REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 25, Pages 166–172 (March 2, 2021)
https://doi.org/10.1090/ert/562

REDUCING MOD p COMPLEX REPRESENTATIONS OF FINITE REDUCTIVE GROUPS

G. LUSZTIG

Dedicated to the memory of Jim Humphreys

Abstract. We state a conjecture on the reduction modulo the defining characteristic of a unipotent representation of a finite reductive group.

Introduction

0.1. Let $\mathbf{k}$ be an algebraic closure of the finite field with p elements (p is a prime number). Let μ be the group of roots of 1 in $\mathbf{C}$. We fix a surjective homomorphism $\phi : \mu \to \mathbf{k}^*$ whose kernel is the set of roots of 1 of order a power of p. Let Γ be a finite group. Let $\mathcal{R}\Gamma$ (resp. $\mathcal{R}_p\Gamma$) be the Grothendieck group of virtual (finite dimensional) representations of Γ over $\mathbf{C}$ (resp. over $\mathbf{k}$) and let $\mathcal{R}^+\Gamma$ (resp. $\mathcal{R}_p^+\Gamma$) be the subset of $\mathcal{R}\Gamma$ (resp. $\mathcal{R}_p\Gamma$) given by actual representations of Γ over $\mathbf{C}$ (resp. over $\mathbf{k}$). Following Brauer and Nesbitt [BN] there is a well defined map $\rho \mapsto \underline{\rho}$ from $\mathcal{R}^+\Gamma$ to $\mathcal{R}_p^+\Gamma$ characterized by the following property: for any $g \in \Gamma$ the eigenvalues of g on $\underline{\rho}$ are obtained by applying ϕ to the eigenvalues of g on ρ. We say that $\underline{\rho}$ is the reduction modulo p of ρ. In the remainder of this paper we assume that $\Gamma = G(F_p)$ is the group of F_p-rational points of an almost simple simply connected linear algebraic group G over $\mathbf{k}$ with a given split F_p-structure with p sufficiently large. Our goal is to present some remarks on the map $\rho \mapsto \underline{\rho}$ in this case.

0.2. Assume that $G = SL_2(\mathbf{k})$. Assume that $\rho \in \mathcal{R}^+\Gamma$ is irreducible. If ρ has dimension $1, p, (p+1)/2$ or $(p-1)/2$, then $\underline{\rho}$ is irreducible. If $\dim \rho = p+1$, then $\underline{\rho}$ has two composition factors, of dimension $c, p+1-c$ with $2 \le c \le (p-1)/2$ (and any such c occurs). If $\dim \rho = p-1$, then either $\underline{\rho}$ has two composition factors, of dimension $c, p-1-c$ with $2 \le c \le (p-3)/2$ (and any such c occurs) or $\underline{\rho}$ is irreducible. These results can be found in the paper [BN] of Brauer and Nesbitt (they actually consider the group $PSL_2(F_p)$ instead of $SL_2(F_p)$ but their method applies also to $SL_2(F_p)$).

0.3. Assume that $G = SL_3(\mathbf{k})$. In the case where ρ is an irreducible representation in $\mathcal{R}^+\Gamma$ which has a line stable under the upper triangular subgroup, the complete description of the composition factors of $\underline{\rho}$ was given in [CL] (written in 1973). For one of the cuspidal irreducible representations ρ of Γ, $\underline{\rho}$ has exactly two composition factors (except when $p = 2$ when $\underline{\rho}$ is irreducible), as stated in [L1] (where the case $p = 2$ was overlooked); one has dimension $p(p-1)(2p-1)/2$ and the other (when $p > 2$) has dimension $(p-1)(p-2)/2$. This is analogous to the cuspidal irreducible

Received by the editors July 31, 2020, and, in revised form, December 4, 2020.
2020 *Mathematics Subject Classification.* Primary 20G99.
The author was supported by NSF grant DMS-1566618.

representation of $SL_2(F_p)$ for which $\underline{\rho}$ is irreducible. In the case where ρ is in one of the three main series of irreducible representations of Γ, a description of $\underline{\rho}$ was given by Humphreys in [H1], [H2].

0.4. For general G let $\mathfrak{U}$ be the set of unipotent representations of Γ (up to isomorphism). A study of the map $\rho \mapsto \underline{\rho}$ in the case where ρ is one of the irreducible representations of Γ attached in [DL, 1.9] to a generic character of a "maximal torus" of Γ appears in Jantzen's paper [J1]; a study of the map $\rho \mapsto \underline{\rho}$ in the case where $\rho \in \mathfrak{U}$ appears in Jantzen's paper [J2]. (The notion of unipotent representation of Γ is defined in [DL, 7.8].)

0.5. In unpublished notes written in 1978, the author gave a conjectural description (on the level of dimensions only) of $\underline{\rho}$ as an explicit linear combination of Weyl modules (see 1.1) in the case where $\rho \in \mathfrak{U}$ and G has type B_2, G_2, A_3, A_4; for types A_1, A_2 this was known earlier, see 0.2, 0.3. Later the author found that this description has been proved to be correct when G has type B_2 by Jantzen [J3] or type G_2 by Mertens [M]. (A copy of [M] was provided to the author by J. Humphreys.) Recently the author understood that the conjectural description in 1978 can be partly explained by a surprising (conjectural) general pattern which will be described in this paper. Namely, there should exist a family of objects $M_w \in \mathcal{R}_p^+\Gamma$ indexed by the "near involutions" (see 1.2) w in W such that for any $\rho \in \mathfrak{U}$, $\underline{\rho}$ is an explicit linear combination of M_w with w near involutions in the two-sided cell determined by ρ; the coefficients are natural numbers whose definition involves among other things the character table of the J-ring associated to the Weyl group.

1. Recollections

1.1. Let B be a Borel subgroup of G defined over F_p; let T be a maximal torus of B defined and split over F_p. Let X be the group of characters $T \to \mathbf{k}^*$ with group operation written as addition. For any $\lambda \in X$ there is (up to isomorphism) at most one irreducible rational G-module $\mathbf{L}(\lambda)$ (over $\mathbf{k}$) such that T acts on some B-stable line in $\mathbf{L}(\lambda)$ through the character λ; this is uniquely defined up to isomorphism. Let X^+ be the set of all $\lambda \in X$ for which $\mathbf{L}(\lambda)$ is defined. There is a unique $\mathbf{Z}$-basis $\{\varpi_i; i \in I\}$ of X such that $X^+ = \sum_{i \in I} \mathbf{N}\varpi_i$. For $I' \subset I$ we set $\lambda_{I'} = \sum_{i \in I'} \varpi_i \in X^+$.

For $\lambda \in X^+$ let $\mathbf{V}(\lambda)$ be a rational G-module (over $\mathbf{k}$) whose character (an element of the group ring $\mathbf{Z}[X]$) is the same as that of the characteristic 0 analogue of $\mathbf{L}(\lambda)$; it is given by the Weyl character formula. Note that $\mathbf{V}(\lambda)$ is well defined up to rearrangement of its composition factors. Let X_p^+ be the set of all $\lambda \in X^+$ of the form $\sum_{i \in I} n_i \varpi_i$ with $0 \le n_i \le p-1$ for all i. For $\lambda \in X_p^+$ we denote by $V(\lambda) \in \mathcal{R}_p^+\Gamma$ and $L(\lambda) \in \mathcal{R}_p^+\Gamma$ the restriction of $\mathbf{V}(\lambda)$ and $\mathbf{L}(\lambda)$ to $\Gamma = G(F_p)$.

1.2. Let $W \subset Aut(X)$ be the Weyl group of G. For any $i \in I$ there is a unique element $s_i \in W$ such that $s_i \ne 1$ and $s_i(\varpi_j) = \varpi_j$ for any $j \in I - \{i\}$. Recall that W is a Coxeter group on the generators $\{s_i; i \in I\}$. Let $w \mapsto l(w)$ be the length function of this Coxeter group. Let w_0 be the longest element of W; let ν be its length. For any $w \in W$ let $\mathcal{L}(w) = \{i \in I; l(s_i w) < l(w)\}$.

Let $u^{1/2}$ be an indeterminate and let H be the free $\mathbf{Q}[u^{1/2}, u^{-1/2}]$-module with basis $\{T_w; w \in W\}$ and with an algebra structure as in [L3, 3.3]. Let $\hat{W}$ be the set of all irreducible W-module E over $\mathbf{Q}$ (up to isomorphism). For $E \in \hat{W}$ let $E(u)$ be an H-module (free as a $\mathbf{Q}[u^{1/2}, u^{-1/2}]$-module) associated to E as in [L2, 1.1]. There is a well defined integer $a_E \geq 0$ such that for $w \in W$ we have

$$\mathrm{tr}(u^{-l(w)/2}T_w, E(u)) = (-1)^{l(w)} c_{w,E} u^{-a_E/2} \mod u^{(-a_E+1)/2}\mathbf{Z}[u^{1/2}]$$

where $c_{w,E} \in \mathbf{Z}$ for all w and $c_{w,E} \neq 0$ for some $w \in W$. (One can interpret $c_{w,E}$ in terms of the character of the irreducible representation associated to E of the J-ring of W at the basis element of the J-ring corresponding to w, see [L4, 3.5(b)].) For $w \in W$ we set $\alpha_w = \sum_{E \in \hat{W}} c_{w,E} E$, a virtual representation of W. Let $\mathcal{J}$ be the set of "near involutions" of W that is the set of all $w \in W$ such that w, w^{-1} are in the same left cell of W. (If W is of classical type, $\mathcal{J}$ is exactly the set of involutions in W.) According to [L4, 3.5] for $w \in W$ we have

$w \in \mathcal{J}$ *if and only if* $\alpha_w \neq 0$.

For $w \in W$ let $R_w \in \mathcal{R}\Gamma$ be as in [DL, 1.5]. By [L3, 6.17], for $w \in W$ there is a well defined object $R_{\alpha_w} \in \mathcal{R}^+\Gamma$ such that

$$\sharp(W) R_{\alpha_w} = \sum_{E \in \hat{W}, y \in W} \mathrm{tr}(y, E) c_{w,E} R_y$$

in $\mathcal{R}\Gamma$. Note that R_{α_w} is zero unless $w \in \mathcal{J}$.

An irreducible representation ρ of Γ (over $\mathbf{C}$) is in $\mathfrak{U}$ if and only if the multiplicity $(\rho : R_{\alpha_w})$ is nonzero for some $w \in \mathcal{J}$.

1.3. In the examples below (types A_1, A_2, B_2, G_2, A_3) we write $I = \{1, 2, \dots\}$ where the notation is such that

(type A_1) if $\lambda = (a-1)\varpi_1$ with $a \geq 1$ then $\dim \mathbf{V}(\lambda) = a$;

(type A_2) if $\lambda = (a-1)\varpi_1 + (b-1)\varpi_2$ with $a \geq 1, b \geq 1$ then $\dim \mathbf{V}(\lambda) = ab(a+b)/2$;

(type B_2) if $\lambda = (a-1)\varpi_1 + (b-1)\varpi_2$ with $a \geq 1, b \geq 1$ then $\dim \mathbf{V}(\lambda) = ab(a+b)(a+2b)/6$;

(type G_2) if $\lambda = (a-1)\varpi_1 + (b-1)\varpi_2$ with $a \geq 1, b \geq 1$ then $\dim \mathbf{V}(\lambda) = ab(a+b)(a+2b)(a+3b)(2a+3b)/120$;

(type A_3) if $\lambda = (a-1)\varpi_1 + (b-1)\varpi_2 + (c-1)\varpi_3$ with $a \geq 1, b \geq 1, c \geq 1$ then $\dim \mathbf{V}(\lambda) = abc(a+b)(b+c)(a+b+c)/12$.

For a sequence $i_1, i_2, \dots$ in I we often write $w = i_1 i_2 \dots$ instead of $w = s_{i_1} s_{i_2} \dots$; we write $\emptyset$ instead of w where w is the unit element of W. We now describe the elements R_{α_w} in several examples. For $\rho \in \mathfrak{U}$ we write $d(\rho) = \dim \rho$.

Type A_1, $I = \{1\}$. We have $\mathcal{J} = \{\emptyset, 1\}$, $\mathfrak{U} = \{1, S\}$ where $d(1) = 1, d(S) = p$ and

$$R_{\alpha_\emptyset} = 1, R_{\alpha_1} = S.$$

Type A_2, $I = \{1, 2\}$. We have $\mathcal{J} = \{\emptyset, 1, 2, 121\}$, $\mathfrak{U} = \{1, r, S\}$ where $d(1) = 1, d(r) = p^2 + p, d(S) = p^3$ and

$$R_{\alpha_\emptyset} = 1, R_{\alpha_1} = R_{\alpha_2} = r, R_{\alpha_{121}} = S.$$

Type B_2, $I = \{1, 2\}$. We have $\mathcal{J} = \{\emptyset, 1, 2, 121, 212, 1212\}$, $\mathfrak{U} = \{1, r, e_1, e_2, \theta, S\}$ where

$$d(1) = 1, d(r) = p(p+1)^2/2, d(e_1) = d(e_2) = p(p^2+1)/2, d(\theta) = p(p-1)^2/2,$$
$$d(S) = p^4,$$

$$R_{\alpha_\emptyset} = 1, R_{\alpha_1} = r + e_1, R_{\alpha_2} = r + e_2, R_{\alpha_{121}} = \theta + e_2, R_{\alpha_{212}} = \theta + e_1, R_{\alpha_{1212}} = S.$$

Type G_2, $I = \{1, 2\}$. We have $\mathcal{J} = \{\emptyset, 1, 2, 121, 212, 12121, 21212, 121212\}$, $\mathfrak{U} = \{1, r, r', e_1, e_2, e', f, g, h, S\}$ where

$$d(1) = 1, d(r) = p(p+1)^2(p^2+p+1)/6, d(r') = p(p+1)^2(p^2-p+1)/2,$$
$$d(e_1) = d(e_2) = p(p^4+p^2+1)/3, d(e') = p(p-1)^2(p^2-p+1)/6,$$
$$d(f) = p(p-1)^2(p^2+p+1)/2, d(g) = d(h) = p(p^2-1)^2/3, d(S) = p^6,$$

$$R_{\alpha_\emptyset} = 1, R_{\alpha_1} = r + r' + e_1, R_{\alpha_2} = r + r' + e_2,$$
$$R_{\alpha_{121}} = r' + e_2 + f + g + h, R_{\alpha_{212}} = r' + e_1 + f + g + h, R_{\alpha_{12121}} = e_1 + e' + f,$$
$$R_{\alpha_{21212}} = e_2 + e' + f, R_{\alpha_{121212}} = S.$$

Type A_3, $I = \{1, 2, 3\}$. We have $\mathcal{J} = \{\emptyset, 1, 2, 3, 13, 121, 232, 2132, 13231, 121321\}$, $\mathfrak{U} = \{1, r, r', r'', S\}$ where

$$d(1) = 1, d(r) : p^3 + p^2 + p, d(r') : p^4 + p^2, d(r'') = p^5 + p^4 + p^3, d(S) : p^6,$$

$$R_{\alpha_\emptyset} = 1, R_{\alpha_1} = R_{\alpha_2} = R_{\alpha_3} = r, R_{\alpha_{13}} = R_{\alpha_{2132}} = r',$$
$$R_{\alpha_{121}} = R_{\alpha_{13231}} = R_{\alpha_{232}} = r'', R_{\alpha_{121321}} = S.$$

2. The elements $M_w \in \mathcal{R}_p^+\Gamma$ for $w \in \mathcal{J}$

2.1. In each of the examples in 1.3 and for any $w \in \mathcal{J}$ we define a virtual representation $M_w \in \mathcal{R}_p\Gamma$ as a certain integer combination of objects $V(\lambda)$. If $I = \{1, 2, \dots, s\}$ we write $V_{n_1,n_2,\dots,n_s}$ instead of $V(\lambda)$ where $\lambda = n_1\varpi_1 + n_2\varpi_2 + \cdots + n_s\varpi_s$. We set $\delta(w) = \dim(M_w)$.

Type A_1: $M_\emptyset = V_0$, $M_1 = V_{p-1}$; $\delta(\emptyset) = 1, \delta(1) = p$.

Type A_2: $M_\emptyset = V_{0,0}$, $M_1 = V_{p-1,0}$, $M_2 = V_{0,p-1}$, $M_{121} = V_{p-1,p-1}$;
$\delta(\emptyset) = 1, \delta(1) = \delta(2) = p(p+1)/2, d(121) = p^3$.

Type B_2: $M_\emptyset = V_{0,0}$, $M_1 = V_{p-1,0}$, $M_2 = V_{0,p-1}$, $M_{121} = V_{p-3,0}$, $M_{212} = V_{0,p-2}$, $M_{1212} = V_{p-1,p-1}$;
$d(\emptyset) = 1$,
$\delta(1) = p(p+1)(p+2)/6$,
$\delta(2) = p(p+1)(2p+1)/6$,
$\delta(121) = p(p-1)(p-2)/6$,
$\delta(212) = p(p-1)(2p-1)/6$,
$\delta(1212) = p^4$.

Type G_2: $M_\emptyset = V_{0,0}$, $M_1 = V_{p-1,0}$, $M_2 = V_{0,p-1}$, $M_{121} = V_{p-4,1}$, $M_{212} = V_{1,p-2}$, $M_{12121} = V_{p-4,0}$, $M_{21212} = V_{0,p-2}$, $M_{121212} = V_{p-1,p-1}$;
$\delta(\emptyset) = 1$,
$\delta(1) = p(p+1)(p+2)(p+3)(2p+3)/120$,
$\delta(2) = p(p+1)(2p+1)(3p+1)(3p+2)/120$,
$\delta(121) = p(p-1)(p+1)(p-3)(p+3)/30$,
$\delta(212) = p(p-1)(p+1)(3p-1)(3p+1)/30$,
$\delta(12121) = p(p-1)(p-2)(p-3)(2p-3)/120$,
$\delta(21212) = p(p-1)(2p-1)(3p-1)(3p-2)/120$,
$\delta(121212) = p^6$.

Type A_3:
$M_\emptyset = V_{0,0,0}$,
$M_1 = V_{p-1,0,0}$,
$M_2 = V_{0,p-1,0} - V_{0,p-3,0}$,

$M_3 = V_{0,0,p-1}$,
$M_{13} = V_{p-1,0,p-1} - V_{p-2,0,p-2}$,
$M_{2132} = V_{0,p-1,0} + V_{0,p-3,0}$,
$M_{121} = V_{p-1,p-1,0}$,
$M_{13231} = V_{p-1,0,p-1} + V_{p-2,0,p-2}$
$M_{232} = V_{0,p-1,p-1}$,
$M_{121321} = V_{p-1,p-1,p-1}$;
$\delta(\emptyset) = 1$,
$\delta(1) = \delta(3) = p(p+1)(p+2)/6$,
$\delta(2) = p(p+1)^2(p+2)/12 - p(p-1)^2(p-2)/12 = p(2p^2+1)/3$,
$\delta(13) = p^2(p+1)(2p+1)/12 - p^2(p-1)(2p-1)/12 = p^2(5p^2+1)/6$,
$\delta(2132) = p(p+1)^2(p+2)/12 + p(p-1)^2(p-2)/12 = p^2(p^2+5)/6$,
$\delta(121) = \delta(232) = p^3(p+1)(2p+1)/6$,
$\delta(13231) = p^2(p+1)(2p+1)/12 + p^2(p-1)(2p-1)/12 = p^3(p^2+2)/3$,
$\delta(121321) = p^6$.

Note that in each of the cases above we have $M_w \in \mathcal{R}_p^+\Gamma$. (This is obvious except for type A_3 and $w = 2$ or $w = 13$ where it can be verified directly.)

2.2. For any $\rho \in \mathfrak{U}$ we write $\underline{\rho}$ (with $\rho \in \mathfrak{U}$) as an $\mathbf{N}$-linear combination of M_w (in $\mathcal{R}_p\Gamma$) in each case in 1.3.

Type A_1: $\underline{1} = M_\emptyset$, $\underline{S} = M_1$.

Type A_2: $\underline{1} = M_\emptyset$, $\underline{r} = M_1 + M_2$, $\underline{S} = M_{121}$.

Type B_2: $\underline{1} = M_\emptyset$, $\underline{r} = M_1 + M_2$, $\underline{e_1} = M_1 + M_{212}$, $\underline{e_2} = M_{121} + M_2$, $\underline{\theta} = M_{121} + M_{212}$, $\underline{S} = M_{1212}$.

Type G_2: $\underline{1} = M_\emptyset$, $\underline{r} = M_1 + M_2$, $\underline{r'} = M_1 + M_{121} + M_2 + M_{212}$, $\underline{e_1} = M_1 + M_{12121} + M_{212}$, $\underline{e_2} = M_{121} + M_2 + M_{21212}$, $\underline{e'} = M_{12121} + M_{21212}$, $\underline{f} = M_{121} + M_{12121} + M_{212} + M_{21212}$, $\underline{g} = \underline{h} = M_{121} + M_{212}$, $\underline{S} = M_{121212}$.

Type A_3: $\underline{1} = M_\emptyset$, $\underline{r} = M_1 + M_2 + M_3$, $\underline{r'} = M_{13} + M_{2132}$, $\underline{r''} = M_{121} + M_{13231} + M_{232}$, $\underline{S} = M_{121321}$.

We return to a general G. We state the following

Conjecture 2.3. *There exist nonzero objects $M_w \in \mathcal{R}_p^+\Gamma$, $(w \in \mathcal{J})$ such that for any $\rho \in \mathfrak{U}$ we have*

(a) $\underline{\rho} = \sum_{w \in \mathcal{J}} (\rho : R_{\alpha_w}) M_w$.

Moreover, we can assume that the following properties hold.

(i) For any $w \in \mathcal{J}$, M_w is a $\mathbf{Z}$-linear combination of V_λ with λ very close to $(p-1)\lambda_{\mathcal{L}(w)}$ (see 1.1*).*

(ii) We have $\dim(M_w) = \pi_w(p)$ *where $\pi_w(t) \in \mathbf{Q}[t]$ (t an indeterminate) is independent of p. There exists an involution $w \leftrightarrow \tilde{w}$ of $\mathcal{J}$ such that $t^\nu \pi_w(1/t) = \pm\pi_{\tilde{w}}(t)$ and $\mathcal{L}(\tilde{w}) = I - \mathcal{L}(w)$ for all $w \in \mathcal{J}$.*

(iii) For $w \in \mathcal{J}$ we write $\pi_w(t) \in t^{c(w)}\mathbf{Q}[t], \pi_w(t) \notin t^{c(w)-1}\mathbf{Q}[t]$ where $c(w) \in \mathbf{N}$ is well defined. Then $c(w)$ depends only on the two-sided cell of W containing w; it is the value of the a-function (see [L4, 3.1]*) of W on that two-sided cell.*

A similar statement can be made when F_p is replaced by the finite field F_{p^n} with p^n elements for some $n \geq 2$.

The conjecture does not say what the M_w are explicitly.

2.4. By the results in 2.1, 2.2 the conjecture holds for types A_1, A_2, B_2, G_2, A_3. For these types, the involution $w \leftrightarrow \tilde{w}$ in 2.3(ii) is given as follows:

Type A_1: $\emptyset \leftrightarrow 1$;
Type A_2: $\emptyset \leftrightarrow 121$, $1 \leftrightarrow 2$;
Type B_2: $\emptyset \leftrightarrow 1212$, $1 \leftrightarrow 2$, $121 \leftrightarrow 212$;
Type G_2: $\emptyset \leftrightarrow 121212$, $1 \leftrightarrow 2$, $121 \leftrightarrow 212$; $12121 \leftrightarrow 21212$;
Type A_3: $\emptyset \leftrightarrow 121321$, $1 \leftrightarrow 232$, $2 \mapsto 13231$, $3 \mapsto 121$, $13 \leftrightarrow 2132$.
Similar evidence exists for type A_4.

2.5. Here is the simplest nontrivial example of objects M_w in 2.3. Assume that V is a three dimensional F_p-vector space and $\Gamma = SL(V)$. Let Z_1 be the set of lines in V. Let Z_2 be the set of planes in V. Let $\mathcal{F}_1$ be the vector space of functions $Z_1 \to \mathbf{k}$ with sum of values equal to 0. Let $\mathcal{F}_2$ be the vector space of functions $Z_2 \to \mathbf{k}$ with sum of values equal to 0. Note that $\mathcal{F}_1$, $\mathcal{F}_2$ are naturally Γ-modules; they both represent $\underline{\rho}$ where $\rho \in \mathfrak{U}$ has dimension $p^2 + p$. Define $\tau : \mathcal{F}_1 \to \mathcal{F}_2$ by $(\tau(f)(P) = \sum_{L \in Z_1; L \subset P} f(L)$ where $f \in \mathcal{F}_1, P \in Z_2$. Define $\tau' : \mathcal{F}_2 \to \mathcal{F}_1$ by $(\tau'(f')(L) = \sum_{P \in Z_2; L \subset P} f'(P)$ where $f' \in \mathcal{F}_2, L \in Z_1$. Note that τ, τ' are well defined Γ-linear maps. Let M be the kernel of τ (it is also the image of τ'). Let M' be the kernel of τ' (it is also the image of τ). Then M, M' are the objects M_w attached to ρ in 2.3.

2.6. If in the sum 2.3(a) we replace each M_w by the basis element t_w of the J-ring of W (see [L4, 3.5]), the resulting element of the J-ring is contained in the centre of that ring.

2.7. A statement similar to 2.3(a) can be made for any, not necessarily unipotent, irreducible representation ρ of Γ. (We replace the J-ring of W and the left cells of W by the analogous ring $H^\infty_{\mathcal{O}}$ and the left cells considered in [L5, 1.9] in terms of an extended Weyl group.) We illustrate this in an example.

Let $\mathcal{O}$ be an orbit for the obvious W-action on $X/(p-1)X$ such that the stabilizer in W of any element of $\mathcal{O}$ is trivial. For any $\zeta \in \mathcal{O}$ there is a unique element $\tilde{\zeta} \in X_p^+$ whose image under $X \to X/(p-1)X$ equals ζ. Let $\zeta_0 \in \mathcal{O}$. Let $\tilde{\zeta}_0'$ be the composition $T \xrightarrow{\tilde{\zeta}_0} \mathbf{k}^* \xrightarrow{\phi'} \mathbf{C}^*$ where ϕ' is the homomorphism such that $\phi(\phi'(x)) = x$ for any $x \in \mathbf{k}^*$ (ϕ as in 0.1). We can restrict $\tilde{\zeta}_0$ to $T(F_p)$ and we regard this restriction as a homomorphism $B(F_p) \to \mathbf{C}^*$ trivial on the Sylow p-subgroup of $B(F_p)$. This last homomorphism can be induced to a representation ρ of Γ over $\mathbf{C}$ which is in fact irreducible and depends only only on $\mathcal{O}$, not on ζ_0. From the results of [CL], $\underline{\rho}$ has each of $L(\tilde{\zeta}), (\zeta \in \mathcal{O})$ as a composition factor (but it may also have other composition factors). We expect that

(a) $\underline{\rho} = \sum_{\zeta \in \mathcal{O}} M_\zeta$

where $M_\zeta \in \mathcal{R}_p^+\Gamma$ is a $\mathbf{Z}$-linear combination of various $V(\lambda)$ with $\lambda \in X_p^+$ very close to $\tilde{\zeta}$.

In the case where $G = SL_3(\mathbf{k})$ such a statement can be deduced from [CL] (in this case we have $M_\zeta = V(\tilde{\zeta})$ for each $\zeta \in \mathcal{O}$); in the case where $G = Sp_4(\mathbf{k})$, a statement like (a) can be deduced from [J3].

References

[BN] R. Brauer and C. Nesbitt, *On the modular characters of groups*, Ann. of Math. (2) **42** (1941), 556–590, DOI 10.2307/1968918. MR4042

[CL] R. W. Carter and G. Lusztig, *Modular representations of finite groups of Lie type*, Proc. London Math. Soc. (3) **32** (1976), no. 2, 347–384, DOI 10.1112/plms/s3-32.2.347. MR396731

[DL] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161, DOI 10.2307/1971021. MR393266

[H1] James E. Humphreys, *Ordinary and modular representations of Chevalley groups*, Lecture Notes in Mathematics, Vol. 528, Springer-Verlag, Berlin-New York, 1976. MR0453884

[H2] J. E. Humphreys, *Ordinary and modular characters of* $\mathrm{SL}(3, p)$, J. Algebra **72** (1981), no. 1, 8–16, DOI 10.1016/0021-8693(81)90309-4. MR634614

[J1] Jens Carsten Jantzen, *Zur Reduktion modulo p der Charaktere von Deligne und Lusztig* (German), J. Algebra **70** (1981), no. 2, 452–474, DOI 10.1016/0021-8693(81)90229-5. MR623819

[J2] Jens Carsten Jantzen, *Zur Reduktion modulo p unipotenter Charaktere endlicher Chevalley-Gruppen* (German), Math. Z. **181** (1982), no. 1, 97–128, DOI 10.1007/BF01214985. MR671718

[J3] J. C. Jantzen, *Representations of Chevalley groups in their own characteristic*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), Proc. Sympos. Pure Math., vol. 47, Amer. Math. Soc., Providence, RI, 1987, pp. 127–146, DOI 10.1016/s0022-4049(99)00142-5. MR933356

[L1] G. Lusztig, *On the discrete representations of the general linear groups over a finite field*, Bull. Amer. Math. Soc. **79** (1973), 550–554, DOI 10.1090/S0002-9904-1973-13198-2. MR315010

[L2] George Lusztig, *Unipotent characters of the symplectic and odd orthogonal groups over a finite field*, Invent. Math. **64** (1981), no. 2, 263–296, DOI 10.1007/BF01389170. MR629472

[L3] George Lusztig, *Characters of reductive groups over a finite field*, Annals of Mathematics Studies, vol. 107, Princeton University Press, Princeton, NJ, 1984. MR742472

[L4] G. Lusztig, *Leading coefficients of character values of Hecke algebras*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), Proc. Sympos. Pure Math., vol. 47, Amer. Math. Soc., Providence, RI, 1987, pp. 235–262, DOI 10.1007/bf01389157. MR933415

[L5] G. Lusztig, *Conjugacy classes in reductive groups and two-sided cells*, Bull. Inst. Math. Acad. Sin. (N.S.) **14** (2019), no. 3, 265–293, DOI 10.21915/bimas.2019301. MR4033057

[M] D. Mertens, *Zur Darstellungstheorie der endlicher Chevalley-Gruppen von typ* G_2, Diplomarbeit (1985), Universität Bonn.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Email address: gyuri@mit.edu