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ON INDUCTION OF CLASS FUNCTIONS

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Abstract. Let G be a connected reductive group defined over a finite field $\mathbf{F}_q$ and let L be a Levi subgroup (defined over $\mathbf{F}_q$) of a parabolic subgroup P of G. We define a linear map from class functions on $L(\mathbf{F}_q)$ to class functions on $G(\mathbf{F}_q)$. This map is independent of the choice of P. We show that for large q this map coincides with the known cohomological induction (whose definition involves a choice of P).

Introduction

0.1. Let G be a connected reductive algebraic group over $\mathbf{k}$, an algebraic closure of the finite field F_q with q elements, with a fixed F_q-rational structure whose Frobenius map is denoted by $F : G \to G$.

Let $\Lambda(G)$ be the set of subgroups M of G such that M is a Levi subgroup of a parabolic subgroup of G; for $M \in \Lambda(G)$ let $\Pi(M)$ be the set of parabolic subgroups P of G for which M is a Levi subgroup. Assume that $M \in \Lambda(G)$ is defined over F_q and that $P \in \Pi(M)$ (not necessarily defined over F_q). Let $\mathcal{K}(G^F)$ (resp. $\mathcal{K}(M^F)$) be the Grothendieck group of representations of G^F (resp. M^F) over an algebraic closure $\bar{\mathbf{Q}}_l$ of the l-adic numbers where l is a prime number not dividing q. (When F acts on a set X we denote by X^F the fixed point set of $F : X \to X$.) Let $R^G_{M,P} : \mathcal{K}(M^F) \to \mathcal{K}(G^F)$ be the "induction" homomorphism defined in [DL] (in the case where M is a maximal torus) and in [L76] (in the general case). Let $cl(G^F)$ (resp. $cl(M^F)$) be the $\bar{\mathbf{Q}}_l$-vector space of class functions $G^F \to \bar{\mathbf{Q}}_l$ (resp. $M^F \to \bar{\mathbf{Q}}_l$). By passage to characters, $R^G_{M,P}$ can be regarded as a $\bar{\mathbf{Q}}_l$-linear map $R^G_{M,P} : cl(M^F) \to cl(G^F)$. In [L76] it was conjectured that

(a) $R^G_{M,P}$ *is independent of the choice of* P.

(At that time it was already known from [DL] that (a) holds when M is a maximal torus of G, so that in that case, the notation R^G_M can be used instead of $R^G_{M,P}$.) As noted in [L76], Deligne had an argument to prove (a) for any M provided that $q \gg 0$ (but his proof has not been published). In [L90, 8.13] a proof of (a) for $q \gg 0$ was given which was based on the theory of character sheaves and thus was quite different from Deligne's proof. (In *loc. cit.* there is also an assumption on the characteristic p of $\mathbf{k}$, but that assumption can be removed in view of the cleanness result for character sheaves in [L12].) In [BM] it is proved that (a) holds assuming only that $q > 2$.

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In this paper we define a $\bar{\mathbf{Q}}_l$-linear map $\mathcal{R}_M^G : cl(M^F) \to cl(G^F)$ with no restriction on q (see 1.7) and we show that

(b) *if* $q \gg 0$ *we have* $\mathcal{R}_M^G = R_{M,P}^G$ *for any* $P \in \Pi(M)$.

(See 1.9 and §2). We expect that the results of [L90] quoted in this paper hold without restriction on q and, as a consequence, that (b) holds without restriction on q.

The definition of $\mathcal{R}_M^G$ is in terms of intersection cohomology; it relies on ideas of [L84]. The proof of (b) relies on the results of [L90] connecting representations of G^F with the character sheaves on G. In §3 we show (based on results of [L90]) that if D is an F-stable conjugacy class of G^F then the function on G^F which is 1 on D^F and 0 on $G^F - D^F$ is a linear combination of characters of $R_T^G(\theta)$ for various F-stable maximal tori of G and various characters θ of T^F. (This was conjectured in [L78].)

0.2. *Notation.* Let ν_G be the dimension of the flag manifold of G. Let $\mathcal{Z}_G$ be the centre of G. For $M \in \Lambda(G)$ let $N_G M$ be the normalizer of M in G. For $g \in G$ we have $g = g_s g_u = g_u g_s$ where $g_s \in G$ is semisimple and $g_u \in G$ is unipotent. For $s \in G$ semisimple we write G_s for the centralizer of s in G. For $g \in G$ let $H_G(g)$ be the smallest subgroup in $\Lambda(G)$ that contains $G^0_{g_s}$. If G' is a subgroup of G, let $Z_G(G')$ be the centralizer of G' in G. Let G_{der} be the derived subgroup of G.

Let X be an algebraic variety over $\mathbf{k}$. Let $ls(X)$ be the collection of $\bar{\mathbf{Q}}_l$-local systems on X. If H is a connected algebraic group acting on X we denote by $ls_H(X)$ the collection of H-equivariant $\bar{\mathbf{Q}}_l$-local systems on X. Let Y be a locally closed, smooth, irreducible subvariety of X and let $\mathcal{E} \in ls(Y)$. Then $\mathcal{E}$ extends canonically as an intersection cohomology complex to the closure $\bar{Y}$ of Y and to X by 0 on $X - \bar{Y}$; the resulting complex on X is denoted by $\mathcal{E}^\sharp$. Assume now that X is defined over F_q with Frobenius map $F : X \to X$ and Y above is F-stable. Assume that $F^*\mathcal{E} \cong \mathcal{E}$ and we are given an isomorphism $\phi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. This induces an isomorphism $\phi^\sharp : F^*\mathcal{E}^\sharp \xrightarrow{\sim} \mathcal{E}^\sharp$. Let $\chi_{\mathcal{E},\phi} : X^F \to \bar{\mathbf{Q}}_l$ be the function whose value at $y \in Y^F$ is the trace of ϕ on the stalk of $\mathcal{E}$ at y and which is zero on $X^F - Y^F$ Let $\chi_{\mathcal{E}^\sharp,\phi^\sharp} : X^F \to \bar{\mathbf{Q}}_l$ be the function whose value at $x \in X^F$ is the alternating sum of traces of the linear maps induced by $\phi^\sharp$ on the stalks at x of the cohomology sheaves of $\mathcal{E}^\sharp$. We have $\chi_{\mathcal{E},\phi}|Y^F = \chi_{\mathcal{E}^\sharp,\phi^\sharp}|Y^F$.

1. The definition of $\mathcal{R}_M^G$

1.1. The conjugacy class of $g \in G$ is said to be isolated if $H(g) = G$. A subset S of G is said to be an isolated stratum if S is the inverse image of an isolated conjugacy class of $G/\mathcal{Z}_G^0$ under the obvious map $G \to G/\mathcal{Z}_G^0$. Let A_G be the set of pairs (L, S) where $L \in \Lambda(G)$ and S is an isolated stratum of L. For $(L, S) \in A_G$ we set $S_r^G = \{g \in S; H_G(g) = L\} = \{g \in S; G^0_{g_s} \subset L\}$ and $Y_{L,S}^G = \cup_{x \in G} x S_r^G x^{-1}$. Then

(a) $Y_{L,S}^G$ *is a smooth locally closed irreducible subvariety of* G *of dimension* $2\nu_G - 2\nu_L + \dim S$.

(see [L84, 3.1]). The subsets $Y_{L,S}^G$ are called the strata of G. We have

$$G = \sqcup_{(L,S) \in A_G \text{ up to G-conjugacy}} Y_{L,S}^G.$$

Note that an isolated stratum S of G is the stratum $Y_{G,S}^G$.

1.2. Let $M \in \Lambda(G)$. Let Y' be a stratum of M. We shall associate to Y' a stratum Y of G as follows.

We set $Y = Y^G_{L,S}$ where $(L,S) \in A_M$ is such that $Y' = Y^M_{L,S} = \cup_{x\in M} xS^M_r x^{-1}$. (We have also $(L,S) \in A_G$.) We set $Y'_r = \cup_{x\in M} xS^G_r x^{-1}$.

Now Y'_r and Y are independent of the choice of (L,S). (Indeed, it is enough to show that if $m \in M$, then $(mSm^{-1})^G_r = mS^G_r m^{-1}$; we use that $H_G(mgm^{-1}) = mH_G(g)m^{-1}$.) We have $Y'_r \neq \emptyset$, $Y'_r \subset Y$ and Y is the unique stratum of G that contains Y'_r. We have also $Y'_r \subset Y'$. (We use that $S^G_r \subset S^M_r$; indeed, if $g \in S$ and $H_G(g) = L$, then $G^0_{g_s} \subset L$ hence $M^0_{g_s} \subset L$ and $H_M(g) = L$.) We show that

(a) *Y'_r is open in Y'.*

Let $(L,S) \in A_M$ be such that $Y' = Y^M_{L,S}$. Now Y' is a locally trivial fibration over the variety of all M-conjugates of L, via $g \mapsto H_M(g)$; let β be the fibre of this map over L. It is enough to show that $Y'_r \cap \beta$ is open in β. We have $\beta = \cup_{n\in N_M L/L} nS^M_r n^{-1}$, $Y'_r \cap \beta = \cup_{n\in N_M L/L} nS^G_r n^{-1}$. It is enough to observe that S^G_r is open in S^M_r; in fact it is open in S.

Let $\tilde{\tilde{Y}} = \{(g,x) \in G \times G; x^{-1}gx \in Y'_r\}$. Define $\sigma : \tilde{\tilde{Y}} \to Y'_r$ by $\sigma(g,x) = x^{-1}gx$. Let $\tilde{Y} = \tilde{\tilde{Y}}/M = \{(g,xM) \in Y \times G/M; x^{-1}gx \in Y'_r\}$. We show:

(b) *$\tilde{Y}$ is a smooth, irreducible variety of dimension equal to* $\dim Y$.

Since Y'_r is smooth, irreducible of dimension equal to $\dim Y'$ (see (a), 1.1(a)), we see that $\tilde{Y}$ is smooth, irreducible of dimension $\dim G/M + \dim Y' = \dim Y$. This proves (b).

If $(g,xM) \in \tilde{Y}$, we have $g \in Y$ (since $Y'_r \subset Y$). Define $\pi : \tilde{Y} \to Y$ by $(g,xM) \mapsto g$. We show:

(c) *π is a finite unramified cover of Y with fibres isomorphic to c_G/c_M where $c_G = \{n \in N_G L, n^{-1}Sn = S\}$, $c_M = \{n \in N_M L, n^{-1}Sn = S\}$ and $(L,S) \in A_M$, $Y' = Y^M_{L,S}$.*

It is enough to show that if $g \in S^G_r$, then $\pi^{-1}(g) \cong c_G$. We can identify $\pi^{-1}(g)$ with $\{xM \in G/M; x^{-1}gx \in Y'_r\}$ hence also with $\{x \in G; \xi^{-1}gx \in S^G_r\}/\{x \in M; \xi^{-1}gx \in S^G_r\}$. It is enough to show that $\{x \in G; \xi^{-1}gx \in S^G_r\} = c_G$ (this would imply $\{x \in M; \xi^{-1}gx \in S^G_r\} = c_M$). Let $x \in G$ be such that $x^{-1}gx \in S^G_r$; then $L = H_G(x^{-1}gx) = x^{-1}H_G(g)x = x^{-1}Lx$ and $x \in N_G L$. Let S', g' be the image of S, g in $L/\mathcal{Z}^0_L$. Now $Ad(x^{-1})$ induces an automorphism of $L/\mathcal{Z}^0_L$ which carries S' to an isolated conjugacy class S'' and $g' \in S'$ to an element of S'; it follows that $S' = S''$ and $Ad(x^{-1})S = S$, so that $x \in c_G$. Conversely, let $x \in N_G L$ be such that $x^{-1}Sx = S$. Then $x^{-1}gx \in S$, $H_G(x^{-1}gx) = x^{-1}H_G(g)x = x^{-1}Lx = L$ and $x^{-1}gx \in S^G_r$. This proves (c).

1.3. We preserve the setup of 1.2. Let $\mathcal{E} \in ls_M(Y')$. We define $j^Y_{Y'}(\mathcal{E}) \in ls_G(Y)$. Note that $\sigma^*(\mathcal{E}|Y'_r) \in ls_{G\times M}(\tilde{\tilde{Y}})$ for the $G \times M$ action $(g_0,m) : (g,x) \mapsto (g_0 g g_0^{-1}, g_0 x m^{-1})$ on $\tilde{\tilde{Y}}$. Hence $\sigma^*(\mathcal{E}|Y'_r) = \sigma_1^* \mathcal{E}_1$ where $\sigma_1 : \tilde{\tilde{Y}} \to \tilde{Y}$ is the obvious map and $\mathcal{E}_1 \in ls(\tilde{Y})$ is well defined. We define $j^Y_{Y'}\mathcal{E} = \pi_*(\mathcal{E}_1) \in ls_G(Y)$; this has rank equal to c_G/c_M times the rank of $\mathcal{E}$.

1.4. We preserve the setup of 1.3. Let $(L,S) \in A_M$ be such that $Y' = Y^M_{L,S}$ and let $\mathcal{E}_0 \in ls_L(S)$. Then $j^{Y'}_S(\mathcal{E}_0) \in ls(Y')$ and $j^Y_S(\mathcal{E}_0) \in ls(Y)$ are defined as in 1.3. From the definition we see that

(a) $j^Y_{Y'}(j^{Y'}_S \mathcal{E}_0) = j^Y_S(\mathcal{E}_0)$.

1.5. Let $\mathcal{F} \in ls_G(Y_1)$ where $Y_1 = Y^G_{L,S}$. We say that $\mathcal{F}$ is *admissible* if it is irreducible and if $\mathcal{F}$ is a direct summand of $j_S^{Y_1}(\mathcal{F}_0) \in ls_G(Y_1)$ for some $\mathcal{F}_0 \in ls_L(S)$ which is cuspidal irreducible (see [L84, 2.4]). (This condition is independent of the choice of (L,S).) Let $\mathcal{A}^G(Y_1)$ be the class of G-equivariant admissible local systems on Y_1. We say that Y_1 is an admissible stratum if $\mathcal{A}^G(Y_1) \neq \emptyset$. In the setup of 1.3 we see (using 1.4(a)) that if $\mathcal{F} \in \mathcal{A}^M(Y')$ then $j^Y_{Y'}(\mathcal{F})$ is a (nonzero) direct sum of objects of $\mathcal{A}^G(Y)$. In particular, if Y' is admissible (for M) then Y is admissible (for G).

1.6. We now assume that M is defined over F_q. If Y' in 1.2 is F-stable then Y'_r in 1.2 is F-stable. Indeed, from $Y' = F(Y')$ and $Y' = Y^M_{L,S}$ we deduce $Y' = F(Y') = Y^M_{F(L),F(S)}$ hence $F(L) = mLm^{-1}$, $F(S) = mSm^{-1}$ for some $m \in M$. Replacing L, S by an M-conjugate we can assume that $F(L) = L, F(S) = S$, so that

$$F(Y'_r) = \cup_{x\in M} xF(S)^G_r x^{-1} = \cup_{x\in M} xS^G_r x^{-1} = Y'_r.$$

A similar argument shows that Y in 1.2 is F-stable; alternatively, this holds since Y is the unique stratum of G containing Y'_r (which is F-stable). Moreover, $\tilde{\tilde{Y}}, \tilde{Y}, \pi$ in 1.2 and σ, σ_1 in 1.3 are defined over F_q. If now $\mathcal{E}$ in 1.3 is such that $F^*\mathcal{E} \cong \mathcal{E}$, then $\sigma^*(\mathcal{E}|Y'_r)$ and $\mathcal{E}_1$ in 1.3 are isomorphic to their inverse image under F hence $\tilde{\mathcal{E}} := j^Y_{Y'}(\mathcal{E})$ satisfies $F^*\tilde{\mathcal{E}} \cong \tilde{\mathcal{E}}$; moreover any isomorphism $\phi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ (of local systems on Y') induces an isomorphism $F^*(\sigma^*(\mathcal{E}|Y'_r)) \xrightarrow{\sim} \sigma^*(\mathcal{E}|Y'_r)$ (of local systems on $\tilde{\tilde{Y}}$), an isomorphism $F^*\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_1$ (of local systems on $\tilde{Y}$) and an isomorphism $\tilde{\phi} : F^*\tilde{\mathcal{E}} \cong \tilde{\mathcal{E}}$ (of local systems on Y). Now $\phi, \tilde{\phi}$ extend to isomorphisms $\phi^\sharp : F^*\mathcal{E}^\sharp \xrightarrow{\sim} \mathcal{E}^\sharp$ (of complexes on M) and $\tilde{\phi}^\sharp : F^*\tilde{\mathcal{E}}^\sharp \xrightarrow{\sim} \tilde{\mathcal{E}}^\sharp$ (of complexes on G). Hence $\chi_{\mathcal{E}^\sharp,\phi^\sharp} : M^F \to \bar{\mathbf{Q}}_l$, $\chi_{\tilde{\mathcal{E}}^\sharp,\tilde{\phi}^\sharp} : G^F \to \bar{\mathbf{Q}}_l$ are well defined. They are class functions on M^F (resp. G^F).

1.7. Let Z_M be the set of all pairs $(Y', \mathcal{E})$ where Y' is an F-stable admissible stratum of M and $\mathcal{E} \in ls_M(Y')$ is admissible (up to isomorphism) such that $F^*\mathcal{E} \cong \mathcal{E}$. For any $(Y', \mathcal{E}) \in Z_M$ we denote by $\mathcal{L}_{Y',\mathcal{E}}$ the subspace of $cl(M^F)$ consisting of the class functions $\chi_{\mathcal{E}^\sharp,\phi^\sharp} : M^F \to \bar{\mathbf{Q}}_l$ where $\phi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is an isomorphism. A different choice for ϕ must be of the form $a\phi$ for some $a \in \bar{\mathbf{Q}}_l^*$ and we have $\chi_{\mathcal{E}^\sharp,(a\phi)^\sharp} = a\chi_{\mathcal{E}^\sharp,\phi^\sharp}$ hence $\mathcal{L}_{Y',\mathcal{E}}$ is well defined line in $cl(M^F)$, independent of any choice. We have

(a) $cl(M^F) = \oplus_{(Y',\mathcal{E})\in Z_M} \mathcal{L}_{Y',\mathcal{E}}$.

A proof is given in [L04, 26.5]. (Alternatively, instead of [L04], one can use [L86, 25.2] complemented by [L12].) Let

(b) $\mathcal{R}^G_M : cl(M^F) \to cl(G^F)$

be the linear map such that for any $(Y', \mathcal{E}) \in Z_M$, the restriction of $\mathcal{R}^G_M$ to the line $\mathcal{L}_{Y',\mathcal{E}}$ sends $\chi_{\mathcal{E}^\sharp,\phi^\sharp} : M^F \to \bar{\mathbf{Q}}_l$ (where $\phi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is an isomorphism) to $\chi_{\tilde{\mathcal{E}}^\sharp,\tilde{\phi}^\sharp} : G^F \to \bar{\mathbf{Q}}_l$ (see 1.6). If ϕ is replaced by $a\phi$ with $a \in \bar{\mathbf{Q}}_l^*$, then $\tilde{\phi}^\sharp$ is replaced by $a\tilde{\phi}^\sharp$. Thus the linear map $\mathcal{R}^G_M$ is well defined (independent of choices).

1.8. In the setup of 1.4 assume that $\mathcal{E}_0 \in ls_L(S)$ is cuspidal irreducible, that L, S are defined over F_q and that we are given an isomorphism $\phi_0 : F^*\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_0$ of local systems on S. Then $\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp} : L^F \to \bar{\mathbf{Q}}_l$ is well defined. Using 1.4(a) and the definitions we see that

(a) $\mathcal{R}^G_M(\mathcal{R}^M_L(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp})) = \mathcal{R}^G_L(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp}))$.

1.9. In the rest of this paper, unless otherwise specified, we assume that $q \gg 0$, so that the results of [L90] can be applied. (As mentioned in 0.1 the assumption of *loc. cit.* on the characteristic of $\mathbf{k}$, can now be removed.) We shall write $R_M^G : cl(M^F) \to cl(G^F)$ instead of $R_{M,P}^G$ (with $P \in \Pi(M)$).

Let $(L, S) \in A_M, \mathcal{E}_0, \phi_0$ be as in 1.8. The following result can be deduced from [L90, 9.2]:

(a) $\mathcal{R}_L^M(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp}) = R_L^M(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp})$.

Let $cl'(M^F)$ be the $\bar{\mathbf{Q}}_l$-subspace of $cl(M^F)$ generated by the elements $R_L^M(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp}) \in cl(M^F)$ for various $L, S, \mathcal{E}_0, \phi_0$ as in (a). The following result will be proved in §2.

(b) *We have* $cl'(M^F) = cl(M^F)$.

We can now prove 0.1(b). By (b), it is enough to show that if $L, S, \mathcal{E}_0, \phi_0$ are as in (a) then

(c) $R_M^G(R_L^M(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp})) = \mathcal{R}_M^G(R_L^M(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp}))$.

By (a), the right hand side of (c) is equal to $\mathcal{R}_M^G(\mathcal{R}_L^M(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp}))$ hence, by 1.8(a), it is equal to $\mathcal{R}_L^G(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp})$. We have $R_M^G R_L^M = R_L^G : cl(L^F) \to cl(G^F)$. (This is proved in [L76, Cor.5] assuming that L is a maximal torus of G; but the same proof works in general.) Thus (c) is equivalent to the equality

$R_L^G(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp}) = \mathcal{R}_L^G(\chi_{\mathcal{E}_0^\sharp,\phi_0^\sharp})$

and this follows from (a) (with M replaced by G). This proves 0.1(b).

2. Proof of 1.9(b)

2.1. Let $\tilde{G}$ be a reductive connected group over $\mathbf{k}$ with an F_q-rational structure (with Frobenius map $F : \tilde{G} \to \tilde{G}$) such that $\tilde{G}_{der}$ is simply connected; assume that we are given a surjective homomorphism of algebraic groups $\tau : \tilde{G} \to G$ defined over F_q whose kernel K is a central torus in $\tilde{G}$. Then $\tilde{M} = \tau^{-1}(M) \in \Lambda(\tilde{G})$ is defined over F_q. Let $P \in \Pi(M)$ and let V be the unipotent radical of P. Then $\tilde{P} = \tau^{-1}(P) \in \Pi(\tilde{M})$. Let $\tilde{V}$ be the unipotent radical of $\tilde{P}$. Let $X = \{gV \in G; g^{-1}F(g) \in F(V)\}$, $\tilde{X} = \{\tilde{g}\tilde{V} \in \tilde{G}; \tilde{g}^{-1}F(\tilde{g}) \in F(\tilde{V})\}$. Now $G^F \times M^F$ acts on X by $(g_0, m_0) : g \mapsto g_0 g m_0^{-1}$ and this induces an action of $G^F \times M^F$ on the l-adic cohomology $H_c^i(X, \bar{\mathbf{Q}}_l)$. Similarly, $\tilde{G}^F \times \tilde{M}^F$ acts on $H_c^i(\tilde{X}, \bar{\mathbf{Q}}_l)$. For $u \in G^F$ unipotent and for $u' \in M^F$ unipotent we set $\gamma_{M,V}^G(u, u') = \sum_i (-1)^i \operatorname{tr}((u, u'), H_c^i(X, \bar{\mathbf{Q}}_l))$. Similarly for $\tilde{u} \in \tilde{G}^F$ unipotent and for $\tilde{u}' \in \tilde{M}^F$ unipotent we set $\gamma_{\tilde{M},\tilde{V}}^{\tilde{G}}(\tilde{u}, \tilde{u}') = \sum_i (-1)^i \operatorname{tr}((\tilde{u}, \tilde{u}'), H_c^i(\tilde{X}, \bar{\mathbf{Q}}_l))$. Assuming that $u = \tau(\tilde{u}), u' = \tau(\tilde{u}')$ we show:

(a) $\gamma_{\tilde{M}}^{\tilde{G}}(\tilde{u}, \tilde{u}') = |K^F| \gamma_M^G(u, v)$.

The restriction of $\tau : \tilde{G} \to G$ defines a map $\tilde{X} \to X$ which is a principal covering with group K^F. Hence we can identify $H_c^i(X, \bar{\mathbf{Q}}_l)$ with the K^F-invariants in $H_c^i(\tilde{X}, \bar{\mathbf{Q}}_l)$. It follows that (a) can be restated as follows:

$\sum_i (-1)^i \operatorname{tr}((\tilde{u}, \tilde{u}'), H_c^i(\tilde{X}, \bar{\mathbf{Q}}_l)) = \sum_{k \in K^F} \sum_i (-1)^i \operatorname{tr}((k\tilde{u}, \tilde{u}'), H_c^i(\tilde{X}, \bar{\mathbf{Q}}_l))$.

By the fixed point formula [DL, 3.2] we have $\sum_i (-1)^i \operatorname{tr}((k\tilde{u}, \tilde{u}'), H_c^i(\tilde{X}, \bar{\mathbf{Q}}_l)) = 0$ for any $k \in K^F - \{1\}$ (since the fixed point of translation by k on $\tilde{X}$ is empty). The desired equality follows.

We shall now omit the symbol V in $\gamma_{M,V}^G(u, u')$; we write instead $\gamma_M^G(u, u')$.

2.2. In the setup of 2.1 we define $a : cl(\tilde{G}^F) \to cl(G^F)$ and $a' : cl(\tilde{M}^F) \to cl(M^F)$ by

$$(a\tilde{f})(g) = \sum_{h\in\tilde{G}^F;\tau(h)=g} \tilde{f}(h),$$

$$(a'\tilde{f})(g) = \sum_{h\in\tilde{M}^F;\tau(h)=g} \tilde{f}(h).$$

For any $f \in cl(\tilde{M}^F)$ we show:

(a) $$a(R^{\tilde{G}}_{\tilde{M},\tilde{P}}(f)) = R^G_{M,P}(a'(f)).$$

We must show that for $g \in G^F$ we have

$$\sum_{h\in\tilde{G}^F;\tau(h)=g} R^{\tilde{G}}_{\tilde{M}}(f)(h) = R^G_M(a'(f))(g)$$

or (using [L90, 1.7(b)]) that

$$\sum_{\substack{h\in\tilde{G}^F;\\ \tau(h)=g}} |\tilde{M}^F|^{-1}|\tilde{G}^{0F}_{h_s}|^{-1} \sum_{\substack{z\in\tilde{G}^F;\\ z^{-1}h_sz\in\tilde{M}}} \sum_{\substack{\tilde{v}\in z\tilde{M}z^{-1}\cap\tilde{G}^{0F}_{h_s}\\ ;unip.}} \gamma^{\tilde{G}^0_{h_s}}_{z\tilde{M}z^{-1}\cap\tilde{G}^0_{h_s}}(h_u,\tilde{v})f(z^{-1}h_s\tilde{v}z)$$

(b)

$$= |M^F|^{-1}|G^{0F}_{g_s}|^{-1} \sum_{\substack{x\in G^F;\\ xi^{-1}g_sx\in M}} \sum_{\substack{v\in xMx^{-1}\cap G^{0F}_{g_s};\\ unip.}} \sum_{\substack{\tilde{m}\in\tilde{M}^F;\\ \tau(\tilde{m})=x^{-1}g_svx}} \gamma^{G^0_{g_s}}_{xMx^{-1}\cap G^0_{g_s}}(g_u,v)f(\tilde{m}).$$

The right hand side of (b) is

$$|K^F|^{-1} \sum_{\substack{h\in\tilde{G}^F;\\ \tau(h)=g}} |\tilde{M}^F|^{-1}|\tilde{G}^{0F}_{h_s}|^{-1}|K_F|^2|K^F|^{-1} \sum_{\substack{z\in\tilde{G}^F;\\ z^{-1}h_sz\in\tilde{M}}} \sum_{\substack{\tilde{v}\in z\tilde{M}z^{-1}\cap\tilde{G}^{0F}_{h_s};\\ unip.}}$$
$$\sum_{\substack{\tilde{m}\in\tilde{M}^F;\\ \tau(\tilde{m})=\tau(z^{-1}h_s\tilde{v}z)}} |K^F|^{-1}\gamma^{\tilde{G}^0_{h_s}}_{z\tilde{M}z^{-1}\cap\tilde{G}^0_{h_s}}(h_u,\tilde{v})f(\tilde{m})$$
$$= |K^F|^{-1} \sum_{\substack{h\in\tilde{G}^F;\\ \tau(h)=g}} |\tilde{M}^F|^{-1}|\tilde{G}^{0F}_{h_s}|^{-1}|K_F|^2|K^F|^{-1}$$
$$\sum_{\substack{z\in\tilde{G}^F;\\ z^{-1}h_sz\in\tilde{M}}} \sum_{\substack{\tilde{v}\in z\tilde{M}z^{-1}\cap\tilde{G}^{0F}_{h_s};\\ unip.}} \sum_{k\in K^F} |K^F|^{-1}\gamma^{\tilde{G}^0_{h_s}}_{z\tilde{M}z^{-1}\cap\tilde{G}^0_{h_s}}(h_u,\tilde{v})f(kz^{-1}h_s\tilde{v}z).$$

(We have used 2.1(a).) This is the same as the left hand side of (b). This proves (a).

2.3. We prove 1.9(b) for G instead of M. Let Θ_G be the set of all pairs $(D,\mathcal{X})$ where D is a conjugacy class of G and $\mathcal{X} \in ls_G(D)$ is irreducible (up to isomorphism). Now F acts on Θ_G by $F(D,\mathcal{X}) = (FD, F^*\mathcal{X})$. For $(D,\mathcal{X}) \in \Theta_G^F$ we denote by $\mathcal{L}_{D,\mathcal{X}}$ the line in $cl(G^F)$ containing the function $\chi_{\mathcal{X}^\sharp,\phi^\sharp} : G^F \to \bar{\mathbf{Q}}_l$ where $\phi : F^*\mathcal{X} \xrightarrow{\sim} \mathcal{X}$ is an isomorphism; note that $\chi_{\mathcal{X}^\sharp,\phi^\sharp}$ is equal to 0 outside the closure of D. (This line is well defined.) It is well known and easy to see that

$cl(G^F) = \oplus_{(D,\mathcal{X})\in\Theta_G^F}\mathcal{L}_{D,\mathcal{X}}$.

Hence to prove that $cl'(G^F) = cl(G^F)$ it is enough to show that

(a) if $(D, \mathcal{X}) \in \Theta_G^F$ and $\phi : F^*\mathcal{X} \xrightarrow{\sim} \mathcal{X}$, $f_0 = \chi_{\mathcal{X}^\sharp, \phi^\sharp}$, then $f_0 \in cl'(G^F)$.

In the special case where G_{der} is simply connected, this follows from [L90, 9.5]. We shall deduce the general case from this special case. We can find $\tau : \tilde{G} \to G, F : \tilde{G} \to \tilde{G}, K$ as in 2.1 such that $\tilde{G}_{der}$ is simply connected. Let $a : cl(\tilde{G}^F) \to cl(G^F)$ be as in 2.2. We define a linear map $b : cl(G^F) \to cl(\tilde{G}^F)$ by $(bf)(\tilde{g}) = f(\tau(\tilde{g}))$; for $f \in cl(G^F)$ we have $abf = |K^F|f$. Since 1.9(b) holds for $\tilde{G}$, we have $bf_0 \in cl'(\tilde{G}^F)$ hence $|K^F|f_0 = abf_0 \in a(cl'(\tilde{G}^F))$. Thus it is enough to show that $a(cl'(\tilde{G}^F)) \subset cl'(G^F)$.

Let $(\tilde{L}, \tilde{S}) \in A_{\tilde{G}}$ be such that $F(\tilde{L}) = \tilde{L}, F(\tilde{S}) = \tilde{S}$ and let $\mathcal{F} \in ls_{\tilde{L}}(\tilde{S})$ be irreducible cuspidal with a given isomorphism $\psi : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. It is enough to show that

(b) $a(R_{\tilde{L}}^{\tilde{G}}(\chi_{\mathcal{F}^\sharp, \psi^\sharp})) \in cl'(G^F)$.

Let $L = \tau(\tilde{L}), S = \tau(\tilde{S})$; we have $(L, S) \in A_M$. Let $\tau' : \tilde{L} \to L$ be the restriction of τ; we define define $a' : cl(\tilde{L}^F) \to cl(L^F)$ by $(a'\tilde{f})(g) = \sum_{\tilde{g} \in \tilde{L}^F; \tau'(\tilde{g}) = g} \tilde{f}(\tilde{g})$. By 2.2(a), for any $f \in cl(\tilde{L}^F)$ we have

(c) $a(R_{\tilde{L}}^{\tilde{G}}(f)) = R_L^G(a'(f))$.

From this we see that the left hand side of (b) is equal to $R_L^G(a'(\chi_{\mathcal{F}^\sharp, \psi^\sharp}))$. From the definitions we see that $a'(\chi_{\mathcal{F}^\sharp, \psi^\sharp})$ is a linear combination of functions of the form $\chi_{\mathcal{E}_0^\sharp, \phi_0^\sharp} : L^F \to \bar{\mathbf{Q}}_l$ where $\mathcal{E}_0 \in ls_L(S)$ is irreducible cuspidal and $\phi_0 : F^*\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_0$. It follows that $R_L^G(a'(\chi_{\mathcal{F}^\sharp, \psi^\sharp})) \in cl'(G^F)$. We see that (b) holds. This completes the proof of 1.9(b) for G.

3. A direct sum decomposition of $cl(G^F)$

3.1. In this section there is no restriction on q. Let $(D, \mathcal{X}) \in \Theta_G$. We associate to $(D, \mathcal{X})$ an admissible stratum of G. Let E be the set of semisimple parts of elements in D; this is a conjugacy class in G. For $s \in E$ let $[s]$ be the set of unipotent conjugacy classes of G_s^0 such that $sC \subset D$. For any $s \in E$ and $C \in [s]$ we define $f_s : C \to D$ by $u \mapsto su$; then $f_s^*\mathcal{X} \in ls_{G_s^0}(C)$. Let $f_s^*\mathcal{X} = \oplus_{\mathcal{Y} \in Q_{s,C}} \mathcal{Y}$ be the isotypic decomposition of $f_s^*\mathcal{X}$; thus each $\mathcal{Y}$ is an isotypic object of $ls_{G_s^0}(C)$. Let D' be the set of all pairs $(g, \mathcal{Y})$ where $g \in D$ and $\mathcal{Y} \in Q_{g_s, C}$ where $C \in [g_s]$ contains g_u. Then D' is naturally an algebraic variety with a transitive action of G such that the map $D' \to D$, $(g, \mathcal{Y}) \mapsto g$ is a G-equivariant unramified finite covering. For $s \in E$, $C \in [s]$, $\mathcal{Y} \in Q_{s,C}$, we choose an irreducible summand η of $\mathcal{Y}$; the generalized Springer correspondence [L84, 6.3] for the reductive connected group G_s^0 associates to the pair (C, η) a triple $(L, S, \mathcal{F}) = (L_{\mathcal{Y}}, S_{\mathcal{Y}}, \mathcal{F}_{\mathcal{Y}})$ (up to G_s^0-conjugacy) where $L \in \Lambda(G_s^0)$, $S = \mathcal{Z}_L^0 c$ with $c = c_{\mathcal{Y}}$ a unipotent class of L and $\mathcal{F} = \bar{\mathbf{Q}}_l \boxtimes \mathcal{F}_0 \in ls_L(S)$ is irreducible cuspidal with $\mathcal{F}_0 \in ls_L(c)$ irreducible; this triple is independent of the choice of η since $\mathcal{Y}$ is isotypic. Let $M = M_{\mathcal{Y}} = Z_G(\mathcal{Z}_L^0) \in \Lambda(G)$. Let $D_{\mathcal{Y}}$ be the conjugacy class in M containing sc. Let $\Sigma = \Sigma_{\mathcal{Y}} = D_{\mathcal{Y}} \mathcal{Z}_M^0$. Since $L \in \Lambda(G_s^0)$, we have $Z_{G_s^0}(\mathcal{Z}_L^0) = L$ hence $(Z_{G_s}(\mathcal{Z}_L^0))^0 = L$. We have $M_s = G_s \cap M = G_s \cap Z_G(\mathcal{Z}_L^0) = Z_{G_s}(\mathcal{Z}_L^0)$ so that $M_s^0 = L$. We have $Z_M(\mathcal{Z}_{M_s^0}^0) = Z_M(\mathcal{Z}_L^0) = Z_G(\mathcal{Z}_L^0) \cap M = M$ hence s is isolated in M and Σ is an isolated stratum of M. Hence we can define $Y = Y_{M,\Sigma}^G$, a stratum of G. If $(L, S, \mathcal{F})$ is replaced by a G_s^0-conjugate or if $(s, C, \mathcal{Y})$ is replaced by a triple in the same G-orbit, then Y is replaced by a G-conjugate hence it remains the same. Thus the stratum Y depends only on $(D, \mathcal{X})$. For

$\mathcal{Y}, (L, S, \mathcal{F}), M, \Sigma$ as above we can find $\mathcal{F}' \in ls_M(\Sigma)$ irreducible such that the inverse image of $\mathcal{F}'$ under $C \to \Sigma, u \mapsto su$ contains $\mathcal{F}$ as a direct summand. By the arguments in [L84, 2.10], $\mathcal{F}'$ is cuspidal. It follows that Y is an admissible stratum. We set $Y = \psi(D, \mathcal{X})$.

Note that if $(D, \mathcal{X}) \in \Theta_G^F$ then $F(Y) = Y$.

3.2. Let Γ'_G be the set of all triples $(L, S, \mathcal{E}_0)$ where $(L, S) \in A_G$ is such that $FL = L, FS = S$ and $\mathcal{E}_0 \in ls_L(S)$ is irreducible cuspidal (up to isomorphism) such that $F^*\mathcal{E}_0 \cong \mathcal{E}_0$. Let Γ_G be the set of orbits of the conjugation action of G^F in Γ'_G. For $(L, S, \mathcal{E}_0) \in \Gamma'_G$ we choose an isomorphism $\phi_0 : F^*\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_0$ of local systems on S. Then $\chi_{\mathcal{E}_0^\sharp, \phi_0^\sharp} : L^F \to \bar{\mathbf{Q}}_l$ is well defined; it is a class function on L^F. Let $\mathcal{L}_{L,S,\mathcal{E}_0}$ be the line in $cl(G^F)$ containing $\mathcal{R}_L^G(\chi_{\mathcal{E}_0^\sharp, \phi_0^\sharp})$ for some/any ϕ_0 as above; this line depends only on the image of $(L, S, \mathcal{E}_0)$ in Γ_G. We have the following result.

Theorem 3.3.

(i) *We have* $cl(G^F) = \oplus_{(L,S,\mathcal{E}_0) \in \Gamma_G} \mathcal{L}_{L,S,\mathcal{E}_0}$.

(ii) *For any F-stable admissible stratum Y of G we define $cl_Y(G^F)$ to be the subspace $\sum_{(L,S,\mathcal{E}_0) \in \Gamma_G; Y_{L,S}^G = Y} \mathcal{L}_{L,S,\mathcal{E}_0}$ of $cl(G^F)$ (this is a direct sum, see (i)); we define $\underline{cl}_Y(G^F)$ to be the subspace $\oplus_{(D,\mathcal{X}) \in \Theta_G^F; \psi(D,\mathcal{X}) = Y} \mathcal{L}_{D,\mathcal{X}}$ of $cl(G^F)$ (see* 2.3, 3.1*). We have $cl_Y(G^F) = \underline{cl}_Y(G^F)$ and $cl(G^F) = \oplus_Y cl_Y(G^F)$ where Y runs over the F-stable admissible strata of G.*

The fact that the sum in (i) is direct follows from the orthogonality relations [L85, 9.9] (its hypotheses are satisfied by the results in [L86] and [L12]). If $(D, \mathcal{X}) \in \Theta_G^F$ and $Y = \psi(D, \mathcal{X})$ then we have

(a) $\mathcal{L}_{D,\mathcal{X}} \subset cl_Y(G^F)$.

When G_{der} is simply connected, (a) follows from [L90, 9.5]. (One can replace R_L^G in *loc.cit.* with q large by $\mathcal{R}_L^G$ without restriction on q.) The general case can be reduced to this special case by passage to $\tilde{G}$ as in the proof in 2.3 (again replacing R_L^G by $\mathcal{R}_L^G$). Since the lines $\mathcal{L}_{D,\mathcal{X}}$ span $cl(G^F)$ we see that (a) implies that the sum in (i) is equal to $cl(G^F)$. Thus (i) holds. From (a) we see that $\underline{cl}_Y(G^F) \subset cl_Y(G^F)$ for any Y. Since $\oplus_Y \underline{cl}_Y(G^F) = \oplus_Y cl_Y(G^F) = cl(G^F)$ (see 2.3 and (i)) it follows that $\underline{cl}_Y(G^F) = cl_Y(G^F)$ for any Y. This proves (ii).

3.4. From 3.3 and the orthogonality relations mentioned in the proof of 3.3 one can deduce that the "Mackey formula" for $R_{L,P}^G$ stated by Deligne (unpublished) in 1976 for q large and in [BM] for $q > 2$ remains valid without restriction on q if $R_{L,P}^G$ is replaced by $\mathcal{R}_L^G$.

3.5. Let $(D, \mathcal{X}) \in \Theta_G$. We use notation of 3.1. We say that $(D, \mathcal{X})$ is of principal type if for $s \in E$, $C \in [s]$, the local system $f_s^*\mathcal{X}$ on C is such that some/any irreducible summand η of $f_s^*\mathcal{X}$ is such that (C, η) appears in the usual Springer correspondence for G_s^0. An equivalent condition is that the stratum $Y = \psi(D, \mathcal{X})$ is the variety of regular semisimple elements in G. For example, $(D, \bar{\mathbf{Q}}_l)$ is of principal type.

Now let $(D, \mathcal{X}) \in \Theta_G^F$ be of principal type; let $\phi : F^*\mathcal{X} \xrightarrow{\sim} \mathcal{X}$ be an isomorphism. From 3.3(a) we deduce

(a) $$\chi_{\mathcal{X}^\sharp, \phi^\sharp} = \sum_{T,\theta} c_{D,\mathcal{X};T,\theta} \mathcal{R}_T^G(\theta)$$

where T runs over the F-stable maximal tori in G, θ runs through the set of characters $T^F \to \bar{\mathbf{Q}}_l$ and the pairs (T,θ) are taken up to G^F-conjugacy; $c_{D,\mathcal{X};T,\theta} \in \bar{\mathbf{Q}}_l$ are uniquely determined. Equivalently, we have

(b) $$\chi_{\mathcal{X}^\sharp,\phi^\sharp} = \sum_{T,\theta} c_{D,\mathcal{X};T,\theta} R_T^G(\theta).$$

Indeed, we have $R_T^G(\theta) = \mathcal{R}_T^G(\theta)$. This follows from the results in [L90] (for large q) and their extension to general q in [Sh]. Moreover, from [L90, 9.5] we see that $c_{D,\mathcal{X};T,\theta}$ are explicitly known (at least if G_{der} is simply connected, but the general case can be reduced to this case as before). Since the multiplicities of irreducible representations of G^F in $R_T^G(\theta)$ are known, it follows that the functions $\chi_{\mathcal{X}^\sharp,\phi^\sharp}$ are computable as explicit linear combinations of irreducible characters.

In particular, (a),(b) hold when D is an F-stable conjugacy class in G^F and $\mathcal{X} = \bar{\mathbf{Q}}_l$.

3.6. Let $(D,\mathcal{X}) \in \Theta_G^F$. Let $\mathcal{Z}$ be the set of all $(D',\mathcal{X}') \in \Theta_G^F$ such that D' is contained in the closure of D. For any $(D',\mathcal{X}') \in \mathcal{Z}$ we choose an isomorphism $\phi_{\mathcal{X}'} : F^*\mathcal{X}' \xrightarrow{\sim} \mathcal{X}'$. We have

(a) $\chi_{\mathcal{X},\phi_\mathcal{X}} = \sum_{(D',\mathcal{X}')\in\mathcal{Z}} d_{D',\mathcal{X}'}\chi_{\mathcal{X}'^\sharp,\phi^\sharp_{\mathcal{X}'}}$

where $d_{D',\mathcal{X}'} \in \bar{\mathbf{Q}}_l$. Assume now that $(D,\mathcal{X})$ is of principal type. Then $d_{D',\mathcal{X}'} = 0$ unless $(D',\mathcal{X}')$ is of principal type. (This can be deduced from the results in [L86] on Green functions.) Using 3.5(b) we deduce

(b) $$\chi_{\mathcal{X},\phi} = \sum_{T,\theta} \tilde{c}_{D,\mathcal{X};T,\theta} R_T^G(\theta)$$

where $\tilde{c}_{D,\mathcal{X};T,\theta} \in \bar{\mathbf{Q}}_l$ is explicitly computable. In particular, (b) holds when $\mathcal{X} = \bar{\mathbf{Q}}_l$. We see that:

(c) *the class function on G^F equal to* 1 *on D^F and equal to* 0 *on $G^F - D^F$ is a linear combination of functions of the form R_T^θ.*

This has been conjectured in [L78, 2.16]. Note that the coefficients in the linear combination above are explicitly computable. Since each R_T^θ is an explicit linear combination of irreducible characters, we deduce that for any D as above the average value on D^F of any irreducible character of G^F is explicitly computable. In the case where D is a semisimple class, a result like (c) appears (in a stronger form) in [DL, 7.5].

Note that (c) also appears in [DM20, Cor.13.3.5] and in [GM20, Cor.2.7.13] (of which the author learned after submitting this paper).

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