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COHERENT IC-SHEAVES ON TYPE A_n AFFINE GRASSMANNIANS AND DUAL CANONICAL BASIS OF AFFINE TYPE A_1

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ABSTRACT. The convolution ring $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ was identified with a quantum unipotent cell of the loop group LSL_2 in Cautis and Williams [J. Amer. Math. Soc. 32 (2019), pp. 709–778]. We identify the basis formed by the classes of irreducible equivariant perverse coherent sheaves with the dual canonical basis of the quantum unipotent cell.

1. INTRODUCTION

1.1. The affine Grassmannian $\mathrm{Gr}_{GL_n} = GL_n(\mathcal{K})/GL_n(\mathcal{O})$ (where $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[\![t]\!]$) is a basic object of the geometric Langlands program. The convolution ring $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ (where $\mathbb{C}^\times$ acts by loop rotations) is a simplest example of the quantized K-theoretic Coulomb branch of a quiver gauge theory (for A_1-quiver). The corresponding non-quantized K-theoretic Coulomb branch $K^{GL_n(\mathcal{O})}(\mathrm{Gr}_{GL_n})$ is a commutative ring, whose spectrum is the trigonometric zastava space ${}^{\dagger}\mathring{Z}^n_{\mathfrak{sl}_2}$ of type A_1 and degree n (alias moduli space of periodic $SU(2)$-monopoles of topological charge n). This space was thoroughly studied in yet another disguise in [GSV11], where its coordinate ring was equipped with a cluster structure.

It is expected by physicists that all the K-theoretic Coulomb branches of gauge theories should carry a (generalized) cluster structure (for trigonometric zastava see [FKR18]). Moreover, it is expected that the quantized K-theoretic Coulomb branches should carry a quantum cluster structure. In the simplest example of $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ such a structure was exhibited in [CW19]. It was identified with a well known cluster structure on a quantum unipotent cell of the loop group LSL_2 (in the non-quantized case, general trigonometric zastava spaces are identified with appropriate affine Richardson varieties in [FKR18]).

Furthermore, all the cluster monomials of this cluster structure have a nice geometric meaning as classes in $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ of certain irreducible $GL_n(\mathcal{O})\rtimes\mathbb{C}^\times$-equivariant perverse coherent sheaves on the affine Grassmannian Gr_{GL_n}. The abelian monoidal category $P^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}_{coh}(\mathrm{Gr}_{GL_n})$ of perverse coherent sheaves was introduced in [BFM05]. Its K-ring coincides with $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$, and it

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is equipped with a distinguished basis formed by the classes of IC-sheaves: irreducible equivariant perverse coherent sheaves. The problem of algebraic computation of this distinguished basis was standing ever since the appearance of [BFM05], and the cluster monomials description of certain IC-classes given in [CW19] was a breakthrough in this direction. However, the IC-classes representable as cluster monomials only form a tip of the iceberg of all the IC-classes; namely, they are IC-extensions of certain equivariant *line* bundles on $GL_n(\mathcal{O})$-orbits in Gr_{GL_n} (so this is similar to the lowest KL-cell in an affine Hecke algebra).

Now the cluster monomials in a quantum unipotent cell of LSL_2 form a part of the dual canonical basis of the quantum group $U_q^+(\widehat{\mathfrak{sl}}_2)$ (more precisely, of a certain localization of a subalgebra of its restricted dual) thanks to [KKKO18]. So it is natural to expect that this dual canonical basis corresponds to the above distinguished basis formed by the IC-classes. This is indeed proved in the present paper.

The dual canonical basis is characterized by two properties: (1) invariance with respect to a certain bar-involution; (2) the fact that the transformation matrix to the dual canonical basis from the dual PBW basis is identity modulo q^{-1} (where $\mathbb{Z}[q^{\pm 1}] = K_{\mathbb{C}^\times}(\mathrm{pt})$). The corresponding bar-involution on $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ (fixing IC-classes) was introduced in [CW19]. The dual PBW basis corresponds to certain convolutions of line bundles on the first minuscule orbit in Gr_{GL_n}. The analogue of property (2) for the usual *constructible* IC-sheaves is very deep (it boils down to the Riemann-Weil conjecture proved by Deligne). In the *coherent* setting of equivariant sheaves on nilpotent cone $P_{coh}^{G\times\mathbb{C}^\times}(\mathcal{N}_{\mathfrak{g}})$ a similar property was proved by Bezrukavnikov in [B06] by making use of his coherent-constructible correspondence and reducing to the above result of Deligne. In this setting the role of dual PBW basis is played by the classes of Andersen-Jantzen sheaves (pushforwards of the dominant line bundles from the Springer resolution of the nilpotent cone).

We are able to check the property (2) by reducing it to Bezrukavnikov's theory for $P_{coh}^{GL_d\times\mathbb{C}^\times}(\mathcal{N}_{\mathfrak{gl}_d})$. Namely, we consider a closed subvariety $\mathrm{Gr}_n^d \subset \mathrm{Gr}_{GL_n}$ formed by all the sublattices of codimension d in the standard lattice $\mathcal{O}^n \subset \mathcal{K}^n$. Then we have a natural smooth morphism of stacks

$$\psi\colon [(GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times)\backslash \mathrm{Gr}_n^d] \to [(GL_d \times \mathbb{C}^\times)\backslash \mathcal{N}_{\mathfrak{gl}_d}],$$

and ψ^* takes coherent IC-sheaves to IC-sheaves, and the Andersen-Jantzen sheaves to the appropriate d-fold convolutions of line bundles on Gr_n^1.

The appearance of the *dual* canonical basis is natural from yet another point of view. According to [FT19], $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ is the homomorphic image of a certain integral form (i.e. a $\mathbb{Q}[q^{\pm 1}]$-subalgebra) of a shifted quantum affine algebra of type A_1. This integral form is spanned by the *dual* Poincaré-Birkhoff-Witt-Drinfeld basis.

Finally, we should note that the problem of algebraic characterization of the IC-basis of $K^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_G)$ makes sense for arbitrary reductive group G, and we have no clue how to approach it for $G \neq GL_n$.

1.2. **Overall convention.** A variety always means a complex algebraic variety. Let X be a variety equipped with an action of a complex algebraic group G. For a (closed) point $x \in X$, we denote by $\mathrm{Stab}_G\, x$ the stabilizer of x in G. We denote by $D^G_{coh}(X)$ (resp. $D^G_{qcoh}(X)$) the derived category of bounded (resp. unbounded) G-equivariant complexes of sheaves on X whose cohomologies are coherent (resp. quasi-coherent). We denote by $\mathbb{D} := \mathbb{R}\mathscr{H}om(-, \omega_X)$ the Grothendieck-Serre duality functor on $D^G_{coh}(X)$, where ω_X is a G-equivariant dualizing complex on X. For a group automorphism ρ of G, we denote by X^ρ the same variety X with a new G-action obtained by twisting the original G-action by ρ. For an object $\mathcal{F} \in D^G_{coh}(X)$, we denote by $\mathcal{F}^\rho$ the sheaf obtained by twisting the G-equivariant structure of $\mathcal{F}$ by ρ. Then $\mathcal{F}^\rho$ is an object of $D^G_{coh}(X^\rho)$.

For an abelian category $\mathcal{A}$, we denote by $\mathsf{Irr}\,\mathcal{A}$ the set of isomorphism classes of simple objects of $\mathcal{A}$. We abbreviate $\mathsf{Irr}\,G := \mathsf{Irr}\,\mathrm{Rep}(G)$, where $\mathrm{Rep}(G)$ is the category of finite-dimensional algebraic representations of G over $\mathbb{C}$.

2. Quantum unipotent cell of LSL_2

2.1. **Quantum algebras of type $\mathsf{A}_1^{(1)}$.** Let $\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ be the generalized Cartan matrix of type $\mathsf{A}_1^{(1)}$ and $\mathsf{Q} := \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1$ be the root lattice (α_0, α_1 are the simple roots). We define a symmetric bilinear form $(-,-)$ on Q by $(\alpha_i, \alpha_j) = a_{ij}$ for $i, j \in \{0, 1\}$. We set $\mathsf{Q}^+ := \mathbb{N}\alpha_0 + \mathbb{N}\alpha_1 \subset \mathsf{Q}$.

Let $U^+ \equiv U_q^+(\widehat{\mathfrak{sl}}_2)$ be the $\mathbb{Q}(q)$-algebra generated by the two generators $\{e_0, e_1\}$ satisfying the quantum Serre relation

$$e_i^3 e_j - (q^2 + q^{-2}) e_i^2 e_j e_i + (q^2 + q^{-2}) e_i e_j e_i^2 - e_j e_i^3 = 0$$

for $\{i, j\} = \{0, 1\}$. The algebra U^+ is the positive (or the upper triangular) part of the quantized enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$. We define the weight grading $U^+ = \bigoplus_{\beta \in \mathsf{Q}^+} U^+_\beta$ by setting $\deg e_i := \alpha_i$.

We equip the tensor square $U^+ \otimes_{\mathbb{Q}(q)} U^+$ with a $\mathbb{Q}(q)$-algebra structure by

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) := q^{(\beta_2, \gamma_1)} x_1 y_1 \otimes x_2 y_2,$$

where $x_i \in U^+_{\beta_i}, y_i \in U^+_{\gamma_i}$. Let $r\colon U^+ \to U^+ \otimes_{\mathbb{Q}(q)} U^+$ be a $\mathbb{Q}(q)$-algebra homomorphism given by

$$r(e_i) := e_i \otimes 1 + 1 \otimes e_i$$

for $i = 0, 1$. Let $A = \bigoplus_{\beta \in \mathsf{Q}^+} A_\beta = \bigoplus_{\beta \in \mathsf{Q}^+} \mathrm{Hom}_{\mathbb{Q}(q)}(U^+_\beta, \mathbb{Q}(q))$ be the restricted dual of U^+ with a $\mathbb{Q}(q)$-algebra structure given by the dual of r.

Following Lusztig [L93, Proposition 1.2.3], we define a nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(-,-)_L$ on U^+ by

$$(1,1)_L = 1, \quad (e_i, e_j)_L = (1 - q^{-2})^{-1} \delta_{ij}, \quad (x, yz)_L = (r(x), y \otimes z)_L,$$

where $(x \otimes y, z \otimes w)_L := (x, z)_L \cdot (y, w)_L$. The bilinear form $(-,-)_L$ induces a $\mathbb{Q}(q)$-algebra isomorphism $\psi_L\colon U^+ \cong A$ defined by $\langle \psi_L(x), y \rangle = (x, y)_L$, where $\langle -, - \rangle\colon A \times U^+ \to \mathbb{Q}(q)$ is the natural pairing.

We define an algebra involution $\mathtt{b}$ of U^+ by

$$\mathtt{b}(q) = q^{-1}, \quad \mathtt{b}(e_i) = e_i.$$

Let $\mathtt{b}^*$ denote the $\mathbb{Q}$-linear involution of A defined as the dual of $\mathtt{b}$, i.e. for $\theta \in A, x \in U^+$, we define

$$\langle \mathtt{b}^*(\theta), x \rangle := \overline{\langle \theta, \mathtt{b}(x) \rangle},$$

where $\overline{f(q)} := f(q^{-1})$ for $f(q) \in \mathbb{Q}(q)$. Then for $\theta_i \in A_{\beta_i}\,(i = 1, 2)$, we have $\mathtt{b}^*(\theta_1\theta_2) = q^{-(\beta_1, \beta_2)} \mathtt{b}^*(\theta_2) \mathtt{b}^*(\theta_1)$.

2.2. **Quantum unipotent subgroup.** We fix $n \in \mathbb{N}$ throughout this paper. Let $w_n = s_{i_1} \cdots s_{i_{2n}} := (s_0 s_1)^n$ be an element of the Weyl group of type $\mathsf{A}_1^{(1)}$ of length $2n$, where s_i is the simple reflection associated with the index $i \in \{0, 1\}$. For each $1 \leq k \leq 2n$, we define the positive root β_k by

$$\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = k\alpha_0 + (k-1)\alpha_1.$$

The roots $\beta_1, \dots, \beta_{2n}$ are all the positive roots α such that $w_n^{-1}(\alpha) < 0$. Define the corresponding root vectors by

$$E(\beta_k) := T_{i_1} \circ \cdots \circ T_{i_{k-1}}(e_{i_k})$$

for $1 \leq k \leq 2n$, where T_i denotes Lusztig's symmetry ($= T'_{i,-1}$ in Lusztig's notation, see [L93, 37.1] for the definition). Let $U_n^+ \subset U^+$ be the $\mathbb{Q}(q)$-subalgebra generated by $\{E(\beta_k) \mid 1 \leq k \leq 2n\}$. The subalgebra $A_n := \psi_L(U_n^+)$ of A is called the *quantum unipotent subgroup* associated with w_n. Both algebras U_n^+ and A_n inherit the Q^+-gradings from U^+ and A:

$$U_n^+ = \bigoplus_{\beta \in \mathsf{Q}^+} (U_n^+)_\beta, \quad A_n = \bigoplus_{\beta \in \mathsf{Q}^+} (A_n)_\beta.$$

2.3. **PBW and dual PBW bases.** For an element $\beta \in \mathsf{Q}^+$, we set

$$\mathrm{KP}_n(\beta) := \{\mathbf{a} = (a_1, \dots, a_{2n}) \in (\mathbb{Z}_{\geq 0})^{2n} \mid a_1\beta_1 + \cdots + a_{2n}\beta_{2n} = \beta\}.$$

For each $\mathbf{a} = (a_1, a_2, \dots, a_{2n}) \in \mathrm{KP}_n(\beta)$, we define the corresponding PBW element as the product of q-divided powers by

$$E(\mathbf{a}) := E(\beta_1)^{(a_1)} E(\beta_2)^{(a_2)} \cdots E(\beta_{2n})^{(a_{2n})},$$

where $x^{(\ell)} := x^\ell / (\prod_{i=1}^{\ell} [i]_q)$, $[i]_q := \frac{q^i - q^{-i}}{q - q^{-1}}$ as usual. Then the set $\{E(\mathbf{a}) \mid \mathbf{a} \in \mathrm{KP}_n(\beta)\}$ forms a $\mathbb{Q}(q)$-basis of $(U_n^+)_\beta$. It is known (cf. [L93, Proposition 38.2.3]) that we have $(E(\mathbf{a}), E(\mathbf{b}))_L = 0$ if $\mathbf{a} \neq \mathbf{b}$ and

$$(E(\mathbf{a}), E(\mathbf{a}))_L = \prod_{k=1}^{2n} \prod_{j=1}^{a_k} (1 - q^{-2j})^{-1}.$$

For each $\mathbf{a} \in \mathrm{KP}_n(\beta)$, we define the dual PBW element in A_n by

$$E^*(\mathbf{a}) := \frac{\psi_L(E(\mathbf{a}))}{(E(\mathbf{a}), E(\mathbf{a}))_L}.$$

By construction, the set $\{E^*(\mathbf{a}) \mid \mathbf{a} \in \mathrm{KP}_n(\beta)\}$ forms a $\mathbb{Q}(q)$-basis of $(A_n)_\beta$ dual to the basis $\{E(\mathbf{a}) \mid \mathbf{a} \in \mathrm{KP}_n(\beta)\}$ of $(U_n^+)_\beta$.

The dual PBW element $E^*(\mathbf{a})$ can be written simply as a product of the dual root vectors $E^*(\beta_k) = (1 - q^{-2})\psi_L(E(\beta_k))$:

$$E^*(\mathbf{a}) = q^{-\sum_{k=1}^{2n} a_k(a_k-1)/2} E^*(\beta_1)^{a_1} E^*(\beta_2)^{a_2} \cdots E^*(\beta_{2n})^{a_{2n}}. \tag{2.1}$$

2.4. **Dual canonical basis.** Let $\mathbf{B} \subset U^+$ be the canonical basis of U^+ constructed in [L93, Part II]. It is characterized up to sign as the set of elements $b \in U^+$ satisfying $\mathtt{b}(b) = b$ and $(b, b)_L \in 1 + q^{-1}\mathbb{Z}[\![q^{-1}]\!]$. The dual canonical basis $\mathbf{B}^*$ is defined as the basis of A dual to the canonical basis $\mathbf{B}$. By definition, each element of $\mathbf{B}^*$ is fixed by the bar involution $\mathtt{b}^*$.

Theorem 2.1 due to Kimura [K12] claims that the dual canonical basis is compatible with the quantum unipotent subgroup A_n and characterized by using the dual PBW basis.

Theorem 2.1 ([K12, Theorem 4.29]). *For each $\beta \in \mathsf{Q}^+$, there exists a unique $\mathbb{Q}(q)$-basis $\mathscr{B}_n(\beta) = \{B^*(\mathbf{a}) \mid \mathbf{a} \in \mathrm{KP}_n(\beta)\}$ of $(A_n)_\beta$ characterized by the following properties:*

(1) $\mathtt{b}^*(B^*(\mathbf{a})) = B^*(\mathbf{a})$;
(2) $B^*(\mathbf{a}) \in E^*(\mathbf{a}) + \sum_{\mathbf{a}' \in \mathrm{KP}_n(\beta)} q^{-1}\mathbb{Z}[q^{-1}]E^*(\mathbf{a}')$.

Moreover we have $\mathscr{B}_n(\beta) = \mathbf{B}^* \cap (A_n)_\beta$.

We refer to the basis $\bigsqcup_{\beta \in \mathsf{Q}^+} \mathscr{B}_n(\beta)$ as the dual canonical basis of A_n. We denote by $A_{n,\mathbb{Z}}$ the integral form of A_n, i.e. the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of A_n spanned by the dual canonical basis (or the dual PBW basis).

Let $\mathsf{P} = \mathbb{Z}\varpi_0 \oplus \mathbb{Z}\varpi_1$ be the weight lattice of type $\mathsf{A}_1^{(1)}$ with ϖ_i being the i-th fundamental weight ($i \in \{0,1\}$). For a dominant weight $\lambda \in \mathsf{P}^+ := \mathbb{N}\varpi_0 + \mathbb{N}\varpi_1$ and Weyl group elements u and v satisfying $u(\lambda) - v(\lambda) \in \mathsf{Q}^+$, we denote by $D_{u(\lambda),v(\lambda)}$ the corresponding *quantum unipotent minor* (see [GLS13, Section 5.2] for the definition). This is an element of $\mathscr{B}_n(u(\lambda)-v(\lambda))$ (see [GLS13, Proposition 6.3]). For each $1 \le k \le 2n$, we set $w_{\le k} := s_{i_1}s_{i_2}\cdots s_{i_k}$ and $w_{<k} := s_{i_1}s_{i_2}\cdots s_{i_{k-1}}$. For each $1 \le b \le d \le 2n$ with $d - b \in 2\mathbb{Z}$, we define $D[b,d] := D_{w_{<b}\varpi_{i_b}, w_{\le d}\varpi_{i_d}}$. By [GLS13, Proposition 7.4], we have $E^*(\beta_k) = D[k,k]$ for each $1 \le k \le 2n$.

2.5. **Quantum cluster structure.** Let $\mathscr{A}_n$ be the quantum cluster algebra (over $\mathbb{Z}[q^{\pm 1/2}]$) associated with the initial quantum seed $\mathscr{S} = ((X_1, \dots, X_{2n}), \widetilde{B}, \Lambda)$ defined as follows (see [BZ05] for the generalities of quantum cluster algebras). The exchange matrix $\widetilde{B}$ is the $2n \times (2n-2)$-matrix given by

$$\widetilde{B} := \left(\begin{array}{cccccccc}
0 & -2 & 1 & & & & & \\
2 & 0 & -2 & 1 & & & & \\
-1 & 2 & 0 & -2 & 1 & & & \\
 & -1 & 2 & 0 & -2 & 1 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & -1 & 2 & 0 & -2 & 1 \\
 & & & & -1 & 2 & 0 & -2 \\
 & & & & & -1 & 2 & 0 \\
\hline
 & & & & & & -1 & 2 \\
 & & & & & & & -1
\end{array}\right),$$

where all blank entries are 0. The skew-symmetric $2n \times 2n$-matrix $\Lambda = (\Lambda_{k\ell})$ is given by $\Lambda_{k\ell} := 2\lceil k/2 \rceil (\lfloor k/2 \rfloor - \lfloor \ell/2 \rfloor)$ for any $1 \le k < \ell \le 2n$ (cf. [CW19, Lemma 6.4]). The quantum cluster algebra $\mathscr{A}_n$ is a $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of the based quantum torus $\mathbb{Z}[q^{\pm 1/2}]\langle X_k^{\pm 1}\ (1 \le k \le 2n) \mid X_k X_\ell = q^{\Lambda_{k\ell}} X_\ell X_k \rangle$.

For each $1 \le b \le d \le 2n$ with $d - b \in 2\mathbb{Z}$, we consider the following normalized element in $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} A_n$:

$$\widetilde{D}[b,d] := q^{(\beta,\beta)/4} D[b,d],$$

where $\beta = w_{<b}\varpi_{i_b} - w_{\le d}\varpi_{i_d}$ is the weight of $D[b,d]$. More generally, for any quantum unipotent minor $D_{u(\lambda),v(\lambda)}$, we define $\widetilde{D}_{u(\lambda),v(\lambda)} := q^{(\beta,\beta)/4} D_{u(\lambda),v(\lambda)}$ with $\beta = u(\lambda) - v(\lambda) \in \mathsf{Q}^+$.

Theorem 2.2 ([GLS13, Theorem 12.3], [KKKO18, Corollary 11.2.8]). *There is a unique isomorphism*

$$\mathscr{A}_n \xrightarrow{\sim} \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} A_{n,\mathbb{Z}}$$

under which the initial cluster variable X_k corresponds to

$$\begin{cases} \widetilde{D}[1,k] & \text{if } k \text{ is odd;} \\ \widetilde{D}[2,k] & \text{if } k \text{ is even,} \end{cases}$$

for each $1 \le k \le 2n$. Moreover the quantum unipotent minor $\widetilde{D}[b,d]$ is the image of a cluster variable for any $1 \le b \le d \le 2n$ with $d - b \in 2\mathbb{Z}$.

Henceforth, we will identify the quantum unipotent subgroup $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} A_{n,\mathbb{Z}}$ with the quantum cluster algebra $\mathscr{A}_n$ via the isomorphism in Theorem 2.2.

2.6. **Berenstein-Zelevinsky's bar involution.** When x, y are q-commuting with each other, say $xy = q^m yx$ for some $m \in \mathbb{Z}$, we write $x \odot y := q^{-m/2} xy$. Note that we have $x \odot y = y \odot x$. Following Berenstein-Zelevinsky [BZ05], let us consider the algebra anti-involution ι of the quantum cluster algebra $\mathscr{A}_n$ defined by

$$\iota(q) = q^{-1}, \quad \iota(X_k) = X_k,$$

for all $1 \le k \le 2n$. If $x, y \in \mathscr{A}_n$ are q-commuting with each other and both are fixed by ι, the element $x \odot y = y \odot x$ is also fixed by ι. In particular, any cluster variables and hence any cluster monomials are fixed by ι.

Lemma 2.3 (cf. [KO19, Remark 7.23])**.** *If $x \in \mathscr{A}_n$ is of weight β, we have $\iota(x) = q^{(\beta,\beta)/2} \mathtt{b}^*(x)$. In particular, x is fixed by $\mathtt{b}^*$ if and only if the rescaled element $q^{(\beta,\beta)/4} x$ is fixed by ι.*

For any $\beta \in \mathsf{Q}^+$ and $\mathbf{a} \in \mathrm{KP}_n(\beta)$, we consider the rescaled elements

$$\widetilde{B}(\mathbf{a}) := q^{(\beta,\beta)/4} B^*(\mathbf{a}), \quad \widetilde{E}(\mathbf{a}) := q^{(\beta,\beta)/4} E^*(\mathbf{a}).$$

Theorem 2.1 yields the following characterization of the rescaled dual canonical basis.

Corollary 2.4. *The basis $\{\widetilde{B}(\mathbf{a}) \mid \mathbf{a} \in \mathrm{KP}_n(\beta)\}$ of $(\mathscr{A}_n)_\beta$ is characterized by the following properties:*

(1) $\iota(\widetilde{B}(\mathbf{a})) = \widetilde{B}(\mathbf{a})$;
(2) $\widetilde{B}(\mathbf{a}) \in \widetilde{E}(\mathbf{a}) + \sum_{\mathbf{a}' \in \mathrm{KP}_n(\beta)} q^{-1}\mathbb{Z}[q^{-1}] \widetilde{E}(\mathbf{a}')$.

Note that in particular the rescaled dual root vector is

$$\widetilde{E}(\beta_k) = q^{1/2} E^*(\beta_k) = \widetilde{D}[k,k]$$

for each $1 \le k \le 2n$. In terms of the rescaled elements, the expression (2.1) is rewritten as

$$\widetilde{E}(\mathbf{a}) = q^{\sum_{k<\ell} a_k a_\ell} \widetilde{E}(\beta_1)^{a_1} \widetilde{E}(\beta_2)^{a_2} \cdots \widetilde{E}(\beta_{2n})^{a_{2n}} \tag{2.2}$$

for each $\mathbf{a} \in \mathrm{KP}_n(\beta)$.

2.7. **Localization.** Note that the frozen variables of the quantum cluster algebra $\mathscr{A}_n$ are the following two elements:

$$D_0 := X_{2n-1} = \widetilde{D}_{\varpi_0, w_n \varpi_0}, \quad D_1 := X_{2n} = \widetilde{D}_{\varpi_1, w_n \varpi_1}.$$

Proposition 2.5 (cf. [KO19, Proposition 4.2])**.** *For each $\lambda \in \mathsf{P}^+$, the unipotent quantum minor $D_{\lambda, w_n \lambda}$ is q-central in A_n. More precisely, for any homogeneous element $x \in (\mathscr{A}_n)_\beta$ of weight $\beta \in \mathsf{Q}^+$, we have $D_{\lambda, w_n\lambda} x = q^{-(\lambda + w_n \lambda, \beta)} x D_{\lambda, w_n \lambda}$. Moreover, for $\lambda, \lambda' \in \mathsf{P}^+$, we have $\widetilde{D}_{\lambda, w_n\lambda} \odot \widetilde{D}_{\lambda', w_n\lambda'} = \widetilde{D}_{\lambda+\lambda', w_n(\lambda+\lambda')}$ in $\mathscr{A}_n$. In particular, we have $\widetilde{D}_{\lambda, w_n\lambda} = D_0^{\odot \ell_0} \odot D_1^{\odot \ell_1}$ for $\lambda = \ell_0 \varpi_0 + \ell_1 \varpi_1 \in \mathsf{P}^+$.*

By Proposition 2.5, the set $\mathcal{D}_n := \{q^{m/2} D_0^{\ell_0} D_1^{\ell_1} \mid m \in \mathbb{Z}, \ell_i \in \mathbb{N}\}$ is an Ore set of the algebra $\mathscr{A}_n$. The localized algebra $\mathscr{A}_n^{loc} := \mathscr{A}_n[\mathcal{D}_n^{-1}]$ is (isomorphic to) the *quantum unipotent cell* associated with w_n (cf. [KO19, Section 4]).

Proposition 2.6 (cf. [KO19, Proposition 4.5])**.** *The set*

$$\widetilde{\mathscr{B}}_n^{loc} := \{\widetilde{B}(\mathbf{a}) \odot D_0^{\odot(-\ell_0)} \odot D_1^{\odot(-\ell_1)} \mid \mathbf{a} \in \mathbb{N}^{2n}, \ell_i \in \mathbb{N}\}$$

forms a $\mathbb{Z}[q^{\pm 1/2}]$*-basis of the quantum unipotent cell* $\mathscr{A}_n^{loc}$*.*

Note that each element in $\widetilde{\mathscr{B}}_n^{loc}$ is fixed by the bar involution ι.

3. Perverse coherent sheaves on type A affine Grassmannian

3.1. **Affine Grassmannian.** Let $T_n \subset GL_n(\mathbb{C})$ be the maximal torus consisting of diagonal matrices and $P^\vee := \mathrm{Hom}(\mathbb{C}^\times, T_n)$ be the coweight lattice. We make the standard identification $P^\vee = \mathbb{Z}^n$ under which the element $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$ corresponds to the 1-parameter group $a \mapsto \mathrm{diag}(a^{\nu_1}, \dots, a^{\nu_n})$. The weight lattice $P = \mathrm{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ is also identified with $\mathbb{Z}^n$ via the standard pairing $\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ given by $\langle \nu, \mu \rangle = \nu_1\mu_1 + \cdots + \nu_n\mu_n$. We say that an element $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$ is dominant if $\nu_1 \geq \cdots \geq \nu_n$. Write $P_+^\vee$ for the set of dominant coweights. For each $1 \leq k \leq n$, we define

$$\omega_k := (\overbrace{1, \dots, 1}^{k}, 0, \dots, 0) \in \mathbb{Z}^n,$$

which is regarded as the k-th fundamental weight or coweight.

Let $\mathcal{K} := \mathbb{C}((t)) \supset \mathcal{O} := \mathbb{C}[\![t]\!]$ be the field of formal Laurent series and its subring of formal power series. We consider the affine Grassmannian of GL_n:

$$\mathrm{Gr}_{GL_n} = GL_n(\mathcal{K})/GL_n(\mathcal{O}) = (GL_n(\mathcal{K}) \rtimes \mathbb{C}^\times)/(GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times),$$

where $\mathbb{C}^\times$ denotes the 4-fold cover of the standard loop rotation. More precisely, we have $(1, a) \cdot (g(t), 1) = (g(a^4 t), a)$ in $GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$ for $g(t) \in GL_n(\mathcal{O})$, $a \in \mathbb{C}^\times$. The affine Grassmannian Gr_{GL_n} decomposes into the union of $GL_n(\mathcal{O})$-orbits (= $GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$-orbits):

$$\mathrm{Gr}_{GL_n} = \bigsqcup_{\nu \in P_+^\vee} \mathrm{Gr}_{GL_n}^\nu,$$

where $\mathrm{Gr}_{GL_n}^\nu$ denotes the orbit of $[t^\nu] \in \mathrm{Gr}_{GL_n}$, $t^\nu := \mathrm{diag}(t^{\nu_1}, \dots, t^{\nu_n}) \in GL_n(\mathcal{K})$. For each $\nu = (\nu_1, \dots, \nu_n) \in P_+^\vee$, we have $\dim \mathrm{Gr}_{GL_n}^\nu = \sum_{k=1}^n (n + 1 - 2k)\nu_k$.

The closure $\overline{\mathrm{Gr}}_{GL_n}^\nu$ of the orbit $\mathrm{Gr}_{GL_n}^\nu$ is called the *Schubert variety*. Let us consider the derived category $D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\overline{\mathrm{Gr}}_{GL_n}^\nu)$ of bounded $GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$-equivariant complexes of sheaves on the reduced scheme $(\overline{\mathrm{Gr}}_{GL_n}^\nu)^{red}$ with coherent cohomologies, formally supported in cohomological degrees $\frac{1}{2} \dim \mathrm{Gr}_{GL_n}^\nu + \mathbb{Z}$ by convention.

The connected components of Gr_{GL_n} are labeled by $\mathbb{Z}$. For each $d \in \mathbb{Z}$, the d-th connected component $\mathrm{Gr}_{GL_n}^{(d)}$ is the union of $GL_n(\mathcal{O})$-orbits $\mathrm{Gr}_{GL_n}^\nu$ with $d = \nu_1 + \cdots + \nu_n$. Note that the parity of $\dim \mathrm{Gr}_{GL_n}^\nu$ is constant on each connected component. Therefore we can define

$$D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n}) := \varinjlim_{\nu} D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\overline{\mathrm{Gr}}_{GL_n}^\nu).$$

3.2. **Convolution product.** For any objects $\mathcal{F}, \mathcal{G} \in D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$, we can define their *convolution product* $\mathcal{F} * \mathcal{G} \in D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ by

$$\mathcal{F} * \mathcal{G} := \overline{m}_*(\mathcal{F} \widetilde{\boxtimes} \mathcal{G}),$$

where $\overline{m} : (GL_n(\mathcal{K}) \rtimes \mathbb{C}^\times) \times^{(GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times)} \mathrm{Gr}_{GL_n} \to \mathrm{Gr}_{GL_n}$ is the multiplication map. Here the sheaf $\mathcal{F} \widetilde{\boxtimes} \mathcal{G}$ is defined by the property $\overline{q}^*(\mathcal{F} \widetilde{\boxtimes} \mathcal{G}) \cong \overline{p}^*(\mathcal{F} \boxtimes \mathcal{G})$, where $\overline{p}$ and $\overline{q}$ are the natural projections:

$$\mathrm{Gr}_{GL_n} \times \mathrm{Gr}_{GL_n} \xleftarrow{\overline{p}} (GL_n(\mathcal{K}) \rtimes \mathbb{C}^\times) \times \mathrm{Gr}_{GL_n} \xrightarrow{\overline{q}} (GL_n(\mathcal{K}) \rtimes \mathbb{C}^\times) \times^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times} \mathrm{Gr}_{GL_n}.$$

If $\mathcal{F}$ (resp. $\mathcal{G}$) is supported on $\overline{\mathrm{Gr}}^\nu_{GL_n}$ (resp. $\overline{\mathrm{Gr}}^{\nu'}_{GL_n}$), the sheaf $\mathcal{F} \widetilde{\boxtimes} \mathcal{G}$ is supported on the finite-dimensional *convolution variety*

$$\overline{\mathrm{Gr}}^\nu_{GL_n} \widetilde{\times} \overline{\mathrm{Gr}}^{\nu'}_{GL_n} \subset GL_n(\mathcal{K}) \times^{GL_n(\mathcal{O})} \mathrm{Gr}_{GL_n}$$

and the convolution product $\mathcal{F} * \mathcal{G}$ is supported on $\overline{\mathrm{Gr}}^{\nu+\nu'}_{GL_n}$.

The convolution product $*$ equips the equivariant K-group $K^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ with a structure of an associative $\mathbb{Z}[q^{\pm 1/2}]$-algebra, where $q^{m/2} \in K^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{pt})$ denotes the class of the pull-back of the 1-dimensional $\mathbb{C}^\times$-module $\mathbb{C}_m$ of weight m along the natural projection $GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times \to \mathbb{C}^\times$.

We use the notation $\{m/2\}$ to denote the $\mathbb{C}^\times$-equivariant twist $- \otimes \mathbb{C}_{-m}$. Thus, for an object $\mathcal{F} \in D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$, we have $[\mathcal{F}\{m/2\}] = q^{-m/2}[\mathcal{F}]$. On the other hand, we denote by $[m/2]$ the cohomological degree shift by $m/2 \in \frac{1}{2}\mathbb{Z}$. It will be convenient to use the notation $\langle m/2 \rangle := [m/2]\{-m/2\}$ for the simultaneous shift and twist by $m/2 \in \frac{1}{2}\mathbb{Z}$. This is the same notation as in [CW19].

3.3. **Perverse coherent sheaves.**

Definition 3.1. A $GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$-equivariant *perverse coherent sheaf* on Gr_{GL_n} is an object $\mathcal{F} \in D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ such that for every orbit $i_\nu \colon \mathrm{Gr}^\nu_{GL_n} \hookrightarrow \mathrm{Gr}_{GL_n}$

(1) $i_\nu^* \mathcal{F} \in D_{qcoh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}^\nu_{GL_n})$ is supported in degrees $\le -\frac{1}{2} \dim \mathrm{Gr}^\nu_{GL_n}$;
(2) $i_\nu^! \mathcal{F} \in D_{qcoh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}^\nu_{GL_n})$ is supported in degrees $\ge -\frac{1}{2} \dim \mathrm{Gr}^\nu_{GL_n}$.

We denote by $P_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n}) \subset D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ the full subcategory of perverse coherent sheaves.

The category $P_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ can be obtained as the core of a finite-length t-structure (called the *perverse t-structure*) of the category $D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ (cf. [AB10]). The convolution product $*$ preserves the category $P_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ and the operation $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} * \mathcal{G}$ is bi-exact (cf. [BFM05]). Thus the equivariant K-group $K^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n}) = K(P_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL_n}))$ becomes an algebra with a canonical $\mathbb{Z}$-basis formed by the classes of simple perverse coherent sheaves.

We say that $(\nu, \mu) \in P^\vee \times P$ is a dominant pair if $\nu \in P^\vee$ is a dominant coweight and $\mu \in P$ is dominant with respect to the Levi quotient of the stabilizer subgroup $\mathrm{Stab}_{GL_n(\mathcal{O})}[t^\nu]$. More explicitly, the set $\mathbf{D}_n$ of dominant pairs is given by

$$\mathbf{D}_n = \left\{ (\nu, \mu) \in P^\vee_+ \times P \;\middle|\; \begin{array}{l} \nu = (\nu_1, \dots, \nu_n), \mu = (\mu_1, \dots, \mu_n), \\ \mu_k \ge \mu_{k+1} \text{ whenever } \nu_k = \nu_{k+1} \end{array} \right\}.$$

To each dominant pair $(\nu,\mu)\in\mathbf{D}_n$, we associate a simple perverse coherent sheaf $\mathcal{P}_{\nu,\mu}$ in the following way. Note that the group

$$\mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu] := \{g\in GL_n(\mathbb{C}) \mid g\cdot t^\nu = t^\nu\cdot g\} \cong \prod_k GL_{m_k}(\mathbb{C})$$

is a Levi subgroup of $\mathrm{Stab}_{GL_n(\mathcal{O})}[t^\nu]$, where m_k is the multiplicity of k in the sequence $\nu\in\mathbb{Z}^n$. Then the group

$$\mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu]\rtimes\mathbb{C}^\times = \mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu]\times\mathbb{C}^\times$$

is a Levi subgroup of $\mathrm{Stab}_{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}[t^\nu]$. Thus we can identify

$$\mathsf{Irr}\,\mathrm{Stab}_{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}[t^\nu] = \mathsf{Irr}(\mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu]\times\mathbb{C}^\times) \supset \mathsf{Irr}\,\mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu],$$

where the set $\mathsf{Irr}\,\mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu]$ is regarded as a subset of $\mathsf{Irr}(\mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu]\times\mathbb{C}^\times)$ consisting of representations with the trivial $\mathbb{C}^\times$-actions. Let $\mathcal{V}_\mu$ denote the simple $GL_n(\mathcal{O})\rtimes\mathbb{C}^\times$-equivariant vector bundle on $\mathrm{Gr}^\nu_{GL_n}$ whose fiber at $[t^\nu]$ is isomorphic to V_μ as a representation of $\mathrm{Stab}_{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}[t^\nu]$. We define the simple perverse coherent sheaf $\mathcal{P}_{\nu,\mu}$ as the following (coherent) IC-extension (cf. [AB10, Theorem 4.2])

$$\mathcal{P}_{\nu,\mu} := (i_\nu)_{!*}\mathcal{V}_\mu\langle\dim\mathrm{Gr}^\nu_{GL_n}/2\rangle\{-\langle\nu,\mu\rangle\},$$

where $i_\nu\colon\mathrm{Gr}^\nu_{GL_n}\hookrightarrow\overline{\mathrm{Gr}}^\nu_{GL_n}$ is the inclusion.

Since each simple perverse coherent sheaf is isomorphic to an IC-extension of a simple vector bundle on some $GL_n(\mathcal{O})\rtimes\mathbb{C}^\times$-orbit (cf. [AB10, Proposition 4.11]), we have a bijection

$$\mathbf{D}_n\times\mathbb{Z}\overset{1:1}{\longleftrightarrow}\mathsf{Irr}\,P^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}_{coh}(\mathrm{Gr}_{GL_n});\quad ((\nu,\mu),m)\longleftrightarrow\mathcal{P}_{\nu,\mu}\{m/2\}.$$

In particular, the set

$$\mathscr{P}_n := \{[\mathcal{P}_{\nu,\mu}] \mid (\nu,\mu)\in\mathbf{D}_n\}$$

forms a $\mathbb{Z}[q^{\pm1/2}]$-basis of the convolution ring $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$.

3.4. **Lattice description.** Recall that the affine Grassmannian Gr_{GL_n} can be interpreted as the moduli space of $\mathcal{O}$-lattices L in $\mathcal{K}^n$. Let $L_0 := \mathcal{O}^n\subset\mathcal{K}^n$ be the standard $\mathcal{O}$-lattice. A coset $[g(t)]\in\mathrm{Gr}_{GL_n} = GL_n(\mathcal{K})/GL_n(\mathcal{O})$ corresponds to a lattice $L = g(t)L_0$. Then for each $\nu\in P^\vee_+$ with $\nu_n\geq 0$, we have

$$\mathrm{Gr}^\nu_{GL_n} = \{L\subset L_0 \mid t|_{L_0/L} \text{ is nilpotent of type } \nu\}.$$

In particular, when $\nu=\omega_k$, we get

$$\mathrm{Gr}^k_{GL_n} := \mathrm{Gr}^{\omega_k}_{GL_n} = \{L\overset{k}{\subset}L_0 \mid tL_0\subset L\},$$

where $L\overset{k}{\subset}L_0$ indicates that $\dim(L_0/L)=k$. In particular, $\mathrm{Gr}^k_{GL_n} = \overline{\mathrm{Gr}}^k_{GL_n}$ is isomorphic to the usual Grassmannian $\mathrm{Gr}(k,n)$ of k-dimensional subspaces in $\mathbb{C}^n$.

For each $1\leq k\leq n$ and $\ell\in\mathbb{Z}$, we put $\mathcal{P}_{k,\ell} := \mathcal{P}_{\omega_k,\ell\omega_k}$ for simplicity. Using the above description of $\mathrm{Gr}^k_{GL_n}$, we see

$$\mathcal{P}_{k,\ell} = i_{\omega_k *}\left(\mathcal{O}_{\mathrm{Gr}^k_{GL_n}}\otimes\det(L_0/L)^\ell\right)\langle k(n-k)/2\rangle\{-k\ell\},$$

where we denote by L_0/L the vector bundle on $\mathrm{Gr}^k_{GL_n}$ whose fiber at L is equal to L_0/L by an abuse of notation.

3.5. **Cautis-Williams' monoidal categorification theorem.** In [CW19], Cautis and Williams proved that, for a general complex reductive group G, the category of $G(\mathcal{O}) \rtimes \mathbb{C}^\times$-equivariant perverse coherent sheaves is a *rigid* monoidal category, i.e. every object $\mathcal{F}$ has its left and right duals. Moreover, they also proved the existence of a *system of renormalized r-matrices* (originated in the settings of the quiver Hecke algebras and the quantum affine algebras, see [KKKO18] for instance), which informally encodes some information about how the category fails to be a braided tensor category. Using these facts, it was successfully proved in the case $G = GL_n$ that the monoidal category $P_{coh}^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ categorifies the quantum unipotent cell $\mathscr{A}_n^{loc}$. More precisely, we have:

Theorem 3.2 (Cautis-Williams [CW19])**.** *There exists an isomorphism of $\mathbb{Z}[q^{\pm 1/2}]$-algebras*

$$\Phi \colon \mathscr{A}_n^{loc} \xrightarrow{\sim} K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n}),$$

which sends each cluster monomial to the class of a simple perverse coherent sheaf. Moreover, for each $1 \le b \le d \le 2n$ with $d - b \in 2\mathbb{Z}$, we have $\Phi(\widetilde{D}[b,d]) = [\mathcal{P}_{k,\ell}]$ with

$$k = 1 + \frac{1}{2}(d-b), \quad \ell = n + 1 - \frac{1}{2}(b+d).$$

In particular, we have

$$\Phi(\widetilde{E}(\beta_k)) = \Phi(\widetilde{D}[k,k]) = [\mathcal{P}_{1,n+1-k}] \tag{3.1}$$

for each $1 \le k \le 2n$.

The bar involution ι of $\mathscr{A}_n^{loc}$ was also categorified in [CW19, Section 6.2]. Let σ_1 be the group involution of $GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$ given by $(g(t), a) \mapsto ({}^{\mathsf{T}}g(t)^{-1}, a)$, where ${}^{\mathsf{T}}(-)$ denotes the transpose of matrices. Then the morphism

$$\eta : \mathrm{Gr}_{GL_n} \to GL_n(\mathcal{K}) \times^{GL_n(\mathcal{O})} (\mathrm{Gr}_{GL_n})^{\sigma_1} = (GL_n(\mathcal{K}) \rtimes \mathbb{C}^\times) \times^{(GL_n(\mathcal{O})\rtimes\mathbb{C}^\times)} (\mathrm{Gr}_{GL_n})^{\sigma_1}$$

defined by $\eta([g(t)]) := [g(t), [{}^{\mathsf{T}}g(t)]]$ becomes $GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$-equivariant. We define an involutive auto-equivalence ι on $D_{coh}^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ by

$$\iota(\mathcal{F}) := \mathbb{D} \circ \mathbb{L} \circ \eta^*(\mathcal{O}_{\mathrm{Gr}_{GL_n}} \widetilde{\boxtimes} \mathcal{F}^{\sigma_1}),$$

where $\mathbb{D}$ is the Grothendieck-Serre duality functor and $\mathbb{L}$ is an auto-equivalence on $D_{coh}^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ which, on the d-th component $\mathrm{Gr}_{GL_n}^{(d)}$, acts by tensoring with

$$\mathcal{O}_{\mathrm{Gr}_{GL_n}}(-n) \otimes \det(L_0/tL_0)^d\{d(n-d)\}.$$

Theorem 3.3 ([CW19, Corollary 6.24 & 6.25])**.** *The involution ι is contravariant with respect to both convolution product $*$ and* Hom*, preserves the category of perverse coherent sheaves and satisfies $\iota(\mathcal{P}_{\nu,\mu}\{m/2\}) \cong \mathcal{P}_{\nu,\mu}\{-m/2\}$ for any $(\nu,\mu) \in \mathbf{D}_n$ and $m/2 \in \frac{1}{2}\mathbb{Z}$. Therefore, for any $\xi \in \mathscr{A}_n^{loc}$, we have $\Phi(\iota\xi) = \iota\Phi(\xi)$.*

Note that the basis $\mathscr{P}_n$ of $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ is nothing but the subset formed by the classes of ι-selfdual simple perverse coherent sheaves.

4. Comparison with nilpotent cones of type A

4.1. **Main result.** The main theorem of this paper is the following.

Theorem 4.1. *Under Cautis-Williams' isomorphism*

$$\Phi\colon \mathscr{A}_n^{loc} \cong K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$$

in Theorem 3.2*, the dual canonical basis* $\widetilde{\mathscr{B}}_n^{loc}$ *of* $\mathscr{A}_n^{loc}$ *bijectively corresponds to the basis* $\mathscr{P}_n$ *of* $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ *formed by the classes of* ι*-selfdual simple perverse coherent sheaves.*

By Theorem 3.2, we have $\Phi(D_0) = [\mathcal{P}_{n,1}]$ and $\Phi(D_1) = [\mathcal{P}_{n,0}]$. Since both $\mathcal{P}_{n,0}$ and $\mathcal{P}_{n,1}$ are invertible objects of the monoidal category $P_{coh}^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$, the operations $-\odot[\mathcal{P}_{n,0}]^{\pm1}$ and $-\odot[\mathcal{P}_{n,1}]^{\pm1}$ induce the self-bijections of the set $\mathscr{P}_n$. Therefore, to verify Theorem 4.1, it suffices to prove the following simpler assertion:

Theorem 4.2. *We have* $\Phi(\widetilde{B}(\mathbf{a})) \in \mathscr{P}_n$ *for any* $\mathbf{a} \in \mathbb{N}^{2n}$.

A proof will be given in the end of Section 4.3.

Remark 4.3. When $n = 2$, Theorem 4.1 can be verified directly by using an explicit computation of the dual canonical basis of the quantum unipotent group A_2 due to Lampe [L14].

4.2. **Perverse coherent sheaves on the nilpotent cone.** Fix $d \in \mathbb{N}$. Let

$$\mathcal{N}^d := \{x \in \mathrm{End}(\mathbb{C}^d) \mid x^d = 0\}$$

be the nilpotent cone of $\mathfrak{gl}_d(\mathbb{C})$. A left action of the group $GL_d(\mathbb{C})\times\mathbb{C}^\times$ on $\mathcal{N}^d$ is given by $(g,a)\cdot x = a^{-4}\mathrm{Ad}(g)x$. The equivariant K-group $K^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d)$ is a module over $\mathbb{Z}[q^{\pm1/2}]$, where $q^{m/2} \in K^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathrm{pt})$ denotes the class of the pull-back of the 1-dimensional $\mathbb{C}^\times$-module $\mathbb{C}_m$ of weight m along the natural projection $GL_d(\mathbb{C})\times\mathbb{C}^\times \to \mathbb{C}^\times$.

Recall that the nilpotent cone $\mathcal{N}^d$ has a finite number of $GL_d(\mathbb{C})$-orbits ($=$ $GL_d(\mathbb{C})\times\mathbb{C}^\times$-orbits) which are parametrized by the set $P(d)$ of partitions of d. The orbit $\mathbb{O}_\nu$ labelled by a partition $\nu = (\nu_1 \ge \nu_2 \ge \cdots) \in P(d)$ consists of nilpotent matrices of Jordan type ν, i.e. whose Jordan normal form is

$$J_\nu := J_{\nu_1}\oplus J_{\nu_2}\oplus\cdots, \quad \text{where } J_m := \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{gl}_m(\mathbb{C}).$$

We can easily compute $\dim\mathbb{O}_\nu = d^2 - \sum_{i\ge1}(2i-1)\nu_i$ and $\mathrm{codim}\,\mathbb{O}_\nu = \dim\mathcal{N}^d - \dim\mathbb{O}_\nu = \sum_{i\ge1}(2i-1)\nu_i - d$, both of which are even numbers.

We can consider the $GL_d(\mathbb{C})\times\mathbb{C}^\times$-equivariant perverse coherent sheaves on the nilpotent cone $\mathcal{N}^d$.

Definition 4.4. A $GL_d(\mathbb{C})\times\mathbb{C}^\times$-equivariant perverse coherent sheaf on $\mathcal{N}^d$ is an object $\mathcal{F} \in D_{coh}^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d)$ such that for every orbit $j_\nu\colon \mathbb{O}_\nu \hookrightarrow \mathcal{N}^d$

(1) $j_\nu^*\mathcal{F} \in D_{qcoh}^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d)$ is supported in degrees $\le \frac{1}{2}\mathrm{codim}\,\mathbb{O}_\nu$;

(2) $j_\nu^!\mathcal{F} \in D_{qcoh}^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d)$ is supported in degrees $\ge \frac{1}{2}\mathrm{codim}\,\mathbb{O}_\nu$.

We denote by $P_{coh}^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d) \subset D_{coh}^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d)$ the full subcategory of perverse coherent sheaves.

The simple perverse coherent sheaves are parametrized by the set

$$\mathbf{O}_d := \{(\nu, V) \mid \nu \in P(d),\ V \in \mathsf{Irr}\,\mathrm{Stab}_{GL_d(\mathbb{C})}(J_\nu)\}$$

up to isomorphism and $\mathbb{C}^\times$-equivariant twist in the following way (cf. [AH19, Section 3]). For each partition $\nu = (\nu_1, \nu_2, \ldots) \in P(d)$, we define a homomorphism $\phi_\nu \colon \mathbb{C}^\times \to GL_d(\mathbb{C})$ by

$$\phi_\nu(a) := \phi_{\nu_1}(a) \oplus \phi_{\nu_2}(a) \oplus \cdots,$$

where $\phi_m(a) := \mathrm{diag}(a^{2(-m+1)}, a^{2(-m+3)}, \ldots, a^{2(m-1)}) \in GL_m(\mathbb{C})$. The homomorphism ϕ_ν is a cocharacter associated to the nilpotent element J_ν in the sense of [J04, Section 5.3]. In particular, we have $\mathrm{Ad}(\phi_\nu(a))J_\nu = a^4 J_\nu$, and the group

$$\mathrm{Stab}^{red}_{GL_d(\mathbb{C})}(J_\nu) := \{g \in \mathrm{Stab}_{GL_d(\mathbb{C})}(J_\nu) \mid g\phi_\nu(a) = \phi_\nu(a)g,\ \forall a \in \mathbb{C}^\times\}$$

is a Levi subgroup of $\mathrm{Stab}_{GL_d(\mathbb{C})}(J_\nu)$. Then the image of the group embedding

$$\mathrm{Stab}^{red}_{GL_d(\mathbb{C})}(J_\nu) \times \mathbb{C}^\times \hookrightarrow \mathrm{Stab}_{GL_d(\mathbb{C})\times\mathbb{C}^\times}(J_\nu); \quad (g, a) \mapsto (g\phi_\nu(a), a)$$

is a Levi subgroup. Via this embedding, we make an identification

$$\mathsf{Irr}(\mathrm{Stab}_{GL_d(\mathbb{C})\times\mathbb{C}^\times}(J_\nu)) = \mathsf{Irr}(\mathrm{Stab}^{red}_{GL_d(\mathbb{C})}(J_\nu) \times \mathbb{C}^\times) \supset \mathsf{Irr}\,\mathrm{Stab}^{red}_{GL_d(\mathbb{C})}(J_\nu),$$

where the set $\mathsf{Irr}(\mathrm{Stab}^{red}_{GL_d(\mathbb{C})}(J_\nu))$ is regarded as a subset of $\mathsf{Irr}(\mathrm{Stab}^{red}_{GL_d(\mathbb{C})}(J_\nu)\times\mathbb{C}^\times)$ consisting of representations with the trivial $\mathbb{C}^\times$-actions. For a pair $(\nu, V) \in \mathbf{O}_d$, let $\mathcal{V}$ be the simple $GL_d(\mathbb{C})\times\mathbb{C}^\times$-equivariant vector bundle on $\mathbb{O}_\nu$ whose fiber at J_ν is isomorphic to V as a representation of $\mathrm{Stab}_{GL_d(\mathbb{C})\times\mathbb{C}^\times}(J_\nu)$. We define the simple perverse coherent sheaf $\mathcal{C}_{\nu,V}$ as the following (coherent) IC-extension

$$\mathcal{C}_{\nu,V} := (j_\nu)_{!*}\mathcal{V}\langle \mathrm{codim}\,\mathbb{O}_\nu/2\rangle,$$

where $j_\nu : \mathbb{O}_\nu \hookrightarrow \overline{\mathbb{O}}_\nu$ is the inclusion.

Under the above notation, we have a bijection

$$\mathbf{O}_d \times \mathbb{Z} \overset{1:1}{\longleftrightarrow} \mathsf{Irr}\,P_{coh}^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d); \quad ((\nu, V), m) \longleftrightarrow \mathcal{C}_{\nu,V}\{m/2\}.$$

Thus the set $\{[\mathcal{C}_{\nu,V}] \mid (\nu, V) \in \mathbf{O}_d\}$ forms a $\mathbb{Z}[q^{\pm 1/2}]$-basis of $K^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d)$.

Next we introduce another basis of $K^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d)$. Let $B_d \subset GL_d(\mathbb{C})$ be the Borel subgroup consisting of invertible *lower* triangular matrices and $\mathcal{B}_d := GL_d(\mathbb{C})/B_d$ be the flag variety. The cotangent bundle $T^*\mathcal{B}_d$ is naturally identified with the space $GL_d(\mathbb{C}) \times^{B_d} \mathfrak{n}_d$, where $\mathfrak{n}_d \subset \mathfrak{gl}_d(\mathbb{C})$ is the Lie algebra of strictly lower triangular matrices (= the nilpotent radical of $\mathrm{Lie}(B_d)$). Let $\pi \colon T^*\mathcal{B}_d \to \mathcal{B}_d$ denote the natural projection $[g, x] \mapsto [g]$ and $\mathtt{Sp} \colon T^*\mathcal{B}_d \to \mathcal{N}^d$ denote the Springer resolution $[g, x] \mapsto \mathrm{Ad}(g)x$. A natural left $GL_d(\mathbb{C})\times\mathbb{C}^\times$-action on $T^*\mathcal{B}_d$ is given by $(h, a)\cdot[g, x] := [hg, a^{-4}x]$. Both morphisms π and $\mathtt{Sp}$ are $GL_d(\mathbb{C})\times\mathbb{C}^\times$-equivariant.

Let $T_d \subset B_d$ be the maximal torus consisting of diagonal matrices. As before, the weight lattice $X := \mathrm{Hom}(T_d, \mathbb{C}^\times) = \mathrm{Hom}(B_d, \mathbb{C}^\times)$ is identified with $\mathbb{Z}^d$. For any $\lambda \in X$, we denote by $\mathcal{O}_{\mathcal{B}_d}(\lambda)$ the corresponding line bundle $GL_d(\mathbb{C}) \times^{B_d} \lambda$ on $\mathcal{B}_d$, which is regarded as a $GL_d(\mathbb{C})\times\mathbb{C}^\times$-equivariant bundle with the trivial $\mathbb{C}^\times$-action. We define the corresponding *Andersen-Jantzen sheaf* $AJ(\lambda)$ by

$$AJ(\lambda) := \mathtt{Sp}_*\pi^*\mathcal{O}_{\mathcal{B}_d}(\lambda).$$

More precisely, $\mathtt{Sp}_*$ denotes the derived push forward and the Andersen-Jantzen sheaf $AJ(\lambda)$ is an object of $D^{GL_d(\mathbb{C})\times\mathbb{C}^\times}_{coh}(\mathcal{N}^d)$ (which may or may not be a genuine sheaf).

As a convention, we regard the weights of $\mathfrak{n}_d$ as the negative roots. Then the set of dominant weights is $X_+ = \{\lambda = (\ell_1, \dots, \ell_d) \in X \mid \ell_1 \geq \cdots \geq \ell_d\}$.

For each dominant weight $\lambda \in X_+$, we define

$$\Delta_\lambda := AJ(w_0\lambda)\{\delta_\lambda\}, \qquad \nabla_\lambda := AJ(\lambda)\{-\delta_\lambda\},$$

where w_0 is the longest element of the Weyl group $\mathfrak{S}_d$ of $GL_d(\mathbb{C})$ and $\delta_\lambda := \min\{\ell(w) \mid w \in \mathfrak{S}_d,\ w\lambda \in -X_+\}$. Explicitly we have

$$\delta_\lambda = \frac{1}{2}(d(d-1) - \sum_k m_k(m_k - 1)), \tag{4.1}$$

where m_k is the multiplicity of $k \in \mathbb{Z}$ in the sequence $\lambda \in \mathbb{Z}^d$.

It is known that both objects Δ_λ and ∇_λ are perverse coherent sheaves. Indeed the family $\{\nabla_\lambda \mid \lambda \in X_+\}$ forms a quasi-exceptional set of the category $D^{GL_d(\mathbb{C})\times\mathbb{C}^\times}_{coh}(\mathcal{N}^d)$ with $\{\Delta_\lambda \mid \lambda \in X_+\}$ being its dual, which yields the above perverse t-structure (cf. [B03]). In particular, there is a canonical morphism $\Delta_\lambda \to \nabla_\lambda$ for each $\lambda \in X_+$. We denote the image of this canonical morphism by $\mathfrak{C}_\lambda$, which is a simple perverse coherent sheaf. The following result due to Achar-Hardesty [AH19] is the graded (or $\mathbb{C}^\times$-equivariant) version of the *Lusztig-Vogan bijection.* The non-graded version was originally established by [A01] (for GL_d) and [B03] (for a general reductive group instead of GL_d).

Theorem 4.5 ([AH19, Theorem 4.5]). *There is a bijection*

$$\mathtt{LV}\colon X_+ \xrightarrow{\sim} \mathbf{O}_d$$

such that we have $\mathfrak{C}_\lambda \cong \mathcal{C}_{\mathtt{LV}(\lambda)}$ *for any* $\lambda \in X_+$.

We define the modified Grothendieck-Serre duality functor $\mathbb{D}_{\mathcal{N}^d}$ on $\mathcal{N}^d$ by

$$\mathbb{D}_{\mathcal{N}^d} := \mathbb{R}\mathscr{H}\!om(-, \mathcal{O}_{\mathcal{N}^d}).$$

Remark 4.6. The usual Grothendieck-Serre duality is defined by using the dualizing complex. The dualizing complex $\omega_{\mathcal{N}^d}$ of the nilpotent cone $\mathcal{N}^d$ is $\mathcal{O}_{\mathcal{N}^d}\langle d(d-1)\rangle$ (see [AH19, Proposition 2.4]).

Let σ be an involution of the group $GL_d(\mathbb{C}) \times \mathbb{C}^\times$ given by $(h, a) \mapsto ({}^{\mathsf{T}}h^{-1}, a)$. Then the transpose map $x \mapsto {}^{\mathsf{T}}x$ induces a $GL_d(\mathbb{C}) \times \mathbb{C}^\times$-equivariant isomorphism $\tau\colon \mathcal{N}^d \xrightarrow{\sim} (\mathcal{N}^d)^\sigma$. We define an involutive auto-equivalence ι of $P^{GL_d(\mathbb{C})\times\mathbb{C}^\times}_{coh}(\mathcal{N}^d)$ by

$$\iota(\mathcal{F}) := \mathbb{D}_{\mathcal{N}^d} \circ \tau^*(\mathcal{F}^\sigma).$$

Then we have $\iota(q^{1/2}) = q^{-1/2}$ at the level of Grothendieck group. The following theorem was originally conjectured by Ostrik [O00] and proved by Bezrukavnikov (see [B03, Introduction]).

Theorem 4.7 (Bezrukavnikov). *The* $\mathbb{Z}[q^{\pm 1/2}]$*-basis* $\{[\mathfrak{C}_\lambda] \mid \lambda \in X_+\}$ *of* $K^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d)$ *is characterized by the following properties:*

(1) $\iota[\mathfrak{C}_\lambda] = [\mathfrak{C}_\lambda]$;
(2) $[\mathfrak{C}_\lambda] \in [\nabla_\lambda] + \sum_{\lambda' \in X_+} q^{-1}\mathbb{Z}[q^{-1}][\nabla_{\lambda'}]$.

4.3. **Comparison with the nilpotent cone.** Towards a proof of Theorem 4.2, let us compare $P_{coh}^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL_n})$ with $P_{coh}^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}^d)$. Fix two positive integers $n,d\in\mathbb{N}$ and consider the Schubert variety

$$\mathrm{Gr}_n^d := \overline{\mathrm{Gr}}_{GL_n}^{d\omega_1} = \{L\subset L_0 \mid \dim(L_0/L)=d\}.$$

This is a finite union of $GL_n(\mathcal{O})$-orbits $\mathrm{Gr}_{GL_n}^\nu$ where ν runs over the set

$$P_n(d) := \{\nu=(\nu_1,\dots,\nu_n)\in\mathbb{Z}^n \mid \nu_1\ge\cdots\ge\nu_n\ge 0,\ \nu_1+\cdots+\nu_n=d\}$$

of partitions of d of length $\le n$, regarded as dominant coweights of GL_n in the same way as before. Let $\mathbf{D}_{n,d}$ be the set of dominant pairs $(\nu,\mu)\in\mathbf{D}_n$ with $\nu\in P_n(d)$. Then the set $\{[\mathcal{P}_{\nu,\mu}] \mid (\nu,\mu)\in\mathbf{D}_{n,d}\}$ forms a $\mathbb{Z}[q^{\pm1/2}]$-basis of $K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_n^d)$.

We define the modified Grothendieck-Serre duality functor $\mathbb{D}_{\mathrm{Gr}_n^d}$ on Gr_n^d by

$$\mathbb{D}_{\mathrm{Gr}_n^d} := \mathbb{R}\mathscr{H}\!om(-,\mathcal{O}_{\mathrm{Gr}_n^d})\langle d(n-1)\rangle.$$

Remark 4.8. The dualizing complex of the Schubert variety Gr_n^d is isomorphic to

$$\mathcal{O}_{\mathrm{Gr}_{GL_n}}(-n)\otimes\det(L_0/tL_0)^d\{d(n-d)\}\langle d(n-1)\rangle$$

(see [CW19, Lemma 6.20]). Therefore we have

$$\mathbb{D}_{\mathrm{Gr}_n^d}(\mathcal{F}) = \mathbb{D}\circ\mathbb{L}(\mathcal{F}) \tag{4.2}$$

for any $\mathcal{F}\in D_{coh}^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_n^d)$ (see Section 3.5 for the definition of $\mathbb{L}$).

Under the above notation, we have the following morphism of quotient stacks:

$$\psi\colon [(GL_n(\mathcal{O})\rtimes\mathbb{C}^\times)\backslash\mathrm{Gr}_n^d]\to[(GL_d(\mathbb{C})\times\mathbb{C}^\times)\backslash\mathcal{N}^d];\quad (L\subset L_0)\mapsto t|_{L_0/L}.$$

Lemma 4.9. *The morphism ψ is formally smooth.*

Proof. For a fixed d and $N\gg0$, $\mathrm{Gr}_n^d=\{L : L_0\supset L\supset t^NL_0\}$. We have an evident morphism $[(GL_n(\mathcal{O}/t^N\mathcal{O})\rtimes\mathbb{C}^\times)\backslash\mathrm{Gr}_n^d]\to[(GL_n(\mathcal{O})\rtimes\mathbb{C}^\times)\backslash\mathrm{Gr}_n^d]$, and by an abuse of notation we will denote by $\psi\colon [(GL_n(\mathcal{O}/t^N\mathcal{O})\rtimes\mathbb{C}^\times)\backslash\mathrm{Gr}_n^d]\to[(GL_d(\mathbb{C})\times\mathbb{C}^\times)\backslash\mathcal{N}^d]$ the composition of the former ψ with the above evident morphism. It suffices to prove that the new ψ is smooth. Moreover, we will keep the same notation ψ for the similar morphism $[GL_n(\mathcal{O}/t^N\mathcal{O})\backslash\mathrm{Gr}_n^d]\to[GL_d(\mathbb{C})\backslash\mathcal{N}^d]$ (disregarding the extra $\mathbb{C}^\times$-equivariance). It suffices to prove that the latter ψ is smooth.

Given an affine test scheme $S=\operatorname{Spec}A$ along with its nilpotent extension $\widetilde{S}=\operatorname{Spec}\widetilde{A}$, $A=\widetilde{A}/I$, and a morphism $\underline{\varphi}\colon S\to[GL_n(\mathcal{O}/t^N\mathcal{O})\backslash\mathrm{Gr}_n^d]$ along with an extension

$$\widetilde{\varphi}\colon\widetilde{S}\to[GL_d(\mathbb{C})\backslash\mathcal{N}^d]\ \text{ of }\ \varphi:=\psi\circ\underline{\varphi}\colon S\to[GL_d(\mathbb{C})\backslash\mathcal{N}^d]$$

we have to find an extension $\underline{\widetilde{\varphi}}\colon\widetilde{S}\to[GL_n(\mathcal{O}/t^N\mathcal{O})\backslash\mathrm{Gr}_n^d]$.

We may and will assume that A and $\widetilde{A}$ are local, hence the projective modules are free. Then an S-point φ is a free A-module M of rank d with a nilpotent endomorphism $\mathsf{t}\in\operatorname{End}_A(M)$. Similarly, an $\widetilde{S}$-point $\widetilde{\varphi}$ is a free $\widetilde{A}$-module $\widetilde{M}$ of rank d with a nilpotent endomorphism $\widetilde{\mathsf{t}}\in\operatorname{End}_{\widetilde{A}}(\widetilde{M})$. An S-point $\underline{\varphi}$ is a free A-module $\underline{M}$ of rank nN with a nilpotent endomorphism $\underline{\mathsf{t}}$ "of Jordan type N^n" and a $\underline{\mathsf{t}}$-invariant (locally) free A-submodule $\underline{M}'\subset\underline{M}$ such that the quotient $\underline{M}/\underline{M}'$ is free of rank d. Finally, an $\widetilde{S}$-point $\underline{\widetilde{\varphi}}$ is a free $\widetilde{A}$-module $\underline{\widetilde{M}}$ of rank nN with a nilpotent endomorphism $\underline{\widetilde{\mathsf{t}}}$ "of Jordan type N^n" and a $\underline{\widetilde{\mathsf{t}}}$-invariant (locally) free $\widetilde{A}$-submodule $\underline{\widetilde{M}}'\subset\underline{\widetilde{M}}$ such that the quotient $\underline{\widetilde{M}}/\underline{\widetilde{M}}'$ is free of rank d.

We have to prove that given $(\widetilde{M},\widetilde{\mathtt{t}})$ and $(\underline{M}'\subset\underline{M},\underline{\mathtt{t}})$ as above giving rise to the same $(\widetilde{M}/I,\widetilde{\mathtt{t}}\pmod I)=(M,\mathtt{t})=(\underline{M}/\underline{M}',\underline{\mathtt{t}}\pmod{\underline{M}'})$, there exists $(\widetilde{\underline{M}}'\subset\widetilde{\underline{M}},\widetilde{\underline{\mathtt{t}}})$ as above such that $(\widetilde{\underline{M}}'\pmod I\subset\widetilde{\underline{M}}\pmod I,\widetilde{\underline{\mathtt{t}}}\pmod I)=(\underline{M}'\subset\underline{M},\underline{\mathtt{t}})$, while $(\widetilde{\underline{M}}/\widetilde{\underline{M}}',\widetilde{\underline{\mathtt{t}}}\pmod{\widetilde{\underline{M}}'})=(\widetilde{M},\widetilde{\mathtt{t}})$. If we disregard the nilpotent operators, then the existence of the desired extension $\widetilde{\underline{M}}'\subset\widetilde{\underline{M}}$ follows from the smoothness of the evident morphism $GL_{nN}\backslash\mathrm{Gr}(d,nN)\to GL_d\backslash\,\mathrm{pt}$ (and is evident by itself).

So it remains to prove that the sequence

$$\mathrm{End}_{\widetilde{A}}(\widetilde{\underline{M}}'\subset\widetilde{\underline{M}})\to\mathrm{End}_{\widetilde{A}}(\widetilde{M})\oplus\mathrm{End}_A(\underline{M}'\subset\underline{M})\to\mathrm{End}_A(M)$$

(see the diagram below) is exact in the middle term:

$$\begin{array}{ccc}\mathrm{End}_{\widetilde{A}}(\widetilde{M}) & \longleftarrow & \mathrm{End}_{\widetilde{A}}(\widetilde{\underline{M}}'\subset\widetilde{\underline{M}})\\ \big\downarrow & & \big\downarrow\\ \mathrm{End}_A(M) & \longleftarrow & \mathrm{End}_A(\underline{M}'\subset\underline{M}).\end{array}$$

This is clear since all our modules are free, and moreover, we can find a complementary free submodule $\widetilde{\underline{M}}''\subset\widetilde{\underline{M}}$ such that $\widetilde{\underline{M}}=\widetilde{\underline{M}}''\oplus\widetilde{\underline{M}}'$. □

Corollary 4.10. *The morphism* $\psi\colon[(GL_n(\mathcal{O})\rtimes\mathbb{C}^\times)\backslash\mathrm{Gr}_n^d]\to[(GL_d(\mathbb{C})\times\mathbb{C}^\times)\backslash\mathcal{N}^d]$ *is flat.*

Proof. This is [EGA IV, Théorème 17.5.1]. □

Therefore the pull-back along ψ^* induces a triangulated functor

$$\psi^*\colon D^{GL_d(\mathbb{C})\times\mathbb{C}^\times}_{coh}(\mathcal{N}^d)\to D^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}_{coh}(\mathrm{Gr}_n^d).$$

In what follows, we restrict ourselves to the open subvariety

$$\mathcal{N}_n^d:=\bigsqcup_{\nu\in P_n(d)}\mathbb{O}_\nu\subset\mathcal{N}^d.$$

Let $\mathbf{O}_{n,d}$ be the set of pairs $(\nu,V)\in\mathbf{O}_d$ with $\nu\in P_n(d)$. Then the set $\{[\mathcal{C}_{\nu,V}]\mid(\nu,V)\in\mathbf{O}_{n,d}\}$ forms a $\mathbb{Z}[q^{\pm1/2}]$-basis of $K^{GL_d(\mathbb{C})\rtimes\mathbb{C}^\times}(\mathcal{N}_n^d)$. We will keep the same notation $AJ(\lambda),\Delta_\lambda,\nabla_\lambda$ for their restrictions to $\mathcal{N}_n^d$.

By construction, the morphism ψ has its image in the open substack $[(GL_d(\mathbb{C})\times\mathbb{C}^\times)\backslash\mathcal{N}_n^d]$. More precisely, for each $\nu\in P_n(d)$, the morphism ψ sends the $GL_n(\mathcal{O})$-orbit $\mathrm{Gr}^\nu_{GL_n}$ to the $GL_d(\mathbb{C})$-orbit $\mathbb{O}_\nu$. Thus we have obtained the triangulated functor:

$$\psi^*\colon D^{GL_d(\mathbb{C})\times\mathbb{C}^\times}_{coh}(\mathcal{N}_n^d)\to D^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}_{coh}(\mathrm{Gr}_n^d).$$

Definition 4.11. We define the triangulated functor

$$\Psi\colon D^{GL_d(\mathbb{C})\times\mathbb{C}^\times}_{coh}(\mathcal{N}_n^d)\to D^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}_{coh}(\mathrm{Gr}_n^d)\quad\text{by }\Psi(-):=\psi^*(-)\langle d(n-1)/2\rangle.$$

Proposition 4.12. *The functor* Ψ *satisfies the following properties:*

(1) Ψ *is* t*-exact with respect to the perverse* t*-structures of both sides. Therefore it induces an exact functor between abelian categories:*

$$\Psi\colon P^{GL_d(\mathbb{C})\times\mathbb{C}^\times}_{coh}(\mathcal{N}_n^d)\to P^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}_{coh}(\mathrm{Gr}_n^d);$$

(2) Ψ *is compatible with the* IC*-extensions, i.e. for any* $\nu\in P_n(d)$*, we have*

$$\Psi\circ(j_\nu)_{!*}\simeq(i_\nu)_{!*}\circ\Psi;$$

(3) Ψ *is compatible with the duality functors, i.e. we have*

$$\Psi \circ \mathbb{D}_{\mathcal{N}_n^d} \simeq \mathbb{D}_{\mathrm{Gr}_n^d} \circ \Psi.$$

Proof. Note that the cohomological degree shift $[d(n-1)/2]$ in the definition of Ψ arises from the fact that $d(n-1) = \operatorname{codim} \mathbb{O}_\nu + \dim \mathrm{Gr}^\nu_{GL_n}$ for any $\nu \in P_n(d)$.

For (1), we apply [AB10, Lemma 3.4]. To do so, we have to check that the morphism ψ is faithfully flat and Gorenstein. For the faithful flatness, thanks to Lemma 4.9, it suffices to show that given a local $\mathbb{C}$-algebra A the morphism ψ yields a surjective map $[(GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times)\backslash \mathrm{Gr}_n^d](A) \twoheadrightarrow [(GL_d(\mathbb{C}) \times \mathbb{C}^\times)\backslash \mathcal{N}_n^d](A)$ of the sets of A-points. (Or instead, we may show that the morphism $\psi_2 \colon \mathcal{M}_n^d \to \mathcal{N}_n^d$ defined below is surjective as a morphism between the schemes associated with the varieties.) This can be proved easily by the definition of $\mathcal{N}_n^d$. For the Gorenstein property, it suffices to show that $\psi^! \mathcal{O}_{\mathcal{N}^d}$ is a cohomological degree shift of an invertible sheaf (see [H66, Exercise V.9.5]). Since the dualizing complex of $\mathcal{N}^d$ is isomorphic to $\mathcal{O}_{\mathcal{N}^d}\langle d(d-1)\rangle$ (see Remark 4.6), the sheaf $\psi^!(\mathcal{O}_{\mathcal{N}^d})$ is isomorphic to the dualizing complex of Gr_n^d up to cohomological degree shift and $\mathbb{C}^\times$-equivariant twist, which is also known to be a cohomological degree shift of an invertible sheaf (see Remark 4.8).

Now (2) can be proved in a similar way by the definition of the minimal extension functors. See [AB10, Theorem 4.2].

The remaining assertion (3) follows from Remarks 4.6 & 4.8 and the fact

$$\psi^!(-) = \psi^*(-) \otimes \omega_{\mathrm{Gr}_n^d/\mathcal{N}_n^d} = (\mathbb{L} \circ \psi^*)(-)\langle d(d-n)\rangle.$$

See [H66, Remark on pp. 143–144] for the first equality. □

For a technical reason, we will introduce an auxiliary space. Let $\mathcal{M}_n^d$ be the variety of pairs (L, γ) such that:

(i) L is a $\mathcal{O}$-lattice of $\mathcal{K}^n$ such that $\dim L_0/L = d$;
(ii) γ is a $\mathbb{C}$-linear isomorphism $\mathbb{C}^d \xrightarrow{\sim} L_0/L$.

We equip the space $\mathcal{M}_n^d$ with a left action of the group $GL_d(\mathbb{C}) \times GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$ by

$$(h, g(t), a) \cdot (L, \gamma) := ((g(t), a)L, (g(t), a) \circ \gamma \circ h^{-1}),$$

where $h \in GL_d(\mathbb{C}),\ g(t) \in GL_n(\mathcal{O}),\ a \in \mathbb{C}^\times$ and in the 2nd entry of the right hand side the element $(g(t), a) \in GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$ is regarded as a $\mathbb{C}$-linear isomorphism $L_0/L \xrightarrow{\sim} L_0/(g(t), a)L$. Then we can consider the following diagram:

$$\mathrm{Gr}_n^d \xleftarrow{\psi_1} \mathcal{M}_n^d \xrightarrow{\psi_2} \mathcal{N}_n^d.$$

Here the morphism ψ_1 is the first projection $(L, \gamma) \mapsto L$ and the morphism ψ_2 is given by

$$\psi_2(L, \gamma) := \gamma^{-1} \circ t|_{L_0/L} \circ \gamma.$$

The morphisms ψ_1 and ψ_2 are equivariant with respect to the actions of the group $GL_d(\mathbb{C}) \times GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$. Here we understand that the group $GL_d(\mathbb{C})$ (resp. $GL_n(\mathcal{O})$) acts trivially on Gr_n^d (resp. $\mathcal{N}_n^d$). Since ψ_1 is a principal $GL_d(\mathbb{C})$-bundle, the pull-back functor gives an equivalence of triangulated categories:

$$\psi_1^* \colon D_{coh}^{GL_d(\mathbb{C}) \times GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathcal{M}_n^d) \xrightarrow{\sim} D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_n^d).$$

We fix a quasi-inverse of ψ_1^* and denote it by $(\psi_1^*)^{-1}$.

Lemma 4.13. *There is an isomorphism of functors* $\psi^* \simeq (\psi_1^*)^{-1} \circ \psi_2^*$.

Proof. This is obvious from the construction. □

For each $\nu = (\nu_1, \dots, \nu_n) \in P_n(d)$, we define a $\mathbb{C}$-linear isomorphism

$$\gamma_\nu \colon \mathbb{C}^d \xrightarrow{\sim} L_0/(t^\nu L_0) \quad \text{by } v_j \mapsto t^{-(\nu_1+\cdots+\nu_{k-1})+j-1} u_k \mod t^\nu L_0$$

for each $1 \le k \le n$ and $\nu_1+\cdots+\nu_{k-1} < j \le \nu_1+\cdots+\nu_k$, where $\{v_j \in \mathbb{C}^d \mid 1 \le j \le d\}$ is the standard $\mathbb{C}$-basis of $\mathbb{C}^d$ and $\{u_k \in \mathcal{K}^n \mid 1 \le k \le n\}$ is the standard $\mathcal{K}$-basis of $\mathcal{K}^n$. The point $p_\nu := (t^\nu L_0, \gamma_\nu) \in \mathcal{M}_n^d$ satisfies $\psi_1(p_\nu) = [t^\nu]$ and $\psi_2(p_\nu) = J_\nu$. Then the natural projections

$$GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times \leftarrow GL_d(\mathbb{C}) \times GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times \to GL_d(\mathbb{C}) \times \mathbb{C}^\times$$

induce the homomorphisms of stabilizers

$$\mathrm{Stab}_{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}[t^\nu] \xleftarrow{\sim} \mathrm{Stab}_{GL_d(\mathbb{C})\times GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(p_\nu) \to \mathrm{Stab}_{GL_d(\mathbb{C})\times\mathbb{C}^\times}(J_\nu),$$

where the left one is an isomorphism because ψ_1 is a $GL_d(\mathbb{C})$-bundle. Let

$$\rho\colon \mathrm{Stab}_{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}[t^\nu] \to \mathrm{Stab}_{GL_d(\mathbb{C})\times\mathbb{C}^\times}(J_\nu)$$

be the group homomorphism obtained by composing the above two homomorphisms. This homomorphism ρ induces an isomorphism between the subgroups

$$\rho_1\colon \mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu] \xrightarrow{\sim} \mathrm{Stab}^{red}_{GL_d(\mathbb{C})}(J_\nu).$$

In particular, the assignment $(\nu, \mu) \mapsto (\nu, V_\mu)$ defines a bijection $\mathbf{D}_{n,d} \xrightarrow{\sim} \mathbf{O}_{n,d}$. Henceforth we identify $\mathbf{O}_{n,d}$ with $\mathbf{D}_{n,d}$ via this bijection and we write $\mathcal{C}_{\nu,\mu}$ instead of $\mathcal{C}_{\nu,V_\mu}$.

Lemma 4.14. *For any $(\nu,\mu) \in \mathbf{D}_{n,d}$, we have an isomorphism*

$$\Psi\left(\mathcal{C}_{\nu,\mu}\{-\langle \omega_n, \mu\rangle\}\right) \cong \mathcal{P}_{\nu,\mu}.$$

In particular, the functor Ψ induces an isomorphism of K-groups

$$[\Psi]\colon K^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}_n^d) \xrightarrow{\sim} K^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_n^d),$$

which gives a bijective correspondence between the classes of simple perverse coherent sheaves.

Proof. By Proposition 4.12(2), it suffices to show that the fiber at $[t^\nu]$ of $\Psi\left(\mathcal{C}_{\nu,\mu}\right)$ is isomorphic to $V_\mu\{-\langle \nu - \omega_n, \mu\rangle - \dim \mathrm{Gr}^\nu_{GL_n}/2\}$ as a representation of $\mathrm{Stab}_{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}[t^\nu]$, disregarding cohomological degree shift. By construction, we observe that the restriction of ρ to the Levi subgroup $\mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu] \times \mathbb{C}^\times \subset \mathrm{Stab}_{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}[t^\nu]$ is given by $(g,a) \mapsto (\rho_1(ga^{2(\nu-\omega_n)})\phi_\nu(a), a)$. Therefore the fiber at $[t^\nu]$ of $\psi^*\left(\mathcal{C}_{\nu,\mu}\right)$ is isomorphic to the pull-back of the representation $V_\mu\{\mathrm{codim}\,\mathbb{O}_\nu /2\}$ along the group homomorphism $\mathrm{Stab}^{red}_{GL_n(\mathcal{O})}[t^\nu] \times \mathbb{C}^\times \to \mathrm{Stab}^{red}_{GL_d(\mathbb{C})}(J_\nu) \times \mathbb{C}^\times$ given by $(g,a) \mapsto (\rho_1(ga^{2(\nu-\omega_n)}), a)$. After the $\mathbb{C}^\times$-equivariant twist $\{-d(n-1)/2\}$, we obtain the desired representation. □

Lemma 4.15. *Let $\lambda = (\ell_1, \dots, \ell_d) \in X = \mathbb{Z}^d$ be a weight of $GL_d(\mathbb{C})$. Then we have an isomorphism*

$$\Psi(AJ(\lambda)\{-\langle \omega_d, \lambda\rangle\}) \cong \mathcal{P}_{1,\ell_1} * \mathcal{P}_{1,\ell_2} * \cdots * \mathcal{P}_{1,\ell_d},$$

where $\omega_d := (1, \dots, 1) \in \mathbb{Z}^d$ and hence $\langle \omega_d, \lambda\rangle = \ell_1 + \cdots + \ell_d$.

Proof. Let $\widetilde{\mathrm{Gr}}_n^d$ be the variety of flags of $\mathcal{O}$-lattices $L_\bullet = (L_d \subset \cdots \subset L_1 \subset L_0)$ satisfying $\dim L_{i-1}/L_i = 1$ for $1 \le i \le d$. This is nothing but the convolution variety $\mathrm{Gr}^1_{GL_n} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}^1_{GL_n}$ (d factors). The multiplication morphism $\overline{m}\colon \widetilde{\mathrm{Gr}}_n^d \to \mathrm{Gr}_n^d$ is given simply by $L_\bullet = (L_d \subset \cdots \subset L_1 \subset L_0) \mapsto (L_d \subset L_0)$. Then we have

$$\mathcal{P}_{1,\ell_1} * \mathcal{P}_{1,\ell_2} * \cdots * \mathcal{P}_{1,\ell_d} = \underline{m}_* \left(\bigotimes_{k=1}^{d} (L_{k-1}/L_k)^{\otimes \ell_k} \right) \langle d(n-1)/2 \rangle \{ -\langle \omega_d, \lambda \rangle \},$$

where we denote by L_{i-1}/L_i the line bundle on $\widetilde{\mathrm{Gr}}_n^d$ whose fiber at $L_\bullet$ is equal to L_{i-1}/L_i by an abuse of notation.

On the other hand, we put $\widetilde{\mathcal{N}}_n^d := \mathtt{Sp}^{-1}(\mathcal{N}_n^d)$, which is an open subvariety of the cotangent bundle $T^*\mathcal{B}_d$. We identify the variety $\widetilde{\mathcal{N}}_n^d$ with the variety of pairs $(V_\bullet, x)$ consisting of a complete flag $V_\bullet = (\{0\} = V_d \subset \cdots \subset V_1 \subset V_0 = \mathbb{C}^d)$ and a nilpotent endomorphism $x \in \mathcal{N}_n^d$ satisfying $x(V_{i-1}) \subset V_i$ for all $1 \le i \le d$. Then we have

$$\pi^* \mathcal{O}_{\mathcal{B}_d}(\lambda)|_{\widetilde{\mathcal{N}}_n^d} = \bigotimes_{k=1}^{d} (V_{k-1}/V_k)^{\otimes \ell_k},$$

where we denote by V_{i-1}/V_i the line bundle on $\widetilde{\mathcal{N}}_n^d$ whose fiber at $(V_\bullet, x)$ is equal to V_{i-1}/V_i by an abuse of notation.

Let $\widetilde{\mathcal{M}}_n^d$ be the variety of pairs $(L_\bullet, \gamma)$ such that:

(i) $L_\bullet = (L_d \subset \cdots \subset L_1 \subset L_0)$ is a flag of $\mathcal{O}$-lattices with $\dim L_{i-1}/L_i = 1$ for $1 \le i \le d$;

(ii) γ is a $\mathbb{C}$-linear isomorphism $\mathbb{C}^d \xrightarrow{\sim} L_0/L_d$.

This space $\widetilde{\mathcal{M}}_n^d$ fits into the following commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathrm{Gr}}_n^d & \xleftarrow{\widetilde{\psi}_1} & \widetilde{\mathcal{M}}_n^d & \xrightarrow{\widetilde{\psi}_2} & \widetilde{\mathcal{N}}_n^d \\ \downarrow{\scriptstyle \overline{m}} & & \downarrow{\scriptstyle \overline{m}'} & & \downarrow{\scriptstyle \mathtt{Sp}} \\ \mathrm{Gr}_n^d & \xleftarrow{\psi_1} & \mathcal{M}_n^d & \xrightarrow{\psi_2} & \mathcal{N}_n^d, \end{array} \tag{4.3}$$

where the morphism $\widetilde{\psi}_1$ is the projection $(L_\bullet, \gamma) \mapsto L_\bullet$ and the morphism $\widetilde{\psi}_2$ is given by

$$(L_\bullet, \gamma) \mapsto (\{0\} \subset \gamma^{-1}(L_{d-1}/L_d) \subset \cdots \subset \gamma^{-1}(L_1/L_d) \subset \mathbb{C}^d, \gamma^{-1} \circ t|_{L_0/L_d} \circ \gamma).$$

All arrows in the diagram (4.3) are $GL_d(\mathbb{C}) \times GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$-equivariant. Moreover, both the left and the right squares are Cartesian. The morphism $\widetilde{\psi}_1$ is a principal $GL_d(\mathbb{C})$-bundle.

Since there is a $GL_d(\mathbb{C}) \times GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$-equivariant isomorphism of line bundles

$$\gamma\colon \widetilde{\psi}_2^* \left(\bigotimes_{k=1}^{d} (V_{k-1}/V_k)^{\otimes \ell_k} \right) \xrightarrow{\sim} \widetilde{\psi}_1^* \left(\bigotimes_{k=1}^{d} (L_{k-1}/L_k)^{\otimes \ell_k} \right),$$

we have

$$\begin{aligned}\psi^*\mathtt{Sp}_*\left(\bigotimes_{k=1}^{d}(V_{k-1}/V_k)^{\otimes \ell_k}\right) &\cong (\psi_1^*)^{-1}\overline{m}'_*\widetilde{\psi}_2^*\left(\bigotimes_{k=1}^{d}(V_{k-1}/V_k)^{\otimes \ell_k}\right)\\ &\cong \overline{m}_*(\widetilde{\psi}_1^*)^{-1}\widetilde{\psi}_2^*\left(\bigotimes_{k=1}^{d}(V_{k-1}/V_k)^{\otimes \ell_k}\right)\\ &\cong \overline{m}_*\left(\bigotimes_{k=1}^{d}(L_{k-1}/L_k)^{\otimes \ell_k}\right),\end{aligned}$$

where we applied the smooth base change formula (cf. [CG97, Proposition 5.3.15]) to the two Cartesian squares in the diagram (4.3). After the shift and twist $\langle d(n-1)/2\rangle\{-\langle \omega_d, \lambda\rangle\}$, we obtain the conclusion. ☐

We fix an element $\beta = d_0\alpha_0 + d_1\alpha_1 \in \mathsf{Q}^+$ (of the root system of type $\mathsf{A}_1^{(1)}$) such that $d_0 - d_1 = d$. For each $\mathbf{a} = (a_1, \dots, a_{2n}) \in \mathrm{KP}_n(\beta)$, we define an object $\mathcal{E}(\mathbf{a}) \in P_{coh}^{GL_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_n^d)$ as the following convolution product:

$$\mathcal{E}(\mathbf{a}) := (\mathcal{P}_{1,n})^{*a_1} * \cdots * (\mathcal{P}_{k,n+1-k})^{*a_k} * \cdots * (\mathcal{P}_{1,1-n})^{*a_{2n}}\{-\textstyle\sum_{k<\ell} a_k a_\ell\}.$$

Then we have $\Phi(\widetilde{E}(\mathbf{a})) = [\mathcal{E}(\mathbf{a})]$ (see (2.2) and (3.1)).

To each $\mathbf{a} \in \mathrm{KP}_n(\beta)$, we attach the unique dominant weight $\lambda_{\mathbf{a}} \in X_+$ which contains the integer $n+1-k$ with multiplicity a_k for each $1 \le k \le 2n$.

Corollary 4.16. *For each* $\mathbf{a} \in \mathrm{KP}_n(\beta)$*, we have*

$$\Psi(\nabla_{\lambda_{\mathbf{a}}}\{-\langle \omega_d, \lambda_{\mathbf{a}}\rangle\}) \cong \mathcal{E}(\mathbf{a}).$$

Proof. This is a consequence of Lemma 4.15 and the fact $\delta_{\lambda_{\mathbf{a}}} = \sum_{k<\ell} a_k a_\ell$, which follows from (4.1). ☐

Let ω denote an automorphism of the group $GL_d(\mathbb{C}) \times \mathbb{C}^\times$ given by $(h, a) \mapsto (ha^2, a)$. Since $\mathcal{N}_n^d = (\mathcal{N}_n^d)^\omega$, the operation $\mathcal{F} \mapsto \mathcal{F}^\omega$ defines an auto-equivalence of $D_{coh}^{GL_d(\mathbb{C})\times C^\times}(\mathcal{N}_n^d)$. Then we have

$$\mathcal{C}_{\nu,\mu}^\omega = \mathcal{C}_{\nu,\mu}\{-\langle \omega_n, \mu\rangle\}, \quad \nabla_\lambda^\omega = \nabla_\lambda\{-\langle \omega_d, \lambda\rangle\}$$

for $(\nu, \mu) \in \mathbf{D}_{n,d}$ and $\lambda \in X_+$ respectively. Thus, Lemma 4.14 and Corollary 4.16 are rewritten as:

$$\Psi(\mathcal{C}_{\nu,\mu}^\omega) \simeq \mathcal{P}_{\nu,\mu}, \tag{4.4}$$

$$\Psi(\nabla_{\lambda_{\mathbf{a}}}^\omega) \simeq \mathcal{E}(\mathbf{a}) \tag{4.5}$$

for any $(\nu, \mu) \in \mathbf{D}_{n,d}$ and $\mathbf{a} \in \mathrm{KP}_n(\beta)$ respectively.

Corollary 4.17. *For any* $\xi \in K^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}_n^d)$*, we have*

$$\iota[\Psi](\xi) = [\Psi]\iota(\xi^{\omega^{-2}}).$$

In particular, $[\Psi](\xi)$ *is fixed by* ι *if and only if* $\xi^{\omega^{-1}}$ *is fixed* ι*.*

Proof. It is enough to consider the case $\xi = [\mathcal{C}_{\nu,\mu}]$ for some $(\nu, \mu) \in \mathbf{D}_{n,d}$. In this case, the assertion is obvious from (4.4). ☐

The following assertion implies Theorem 4.2.

Theorem 4.18. *For each* $\mathbf{a} \in \mathrm{KP}_n(\beta)$*, we have* $\mathtt{LV}(\lambda_{\mathbf{a}}) \in \mathbf{D}_{n,d}$ *and*

$$\Phi(\widetilde{B}(\mathbf{a})) = [\mathcal{P}_{\mathtt{LV}(\lambda_{\mathbf{a}})}],$$

where $\mathtt{LV}$ *is the Lusztig-Vogan bijection (see Theorem* 4.5*).*

Proof. Let $\mathcal{L} := \bigoplus_{(\nu,\mu)\in \mathbf{D}_{n,d}} \mathbb{Z}[q^{-1/2}][\mathcal{C}_{\nu,\mu}]$ be a $\mathbb{Z}[q^{-1/2}]$-lattice of $K^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(\mathcal{N}_n^d)$. Since the elements $[\mathcal{C}_{\nu,\mu}]$ are fixed by ι (cf. Theorem 4.5 and Theorem 4.7), we have $\mathcal{L} \cap \iota(\mathcal{L}) = \bigoplus_{(\nu,\mu)\in \mathbf{D}_{n,d}} \mathbb{Z}[\mathcal{C}_{\nu,\mu}]$.

By Corollary 2.4 and (4.5), we have

$$([\Psi]^{-1}\Phi(\widetilde{B}(\mathbf{a})))^{\omega^{-1}} - [\nabla_{\lambda_{\mathbf{a}}}] \in \sum_{\mathbf{a}'\in \mathrm{KP}_n(\beta)} q^{-1}\mathbb{Z}[q^{-1}][\nabla_{\lambda_{\mathbf{a}'}}].$$

Combining this with Theorem 4.7, we obtain

$$([\Psi]^{-1}\Phi(\widetilde{B}(\mathbf{a})))^{\omega^{-1}} - [\mathfrak{C}_{\lambda_{\mathbf{a}}}|_{\mathcal{N}_n^d}] \in q^{-1/2}\mathcal{L}.$$

By Corollary 4.17, the element $([\Psi]^{-1}\Phi(\widetilde{B}(\mathbf{a})))^{\omega^{-1}}$ is ι-invariant. Thus we have $([\Psi]^{-1}\Phi(\widetilde{B}(\mathbf{a})))^{\omega^{-1}} = [\mathfrak{C}_{\lambda_{\mathbf{a}}}|_{\mathcal{N}_n^d}]$.

Note that $\mathfrak{C}_{\lambda_{\mathbf{a}}}|_{\mathcal{N}_n^d} = \mathcal{C}_{\mathtt{LV}(\lambda_{\mathbf{a}})}$ if $\mathtt{LV}(\lambda_{\mathbf{a}}) \in \mathbf{D}_{n,d}$, and $\mathfrak{C}_{\lambda_{\mathbf{a}}}|_{\mathcal{N}_n^d} = 0$ otherwise. However the latter case can not happen because we know that $[\Psi]^{-1}\Phi(\widetilde{B}(\mathbf{a}))$ is nonzero. Therefore we have $\Phi(\widetilde{B}(\mathbf{a})) = [\Psi(\mathcal{C}^{\omega}_{\mathtt{LV}(\lambda_{\mathbf{a}})})] = [\mathcal{P}_{\mathtt{LV}(\lambda_{\mathbf{a}})}]$. ☐

4.4. **Comparison of the bar involutions.** In this complementary subsection, we prove the following categorical version of Corollary 4.17.

Proposition 4.19. *For any* $\mathcal{F} \in D^{GL_d(\mathbb{C})\times\mathbb{C}^\times}_{coh}(\mathcal{N}_n^d)$*, we have*

$$\iota \circ \Psi(\mathcal{F}) \cong \Psi \circ \iota(\mathcal{F}^{\omega^{-2}}).$$

For a proof, we need to introduce some new notation.

We define an automorphism σ_1 of the group $GL_d(\mathbb{C}) \times GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$ by

$$\sigma_1(h, g(t), a) := ({}^{\mathsf{T}}h^{-1}a^{-4}, {}^{\mathsf{T}}g(t)^{-1}, a),$$

where ${}^{\mathsf{T}}(-)$ denotes the transpose of matrices. We will use the same notation σ_1 for its restrictions to the subgroups $GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$ or $GL_d(\mathbb{C}) \times \mathbb{C}^\times$. For the subgroup $GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$, this notation is consistent with σ_1 defined in Section 3.5. For the subgroup $GL_d(\mathbb{C}) \times \mathbb{C}^\times$, we have a relation $\sigma_1 = \sigma \circ \omega^{-2} = \omega^{-2} \circ \sigma$.

Now we consider a morphism

$$\eta' \colon \mathcal{M}_n^d \to GL_n(\mathcal{K}) \times^{GL_n(\mathcal{O})} (\mathcal{M}_n^d)^{\sigma_1} = (GL_n(\mathcal{K}) \rtimes \mathbb{C}^\times) \times^{(GL_n(\mathcal{O})\rtimes\mathbb{C}^\times)} (\mathcal{M}_n^d)^{\sigma_1}$$

given by

$$\eta'(L, \gamma) := [g(t), ({}^{\mathsf{T}}g(t)L_0, ({}^{\mathsf{T}}g(t)t^{-1}) \circ {}^{\mathsf{T}}\gamma^{-1})].$$

Here $g(t) \in GL_n(\mathcal{K})$ is an element such that $L = g(t)L_0$, and the $\mathbb{C}$-linear isomorphism ${}^{\mathsf{T}}\gamma \colon (t\,{}^{\mathsf{T}}g(t)^{-1}L_0)/tL_0 \xrightarrow{\sim} \mathbb{C}^d$ is determined so that the following diagram commutes

$$\begin{CD}
(L_0/g(t)L_0)^* @>{\gamma^*}>> (\mathbb{C}^d)^* \\
@V{\cong}VV @VV{\cong}V \\
(t\,{}^{\mathsf{T}}g(t)^{-1}L_0)/tL_0 @>{{}^{\mathsf{T}}\gamma}>> \mathbb{C}^d,
\end{CD}$$

where the isomorphism $(L_0/g(t)L_0)^* \xrightarrow{\cong} (t^{\mathsf{T}}g(t)^{-1}L_0)/tL_0$ is given by the residue pairing

$$\langle -,-\rangle\colon \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{C}; \quad \langle v(t), w(t)\rangle := \operatorname*{Res}_{t=0} \frac{{}^{\mathsf{T}}v(t)\cdot w(t)}{t}\mathrm{d}t,$$

and the isomorphism $(\mathbb{C}^d)^* \xrightarrow{\sim} \mathbb{C}^d$ is given by the standard pairing $\langle v, w\rangle := {}^{\mathsf{T}}v \cdot w$. We can easily check that the resulting coset $[g(t), ({}^{\mathsf{T}}g(t)L_0, ({}^{\mathsf{T}}g(t)t^{-1}) \circ {}^{\mathsf{T}}\gamma^{-1})]$ is independent of the choice of $g(t) \in GL_n(\mathcal{K})$ such that $L = g(t)L_0$.

Lemma 4.20. *The morphism η' is $GL_d(\mathbb{C}) \times GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$-equivariant.*

Proof. This is proved by the following straightforward computation. For $(g(t)L_0, \gamma) \in \mathcal{M}_n^d$ and $(h, g_1(t), a) \in GL_d(\mathbb{C}) \times GL_n(\mathcal{O}) \rtimes \mathbb{C}^\times$, we compute:

$$\begin{aligned}
&\eta'((h, g_1(t), a)\cdot(g(t)L_0,\gamma)) \\
=\ &\eta'(g_1(t)g(a^4t)L_0, (g_1(t),a)\circ\gamma\circ h^{-1}) \\
=\ &[g_1(t)g(a^4t), ({}^{\mathsf{T}}(g_1(t)g(a^4t))L_0, ({}^{\mathsf{T}}(g_1(t)g(a^4t))t^{-1})\circ {}^{\mathsf{T}}((g_1(t),a)\circ\gamma\circ h^{-1})^{-1}] \\
=\ &[g_1(t)g(a^4t), ({}^{\mathsf{T}}g(a^4t){}^{\mathsf{T}}g_1(t)L_0, ({}^{\mathsf{T}}g(a^4t){}^{\mathsf{T}}g_1(t)t^{-1})\circ({}^{\mathsf{T}}g_1(t)^{-1},a)\circ {}^{\mathsf{T}}\gamma^{-1}\circ {}^{\mathsf{T}}h)] \\
=\ &[g_1(t)g(a^4t), ({}^{\mathsf{T}}g(a^4t)L_0, (({}^{\mathsf{T}}g(a^4t)t^{-1},a)\circ {}^{\mathsf{T}}\gamma^{-1}\circ {}^{\mathsf{T}}h)] \\
=\ &[g_1(t)g(a^4t), ((1,a){}^{\mathsf{T}}g(t)L_0, ((1,a){}^{\mathsf{T}}g(t)t^{-1}a^4)\circ {}^{\mathsf{T}}\gamma^{-1}\circ {}^{\mathsf{T}}h)] \\
=\ &[g_1(t)g(a^4t), ((1,a){}^{\mathsf{T}}g(t)L_0, ((1,a){}^{\mathsf{T}}g(t)t^{-1})\circ {}^{\mathsf{T}}\gamma^{-1}\circ({}^{\mathsf{T}}ha^4))] \\
=\ &[g_1(t)g(a^4t), (h,1,a)\cdot({}^{\mathsf{T}}g(t)L_0, ({}^{\mathsf{T}}g(t)t^{-1})\circ {}^{\mathsf{T}}\gamma^{-1})] \\
=\ &(h, g_1(t), a)\cdot[g(t), ({}^{\mathsf{T}}g(t)L_0, {}^{\mathsf{T}}g(t)t^{-1}\circ {}^{\mathsf{T}}\gamma^{-1})].
\end{aligned}$$

Therefore we have $\eta'((h,g_1(t),a)\cdot(g(t)L_0,\gamma)) = (h,g_1(t),a)\cdot\eta'(g(t)L_0,\gamma)$. ☐

Let $\psi_2'\colon GL_n(\mathcal{K}) \times^{GL_n(\mathcal{O})} (\mathcal{M}_n^d)^{\sigma_1} \to (\mathcal{N}_n^d)^{\sigma_1}$ be a morphism defined by the assignment $[g(t),(L,\gamma)] \mapsto \gamma^{-1}\circ t|_{L_0/L}\circ\gamma$. We can easily check that this is well-defined and $GL_d(\mathbb{C})\times GL_n(\mathcal{O})\rtimes\mathbb{C}^\times$-equivariant. On the other hand, the transpose map $x \mapsto {}^{\mathsf{T}}x$ induces a $GL_d(\mathbb{C})\times\mathbb{C}^\times$-equivariant morphism $\tau\colon \mathcal{N}_n^d \to (\mathcal{N}_n^d)^{\sigma_1}$.

Lemma 4.21. *The following diagram commutes:*

(4.6)
$$\begin{array}{ccccc}
\mathrm{Gr}_n^d & \xleftarrow{\psi_1} & \mathcal{M}_n^d & \xrightarrow{\psi_2} & \mathcal{N}_n^d \\
\downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta'} & & \downarrow{\scriptstyle\tau} \\
GL_n(\mathcal{K})\times^{GL_n(\mathcal{O})}(\mathrm{Gr}_n^d)^{\sigma_1} & \xleftarrow{(\mathrm{id}\times\psi_1)} & GL_n(\mathcal{K})\times^{GL_n(\mathcal{O})}(\mathcal{M}_n^d)^{\sigma_1} & \xrightarrow{\psi_2'} & (\mathcal{N}_n^d)^{\sigma_1},
\end{array}$$

where all arrows are $GL_d(\mathbb{C})\times GL_n(\mathcal{O})\rtimes\mathbb{C}^\times$-equivariant.

Proof. The commutativity of the left square is obvious from the definitions. The commutativity of the right square follows from the relation

$${}^{\mathsf{T}}(\gamma^{-1}\circ t|_{L_0/g(t)L_0}\circ\gamma) = {}^{\mathsf{T}}\gamma\circ t|_{t^{\mathsf{T}}g(t)^{-1}L_0/L_0}\circ {}^{\mathsf{T}}\gamma^{-1},$$

which holds for any $(g(t)L_0,\gamma)\in\mathcal{M}_n^d$. ☐

Proof of Proposition 4.19. Using the commutative diagram (4.6), we have

$$\begin{aligned}\eta^*(\mathcal{O}_{\mathrm{Gr}_n^d}\widetilde{\boxtimes}\psi^*(\mathcal{F})^{\sigma_1}) &\cong (\psi_1^*)^{-1}(\eta')^*(\mathcal{O}_{\mathcal{M}_n^d}\widetilde{\boxtimes}\psi_2^*(\mathcal{F}^{\sigma_1}))\\ &= (\psi_1^*)^{-1}(\eta')^*(\psi_2')^*\mathcal{F}^{\sigma_1}\\ &= (\psi_1^*)^{-1}\psi_2^*\tau^*(\mathcal{F}^{\omega^{-2}})^{\sigma}\\ &\cong \psi^*\tau^*(\mathcal{F}^{\omega^{-2}})^{\sigma},\end{aligned}$$

and hence $\eta^*(\mathcal{O}_{\mathrm{Gr}_n^d}\widetilde{\boxtimes}\Psi(\mathcal{F})^{\sigma_1}) \simeq \Psi\circ\tau^*(\mathcal{F}^{\omega^{-2}})^{\sigma}$. Applying the duality functor $\mathbb{D}_{\mathrm{Gr}_n^d}$, we have

$$\iota\circ\Psi(\mathcal{F}) = \mathbb{D}_{\mathrm{Gr}_n^d}\circ\eta^*(\mathcal{O}_{\mathrm{Gr}_n^d}\widetilde{\boxtimes}\Psi(\mathcal{F})^{\sigma_1}) \cong \Psi\circ\mathbb{D}_{\mathcal{N}_n^d}\circ\tau^*(\mathcal{F}^{\omega^{-2}})^{\sigma} = \Psi\circ\iota(\mathcal{F}^{\omega^{-2}}),$$

where we used Proposition 4.12(3) and the relation (4.2). □

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