Proposition 12.8], there exists a G-invariant measure on the quotient $Z_G H\backslash G$.

Definition 2.5. Let ω be a unitary character of Z_G. Let (π, V) be an H-distinguished ω-representation of G. Then (π, V) is said to be

(1) (H,λ)-relatively square integrable if and only if all of the λ-relative matrix coefficients are square integrable modulo $Z_G H$.
(2) H-relatively square integrable if and only if π is (H,λ)-relatively square integrable for every $\lambda \in \operatorname{Hom}_H(\pi, 1)$.

2.3. **Exponents and the relative Casselman criterion.** Let (π, V) be a finitely generated admissible representation of G. Let $\mathcal{E}xp_{Z_G}(\pi)$ be the (finite) set of quasi-characters of Z_G that occur as the central characters of the irreducible subquotients of π. We refer to the characters that appear in $\mathcal{E}xp_{Z_G}(\pi)$ as the exponents of π. By [Cas95, Proposition 2.1.9], the quasi-character $\chi : Z_G \to \mathbb{C}^\times$ occurs in $\mathcal{E}xp_{Z_G}(\pi)$ if and only if the generalized χ-eigenspace for the action of Z_G on V is nonzero. Let Z be a closed subgroup of Z_G. The exponents $\mathcal{E}xp_Z(\pi)$ with respect to the action of Z on V are defined analogously. If $Z_1 \supset Z_2$ are two closed subsets of Z_G, then the map $\mathcal{E}xp_{Z_1}(\pi) \to \mathcal{E}xp_{Z_2}(\pi)$ defined by restriction of quasi-characters is surjective (see, for instance, [Smi18b, Lemma 4.15]).

Let $P = MN$ be a parabolic subgroup of G with unipotent radical N and Levi factor M. The normalized Jacquet module (π_N, V_N) of (π, V) along P is also finitely generated and admissible [Cas95, Theorem 3.3.1]. The set $\mathcal{E}xp_{A_M}(\pi_N)$ of exponents of π_N, with respect to the action of the F-split component A_M of M, is referred to as the set of exponents of π along P.

Lemma 2.6 ([Smi18b, Lemma 4.16])**.** *Let $P = MN$ be a parabolic subgroup of G, let (ρ, W) be a finitely generated admissible representation of M, and let $\pi = \iota_P^G \rho$. The exponents $\mathcal{E}xp_{A_G}(\pi)$ of π are the restrictions to A_G of the exponents $\mathcal{E}xp_{A_M}(\rho)$ of ρ.*

Let (π, V) be a finitely generated admissible H-distinguished representation of G. Let $\lambda \in \operatorname{Hom}_H(\pi, 1)$ be a nonzero H-invariant linear form on (π, V). In [KT10], Kato and Takano defined

$$\mathcal{E}xp_Z(\pi, \lambda) = \{\chi \in \mathcal{E}xp_Z(\pi) : \lambda|_{V_\chi} \neq 0\},$$

for any closed subgroup Z of Z_G, where

$$V_\chi = \bigcup_{n=1}^{\infty} \{v \in V : (\pi(z) - \chi(z))^n v = 0, \forall z \in Z\}$$

is the generalized χ-eigenspace for the Z action on V. The elements of $\mathcal{E}xp_{A_G}(\pi, \lambda)$ are referred to as the exponents of π relative to λ. Let P be a θ-split parabolic subgroup of G with unipotent radical N and θ-stable Levi subgroup $M = P \cap \theta(P)$. Using Casselman's Canonical Lifting [Cas95, Proposition 4.1.4], Kato–Takano [KT08] and Lagier [Lag08] defined an M^θ-invariant linear functional $\lambda_N \in \operatorname{Hom}_{M^\theta}(\pi_N, 1)$, canonically associated to λ, on the Jacquet module π_N of π along P. We refer the reader to [KT08, Proposition 5.6] for details of the construction and additional properties of the map $\lambda \mapsto \lambda_N$. We may now state the Relative Casselman Criterion [KT10, Theorem 4.7].

Theorem 2.7 (Relative Casselman Criterion)**.** *Let ω be a unitary character of Z_G. Let (π, V) be a finitely generated admissible H-distinguished ω-representation* *of G. Fix a nonzero element λ in $\mathrm{Hom}_H(\pi, V)$. The representation (π, V) is (H, λ)-relatively square integrable if and only if the condition*

$$|\chi(s)| < 1, \qquad \forall \chi \in \mathcal{E}xp_{S_M}(\pi_N, \lambda_N), \forall s \in S_M^- \setminus S_G S_M^1 \tag{2.2}$$

is satisfied for every proper θ-split parabolic subgroup $P = MN$ of G.

2.4. **Conventions regarding $\mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F)$.** From now on, unless otherwise specified, we let $G = \mathbf{GL}_{2n}(F)$ and let $H = \mathbf{Sp}_{2n}(F)$, where $n \geq 2$. We will realize the symplectic group H explicitly as the subgroup of G fixed pointwise by the involution θ given by

$$\theta(g) = \varepsilon_{2n}^{-1}{}^t g^{-1} \varepsilon_{2n},$$

where ${}^t g$ denotes the transpose of $g \in G$,

$$\varepsilon_{2n} = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \in \mathbf{GL}_{2n}(F) \qquad \text{and} \qquad J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathbf{GL}_n(F).$$

Note that ε_{2n} is a nonsingular skew-symmetric element of G; moreover, $\varepsilon_{2n}^{-1} = {}^t\varepsilon_{2n}$. With this choice of involution, the subgroup of upper triangular matrices in H is a Borel subgroup (over F). Let A_0 be the maximal diagonal F-split torus of G.

There is a right G-action on the set of involutions of G given by

$$g \cdot \theta(x) = g^{-1}\theta(gxg^{-1})g, \tag{2.3}$$

for any $x, g \in G$. Any involution of the form $g \cdot \theta$ is said to be G-equivalent to θ. If $x \in G$ is skew-symmetric, then we obtain a realization of the symplectic group as the fixed points of the involution θ_x defined by

$$\theta_x(g) = x^{-1}{}^t g^{-1} x.$$

Moreover, the G-action on involutions is compatible with the right G-action on the set of skew-symmetric matrices given by $x \cdot g = {}^t gxg$, for any $g \in G$ and any skew-symmetric matrix $x \in G$. Indeed, if $x, g \in G$ and x is skew-symmetric, then $g \cdot \theta_x = \theta_{x \cdot g}$.

We will write $\mathrm{diag}(a_1, \ldots, a_m)$ to denote the diagonal $m \times m$ matrix with entries $a_1, \ldots, a_m$ on the main diagonal. Given a partition $(m_1, \ldots, m_k)$ of a positive integer m, write $P_{(m_1,\ldots,m_k)}$ for the standard block-upper triangular parabolic subgroup of $\mathbf{GL}_m(F)$ with Levi factor $M_{(m_1,\ldots,m_k)}$ and unipotent radical $N_{(m_1,\ldots,m_k)}$. Write ν for the unramified character $|\det(\cdot)|_F$ of $\mathbf{GL}_m(F)$, where m is understood from context.

3. Speh representations

Recall that a representation π of $\mathbf{GL}_m(F)$ is said to be generic if it admits a Whittaker model, that is, if there exists a nonzero intertwining operator in the space $\mathrm{Hom}_{N_m}(\pi, \mathrm{Ind}_{N_m}^{\mathbf{GL}_m(F)} \Psi_m)$, where N_m is the subgroup of $\mathbf{GL}_m(F)$ consisting of upper triangular unipotent matrices and Ψ_m is a non-degenerate character of N_m.

Let δ be an irreducible square integrable representation of $\mathbf{GL}_n(F)$. The parabolically induced representation $\nu^{1/2}\delta \times \nu^{-1/2}\delta$ has length two and admits a unique irreducible generic subrepresentation $\mathcal{Z}(\delta, 2)$ and a unique irreducible quotient $\mathcal{U}(\delta, 2)$

[BZ77, Zel80]. In particular, we have the following short exact sequence of G-representations

$$0 \to \mathcal{Z}(\delta, 2) \to \nu^{1/2}\delta \times \nu^{-1/2}\delta \to \mathcal{U}(\delta, 2) \to 0. \tag{3.1}$$

The representations $\mathcal{U}(\delta, 2)$ are the Speh representations.

Heumos and Rallis proved that the Speh representations $\mathcal{U}(\delta, 2)$ are H-distinguished by constructing a nonzero H-invariant linear functional on the full induced representation $\nu^{1/2}\delta \times \nu^{-1/2}\delta$ and then appealing to [HR90, Theorem 3.2.2] to show that the generic subrepresentation $\mathcal{Z}(\delta, 2)$ cannot be H-distinguished. One can then use the exact sequence (3.1) to conclude that the invariant functional on $\nu^{1/2}\delta \times \nu^{-1/2}\delta$ descends to a well-defined nonzero H-invariant linear functional on $\mathcal{U}(\delta, 2)$. Note that the invariant form on $\nu^{1/2}\delta \times \nu^{-1/2}\delta$ may be realized via [Off17, Proposition 7.1], (cf. [Smi20, Lemma 1.3.4]).

Next, let us recall descriptions of the generalized Steinberg representations and generalized Speh representations of general linear groups. Let ρ be an irreducible unitary supercuspidal representation of $\mathbf{GL}_r(F)$, $r \geq 1$. Let $k \geq 2$ be an integer. By [Zel80, Theorem 9.3], the induced representation

$$\nu^{\frac{k-1}{2}}\rho \times \ldots \times \nu^{\frac{1-k}{2}}\rho$$

of $\mathbf{GL}_{kr}(F)$ admits a unique irreducible subrepresentation $\mathcal{Z}(\rho, k)$; moreover, $\mathcal{Z}(\rho, k)$ is square integrable. Often in the literature, $\mathcal{Z}(\rho, k)$ is denoted by $\mathrm{St}(\rho, k)$ and these are the generalized Steinberg representations of $\mathbf{GL}_{kr}(F)$. Let δ be a discrete series representation of $\mathbf{GL}_d(F)$, $d \geq 2$. Let $m \geq 2$ be an integer. By [Zel80, Theorem 6.1(a)], the induced representation

$$\nu^{\frac{m-1}{2}}\delta \times \ldots \times \nu^{\frac{1-m}{2}}\delta$$

admits a unique irreducible (unitary) quotient $\mathcal{U}(\delta, m)$.

Remark 3.1. The Speh representations, and generalized Speh representations, feature prominently in the classification of the unitary dual of general linear groups carried out by Tadić [Tad85, Tad86].

Remark 3.2. The method used by Heumos and Rallis to demonstrate the $\mathbf{Sp}_{2n}(F)$-distinction of the Speh representations $\mathcal{U}(\delta, 2)$ does not immediately extend to the generalized Speh representations $\mathcal{U}(\delta, m)$, $m > 2$. However, Offen and Sayag [OS07] study the distinction of the generalized Speh representations by utilizing work of Jacquet and Rallis [JR96] and Bernstein's meromorphic continuation. The method of Offen and Sayag, used to prove the "hereditary property of symplectic periods," is a special case of the method of Blanc and Delorme [BD08]. We refer the reader to [OS07] for more details.

4. X-distinguished Arthur parameters

In the following discussion, $\mathbf{G}$ can be taken to be an arbitrary connected reductive group that is split over F. We return to $G = \mathbf{GL}_{2n}(F)$ in Section 4.1.

Let $\mathcal{W}_F$ be the Weil group of F and let $\mathcal{L}_F = \mathcal{W}_F \times \mathrm{SL}(2, \mathbb{C})$ be the Weil–Deligne group of F. Since $\mathbf{G}$ is split over F, $\mathcal{W}_F$ acts trivially on the complex dual group $G^\vee$, and the L-group of $\mathbf{G}$ can be identified with the dual group ${}^LG = G^\vee$. Recall that an Arthur parameter, or an A-parameter, for G is a continuous homomorphism

$\psi : \mathcal{L}_F \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$ such that

- the restriction $\psi|_{\mathcal{W}_F}$ of ψ to the Weil group $\mathcal{W}_F$ is bounded,
- the image of $\psi|_{\mathcal{W}_F}$ consists of semisimple elements of $G^\vee$,
- and the restriction of ψ to each of the two $\mathrm{SL}(2,\mathbb{C})$ factors is algebraic.

A Langlands parameter, or an L-parameter, for G is a continuous homomorphism $\phi : \mathcal{L}_F \to G^\vee$ such that

- the image of $\phi|_{\mathcal{W}_F}$ consists of semisimple elements of $G^\vee$,
- and the restriction of ϕ to $\mathrm{SL}(2,\mathbb{C})$ is algebraic.

Inspired by work of Gaitsgory and Nadler, Sakellaridis and Venkatesh have associated to any $\mathbf{G}$-spherical F-variety $\mathbf{X}$ a complex dual group $G_X^\vee$, see [SV17, Sections 2–3] (provided that the assumption of [SV17, Proposition 2.2.2] on the spherical roots is satisfied). In addition, they described a *distinguished morphism* $\varrho : G_X^\vee \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$ satisfying certain properties and unique up to conjugation by a canonical maximal torus in $G^\vee$ [SV17, Section 3.2]. Existence of distinguished morphisms has been proven in full generality by Knop and Schalke [KS17].

Definition 4.1. An A-parameter $\psi : \mathcal{L}_F \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$ is X-distinguished if it factors through the distinguished morphism $\varrho : G_X^\vee \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$, that is, if and only if there exists a tempered (that is, bounded on $\mathcal{W}_F$) L-parameter $\phi_X : \mathcal{L}_F \to G_X^\vee$ such that $\psi = \varrho \circ (\phi_X \times \mathrm{Id})$, where $\mathrm{Id} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ is the identity.

Definition 4.2. An X-distinguished A-parameter is X-elliptic if it factors through ϱ via an elliptic L-parameter $\phi_X : \mathcal{L}_F \to G_X^\vee$, that is, the image of ϕ_X is not contained in any proper Levi subgroup of $G_X^\vee$.

We recall the following conjecture [SV17, Conjecture 1.3.1].

Conjecture 4.3 (Sakellaridis–Venkatesh)**.** *The support of the Plancherel measure for $L^2(X)$, as a representation of G, is contained in the union of Arthur packets attached to X-distinguished A-parameters.*

In fact, Sakellaridis and Venkatesh give much more refined conjectures that predict a direct integral decomposition of $L^2(X)$ over X-distinguished A-parameters [SV17, Conjecture 16.2.2]. In addition, the refined conjectures make the following prediction about the X-distinguished A-parameters of the relative discrete series representations.

Conjecture 4.4 (Sakellaridis–Venkatesh)**.** *A relative discrete series representation π in $L^2(X)$ is contained in an Arthur packet corresponding to an X-distinguished X-elliptic A-parameter.*

Of course, in the setting we are concerned with, the situation is greatly simplified by the fact that Arthur packets (and L-packets) for the general linear group are singleton sets.

4.1. **Distinguished A-parameters for $\mathbf{Sp}_{2n}\backslash\mathbf{GL}_{2n}$.** Let $\mathbf{X}$ be the symmetric variety $\mathbf{Sp}_{2n}\backslash\mathbf{GL}_{2n}$. The dual group of $G = \mathbf{GL}_{2n}(F)$ is $G^\vee = \mathrm{GL}(2n,\mathbb{C})$. The dual group $G_X^\vee$ of $\mathbf{X}$ is the rank-n complex general linear group, that is, $G_X^\vee = \mathrm{GL}(n,\mathbb{C})$ (see Lemma 5.6).

Let π be an irreducible unitary $\mathbf{Sp}_{2n}(F)$-distinguished representation of G. Let $\psi_\pi : \mathcal{L}_F \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{C})$ be the A-parameter of π. We refer the reader to

[Art13, Xu17] for the description of the A-parameters of the representations in the unitary dual of G, including the generalized Steinberg and Speh representations. The distinguished morphism $\varrho : \mathrm{GL}(n,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{C})$ is given by the tensor product of the standard n-dimensional representation of $\mathrm{GL}(n,\mathbb{C})$ with the standard 2-dimensional representation $\mathcal{S}(2)$ of $\mathrm{SL}(2,\mathbb{C})$ [SV17, Example 1.3.2]. Thus, if $\phi : \mathcal{L}_F \to \mathrm{GL}(n,\mathbb{C})$ is an L-parameter, then $\varrho \circ (\phi \times \mathrm{Id}) = \phi \otimes \mathcal{S}(2)$. By Conjecture 4.4, we expect π to be relatively square integrable if its A-parameter ψ_π has the following two properties:

- **P1:** The A-parameter ψ_π is X-distinguished, that is, ψ_π factors through the distinguished morphism ϱ; in particular, $\psi_\pi = \phi_{\pi,X} \otimes \mathcal{S}(2)$, where $\phi_{\pi,X} : \mathcal{L}_F \to \mathrm{GL}(n,\mathbb{C})$ is a tempered L-parameter.
- **P2:** The L-parameter $\phi_{\pi,X} : \mathcal{L}_F \to \mathrm{GL}(n,\mathbb{C})$ is elliptic, that is, the image of $\phi_{\pi,X}$ is not contained in any proper parabolic subgroup of $\mathrm{GL}(n,\mathbb{C})$.

Recall that, under the Local Langlands Correspondence, a tempered elliptic parameter $\phi : \mathcal{L}_F \to \mathrm{GL}(n,\mathbb{C})$ corresponds to a discrete series representation of $\mathbf{GL}_n(F)$.

Proposition 4.5. *Let π be an irreducible unitary $\mathbf{Sp}_{2n}(F)$-distinguished representation of $\mathbf{GL}_{2n}(F)$. Let $\psi_\pi : \mathcal{L}_F \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{C})$ be the A-parameter of π. The A-parameter ψ_π is X-distinguished and X-elliptic if and only if π is isomorphic to a Speh representation $\mathcal{U}(\delta,2)$ for some discrete series representation δ of $\mathbf{GL}_n(F)$.*

Proof. Let π be an irreducible unitary $\mathbf{Sp}_{2n}(F)$-distinguished representation of $\mathbf{GL}_{2n}(F)$. Following the notation of [OS07], let $\pi(\sigma,\alpha) = \nu^\alpha\sigma \times \nu^{-\alpha}\sigma$, where $\alpha \in \mathbb{R}$ so that $|\alpha| < 1/2$, and σ is a smooth representation of $\mathbf{GL}_d(F)$, for some $d \geq 1$. Offen and Sayag [OS07, OS08b] have shown that π must be equivalent to a representation of the form

$$\left(\bigtimes_{i=1}^{l} \mathcal{U}(\mathcal{Z}(\rho_i,k_i),2m_i)\right) \times \left(\bigtimes_{i=l+1}^{t} \pi(\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2m_i),\alpha_i)\right),$$

where $2n = \sum_{i=1}^{l} 2k_ir_im_i + \sum_{i=l+1}^{t} 4k_ir_im_i$, the representations ρ_i of $\mathbf{GL}_{r_i}(F)$ are irreducible, unitary and supercuspidal, and $\alpha_i \in \mathbb{R}$ with $|\alpha_i| < 1/2$.

Let $\phi_{\rho_i} : \mathcal{W}_F \to \mathrm{GL}(r_i,\mathbb{C})$ be the L-parameter of the supercuspidal representation ρ_i. Write $\mathcal{S}(d) \cong \mathrm{Sym}^{d-1}(\mathbb{C}^2)$ for the unique (up to isomorphism) d-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. Let $|\cdot|_{\mathcal{W}_F} : \mathcal{W}_F \to \mathbb{R}_{>0}$ denote the absolute value on the Weil group given by $|\cdot|_{\mathcal{W}_F} = |\,\mathrm{Art}_F^{-1}(\cdot)|_F$, where $\mathrm{Art}_F : F^\times \to \mathcal{W}_F^{\mathrm{ab}}$ is the Artin map and $|\cdot|_F$ is the (normalized) absolute value on F. Up to equivalence, the A-parameter ψ_π of π is equal to

(4.1)
$$\bigoplus_{i=1}^{l} \psi_{\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2m_i)} \oplus \bigoplus_{i=l+1}^{t} \left(|\cdot|_{\mathcal{W}_F}^{\alpha_i}\psi_{\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2m_i)} \oplus |\cdot|_{\mathcal{W}_F}^{-\alpha_i}\psi_{\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2m_i)}\right),$$

where

$$\psi_{\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2m_i)} = \phi_{\mathcal{Z}(\rho_i,k_i)} \otimes \mathcal{S}(2m_i) : \mathcal{L}_F \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(2k_ir_im_i,\mathbb{C}) \tag{4.2}$$

is the A-parameter of the generalized Speh representation $\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2m_i)$, and

$$\phi_{\mathcal{Z}(\rho_i,k_i)} = \phi_{\rho_i} \otimes \mathcal{S}(k_i) : \mathcal{L}_F \to \mathrm{GL}(k_ir_i,\mathbb{C})$$

is the L-parameter of the generalized Steinberg representation $\mathcal{Z}(\rho_i,k_i)$.

The A-parameter ψ_π is X-distinguished (**P1**) if and only if $\psi_\pi = \phi_{\pi,X} \otimes \mathcal{S}(2)$, for some tempered L-parameter $\phi_{\pi,X} : \mathcal{L}_F \to \mathrm{GL}(n,\mathbb{C})$. In light of (4.1) and (4.2), we first notice that ψ_π factors through the distinguished morphism ϱ if and only if $m_i = 1$, for all $1 \leq i \leq t$. In this case,

$$\pi \cong \left(\bigtimes_{i=1}^{l} \mathcal{U}(\mathcal{Z}(\rho_i,k_i),2) \right) \times \left(\bigtimes_{i=l+1}^{t} \pi(\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2),\alpha_i) \right),$$

and

$$\begin{aligned}\psi_\pi &= \bigoplus_{i=1}^{l} \psi_{\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2)} \oplus \bigoplus_{i=l+1}^{t} \left(|\cdot|_{\mathcal{W}_F}^{\alpha_i} \psi_{\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2)} \oplus |\cdot|_{\mathcal{W}_F}^{-\alpha_i} \psi_{\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2)} \right) \\ &= \left(\bigoplus_{i=1}^{l} \phi_{\mathcal{Z}(\rho_i,k_i)} \oplus \bigoplus_{i=l+1}^{t} |\cdot|_{\mathcal{W}_F}^{\alpha_i} \phi_{\mathcal{Z}(\rho_i,k_i)} \oplus |\cdot|_{\mathcal{W}_F}^{-\alpha_i} \phi_{\mathcal{Z}(\rho_i,k_i)} \right) \otimes \mathcal{S}(2) \\ &= \phi_{\pi,X} \otimes \mathcal{S}(2),\end{aligned}$$

where $\phi_{\pi,X} : \mathcal{L}_F \to \mathrm{GL}(n,\mathbb{C})$ is the L-parameter

$$\bigoplus_{i=1}^{l} \phi_{\mathcal{Z}(\rho_i,k_i)} \oplus \bigoplus_{i=l+1}^{t} |\cdot|_{\mathcal{W}_F}^{\alpha_i} \phi_{\mathcal{Z}(\rho_i,k_i)} \oplus |\cdot|_{\mathcal{W}_F}^{-\alpha_i} \phi_{\mathcal{Z}(\rho_i,k_i)}.$$

Moreover, the L-parameter $\phi_{\pi,X}$ is tempered if and only if $\phi_{\pi,X}$ restricted to $\mathcal{W}_F$ has bounded image, that is, if and only if $l = t$ (or, equivalently, $\alpha_i = 0$ for all $l+1 \leq i \leq t$). In particular, the representations $\pi(\mathcal{U}(\mathcal{Z}(\rho_i,k_i),2),\alpha_i)$ cannot appear in the inducing data of π. Thus, ψ_π is X-distinguished (**P1**) if and only if $\psi_\pi = \phi_{\pi,X} \otimes \mathcal{S}(2)$, where

$$\phi_{\pi,X} = \bigoplus_{i=1}^{l} \phi_{\mathcal{Z}(\rho_i,k_i)}. \tag{4.3}$$

The L-parameter $\phi_{\pi,X}$ is X-elliptic (**P2**) if and only if there is precisely one direct summand in (4.3) (that is, $l = 1$) and $\phi_{\pi,X} = \phi_{\mathcal{Z}(\rho,k)}$ corresponds to the discrete series representation $\delta = \mathcal{Z}(\rho,k)$ of $\mathbf{GL}_n(F)$. In particular, ψ_π is X-distinguished (**P1**) and X-elliptic (**P2**) if and only if $\psi_\pi = \phi_{\mathcal{Z}(\rho,k)} \otimes \mathcal{S}(2)$, in which case, by the Local Langlands Correspondence, $\pi \cong \mathcal{U}(\delta,2)$ is a Speh representation. □

In summary, Proposition 4.5 allows us to interpret Conjecture 4.4 as predicting that only the Speh representations $\mathcal{U}(\delta,2)$, where $\delta = \mathcal{Z}(\rho,k)$ is a discrete series representation of $\mathbf{GL}_n(F)$, appear in the discrete spectrum of $X = \mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F)$. The goal of the rest of this paper is to prove that the representations $\mathcal{U}(\delta,2)$ do indeed appear in $L^2_{\mathrm{disc}}(X)$.

Remark 4.6. We do not show that generalized Speh representations $\mathcal{U}(\delta,2m)$, $m \geq 2$, are not relatively square integrable despite the fact that Conjecture 4.4 predicts that these representations do not appear in $L^2_{\mathrm{disc}}(X)$. See [Smi18b, Remark 6.6] for a discussion of the difficulties therein.

5. Tori and parabolic subgroups: Structure of $\mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F)$

In this section, we identify the θ-split parabolic subgroups required for our application of the Relative Casselman Criterion. First we introduce a second involution that is G-equivalent to θ (*cf.* Section 2.4). Let $w_+ \in G$ be the permutation matrix associated to the permutation

$$\begin{cases} 2i-1 \mapsto i & 1 \le i \le n \\ 2i \mapsto 2n+1-i & 1 \le i \le n \end{cases}$$

of $\{1, \dots, 2n\}$. We have chosen w_+ such that

$$\varepsilon_{2n} \cdot w_+ = {}^t w_+ \varepsilon_{2n} w_+ = w_+^{-1} \varepsilon_{2n} w_+ = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Let x_{2n} denote the nonsingular skew-symmetric matrix $\varepsilon_{2n} \cdot w_+$ and let $\theta_{x_{2n}}$ be the associated involution of G. As above, we have that $\theta_{x_{2n}} = \theta_{\varepsilon_{2n} \cdot w_+} = w_+ \cdot \theta$, and $\theta_{x_{2n}}$ is G-equivalent to θ.

Lemma 5.1. *Let A_0 be the maximal diagonal F-split torus of G. The torus A_0 is θ-stable and contains the maximal (θ, F)-split torus S_0, where*

$$S_0 = \{\mathrm{diag}(a_1, \dots, a_n, a_n, \dots, a_1) : a_i \in F^\times, 1 \le i \le n\}.$$

Proof. Let $a = \mathrm{diag}(a_1, \dots, a_{2n}) \in A_0$. First note that

$$\theta(a) = \mathrm{diag}(a_{2n}^{-1}, \dots, a_1^{-1}).$$

In particular, a is θ-split if and only if $a_{2n+1-i} = a_i$ for all $1 \le i \le n$. The torus S_0 is the (θ, F)-split component of A_0. Thus, it is sufficient to show that S_0 is a maximal (θ, F)-split torus in G. To do so, we will prove that the block-upper triangular parabolic $P_{(\underline{2})}$ corresponding to the partition $(\underline{2}) = (2, \dots, 2)$ of $2n$ is a minimal $\theta_{x_{2n}}$-split parabolic, and then use the G-equivalence of $\theta_{x_{2n}}$ and θ to conclude that $P_0 = w_+ P_{(\underline{2})} w_+^{-1}$ is a minimal θ-split parabolic subgroup of G. The desired result then follows from [HW93, Proposition 4.7(iv)].

To see that $P_{(\underline{2})}$ is $\theta_{x_{2n}}$-split, first note that $x_{2n} \in M_{(\underline{2})}$; therefore, the block-diagonal Levi $M_{(\underline{2})}$ is $\theta_{x_{2n}}$-stable. The unipotent radical $N_{(\underline{2})}$ of $P_{(\underline{2})}$ is mapped to the opposite unipotent radical $N_{(\underline{2})}^{\mathrm{op}}$ (with respect to $M_{(\underline{2})}$) by taking conjugate-transpose, and both $N_{(\underline{2})}$ and $N_{(\underline{2})}^{\mathrm{op}}$ are normalized by $M_{(\underline{2})}$. It follows that $\theta_{x_{2n}}(P_{(\underline{2})}) = M_{(\underline{2})} N_{(\underline{2})}^{\mathrm{op}} = P_{(\underline{2})}^{\mathrm{op}}$ and $P_{(\underline{2})}$ is $\theta_{x_{2n}}$-split. It only remains to show that $P_{(\underline{2})}$ is a minimal $\theta_{x_{2n}}$-split parabolic subgroup. Suppose that $P = MN \subsetneq P_{(\underline{2})}$ is a $\theta_{x_{2n}}$-split parabolic subgroup of G that is properly contained in $P_{(\underline{2})}$. The parabolic subgroup $P \cap M_{(\underline{2})}$ of $M_{(\underline{2})}$ is $\theta_{x_{2n}}$-split in $M_{(\underline{2})}$. Notice that the GL-blocks of $M_{(\underline{2})}$ are not interchanged by $\theta_{x_{2n}}$. In fact, $\theta_{x_{2n}}$ restricted to $M_{(\underline{2})}$ is equal to the product $\theta_{x_2} \times \dots \times \theta_{x_2}$. It follows that $P \cap M_{(\underline{2})}$ is a product of θx_2-split parabolic subgroups in $\mathbf{GL}_2(F)$. Notice that the F-split component of the centre of $M_{(\underline{2})}$ is $(\theta_{x_{2n}}, F)$-split. By [HW93, Proposition 4.7(iv)], no proper parabolic subgroup of $\mathbf{GL}_2(F)$ can be θx_2-split, and it follows that $M_{(\underline{2})}$ has no proper $\theta_{x_{2n}}$-split parabolic subgroups. In particular, $P_{(\underline{2})}$ is a minimal $\theta_{x_{2n}}$-split parabolic subgroup of G. □

The torus $S_{0,x_{2n}} = \{\mathrm{diag}(a_1, a_1, \dots, a_n, a_n) : a_i \in F^\times\}$ is a maximal $(\theta_{x_{2n}}, F)$-split torus of G, it is the $(\theta_{x_{2n}}, F)$-split component of $P_{(\underline{2})}$ and the F-split component of $M_{(\underline{2})}$. The torus S_0 is the w_+-conjugate of $S_{0,x_{2n}}$. We also note explicitly that $P_0 = w_+ P_{(\underline{2})} w_+^{-1}$ is θ-split:

$$\begin{aligned}\theta(P_0) &= \theta(w_+ P_{(\underline{2})} w_+^{-1}) \\ &= w_+ w_+^{-1} \theta(w_+ P_{(\underline{2})} w_+^{-1}) w_+ w_+^{-1} \\ &= w_+ \theta_{x_{2n}}(P_{(\underline{2})}) w_+^{-1} \\ &= w_+ (P_{(\underline{2})}^{\mathrm{op}}) w_+^{-1} \\ &= P_0^{\mathrm{op}},\end{aligned}$$

where the opposite is taken with respect to the θ-stable Levi factor $M_0 = w_+ M_{(\underline{2})} w_+^{-1}$. Let $N_0 = w_+ N_{(\underline{2})} w_+^{-1}$ denote the unipotent radical of P_0. We emphasize that $P_0 = M_0 N_0$ is a minimal θ-split parabolic subgroup of G.

5.1. **The restricted root system and θ-split parabolic subgroups.**

Definition 5.2. Let Δ be a base of a root system Φ. The Δ-positive (respectively, Δ-negative) roots in Φ consist of the collection of positive (respectively, negative) roots in Φ with respect to Δ; in particular, the set of Δ-positive roots is equal to $\Phi \cap \mathrm{span}_{\mathbb{Z}_{\geq 0}} \Delta$.

Let $\Phi_0 = \Phi(\mathbf{G}, \mathbf{A}_0)$ be the root system of G with respect to A_0. Since A_0 is θ-stable, the involution θ acts on $X^*(A_0)$ and Φ_0 is θ-stable under this action. Let $\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq 2n-1\}$ be the standard base for Φ_0, where ϵ_i denotes the i-th F-rational coordinate character of A_0. Define $\Delta_0 = w_+ \Delta$ to be the Weyl group translate of Δ by the permutation matrix $w_+ \in W_0$, where $W_0 \cong N_G(A_0)/A_0$ is the Weyl group of G (with respect to A_0). We identify W_0 with the subgroup of G consisting of all permutation matrices.

Lemma 5.3. *The set Φ_0^θ of θ-fixed roots in Φ_0 is equal to the set*

$$\Phi_0^\theta = \{\epsilon_i - \epsilon_{2n+1-i} : 1 \leq i \leq 2n\},$$

corresponding to the root spaces on the main anti-diagonal in $\mathfrak{gl}_{2n}$.

Proof. For any $1 \leq i \neq j \leq 2n$, we have that $\theta(\epsilon_i - \epsilon_j) = \epsilon_{2n+1-j} - \epsilon_{2n+1-i}$. Note that $2n+1-(2n+1-i) = i$; therefore, the root $\epsilon_i - \epsilon_j$ is θ-fixed if and only if $j = 2n+1-i$. □

Lemma 5.4. *The set of simple roots $\Delta_0 = w_+\Delta$ is a θ-base of Φ_0.*

Proof. The set Φ_0^+ of Δ_0-positive roots is equal to $w_+\Phi_\Delta^+$, where Φ_Δ^+ is the set of Δ-positive roots. Moreover, the set of Δ_0-negative roots in Φ_0 is $\Phi_0^- = -\Phi_0^+ = w_+\Phi_\Delta^-$. Let $\alpha = \epsilon_i - \epsilon_j \in \Phi_\Delta^+$, that is, $1 \leq i < j \leq 2n$ and $w_+\alpha \in \Phi_0^+$. Suppose that $w_+\alpha$ is not θ-fixed. Note that $w_+\epsilon_i = \epsilon_{w_+(i)}$ and thus

$$\theta(w_+(\epsilon_i - \epsilon_j)) = \epsilon_{2n+1-w_+(j)} - \epsilon_{2n+1-w_+(i)}.$$

We consider the image of $w_+\alpha$ under θ in the following four cases.

Case (i): i, j both odd. We can write $i = 2k-1$ and $j = 2l-1$ with $1 \leq k < l \leq n$. It follows that

$$\theta(w_+\alpha) = \epsilon_{2n+1-l} - \epsilon_{2n+1-k} = w_+(\epsilon_{2l} - \epsilon_{2k});$$

moreover, since $2l > 2k$, we have that $w_+(\epsilon_{2l} - \epsilon_{2k}) \in \Phi_0^-$.

Case (ii): i odd, j even. Let $i = 2k-1$ and $j = 2l$ with $1 \le k \le l \le n$. As above,

$$\theta(w_+\alpha) = \epsilon_l - \epsilon_{2n+1-k} = w_+(\epsilon_{2l-1} - \epsilon_{2k}).$$

Observe that $k \neq l$, since otherwise $w_+\alpha = \theta(w_+\alpha) \in \Phi_0^\theta$ and we have assumed that $w_+\alpha$ is not θ-fixed. Since $l > k$, we have $2l-1 > 2k$ and $w_+(\epsilon_{2l} - \epsilon_{2k}) \in \Phi_0^-$.

Case (iii): i even, j odd. Let $i = 2k$ and $j = 2l-1$ where $1 \le k < l \le n$. It follows that

$$\theta(w_+\alpha) = \epsilon_{2n+1-l} - \epsilon_{2n+1-(2n+1-k)} = w_+(\epsilon_{2l} - \epsilon_{2k-1});$$

moreover, since $l > k$, we have $2l > 2k-1$ and $w_+(\epsilon_{2l} - \epsilon_{2k-1}) \in \Phi_0^-$.

Case (iv): i, j both even. Let $i = 2k$ and $j = 2l$ for $1 \le k < l \le n$. We have

$$\theta(w_+(\epsilon_{2k} - \epsilon_{2l})) = \epsilon_{2n+1-(2n+1-l)} - \epsilon_{2n+1-(2n+1-k)} = w_+(\epsilon_{2l-l} - \epsilon_{2k-1});$$

moreover, since $l > k$, we have that $2l-1 > 2k-1$ and $w_+(\epsilon_{2l-1} - \epsilon_{2k-1}) \in \Phi_0^-$.

It follows that if $\beta \in \Phi_0^+$ is not θ-fixed, then $\theta(\beta) \in \Phi_0^-$; therefore, Δ_0 is a θ-base of Φ_0. □

Observation 5.5. From the proof of Lemma 5.4, we see that the set of θ-fixed Δ_0-positive roots are the translates of $\{\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4, \ldots, \epsilon_{2n-1} - \epsilon_{2n}\}$ by w_+. The subset $\{\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4, \ldots, \epsilon_{2n-1} - \epsilon_{2n}\}$ of Δ consists of $\theta_{x_{2n}}$-fixed roots and determines the (minimal $\theta_{x_{2n}}$-split) parabolic subgroup $P_{(\underline{2})}$.

To aid in our understanding of the structure of Δ_0, we partition the roots in the standard base Δ into the disjoint subsets

$$\Delta_{\text{odd}} = \{\epsilon_{2i-1} - \epsilon_{2i} : 1 \le i \le n\}$$

and

$$\Delta_{\text{even}} = \{\epsilon_{2j} - \epsilon_{2j+1} : 1 \le j \le n-1\}.$$

Notice that the set of θ-fixed simple roots in Δ_0 is equal to $\Delta_0^\theta = w_+\Delta_{\text{odd}}$. Moreover, Δ_0 is the disjoint union $\Delta_0 = \Delta_0^\theta \sqcup w_+\Delta_{\text{even}}$. Explicitly,

$$\Delta_0^\theta = w_+\Delta_{\text{odd}} = \{\epsilon_i - \epsilon_{2n+1-i} : 1 \le i \le n\}$$

and

$$w_+\Delta_{\text{even}} = \{\epsilon_{2n+1-j} - \epsilon_{j+1} : 1 \le j \le n-1\}.$$

Let $r : X^*(A_0) \to X^*(S_0)$ be the surjective homomorphism defined by restricting F-rational characters of A_0 to S_0. The θ-fixed simple roots are trivial on S_0. It follows that

$$\overline{\Delta}_0 = r(\Delta_0 \setminus \Delta_0^\theta) = r(w_+\Delta_{\text{even}}) = \{\bar{\epsilon}_i - \bar{\epsilon}_{i+1} : 1 \le i \le n-1\},$$

where $\bar{\epsilon}_i$ is the i-th F-rational coordinate character of S_0 given by

$$\bar{\epsilon}_i(\text{diag}(a_1, \ldots, a_n, a_n, \ldots, a_1)) = a_i,$$

for $1 \le i \le n$. In addition, the full set of restricted roots is

$$\overline{\Phi}_0 = r(\Phi_0) \setminus \{0\} = r(\Phi_0 \setminus \Phi_0^\theta) = \{\bar{\epsilon}_i - \bar{\epsilon}_j : 1 \le i \neq j \le n\}.$$

We have established the following.

Lemma 5.6. *The restricted root system associated to* $\mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F)$ *is of type* A_{n-1} *and the dual group* $G_X^\vee$ *of* $X = \mathbf{Sp}_{2n}(F)\backslash\mathbf{GL}_{2n}(F)$ *is* $\mathrm{GL}(n, \mathbb{C})$.

Proper (Δ_0-)standard θ-split parabolic subgroups of G are parametrized by proper θ-split subsets of Δ_0, where a subset Θ of Δ_0 is θ-split if it is of the form

$$\Theta = [\overline{\Theta}] := r^{-1}(\overline{\Theta}) \cup \Delta_0^\theta,$$

and $\overline{\Theta}$ is a subset of $\overline{\Delta}_0$. The subset Δ_0^θ of θ-fixed simple roots determines the minimal standard θ-split parabolic $P_0 = M_0N_0$ of G, with Levi factor $M_0 = C_G(S_0)$ and unipotent radical N_0. By [KT08, Lemma 2.5], any θ-split parabolic subgroup of G is $(\mathbf{HM}_0)(F)$-conjugate to a standard θ-split parabolic. In the current setting, the Galois cohomology of $\mathbf{M}_0 \cap \mathbf{H}$ over F is trivial and it follows that $(\mathbf{HM}_0)(F) = HM_0$; moreover, any θ-split parabolic subgroup is H-conjugate to a standard θ-split parabolic subgroup. For completeness, we give a proof.

Lemma 5.7. *The first Galois cohomology of* $\mathbf{M}_0 \cap \mathbf{H}$ *over* F *is trivial and* $(\mathbf{HM}_0)(F) = HM_0$.

Proof. First, one may readily verify that

$$\mathbf{M}_0 \cap \mathbf{H} = w_+ \left(\mathbf{M}_{(\underline{2})} \cap \mathbf{G}^{\theta_{x_{2n}}}\right) w_+^{-1} \cong \prod_1^n \mathbf{SL}_2.$$

By Hilbert's Theorem 90, it follows that

$$H^1(\mathbf{M}_0 \cap \mathbf{H}, F) \cong \bigoplus_1^n H^1(\mathbf{SL}_2, F) = 0.$$

Let $\bar{F}$ denote the algebraic closure of F. By considering the long exact sequence in Galois cohomology obtained from the short exact sequence

$$1 \to \mathbf{M}_0(\bar{F}) \cap \mathbf{H}(\bar{F}) \to \mathbf{H}(\bar{F}) \times \mathbf{M}_0(\bar{F}) \to \mathbf{H}(\bar{F})\mathbf{M}_0(\bar{F}) \to 1$$

of pointed sets, it follows that $(\mathbf{HM}_0)(F) = HM_0$, as claimed. ☐

Proposition 5.8. *Let P be a θ-split parabolic subgroup of G. There exists a θ-split subset Θ of Δ_0 and an element $h \in H$ such that $P = hP_\Theta h^{-1}$. Moreover, P has unipotent radical $N = hN_\Theta h^{-1}$ and θ-stable Levi factor $M = hM_\Theta h^{-1}$.*

Proof. Apply Lemma 5.7 and [KT08, Lemma 2.5]. ☐

With the last result in hand, we explicitly determine the maximal proper standard θ-split parabolic subgroups of G which correspond to the maximal proper θ-split subsets of Δ_0. A maximal proper θ-split subset of Δ_0 has the form $[\overline{\Delta}_0 \setminus \{\bar{\alpha}\}] = r^{-1}(\overline{\Delta}_0 \setminus \{\bar{\alpha}\}) \cup \Delta_0^\theta$, where $\bar{\alpha} \in \overline{\Delta}_0$. Observe that for each $\bar{\alpha} \in \overline{\Delta}_0$ there is a unique $\alpha \in w_+\Delta_{\mathrm{even}}$ such that $r(\alpha) = \bar{\alpha}$. Precisely, the pre-image of $\bar{\epsilon}_i - \bar{\epsilon}_{i+1}$ under the restriction map $r : X^*(A_0) \to X^*(S_0)$ is $r^{-1}(\bar{\epsilon}_i - \bar{\epsilon}_{i+1}) = w_+(\epsilon_{2i} - \epsilon_{2i+1})$, for each $1 \leq i \leq n-1$. It follows that for each $1 \leq k \leq n-1$ we have a maximal θ-split subset of Δ_0 given by

$$\begin{aligned} \Theta_k &= r^{-1}(\overline{\Delta}_0 \setminus \{\bar{\epsilon}_k - \bar{\epsilon}_{k+1}\}) \cup \Delta_0^\theta \\ &= w_+(\Delta \setminus \{\epsilon_{2k} - \epsilon_{2k+1}\}) \\ &= \Delta_0 \setminus \{\epsilon_{2n+1-k} - \epsilon_{k+1}\}. \end{aligned} \tag{5.1}$$

To each Θ_k, $1 \leq k \leq n-1$, we associate the maximal Δ_0-standard θ-split parabolic subgroup

$$P_{\Theta_k} = w_+ P_{(2k,2n-2k)} w_+^{-1},$$

with θ-stable Levi factor $M_{\Theta_k} = w_+ M_{(2k,2n-2k)} w_+^{-1}$ and unipotent radical $N_{\Theta_k} = w_+ N_{(2k,2n-2k)} w_+^{-1}$. Notice that P_{Θ_k} does indeed contain the minimal standard θ-split parabolic subgroup $P_0 = w_+ P_{(\underline{2})} w_+^{-1}$, corresponding to Δ_0^θ (or the partition $(\underline{2}) = (2, \ldots, 2)$ of $2n$). Moreover, by Lemma 5.1, the (θ, F)-split component S_{Θ_k} of P_{Θ_k} is equal to its F-split component A_{Θ_k}.

Note. It may be helpful to observe that the maximal $\theta_{x_{2n}}$-split subsets of Δ are thus given by $\Delta \setminus \{\epsilon_{2k} - \epsilon_{2k+1}\}$, where $1 \leq k \leq n-1$. It follows that the standard block-upper-triangular parabolic subgroups $P_{(2k,2n-2k)}$, with even sized blocks, are the maximal Δ-standard $\theta_{x_{2n}}$-split parabolic subgroups.

5.2. **Inducing from distinguished representations of θ-elliptic Levi subgroups.** We recall the following definition.

Definition 5.9. A θ-stable Levi subgroup of G is θ-elliptic if L is not contained in any proper θ-split parabolic subgroup of G.

In order to place the Speh representations within the context of the relative discrete series constructed in [Smi18b, Smi18a], we show that $\mathcal{U}(\delta, 2)$ can be realized as the quotient of a representation induced from a distinguished representation of a θ-elliptic Levi subgroup.

Lemma 5.10. *The block-upper triangular parabolic subgroup $P_{(n,n)}$, corresponding to $\Omega^{\mathrm{ell}} = \Delta \setminus \{\epsilon_n - \epsilon_{n-1}\} \subset \Delta$, is θ-stable and the Δ-standard block-diagonal Levi subgroup $M_{(n,n)}$ is θ-elliptic.*

Proof. First, it is clear that $P_{(n,n)}$ and $M_{(n,n)}$ are θ-stable subgroups of G. It is readily verified that the (θ, F)-split component of $M_{(n,n)}$ is equal to the (θ, F)-split component S_G of G; moreover, $S_G = A_G$, that is, the (θ, F)-split component of G is equal to the F-split component of G. By [Smi18b, Lemma 3.8], the θ-stable Levi subgroup $M_{(n,n)}$ is θ-elliptic. ☐

In what follows, we let $Q = P_{(n,n)} = P_{\Omega^{\mathrm{ell}}}$, $L = M_{(n,n)} = M_{\Omega^{\mathrm{ell}}}$, and $U = N_{(n,n)} = N_{\Omega^{\mathrm{ell}}}$. Define $\Omega = w_+ \Omega^{\mathrm{ell}} \subset \Delta_0$. We then have that $Q = w_+^{-1} P_\Omega w_+$ or, equivalently, that $P_\Omega = w_+ Q w_+^{-1}$.

Definition 5.11. An ordered partition $(m_1, \ldots, m_k)$ of an integer $m \geq 2$ is balanced if $(m_1, \ldots, m_k)$ is equal to the opposite partition $(m_1, \ldots, m_k)^{\mathrm{op}} = (m_k, \ldots, m_1)$.

Lemma 5.12. *Let P be a block-upper triangular (Δ-standard) parabolic subgroup of G. The subgroup P is θ-stable if and only if P corresponds to a balanced partition of $2n$. In addition, the only θ-stable Δ-standard maximal parabolic that admits a θ-elliptic Levi subgroup is $P_{(n,n)}$.*

Proof. The proof is the same as that of [Smi18a, Lemma 4.15]. ☐

Recall that a parabolic subgroup P is A_0-semi-standard if P contains the maximal F-split torus A_0. In particular, the Δ- and Δ_0-standard parabolic subgroups are A_0-semi-standard. The next result is the analogue of [Smi18a, Lemma 4.21]; the proof is the same.

Lemma 5.13. *Let P be any θ-stable parabolic subgroup of G. The subgroup P is H-conjugate to a θ-stable A_0-semi-standard parabolic subgroup.*

Lemma 5.14. *The θ-stable Levi subgroup $L = M_{(n,n)}$ is the only proper θ-elliptic A_0-semi-standard Levi subgroup of G up to conjugacy by Weyl group elements $w \in W_0 = W(G, A_0) = N_G(A_0)/A_0$ such that $w^{-1}\varepsilon_{2n}w \in N_G(L) \setminus L$.*

Proof. See the proof of [Smi18a, Lemma 4.20(2)]. ☐

Lemma 5.15. *The group L^θ of θ-fixed points in $L = M_{(n,n)}$ is isomorphic to $\mathbf{GL}_n(F)$ embedded in L as follows:*

$$L^\theta = \left\{ \begin{pmatrix} g & 0 \\ 0 & J_n^{-1}{}^t g^{-1} J_n \end{pmatrix} : g \in \mathbf{GL}_n(F) \right\}.$$

Proof. We omit the straightforward calculation. ☐

Proposition 5.16. *Let $\tau_1 \otimes \tau_2$ be an irreducible admissible representation of $L = M_{(n,n)}$. Then $\tau_1 \otimes \tau_2$ is L^θ-distinguished if and only if $\tau_2 \cong \tau_1$.*

Proof. First, one can show that $\tau_1 \otimes \tau_2$ is L^θ-distinguished if and only if $\tau_2 \cong \widetilde{\tau}_1 \circ \theta_{J_n}$, where θ_{J_n} is the involution on $\mathbf{GL}_n(F)$ given by $\theta_{J_n}(g) = J_n^{-1}{}^t g^{-1} J_n$, for $g \in \mathbf{GL}_n(F)$. Now, the lemma is a simple consequence of [GK75, Theorem 2] which implies that $\widetilde{\tau}_1 \cong \tau_1 \circ {}^t(\cdot)^{-1}$ and the fact that $J_n^{-1} = J_n = {}^t J_n$ (see [Smi18a, Lemma 5.3] for additional details). ☐

Let τ be an irreducible admissible representation of $\mathbf{GL}_n(F)$. The representation $\tau \otimes \tau$ of L is L^θ-distinguished by Proposition 5.16. Moreover, the L^θ-invariant linear form on $\tau \otimes \tau$ can be realized via the standard pairing between τ and its contragredient $\widetilde{\tau}$. Indeed, this follows from [GK75, Theorem 2] and the fact that $\widetilde{\tau} \cong \tau \circ \theta_{J_n}$. Let $\lambda_\tau \in \mathrm{Hom}_{L^\theta}(\tau \otimes \tau, 1)$ be the (nonzero) invariant form that arises via the pairing on $\tau \otimes \widetilde{\tau}$. Let $l = \mathrm{diag}(x, \theta_{J_n}(x)) \in L^\theta$ and consider the value of $\delta_{Q^\theta}\delta_Q^{-1/2}$ on l. It is straightforward to check that

$$\left(\delta_{Q^\theta}\ \delta_Q^{-1/2}\Big|_{L^\theta} \right)(l) = |\det(x)|^{n+1}|\det(x)|^{-n} = |\det(x)| = \nu(x),$$

that is, $\delta_{Q^\theta}\delta_Q^{-1/2}$ agrees with the character ν on $\mathbf{GL}_n(F) \cong L^\theta$. Since the contragredient of ν is ν^{-1}, it follows that $\lambda_\tau \in \mathrm{Hom}_{L^\theta}(\nu^{1/2}\tau \otimes \nu^{-1/2}\tau, \nu) \cong \mathrm{Hom}_{L^\theta}(\delta_Q^{1/2}\tau \otimes \tau, \delta_{Q^\theta})$. By [Off17, Proposition 7.1], λ_τ maps to a nonzero H-invariant linear form $\lambda \in \mathrm{Hom}_H(\nu^{1/2}\tau \times \nu^{-1/2}\tau, 1)$, and the parabolically induced representation $\nu^{1/2}\tau \times \nu^{-1/2}\tau = \iota_Q^G\left(\nu^{1/2}\tau \otimes \nu^{-1/2}\tau\right)$ is H-distinguished. We now state a result of Heumos and Rallis [HR90, Theorem 11.1] (*cf.* Section 3). We give a sketch of the proof (still appealing to the main results of [HR90]).

Proposition 5.17 (Heumos–Rallis)**.** *Let δ be an irreducible square integrable representation of $\mathbf{GL}_n(F)$. The parabolically induced representation $\nu^{1/2}\delta \times \nu^{-1/2}\delta = \iota_Q^G\left(\nu^{1/2}\delta \otimes \nu^{-1/2}\delta\right)$ is H-distinguished. Moreover, the unique irreducible quotient $\mathcal{U}(\delta, 2)$ of $\nu^{1/2}\delta \times \nu^{-1/2}\delta$ is H-distinguished.*

Proof. As above, $\nu^{1/2}\delta \times \nu^{-1/2}\delta$ is H-distinguished by Proposition 5.16 and [Off17, Proposition 7.1]. The parabolically induced representation $\nu^{1/2}\delta \times \nu^{-1/2}\delta$ has length two [BZ77, Zel80]. Let $\mathcal{Z}(\delta, 2)$ be the unique irreducible subrepresentation and let $\mathcal{U}(\delta, 2)$ be the unique irreducible quotient of $\nu^{1/2}\delta \times \nu^{-1/2}\delta$. The subrepresentation $\mathcal{Z}(\delta, 2)$ is tempered and thus generic [Zel80, Theorem 9.3]. Therefore, by [HR90, Theorem 3.2.2], $\mathcal{Z}(\delta, 2)$ cannot be H-distinguished. It follows that any nonzero

H-invariant linear form on $\nu^{1/2}\delta \times \nu^{-1/2}\delta$ descends to a well-defined nonzero H-invariant linear functional on the quotient $\mathcal{U}(\delta, 2)$. □

Remark 5.18. By the multiplicity-one result [HR90, Theorem 2.4.2], the H-invariant linear form on $\nu^{1/2}\tau \times \nu^{-1/2}\tau$ constructed via [Off17, Proposition 7.1] is a scalar multiple of the invariant form produced by Heumos and Rallis in [HR90, §11.3.1.2] (*cf.* [Smi20, Lemma 1.3.4]).

6. Application of the relative Casselman criterion

We now come to the main result of the paper.

Theorem 6.1. *Let δ be a discrete series representation of $\mathbf{GL}_n(F)$. The Speh representation $\mathcal{U}(\delta, 2)$ of $\mathbf{GL}_{2n}(F)$ is $\mathbf{Sp}_{2n}(F)$-relatively square integrable.*

Proof. Let $\lambda \in \mathrm{Hom}_H(\mathcal{U}(\delta, 2), 1)$ be nonzero. Let $\pi = \nu^{1/2}\delta \times \nu^{-1/2}\delta$. Recall from Section 3 that $\mathcal{U}(\delta, 2)$ is the unique irreducible quotient of π. By Proposition 5.8 and [Smi18b, Proposition 4.22], it is enough to consider exponents along maximal standard θ-split parabolic subgroups of G when applying Theorem 2.7 ([KT10, Theorem 4.7]). Let $P = MN$ be a maximal Δ_0-standard θ-split parabolic subgroup of G with unipotent radical N and θ-stable Levi factor $M = P \cap \theta(P)$. By [Smi18b, Proposition 4.23], only exponents corresponding to irreducible M^θ-distinguished subquotients of the Jacquet module $\mathcal{U}(\delta, 2)_N$ may appear in $\mathcal{E}xp_{S_M}(\mathcal{U}(\delta, 2)_N, \lambda_N)$. By Proposition 6.2, the irreducible unitary subquotients of π_N, and also $\mathcal{U}(\delta, 2)_N$, are not M^θ-distinguished. By Proposition 6.5, all exponents that appear in $\mathcal{E}xp_{S_M}(\mathcal{U}(\delta, 2)_N, \lambda_N)$ satisfy (2.2). By Theorem 2.7, $\mathcal{U}(\delta, 2)$ is (H, λ)-relatively square integrable. Multiplicity-one holds by [HR90, Theorem 2.4.2], thus $\dim \mathrm{Hom}_H (\mathcal{U}(\delta, 2), 1) = 1$ and $\mathcal{U}(\delta, 2)$ is H-relatively square integrable. □

The remainder of the paper is dedicated to proving Proposition 6.2 and Proposition 6.5.

Let δ be an irreducible admissible square integrable (discrete series) representation of $\mathbf{GL}_n(F)$. Let $\pi = \nu^{1/2}\delta \times \nu^{-1/2}\delta$. The sequence

$$0 \to \mathcal{Z}(\delta, 2) \to \pi \to \mathcal{U}(\delta, 2) \to 0$$

of G-modules is exact, where $\mathcal{Z}(\delta, 2)$ is the unique irreducible generic subrepresentation of π (see Section 3). We keep the notation of Section 5 and let $Q = P_{(n,n)}$, $L = M_{(n,n)}$, and $U = N_{(n,n)}$. Let $P = MN$ be a maximal Δ_0-standard θ-split parabolic subgroup of G, with unipotent radical N and θ-stable Levi factor $M = P \cap \theta(P)$. The Jacquet restriction functor (along P) is exact; therefore, we have an exact sequence of M-modules

$$0 \to \mathcal{Z}(\delta, 2)_N \to \pi_N \to \mathcal{U}(\delta, 2)_N \to 0. \tag{6.1}$$

Our goal is to understand the irreducible subquotients, and the exponents, of $\mathcal{U}(\delta, 2)_N$ by applying the Geometric Lemma [BZ77, Lemma 2.12] to π_N. If $\chi \in \mathcal{E}xp_{A_M}(\mathcal{U}(\delta, 2)_N)$, then χ is the central quasi-character of an irreducible subquotient of $\mathcal{U}(\delta, 2)_N$ and thus of π_N, that is, χ appears in $\mathcal{E}xp_{A_M}(\pi_N)$. Recall that we can realize $Q = w_+^{-1} P_\Omega w_+$, where $\Omega = \Delta_0 \setminus \{w_+(\epsilon_n - \epsilon_{n+1})\}$, and $P = P_\Theta$, for some $1 \le k \le n-1$, where $\Theta = \Theta_k$ is described in (5.1). In particular, Ω and Θ are subsets of the θ-base Δ_0. Let

$$[W_\Theta \backslash W_0 / W_\Omega] = \{w \in W_0 : w\Omega \subset \Phi_0^+, w^{-1}\Theta \subset \Phi_0^{-1}\},$$

where Φ_0^+ is the set of Δ_0-positive roots. By [Cas95, Propositions 1.3.1 and 1.3.3], the set $[W_\Theta\backslash W_0/W_\Omega]\cdot w_+$ is a system of representatives for $P\backslash G/Q$. By the Geometric Lemma [BZ77, Lemma 2.12], there exists a filtration of the space of π_N such that the associated graded object $\mathrm{gr}(\pi_N)$ is isomorphic to

$$\bigoplus_{y\in[W_\Theta\backslash W_0/W_\Omega]\cdot w_+} \iota_{M\cap {}^yQ}^M \left({}^y(\nu^{1/2}\delta\otimes\nu^{-1/2}\delta)_{N\cap {}^yL}\right). \tag{6.2}$$

Write $\mathcal{F}_N^y(\delta,2)$ to denote the representation $\iota_{M\cap {}^yQ}^M\left({}^y(\nu^{1/2}\delta\otimes\nu^{-1/2}\delta)_{N\cap {}^yL}\right)$. Thus

$$\mathrm{gr}(\pi_N)\cong\bigoplus_{w\in[W_\Theta\backslash W_0/W_\Omega]}\mathcal{F}_N^{ww_+}(\delta,2).$$

The exponents of π along P are the central characters of the irreducible subquotients of π_N; moreover, the exponents of $\mathcal{U}(\delta,2)$ along P are a subset of the exponents of π along P. Recall that, by Lemma 5.1, the (θ,F)-split component S_M of M is equal to its F-split component A_M; precisely,

$$\begin{aligned} A_M &= w_+\{\mathrm{diag}(\underbrace{a,\dots,a}_{2k},\underbrace{b,\dots,b}_{2n-2k}) : a,b\in F^\times\}w_+^{-1} \\ &= \{\mathrm{diag}(\underbrace{a,\dots,a}_{k},\underbrace{b,\dots,b}_{2n-2k},\underbrace{a,\dots,a}_{k}) : a,b\in F^\times\}. \end{aligned}$$

By [Cas95, Proposition 1.3.3], with our choice $[W_\Theta\backslash W_0/W_\Omega]\cdot w_+$ of representatives for $P\backslash G/Q$, if $y=ww_+$ where $w\in[W_\Theta\backslash W_0/W_\Omega]$, then $M\cap {}^yQ$ is a parabolic subgroup of M with Levi factor $M\cap {}^yL$ and unipotent radical $M\cap {}^yU$. Similarly, $P\cap {}^yL$ is a parabolic subgroup of L with Levi subgroup $M\cap {}^yL$ and unipotent radical $N\cap {}^yL$. Explicitly, since $P=P_\Theta$ and $Q=w_+^{-1}P_\Omega w_+$, we see that

$$M\cap {}^yL = M_\Theta\cap wM_\Omega w^{-1} = M_\Theta\cap M_{w\Omega} = M_{\Theta\cap w\Omega},$$

$$N\cap {}^yL = N_\Theta\cap wM_\Omega w^{-1} = N_\Theta\cap M_{w\Omega},$$

and

$$M\cap {}^yU = M_\Theta\cap wN_\Omega w^{-1} = M_\Theta\cap N_{w\Omega}.$$

Let $w\in[W_\Theta\backslash W_0/W_\Omega]$. To achieve our goal, there are two cases that we need to consider.

Case 1: $P_\Theta\cap {}^wM_\Omega = {}^wM_\Omega$.

Case 2: $P_\Theta\cap {}^wM_\Omega \subsetneq {}^wM_\Omega$ is a proper parabolic subgroup of ${}^wM_\Omega$.

In Case 1, we show that the associated irreducible subquotients of π_N are not M^θ-distinguished. In Case 2, we show that the corresponding exponents of π_N satisfy condition (2.2). The exact sequence (6.1) allows us to conclude that the same holds for $\mathcal{U}(\delta,2)_N$.

6.1. **Case 1: no distinction.** Assume that $w\in[W_\Theta\backslash W_0/W_\Omega]$ is such that $P_\Theta\cap {}^wM_\Omega = {}^wM_\Omega$. Then $N_\Theta\cap {}^wM_\Omega=\{e\}$ and $M_\Theta\cap {}^wM_\Omega = {}^wM_\Omega = M_{w\Omega}$. In particular, ${}^wM_\Omega\subset M_\Theta$, and since ${}^wM_\Omega$ is maximal it follows that ${}^wM_\Omega = M_\Theta\cong \mathbf{GL}_n(F)\times\mathbf{GL}_n(F)$. That is, M_Θ and M_Ω are associate standard Levi subgroups isomorphic to $\mathbf{GL}_n(F)\times\mathbf{GL}_n(F)$. It follows that n must be even, $k=n/2$, and $\Theta=\Theta_{n/2}=w_+(\Delta\setminus\{\epsilon_n-\epsilon_{n+1}\})=\Omega$. That is, $M_\Theta=M_\Omega$ and w lies in

$[W_\Omega\backslash W_0/W_\Omega]\cap W(\Omega,\Omega)$, where $W(\Theta,\Omega)=\{w\in W_0 : w\Omega=\Theta\}$. Set $y=ww_+$. Then $M_\Omega\cap {}^yQ=M_{w\Omega}=M_\Omega$ and $P_\Omega\cap {}^yL=M_{w\Omega}=M_\Omega$. In this setting,

$$\mathscr{F}_\Omega^y(\delta,2)=\iota_{M_\Omega}^{M_\Omega}({}^y(\nu^{1/2}\delta\otimes\nu^{-1/2}\delta)_{\{e\}})={}^y(\nu^{1/2}\delta\otimes\nu^{-1/2}\delta),$$

since $N_\Omega\cap {}^wM_\Omega=N_\Omega\cap M_\Omega=\{e\}$.

Proposition 6.2. *Let $w\in[W_\Omega\backslash W_0/W_\Omega]\cap W(\Omega,\Omega)$ and set $y=ww_+$. Let τ be an irreducible admissible generic representation of $\mathbf{GL}_n(F)$. The representation ${}^y(\nu^{1/2}\tau\otimes\nu^{-1/2}\tau)$ of M_Ω is not M_Ω^θ-distinguished, that is, $\operatorname{Hom}_{M_\Omega^\theta}({}^y(\nu^{1/2}\tau\otimes\nu^{-1/2}\tau),1)=0$.*

Proof. First, recall that n is even, and observe that $M_\Omega^\theta\cong\mathbf{Sp}_n(F)\times\mathbf{Sp}_n(F)$. Indeed, $M_\Omega=w_+M_{(n,n)}w_+^{-1}$ and $m=w_+\underline{m}w_+^{-1}\in M_\Omega$ is θ-fixed if and only if $\underline{m}\in M_{(n,n)}$ is fixed by $w_+\cdot\theta=\theta_{x_{2n}}$. Recall (see Section 5) that

$$x_{2n}=\varepsilon_{2n}\cdot w_+=\begin{pmatrix}0&1&&&\\-1&0&&&\\&&\ddots&&\\&&&0&1\\&&&-1&0\end{pmatrix}\in M_{(n,n)}$$

and $\theta_{x_{2n}}(g)=x_{2n}^{-1}{}^tg^{-1}x_{2n}$. One may readily verify that the image of $\underline{m}=\operatorname{diag}(m_1,m_2)\in M_{(n,n)}$ under $\theta_{x_{2n}}$ is given by

$$\theta_{x_{2n}}(\underline{m})=\operatorname{diag}(x_n^{-1}{}^tm_1^{-1}x_n,x_n^{-1}{}^tm_2^{-1}x_n)=\operatorname{diag}(\theta_{x_n}(m_1),\theta_{x_n}(m_2)).$$

It follows that $\underline{m}$ is $\theta_{x_{2n}}$-fixed if and only if $m_i=\theta_{x_n}(m_i)$, for each $i=1,2$. Moreover,

$$\begin{aligned}M_\Omega^\theta&=w_+\left(M_{(n,n)}^{\theta_{x_{2n}}}\right)w_+^{-1}\\&=w_+\left(\mathbf{GL}_n(F)^{\theta_{x_n}}\times\mathbf{GL}_n(F)^{\theta_{x_n}}\right)w_+^{-1}\\&\cong\left(\mathbf{Sp}_n(F)\times\mathbf{Sp}_n(F)\right),\end{aligned}$$

since $x_n\in\mathbf{GL}_n(F)$ is nonsingular and skew symmetric, and $\mathbf{GL}_n^{\theta_{x_n}}\cong\mathbf{Sp}_n$.

Next, we note that $[W_\Omega\backslash W_0/W_\Omega]\cap W(\Omega,\Omega)$ consists of two elements: the identity e and $w_+w_{(n,n)}w_+^{-1}$, where

$$w_{(n,n)}=\begin{pmatrix}0&I_n\\I_n&0\end{pmatrix}.$$

First, realize $[W_\Omega\backslash W_0/W_\Omega]=w_+[W_{\Omega^{\mathrm{ell}}}\backslash W_0/W_{\Omega^{\mathrm{ell}}}]w_+^{-1}$, and $W(\Omega,\Omega)=w_+W(\Omega^{\mathrm{ell}},\Omega^{\mathrm{ell}})w_+^{-1}$, where $\Omega^{\mathrm{ell}}=\Delta\setminus\{\epsilon_n-\epsilon_{n+1}\}=w_+^{-1}\Omega$. If $w\in W_0$, then we identify w with a permutation of $\{1,\dots,2n\}$ and note that $w(\epsilon_i)=\epsilon_{w(i)}$. The set of Δ-positive roots in Φ_0 is $\Phi_\Delta^+=\{\epsilon_i-\epsilon_j : 1\le i<j\le 2n\}$. Thus, by definition, $w\in W_0$ lies in the set $[W_{\Omega^{\mathrm{ell}}}\backslash W_0/W_{\Omega^{\mathrm{ell}}}]$ if and only if $w(i)<w(i+1)$ and $w^{-1}(i)<w^{-1}(i+1)$, for all $1\le i\le n-1$ and $n+1\le i\le 2n-1$ (with $i\ne n$ since $\epsilon_n-\epsilon_{n+1}\notin\Omega^{\mathrm{ell}}$). It is not difficult to verify that $[W_{\Omega^{\mathrm{ell}}}\backslash W_0/W_{\Omega^{\mathrm{ell}}}]$ consists of the $n+1$ permutation matrices of the form

$$\begin{pmatrix}I_j&0&0&0\\0&0&I_{n-j}&0\\0&I_{n-j}&0&0\\0&0&0&I_j\end{pmatrix},\tag{6.3}$$

where $0 \leq j \leq n$. Notice that $j = 0$ corresponds to $w_{(n,n)}$ and $j = n$ corresponds to the identity matrix $e = I_{2n}$. On the other hand, the elements of the set $W(\Omega^{\mathrm{ell}}, \Omega^{\mathrm{ell}})$ satisfy $w\Omega^{\mathrm{ell}} = \Omega^{\mathrm{ell}}$ and thus normalize the block-diagonal Levi subgroup $M_{(n,n)} = M_{\Omega^{\mathrm{ell}}}$. One may quickly check that, of the elements of the form in (6.3), only the identity e and $w_{(n,n)}$ normalize $M_{(n,n)}$. It follows that $[W_{\Omega^{\mathrm{ell}}}\backslash W_0/W_{\Omega^{\mathrm{ell}}}] \cap W(\Omega^{\mathrm{ell}}, \Omega^{\mathrm{ell}})$ consists of precisely e and $w_{(n,n)}$, proving the claim.

We now turn to studying the M_Ω^θ-distinction of $\mathcal{F}_\Omega^y(\tau, 2) = {}^y(\nu^{1/2}\tau \otimes \nu^{-1/2}\tau)$, where $y = ww_+$. There are two sub-cases to consider, either $w = e$ or $w = w_+w_{(n,n)}w_+^{-1}$. If $w = e$, then $y = w_+ \in [W_\Omega\backslash W_0/W_\Omega] \cap W(\Omega, \Omega)$. As above, $\mathcal{F}_\Omega^y(\tau, 2) = {}^{w_+}(\nu^{1/2}\tau \otimes \nu^{-1/2}\tau)$. If $w = w_{(n,n)}$, then $y = ww_+ = w_+w_{(n,n)}w_+^{-1}w_+ = w_+w_{(n,n)}$. It follows that $\mathcal{F}_\Omega^y(\tau, 2) = {}^{w_+w_{(n,n)}}(\nu^{1/2}\tau \otimes \nu^{-1/2}\tau)$. Conjugation by $w_{(n,n)}$ interchanges the two GL-blocks of $M_\Omega = M_{(n,n)}$; therefore, twisting a representation $\pi_1 \otimes \pi_2$ of M_Ω by $w_{(n,n)}$ interchanges the two representations, that is, ${}^{w_{(n,n)}}(\pi_1 \otimes \pi_2) \cong \pi_2 \otimes \pi_1$. Therefore, ${}^{w_+w_{(n,n)}}(\nu^{1/2}\tau \otimes \nu^{-1/2}\tau) = {}^{w_+}(\nu^{-1/2}\tau \otimes \nu^{1/2}\tau)$. We have seen above that $M_\Omega^\theta \cong (\mathbf{Sp}_n(F) \times \mathbf{Sp}_n(F))$. In both cases $(w = e, w = w_{(n,n)})$, it follows that $\mathcal{F}_\Omega^y(\tau, 2)$ is M_Ω^θ-distinguished if and only if $\nu^{1/2}\tau$ and $\nu^{-1/2}\tau$ are $\mathbf{Sp}_n(F)$-distinguished. By assumption, τ is an irreducible generic representation; therefore, by [HR90, Theorem 3.2.2], $\mathrm{Hom}_{\mathbf{Sp}_n(F)}(\tau, 1) = \{0\}$. It follows, since ν is trivial on (maximal) unipotent subgroups of $\mathbf{GL}_n(F)$, that $\nu^s\tau$ is generic and $\mathrm{Hom}_{\mathbf{Sp}_n(F)}(\nu^s\tau, 1) = \{0\}$, for every $s \in \mathbb{C}$. Moreover, if w is equal to either e or $w_{(n,n)}$, then $\mathrm{Hom}_{M_\Omega^\theta}(\mathcal{F}_\Omega^y(\tau, 2), 1) = 0$, as claimed. □

6.2. **Case 2: 'good' exponents.** Assume that $w \in [W_\Theta\backslash W_0/W_\Omega]$ is such that $P_\Theta \cap {}^wM_\Omega$ is a proper parabolic subgroup of ${}^wM_\Omega$. First, we show that $M_\Theta \cap {}^wP_\Omega$ is also a proper parabolic subgroup of M_Θ. We argue by contradiction, and suppose that $M_\Theta \cap {}^wP_\Omega = M_\Theta$. By [Cas95, Proposition 1.3.3], $M_\Theta \cap {}^wN_\Omega = \{e\}$ and $M_\Theta \cap {}^wM_\Omega = M_\Theta$. In particular, $M_\Theta \subset {}^wM_\Omega = M_{w\Omega}$. However, both M_Ω and M_Θ are maximal Levi subgroups of G, and it follows that $M_\Theta = {}^wM_\Omega$. This, in turn, implies that $P_\Theta \cap {}^wM_\Omega = M_\Theta = {}^wM_\Omega$, which contradicts our assumption that $P_\Theta \cap {}^wM_\Omega$ is a proper parabolic subgroup of ${}^wM_\Omega$. We conclude that $M_\Theta \cap {}^wP_\Omega$ is a proper parabolic subgroup of M_Θ.

It follows from this last observation that if $y = ww_+$, then the representation $\mathcal{F}_N^y(\delta, 2) = \iota_{M\cap {}^yQ}^M\left({}^y(\nu^{1/2}\delta \otimes \nu^{-1/2}\delta)_{N\cap {}^yL}\right)$ is induced from ${}^y(\nu^{1/2}\delta \otimes \nu^{-1/2}\delta)_{N\cap {}^yL}$ along the proper parabolic $M \cap {}^yQ = M_\Theta \cap {}^wP_\Omega$ of $M = M_\Theta$; moreover, the Jacquet module ${}^y(\nu^{1/2}\delta \otimes \nu^{-1/2}\delta)_{N\cap {}^yL}$ is taken along the proper parabolic $P \cap {}^yL = P_\Theta \cap {}^wM_\Omega$ of ${}^yL = {}^wM_\Omega$. That is, both the Jacquet restriction and parabolic induction steps appearing in $\mathcal{F}_N^y(\delta, 2)$ are along proper parabolic subgroups. To be completely explicit, we note that

$$\mathcal{F}_N^{ww_+}(\delta, 2) = \iota_{M_\Theta\cap {}^wP_\Omega}^{M_\Theta}\left({}^{ww_+}(\nu^{1/2}\delta \otimes \nu^{-1/2}\delta)_{N_\Theta\cap {}^wM_\Omega}\right).$$

In this subsection, we will use the shorthand notation $[\tau] = \nu^{1/2}\tau \otimes \nu^{-1/2}\tau$, where τ is an irreducible admissible representation of $\mathbf{GL}_n(F)$. Our goal is to compute the exponents of $\pi = \nu^{1/2}\delta \times \nu^{-1/2}\delta$ along $P = P_\Theta$; therefore, we need to understand the central characters of the irreducible subquotients of the $\mathcal{F}^{ww_+}(\delta, 2)$. By [Smi18b, Lemma 4.16], the quasi-characters appearing in $\mathcal{E}xp_{A_\Theta}(\mathcal{F}^{ww_+}(\delta, 2))$ are the restrictions to A_Θ of the quasi-characters appearing in $\mathcal{E}xp_{A_{\Theta\cap w\Omega}}({}^{ww_+}[\delta]_{N_\Theta\cap {}^wM_\Omega})$, where the F-split component of $M_\Theta \cap {}^wM_\Omega = M_{\Theta\cap w\Omega}$ is $A_{\Theta\cap w\Omega}$. Thus, our problem reduces to understanding the exponents of ${}^{ww_+}[\delta]$ along $P_\Theta \cap {}^wM_\Omega$.

Since $L = M_{(n,n)} \cong \mathbf{GL}_n(F)\times\mathbf{GL}_n(F)$, we have that $P_\Theta\cap{}^wM_\Omega \cong P_1\times P_2$, where P_1 and P_2 are parabolic subgroups of $\mathbf{GL}_n(F)$, at least one of which is proper. We can realize $w = w_+w'w_+^{-1} \in [W_\Theta\backslash W_0/W_\Omega]$, where $w' \in [W_{(2k,2n-2k)}\backslash W_0/W_{(n,n)}]$. Then, with $w = w_+w'w_+^{-1}$,

$$\begin{aligned}
\left({}^{ww_+}[\delta]\right)_{N_\Theta\cap{}^wM_\Omega} &= \left({}^{w_+w'}[\delta]\right)_{w_+N_{(2k,2n-2k)}w_+^{-1}\cap w_+w'M_{(n,n)}w'^{-1}w_+^{-1}}\\
&= \left({}^{w_+w'}[\delta]\right)_{w_+\left(N_{(2k,2n-2k)}\cap w'M_{(n,n)}w'^{-1}\right)w_+^{-1}}\\
&= {}^{w_+}\left({}^{w'}[\delta]_{N_{(2k,2n-2k)}\cap w'M_{(n,n)}w'^{-1}}\right)\\
&= {}^{w_+w'}\left([\delta]_{w'^{-1}N_{(2k,2n-2k)}w'\cap M_{(n,n)}}\right)\\
&= {}^{w_+w'}\left([\delta]_{N_1\times N_2}\right),
\end{aligned}$$

where we identify $P_\Theta \cap {}^wM_\Omega \cong P_1 \times P_2$ with a parabolic subgroup of $M_{(n,n)} \cong \mathbf{GL}_n(F) \times \mathbf{GL}_n(F)$ via

$$\begin{aligned}
P_1 \times P_2 &= w'^{-1}N_{(2k,2n-2k)}w' \cap M_{(n,n)}\\
&= w'^{-1}\left(N_{(2k,2n-2k)} \cap w'M_{(n,n)}w'^{-1}\right)w'\\
&= w'^{-1}w_+^{-1}\left(w_+N_{(2k,2n-2k)}w_+^{-1} \cap w_+w'M_{(n,n)}w'^{-1}w_+^{-1}\right)w_+w'\\
&= w'^{-1}w_+^{-1}\left(w_+N_{(2k,2n-2k)}w_+^{-1} \cap w_+w'w_+^{-1}w_+M_{(n,n)}w_+^{-1}w_+w'^{-1}w_+^{-1}\right)w_+w'\\
&= w'^{-1}w_+^{-1}\left(w_+N_{(2k,2n-2k)}w_+^{-1} \cap ww_+M_{(n,n)}w_+^{-1}w^{-1}\right)w_+w'\\
&= w'^{-1}w_+^{-1}\left(N_\Theta \cap wM_\Omega w^{-1}\right)w_+w'\\
&= w_+^{-1}w^{-1}\left(N_\Theta \cap wM_\Omega w^{-1}\right)w_+w,
\end{aligned}$$

using that $ww_+ = w_+w'w_+^{-1}w_+ = w_+w'$. It follows that

$$\begin{aligned}
\left({}^{ww_+}[\delta]\right)_{N_\Theta\cap{}^wM_\Omega} &= {}^{w_+w'}\left([\delta]_{N_1\times N_2}\right)\\
&= {}^{w_+w'}\left((\nu^{1/2}\delta \otimes \nu^{-1/2}\delta)_{N_1\times N_2}\right)\\
&= {}^{w_+w'}\left(\nu^{1/2}\delta_{N_1} \otimes \nu^{-1/2}\delta_{N_2}\right)\\
&= {}^{ww_+}\left(\nu^{1/2}\delta_{N_1} \otimes \nu^{-1/2}\delta_{N_2}\right),
\end{aligned}$$

where in the final equality we have again used that $ww_+ = w_+w'$. In the above calculation of $\left({}^{ww_+}[\delta]\right)_{N_\Theta\cap{}^wM_\Omega}$, we also implicitly used the following basic fact.

Lemma 6.3. *Let (π, V) be a smooth representation of $G = \mathbf{GL}_m(F)$. Let $P = MN$ be a (proper) parabolic subgroup of G with Levi factor M and unipotent radical N. Let $s \in \mathbb{C}$. Then the Jacquet module $(\nu^s \otimes \pi)_N$ is equivalent to the twisted Jacquet module $\nu^s|_M \otimes \pi_N$.*

Proof. The lemma follows immediately from the fact that ν is trivial on the unipotent group N. Indeed, the space of both representations π and $\nu^s \otimes \pi = \nu^s\pi$ is V. The space of the Jacquet module of π, respectively $\nu^s\pi$, is the quotient of V by the subspace $V(N) = \mathrm{span}\{v - \pi(n)v : v \in V, n \in N\}$, respectively $\mathrm{span}\{v - \nu^s(n)\pi(n)v : v \in V, n \in N\}$. Since $\nu^s(n) = 1$ for every $n \in N$, we see that the space of both π_N and $(\nu^s\pi)_N$ is $V_N = V/V(N)$. Finally, observe that for any $m \in M$ and $v + V(N) \in V_N$ we have

$$\begin{aligned}(\nu^s\pi)_N(m)(v+V(N)) &= \delta_P^{-1/2}(m)\nu^s(m)\pi(m)v + V(N)\\ &= \nu^s(m)\left(\delta_P^{-1/2}(m)\pi(m)v + V(N)\right)\\ &= \nu^s(m)\pi_N(m)(v+V(N));\end{aligned}$$

therefore $(\nu^s\pi)_N = \nu^s|_M \otimes \pi_N$, as claimed. □

In order to understand the exponents of $({}^{ww_+}[\delta])_{N_\Theta \cap {}^w M_\Omega}$, we require the following proposition.

Proposition 6.4. *Let G and G' be two connected reductive groups over F. Let (π, V), respectively (σ, W), be a finitely generated admissible representation of G, respectively G'. The set of exponents of the (external) tensor product $\pi \otimes \sigma$ consists of all pairwise products $\chi \otimes \chi'$, where $\chi \in \mathcal{E}xp_{Z_G}(\pi)$ and $\chi' \in \mathcal{E}xp_{Z_{G'}}(\sigma)$ are exponents of π and σ respectively. That is,*

$$\begin{aligned}\mathcal{E}xp_{Z_G \times Z_{G'}}(\pi\otimes\sigma) &= \{\chi\otimes\chi' : \chi \in \mathcal{E}xp_{Z_G}(\pi), \chi' \in \mathcal{E}xp_{Z_{G'}}(\sigma)\}\\ &\cong \mathcal{E}xp_{Z_G}(\pi)\times \mathcal{E}xp_{Z_{G'}}(\sigma).\end{aligned} \tag{6.4}$$

Proof. The exponents $\mathcal{E}xp_{Z_G\times Z_{G'}}(\pi\otimes\sigma)$ of $\pi\otimes\sigma$ are precisely the central characters of the irreducible subquotients of $\pi\otimes\sigma$ (*cf.* [Cas95, Proposition 2.1.9], [Smi18b, Lemma 4.14]). To prove the proposition, it is sufficient to show that the irreducible subquotients of $\pi\otimes\sigma$ are of the form $V^j \otimes W^k$, where V^j, respectively W^k, is an irreducible subquotient of (π, V), respectively (σ, W). Indeed, if V^j (resp. W^k) is irreducible, then it admits a central character χ_j (resp. χ_k); moreover, $V^j\otimes W^k$ has central character $\chi_j\otimes\chi_k : Z_G \times Z_{G'} \to \mathbb{C}^\times$. We omit the proof of the elementary fact regarding the subquotients of the external tensor product $(\pi\otimes\sigma, V\otimes W)$. □

Note. To clarify the following calculations we introduce some additional notation for certain subsets of Δ. For and $1 \le j \le 2n-1$, let $\Xi_j = \Delta \setminus \{\epsilon_j - \epsilon_{j+1}\}$. We will be particularly interested in Ξ_{2k} and $\Xi_n = \Omega^{\mathrm{ell}}$ since $\Theta = w_+\Xi_{2k}$ and $\Omega = w_+\Xi_n$.

Recall that the (θ, F)-split component S_Θ of M_Θ is equal to the F-split component A_Θ. In particular, the (θ, F)-split component of G is $S_G = A_G$. We now consider the exponents of $({}^{ww_+}[\delta])_{N_\Theta\cap {}^wM_\Omega} = {}^{ww_+}\left(\nu^{1/2}\delta_{N_1}\otimes\nu^{-1/2}\delta_{N_2}\right)$ restricted to $S_\Theta^- \setminus S_\Theta^1 S_G = A_\Theta^- \setminus A_\Theta^1 A_G$. Let $s \in S_\Theta = A_\Theta$. Since $A_\Theta = w_+ A_{(2k,2n-2k)} w_+^{-1}$, we can write $s = w_+ a w_+^{-1}$, where $a = \mathrm{diag}(a_1 I_{2k}, a_2 I_{2n-2k})$ lies in $A^-_{(2k,2n-2k)} \setminus A^1_{(2k,2n-2k)}A_G$. In particular, $A_{(2k,2n-2k)} = A_{\Xi_{2k}}$ and a has the property that $|\epsilon_{2k} - \epsilon_{2k+1}(a)| = |a_1a_2^{-1}| < 1$. By Proposition 6.4 and Lemma 6.3, the exponents of $({}^{ww_+}[\delta])_{N_\Theta\cap {}^wM_\Omega} = {}^{ww_+}\left(\nu^{1/2}\delta_{N_1}\otimes\nu^{-1/2}\delta_{N_2}\right)$ are all of the form ${}^{ww_+}\left(\nu^{1/2}\chi_1\otimes\nu^{-1/2}\chi_2\right)$, where $\chi_1 \in \mathcal{E}xp_{A_1}(\delta_{N_1})$, and $\chi_2 \in \mathcal{E}xp_{A_2}(\delta_{N_2})$. Here we write A_i for the F-split component of $M_i \subset P_i \subset \mathbf{GL}_n(F)$, $i = 1, 2$. In particular,

$$\begin{aligned}{}^{ww_+}\left(\nu^{1/2}\chi_1\otimes\nu^{-1/2}\chi_2\right)(s) &= {}^{ww_+}\left(\nu^{1/2}\chi_1\otimes\nu^{-1/2}\chi_2\right)(w_+aw_+^{-1})\\ &= \left(\nu^{1/2}\chi_1\otimes\nu^{-1/2}\chi_2\right)(w_+^{-1}w^{-1}w_+aw_+^{-1}ww_+)\\ &= \left(\nu^{1/2}\chi_1\otimes\nu^{-1/2}\chi_2\right)(w'^{-1}aw'),\end{aligned}$$

where $w' = w_+^{-1} w w_+ \in [W_{(2k,2n-2k)} \backslash W_0 / W_{(n,n)}]$ and

$$\begin{aligned} {w'}^{-1} a w' &\in {w'}^{-1} A^-_{(2k,2n-2k)} w' \setminus {w'}^{-1} A^1_{(2k,2n-2k)} w' A_G \\ &\subset A^-_{{w'}^{-1} M_{(2k,2n-2k)} w' \cap M_{(n,n)}} \setminus A^1_{{w'}^{-1} M_{(2k,2n-2k)} w' \cap M_{(n,n)}} A_{(n,n)} \\ &= A^-_{({w'}^{-1}\Xi_{2k}) \cap \Xi_n} \setminus A^1_{({w'}^{-1}\Xi_{2k}) \cap \Xi_n} A_{(n,n)} \\ &= A^-_{M_1 \times M_2} \setminus A^1_{M_1 \times M_2} A_{(n,n)} \\ &= A_1^- \times A_2^- \setminus (A_1^1 \times A_2^1) A_{(n,n)}, \end{aligned}$$

where the containment in the second line follows as in the proof of [Smi18b, Lemma 8.4]. By assumption, δ is a discrete series representation of $\mathbf{GL}_n(F)$; therefore, the exponents χ_1 and χ_2 of δ satisfy Casselman's Criterion ([Cas95, Theorem 6.5.1]) and

$$|\chi_1 \otimes \chi_2({w'}^{-1} a w')| < 1.$$

To ensure that the exponents ${}^{ww_+}\left(\nu^{1/2}\chi_1 \otimes \nu^{-1/2}\chi_2\right)$ of $\left({}^{ww_+}[\delta]\right)_{N_\Theta \cap {}^w M_\Omega} = {}^{ww_+}\left(\nu^{1/2}\delta_{N_1} \otimes \nu^{-1/2}\delta_{N_2}\right)$ satisfy the Relative Casselman's Criterion ([KT10, Theorem 4.7]), we need to ensure that

$$|\nu^{1/2} \otimes \nu^{-1/2}({w'}^{-1} a w')| \leq 1. \tag{6.5}$$

We can realize the restriction of the unramified character $\nu^{1/2} \otimes \nu^{-1/2}$ to the maximal (diagonal) F-split torus A_0 as the composition of $|\cdot|_F^{1/2}$ with the sum over all roots in Δ with positive integral coefficients, that is,

$$(\nu^{1/2} \otimes \nu^{-1/2})|_{A_0} = |\cdot|_F^{1/2} \circ \left(\sum_{\alpha \in \Delta} c_\alpha \cdot \alpha \right), \tag{6.6}$$

where $c_{\epsilon_i - \epsilon_{i+1}} = i$, for $1 \leq i \leq n$, and $c_{\epsilon_{n+j} - \epsilon_{n+j+1}} = n - j$, for $1 \leq j \leq n-1$. To compute $(\nu^{1/2} \otimes \nu^{-1/2})({w'}^{-1} a w')$ it is helpful to partition Δ as the disjoint union of $({w'}^{-1}\Xi_{2k}) \cap \Xi_n$ and $\Delta \setminus (({w'}^{-1}\Xi_{2k}) \cap \Xi_n)$. Indeed, since $A_{{w'}^{-1}\Xi_{2k}} \subset A_{({w'}^{-1}\Xi_{2k}) \cap \Xi_n}$, it follows that $\alpha({w'}^{-1} a w') = 1$, for all $\alpha \in ({w'}^{-1}\Xi_{2k}) \cap \Xi_n$. On the other hand, since ${w'}^{-1} a w' \in A^-_{({w'}^{-1}\Xi_{2k}) \cap \Xi_n}$ we have that $|\beta({w'}^{-1} a w')|_F \leq 1$, for all $\beta \in \Delta \setminus (({w'}^{-1}\Xi_{2k}) \cap \Xi_n)$. From (6.6), it follows that

$$\begin{aligned} &(\nu^{1/2} \otimes \nu^{-1/2})({w'}^{-1} a w') = \prod_{\alpha \in \Delta} |\alpha({w'}^{-1} a w')|_F^{c_\alpha/2} \\ &= \left(\prod_{\alpha \in ({w'}^{-1}\Xi_{2k}) \cap \Xi_n} |\alpha({w'}^{-1} a w')|_F^{c_\alpha/2} \right) \left(\prod_{\beta \in \Delta \setminus (({w'}^{-1}\Xi_{2k}) \cap \Xi_n)} |\beta({w'}^{-1} a w')|_F^{c_\beta/2} \right) \\ &= \prod_{\beta \in \Delta \setminus (({w'}^{-1}\Xi_{2k}) \cap \Xi_n)} |\beta({w'}^{-1} a w')|_F^{c_\beta/2} \\ &\leq 1, \end{aligned}$$

which establishes the truth of (6.5). Moreover, we now have that

$$\begin{aligned} \left| {}^{ww_+}\left(\nu^{1/2}\chi_1 \otimes \nu^{-1/2}\chi_2\right)(s) \right| &= \left(\nu^{1/2}\chi_1 \otimes \nu^{-1/2}\chi_2\right)({w'}^{-1} a w') \\ &= |\chi_1 \otimes \chi_2({w'}^{-1} a w')| |\nu^{1/2} \otimes \nu^{-1/2}({w'}^{-1} a w')| \\ &< 1, \end{aligned}$$

for all $\chi_1 \in \mathcal{E}xp_{A_1}(\delta_{N_1})$, $\chi_2 \in \mathcal{E}xp_{A_2}(\delta_{N_2})$, and $s = w_+ a w_+^{-1} \in S_\Theta \setminus S^1_\Theta S_G$, where $w' = w_+^{-1} w w_+$ as above. Finally, we have established the desired result:

Proposition 6.5. *Let $\Theta = \Theta_k$, $1 \leq k \leq n-1$, be a maximal θ-split subset of Δ_0. Let $w \in [W_\Theta \backslash W_0 / W_\Omega]$ be such that $P_\Theta \cap {}^w M_\Omega$ is a proper parabolic subgroup of ${}^w M_\Omega$. Let δ be an irreducible admissible square integrable representation of $\mathbf{GL}_n(F)$. The exponents of π_N, and $\mathcal{U}(\delta, 2)_N$, corresponding to the irreducible subquotients of $\mathcal{F}_N^{ww_+}(\delta, 2) = ({}^{ww_+}[\delta])_{N_\Theta \cap {}^w M_\Omega}$ satisfy condition (2.2) of Theorem 2.7.*

Acknowledgments

The author would like to thank Omer Offen and Yiannis Sakellaridis for many helpful discussions. The author also thanks the anonymous referee for several helpful suggestions.

References

[Art13] James Arthur, *The endoscopic classification of representations: Orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. MR3135650

[BD08] Philippe Blanc and Patrick Delorme, *Vecteurs distributions H-invariants de représentations induites, pour un espace symétrique réductif p-adique G/H* (French, with English and French summaries), Ann. Inst. Fourier (Grenoble) **58** (2008), no. 1, 213–261. MR2401221

[BZ77] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive $\mathfrak{p}$-adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472. MR579172

[Cas95] William Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, Unpublished manuscript, draft prepared by the Séminaire Paul Sally, 1995, Available at www.math.ubc.ca/∼cass/research/publications.html.

[GK75] I. M. Gel′fand and D. A. Kajdan, *Representations of the group* $\mathrm{GL}(n, K)$ *where K is a local field*, Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), Halsted, New York, 1975, pp. 95–118. MR0404534

[HH98] A. G. Helminck and G. F. Helminck, *A class of parabolic k-subgroups associated with symmetric k-varieties*, Trans. Amer. Math. Soc. **350** (1998), no. 11, 4669–4691, DOI 10.1090/S0002-9947-98-02029-7. MR1443876

[HR90] Michael J. Heumos and Stephen Rallis, *Symplectic-Whittaker models for* Gl_n, Pacific J. Math. **146** (1990), no. 2, 247–279. MR1078382

[HW93] A. G. Helminck and S. P. Wang, *On rationality properties of involutions of reductive groups*, Adv. Math. **99** (1993), no. 1, 26–96, DOI 10.1006/aima.1993.1019. MR1215304

[JR96] Hervé Jacquet and Stephen Rallis, *Uniqueness of linear periods*, Compositio Math. **102** (1996), no. 1, 65–123. MR1394521

[KS17] F. Knop and B. Schalke, *The dual group of a spherical variety*, Trans. Moscow Math. Soc. **78** (2017), 187–216, DOI 10.1090/mosc/270. MR3738085

[KT08] Shin-ichi Kato and Keiji Takano, *Subrepresentation theorem for p-adic symmetric spaces*, Int. Math. Res. Not. IMRN **11** (2008), Art. ID rnn028, 40, DOI 10.1093/imrn/rnn028. MR2428854

[KT10] Shin-ichi Kato and Keiji Takano, *Square integrability of representations on p-adic symmetric spaces*, J. Funct. Anal. **258** (2010), no. 5, 1427–1451, DOI 10.1016/j.jfa.2009.10.026. MR2566307

[Lag08] Nathalie Lagier, *Terme constant de fonctions sur un espace symétrique réductif p-adique* (French, with English and French summaries), J. Funct. Anal. **254** (2008), no. 4, 1088–1145, DOI 10.1016/j.jfa.2007.07.012. MR2381204

[Off17] Omer Offen, *On parabolic induction associated with a p-adic symmetric space*, J. Number Theory **170** (2017), 211–227, DOI 10.1016/j.jnt.2016.06.014. MR3541705

[OS07] Omer Offen and Eitan Sayag, *On unitary representations of* GL_{2n} *distinguished by the symplectic group*, J. Number Theory **125** (2007), no. 2, 344–355, DOI 10.1016/j.jnt.2006.10.018. MR2332593

[OS08a] Omer Offen and Eitan Sayag, *Global mixed periods and local Klyachko models for the general linear group*, Int. Math. Res. Not. IMRN **1** (2008), Art. ID rnm 136, 25, DOI 10.1093/imrn/rnm136. MR2417789

[OS08b] Omer Offen and Eitan Sayag, *Uniqueness and disjointness of Klyachko models*, J. Funct. Anal. **254** (2008), no. 11, 2846–2865, DOI 10.1016/j.jfa.2008.01.004. MR2414223

[Rob83] Alain Robert, *Introduction to the representation theory of compact and locally compact groups*, London Mathematical Society Lecture Note Series, vol. 80, Cambridge University Press, Cambridge-New York, 1983. MR690955

[Rod73] François Rodier, *Whittaker models for admissible representations of reductive p-adic split groups*, Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), Amer. Math. Soc., Providence, R.I., 1973, pp. 425–430. MR0354942

[Smi18a] Jerrod Manford Smith, *Local unitary periods and relative discrete series*, Pacific J. Math. **297** (2018), no. 1, 225–256, DOI 10.2140/pjm.2018.297.225. MR3864235

[Smi18b] Jerrod Manford Smith, *Relative discrete series representations for two quotients of p-adic* $\mathbf{GL}_n$, Canad. J. Math. **70** (2018), no. 6, 1339–1372, DOI 10.4153/CJM-2017-047-7. MR3850546

[Smi20] Jerrod Manford Smith, *The support of closed orbit relative matrix coefficients*, manuscripta math. (2020).

[SV17] Yiannis Sakellaridis and Akshay Venkatesh, *Periods and harmonic analysis on spherical varieties* (English, with English and French summaries), Astérisque **396** (2017), viii+360. MR3764130

[Tad85] Marko Tadić, *Unitary dual of p-adic* $\mathrm{GL}(n)$*. Proof of Bernstein conjectures*, Bull. Amer. Math. Soc. (N.S.) **13** (1985), no. 1, 39–42, DOI 10.1090/S0273-0979-1985-15355-8. MR788387

[Tad86] Marko Tadić, *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case)*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 335–382. MR870688

[Xu17] Bin Xu, *On Mœglin's parametrization of Arthur packets for p-adic quasisplit* $Sp(N)$ *and* $SO(N)$, Canad. J. Math. **69** (2017), no. 4, 890–960, DOI 10.4153/CJM-2016-029-3. MR3679701

[Zel80] A. V. Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*-adic groups. II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210. MR584084

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