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QUIVERS FOR SL_2 TILTING MODULES

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Abstract. Using diagrammatic methods, we define a quiver with relations depending on a prime p and show that the associated path algebra describes the category of tilting modules for SL_2 in characteristic p. Along the way we obtain a presentation for morphisms between p-Jones–Wenzl projectors.

Contents

1. Introduction

Let $\mathbb{K}$ denote an algebraically closed field and $\mathbf{Tilt} = \mathbf{Tilt}\big(\mathrm{SL}_2(\mathbb{K})\big)$ the additive, $\mathbb{K}$-linear category of (left-)tilting modules for the algebraic group $\mathrm{SL}_2(\mathbb{K})$. This category can be described as the full subcategory of $\mathrm{SL}_2(\mathbb{K})$-modules which is monoidally generated by the vector representation $\mathtt{T}(1) \cong \mathbb{K}^2$, and which is closed under taking finite direct sums and direct summands.

The purpose of this paper is to give a *generators and relations presentation* of **Tilt** by identifying it with the category of projective modules for the path algebra of an explicitly described quiver with relations. This quiver can be interpreted as the *semi-infinite Ringel dual* of $\mathrm{SL}_2(\mathbb{K})$ in the sense of [BS18]. For $\mathbb{K}$ of characteristic zero this is trivial as **Tilt** is semisimple, and the indecomposable tilting modules are indeed the simple modules. The quantum analog at a complex root of unity is related to the zigzag algebra with vertex set $\mathbb{N}_0$ and a starting condition, see e.g. [AT17].

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The focus of this paper is on the case of positive characteristic p, for which we represent **Tilt** as a quotient $\mathtt{Z} = \mathtt{Z}_{\mathsf{p}}$ of the path algebra of an infinite, fractal-like quiver, a truncation of which is illustrated for $\mathsf{p} = 3$ in Figure 1.

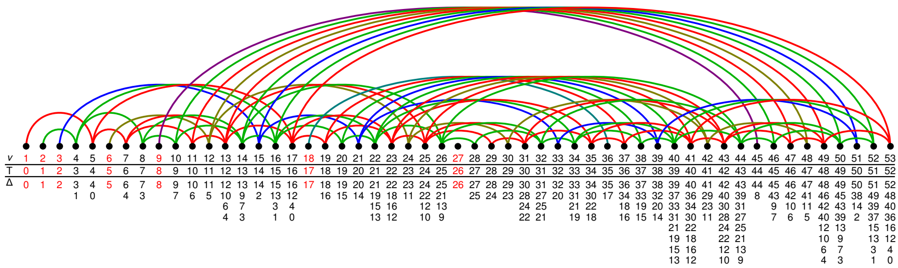


FIGURE 1. The full subquiver containing the first 53 vertices of the quiver underlying $\mathtt{Z}_3$.

The main result. From now on let $\mathbb{K}$ be an algebraically closed field of characteristic $\mathsf{p} > 0$, and let $\mathrm{SL}_2(\mathbb{K})$ be the corresponding special linear group. Recall that the indecomposable tilting modules for $\mathrm{SL}_2(\mathbb{K})$ are classified (up to isomorphism) by their highest weight $v - 1 \in \mathbb{N}_0$, and we pick a collection of representatives denoted by $\mathtt{T}(v-1)$.

Theorem A. *There is an algebra isomorphism*

$$\mathscr{F} \colon \mathtt{Z} \xrightarrow{\cong} \bigoplus_{v,w \in \mathbb{N}} \mathrm{Hom}_{\mathbf{Tilt}}\big(\mathtt{T}(v-1), \mathtt{T}(w-1)\big),$$

which sends the constant path on the vertex $v-1$ to the idempotent for the summand $\mathtt{T}(v-1)$.

Let $p\mathbf{Mod}$-$\mathtt{Z}$ denote the category of finitely-generated, projective (right-)modules for $\mathtt{Z}$. By semi-infinite Ringel duality [BS18, Section 4], we have the following consequence of Theorem A.

Corollary A. *There is an equivalence of additive, $\mathbb{K}$-linear categories*

$$\mathscr{F}' \colon \mathbf{Tilt} \xrightarrow{\cong} p\mathbf{Mod}\text{-}\mathtt{Z},$$

sending indecomposable tilting modules to indecomposable projectives.

Several classical facts about $\mathrm{SL}_2(\mathbb{K})$-modules are reflected in the presentation of the algebra $\mathtt{Z}$. For example, a path from $v-1$ to $w-1$ can only be non-zero in $\mathtt{Z}$ if $\mathtt{T}(v-1)$ and $\mathtt{T}(w-1)$ share a common Weyl factor. More specifically, if the p-adic expansion $v = [a_j, \dots, a_0]_{\mathsf{p}} = \sum_{i=0}^{j} a_i \mathsf{p}^i$ has exactly $r+1$ non-zero digits, then $\mathtt{T}(v-1)$ has exactly 2^r Weyl factors $\Delta(w-1)$ where w is obtained by negating some of the non-zero digits a_i for $i < j$. In this case, $\mathtt{Z}$ contains r arrows from $v-1$ to those $w-1 < v-1$ that are obtained by negating a single digit.

Our assignment of morphisms to arrows uses the Temperley–Lieb category. In contrast to other descriptions of morphisms between indecomposable tilting modules for $\mathrm{SL}_2(\mathbb{K})$, this presentation of $\mathtt{Z}$ is well-adapted to study **Tilt** as a monoidal category.

The Weyl factors in indecomposable tilting modules are illustrated in the lines v, $\mathtt{T}$, Δ in Figure 1, where the colors distinguish arrows in different blocks, the connected components of the quiver, and each reddish number corresponds to the unique simple tilting module in its block.

The algebra $\mathtt{Z}$ in a nutshell. We define the algebra $\mathtt{Z}$ as a quotient of the path algebra of an infinite, fractal-like quiver over the prime field $\mathbb{F}_{\mathsf{p}} \subset \mathbb{K}$. (In particular, we can always extend the algebra $\mathtt{Z}$ to an algebra over $\mathbb{K}$.) We will use this introduction to sketch the main features of $\mathtt{Z}$ and relegate the precise statement to Theorem 3.2.

- *The underlying quiver.* We identify the vertex set with $\mathbb{N}_0$ and the constant path at the vertex $v-1$ will be denoted e_{v-1} (it corresponds to $\mathtt{T}(v-1)$). If $v = [a_j, \ldots, a_0]_{\mathsf{p}}$, then for every digit $a_i \neq 0$ with $i \neq j$ there is a pair of arrows

  $$\mathrm{D}_i e_{v-1} \colon (v-1) \to (v[i]-1), \qquad \mathrm{U}_i e_{v[i]-1} \colon (v[i]-1) \to (v-1),$$

  where $v[i] = [a_j, \ldots, a_{i+1}, -a_i, a_{i-1}, \ldots, a_0]_{\mathsf{p}} = v - 2a_i\mathsf{p}^i$.
- *Some relations.* Up to some additional rules in special cases (which we ignore for the sake of this introduction), there are five types of relations among paths, which hold whenever both sides are defined and satisfy certain admissibility conditions.
  1. *Idempotents.* $e_{v-1}e_{w-1} = \delta_{v,w}e_{v-1}$, $e_{w-1}\mathrm{F}e_{v-1} = \mathrm{F}e_{v-1}$ and $e_{w-1}\mathrm{F}e_{v-1} = e_{w-1}\mathrm{F}$, where F is a word in the generators starting at $v-1$ and ending at $w-1$. (Throughout, we use such relations to absorb all but one idempotent in each string of generators.)
  2. *Nilpotency.* $\mathrm{D}_i^2 e_{v-1} = \mathrm{U}_i^2 e_{v-1} = 0$.
  3. *Far-commutativity.* $\mathrm{D}_i\mathrm{D}_j e_{v-1} = \mathrm{D}_j\mathrm{D}_i e_{v-1}$, $\mathrm{U}_i\mathrm{D}_j e_{v-1} = \mathrm{D}_j\mathrm{U}_i e_{v-1}$, as well as $\mathrm{U}_i\mathrm{U}_j e_{v-1} = \mathrm{U}_j\mathrm{U}_i e_{v-1}$ whenever $|i-j| > 1$.
  4. *Adjacency relations.* $\mathrm{D}_{i+1}\mathrm{U}_i e_{v-1} = \mathrm{D}_i\mathrm{D}_{i+1} e_{v-1}$ and $\mathrm{D}_i\mathrm{U}_{i+1} e_{v-1} = \mathrm{U}_{i+1}\mathrm{U}_i e_{v-1}$, and scaled versions $\mathrm{D}_{i+1}\mathrm{D}_i e_{v-1} = \mathtt{g}'\mathrm{U}_i\mathrm{D}_{i+1} e_{v-1}$ and $\mathrm{U}_i\mathrm{U}_{i+1} e_{v-1} = \mathtt{g}''\mathrm{U}_{i+1}\mathrm{D}_i e_{v-1}$.
  5. *Zigzag.* $\mathrm{D}_i\mathrm{U}_i e_{v-1} = \mathtt{g}\mathrm{U}_i\mathrm{D}_i e_{v-1} + \mathtt{f}\mathrm{U}_{i+1}\mathrm{U}_i\mathrm{D}_i\mathrm{D}_{i+1} e_{v-1}$.

  Here $\mathtt{g}$, $\mathtt{g}'$, $\mathtt{g}''$ and $\mathtt{f}$ are scalars that depend on p and the digit a_{i+1}.
- *Hom spaces.* For $v, w \in \mathbb{N}$ the $\mathbb{F}_{\mathsf{p}}$-vector space $e_{w-1}\mathtt{Z}e_{v-1}$ is spanned by paths of the form $e_{w-1}\mathrm{U}_{j_k}\cdots\mathrm{U}_{j_1}\mathrm{D}_{i_1}\cdots\mathrm{D}_{i_l}e_{v-1}$ with $j_k > \cdots > j_1, i_1 < \cdots < i_l$, i.e. paths that descend before ascending again. In particular, we have $e_{v-1}\mathtt{Z}e_{v-1} \cong \mathbb{F}_{\mathsf{p}}$ whenever $v = a\mathsf{p}^j$ for $1 \leq a < \mathsf{p}$, which reflects the fact that the corresponding tilting module $\mathtt{T}(v-1)$ is simple.
- *Endomorphism algebras.* Let $v > 0$ have $r+1$ non-zero digits with indices $i_{r+1} > \cdots > i_1$. Then we have the following identifications of $\mathbb{F}_{\mathsf{p}}$-algebras

  $$e_{v-1}\mathtt{Z}e_{v-1} \cong \mathbb{F}_{\mathsf{p}}[\mathrm{U}_{i_r}\mathrm{D}_{i_r}, \ldots, \mathrm{U}_{i_1}\mathrm{D}_{i_1}] \Big/ \big\langle (\mathrm{U}_{i_r}\mathrm{D}_{i_r})^2, \ldots, (\mathrm{U}_{i_1}\mathrm{D}_{i_1})^2 \big\rangle.$$

  This leads to a description of the endomorphism algebra of $\mathtt{T}(v-1)$, which could have been expected from Donkin's tensor product theorem [Don93, Proposition 2.1].

We would like to highlight that we will meet a *law of small primes* (losp) repeatedly. By this we mean the appearance of exceptional relations in cases of p-adic expansions with digits 0, 1, $\mathsf{p}-2$, or $\mathsf{p}-1$. These relations are exceptional in the sense that they contrast with the relations shown above, which describe the behavior of *generic* p-adic expansions for large primes p. Nevertheless, exceptional relations are relevant for all primes, and for $\mathsf{p} = 2$ only exceptional relations apply.

A word about the proof of Theorem A. The basis for our work is the classical fact that the *Temperley–Lieb* algebra controls the finite-dimensional representation theory of $\mathrm{SL}_2(\mathbb{K})$. The second main ingredient is an explicit description of p-Jones–Wenzl projectors [BLS19], which are characteristic p analogs of the classical Jones–Wenzl projectors, that diagrammatically encode the projections

$$\mathtt{T}(1)^{\otimes(v-1)} \to \mathtt{T}(v-1) \to \mathtt{T}(1)^{\otimes(v-1)}.$$

The bulk of this paper is devoted to a careful study of morphisms between p-Jones–Wenzl projectors over $\mathbb{F}_{\mathsf{p}}$ and the linear relations between them. This work was supported by extensive computer experimentation using Mathematica and SageMATH.

Relations to other work. To the best of our knowledge, the quiver underlying the tilting category is new: We study **Tilt** as a finitely presented category. So our main concern are the *relations among composites of generating morphisms*, rather than just the *combinatorics of objects* or the *dimensions of morphism spaces*, which appear in the classical literature.

We would like to acknowledge and reinforce that the $\mathrm{SL}_2(\mathbb{K})$ representation theory is, of course, well-understood on the level of the modules, see e.g. [CC76], [AJL83], [Don93], [EH02a], [EH02b] and [DH05]. Further, various other quivers associated to $\mathrm{SL}_2(\mathbb{K})$ are known, describing e.g. rational modules [MT11] or the extension algebra for simple [MT15] or Weyl modules [MT13].

A graded extension and translation functors. It is possible to give a similar quiver description of **Tilt** as a positively graded module category of the diagrammatic Soergel category $\mathbb{K}\mathbf{S}$ for the Weyl group of type $\tilde{\mathrm{A}}_1$, acting by translation functors. The first step in such an extension uses the quantum Satake equivalence (at $q = 1$) [Eli17] to connect the Temperley–Lieb diagrammatic calculus to $\mathbb{K}\mathbf{S}$. In fact, $\mathtt{Z}$ faithfully describes the degree zero part of the antispherical module category for $\mathbb{K}\mathbf{S}$. The second step uses ideas from [RW18] to relate $\mathbb{K}\mathbf{S}$ and the principal block $\mathbf{Tilt}_0 \subset \mathbf{Tilt}$ as long as $\mathsf{p} > 2$. Along this route, **Tilt** also inherits a grading from $\mathbb{K}\mathbf{S}$.

In this case, the algebra $\mathtt{Z}$ essentially describes the degree zero part of the principal block $\mathbf{Tilt}_0$, while the positive degree part is generated by additional degree 1 arrows $\mathrm{U}_\flat\colon v \to v+1$ and $\mathrm{D}_\flat\colon v+1 \to v$, which interact non-trivially with other paths. Note another fractal-like structure: $\mathtt{Z}$ describes **Tilt**, but also the degree zero part of $\mathbf{Tilt}_0 \subset \mathbf{Tilt}$. We will not pursue this extension in the present paper.

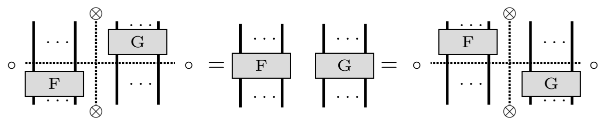
Characteristic zero and higher rank cases. Throughout we could allow the case of characteristic zero, for which **Tilt** is semisimple. In a more interesting variant, one replaces $\mathrm{SL}_2(\mathbb{K})$ by its quantum group analog at a complex root of unity, using the Jones–Wenzl projectors from [GW93]. The role of $\mathtt{Z}$ is then played by the zigzag algebra with vertex set $\mathbb{N}_0$ and a starting condition, and we would recover a result of [AT17]. In this sense we think of $\mathtt{Z}$ as a positive characteristic version of the zigzag algebra.

We also like to highlight that, to the best of our knowledge, a quiver underlying tilting modules for higher rank groups is still unknown, even for the quantum group analog in characteristic zero, cf. [MMMT20, Section 5C] for some first steps in this direction.

We expect the diagrammatic methods used in this paper to generalize to $\mathrm{SL}_N(\mathbb{K})$ and $\mathrm{GL}_N(\mathbb{K})$. This would involve developing characteristic p analogs of so-called *clasps*, living in the corresponding web calculus, see e.g. [CKM14] or [TVW17], defined over $\mathbb{F}_{\mathsf{p}}$.

2. The Temperley–Lieb calculus

Let $\mathbf{C} = (\mathbf{C}, \otimes, \mathbb{1}_{\mathbf{C}}, {}^{\star})$ be a pivotal category with (strict) monoidal composition $\otimes$, unit $\mathbb{1}_{\mathbf{C}}$, and duality *. We usually write $\mathrm{FG} := \mathrm{F} \circ \mathrm{G}$ for the composition of morphisms. We read string diagrams for morphisms in $\mathbf{C}$ from bottom to top and left to right, e.g.

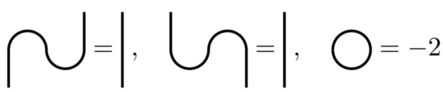
$(\mathbb{1} \otimes \mathrm{G})(\mathrm{F} \otimes \mathbb{1}) =$  $= (\mathrm{F} \otimes \mathbb{1})(\mathbb{1} \otimes \mathrm{G}).$

The duality maps are pictured as cup and cap string diagrams, subject to the expected string-straightening relations. The pivotal structure additionally allows the rotation of string diagrams and guarantees that planar-isotopic diagrams represent the same morphism.

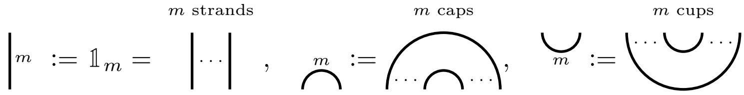
Let $\mathbb{S}$ be any commutative and unital ring. (For us $\mathbb{S}$ will usually be $\mathbb{Q}$ or $\mathbb{F}_{\mathsf{p}} \subset \mathbb{K}$, the prime field of $\mathbb{K}$. However, it also makes sense to formulate everything for $\mathbb{Q}_{\mathsf{p}}$ and $\mathbb{Z}_{\mathsf{p}}$.)

The category $\mathbb{S}\mathbf{TL}$ (see e.g. [KL94]) can be described as the pivotal $\mathbb{S}$-linear category with objects indexed by $m \in \mathbb{N}_0$, and with morphisms from m to n being $\mathbb{S}$-linear combinations of unoriented string diagrams drawn in a horizontal strip $\mathbb{R} \times [0,1]$ between m marked points on the lower boundary $\mathbb{R} \times \{0\}$ and n marked points on the upper boundary $\mathbb{R} \times \{1\}$, considered up to planar isotopy relative to the boundary and the relation that a circle evaluates to -2. The composition and tensor product operations are as described above.

Particular cases of the isotopy and circle relations are



In the following we will use labeled strands as shorthand notation for bundles of parallel strands:



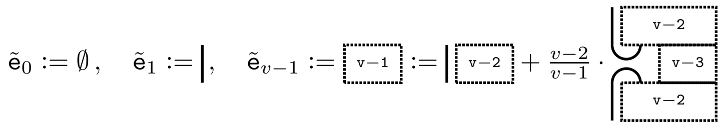
We even omit these numbers or the lines altogether if no confusion can arise.

The category $\mathbb{S}\mathbf{TL}$ furthermore admits a contravariant, $\mathbb{S}$-linear involution which reflects string diagrams in a horizontal line. Several arguments in the following will use this up-down symmetry. However, we will usually not have a left-right symmetry.

Recall that $\mathrm{Hom}_{\mathbb{S}\mathbf{TL}}(m,n)$ is a free $\mathbb{S}$-module with a basis B given by crossingless matchings. The *through-degree* $\mathrm{td}(\mathrm{X}_i)$ of $\mathrm{X}_i \in \mathrm{B}$ is the number of strands connecting the bottom to the top. More generally, the through-degree of a general morphism $F = \sum_{X_i \in B} x_i X_i$ is defined via $td(F) := \max\{td(X_i) \mid x_i \neq 0\}$. Note that $td(FG) \leq \min\big(td(F), td(G)\big)$, and thus, $\mathbf{td}_i(m,n) := \{F \in \mathrm{Hom}_{\mathbb{S}\mathbf{TL}}(m,n) \mid td(F) \leq i\}$ form a sequence of nested ($\circ$-)ideals in $\mathbb{S}\mathbf{TL}$.

Instead of m, the number of strands, let us now use $v = m+1 \in \mathbb{N}$, which will be crucial number for everything that follows.

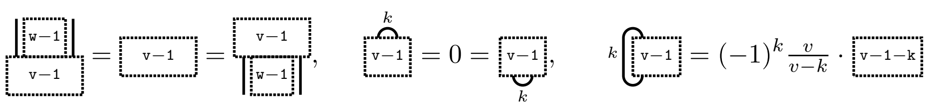
Definition 2.1. For $v \in \mathbb{N}$ the JW projectors $\tilde{\mathsf{e}}_{v-1} \in \mathrm{Hom}_{\mathbb{Q}\mathbf{TL}}(v-1, v-1)$ are the morphisms, which are recursively defined by

(2-1)


if $v > 2$,

where we use a box with $v-1$ bottom and top strands to indicate $\tilde{\mathsf{e}}_{v-1}$.

Lemma 2.2 (See e.g. [KL94, Section3]). *We have* $(\tilde{\mathsf{e}}_{v-1})^{\star} = \tilde{\mathsf{e}}_{v-1}$ *and* $td(\tilde{\mathsf{e}}_{v-1}) = v-1$. *Furthermore, the following properties hold, which are best expressed diagrammatically.*

(2-2) (2-3) (2-4)


Here $1 \leq w \leq v$ *and the projector of thickness* $w-1$ *in* (2-2) *can be at any place. Similarly, the cap or cup in* (2-3) *can be at any place and of any thickness.* $\square$

2A. **Characteristic p notions.** As already suggested by the recursion (2-1), the JW projectors have rational coefficients with respect to B and typically cannot be defined in $\mathbb{F}_{\mathsf{p}}\mathbf{TL}$. To formalize this, consider the p-adic valuation $\nu_{\mathsf{p}} \colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$, defined for $n \in \mathbb{Z}$ as $\nu_{\mathsf{p}}(n) = \max\{m \in \mathbb{N}_0 \mid \mathsf{p}^m | n\}$ (including $\nu_{\mathsf{p}}(0) = \infty$) and for $c = {}^{r}/_{s} \in \mathbb{Q}$ as $\nu_{\mathsf{p}}(c) := \nu_{\mathsf{p}}(r) - \nu_{\mathsf{p}}(s)$.

Definition 2.3. For a non-zero $F = \sum_{X_i \in B} x_i X_i \in \mathbb{Q}\mathbf{TL}$ we let $\nu_{\mathsf{p}}(F) := \min_i\{\nu_{\mathsf{p}}(x_i)\}$. We call such a morphism p-admissible if $\nu_{\mathsf{p}}(F) \geq 0$.

To highlight morphisms that might not be p-admissible, we use $\tilde{}$ as e.g. in (2-1). Note that $F = \sum_{X_i \in B} x_i X_i \in \mathbb{Q}\mathbf{TL}$ is p-admissible if and only if every coefficient x_i can be presented as a reduced fraction ${}^{r}/_{s}$ with $\mathsf{p} \nmid s$. In this case, F represents an element $\overline{F}$ of $\mathbb{F}_{\mathsf{p}}\mathbf{TL}$, which is zero if and only if $\nu_{\mathsf{p}}(F) > 0$. If we write $F = F_0 + F_{>0}$ with $\nu_{\mathsf{p}}(F_0) = 0$ and $\nu_{\mathsf{p}}(F_{>0}) > 0$, then $\overline{F} = \overline{F_0}$.

Example 2.4. We have $\nu_{\mathsf{p}}(\tilde{\mathsf{e}}_{v-1}) = 0$ for $v \leq \mathsf{p}$, which corresponds to the fact that the characteristic zero Weyl module $\Delta(v-1) = \mathtt{T}(v-1)$ stays simple when reduced modulo p. However, for $v > \mathsf{p}$, one typically has $\nu_{\mathsf{p}}(\tilde{\mathsf{e}}_{v-1}) < 0$, and in such cases the projectors $\tilde{\mathsf{e}}_{v-1}$ cannot be defined in $\mathbb{F}_{\mathsf{p}}\mathbf{TL}$.

However, there are alternative idempotents $\overline{\mathsf{e}}_{v-1} \in \mathbb{Q}\mathbf{TL}$ satisfying $\nu_{\mathsf{p}}(\overline{\mathsf{e}}_{v-1}) \geq 0$ and we will consider their specializations $\mathsf{e}_{v-1} := \overline{\overline{\mathsf{e}}_{v-1}} \in \mathbb{F}_{\mathsf{p}}\mathbf{TL}$. To this end, recall that we write $v = [a_j, \dots, a_0]_{\mathsf{p}} = \sum_i a_i p^i$ for the p-adic expansion of $v \in \mathbb{N}$ with digits $0 \leq a_i < \mathsf{p}$ and $a_j \neq 0$. (More generally, we also write $[b_j, \dots, b_0]_{\mathsf{p}} := \sum_i b_i p^i$ for any $b_i \in \mathbb{Z}$.)

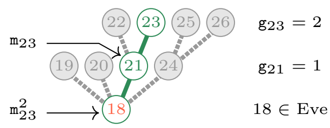
Definition 2.5. If $v = [a_j, \dots, a_0]_{\mathsf{p}} \in \mathbb{N}$ has only a single non-zero digit, then v is called an eve. The set of eves is denoted by Eve. If $v \notin$ Eve, then the mother $\mathtt{m}_v$

of v is obtained by setting the rightmost non-zero digit of v to zero. We will also consider the set $\mathrm{A}(v) := \{\mathtt{m}_v, \mathtt{m}_v^2 := \mathtt{m}_{\mathtt{m}_v}, \ldots\}$ of (matrilineal) ancestors of v, whose size $\mathtt{g}_v$ is called the generation of v.

Note that $\mathrm{A}(v) = \emptyset$ if and only if $v \in \mathrm{Eve}$, and for $v \notin \mathrm{Eve}$ we write $\mathrm{Eve}(v)$ for its eve.

Definition 2.6. For $v = [a_j, \ldots, a_0]_{\mathtt{p}}$, the support $\mathrm{supp}(v) \subset \mathbb{N}$ is the set of the $2^{\mathtt{g}_v}$ integers of the form $w = [a_j, \pm a_{j-1}, \ldots, \pm a_0]_{\mathtt{p}}$. The integers $v[i] = [a_j, \ldots, a_{i+1}, -a_i, a_{i-1}, \ldots, a_0]_{\mathtt{p}}$ for $a_i \neq 0$ form the fundamental support $\mathrm{fsupp}(v) \subset \mathrm{supp}(v)$ of v.

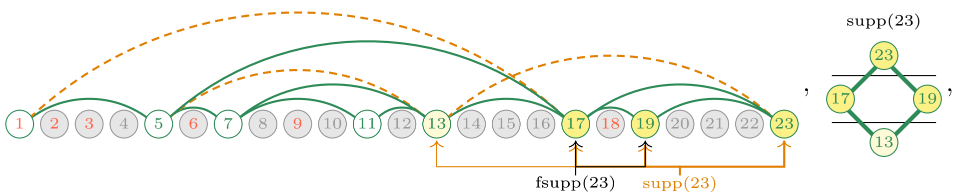
Example 2.7. Let $\mathtt{p} = 3$. Then $v = 23 = [2, 1, 2]_3$ has $\mathtt{g}_{23} = 2$, and $\mathtt{m}_{23} = 21 = [2, 1, 0]_3$ and $\mathtt{m}_{23}^2 = \mathrm{Eve}(23) = 18 = [2, 0, 0]_3$. Hence, the ancestry chart of 23 is

$\mathrm{A}(23) =$ 

.

Moreover, $\mathrm{supp}(23) = \{23 = [2, 1, 2]_3, 19 = [2, 1, -2]_3, 17 = [2, -1, 2]_3, 13 = [2, -1, -2]_3\}$ and $\mathrm{fsupp}(23) = \{19, 17\}$. In pictures,

(2-5)



where we have highlighted in yellow the support of 23. The solid green arcs indicate successive inclusions in fundamental supports, and dashed orange arcs indicate successive inclusions in non-fundamental supports, all starting from 23.

To account for losp we need the following admissibility conditions.

Definition 2.8. Let $S \subset \mathbb{N}_0$ be a finite set. We consider partitions $S = \bigsqcup_i S_i$ of S into subsets S_i of consecutive integers, which we call *stretches* (in the $\mathtt{p}$-adic expansion of v). For the purpose of this definition, we fix the *coarsest* such partition.

The set S is called down-admissible for $v = [a_j, \ldots, a_0]_{\mathtt{p}}$ if:

(d1) $a_{\min(S_i)} \neq 0$ for every i, and
(d2) if $s \in S$ and $a_{s+1} = 0$, then $s + 1 \in S$.

If $S \subset \mathbb{N}_0$ is down-admissible for $v = [a_j, \ldots, a_0]_{\mathtt{p}}$, then we define

$$v[S] := [a_j, \epsilon_{j-1} a_{j-1}, \ldots, \epsilon_0\, a_0]_{\mathtt{p}}, \quad \epsilon_k = \begin{cases} 1 & \text{if } k \notin S, \\ -1 & \text{if } k \in S. \end{cases}$$

Conversely, S is up-admissible for $v = [a_j, \dots, a_0]_{\mathsf{p}}$ if the following conditions are satisfied:

(u1) $a_{\min(S_i)} \neq 0$ for every i, and
(u2) if $s \in S$ and $a_{s+1} = \mathsf{p} - 1$, then $s + 1 \in S$.

If $S \subset \mathbb{N}_0$ is up-admissible for $v = [a_j, \dots, a_0]_{\mathsf{p}}$, then we define

$$v(S) := [a'_{r(S)}, \dots, a'_0]_{\mathsf{p}}, \quad a'_k = \begin{cases} a_k & \text{if } k \notin S, k-1 \notin S, \\ a_k + 2 & \text{if } k \notin S, k-1 \in S, \\ -a_k & \text{if } k \in S, \end{cases}$$

where we extend the digits of v by $a_h = 0$ for $h > j$ if necessary.

If S is up-admissible, then we denote by $\overline{S} \subset \mathbb{N}_0$ the *down-admissible hull* of S, the smallest down-admissible set containing S, if it exists.

Example 2.9. Let $\mathsf{p} = 7$. The set $S = \{5, 4, 3|0\}$ (here and in the following, we use vertical bars to highlight the coarsest partition into stretches) is down-admissible but not up-admissible for $v = [4, 5, 0, 2, 0, 6, 1]_7$. On the other hand, $S' = \{5, 4, 3|1, 0\}$ is up-admissible, but not down-admissible for v, and we get

$$v[5,4,3|0] = [4, \underline{5,0,2}, 0, 6, \underline{1}]_7 = [4, -5, 0, -2, 0, 6, -1]_7,$$
$$v(5,4,3|1,0) = [4, \utilde{5,0,2}, 0, \utilde{6,1}]_7 = [6, -5, 0, -2, 2, -6, -1]_7.$$

Here we have underlined the stretches of digits in $\underline{S}$ and $\utilde{S'}$. Furthermore, $\overline{S'} = \{5, 4, 3, 2, 1, 0\}$.

Example 2.10. We think of the operations $v \mapsto v(S)$ and $v \mapsto v[S]$ as reflecting v down and up along S, respectively. The admissibility restrictions ensure that the down-admissible sets S are in bijection with the elements $v[S] \in \operatorname{supp}(v)$ and that reflecting down and up are inverse operations as we will see in Lemma 2.14. Explicitly, for $\mathsf{p} = 3$ and $S = \{1, 0\}$ one gets

$$13(1,0) = [1, \utilde{1,1}]_3 = [3, -1, -1]_3 = 23, \quad 23[1,0] = [2, \underline{1,2}]_3 = [2, -1, -2]_3 = 13.$$

See also (2-5).

Definition 2.11. For two non-empty sets $A, B \subset \mathbb{N}_0$ we define

$$\mathsf{d}(A, B) = \min(|a - b| \mid a \in A, b \in B).$$

We say A and B are *distant* if $\mathsf{d}(A, B) > 1$, *adjacent* if $\mathsf{d}(A, B) = 1$, or *overlapping* if $\mathsf{d}(A, B) = 0$.

If S and S' are down- or up-admissible for v and $S \cap S' = \emptyset$, then $S \cup S'$ will also be down- or up-admissible, respectively. Conversely, if S is down- or up-admissible for v and $S' \subset S$, then S' need not be down- or up-admissible for v. For down- or up-admissible sets S, a central object in the following will be the *finest partition* into down- or up-admissible subsets $S = \bigsqcup_{k=0}^{r(S)} S_k$ (the number $r(S) + 1$ is the size of this partition), which we order by size of their elements $S_k > S_{k-1}$. Note that the elements of S_k are necessarily consecutive integers, and that this partition is typically finer than the partition considered in Definition 2.8. We call the S_k *minimal down-* or *up-admissible stretches* of v, respectively. It is easy to check that

$$v[S] = v[S_{r(S)}] \cdots [S_0], \quad v(S) = v(S_0) \cdots (S_{r(S)}),$$

for down- or up-admissible S, respectively.

Example 2.12. For the set $S = \{5,4,3|0\}$ (partitioned into stretches by the bar) and v as in Example 2.9 the finest down-admissible partition is $S = \{5|4,3|0\} = S_2 \cup S_1 \cup S_0$ where $v[S_0], v[S_1], v[S_2] \in \mathrm{fsupp}(v)$. More generally, the down-admissible sets S with $v[S] \in \mathrm{fsupp}(v)$ are exactly the minimal down-admissible stretches for v.

If S' is also down- or up-admissible and distant from S, i.e. $\mathsf{d}(S,S') > 1$, then we have:

$$v[S][S'] = v[S'][S], \quad v(S)(S') = v(S')(S), \quad v(S)[S'] = v[S'](S). \tag{2-6}$$

If S and T are subsets of $\mathbb{N}_0$, we write $T > S$ to indicate the requirement that every element in T be strictly greater than every element in S. We have the following equivalences of admissibilities.

Lemma 2.13. *Consider stretches $S' > S$ with $\mathsf{d}(S,S') = 1$.*

(a) *S is down-admissible for v and S' is down-admissible for $v[S]$ if and only if S' is down-admissible for v and S is up-admissible for $v[S']$. In this case we have $v[S][S'] = v[S'](S)$.*
(b) *S' is up-admissible for v and S is up-admissible for $v(S')$ if and only if S is down-admissible for v and S' is up-admissible for $v[S]$. In this case we have $v(S')(S) = v[S](S')$.*

Proof. We prove (a). For this we write $v = [a_j, \dots, a_0]_{\mathsf{p}}$, $S = \{s, s+1, \dots, s'-1\}$ and $S' = \{s', s'+1, \dots, t-1\}$.

S is down-admissible for v if and only if $a_s \neq 0$ and $a_{s'} \neq 0$, and we get

$$v[S] = [a_j, \dots, a_t, a_{t-1}, \dots, a_{s'+1}, a_{s'} - 1, \mathsf{p} - a_{s'-1} - 1, \dots, \mathsf{p} - a_s, a_{s-1}, \dots, a_0]_{\mathsf{p}}.$$

Now S' is down-admissible for $v[S]$ if and only if $a_{s'} \neq 1$ and $a_t \neq 0$, and we get

$$v[S][S'] = [a_j, \dots, a_t - 1, \mathsf{p} - a_{t-1} - 1, \dots, \mathsf{p} - a_{s'} + 1, \mathsf{p} - a_{s'-1} - 1, \dots, \mathsf{p} - a_s, a_{s-1}, \dots, a_0]_{\mathsf{p}}.$$

Conversely, S' is down-admissible for v if and only if $a_{s'} \neq 0$ and $a_t \neq 0$, and we get

$$v[S'] = [a_j, \dots, a_t - 1, \mathsf{p} - a_{t-1} - 1, \dots, \mathsf{p} - a_{s'}, a_{s'-1}, \dots, a_s, a_{s-1}, \dots, a_0]_{\mathsf{p}}.$$

Now S is up-admissible for S' if and only if $a_s \neq 0$ and $a_{s'} \neq 1$. This shows the equivalence of admissibilities. Furthermore, by reflecting $v[S']$ up along S, it is easy to see $v[S'](S) = v[S][S']$. The case of (b) is analogous. □

Lemma 2.14. *Let $v \in \mathbb{N}$ and $S \subset \mathbb{N}_0$ finite.*

(a) *If S is up-admissible for v, then S is down-admissible for $w = v(S)$ and $v = w[S]$.*
(b) *If S is down-admissible for v, then S is up-admissible for $u = v[S]$ and $v = u(S)$.*

Proof. Let $v = [a_j, \dots, a_0]_{\mathsf{p}}$. By (2-6) it suffices to consider the case where $S = \{s, s+1, \dots, s'-1\}$ is a single stretch. For Lemma 2.14.(a), suppose that S is up-admissible for v, i.e. $a_s \neq 0$ and $a_{s'} \neq \mathsf{p} - 1$. We get

$$\begin{aligned} w = v(S) &= [\dots, a_{s'+1}, a_{s'} + 2, -a_{s'-1}, \dots, -a_{s+1}, -a_s, a_{s-1}, \dots, a_0]_{\mathsf{p}} \\ &= [\dots, a_{s'+1}, a_{s'} + 1, \mathsf{p} - a_{s'-1} - 1, \dots, \mathsf{p} - a_{s+1} - 1, \mathsf{p} - a_s, a_{s-1}, \dots, a_0]_{\mathsf{p}}. \end{aligned}$$

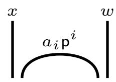
Since $a_{s'} + 1 \neq 0$ and $\mathsf{p} - a_s \neq 0$, S is down-admissible for w and we have:

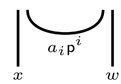
$$v(S)[S] = [\dots, a_{s'+1}, a_{s'} + 1, a_{s'-1} - \mathsf{p} + 1, \dots, a_{s+1} - \mathsf{p} + 1, a_s - \mathsf{p}, a_{s-1}, \dots, a_0]_{\mathsf{p}} = v.$$

The proof of (b) is completely analogous. □

2B. **Bookkeeping for caps and cups.** For this section, we fix $v = [a_j, \dots, a_0]_{\mathsf{p}}$.

Definition 2.15. For $0 \le i \le j$ we consider $w = [a_j, \dots, a_{i+1}, -a_i, 0, \dots, 0]_{\mathsf{p}} - 1$ and $x = [a_{i-1}, \dots, a_0]_{\mathsf{p}}$ to define (down and up) diagrams in $\mathbb{Q}\mathbf{TL}$ via

$$\mathrm{d}_i \mathbb{1}_{v-1} := \mathbb{1}_{x+w}\mathrm{d}_i \mathbb{1}_{v-1} :=$$ 

$$, \quad \mathbb{1}_{v-1}\mathrm{u}_i := \mathbb{1}_{v-1}\mathrm{u}_i\mathbb{1}_{x+w} :=$$ 

.

This includes the case of $a_i = 0$, for which we have $\mathrm{d}_i \mathbb{1}_{v-1} = \mathbb{1}_{v-1}\mathrm{u}_i = \mathbb{1}_{v-1}$. Note that we use symbols such as $\mathbb{1}_{v-1}$ to indicate the source or target of these morphisms.

Now, suppose that $S = \{s_k > \dots > s_1 > s_0\}$ and $S' = \{s'_l > \dots > s'_1 > s'_0\}$ are down-, respectively, up-admissible for v. Then we set

$$\begin{aligned} \mathrm{d}_S \mathbb{1}_{v-1} &:= \mathbb{1}_{v[S]-1}\mathrm{d}_S := \mathbb{1}_{v[S]-1}\mathrm{d}_{s_0}\cdots\mathrm{d}_{s_k}\mathbb{1}_{v-1}, \\ \mathrm{u}_{S'}\mathbb{1}_{v-1} &:= \mathbb{1}_{v(S)-1}\mathrm{u}_{S'} := \mathbb{1}_{v(S)-1}\mathrm{u}_{s'_l}\cdots\mathrm{u}_{s'_0}\mathbb{1}_{v-1}. \end{aligned} \tag{2-7}$$

In (2-7) and in the following we use the usual notation of idempotented algebras to drop some of the involved idempotents. Further, the different orderings of the factors in d_S and $\mathrm{u}_{S'}$ ensure that stretches of consecutive integers in S and S' give rise to nested caps and cups, respectively.

Lemma 2.16. *For $S' > S$ with $\mathsf{d}(S', S) = 1$ the following hold.*

(a) *S' is down-admissible for v and S is down-admissible for $v[S']$ if and only if S and $S \cup S'$ are down-admissible for v. In this case we have* $\mathrm{d}_S\mathrm{d}_{S'}\mathbb{1}_{v-1} = \mathrm{d}_{S\cup S'}\mathbb{1}_{v-1}$.

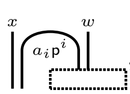
(b) *S is up-admissible for v and S' is up-admissible for $v(S)$ if and only if S' and $S' \cup S$ are up-admissible for v. In this case we have* $\mathrm{u}_{S'}\mathrm{u}_S\mathbb{1}_{v-1} = \mathrm{u}_{S'\cup S}\mathbb{1}_{v-1}$.

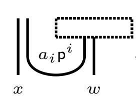
(c) *If S' is up-admissible for v and S is down-admissible for $v(S')$, then $S' \cup S$ is up-admissible for v. In this case we have* $\mathrm{d}_S\mathrm{u}_{S'}\mathbb{1}_{v-1} = \mathrm{u}_{S'\cup S}\mathbb{1}_{v-1}$.

(d) *If S is up-admissible for v and S' is down-admissible for $v(S)$, then $S \cup S'$ is down-admissible for v. In this case we have* $\mathrm{d}_{S'}\mathrm{u}_S\mathbb{1}_{v-1} = \mathrm{d}_{S\cup S'}\mathbb{1}_{v-1}$.

Proof. The claims about admissibility are not hard to prove and follow, *mutatis mutandis*, as in the proof of Lemma 2.13 given above. Finally, the equalities as e.g. $\mathrm{d}_S\mathrm{d}_{S'}\mathbb{1}_{v-1} = \mathrm{d}_{S\cup S'}\mathbb{1}_{v-1}$ are isotopies. □

Definition 2.17. Using the same notation as in Definition 2.15, we define diagrams in $\mathbb{Q}\mathbf{TL}$

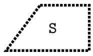
$$\tilde{\mathrm{d}}_i \mathbb{1}_{v-1} := \mathbb{1}_{x+w}\tilde{\mathrm{d}}_i \mathbb{1}_{v-1} :=$$ 

$$, \quad \mathbb{1}_{v-1}\tilde{\mathrm{u}}_i := \mathbb{1}_{v-1}\tilde{\mathrm{u}}_i\mathbb{1}_{x+w} :=$$ 

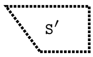
.

The boxes represent JW projectors of the size implicit in the diagram, namely $w + a_i\mathsf{p}^i = v - a_i\mathsf{p}^i - x$.

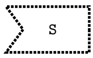
Definition 2.18. Suppose that $S = \{s_k > \dots > s_1 > s_0\}$ is down-admissible for v and $S' = \{s'_l > \dots > s'_1 > s'_0\}$ is up-admissible for v. Then we define *trapezes* and *standard loops*



$:= \tilde{\mathrm{d}}_S \mathbb{1}_{v-1} := \tilde{\mathsf{e}}_{v[S]-1}\tilde{\mathrm{d}}_{s_0}\cdots\tilde{\mathrm{d}}_{s_k}\mathbb{1}_{v-1},$



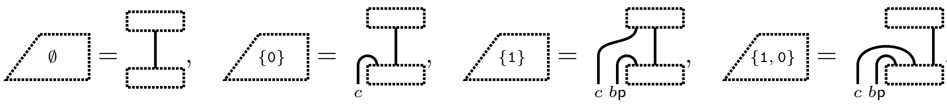
$:= \tilde{\mathrm{u}}_{S'}\mathbb{1}_{v-1} := \mathbb{1}_{v(S')-1}\tilde{\mathrm{u}}_{s'_l}\cdots\tilde{\mathrm{u}}_{s'_0}\tilde{\mathsf{e}}_{v-1},$



$:= \tilde{\mathrm{L}}^S_{v-1} := \tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{v[S]-1}\tilde{\mathrm{d}}_S.$

Note that the diagrams defined in Definition 2.18 are not left-right symmetric.

Example 2.19. For $v = [a, b, c]_{\mathsf{p}}$ we have:



We record that $\mathrm{td}(\tilde{\mathrm{d}}_S\mathbb{1}_{v-1}) = v[S] - 1$, $\mathrm{td}(\tilde{\mathrm{u}}_S\mathbb{1}_{v-1}) = v - 1$, and $\mathrm{td}(\tilde{\mathrm{L}}^S_{v-1}) = v[S] - 1$.

2C. **The p-Jones–Wenzl projectors.** For $v \in \mathbb{N}$ and $s \in \mathbb{N}_0$ let $a_{v,s}$ denote the youngest ancestor of v whose sth digit is zero. (By convention, $a_{v,-1} = v$.) For each down-admissible S for v we let

$$\lambda_{v,S} := \prod_{s\in S} (-1)^{a_s\mathsf{p}^s}\,\frac{a_{v,s-1}[S]}{a_{v,s}[S]} \in \mathbb{Q}. \tag{2-8}$$

Note that $\nu_{\mathsf{p}}(\lambda_{v,S}) = -|S|$.

Example 2.20. Let $v = [1, 2, 6, 4, 0, 6, 6]_7$ and $S = \{5|3, 2, 1, 0\}$. Then we have $\sum_{s\in S} a_s = 18$, so the overall sign is positive. The relevant reflected ancestors in the telescoping product (2-8) are $a_{v,-1}[S] = [1, -2, 6, -4, 0, -6, -6]_7$, $a_{v,3}[S] = [1, -2, 6, 0, 0, 0, 0]_7$, $a_{v,4}[S] = [1, -2, 0, 0, 0, 0, 0]_7$, and $a_{v,5}[S] = [1, 0, 0, 0, 0, 0, 0]_7$. So we get

$$\lambda_{v,S} = \frac{[1,-2,0,0,0,0,0]_7[1,-2,6,-4,0,-6,-6]_7}{[1,0,0,0,0,0,0]_7[1,-2,6,0,0,0,0]_7} = \frac{485105}{689087}, \quad \nu_7(\lambda_{v,S}) = -5.$$

The following is immediate from (2-8).

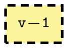
Lemma 2.21. *If $S' > S$ are down-admissible for v, then $\lambda_{v,S\cup S'} = \lambda_{v[S'],S}\lambda_{v,S'}$.* □

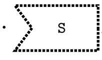
As we will see below, the following definition is a reformulation of [BLS19, Section 2.3].

Definition 2.22. For $v - 1 \in \mathbb{N}_0$ the *rational* p*JW projector*

$$\overline{\mathsf{e}}_{v-1} \in \mathrm{Hom}_{\mathbb{Q}\mathbf{TL}}(v-1, v-1)$$

is defined to be

(2-9) 

$$:= \overline{\mathsf{e}}_{v-1} := \sum_{v[S]\in\mathrm{supp}(v)} \lambda_{v,S}\,\tilde{\mathrm{L}}^S_{v-1} = \sum_{v[S]\in\mathrm{supp}(v)} \lambda_{v,S}\cdot$$


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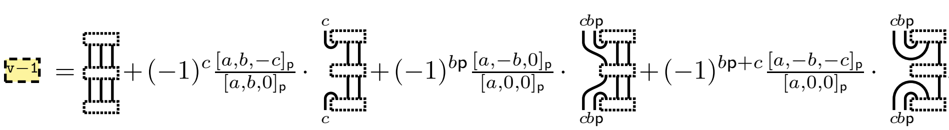
Example 2.23. For $v = [a, b, c]_{\mathsf{p}}$ we have

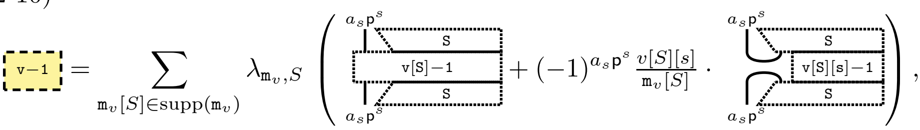


$$= \ldots + (-1)^c\frac{[a,b,-c]_{\mathsf{p}}}{[a,b,0]_{\mathsf{p}}}\cdot\ldots + (-1)^{b\mathsf{p}}\frac{[a,-b,0]_{\mathsf{p}}}{[a,0,0]_{\mathsf{p}}}\cdot\ldots + (-1)^{b\mathsf{p}+c}\frac{[a,-b,-c]_{\mathsf{p}}}{[a,0,0]_{\mathsf{p}}}\cdot\ldots.$$

Lemma 2.24. *The elements* $\overline{\mathtt{e}}_{v-1}$ *agree with the ones defined in* [BLS19, Section 2.3]*. (Note however that we have mirrored their definition.)*

Proof. Careful inspection of the recursive definition in [BLS19, Section 2.3]. More precisely, in our notation their recursion works as follows. If $v \in \mathrm{Eve}$, then $\overline{\mathtt{e}}_{v-1} = \tilde{\mathtt{e}}_{v-1}$. Otherwise,

(2-10)



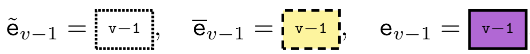
where a_s is the first non-zero digit of v. □

By Lemma 2.24, we can refer to results of [BLS19] without further notice.

Proposition 2.25 ([BLS19, Theorem 2.6])**.** *For any* $v \in \mathbb{N}$ *we have* $\nu_{\mathtt{p}}(\overline{\mathtt{e}}_{v-1}) \geq 0$. □

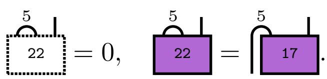
Definition 2.26. We define the $\mathtt{p}$*JW projectors* $\mathtt{e}_{v-1} := \overline{\overline{\mathtt{e}}_{v-1}} \in \mathrm{End}_{\mathbb{F}_{\mathtt{p}}\mathbf{TL}}(v-1)$.

In illustrations we distinguish the three types of JW projectors via



called JW, rational $\mathtt{p}$JW and $\mathtt{p}$JW projectors, respectively.

Example 2.27. Note that these projectors behave quite differently, e.g. for the projectors as in Example 2.23 we have



Proposition 2.28. *We have a pivotal,* $\mathbb{K}$*-linear functor*

$$\mathscr{D}\colon \mathbb{K}\mathbf{TL} \to \mathbf{Tilt}, \quad \mathscr{D}(v-1) = \mathtt{T}(1)^{\otimes(v-1)},$$

which sends the idempotent $\mathtt{e}_{v-1}$ *to the projection* $\mathtt{T}(1)^{\otimes(v-1)} \to \mathtt{T}(v-1) \to \mathtt{T}(1)^{\otimes(v-1)}$*. This functor induces an equivalence of pivotal* $\mathbb{K}$*-linear categories upon additive Karoubi completion.*

Proof. By Proposition 2.25 and the construction of $\mathbb{K}\mathbf{TL}$, the only non-trivial statement is the fully-faithfulness of $\mathscr{D}$. This is known; however, for completeness, let us give a short (but not new, cf. [Eli15, Theorem 2.58] or [AST17, Proposition 2.3]) argument for this. First, the same statement over $\mathbb{C}$ is a classical result and dates back to work of Rumer–Teller–Weyl. This implies that hom spaces on both sides have the same dimension over $\mathbb{C}$. The point is now the *flatness* of both sides. Precisely, the standard basis B works for $\mathbb{Z}\mathbf{TL}$, showing that the dimensions of hom spaces in $\mathbb{K}\mathbf{TL}$ are independent of the characteristic. The same is true in the image of $\mathscr{D}$: The module $\mathtt{T}(1)$ is a tilting module regardless of the characteristic, and the same thus holds for $\mathtt{T}(1)^{\otimes(v-1)}$. This implies that the hom spaces in $\mathscr{D}(\mathbb{K}\mathbf{TL})$ are also independent of the characteristic. Finally, one can check that $\mathscr{D}(\mathrm{B})$ remains linear independent, and the claim follows since all involved dimensions are finite and the same on both sides. □

3. The quiver algebra

3A. Generators and relations. In order to prove Theorem A we have to give a presentation of the algebra

$$\mathtt{Z} := \bigoplus_{v,w\in\mathbb{N}} \mathrm{Hom}_{\mathbb{F}_\mathtt{p}\mathbf{TL}}\big(\mathtt{e}_{v-1},\mathtt{e}_{w-1}\big) \tag{3-1}$$

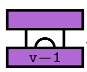
by generators and relations. To this end, we first introduce notation for certain elements.

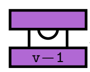
Definition 3.1. Let S and S' be down- and up-admissible for v, respectively. Then we define

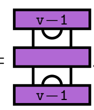
$$\begin{aligned} \mathrm{D}_S\mathtt{e}_{v-1} &:= \mathtt{e}_{v[S]-1}\mathrm{d}_S\mathtt{e}_{v-1},\\ \mathrm{U}_{S'}\mathtt{e}_{v-1} &:= \mathtt{e}_{v(S')-1}\mathrm{u}_{S'}\mathtt{e}_{v-1},\\ \mathrm{L}^S_{v-1}\mathtt{e}_{v-1} &:= \mathtt{e}_{v-1}\mathrm{u}_S\mathrm{d}_S\mathtt{e}_{v-1}. \end{aligned} \tag{3-2}$$

We call the latter the $\mathtt{p}$*loop* on $v-1$ down through $v[S]-1$.

We will consider the morphisms $\mathrm{D}_S\mathtt{e}_{v-1}$ and $\mathrm{U}_S\mathtt{e}_{v-1}$ as generators for $\mathtt{Z}$, but restrict to the cases when S and S' are minimal admissible stretches of consecutive integers. Then these morphisms can be pictured as

$\mathrm{D}_S\mathtt{e}_{v-1} =$ 

, $\mathrm{U}_{S'}\mathtt{e}_{v-1} =$ 

, $\mathrm{L}^S_{v-1}\mathtt{e}_{v-1} =$ 

.

We define two functions $\mathtt{f},\mathtt{g}\colon \mathbb{F}_\mathtt{p}\to\mathbb{F}_\mathtt{p}$ (where we again see losp) via

$$\mathtt{f}(a) = \begin{cases} (-1)^a\frac{2}{a} & \text{if } 1\leq a\leq \mathtt{p}-2,\\ 0 & \text{if } a=0 \text{ or } a=\mathtt{p}-1,\end{cases} \qquad \mathtt{g}(a) = \begin{cases} -(\frac{a+1}{a}) & \text{if } 1\leq a\leq \mathtt{p}-1,\\ -2 & \text{if } a=0.\end{cases}$$

Note that $\mathtt{f}(\mathtt{p}-1)=\mathtt{g}(\mathtt{p}-1)=0$ and $\mathtt{g}(a)=\mathtt{g}(\mathtt{p}-a-1)^{-1}$ for $a\neq 0,\mathtt{p}-1$. Then for each finite $S\subset\mathbb{N}_0$ we define scaling operators $\mathtt{f}_S,\mathtt{g}_S,\mathtt{h}_S\in\mathtt{Z}$ on $v=[a_j,\dots,a_0]_\mathtt{p}$ as

$$\mathtt{f}_S\mathtt{e}_{v-1} = \mathtt{f}(a_{\max(S)+1})\mathtt{e}_{v-1},\quad \mathtt{g}_S\mathtt{e}_{v-1} = \mathtt{g}(a_{\max(S)+1})\mathtt{e}_{v-1},$$
$$\mathtt{h}_S\mathtt{e}_{v-1} = \mathtt{g}(a_{\max(S)+1}-1)\mathtt{e}_{v-1}.$$

These are not considered as generators of $\mathtt{Z}$, but as mere bookkeeping devices for the appearing scalars.

Theorem 3.2 (Generators and relations)**.** *The algebra* $\mathtt{Z}$ *is generated by* $\mathtt{e}_{v-1}$ *for* $v\in\mathbb{N}$*, and elements* $\mathrm{D}_S\mathtt{e}_{v-1}$ *and* $\mathrm{U}_{S'}\mathtt{e}_{v-1}$*, where* S *and* S' *denote minimal down- and up-admissible stretches for* v*, respectively. These generators are subject to the following complete set of relations.*

(1) Idempotents.

$$\mathtt{e}_{v-1}\mathtt{e}_{w-1}=\delta_{v,w}\mathtt{e}_{v-1},\quad \mathtt{e}_{v[S]-1}\mathrm{D}_S\mathtt{e}_{v-1}=\mathrm{D}_S\mathtt{e}_{v-1},\quad \mathtt{e}_{v(S')-1}\mathrm{U}_{S'}\mathtt{e}_{v-1}=\mathrm{U}_{S'}\mathtt{e}_{v-1}.$$

(2) Containment. *If* $S'\subset S$*, then we have*

$$\mathrm{D}_{S'}\mathrm{D}_S\mathtt{e}_{v-1}=0,\quad \mathrm{U}_S\mathrm{U}_{S'}\mathtt{e}_{v-1}=0.$$

(3) Far-commutativity. *If* $\mathtt{d}(S,S')>1$*, then*

$$\mathrm{D}_S\mathrm{D}_{S'}\mathtt{e}_{v-1}=\mathrm{D}_{S'}\mathrm{D}_S\mathtt{e}_{v-1},\quad \mathrm{D}_S\mathrm{U}_{S'}\mathtt{e}_{v-1}=\mathrm{U}_{S'}\mathrm{D}_S\mathtt{e}_{v-1},\quad \mathrm{U}_S\mathrm{U}_{S'}\mathtt{e}_{v-1}=\mathrm{U}_{S'}\mathrm{U}_S\mathtt{e}_{v-1}.$$

(4) Adjacency relations. *If* $\mathtt{d}(S,S')=1$ *and* $S'>S$*, then*

$$\mathrm{D}_{S'}\mathrm{U}_S\mathtt{e}_{v-1}=\mathrm{D}_{S\cup S'}\mathtt{e}_{v-1},\quad \mathrm{D}_S\mathrm{U}_{S'}\mathtt{e}_{v-1}=\mathrm{U}_{S'\cup S}\mathtt{e}_{v-1},$$
$$\mathrm{D}_{S'}\mathrm{D}_S\mathtt{e}_{v-1}=\mathrm{U}_S\mathrm{D}_{S'}\mathtt{h}_S\mathtt{e}_{v-1},\quad \mathrm{U}_S\mathrm{U}_{S'}\mathtt{e}_{v-1}=\mathtt{h}_S\mathrm{U}_{S'}\mathrm{D}_S\mathtt{e}_{v-1}.$$

(5) Overlap relations. *If* $S'\geq S$ *with* $S'\cap S=\{s\}$ *and* $S'\not\subset S$*, then we have*

$$\mathrm{D}_{S'}\mathrm{D}_S\mathtt{e}_{v-1}=\mathrm{U}_{\{s\}}\mathrm{D}_S\mathrm{D}_{S'\setminus\{s\}}\mathtt{e}_{v-1},\quad \mathrm{U}_S\mathrm{U}_{S'}\mathtt{e}_{v-1}=\mathrm{U}_{S'\setminus\{s\}}\mathrm{U}_S\mathrm{D}_{\{s\}}\mathtt{e}_{v-1}.$$

(6) Zigzag.

$$\mathrm{D}_S\mathrm{U}_S\mathtt{e}_{v-1}=\mathrm{U}_{\overline{S}}\mathrm{D}_{\overline{S}}\mathtt{g}_S\mathtt{e}_{v-1}+\mathrm{U}_T\mathrm{U}_{\overline{S}}\mathrm{D}_{\overline{S}}\mathrm{D}_T\mathtt{f}_S\mathtt{e}_{v-1}.$$

Here, if the down-admissible hull $\overline{S}$*, or the smallest minimal down-admissible stretch* T *with* $T>\overline{S}$ *does not exist, then the involved symbols are zero by definition.*

(Basis) *The elements of the form*

$$\mathtt{e}_{w-1}\mathrm{U}_{S'_{i_l}}\cdots\mathrm{U}_{S'_{i_0}}\mathrm{D}_{S_{i_0}}\cdots\mathrm{D}_{S_{i_k}}\mathtt{e}_{v-1},$$

with $S'_{i_l}>\cdots>S'_{i_0}$*, and* $S_{i_0}<\cdots<S_{i_k}$*, form a basis for* $\mathtt{e}_{w-1}\mathtt{Z}\mathtt{e}_{v-1}$.

(Complete) *Any word* $\mathtt{e}_{w-1}\mathrm{F}\mathtt{e}_{v-1}$ *in the generators of* $\mathtt{Z}$ *can be rewritten as a linear combination of basis elements from* (Basis) *using only the relations* (1)–(6)*.*

Remark 3.3. The algebra $\mathtt{Z}$ is a path algebra of an underlying quiver as follows. The idempotents $\mathtt{e}_{v-1}$ correspond to vertices of a quiver, call these $v-1$. The elements $\mathrm{D}_S\mathtt{e}_{v-1}$ and $\mathrm{U}_{S'}\mathtt{e}_{v-1}$ correspond to arrows starting at the vertex $v-1$, and either pointing to *downwards* or *upwards* (which is leftwards respectively rightwards in Figure 1) to $v[S]-1$ or $v(S)-1$.

Remark 3.4. In the special case of $v=w$, Theorem 3.2.(Basis) says that $\mathtt{ploops}$ form a basis of the endomorphism spaces. Furthermore, we will see in Lemma 3.25 that all $\mathtt{ploops}$ are products of $\mathtt{ploops}$ $\mathrm{L}^S_{v-1}\mathtt{e}_{v-1}$ for S minimal down-admissible.

Remark 3.5. In Theorem 3.2.(4) and (6), the right-hand sides of the shown relations feature morphisms indexed by admissible subsets that are not necessarily minimal. We shall see in Lemma 3.16 that such morphisms decompose into products of generators

$$\mathrm{D}_S\mathtt{e}_{v-1}:=\mathrm{D}_{S_{i_1}}\cdots\mathrm{D}_{S_{i_k}}\mathtt{e}_{v-1},\quad \mathrm{U}_{S'}\mathtt{e}_{v-1}:=\mathrm{U}_{S'_{i_l}}\cdots\mathrm{U}_{S'_{i_1}}\mathtt{e}_{v-1},\tag{3-3}$$

where the products are taken over the minimal down- respectively up-admissible stretches S_{i_j} and S'_{i_j}, such that $S=\bigsqcup_j S_{i_j}$ and $S'=\bigsqcup_j S'_{i_j}$, with $S_{i_1}<\cdots<S_{i_k}$ and $S'_{i_l}>\cdots>S'_{i_1}$.

In Theorem 3.2 we use (3-3) as a *shorthand notation*, but one could also take $\mathrm{D}_S\mathtt{e}_{v-1}$ and $\mathrm{U}_{S'}\mathtt{e}_{v-1}$ for (not necessarily minimal) admissible S and S' as generators for $\mathtt{Z}$. This requires listing the additional relations

$$\mathrm{D}_S\mathtt{e}_{v-1}=\mathrm{D}_{S_1}\mathrm{D}_{S_2}\mathtt{e}_{v-1},\quad \mathrm{U}_{S'}\mathtt{e}_{v-1}=\mathrm{U}_{S'_2}\mathrm{U}_{S'_1}\mathtt{e}_{v-1},\tag{3-4}$$

for down-admissible $S_1<S_2$ with $S=S_1\cup S_2$ and up-admissible $S'_2>S'_1$ with $S'=S'_2\cup S'_1$, in addition to the relations Theorem 3.2.(1)-(6) among minimal generators. One advantage of such a presentation is that it exhibits $\mathtt{Z}$ as a quadratic algebra, since relations Theorem 3.2.(4)-(6) now turn into quadratic relations with respect to the enlarged generating set.

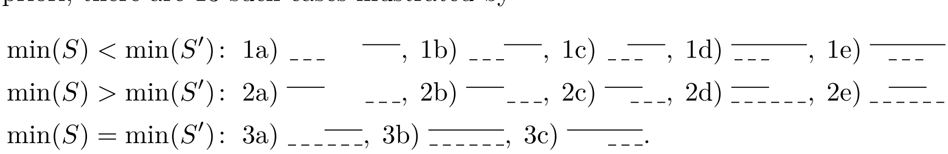
The proof of Theorem 3.2 will occupy the remainder of this paper. However, we already note that Theorem 3.2.(1) holds by the definition of $\mathtt{Z}$ as the endomorphism algebra of a direct sum. Moreover, assuming the relations Theorem 3.2.(1)-(6), we get:

Lemma 3.6 (Completeness—Theorem 3.2.(Complete))**.** *Let* $\mathtt{e}_{w-1}\mathrm{F}\mathtt{e}_{v-1} \in \mathtt{Z}$*. Then there is a finite sequence of relations Theorem 3.2.(1)-(6) rewriting it as a linear combination of elements of the form Theorem 3.2.(Basis).*

Proof. We can immediately restrict to the case where $\mathtt{e}_{w-1}\mathrm{F}\mathtt{e}_{v-1}$ is a product of generators of $\mathtt{Z}$ (rather than a linear combination of such). In order to prove the claim, we will show that, if $\mathtt{e}_{w-1}\mathrm{F}\mathtt{e}_{v-1}$ is not of the desired form, then we can measure its complexity by counting *out-of-order* pairs of the following forms, all other pairs are called *in-order*.

(i) $\mathrm{D}_{S'}\mathrm{D}_{S}$ or $\mathrm{U}_{S}\mathrm{U}_{S'}$ for $S' \geq S$.
(ii) $\mathrm{D}_{S}\mathrm{U}_{S'}$.

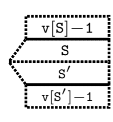
A case-by-case check will verify that we can use our relations to reduce these to in-order pairs, which then inductively shows the claim. For the case-by-case check we write down the list of all combinations how stretches S and S' can meet. A priori, there are 13 such cases illustrated by

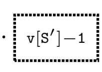


where the solid line represents S and the dashed line S', with smaller entries appearing further to the right. Some of the illustrated cases never arise when considering minimal admissible stretches and the remaining cases are precisely covered by our relations. Let us do this in detail for the out-of-order pair $\mathrm{D}_{S'}\mathrm{D}_{S}$. First, the cases 2a)–2e) as well as 1e) and 3c) are ruled out by the assumption $S' \geq S$. The case 1a) is far-commutativity, the case 1b) adjacency, while 1d) and 3b) are covered by containment. The relation 3a) does not occur as S' would not be minimal. The remaining case 1c) is only possible if $S' \cap S = \{\min(S')\}$, in which case we can apply the overlap relation. □

3B. **Basic properties of pJW projectors.** We invite the reader to illustrate the statements and proofs of the next lemmas using the explicit diagrammatic examples of trapezes from Example 2.19 and of pJW projectors from Example 2.23.

Lemma 3.7 (See [BLS19, Proposition 3.2])**.** *Suppose that S and S' are down-admissible for v. Then we have*

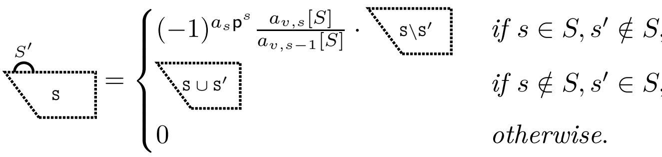
$\tilde{\mathtt{e}}_{v[S]-1}\tilde{\mathrm{d}}_{S}\tilde{\mathrm{u}}_{S'}\tilde{\mathtt{e}}_{v[S']-1} =$ 

$= \delta_{S,S'}\lambda_{v,S'}^{-1} \cdot$ 

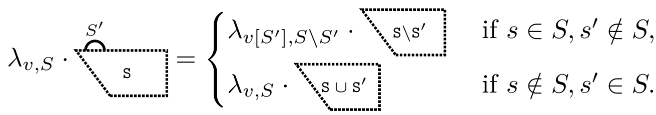
$= \delta_{S,S'}\lambda_{v,S'}^{-1}\tilde{\mathtt{e}}_{v[S']-1}.$

Thus, the summands $\lambda_{v,S}\tilde{\mathrm{L}}_{v-1}^{S}$ in (2-9) *are orthogonal idempotents in* $\mathbb{Q}\mathbf{TL}$*.* □

Lemma 3.8. *Suppose S is down-admissible for v, and $S' = \{s, \dots, s'-1\}$ is a minimal down-admissible stretch for v. Then we have*



We will also use the non-zero cases in the form:

(3-5)


Proof. Relation (2-3) implies that $\mathrm{d}_{S'}\tilde{\mathrm{u}}_S\tilde{\mathrm{e}}_{v[S]-1} = 0$ if either $s \in S$, $s' \in S$ or $s \notin S$, $s' \notin S$. For the other cases, we define $S_+ = \{t \in S \mid t > s'\}$ and $S_- = \{t \in S \mid t < s\}$. If $s \in S$ and $s' \notin S$, then we use far commutativity, relation (2-4), and $\mathrm{d}_{S'} = \mathrm{d}_s$ to compute

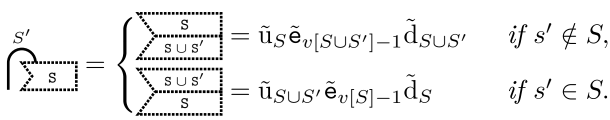
$$\mathrm{d}_{S'}\tilde{\mathrm{u}}_S\tilde{\mathrm{e}}_{v[S]-1} = \tilde{\mathrm{u}}_{S_+}\mathrm{d}_{S'}\tilde{\mathrm{u}}_{S'}\tilde{\mathrm{u}}_{S_-}\tilde{\mathrm{e}}_{v[S]-1} = (-1)^{a_s \mathrm{p}^s}\frac{a_{v,s}[S]}{a_{v,s-1}[S]}\tilde{\mathrm{u}}_{S_+}\tilde{\mathrm{u}}_{S_-}\tilde{\mathrm{e}}_{v[S]-1}$$
$$= (-1)^{a_s \mathrm{p}^s}\frac{a_{v,s}[S]}{a_{v,s-1}[S]}\tilde{\mathrm{u}}_{S\setminus S'}\tilde{\mathrm{e}}_{v[S]-1}.$$

Similarly, if $s \notin S$ but $s' \in S$, we use far commutativity and an isotopy to compute

$$\mathrm{d}_{S'}\tilde{\mathrm{u}}_S\tilde{\mathrm{e}}_{v[S]-1} = \mathrm{d}_{S'}\tilde{\mathrm{u}}_{S_+\cup\{s'\}}\tilde{\mathrm{u}}_{S_-}\tilde{\mathrm{e}}_{v[S]-1} = \tilde{\mathrm{u}}_{S_+\cup\{s'\}}\tilde{\mathrm{u}}_{S'}\tilde{\mathrm{u}}_{S_-}\tilde{\mathrm{e}}_{v[S]-1} = \tilde{\mathrm{u}}_{S'\cup S}\tilde{\mathrm{e}}_{v[S]-1},$$

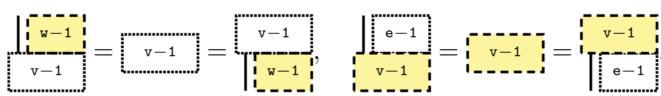
which finishes the proof. □

Lemma 3.9. *Suppose that $S' = \{s, \dots, s'-1\}$ is the smallest minimal down-admissible stretch for v and let S be down-admissible for $a_{v,s} = \mathtt{m}_v$. Then we have:*

(3-6)


Proof. Similar, but easier than the proof of Lemma 3.8. □

Lemma 3.10. *Let $e = \mathrm{Eve}(v)$ and $w \le v = [a_j, \dots, a_0]_{\mathrm{p}}$. Then we have*



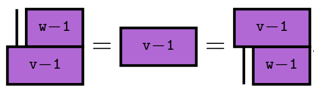
Proof. The first pair of equalities is clear since $\overline{\mathrm{e}}_{w-1}$ contains $\mathbb{1}_{w-1}$ with coefficient 1 and otherwise only cap and cup diagrams, which are killed by (2-3). For a down-admissible set S, let $i(S) = \max\{s \in S \mid a_s \neq 0\}$. For the second pair of equalities we express $\overline{\mathrm{e}}_{v-1}$ as

$$\overline{\mathrm{e}}_{v-1} = \tilde{\mathrm{e}}_{v-1} + \sum_{i=0}^{j-1} \underbrace{\left(\sum_{v[S]\in\mathrm{supp}(v), i=i(S)} \lambda_{v,S}\,\tilde{\mathrm{L}}_{v-1}^S \right)}_{:=\overline{\mathrm{e}}_{v-1}(i)}.$$

It follows from Lemma 3.7 that the summands $\overline{\mathsf{e}}_{v-1}(i)$ are orthogonal idempotents. Note that we can write $\overline{\mathsf{e}}_{v-1}(i) = \tilde{\mathrm{u}}_i \mathrm{F}(v,i)\tilde{\mathrm{d}}_i$ for some morphism $\mathrm{F}(v,i)$. In particular $\overline{\mathsf{e}}_{v-1}(i)$ absorbs $\tilde{\mathsf{e}}_{a_{v,i}-1}$ or smaller, and it annihilates all $\tilde{\mathsf{e}}_k$ for $k > a_{v,i}-1$. In particular, it absorbs $\tilde{\mathsf{e}}_{e-1}$. □

We prove now a significant generalization of [BLS19, Proposition 3.3] and the analog of (2-2).

Proposition 3.11 (Classical absorption)**.** *Let $w \leq v$. Then we have*



Proof. We distinguish two cases. If $w \leq e = \mathrm{Eve}(v)$, then we have

$$\overline{\mathsf{e}}_{w-1}\overline{\mathsf{e}}_{v-1} = \overline{\mathsf{e}}_{w-1}\tilde{\mathsf{e}}_{e-1}\overline{\mathsf{e}}_{v-1} = \tilde{\mathsf{e}}_{e-1}\overline{\mathsf{e}}_{v-1} = \overline{\mathsf{e}}_{v-1}$$

and the other equation follows by reflection.

On the other hand, if $w \geq \mathrm{Eve}(v)$, then $\mathrm{A}(v) \cap \mathrm{A}(w) \neq \emptyset$. Let $z = a_{v,s} = a_{w,t}$ denote the youngest common ancestor of v and w. It follows that $u := a_{v,s-1}$ is the oldest ancestor of v with $u \geq w$. Now, we have $\tilde{\mathsf{e}}_{v-1}\tilde{\mathsf{e}}_{w-1} = \tilde{\mathsf{e}}_{v-1}$ and $\tilde{\mathsf{e}}_{v-1}\overline{\mathsf{e}}_{w-1}(j) = 0$ for any j, as well as

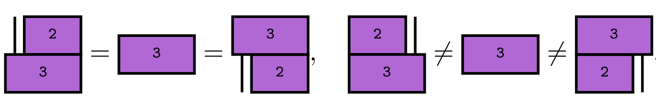
$$\overline{\mathsf{e}}_{v-1}(i)\tilde{\mathsf{e}}_{w-1} = \begin{cases} \overline{\mathsf{e}}_{v-1}(i) & \text{if } i < s, \\ 0 & \text{if } i \geq s, \end{cases} \qquad \overline{\mathsf{e}}_{v-1}(i)\overline{\mathsf{e}}_{w-1}(j) = \delta_{a_{v,i},a_{w,j}}\overline{\mathsf{e}}_{v-1}(i).$$

The latter is a consequence of Lemma 3.7. Moreover, for each $i \geq s$, there exists exactly one j, such that $a_{v,i} = a_{w,j}$. Thus, we have

$$\begin{aligned} \overline{\mathsf{e}}_{v-1}\overline{\mathsf{e}}_{w-1} &= \tilde{\mathsf{e}}_{v-1}\overline{\mathsf{e}}_{w-1} + \sum_i \overline{\mathsf{e}}_{v-1}(i)\tilde{\mathsf{e}}_{w-1} + \sum_{i,j} \overline{\mathsf{e}}_{v-1}(i)\overline{\mathsf{e}}_{w-1}(j) \\ &= \tilde{\mathsf{e}}_{v-1} + \sum_{i<s} \overline{\mathsf{e}}_{v-1}(i) + \sum_{i \geq s} \overline{\mathsf{e}}_{v-1}(i) = \overline{\mathsf{e}}_{v-1}. \end{aligned}$$

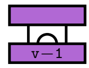
The computation for $\overline{\mathsf{e}}_{w-1}\overline{\mathsf{e}}_{v-1}$ is analogous. □

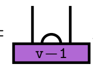
Example 3.12. For $\mathsf{p} = 3$ we have



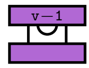
We also have the following relations with no classical analog.

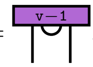
Proposition 3.13 (Non-classical absorption and shortening). *Let S be a down-admissible stretch for v. Then we have*

$\mathsf{e}_{v[S]-1}\mathrm{d}_S\mathsf{e}_{v-1} = \mathrm{d}_S\mathsf{e}_{v-1}$ ⇝ 

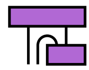
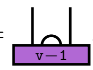
= 

,

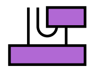
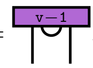
$\mathsf{e}_{v-1}\mathrm{u}_S\mathsf{e}_{v[S]-1} = \mathsf{e}_{v-1}\mathrm{u}_S$ ⇝ 

= 

,

$\mathsf{e}_{v[S]-1}\mathrm{d}_S(\mathbb{1}_{v-a_{v,s}} \otimes \mathsf{e}_{a_{v,s}-1}) = \mathrm{d}_S\mathsf{e}_{v-1}$ ⇝ 
= 

,

$(\mathbb{1}_{v-a_{v,s}} \otimes \mathsf{e}_{a_{v,s}-1})\mathrm{u}_S\mathsf{e}_{v[S]-1} = \mathsf{e}_{v-1}\mathrm{u}_S$ ⇝ 
= 

.

Here $a_{v,s}$ denotes the youngest ancestor of v for which the sth digit is zero.

Proof. If suffices to prove these relations in the case of minimal down-admissible stretches. To be consistent with the above notation, let us write $S' = \{s, \dots, s'-1\}$ instead of S.

In order to verify the first relation we compute, using (3-5), that

$$\begin{aligned} \mathrm{d}_{S'}\overline{\mathsf{e}}_{v-1} &= \mathrm{d}_{S'} \sum_{v[S]\in\mathrm{supp}(v)} \lambda_{v,S}\tilde{\mathrm{L}}^S_{v-1} \\ &= \sum_{s\in S, s'\notin S} \lambda_{v[S'],S\setminus S'}\,\tilde{\mathrm{u}}_{S\setminus S'}\tilde{\mathsf{e}}_{v[S]-1}\tilde{\mathrm{d}}_S + \sum_{s\notin S, s'\in S} \lambda_{v,S}\tilde{\mathrm{u}}_{S\cup S'}\tilde{\mathsf{e}}_{v[S]-1}\tilde{\mathrm{d}}_S. \end{aligned} \tag{3-7}$$

For s with $s \in S, s' \notin S$, we define $S_- = S \setminus S'$. For S with $s \notin S, s' \in S$, we define $S_+ = S \cup S'$. It is easy to verify that the sets S_- and S_+ are down-admissible for $v[S']$.

Then Lemma 3.7 implies that each summand in (3-7) is invariant under left multiplication by a unique summand in $\overline{\mathsf{e}}_{v[S']-1} = \sum_{v[S'][X]\in\mathrm{supp}(v[S'])} \lambda_{v[S'],X}\,\tilde{\mathrm{L}}^X_{v[S']-1}$, while it is killed under left multiplication by any other summand. This proves the first equation; the second absorption equation follows by reflection symmetry.

For later use, note that the relevant summands of $\overline{\mathsf{e}}_{v[S']-1}$ are the $\lambda_{v[S'],X}\tilde{\mathrm{L}}^X_{v[S']-1}$ for which $s, s' \in X$ or $s, s' \notin X$.

Now we are ready to prove the projector shortening relations. We start by expanding

$$\mathrm{d}_{S'}(\mathbb{1}_{v-a_{v,s}} \otimes \overline{\mathsf{e}}_{a_{v,s}-1}) = \mathrm{d}_{S'} \sum_{a_{v,s}[S]\in\mathrm{supp}(a_{v,s})} \lambda_{a_{v,s},S}\,(\mathbb{1}_{v-a_{v,s}} \otimes \tilde{\mathrm{L}}^S_{a_{v,s}-1}).$$

Note that the down-admissible sets S for $a_{v,s} = [a_j, \dots, a_{s+1}, 0, 0, \dots, 0]_{\mathsf{p}}$ are exactly the down-admissible sets of v which are contained in $\{s+1, \dots, j-1\}$ and that $\lambda_{a_{v,s},S} = \lambda_{v,S}$. Recall that, by (3-6), we have

$$\mathrm{d}_{S'}(\mathbb{1}_{v-a_{v,s}} \otimes \tilde{\mathrm{L}}^S_{a_{v,s}-1}) = \begin{cases} \tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{a_{v,s-1}[S\cup S']-1}\tilde{\mathrm{d}}_{S\cup S'} & \text{if } s' \notin S, \\ \tilde{\mathrm{u}}_{S\cup S'}\tilde{\mathsf{e}}_{a_{v,s-1}[S]-1}\tilde{\mathrm{d}}_S & \text{if } s' \in S. \end{cases} \tag{3-8}$$

In the resulting elements we either see $\tilde{\mathrm{u}}_S$, with $s, s' \notin S$ or $\tilde{\mathrm{u}}_{S\cup S'}$ with $s, s' \in S \cup S'$.

Now, if X is down-admissible for $v[S']$, we compute

$$(\lambda_{v[S'],X}\,\tilde{\mathrm{L}}^{X}_{v[S']-1})\mathrm{d}_{S'}(\lambda_{a_{v,s},S}\,\mathbb{1}_{v-a_{v,s}}\otimes\tilde{\mathrm{L}}^{S}_{a_{v,s}-1})$$

$$=\begin{cases} c_1(v,Y)\,\tilde{\mathrm{u}}_{S_-}\,\tilde{\mathsf{e}}_{v[Y]-1}\tilde{\mathrm{d}}_Y & \text{if } s\notin X=:S_-,\ s'\notin S,\ X(\geq s')=S,\ Y:=S_-\cup S',\\ c_2(v,Y)\,\tilde{\mathrm{u}}_{S_+}\tilde{\mathsf{e}}_{v[Y]-1}\tilde{\mathrm{d}}_Y & \text{if } s\in X=:S_+,\ s'\in S,\ X(\geq s')=S,\ Y:=S_+\setminus S',\\ 0 & \text{otherwise,}\end{cases}$$

where the scalars $c_1(v,Y)$ and $c_2(v,Y)$ are computed as follows.

$$\begin{aligned} c_1(v,Y) &= \lambda_{v[S'],X}\,\lambda_{a_{v,s},S}\,\lambda^{-1}_{a_{v[S'],s-1},S}\\ &= \lambda_{v[S'],X}\,\lambda_{a_{v,s},S}\,\lambda^{-1}_{a_{v,s},S} = \lambda_{v[S'],Y\setminus S'}.\\ c_2(v,Y) &= \lambda_{v[S'],X}\,\lambda_{a_{v,s},S}\,\lambda^{-1}_{a_{v[S'],s-1},S\cup S'}\\ &= (-1)^{a_s\mathsf{p}^s}\tfrac{a_{v,s-1}[Y]}{a_{v,s}[Y]}\lambda_{v,Y}\,\lambda_{a_{v,s},S}\,(-1)^{a_s\mathsf{p}^s}\tfrac{a_{v,s}[Y]}{a_{v,s-1}[Y]}\lambda^{-1}_{a_{v,s},S} = \lambda_{v,Y}.\end{aligned}$$

Thus, by (3-8), we have

$$\begin{aligned}\overline{\mathsf{e}}_{v[S']-1}\mathrm{d}_{S'}(\mathbb{1}_{v-a_{v,s}}\otimes\overline{\mathsf{e}}_{a_{v,s}-1}) &= \sum_{X,S}(\lambda_{v[S'],X}\,\tilde{\mathrm{L}}^{X}_{v[S']-1})\mathrm{d}_{S'}(\lambda_{a_{v,s},S}\,\mathbb{1}_{v-a_{v,s}}\otimes\tilde{\mathrm{L}}^{S}_{a_{v,s}-1})\\ &= \mathrm{d}_{S'}\overline{\mathsf{e}}_{v-1}.\end{aligned}$$

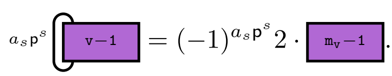
This establishes the third relation. The analogous relation for cups follow by reflection. ☐

The characteristic p analog of (2-4) is:

Proposition 3.14. *(Partial trace.)*

(a) *For $v\notin$ Eve, a_s being the first non-zero digit of v, we have*

(3-9)



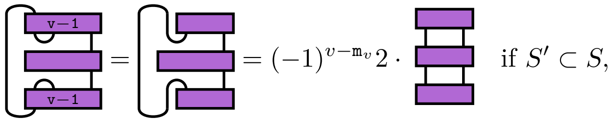
On the other hand, if $v\in$ Eve and $v\geq\mathsf{p}$, then the (partial) trace of e_{v-1} is zero.

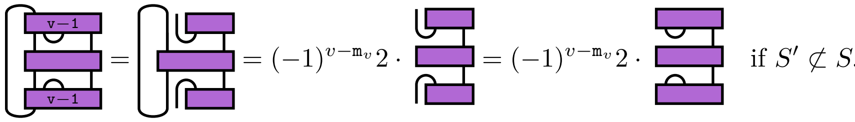
(b) *Let S be down-admissible for v and S' the smallest minimal down-admissible stretch for v. Then the partial trace on the bundle of strands specified by S' evaluates as:*

$$\mathrm{pTr}_{S'}(\mathrm{L}^{S}_{v-1})=\begin{cases}\mathrm{L}^{S\setminus S'}_{\mathtt{m}_v-1} & \text{if } S'\subset S,\\ (-1)^{v-\mathtt{m}_v}2\cdot\mathrm{L}^{S}_{\mathtt{m}_v-1} & \text{if } S'\not\subset S.\end{cases}$$

Proof. The second claim in Proposition 3.14.(a), concerning the case of $v\in$ Eve, follows from $\overline{\mathsf{e}}_{v-1}=\tilde{\mathsf{e}}_{v-1}$ and (2-4), which produces a scalar $a\in\mathbb{Q}$ with $\nu_{\mathsf{p}}(a)>0$. The case $v\notin$ Eve follows immediately by applying (2-4) to the two expressions in the bracket in (2-10).

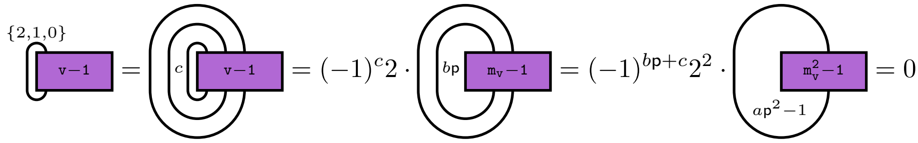
In Proposition 3.14.(a) we have already seen the case $S=\emptyset$, so we assume that $S\neq\emptyset$. We then apply the projector shortening relations from Proposition 3.13, and get the following two cases for $\mathrm{pTr}_{S'}(\mathrm{L}^S_{v-1})$, depending on whether $S' \subset S$ or $S' \not\subset S$.





Here we have used Proposition 3.14 for the second equation in the bottom row. □

Example 3.15. Note that (3-9) and (2-4) (for eves) give a recursive way to compute traces. For example, for $v = [a, b, c]_{\mathtt{p}}$ we have $\mathtt{m}_v = [a, b, 0]_{\mathtt{p}}$, $\mathtt{m}^2_v = [a, 0, 0]_{\mathtt{p}}$, and



The proposition also implies that the (full) trace of the pJW projectors are zero unless $v < \mathtt{p}$.

3C. **Morphisms between pJW projectors—the linear structure.** First, we state direct consequences of classical absorption, see Proposition 3.11, and non-classical absorption, see Proposition 3.13.

Lemma 3.16.

(a) *If $S = \bigsqcup_{j=0}^{k} S_{i_j}$ with $S_{i_k} > \cdots > S_{i_1} > S_{i_0}$, each down-admissible for v, and $S' = \bigsqcup_{j=0}^{l} S'_{i_j}$ with $S'_{i_l} > \cdots > S'_{i_1} > S'_{i_0}$, each up-admissible for v, then*

$$\mathrm{D}_S \mathtt{e}_{v-1} = \mathrm{D}_{S_{i_0}} \cdots \mathrm{D}_{S_{i_k}} \mathtt{e}_{v-1}, \qquad \mathrm{U}_{S'} \mathtt{e}_{v-1} = \mathrm{U}_{S'_{i_l}} \cdots \mathrm{U}_{S'_{i_0}} \mathtt{e}_{v-1}.$$

(b) *Let S and S' be down- and up-admissible for v, respectively. Then we have*

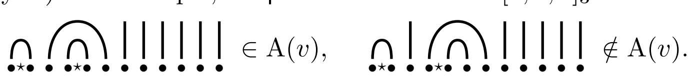
$$\mathtt{e}_{v[S]-1} \mathrm{d}_S \mathtt{e}_{v-1} = \mathrm{d}_S \mathtt{e}_{v-1} = \mathtt{e}_{v[S]-1} \mathrm{d}_S (\mathbb{1}_{v-a_{v(S),S}} \otimes \mathtt{e}_{a_{v,S}-1}).$$

$$\mathtt{e}_{v(S')-1} \mathrm{u}_{S'} \mathtt{e}_{v-1} = \mathtt{e}_{v(S')-1} \mathrm{u}_{S'} = (\mathbb{1}_{v-a_{v(S'),S'}} \otimes \mathtt{e}_{a_{v(S'),S'}-1}) \mathrm{u}_{S'} \mathtt{e}_{v-1}.$$

Here $a_{v,S}$ denotes the youngest ancestor of v for which all digits with indices in S are zero. □

Let F be a cap (or cup) configuration, i.e. a Temperley–Lieb diagram consisting only of caps (or cups), with source (target) $v-1$. We say that F is ancestor-centered for v, and write $\mathrm{F} \in \mathrm{A}(v)$, if each cap or cup has its center immediately to the right of an ancestor strand of v.

Example 3.17. A way to illustrate ancestor-centering is imagine a line with a marker $\star$ to the right of each ancestor strand of v. (There are $v-1$ strands in total and $\mathtt{g}_v$ many $\star$.) For example, for $\mathtt{p} = 3$ and $v = 13 = [1, 1, 1]_3$ we have



Moreover, we have $\mathrm{d}_i \mathbb{1}_{v-1}, \mathbb{1}_{v-1}\mathrm{u}_i \in \mathrm{A}(v)$, with mid point right to $[a_j, \dots, a_{i+1}, 0, \dots, 0]_{\mathsf{p}} \in \mathrm{A}(v)$. More generally, the morphisms $\mathrm{d}_S \mathbb{1}_{v-1}$ and $\mathbb{1}_{v-1}\mathrm{u}_S$ are ancestor-centered if S is down- or up-admissible, respectively.

The following is the analog of (2-3).

Lemma 3.18. *For a cap configuration* d *we have* $\mathrm{d}\mathsf{e}_{v-1} = 0$ *unless* $\mathrm{d} \in \mathrm{A}(v)$. *Analogously for cup configurations.*

Proof. By assumption, d contains a cap which is not centered around an ancestor of v. By expanding $\mathrm{d}\overline{\mathsf{e}}_{v-1}$ along (2-9), we see that this cap hits a JW projector in every summand in (2-9), and thus annihilates $\overline{\mathsf{e}}_{v-1}$. □

Lemma 3.19.

(a) *Suppose that* S *and* S' *are down-admissible for* w *and* v*, respectively, with* $w[S] = v[S']$. *Then we have*

$$\overline{\mathsf{e}}_{w-1}\mathrm{U}_S\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1} = \tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{v[S']-1}\tilde{\mathrm{d}}_{S'} + \sum_{X,Y} c_{X,Y}\tilde{\mathrm{u}}_X\tilde{\mathsf{e}}_{v[Y]-1}\tilde{\mathrm{d}}_Y, \tag{3-10}$$

for some coefficients $c_{X,Y} \in \mathbb{Q}$*, where* X *and* Y *are down-admissible for* w *and* v*, respectively, and* $w[X] = v[Y] < v[S']$. *(In other words* $\overline{\mathsf{e}}_{w-1}\mathrm{U}_S\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1} - \tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{v[S']-1}\tilde{\mathrm{d}}_{S'} \in \mathbf{td}_>$.*)*

(b) *We have isomorphisms of* $\mathbb{Q}$*-vector spaces*

$$\mathrm{Hom}_{\mathbb{Q}\mathbf{TL}}(\overline{\mathsf{e}}_{v-1}, \overline{\mathsf{e}}_{w-1}) \cong \mathrm{span}_{\mathbb{Q}}(\tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{v[S]-1}\tilde{\mathrm{d}}_{S'}) \cong \mathrm{span}_{\mathbb{Q}}(\overline{\mathsf{e}}_{w-1}\mathrm{U}_S\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1}), \tag{3-11}$$

where (S, S') *ranges over pairs of sets that are down-admissible for* w *and* v*, respectively, such that* $w[S] = v[S']$. *In particular,* $\mathrm{End}_{\mathbb{Q}\mathbf{TL}}(\overline{\mathsf{e}}_{v-1}) \cong \mathrm{span}_{\mathbb{Q}}(\mathrm{L}^S_{v-1})$.

Note that the second isomorphism in (3-11) is unitriangular by (3-10). We will refer to morphisms of the form $\tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{v[S]-1}\tilde{\mathrm{d}}_{S'}$ as *standard morphisms* and to morphisms of the form $\overline{\mathsf{e}}_{w-1}\mathrm{U}_S\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1}$ as p*-morphisms*.

Proof. The proof of (a) proceeds by iterating Lemma 3.8. Let $S = \bigsqcup_i S_i$ and $S' = \bigsqcup_j S'_j$ be the partitions into minimal admissible stretches of consecutive integers with the usual ordering. Then we expand

$$\begin{aligned}\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1} = \mathrm{D}_{S'_1}\cdots\mathrm{D}_{S'_l}\overline{\mathsf{e}}_{v-1} &= \sum_X \mathrm{d}_{S'_1}\cdots\mathrm{d}_{S'_{l-1}}\cdot\lambda_{v,X}\cdot\mathrm{d}_{S'_l}\tilde{\mathrm{u}}_X\tilde{\mathsf{e}}_{v[X]-1}\tilde{\mathrm{d}}_X \\ &\in \sum_{X\supset S'_l} \mathrm{d}_{S'_1}\cdots\mathrm{d}_{S'_{l-2}}\cdot\lambda_{v[X\setminus S'_l],X\setminus S'_l}\cdot\mathrm{d}_{S'_{l-1}}\tilde{\mathrm{u}}_{R\setminus S'_l}\tilde{\mathsf{e}}_{v[X]-1}\tilde{\mathrm{d}}_X + \mathbf{td}_>,\end{aligned}$$

since by (3-5), we have $\lambda_{v,X}\cdot\mathrm{d}_{S'_l}\tilde{\mathrm{u}}_X = \lambda_{v,X\setminus S'_l}\cdot\tilde{\mathrm{u}}_{X\setminus S'_l}$ if $S'_l \subset X$, and otherwise $\max(S'_l)+1 \in X$ and thus $v[X] < v[S']$. Here we write $\mathbf{td}_>$ for the ideal of morphisms of smaller through-degree than the leading term. We now iterate this argument to find

$$\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1} \in \tilde{\mathsf{e}}_{v[S']-1}\tilde{\mathrm{d}}_{S'} + \mathbf{td}_>, \quad \overline{\mathsf{e}}_{w-1}\mathrm{U}_S \in \tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{w[S]-1} + \mathbf{td}_>,$$

which together imply (3-10).

To see the first isomorphism in (3-11): For a given $\mathrm{F} \in \mathrm{Hom}_{\mathbb{Q}\mathbf{TL}}(v-1, w-1)$, we compute

$$\begin{aligned}\overline{\mathsf{e}}_{w-1}\mathrm{F}\,\overline{\mathsf{e}}_{v-1} &= \sum_{S,S'} (\lambda_{w,S}\tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{w[S]-1}\tilde{\mathrm{d}}_S)\mathrm{F}(\lambda_{v,S'}\tilde{\mathrm{u}}_{S'}\tilde{\mathsf{e}}_{v[S']-1}\tilde{\mathrm{d}}_{S'}) \\ &= \sum_{S,S'} \delta_{w[S],v[S']}\tilde{\mathrm{u}}_S(\lambda_{w,S}\lambda_{v,S'}\tilde{\mathsf{e}}_{w[S]-1}\tilde{\mathrm{d}}_S\mathrm{F}\tilde{\mathrm{u}}_{S'}\tilde{\mathsf{e}}_{v[S']-1})\tilde{\mathrm{d}}_{S'} \\ &= \sum_{S,S'} \delta_{w[S],v[S']}c_{X,S,S'}\tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{v[S']-1}\tilde{\mathrm{d}}_{S'},\end{aligned}$$

where $c_{X,S,S'} \in \mathbb{Q}$. In the last two lines, we have used Lemma 3.7 and the fact the JW projectors have no endomorphisms besides scalar multiples of the identity, cf. (2-3). Finally, (3-10) implies then the second isomorphism in (3-11). □

Lemma 3.20 (Theorem 3.2.(Basis))**.**

(a) *Suppose that a* p*-admissible morphism is expressed as*

$$\sum_{S,S'} r_{S,S'} \cdot \overline{\mathsf{e}}_{w-1}\mathrm{U}_S\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1} \in \mathrm{Hom}_{\mathbb{Q}\mathbf{TL}}(v-1, w-1), \tag{3-12}$$

where $r_{S,S'} \in \mathbb{Q}$ *and the sum* (S, S') *ranges over pairwise distinct pairs of sets that are down-admissible for* w *and* v*, respectively, such that* $w[S] = v[S']$*. Then every coefficient* $r_{S,S'}$ *is* p*-admissible.*

(b) *We have the* $\mathbb{F}_{\mathsf{p}}$*-vector space isomorphisms*

$$\mathrm{Hom}_{\mathbb{F}_{\mathsf{p}}\mathbf{TL}}(\mathsf{e}_{v-1}, \mathsf{e}_{w-1}) \cong \mathrm{span}_{\mathbb{F}_{\mathsf{p}}}(\mathsf{e}_{w-1}\mathrm{U}_S\mathrm{D}_{S'}\mathsf{e}_{v-1}),$$

where (S, S') *ranges over the same set as above. In particular*

$$\mathrm{End}_{\mathbb{F}_{\mathsf{p}}\mathbf{TL}}(\mathsf{e}_{v-1}) \cong \mathrm{span}_{\mathbb{F}_{\mathsf{p}}}(\mathrm{L}^S_{v-1} | S \textit{ down-admissible for } v).$$

Proof. For the first claim, we proceed by induction on the through-degree. Note that the through-degree of $\overline{\mathsf{e}}_{w-1}\mathrm{U}_S\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1}$ is $w[S] = v[S']$. Let (S, S') be the pair labeling the summand with maximal through-degree. Then $r_{S,T}$ is p-admissible since it is the coefficient of the (maximal through-degree) basis element $\mathrm{u}_S\,\mathbb{1}_{v[S']-1}\mathrm{d}_{S'}$ in (3-12). Thus, we can subtract $r_{S,S'} \cdot \overline{\mathsf{e}}_{w-1}\mathrm{U}_S\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1}$ to obtain another p-admissible sum, which now has strictly lower through-degree since

$$r_{S,S'} \cdot \overline{\mathsf{e}}_{w-1}\mathrm{U}_S\mathrm{D}_{S'}\overline{\mathsf{e}}_{v-1}$$

was the only summand with this maximal through-degree. If the resulting sum is non-zero, then the remaining coefficients are now p-admissible by the induction hypothesis. The basis step for the induction concerns the morphism of minimal possible through-degree, which is p-admissible (and thus also its coefficient) since there are no correction terms in (3-10).

To see (b), for any given $\mathrm{F} \in \mathrm{Hom}_{\mathbb{F}_{\mathsf{p}}\mathbf{TL}}(v-1, w-1)$, we choose a lift $\tilde{\mathrm{F}} \in \mathrm{Hom}_{\mathbb{Z}\mathbf{TL}}(v-1, w-1) \subset \mathrm{Hom}_{\mathbb{Q}\mathbf{TL}}(v-1, w-1)$. By (3-11), the p-admissible morphism $\overline{\mathsf{e}}_{w-1}\tilde{\mathrm{F}}\,\overline{\mathsf{e}}_{v-1}$ can be expanded in the p-morphism basis over $\mathbb{Q}$. By (a), all appearing coefficients are p-admissible and can be specialized to $\mathbb{F}_{\mathsf{p}}$. This results in an expansion of $\mathsf{e}_{w-1}\mathrm{F}\,\mathsf{e}_{v-1}$ in terms of the p-morphisms over $\mathbb{F}_{\mathsf{p}}$. Note that all such morphisms are still linearly independent, since they have distinct through-degrees. □

3D. Morphisms between pJW projectors—the algebra structure.

Lemma 3.21.

(a) *The algebra* $\mathrm{End}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1})$ *is commutative.*

(b) *Every* L^S_{v-1} *is nilpotent. As a consequence, every element of non-maximal through-degree in* $\mathrm{End}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1})$ *is nilpotent.*

Proof. By Lemma 3.20.(b), $\mathrm{End}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1})$ has a basis that is invariant under reflection. Thus, for all $a, b \in \mathrm{End}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1})$ we have $a^\star = a$ and $b^\star = b$, and then $ab = a^\star b^\star = (ba)^\star = ba$. This implies that $\mathrm{End}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1})$ is commutative.

To see (b), we shall use induction on $\mathrm{td}(\mathrm{L}^S_{v-1})$. We work over $\mathbb{Q}$ and start by expanding $\mathrm{L}^S_{v-1} \in \tilde{\mathrm{L}}^S_{v-1} + \mathbf{td}_>$ into a sum of orthogonal quasi-idempotents and noting that $\tilde{\mathrm{L}}^S_{v-1}$ has eigenvalue divisible by p. If S was maximal, then we have $(\mathrm{L}^S_{v-1})^2 = (\tilde{\mathrm{L}}^S_{v-1})^2 = 0$ in $\mathrm{End}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1})$. Otherwise, if $S \neq \emptyset$, we conclude $\mathrm{td}((\mathrm{L}^S_{v-1})^2) < \mathrm{td}(\mathrm{L}^S_{v-1})$. By Lemma 3.20.(b), $(\mathrm{L}^S_{v-1})^2$ is a linear combination of $\mathsf{p}$loops $\mathrm{L}^{S'}_{v-1}$ with $\mathrm{td}(\mathrm{L}^{S'}_{v-1}) < \mathrm{td}(\mathrm{L}^S_{v-1})$. Then the induction hypothesis implies that $(\mathrm{L}^S_{v-1})^2$, and thus also L^S_{v-1}, is nilpotent. □

Lemma 3.22 (Containment—Theorem 3.2.(2).)**.** *Let S be a stretch that is down- or up-admissible for v and $S' \subset S$ down-admissible for $v[S]$ or up-admissible for $v(S)$ respectively. Then we have*

$$\mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1} = 0, \quad \mathrm{U}_S\mathrm{U}_{S'}\mathsf{e}_{v-1} = 0.$$

Proof. Note that by projector absorption, we have $\mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1} = \mathrm{d}_{S'}\mathrm{d}_S\mathsf{e}_{v-1}$. This is a cap configuration consisting of a pair of collections of concentric caps. The right one is not ancestor-centered and, thus, kills e_{v-1} by Lemma 3.18. □

Lemma 3.23 (Far-commutativity—Theorem 3.2.(3))**.** *Suppose that S and S' are down-admissible, T and T' up-admissible and $\mathsf{d}(S,S') > 1$, $\mathsf{d}(S,T) > 1$, and $\mathsf{d}(T,T') > 1$. The following hold.*

$$\mathrm{D}_S\mathrm{D}_{S'}\mathsf{e}_{v-1} = \mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1}, \quad \mathrm{D}_S\mathrm{U}_T\mathsf{e}_{v-1} = \mathrm{U}_T\mathrm{D}_S\mathsf{e}_{v-1}, \quad \mathrm{U}_T\mathrm{U}_{T'}\mathsf{e}_{v-1} = \mathrm{U}_{T'}\mathrm{U}_T\mathsf{e}_{v-1}.$$

Proof. These relations follow from projector absorption. For example, for the first relation we compute

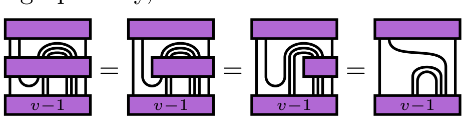
$$\mathrm{D}_S\mathrm{D}_{S'}\mathsf{e}_{v-1} = \mathrm{d}_S\mathsf{e}_w\mathrm{d}_{S'}\mathsf{e}_{v-1} = \mathrm{d}_S\mathrm{d}_{S'}\mathsf{e}_{v-1} = \mathrm{d}_{S'}\mathrm{d}_S\mathsf{e}_{v-1} = \mathrm{d}_{S'}\mathsf{e}_z\mathrm{d}_S\mathsf{e}_{v-1} = \mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1}.$$

Here we have used an isotopy of caps in the third equality. □

Lemma 3.24 (Adjacency relations 1—Theorem 3.2.(4))**.** *If $\mathsf{d}(S,S') = 1$ and $S' > S$, then the following equations hold whenever one side, and thus also the other one, is admissible*

$$\mathrm{D}_{S'}\mathrm{U}_S\mathsf{e}_{v-1} = \mathrm{D}_S\mathrm{D}_{S'}\mathsf{e}_{v-1}, \quad \mathrm{D}_S\mathrm{U}_{S'}\mathsf{e}_{v-1} = \mathrm{U}_{S'}\mathrm{U}_S\mathsf{e}_{v-1}.$$

Proof. The first relation follows from projector shortening and absorption, as can be best verified graphically, i.e.

$\mathrm{D}_{S'}\mathrm{U}_S\mathsf{e}_{v-1} =$  $= \mathrm{D}_{S\cup S'}\mathsf{e}_{v-1} = \mathrm{D}_S\mathrm{D}_{S'}\mathsf{e}_{v-1}.$

Here we have used projector shortening twice, then projector absorption and an isotopy. The second relation is analogous. □

The following four statements will be proved jointly by induction in v. The proofs depend on each other in a non-trivial way.

Lemma 3.25 (The endomorphisms). *Let $v \in \mathbb{N}$ with minimal down-admissible stretches $S_j, \dots, S_0$. Then we have the algebra isomorphism*

$$\mathrm{End}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1}) \cong \mathbb{F}_\mathsf{p}\big[\mathrm{L}^{S_j}_{v-1}, \dots, \mathrm{L}^{S_0}_{v-1}\big] \Big/ \big\langle (\mathrm{L}^{S_j}_{v-1})^2, \dots, (\mathrm{L}^{S_0}_{v-1})^2 \big\rangle,$$

and if S is down-admissible for v, then $\mathrm{L}^S_{v-1} = \prod_{k|S_k \subset S} \mathrm{L}^{S_k}_{v-1}$. Furthermore, if S is down-admissible for v, then we have

$$\mathrm{D}_S\mathrm{U}_S\mathrm{D}_S\mathsf{e}_{v-1} = 0, \quad \mathsf{e}_{v-1}\mathrm{U}_S\mathrm{D}_S\mathrm{U}_S = 0. \tag{3-13}$$

Lemma 3.26 (Adjacency relations 2—Theorem 3.2.(4)). *Let $S' > S$ be down-admissible stretches of consecutive integers for v with $\mathsf{d}(S, S') = 1$. Then we have*

$$\mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1} = \mathrm{U}_S\mathrm{D}_{S'}\mathtt{h}_{S}\mathsf{e}_{v-1}, \quad \mathsf{e}_{v-1}\mathrm{U}_S\mathrm{U}_{S'} = \mathsf{e}_{v-1}\mathtt{h}_S\mathrm{U}_{S'}\mathrm{D}_S.$$

Lemma 3.27 (Overlap relations—Theorem 3.2.(5)). *Suppose that S is a minimal down-admissible stretch for v and $S' \geq S$ a minimal down-admissible stretch for $v[S]$ with $S' \cap S = \{s\}$ and $S' \not\subset S$, then we have*

$$\mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1} = \mathrm{U}_{\{s\}}\mathrm{D}_S\mathrm{D}_{S'\setminus\{s\}}\mathsf{e}_{v-1}, \quad \mathsf{e}_{v-1}\mathrm{U}_S\mathrm{U}_{S'} = \mathsf{e}_{v-1}\mathrm{U}_{S'\setminus\{s\}}\mathrm{U}_S\mathrm{D}_{\{s\}}.$$

Lemma 3.28 (Zigzag—Theorem 3.2.(6)). *Suppose that S is an up-admissible stretch for v. If S is also down-admissible for v, then we have*

$$\mathrm{D}_S\mathrm{U}_S\mathsf{e}_{v-1} = \mathrm{U}_S\mathrm{D}_S\mathtt{g}_S\mathsf{e}_{v-1} + \mathrm{U}_T\mathrm{U}_S\mathrm{D}_S\mathrm{D}_T\mathtt{f}_S\mathsf{e}_{v-1}.$$

Here T denotes the smallest minimal down-admissible stretch with $T > S$, provided it exists. If not, then the equation holds without the second term on the right-hand side.

Furthermore, if S is not down-admissible for v, then we have

$$\mathrm{D}_S\mathrm{U}_S\mathsf{e}_{v-1} = -2\mathrm{U}_{\overline{S}}\mathrm{D}_{\overline{S}}\mathsf{e}_{v-1}.$$

Here $\overline{S}$ denotes the down-admissible hull of S, if it exists. If not, then the right-hand side is defined to be zero.

4. Inductive proof of the relations

In this section we will use the far-commutativity relations from Lemma 3.23, the containment relations from Lemma 3.22, and the adjacency relations from Lemma 3.24, sometimes without explicitly mentioning them. Further, we only prove Lemmas 3.26 and 3.27, and (3-13) for the first shown relations as the other ones are equivalent by reflection.

Convention 4.1. Throughout this section, unless stated otherwise, we use the convention that S denotes either a minimal down- or up-admissible stretch for v, and $U > T > \overline{S}$ are the following minimal down-admissible stretches for v. To declutter the notation, we will suppress $\cup$ symbols in many expressions, for example $\mathrm{D}_{STU} := \mathrm{D}_{S\cup T\cup U}$. Further, we introduce shorthand notation for the states where we have already proven the above Lemmas for certain $v \in N$.

- $Z_-(v)$ means Lemma 3.28 holds for all zigzags of the form $\mathrm{D}_X\mathsf{e}_{w-1}\mathrm{U}_X$ where $w \leq v$ and X is down-admissible for w, except possibly for the case $w = v$ and $X = S$, the smallest minimal down-admissible stretch for v.

- $A(v)$ means Lemma 3.26 on adjacent generators holds for all $w \leq v$.
- $O(v)$ means Lemma 3.27 on overlapping generators holds for all $w \leq v$.
- $Z(v)$ means Lemma 3.28, holds for all zigzags of the form $\mathrm{D}_X \mathtt{e}_{w-1} \mathrm{U}_X$ where $w \leq v$.
- $E(v)$ means Lemma 3.25, which describes $\mathrm{End}_{\mathbb{F}_{\mathsf{p}}\mathbf{TL}}(\mathtt{e}_{w-1})$, holds for all $w \leq v$.

Here we would like to draw the readers attention to the fact that the relevant quantity for zigzags is not where they start, but how high they reach.

The inductive proof of these conditions will proceed in the order shown. As base cases we observe that $A(v)$, $O(v)$, $E(v)$ and $Z(v)$ are all vacuously satisfied for $1 \leq v \leq \mathsf{p}$. Then, assuming that these conditions all hold for $v-1$, we will first deduce $Z_{-}(v)$, then $A(v)$ and $O(v)$, followed by $Z(v)$, and finally $E(v)$.

Lemma 4.2. *$Z_{-}(v)$ follows if we have $Z(v-1)$.*

Proof. We need to show that we can resolve all zigzags of the form $\mathrm{D}_Y \mathrm{U}_Y \mathtt{e}_{v[Y]-1}$ where Y denotes a down-admissible stretch for v such that $Y \neq S$, the smallest minimal down-admissible stretch for v. If $S \not\subset Y$, then this is possible using projector absorption and $Z(v-1)$. In the remaining cases we write $Y_{+} := Y \setminus S$ and employ the same trick, but for $Y_{+} \not\supset S$. If Y_{+} is down-admissible for v, we get

$$\begin{aligned}\mathrm{D}_Y \mathrm{U}_Y \mathtt{e}_{v[Y]-1} &= \mathrm{D}_S \underset{\sim\sim\sim}{\mathrm{D}_{Y_+} \mathrm{U}_{Y_+}} \mathrm{U}_S \mathtt{e}_{v[Y]-1} \\ &= \underset{\sim\sim\sim}{\mathrm{D}_S \mathrm{U}_{Y_+}} \underset{\sim\sim\sim}{\mathrm{D}_{Y_+} \mathrm{U}_S} \mathtt{g}_Y \mathtt{e}_{v[Y]-1} + \underset{\sim\sim\sim}{\mathrm{D}_S \mathrm{U}_T \mathrm{U}_{Y_+}} \underset{\sim\sim\sim}{\mathrm{D}_{Y_+} \mathrm{D}_T \mathrm{U}_S} \mathtt{f}_Y \mathtt{e}_{v[Y]-1} \\ &= \mathrm{U}_{Y_+} \mathrm{U}_S \mathrm{D}_S \mathrm{D}_{Y_+} \mathtt{g}_Y \mathtt{e}_{v[Y]-1} + \mathrm{U}_T \mathrm{U}_{Y_+} \mathrm{U}_S \mathrm{D}_S \mathrm{D}_{Y_+} \mathrm{D}_T \mathtt{f}_Y \mathtt{e}_{v[Y]-1} \\ &= \mathrm{U}_Y \mathrm{D}_Y \mathtt{g}_Y \mathtt{e}_{v[Y]-1} + \mathrm{U}_T \mathrm{U}_Y \mathrm{D}_Y \mathrm{D}_T \mathtt{f}_Y \mathtt{e}_{v[Y]-1}.\end{aligned}$$

Here T denotes the smallest minimal down-admissible stretch $T > Y$ for $v[Y]$, if it exists. We have also underlined the locations where relations are applied. If Y_{+} is not down-admissible for v, then we instead get

$$\begin{aligned}\mathrm{D}_Y \mathrm{U}_Y \mathtt{e}_{v[Y]-1} = \mathrm{D}_S \underset{\sim\sim\sim}{\mathrm{D}_{Y_+} \mathrm{U}_{Y_+}} \mathrm{U}_S \mathtt{e}_{v[Y]-1} &= -2 \underset{\sim\sim\sim}{\mathrm{D}_S \mathrm{U}_{\overline{Y_+}}} \underset{\sim\sim\sim}{\mathrm{D}_{\overline{Y_+}} \mathrm{U}_S} \mathtt{e}_{v[Y]-1} \\ &= -2 \mathrm{U}_{\overline{Y_+}} \mathrm{U}_S \mathrm{D}_S \mathrm{D}_{\overline{Y_+}} \mathtt{e}_{v[Y]-1} = -2 \mathrm{U}_{\overline{Y}} \mathrm{D}_{\overline{Y}} \mathtt{e}_{v[Y]-1},\end{aligned}$$

or zero, if $\overline{Y_+}$ (and thus $\overline{Y}$) does not exist. □

4A. **Adjacency relations.** Next we focus on establishing $A(v)$. These relations are irrelevant for $\mathsf{p} = 2$, so we will assume $\mathsf{p} > 2$ in this subsection. For this we need an approximate result first.

Lemma 4.3. *Suppose that $S < T < U$ are adjacent minimal down-admissible stretches for v. Then we have*

$$\mathrm{D}_T \mathrm{D}_S \mathtt{e}_{v-1} \in \mathrm{U}_S \mathrm{D}_T \mathtt{h}_S \mathtt{e}_{v-1} + V_{>U}.$$

Here $V_{>U} = \mathrm{span}_{\mathbb{F}_{\mathsf{p}}}(\mathrm{U}_S \mathrm{D}_T \mathtt{e}_{v-1} \mid \exists t \in T \text{ such that } t > U)$ is the span of morphisms with T exceeding U.

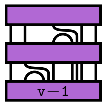
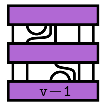
Similarly, if the stretches are up-admissible for v, then we have

$$\mathrm{U}_S \mathrm{U}_T \mathtt{e}_{v-1} \in \mathtt{h}_S \mathrm{U}_T \mathrm{D}_S \mathtt{e}_{v-1} + W_{>U}.$$

where $W_{>U} = \mathrm{span}_{\mathbb{F}_{\mathsf{p}}}(\mathrm{U}_S \mathrm{D}_T \mathtt{e}_{v-1} \mid \exists s \in S \text{ such that } s > U)$.

In either case, if U is a largest down-admissible stretch for v, or if no down-admissible stretch exists above T, then the relations from Lemma 3.26 hold on the nose.

Proof. Let us write $\mathtt{h}$ for the scalar appearing in $\mathtt{h}\mathsf{e}_{v-1} = \mathtt{h}_S\mathsf{e}_{v-1}$. (The functions $\mathtt{f}$, $\mathtt{g}$, and $\mathtt{h}$ were defined in Subsection 3A.) We would like to identify

$$\mathsf{e}_{w-1}\mathrm{D}_T\mathrm{D}_S\mathsf{e}_{v-1} =$$  and $\mathtt{h}\cdot$  $= \mathtt{h}\mathsf{e}_{w-1}\mathrm{U}_S\mathrm{D}_T\mathsf{e}_{v-1}.$

By projector absorption it suffices to do this in the case when S is the smallest minimal down-admissible stretch of v. We will start by computing the characteristic zero analogs of both sides.

Suppose that $V \subset \mathbb{N}_0$ is down-admissible for v, then by Lemma 3.8 we have

$$\mathrm{d}_S\lambda_{v,V}\,\tilde{\mathrm{L}}^V_{v-1} = \begin{cases} \lambda_{v[S],V\setminus S}\,\tilde{\mathrm{u}}_{V\setminus S}\tilde{\mathsf{e}}_{v[V]-1}\tilde{\mathrm{d}}_V & \text{if } S \subset V, T \not\subset V,\\ \lambda_{v,V}\,\tilde{\mathrm{u}}_{VS}\tilde{\mathsf{e}}_{v[V]-1}\tilde{\mathrm{d}}_V & \text{if } S \not\subset V, T \subset V,\\ 0 & \text{otherwise.}\end{cases}$$

After another application of Lemma 3.8 we get

(4-1)

$$\mathrm{d}_T\mathrm{d}_S\lambda_{v,V}\,\tilde{\mathrm{L}}^V_{v-1} = \begin{cases} \lambda_{v[S],V\setminus S}\,\tilde{\mathrm{u}}_{VT\setminus S}\tilde{\mathsf{e}}_{v[V]-1}\tilde{\mathrm{d}}_V & \text{if } S \subset V, T \not\subset V, U \subset V,\\ \lambda_{v,V}\,\lambda^{-1}_{v[S],VS}\,\lambda_{v[T](S),VS\setminus T}\,\tilde{\mathrm{u}}_{VS\setminus T}\tilde{\mathsf{e}}_{v[V]-1}\tilde{\mathrm{d}}_V & \text{if } S \not\subset V, T \subset V, U \not\subset V,\\ 0 & \text{otherwise.}\end{cases}$$

These possibilities for V index the p-morphism basis for $\mathrm{Hom}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1}, \mathsf{e}_{w-1})$ from Lemma 3.20. The term of maximal through-degree arises for $V = T$. Now, by the unitriangularity of the basis change between p-morphisms and standard morphisms, we can read off the coefficient of $\mathrm{U}_S\mathrm{D}_T\mathsf{e}_{v-1}$ in the p-morphism expansion of $\mathrm{D}_T\mathrm{D}_S\mathsf{e}_{v-1}$ as the coefficient of $\tilde{\mathrm{u}}_S\tilde{\mathsf{e}}_{v[T]-1}\tilde{\mathrm{d}}_T$ in (4-1). Writing $S = \{i, i+1, \dots, i_1 - 1\}$ and $T = \{i_1, \dots, i_2 - 1\}$, we compute this coefficient as

$$\text{(4-2)}\quad q = \frac{\lambda_{v,T}\,\lambda_{v[T](S),S}}{\lambda^{-1}_{v[S],ST}} = \frac{\lambda_{v,T}}{\lambda_{v[S],T}} = (-1)^{\mathsf{p}^{i_1}}\,\frac{[a_j,\dots,a_{i_2},-a_{i_2-1},\dots,-a_{i_1+1},-a_{i_1},0,\dots,0]_\mathsf{p}}{[a_j,\dots,a_{i_2},-a_{i_2-1},\dots,-a_{i_1+1},1-a_{i_1},0,\dots,0]_\mathsf{p}}$$
$$= (-1)^{\mathsf{p}^{i_1}}\,\frac{a_{v,S}[T]}{a_{v,S}[T]+\mathsf{p}^{i_1}}.$$

From this, we immediately get $q \equiv \mathtt{g}(a_{i_1} - 1) = \mathtt{h} \bmod \mathsf{p}$, as desired.

To finish the proof, we also need to show that $\mathrm{U}_{UT}\mathrm{D}_{SU}\mathsf{e}_{v-1}$ (the basis morphism of second highest through-degree in $\mathrm{Hom}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1}, \mathsf{e}_{w-1})$) does not occur in the p-morphism expansion of $\mathrm{D}_T\mathrm{D}_S\mathsf{e}_{v-1}$. Thanks to triangularity of the basis change, this can be verified by computing the coefficient q' of $\tilde{\mathrm{u}}_{UT}\tilde{\mathsf{e}}_{v[SU]-1}\tilde{\mathrm{d}}_{SU}$ in the difference $\overline{\mathsf{e}}_{w-1}\mathrm{d}_T\mathrm{d}_S\overline{\mathsf{e}}_{v[T]-1} - q\overline{\mathsf{e}}_{w-1}\tilde{\mathrm{u}}_S\tilde{\mathrm{d}}_T\overline{\mathsf{e}}_{v-1}$, and showing that it reduces to zero modulo p.

To this end, we again use Lemma 3.8 to expand

(4-3)

$$\overline{\mathsf{e}}_{w-1}\tilde{\mathrm{u}}_S\tilde{\mathrm{d}}_T\lambda_{v,V}\,\tilde{\mathrm{L}}^V_{v-1} = \begin{cases} \lambda_{v,V}\,\lambda^{-1}_{v[T],VT}\,\lambda_{v[T](S),VT\setminus S}\,\tilde{\mathrm{u}}_{VT\setminus S}\tilde{\mathsf{e}}_{v[V]-1}\tilde{\mathrm{d}}_V & \text{if } S \subset V, T \not\subset V, U \subset V,\\ \lambda_{v[T],V\setminus T}\,\tilde{\mathrm{u}}_{VS\setminus T}\tilde{\mathsf{e}}_{v[V]-1}\tilde{\mathrm{d}}_V & \text{if } S \not\subset V, T \subset V, U \not\subset V,\\ 0 & \text{otherwise.}\end{cases}$$

Focusing on the case $V = S \cup U$, we compute the crucial coefficient q' as

$$
\begin{aligned}
q' = \lambda_{v[S],U} - \tfrac{\lambda_{v,T}}{\lambda_{v[S],T}} \tfrac{\lambda_{v[S],SU}\,\lambda_{v[T](S),TU}}{\lambda_{v[T],SUT}} &= \lambda_{v,U}\left(1 - \tfrac{\lambda_{v,T}\,\lambda_{v[T](S),TU}}{\lambda_{v[S],T}\,\lambda_{v[T],TU}}\right) \\
&= \lambda_{v,U}\left(1 - \tfrac{\lambda_{v,T}\,\lambda_{v[SU](T),T}}{\lambda_{v[S],T}\,\lambda_{v[U](T),T}}\right),
\end{aligned}
\tag{4-4}
$$

where we have used $\lambda_{v[S],U} = \lambda_{v,U}$, $\lambda_{v[S],SU} = \lambda_{v,U}\,\lambda_{v[U],S}$ and $\lambda_{v[T],SUT} = \lambda_{v[T],UT}\,\lambda_{v[U],S}$ in the first line, and in the second line

$$
\begin{aligned}
\lambda_{v[T](S),TU} &= \lambda_{v[T](S),U}\,\lambda_{v[T](S)[U],T} = \lambda_{v[T],U}\,\lambda_{v[SU](T),T}, \\
\lambda_{v[T],TU} &= \lambda_{v[T],U}\,\lambda_{v[T][U],T} = \lambda_{v[T],U}\,\lambda_{v[U](T),T}.
\end{aligned}
$$

Now we note that

$$
\tfrac{\lambda_{v[SU](T),T}}{\lambda_{v[U](T),T}} = \tfrac{[a_j,\dots,a_{i_3},-a_{i_3-1},\dots,-a_{i_2},0,\dots,0,a_{i_1}-1,0,\dots,0]_{\mathsf{p}}}{[a_j,\dots,a_{i_3},-a_{i_3-1},\dots,-a_{i_2},0,\dots,0,a_{i_1},0,\dots,0]_{\mathsf{p}}} = \tfrac{a_{v[U],S}-\mathsf{p}^{i_1}}{a_{v[U],S}},
$$

and together with (4-2) we can continue

$$
\begin{aligned}
(4\text{-}4) &= \lambda_{v,U}\left(\tfrac{(a_{v,S}[T]+\mathsf{p}^{i_1})a_{v[U],S} - a_{v,S}[T](a_{v[U],S}-\mathsf{p}^{i_1})}{(a_{v,S}[T]+\mathsf{p}^{i_1})a_{v[U],S}}\right) \\
&= \lambda_{v,U}\left(\tfrac{\mathsf{p}^{i_1}(a_{v[U],S}+a_{v,S}[T])}{(a_{v,S}[T]+\mathsf{p}^{i_1})a_{v[U],S}}\right).
\end{aligned}
$$

This is divisible by $\frac{\mathsf{p}^{-|U|}\mathsf{p}^{|S|}\mathsf{p}^{|S|+|T|+|U|}}{\mathsf{p}^{|S|}\mathsf{p}^{|S|}} = \mathsf{p}^{|T|}$ and, thus, q' is zero modulo p. This completes the proof of the first claim of the lemma. The second one is analogous. □

Lemma 4.4. *$A(v)$ follows if we have $A(v-1)$, $E(v-1)$ and $Z_{-}(v)$.*

The proof will be split into two parts. First we give a proof that works under a technical assumption, which is generically satisfied. In the second part, we treat the remaining cases.

Proof, with caveat. By $A(v-1)$ and projector absorption we may assume that S is a smallest down-admissible stretch. At first, we will also assume that $S < T$ are minimal down-admissible stretches for v and that T is also down-admissible for $v[S]$. By Lemma 2.16, this implies that S is up-admissible for v and T is down-admissible for $v(S)$.

We already know that the desired equation holds up to certain potential error terms, i.e.

$$
\mathrm{D}_T\mathrm{D}_S\mathsf{e}_{v-1} = \mathtt{h}_1\mathrm{U}_S\mathrm{D}_T\mathsf{e}_{v-1} + \sum_X \big(c_X\mathrm{U}_{XUT}\mathrm{D}_{SUX} + d_X\mathrm{U}_{XS}\mathrm{D}_{TX}\big)\mathsf{e}_{v-1}, \tag{4-5}
$$

where the summation runs over down-admissible subsets $X > U$, $c_X, d_X \in \mathbb{F}_{\mathsf{p}}$ and where we write $\mathtt{h}_1 := \mathtt{g}(a_{\max(S)+1} - 1)$ for $v = [a_j, \dots, a_0]_{\mathsf{p}}$. We now multiply this equation with D_S on the left and with U_T on the right and rewrite it using $w = v[T]$ and Lemma 3.24 into

$$
\begin{aligned}
\mathrm{D}_{ST}\mathrm{U}_{TS}\mathsf{e}_{w-1} = {}&\mathtt{h}_1\mathrm{D}_S\mathrm{U}_S\mathrm{D}_T\mathrm{U}_T\mathsf{e}_{w-1} \\
&+ \sum_X \big(c_X\mathrm{L}^{XUTS}_{w-1} + d_X\mathrm{L}^{X}_{w-1}\mathrm{D}_S\mathrm{U}_S\mathrm{D}_T\mathrm{U}_T\big)\mathsf{e}_{w-1}.
\end{aligned}
\tag{4-6}
$$

This equation can be simplified using $Z_{-}(v)$. In this proof attempt, we only consider the *generic case* where T (and thus also TS) is down-admissible for w. So, using $Z_{-}(v)$ for the pair (v, ST) we get:

$$
\mathrm{D}_{ST}\mathrm{U}_{TS}\mathsf{e}_{v[T]-1} = \mathtt{g}_1\mathrm{L}^{TS}_{v[T]-1} + \mathtt{f}_1\mathrm{L}^{UTS}_{v[T]-1},
$$

where $\mathtt{g}_1 := \mathtt{g}(b_{\max(S\cup T)+1})$ and $\mathtt{f}_1 := \mathtt{f}(b_{\max(S\cup T)+1})$ are computed from $v[T] = [b_i, \dots, b_0]_{\mathsf{p}}$. Further, using $Z_-(v)$ for $(v[T](S), S)$ and (v, T) as well as $E(v-1)$ we compute

$$\begin{aligned}\mathrm{D}_S\mathrm{U}_S\mathrm{D}_T\mathrm{U}_T\mathtt{e}_{v[T]-1} &= (\mathtt{g}_2\mathrm{L}^{S}_{v[T]-1} + \mathtt{f}_2\mathrm{L}^{TS}_{v[T]-1})(\mathtt{g}_1\mathrm{L}^{T}_{v[T]-1} + \mathtt{f}_1\mathrm{L}^{UT}_{v[T]-1})\\ &= \mathtt{g}_2\mathtt{g}_1\mathrm{L}^{TS}_{v[T]-1} + \mathtt{g}_2\mathtt{f}_1\mathrm{L}^{UTS}_{v[T]-1},\end{aligned}$$

where $\mathtt{g}_1$ and $\mathtt{f}_1$ are as above, while $\mathtt{f}_2 := \mathtt{f}(b_{\max(S)+1}) = \mathtt{f}(\mathsf{p} - a_{\max(S)+1})$. We also have

$$\mathtt{g}_2 := \mathtt{g}(b_{\max(S)+1}) = \mathtt{g}(\mathsf{p} - a_{\max(S)+1}) = \mathtt{h}_1^{-1}.$$

(Note that $\mathtt{h}_1$ is invertible since $\mathsf{p} > 2$.) Using these two computations and $E(v-1)$, the equation (4-6) transforms into

$$0 = 0 + \sum_X \big((c_X + \mathtt{g}_2\mathtt{f}_1 d_X)\mathrm{L}^{XUTS}_{w-1} + \mathtt{g}_2\mathtt{g}_1 d_X \mathrm{L}^{XTS}_{w-1}\big).$$

Since the p-loops L^{Y}_{w-1} form a basis of $\mathrm{End}_{\mathbb{F}_{\mathsf{p}}\mathbf{TL}}(\mathtt{e}_{w-1})$ and the scalars $\mathtt{g}_1$ and $\mathtt{g}_2$ are non-zero by admissibility and $\mathsf{p} > 2$, we conclude $d_X = 0$ and then $c_X = 0$. Thus all error terms in (4-5) vanish. This completes the proof in the case where S and T are minimal.

In the general case, we partition S and S' into minimal down-admissible stretches $S_1 < \cdots < S_k$ and $S'_1 < \cdots < S'_l$, respectively. Then we have

$$\begin{aligned}\mathrm{D}_{S'}\mathrm{D}_S\mathtt{e}_{v-1} &= \mathrm{D}_{S'_1}\mathrm{D}_{S'_2}\cdots\mathrm{D}_{S'_l}\mathrm{D}_{S_1}\cdots\mathrm{D}_{S_{k-1}}\mathrm{D}_{S_k}\mathtt{e}_{v-1}\\ &= \mathrm{D}_{S_1}\cdots\mathrm{D}_{S_{k-1}}\mathrm{D}_{S'_1}\mathrm{D}_{S_k}\mathrm{D}_{S'_2}\cdots\mathrm{D}_{S'_l}\mathtt{e}_{v-1}\\ &= \mathrm{D}_{S_1}\cdots\mathrm{D}_{S_{k-1}}\mathrm{U}_{S_k}\mathrm{D}_{S'_1}\mathtt{h}_{S_k}\mathrm{D}_{S'_2}\cdots\mathrm{D}_{S'_l}\mathtt{e}_{v-1}\\ &= \mathrm{U}_{S_k}\cdots\mathrm{U}_{S_1}\mathrm{D}_{S'_1}\cdots\mathrm{D}_{S'_l}\mathtt{h}_{S_k}\mathtt{e}_{v-1} = \mathrm{U}_S\mathrm{D}_{S'}\mathtt{h}_S\mathtt{e}_{v-1}.\end{aligned}$$

Here we have first used far-commutativity, then $A(v-1)$ on the adjacent minimal stretches $S_k < S'_1$, and finally Lemma 3.24. Note also that $\mathtt{h}_{S_k} = \mathtt{h}_S$ far-commutes with $\mathrm{D}_{S'_2}\cdots\mathrm{D}_{S'_l}$. □

Proof of the remaining cases. In the previous proof we made the assumption that T, and thus also $T \cup S$, is down-admissible for $w = v[T]$. Now suppose this is not the case. At first we can proceed in a very similar way as in the previous proof. Whenever we use zigzag relations, we have to replace T by $\overline{T} = T \cup U$ and set the $\mathtt{f}$-term to zero. Hence, we get

$$\begin{aligned}\mathrm{D}_{ST}\mathrm{U}_{TS}\mathtt{e}_{w-1} &= \mathtt{g}_1\mathrm{L}^{UTS}_{w-1},\\ \mathrm{D}_S\mathrm{U}_S\mathrm{D}_T\mathrm{U}_T\mathtt{e}_{w-1} &= (\mathtt{g}_2\mathrm{L}^{S}_{w-1} + \mathtt{f}_2\mathrm{L}^{UTS}_{w-1})(\mathtt{g}_1\mathrm{L}^{UT}_{w-1}) = \mathtt{g}_2\mathtt{g}_1\mathrm{L}^{UTS}_{w-1},\end{aligned}$$

and the equation (4-6) transforms into

$$0 = 0 + \sum_X \big((c_X + \mathtt{g}_2\mathtt{g}_1 d_X)\mathrm{L}^{XUTS}_{w-1}\big).$$

This implies that the coefficients c_X and d_X are unit multiples of each other for every X. Next we will use a different strategy to show that $d_X = 0$, which thus implies $c_X = 0$ and finishes the proof. The strategy is to multiply both sides of (4-5) by L^{U}_{v-1} on the right, to equate the first two terms, to kill all terms with coefficients c_X, and to preserve all terms with coefficients d_X.

The first two terms are rewritten as

$$\mathrm{D}_T\mathrm{D}_S\mathrm{L}^U_{v-1} = \mathrm{D}_T\mathrm{U}_U\mathrm{D}_S\mathrm{D}_U\mathtt{e}_{v-1} = \mathrm{U}_U\mathrm{U}_T\mathrm{D}_S\mathrm{D}_U\mathtt{e}_{v-1}$$
$$\mathtt{h}_1\mathrm{U}_S\mathrm{D}_T\mathrm{L}^U_{v-1} = \mathtt{h}_1\mathrm{U}_S\mathrm{U}_U\mathrm{U}_T\mathrm{D}_U\mathtt{e}_{v-1} = \mathtt{h}_1\mathrm{U}_U\mathrm{U}_S\mathrm{U}_T\mathrm{D}_U\mathtt{e}_{v-1},$$

which are equal by virtue of $A(v-1)$ since $v[T](S)[U] < v$. We also note that the scalar that appears is exactly $\mathtt{h}_1^{-1}$. After subtracting these terms from the multiple of (4-5), we are left with

$$0 = \sum_X \big(c_X\mathrm{U}_{XUT}\mathrm{D}_{SUX}\mathrm{L}^U_{v-1} + d_X\mathrm{U}_{XS}\mathrm{D}_{TX}\mathrm{L}^U_{v-1}\big). \tag{4-7}$$

We first claim that $\mathrm{U}_{XUT}\mathrm{D}_{SUX}\mathrm{L}^U_{v-1} = 0$. To verify this, we distinguish between the two cases in which X is distant or adjacent to U. In the first case, we get

$$\mathrm{U}_{XUT}\mathrm{D}_{SUX}\mathrm{L}^U_{v-1} = \mathrm{U}_{XUT}\mathrm{D}_S\mathrm{D}_U\mathrm{U}_U\mathrm{D}_U\mathrm{D}_X\mathtt{e}_{v-1} = 0,$$

since $\mathrm{D}_U\mathrm{U}_U\mathrm{D}_U\mathrm{D}_X\mathtt{e}_{v-1} = 0$ thanks to $E(v-1)$ as $v[X] < v$. In the second case, we get

$$\mathrm{U}_{XUT}\mathrm{D}_{SUX}\mathrm{L}^U_{v-1} = \mathrm{U}_{XUT}\mathrm{D}_S\mathrm{D}_{UX}\mathrm{U}_U\mathrm{D}_U\mathtt{e}_{v-1} = \mathrm{U}_{XUT}\mathrm{D}_S\mathrm{D}_X\mathrm{U}_U\mathrm{U}_U\mathrm{D}_U\mathtt{e}_{v-1} = 0,$$

since $\mathrm{U}_U^2\mathtt{e}_{v-1} = 0$. This proves the claim.

Our second claim is that $\mathrm{U}_{XS}\mathrm{D}_{TX}\mathrm{L}^U_{v-1} \neq 0$ for every X and that these morphisms are linearly independent. Again it matters whether X is distant or adjacent to U. In the first case we get

$$\begin{aligned}\mathrm{U}_{XS}\mathrm{D}_{TX}\mathrm{L}^U_{v-1} = \mathrm{U}_{XS}\mathrm{D}_T\mathrm{U}_U\mathrm{D}_U\mathrm{D}_X\mathtt{e}_{v-1} &= \mathrm{U}_X\mathrm{U}_U\mathrm{U}_S\mathrm{U}_T\mathrm{D}_U\mathrm{D}_X\mathtt{e}_{v-1}\\ &\sim \mathrm{U}_X\mathrm{U}_U\mathrm{U}_T\mathrm{D}_S\mathrm{D}_U\mathrm{D}_X\mathtt{e}_{v-1}.\end{aligned}$$

Here we have used $A(v-1)$ for $v[U\cup X] < v$ to proceed to the second line. (We use $\sim$ to indicate unit proportionality.) In the second case we compute

$$\begin{aligned}\mathrm{U}_{XS}\mathrm{D}_{TX}\mathrm{L}^U_{v-1} &= \mathrm{U}_{XS}\mathrm{D}_{TU}\mathrm{D}_X\mathrm{D}_U\mathtt{e}_{v-1}\\ &\sim \mathrm{U}_{XS}\mathrm{D}_T\mathrm{D}_U\mathrm{U}_U\mathrm{D}_X\mathtt{e}_{v-1}\\ &\sim \mathrm{U}_{XS}\mathrm{D}_T\mathrm{U}_U\mathrm{D}_{UX}\mathtt{e}_{v-1} \quad = \mathrm{U}_{XS}\mathrm{U}_{UT}\mathrm{D}_{UX}\mathtt{e}_{v-1}\\ &\qquad\qquad\qquad\qquad\quad \sim \mathrm{U}_{XUT}\mathrm{D}_{SUX}\mathtt{e}_{v-1}\end{aligned}$$

This time we have used $A(v-1)$, namely on the ancestor $a_{v,T} < v$ using projector absorption, to get to the second and the fourth line, and $Z_-(v)$ in the form of a zigzag relation for $v[X](U) < v$, noting that U is down-admissible for $v[X]$, to get to the third line. The proportionality constants that appear in these steps are units and $\mathrm{U}_{XUT}\mathrm{D}_{SUX}\mathtt{e}_{v-1}$ are linearly independent as X varies.

Finally, the two claims and equation (4-7) imply that $d_X = 0$ for every X, and thus also $c_X = 0$, which finishes the proof of $A(v)$. □

4B. **Overlap relations.** Next, we focus on establishing $O(v)$. We again start with an approximate version.

Lemma 4.5. *Suppose that $S < T$ are adjacent minimal down-admissible stretches for v and $S' \geq S$ is a minimal down-admissible stretch for $v[S]$ with $S' \cap S = \{s\}$ and $S' \not\subset S$, then we have*

$$\mathrm{D}_{S'}\mathrm{D}_S\mathtt{e}_{v-1} = \mathrm{U}_{\{s\}}\mathrm{D}_S\mathrm{D}_{S'\setminus\{s\}}\mathtt{e}_{v-1} + V_{>T}, \quad \mathtt{e}_{v-1}\mathrm{U}_S\mathrm{U}_{S'} = \mathtt{e}_{v-1}\mathrm{U}_{S'\setminus\{s\}}\mathrm{U}_S\mathrm{D}_{\{s\}} + W_{>T}.$$

Here we use the notations $V_{>T} = \mathrm{span}_{\mathbb{F}_\mathsf{p}}(\mathrm{U}_X\mathrm{D}_Y\mathtt{e}_{v-1} \mid \exists y \in Y$ such that $y > T)$ and $W_{>T} = \mathrm{span}_{\mathbb{F}_\mathsf{p}}(\mathtt{e}_{v-1}\mathrm{U}_Y\mathrm{D}_X \mid \exists y \in Y$ such that $y > T)$. In either case, if T is *a largest down-admissible stretch for v then the relations from Lemma 3.27 hold on the nose.*

Proof. We will use the notation $w=v[T](R)$ and $\{s\}=S\cap S'$, $R=S\setminus\{s\}$, and note $S'=T\cup\{s\}$. We will also consider the minimal down-admissible stretch $U>T$ for v, if it exists. For the purpose of this proof it is useful to explicitly write down the relevant parts of the continued fraction expansions of v, w and other entities

$$\begin{aligned}
v &= [\dots,0,a_x,0,\dots,a_u,0,\dots,0,1,0,0,\dots,0,a_r,\dots]_{\mathsf{p}},\\
v[S] &= [\dots,0,a_x,0,\dots,a_u,0,\dots,0,0,\mathsf{p}-1,\mathsf{p}-1,\dots,\mathsf{p}-1,\mathsf{p}-a_r,\dots]_{\mathsf{p}},\\
v[T] &= [\dots,0,a_x,0,\dots,a_u-1,\mathsf{p}-1,\dots,\mathsf{p}-1,\mathsf{p}-1,0,0,\dots,0,a_r,\dots]_{\mathsf{p}} = w[R],\\
v[T][S] &= [\dots,0,a_x,0,\dots,a_u-1,\mathsf{p}-1,\dots,\mathsf{p}-1,\mathsf{p}-2,\mathsf{p}-1,\mathsf{p}-1,\dots,\mathsf{p}-1,\mathsf{p}-a_r,\dots]_{\mathsf{p}},\\
v[S][S'] &= [\dots,0,a_x,0,\dots,a_u-1,\mathsf{p}-1,\dots,\mathsf{p}-1,\mathsf{p}-1,1,\mathsf{p}-1,\dots,\mathsf{p}-1,\mathsf{p}-a_r,\dots]_{\mathsf{p}} = w.
\end{aligned}$$

Here we have highlighted the digit in position s in red.

From this description, it is straightforward to see that $\mathrm{Hom}_{\mathbb{F}_{\mathsf{p}}\mathbf{TL}}(\mathsf{e}_{v-1},\mathsf{e}_{w-1})$ is spanned by morphisms of the following four different types

$$\mathrm{U}_{XR}\mathrm{D}_{TX}\mathsf{e}_{v-1},\quad \mathrm{U}_{XUS'}\mathrm{D}_{SUX}\mathsf{e}_{v-1},\quad \mathrm{U}_{X\{s\}}\mathrm{D}_{STX}\mathsf{e}_{v-1},\quad \mathrm{U}_{XUTR}\mathrm{D}_{UX}\mathsf{e}_{v-1},$$

where X denotes a down-admissible subset for v with $X>U$, which may be empty. The basis elements of highest and second highest through-degree among the above are $\mathrm{U}_R\mathrm{D}_T\mathsf{e}_{v-1}$ and $\mathrm{U}_{\{s\}}\mathrm{D}_{ST}\mathsf{e}_{v-1}$, and all other basis elements are in the subspace $V_{>T}$.

Our task is to show that $\mathrm{U}_R\mathrm{D}_T\mathsf{e}_{v-1}$ appears with coefficient 0 and $\mathrm{U}_{\{s\}}\mathrm{D}_{ST}\mathsf{e}_{v-1}$ appears with coefficient 1 if we expand $\mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1}$ in this basis. We again start with a computation in characteristic zero.

Two applications of Lemma 3.8 establish

$$\mathrm{d}_{S'}\mathrm{d}_S\lambda_{v,V}\,\tilde{\mathrm{L}}^V_{v-1}$$
$$= \begin{cases} \lambda_{v[S],V\setminus S}\,\tilde{\mathrm{u}}_{VS'\setminus S}\tilde{\mathrm{e}}_{v[V]-1}\tilde{\mathrm{d}}_V & \text{if } S\subset V, T\not\subset V, U\subset V,\\ \lambda_{v,V}\,\lambda^{-1}_{v[S],VS}\,\lambda_{w,VR\setminus T}\,\tilde{\mathrm{u}}_{VR\setminus T}\tilde{\mathrm{e}}_{v[V]-1}\tilde{\mathrm{d}}_V & \text{if } S\not\subset V, T\subset V, U\not\subset V,\\ 0 & \text{otherwise.}\end{cases} \tag{4-8}$$

where we have used $V\cup S\setminus S'=V\cup R\setminus T$ in the second case.

The coefficient q of the maximal through-degree basis element $\mathrm{U}_R\mathrm{D}_T\mathsf{e}_{v-1}$ in $\mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1}$ is equal to the coefficient shown for $\tilde{\mathrm{u}}_R\tilde{\mathrm{e}}_{v[T]-1}\tilde{\mathrm{d}}_T$ in (4-8). This is

$$\begin{aligned} q=\lambda_{v,T}\,\lambda^{-1}_{v[S],TS}\,\lambda_{w,R} &\equiv (-1)^{\mathsf{p}^s}\frac{[\dots,0,a_x,0,\dots,a_u-1,\mathsf{p}-1,\dots,\mathsf{p}-1,\mathsf{p}-1,0]_{\mathsf{p}}}{[\dots,0,a_x,0,\dots,a_u-1,\mathsf{p}-1,\dots,\mathsf{p}-1,\mathsf{p}-1,1]_{\mathsf{p}}}\\ &\equiv 0 \bmod \mathsf{p}.\end{aligned}$$

This shows that $\mathrm{U}_R\mathrm{D}_T\mathsf{e}_{v-1}$ appears with coefficient 0 in $\mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1}$. The term $\mathrm{U}_{\{s\}}\mathrm{D}_{ST}\mathsf{e}_{v-1}$, however, does not seem to appear at all in (4-8). Since it is of second highest through-degree in $\mathrm{Hom}_{\mathbb{F}_{\mathsf{p}}\mathbf{TL}}(\mathsf{e}_{v-1},\mathsf{e}_{w-1})$, its coefficient is congruent to the coefficient of $\mathrm{u}_{\{s\}}\overline{\mathrm{e}}_{v[ST]-1}\mathrm{d}_{ST}$ in the p-morphism expansion of $q\tilde{\mathrm{u}}_R\tilde{\mathrm{e}}_{v[T]-1}\tilde{\mathrm{d}}_T$. Using Lemma 3.8, it is straightforward to compute that $\tilde{\mathrm{u}}_R\tilde{\mathrm{e}}_{v[T]-1}\tilde{\mathrm{d}}_T$ equals $\mathrm{u}_R\overline{\mathrm{e}}_{v[T]-1}\mathrm{d}_T-\lambda_{w,\{s\}}\mathrm{u}_{\{s\}}\overline{\mathrm{e}}_{v[ST]-1}\mathrm{d}_{ST}$ up to terms of lower through-degree. Thus, we compute the coefficient of interest as

$$\begin{aligned} -\lambda_{w,\{s\}}q &\equiv -\frac{[\dots,a_u-1,\mathsf{p}-1,\dots,\mathsf{p}-1,-1]_{\mathsf{p}}}{[\dots,a_u-1,\mathsf{p}-1,\dots,\mathsf{p}-1,0]_{\mathsf{p}}}\,\frac{[\dots,a_u-1,\mathsf{p}-1,\dots,\mathsf{p}-1,0]_{\mathsf{p}}}{[\dots,a_u-1,\mathsf{p}-1,\dots,\mathsf{p}-1,1]_{\mathsf{p}}}\\ &\equiv 1 \bmod \mathsf{p},\end{aligned}$$

and this finishes the proof. $\square$

Lemma 4.6. *$O(v)$ follows if we have $E(v-1)$, $A(v-1)$, $O(v-1)$, and $Z_{-}(v)$.*

Proof, with caveat. As usual, $O(v-1)$ and projector absorption allows us to restrict to the case when S is the smallest minimal down-admissible stretch for v. By Lemma 4.5 we then have

$$\begin{aligned}\mathrm{D}_{S'}\mathrm{D}_{S}\mathtt{e}_{v-1} &= \mathrm{U}_{\{s\}}\mathrm{D}_{S}\mathrm{D}_{T}\mathtt{e}_{v-1}\\ &\quad+\sum_{X\neq\emptyset} c_X\mathrm{U}_{XR}\mathrm{D}_{TX}\mathtt{e}_{v-1} \qquad +\sum_{X} d_X\mathrm{U}_{XUS'}\mathrm{D}_{SUX}\mathtt{e}_{v-1}\\ &\quad+\sum_{X\neq\emptyset} e_X\mathrm{U}_{X\{s\}}\mathrm{D}_{STX}\mathtt{e}_{v-1} \quad +\sum_{X} f_X\mathrm{U}_{XUTR}\mathrm{D}_{UX}\mathtt{e}_{v-1}.\end{aligned}\tag{4-9}$$

Here $U > T$ denotes another adjacent minimal down-admissible stretch for v, if it exists, and X ranges over down-admissible subsets $X > U$ for v. Our task is to show that the scalars $c_X, d_X, e_X, f_X \in \mathbb{F}_{\mathsf{p}}$ are all zero.

We start by multiplying both sides of (4-9) by U_T on the right. After rearranging, we get

$$\begin{aligned}\mathrm{D}_{S'}\mathrm{U}_{ST}\mathtt{e}_{v[T]-1} &= \mathrm{U}_{\{s\}}\mathrm{D}_{S}\mathrm{D}_{T}\mathrm{U}_{T}\mathtt{e}_{v[T]-1}\\ &\quad+\sum_{X\neq\emptyset} c_X\mathrm{U}_{XR}\mathrm{D}_{X}\mathrm{D}_{T}\mathrm{U}_{T}\mathtt{e}_{v[T]-1}\\ &\quad+\sum_{X} d_X\mathrm{U}_{XUS'}\mathrm{D}_{STUX}\mathtt{e}_{v[T]-1}\\ &\quad+\sum_{X\neq\emptyset} e_X\mathrm{U}_{X\{s\}}\mathrm{D}_{SX}\mathrm{D}_{T}\mathrm{U}_{T}\mathtt{e}_{v[T]-1}\\ &\quad+\sum_{X} f_X\mathrm{U}_{XUTR}\mathrm{D}_{TUX}\mathtt{e}_{v[T]-1}.\end{aligned}\tag{4-10}$$

The next step is to apply the zigzag relations and for this we shall assume that we are in the *generic case*, where T is down-admissible for $v[T]$ (and thus $\mathsf{p} > 2$). This also implies that S' is down-admissible for $v[T](R)$, and using the zigzag relations provided by $Z_{-}(v)$ for $v[T](R)$ we compute

$$\begin{aligned}\mathrm{D}_{S'}\mathrm{U}_{ST}\mathtt{e}_{v[T]-1} &= \mathrm{D}_{S'}\mathrm{U}_{S'}\mathrm{U}_{R}\mathtt{e}_{v[T]-1}\\ &= \mathtt{g}(a_u)\mathrm{U}_{S'}\mathrm{D}_{S'}\mathrm{U}_{R}\mathtt{e}_{v[T]-1}\\ &\quad+\mathtt{f}(a_u)\mathrm{U}_{US'}\mathrm{D}_{S'U}\mathrm{U}_{R}\mathtt{e}_{v[T]-1}\\ &= \mathtt{g}(a_u)\mathrm{U}_{S'}\mathrm{D}_{ST}\mathtt{e}_{v[T]-1}\\ &\quad+\mathtt{f}(a_u)\mathrm{U}_{US'}\mathrm{D}_{STU}\mathtt{e}_{v[T]-1}.\end{aligned}$$

Similarly we compute

(4-11)

$$\begin{aligned}\mathrm{U}_{\{s\}}\mathrm{D}_{S}\mathrm{D}_{T}\mathrm{U}_{T}\mathtt{e}_{v[T]-1} &= \mathtt{g}\mathrm{U}_{\{s\}}\mathrm{D}_{S}\mathrm{U}_{T}\mathrm{D}_{T}\mathtt{e}_{v[T]-1} + \mathtt{f}\mathrm{U}_{\{s\}}\mathrm{D}_{S}\mathrm{U}_{UT}\mathrm{D}_{TU}\mathtt{e}_{v[T]-1}\\ &= \mathtt{g}\mathrm{U}_{\{s\}}\mathrm{U}_{TS}\mathrm{D}_{T}\mathtt{e}_{v[T]-1} + \mathtt{f}\mathrm{U}_{\{s\}}\mathrm{U}_{UTS}\mathrm{D}_{TU}\mathtt{e}_{v[T]-1}\\ &= \mathtt{g}(\mathsf{p}-2)\mathtt{g}\mathrm{U}_{T}\mathrm{D}_{\{s\}}\mathrm{U}_{\{s\}}\mathrm{U}_{R}\mathrm{D}_{T}\mathtt{e}_{v[T]-1}\\ &\quad+\mathtt{g}(\mathsf{p}-2)\mathtt{f}\mathrm{U}_{UT}\mathrm{D}_{\{s\}}\mathrm{U}_{\{s\}}\mathrm{U}_{R}\mathrm{D}_{TU}\mathtt{e}_{v[T]-1}\\ &= \mathtt{g}\mathrm{U}_{S'}\mathrm{D}_{\{s\}}\mathrm{U}_{R}\mathrm{D}_{T}\mathtt{e}_{v[T]-1} + \mathtt{f}\mathrm{U}_{US'}\mathrm{D}_{\{s\}}\mathrm{U}_{R}\mathrm{D}_{TU}\mathtt{e}_{v[T]-1}\\ &= \mathtt{g}\mathrm{U}_{S'}\mathrm{D}_{ST}\mathtt{e}_{v[T]-1} + \mathtt{f}\mathrm{U}_{US'}\mathrm{D}_{STU}\mathtt{e}_{v[T]-1},\end{aligned}$$

where we write $\mathtt{g} = \mathtt{g}(a_u - 1)$ and $\mathtt{f} = \mathtt{f}(a_u - 1)$, and we have used $Z_-(v)$ on the pair (v, T) and smaller instances, as well as $A(v-1)$. Thus, we have equated the first two terms in (4-10). We also simplify the c_X terms

$$\begin{aligned}\mathrm{U}_{XR}\mathrm{D}_X\mathrm{D}_T\mathrm{U}_T\mathtt{e}_{v[T]-1} &= \mathtt{g}\mathrm{U}_{XR}\mathrm{D}_X\mathrm{U}_T\mathrm{D}_T\mathtt{e}_{v[T]-1} + \mathtt{f}\mathrm{U}_{XR}\mathrm{D}_X\mathrm{U}_{UT}\mathrm{D}_{TU}\mathtt{e}_{v[T]-1}\\ &= \mathtt{g}\mathrm{U}_{XTR}\mathrm{D}_{TX}\mathtt{e}_{v[T]-1} + \mathtt{f}\mathrm{U}_{XUTR}\mathrm{D}_{TUX}\mathtt{e}_{v[T]-1},\end{aligned}$$

where we have again used $Z_-(v)$ on the pair (v, T), then $A(v-1)$, and smaller instances of zigzag relations in the case when $X \neq \emptyset$ is adjacent to U for the final step. To be explicit, the sequence of transformations is

$$\begin{aligned}&\mathrm{D}_X\mathrm{U}_{UT}\mathrm{D}_{TU}\mathtt{e}_{v[T]-1} = \mathrm{D}_{TU}\mathrm{D}_X\mathrm{D}_{TU}\mathtt{e}_{v[T]-1}\\ &\sim \mathrm{D}_{TU}\mathrm{U}_{UT}\mathrm{D}_X\mathtt{e}_{v[T]-1} \sim \mathrm{U}_{UT}\mathrm{D}_{TUX}\mathtt{e}_{v[T]-1}.\end{aligned}$$

The simplification of the f_X term proceeds in complete analogy to (4-11) and we get

$$\mathrm{U}_{X\{s\}}\mathrm{D}_{SX}\mathrm{D}_T\mathrm{U}_T\mathtt{e}_{v[T]-1} = \mathtt{g}\mathrm{U}_{XS'}\mathrm{D}_{STX}\mathtt{e}_{v[T]-1} + \mathtt{f}\mathrm{U}_{XUS'}\mathrm{D}_{STUX}\mathtt{e}_{v[T]-1},$$

having again used only $Z_-(v)$ and $A(v-1)$.

Finally, after all these simplifications, (4-10) gives the following linear system

$$0 = \mathtt{g}c_X, \quad 0 = \mathtt{f}c_X + f_X, \quad 0 = d_X + \mathtt{f}e_X, \quad 0 = \mathtt{g}e_X,$$

which, since $\mathtt{g} \neq 0$, implies that all unwanted scalars are zero. □

Proof of the remaining cases. Now suppose that T is not down-admissible for $v[T]$, which happens exactly if $a_u = 1$ in the notation from above. In this case we have $\overline{T} = T \cup U$ for $v[T]$ and $\overline{S'} = S' \cup U$ for $v[T](R)$.

We proceed exactly as above, with the only differences being that no $\mathtt{g}$ terms arise and $\mathtt{f} = -2$. The linear system resulting from (4-10) is

$$0 = -2c_X + f_X, \quad 0 = d_X - 2e_X.$$

(Note that if $\mathtt{p} = 2$, we immediately see $d_X = 0 = f_X$.) To see that all coefficients are zero, we multiply (4-9) by L^U_{v-1}, expecting that this should allow us to equate the first two terms, kill the d_X and f_X terms, and not hurt the c_X and e_X terms. Let us check these assertions in turn.

For the first term we get

$$\mathrm{D}_{S'}\mathrm{D}_S\mathrm{U}_U\mathrm{D}_U\mathtt{e}_{v-1} = \mathrm{D}_{S'}\mathrm{U}_U\mathrm{D}_S\mathrm{D}_U\mathtt{e}_{v-1} = \mathrm{U}_{US'}\mathrm{D}_{SU}\mathtt{e}_{v-1}.$$

For the second term we compute

$$\mathrm{U}_{\{s\}}\mathrm{D}_S\mathrm{D}_T\mathrm{U}_U\mathrm{D}_U\mathtt{e}_{v-1} = \mathrm{U}_{\{s\}}\mathrm{U}_{UTS}\mathrm{D}_U\mathtt{e}_{v-1} = \mathrm{U}_{US'}\mathrm{D}_{SU}\mathtt{e}_{v-1},$$

where the second step works as in (4-11) and requires $A(v-1)$ and $Z_-(v)$. This equates the first two terms.

Now we claim that the d_X and f_X terms are killed by the loop along U

$$\mathrm{U}_{XUS'}\mathrm{D}_{SUX}\mathrm{U}_U\mathrm{D}_U\mathtt{e}_{v-1} = 0 = \mathrm{U}_{XUTR}\mathrm{D}_{UX}\mathrm{U}_U\mathrm{D}_U\mathtt{e}_{v-1}.$$

If $X \neq \emptyset$ is adjacent to U, then both assertions follow from

$$\mathrm{D}_{UX}\mathrm{U}_U\mathrm{D}_U\mathtt{e}_{v-1} = (\mathrm{D}_U)^2\mathrm{D}_X\mathrm{D}_U\mathtt{e}_{v-1} = 0.$$

If X is distant from U or empty, then we use far-commutativity to see substrings of the form $\mathrm{D}_U\mathrm{U}_U\mathrm{D}_U\mathrm{D}_X\mathtt{e}_{v-1} = 0$ by $E(v-1)$.

Now we claim that the c_X and e_X terms survive the multiplication by the loop along U:

$$\mathrm{U}_{XR}\mathrm{D}_{TX}\mathrm{U}_U\mathrm{D}_U\mathsf{e}_{v-1} \neq 0 \neq \mathrm{U}_{X\{s\}}\mathrm{D}_{STX}\mathrm{U}_U\mathrm{D}_U\mathsf{e}_{v-1}.$$

To see this, let us first observe $\mathrm{D}_X\mathrm{U}_U\mathrm{D}_U\mathsf{e}_{v-1} = \mathrm{U}_U\mathrm{D}_U\mathrm{D}_X\mathsf{e}_{v-1}$. This is clear if X is distant from U, and it follows from $A(v-1)$ and $Z_-(v)$, otherwise. Using this observation, we compute

$$\begin{aligned}\mathrm{U}_{XR}\mathrm{D}_{TX}\mathrm{U}_U\mathrm{D}_U\mathsf{e}_{v-1} = \mathrm{U}_{XR}\mathrm{D}_T\mathrm{U}_U\mathrm{D}_{UX}\mathsf{e}_{v-1} &= \mathrm{U}_{XUTR}\mathrm{D}_{UX}\mathsf{e}_{v-1} \neq 0\\ \mathrm{U}_{X\{s\}}\mathrm{D}_{STX}\mathrm{U}_U\mathrm{D}_U = \mathrm{U}_{X\{s\}}\mathrm{D}_{ST}\mathrm{U}_U\mathrm{D}_{UX} &= \mathrm{U}_{X\{s\}}\mathrm{U}_{UTS}\mathrm{D}_{UX}\\ &= \mathrm{U}_{XUS'}\mathrm{D}_{SUX} \neq 0\end{aligned}$$

where the last step works as in (4-11) and requires $A(v-1)$ and $Z_-(v)$.

After these simplifications, we see that (4-9) multiplied by L^U_{v-1} shows $c_X = 0 = e_X$, which (for $\mathsf{p} > 2$) in turn implies $d_X = 0 = f_X$. This completes the proof of $O(v)$. □

Let us also note the following consequence.

Lemma 4.7. *Suppose that a minimal stretch S is down-admissible for v but not for $v[S]$, and suppose the down-admissible hull $\overline{S}$ exists. Then $O(v)$ implies*

$$\mathrm{D}_{\overline{S}}\mathrm{D}_S\mathsf{e}_{v-1} = \mathrm{U}_S\mathrm{D}_{\overline{S}}\mathsf{e}_{v-1}, \quad \mathsf{e}_{v-1}\mathrm{U}_S\mathrm{U}_{\overline{S}} = \mathsf{e}_{v-1}\mathrm{U}_{\overline{S}}\mathrm{D}_S. \tag{4-12}$$

Proof. Let $s = \max(S)$ and $S' = \{s\} \cup \overline{S} \setminus S$ and $R = S \setminus \{s\}$. Then S' is down-admissible for $v[S]$ and we use $O(v)$ to compute

$$\begin{aligned}\mathrm{D}_{\overline{S}}\mathrm{D}_S\mathsf{e}_{v-1} &= \mathrm{D}_R\mathrm{D}_{S'}\mathrm{D}_S\mathsf{e}_{v-1} = \mathrm{D}_R\mathrm{U}_{\{s\}}\mathrm{D}_S\mathrm{D}_{S'\setminus\{s\}}\mathsf{e}_{v-1} = \mathrm{U}_{\{s\}}\mathrm{U}_R\mathrm{D}_{\overline{S}}\mathsf{e}_{v-1}\\ &= \mathrm{U}_S\mathrm{D}_{\overline{S}}\mathsf{e}_{v-1}.\end{aligned}$$

The other relation follows by reflection. □

4C. **Zigzag relations.**

Lemma 4.8. *The zigzag relations from Lemma 3.28 hold in generation* 2.

Proof. Suppose that S' is a down-admissible stretch for v such that $w = v[S']$ is of generation 2. Then, using the projector shortening property from Proposition 3.13, we get

$$\mathrm{D}_{S'}\mathrm{U}_{S'}\mathsf{e}_{w-1} = \mathrm{pTr}_{(v-w)/2}(\mathsf{e}_{(v+w)/2-1}).$$

This partial trace is not covered by Proposition 3.14, but since $(v+w)/2$ is of generation at most 2, it can be straightforwardly computed: One first expands $\overline{\mathsf{e}}_{(v+w)/2-1}$ into a linear combination of standard loops and computes their partial traces using (2-4). The result follows by changing back into the $\mathsf{p}$loop basis of $\mathrm{End}_{\mathbb{Q}\mathbf{TL}}(\overline{\mathsf{e}}_{w-1})$ and reducing the coefficients to $\mathbb{F}_{\mathsf{p}}$.

The basis change from ploops to standard loops for w of generation 2 with minimal down-admissible stretches $S < T$ is

$$\begin{aligned}
\mathrm{L}^{\emptyset}_{w-1} &= \tilde{\mathrm{L}}^{\emptyset}_{w-1} + (-1)^{w-\mathtt{m}_w} \tfrac{w[S]}{\mathtt{m}_w} \cdot \tilde{\mathrm{L}}^{S}_{w-1} + (-1)^{\mathtt{m}_w - \mathtt{m}_w^2} \tfrac{\mathtt{m}_w[T]}{\mathtt{m}_w^2} \cdot \tilde{\mathrm{L}}^{T}_{w-1} \\
&\quad + (-1)^{w-\mathtt{m}_w^2} \tfrac{w[ST]}{\mathtt{m}_w^2} \cdot \tilde{\mathrm{L}}^{ST}_{w-1}, \\
\mathrm{L}^{S}_{w-1} &= \tilde{\mathrm{L}}^{S}_{w-1} + (-1)^{w-\mathtt{m}_w^2} \tfrac{(\mathtt{m}_w[T])^2}{w[T]\mathtt{m}_w^2} \cdot \tilde{\mathrm{L}}^{T}_{w-1}, \\
\mathrm{L}^{T}_{w-1} &= \tilde{\mathrm{L}}^{T}_{w-1} + (-1)^{w-\mathtt{m}_w} \tfrac{w[ST]}{\mathtt{m}_w[T]} \cdot \tilde{\mathrm{L}}^{ST}_{w-1}, \\
\mathrm{L}^{ST}_{w-1} &= \tilde{\mathrm{L}}^{ST}_{w-1}.
\end{aligned}$$

The inverse basis change can be readily computed from this. The basis change in generation 1 is easier and left as an exercise for the reader. ☐

Lemma 4.9. *$Z(v)$ follows if we have $Z_{-}(v)$, $E(v-1)$, $A(v)$ and $O(v)$.*

The proof again splits into two parts. First we give a proof that works under a technical assumption, which is generically satisfied. In the second part, we refine this proof to work in all cases.

Proof, with caveat. We need to consider the zigzag $\mathrm{D}_S \mathtt{e}_{v-1} \mathrm{U}_S$ where S is the smallest minimal down-admissible stretch of v. Let us also assume that we are in the *generic case*, where S is also down-admissible for $v[S]$ (and thus $\mathtt{p} > 2$), and we denote by T the minimal down-admissible stretch for $v[S]$ that is adjacent and $T > S$.

By the unitriangularity of the basis change between the ploops basis and the standard loops basis for $\mathrm{End}_{\mathbb{Q}\mathbf{TL}}(\overline{\mathtt{e}}_{v[S]-1})$ and by the generation 2 case in Lemma 4.8, we may assume that

$$\mathrm{D}_S \mathrm{U}_S \mathtt{e}_{v[S]-1} = \mathtt{g}_S \mathrm{L}^{S}_{v[S]-1} + \mathtt{f}_S \mathrm{L}^{ST}_{v[S]-1} + \sum_{U \not\subset S \cup T} x_U \mathrm{L}^{U}_{v[S]-1}, \tag{4-13}$$

with error terms $x_U \mathrm{L}^{U}_{v[S]-1}$ with $x_U \in \mathbb{F}_{\mathtt{p}}$. Our job is to show that we have $x_U = 0$ for all such U. If we multiply (4-13) by $\mathrm{L}^{S}_{v[S]-1}$, then $E(v-1)$ implies $0 = 0 + 0 + \sum_X x_X \mathrm{L}^{SX}_{v[S]-1}$ and thus $x_X = 0$, where X runs over all U as above, for which $S \not\subset U$. On the other hand, if we multiply (4-13) by $\mathrm{L}^{T}_{v[T]-1}$, then we get

$$\mathrm{D}_S \mathrm{U}_S \mathrm{U}_T \mathrm{D}_T \mathtt{e}_{v[S]-1} = \mathtt{g}_S \mathrm{L}^{ST}_{v-1} \mathtt{e}_{v[S]-1} + \sum_X x_X \mathrm{L}^{TX}_{v[S]-1},$$

where now X runs over all remaining U such that $T \not\subset X$. Then, by $A(v)$, we also get

$$\mathrm{D}_S \mathrm{U}_S \mathrm{U}_T \mathrm{D}_T \mathtt{e}_{v[S]-1} = \mathtt{g}_S \mathrm{L}^{ST}_{v[S]-1}.$$

This implies $x_X = 0$ for such X. The only coefficients x_U that are left to be considered are the ones for which $S \cup T \subset U$. Now we apply the partial trace $\mathrm{pTr}_{ST} := \mathrm{pTr}_{(v[S]-a_{v[S],ST})}$ to both sides of (4-13). For this we will use the notation $w = a_{v[S],S}$ and $u = a_{v[S],ST}$, and we get

$$(-1)^{w+1-u} 2\mathtt{e}_{u-1} = \mathtt{g}_S (-1)^{w-u} 2\mathtt{e}_{u-1} + \mathtt{f}_S \mathtt{e}_{u-1} + \sum_{U \neq \emptyset} x_U \mathrm{L}^{U}_{u-1}.$$

This is because $\mathrm{pTr}_{ST}(\mathrm{D}_S\mathrm{U}_S\mathtt{e}_{v-1}) = \mathrm{pTr}_T(\mathtt{e}_{z-1})$ where $z = a_{v,S}$, which differs from w by increasing its first non-zero digit a by one. The coefficient of $\mathtt{e}_{u-1}$ on the right-hand side is computed as follows

$$-(-1)^{w-u}2\left(\tfrac{a+1}{a}\right) + (-1)^{w-u}\tfrac{2}{a} = (-1)^{w+1-u}2.$$

After subtracting the multiples of $\mathtt{e}_{u-1}$ from both sides, we conclude $x_U = 0$. □

Proof of the remaining cases. Now suppose that S is smallest minimal down-admissible stretch for v, but not down-admissible for $v[S]$. Then (4-13) takes the form

$$\mathrm{D}_S\mathrm{U}_S\mathtt{e}_{v[S]-1} = -2\mathrm{L}^{\overline{S}}_{v[S]-1} + \sum_{U\not\subset\overline{S}} x_U\mathrm{L}^{U}_{v[S]-1}.$$

We first multiply this by $\mathrm{L}^{\overline{S}}_{v[S]-1}$ and deduce

$$\mathrm{D}_S\mathrm{U}_S\mathrm{U}_{\overline{S}}\mathrm{D}_{\overline{S}}\mathtt{e}_{v[S]-1} = \mathrm{D}_S\mathrm{U}_{\overline{S}}\mathrm{D}_S\mathrm{D}_{\overline{S}}\mathtt{e}_{v[S]-1} = 0$$

from $O(v)$. Since $(\mathrm{L}^{\overline{S}}_{v[S]-1})^2 = 0$ by $E(v-1)$, we get $x_U = 0$ unless $\overline{S}\subset U$. Then a partial trace argument as above shows that all remaining x_U are also zero. □

Lemma 4.10. *$E(v)$ follows if we have $E(v-1)$, $A(v)$, $O(v)$, and $Z(v)$.*

Proof. We first prove (3-13). By $E(v-1)$ and projector absorption, we may assume that S is a smallest minimal down-admissible stretch. Suppose first that S is down-admissible for $v[S]$. Then we have

$$\begin{aligned}\mathrm{D}_S\mathrm{U}_S\mathrm{D}_S\mathtt{e}_{v-1} &= \mathtt{g}\mathrm{U}_S\mathrm{D}_S\mathrm{D}_S\mathtt{e}_{v-1} + \mathtt{f}\mathrm{U}_T\mathrm{U}_S\mathrm{D}_S\mathrm{D}_T\mathrm{D}_S\mathtt{e}_{v-1}\\ &= 0 + \mathtt{hf}\mathrm{U}_T\mathrm{U}_S\mathrm{D}_S\mathrm{U}_S\mathrm{D}_T\mathtt{e}_{v-1} = 0.\end{aligned}$$

where we have used $Z(v)$, $A(v)$ and finally $E(v-1)$ to deduce $\mathtt{e}_{v[T](S)-1}\mathrm{U}_S\mathrm{D}_S\mathrm{U}_S = 0$. Now, suppose that S is not down-admissible for $v[S]$. Then we instead get

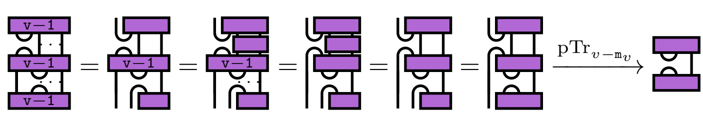
$$\begin{aligned}\mathrm{D}_S\mathrm{U}_S\mathrm{D}_S\mathtt{e}_{v-1} &= -2\mathrm{U}_{\overline{S}}\mathrm{D}_{\overline{S}}\mathrm{D}_S\mathtt{e}_{v-1}\\ &= -2\mathrm{U}_{\overline{S}}\mathrm{U}_S\mathrm{D}_{\overline{S}}\mathtt{e}_{v-1} = 0.\end{aligned}$$

Here we have used $O(v)$ and Lemma 3.22.

Next we need to prove that $(\mathrm{L}^{S_k}_{v-1})^2 = 0$ and $\mathrm{L}^X_{v-1} = \prod_{k|S_k\subset X}\mathrm{L}^{S_k}_{v-1}$. The first relation simply follows from (3-13), i.e.

$$(\mathrm{L}^{S_k}_{v-1})^2 = \mathrm{U}_{S_k}\mathrm{D}_{S_k}\mathrm{U}_{S_k}\mathrm{D}_{S_k}\mathtt{e}_{v-1} = 0.$$

Next, suppose we already know that $\mathrm{L}^X_{v-1} = \prod_{S_k\subset X}\mathrm{L}^{S_k}_{v-1}$ for subsets X that decompose into $l-1$ minimal stretches, and let us now consider $X\cup S_i$ where $X > S_i$. The result is clear if X and S_i are distant, so we will assume that S_i is adjacent to X. By projector absorption and (3-13), we may further assume that $S_i = S$ is the smallest minimal down-admissible stretch for v. Then we compute

$\mathrm{L}^S_{v-1}\mathrm{L}^X_{v-1} =$  $= \mathrm{L}^X_{\mathtt{m}_v-1}.$

The equation $\mathrm{pTr}_{v-\mathtt{m}_v}(\mathrm{L}^S_{v-1}\mathrm{L}^X_{v-1}) = \mathrm{L}^X_{\mathtt{m}_v-1}$ and Proposition 3.14 imply $\mathrm{L}^S_{v-1}\mathrm{L}^X_{v-1} = x\mathrm{L}^X_{v-1} + y\mathrm{L}^{XS}_{v-1}$ where $(-1)^{v-\mathtt{m}_v}2x + y = 1$. Equivalently, we can write $\mathrm{L}^X_{v-1}(-x+$

$\mathrm{L}^{S}_{v-1}) = y\mathrm{L}^{XS}_{v-1}$. Suppose that $x \neq 0$, then $(-x + \mathrm{L}^{S}_{v-1})$ would be a unit in $\mathrm{End}_{\mathbb{F}_\mathsf{p}\mathbf{TL}}(\mathsf{e}_{v-1})$, so we can write

$$\mathrm{L}^{X}_{v-1} = (-x + \mathrm{L}^{S}_{v-1})^{-1} y \mathrm{L}^{XS}_{v-1}.$$

However, the left-hand side has through-degree $v[X]$, while the right-hand side has through-degree at most $v[X \cup S] < v[X]$, a contradiction. Thus, we have $x = 0$ and $y = 1$, and consequently $\mathrm{L}^{S}_{v-1}\mathrm{L}^{X}_{v-1} = \mathrm{L}^{XS}_{v-1}$. □

This completes the proof of Theorem 3.2, which by Proposition 2.28, completes the proof of Theorem A.

Remark 4.11. In addition to the eve base cases $1 \leq v \leq \mathsf{p}$ for the induction we have explicitly seen certain relations in cases of low generation. For example, for v of generation 1, the description of the endomorphism algebra can be deduced from the proof of Lemma 3.21 while the adjacency and overlap relations hold vacuously. For v of generation 2, we have seen the adjacency relations in Lemma 4.3 and the overlap relations in Lemma 4.5. Finally, zigzag relations for loops based at w of generation 2 were treated in Lemma 4.8.

5. Some conclusions

5A. **The fractal nature of Z.** The quiver underlying Z is a graph with countably infinitely many connected components. In each connected component there is a unique vertex $e - 1$ with $e \in \mathrm{Eve}$, and we denote the vertex set of this component by $(e)_\mathsf{p}$.

Example 5.1. We have $(1)_3 = \{0 < 4 < 6 < 10 < 12 < 16 < 18 < 22 < \cdots\}$, cf. (2-5).

The decomposition of the quiver implies that the algebra Z decomposes as

$$\mathsf{Z} = \coprod_{e \in \mathrm{Eve}} \mathsf{Z}_{e-1}, \qquad \mathsf{Z}_{e-1} := \bigoplus_{v,w \in (e)_\mathsf{p}} e_{w-1}\mathsf{Z}e_{v-1}.$$

Let M_p denote the free monoid on the set $L = \{0, \dots, \mathsf{p}-1\}$. We represent the elements of M_p by words $[b_k, \dots, b_0]$ for $b_i \in L$ and the multiplication is given by

$$[b_k, \dots, b_0] \odot [a_j, \dots, a_0] := [a_j, \dots, a_0, b_k, \dots, b_0].$$

(Note that the empty word $\emptyset$ is the neutral element.) Elements of M_p with differing numbers of leading zeros are considered as distinct, and so they should not be interpreted as p-adic expansions of natural numbers. However, $\mathbb{N}$ carries an action of M_p defined by

$$M_\mathsf{p} \times \mathbb{N} \to \mathbb{N}, \quad [b_k, \dots, b_0] \odot [a_j, \dots, a_0]_\mathsf{p} := [a_j, \dots, a_0, b_k, \dots, b_0]_\mathsf{p}.$$

The fractal nature, i.e. the self-similarity, of Z is now captured by the following proposition.

Proposition 5.2. *The monoid M_p acts on Z by algebra endomorphisms: For each $w \in M_\mathsf{p}$, there is an algebra endomorphism $\phi_w \in \mathrm{End}(\mathsf{Z})$ acting on idempotents by*

$$\phi_w(\mathsf{e}_{v-1}) := \mathsf{e}_{w \odot v-1},$$

and on arrows by reindexation. Moreover, we have

$$\phi_z\phi_w = \phi_{z\odot w} \text{ for } w, z \in M_{\mathsf{p}}.$$

Finally, if $w = [0, \dots, 0]$*, then* ϕ_w *maps any* $\mathtt{Z}_{e-1}$ *isomorphically to* $\mathtt{Z}_{w\odot e-1}$*. In this sense,* $\mathtt{Z}$ *is generated under the action of* M_{p} *by the summands* $\mathtt{Z}_{e-1}$ *for* $e \in \{1, \dots, \mathsf{p}-1\}$*.*

Proof. This is a direct consequence of Theorem 3.2. □

We also note that, since $\mathtt{Z}$ is the direct sum of $\mathbb{N}$ many copies of $\mathtt{Z}_{e-1}$ for $e \in \{1, \dots, \mathsf{p}-1\}$, the underlying quiver of $\mathtt{Z}$ is a *fractal graph* in the sense of Ille–Woodrow [IW19], albeit in the trivial sense that any countable graph without edges and more than one vertex can be considered as a fractal factor.

5B. **A few words about tilting modules.** Let us work over the ground field $\mathbb{K}$. First, recall the category of finite-dimensional modules for $\mathrm{SL}_2(\mathbb{K})$ has simple $\mathtt{L}(v-1)$, Weyl $\Delta(v-1)$, dual Weyl $\nabla(v-1)$ and indecomposable tilting modules $\mathtt{T}(v-1)$ for $v \in \mathbb{N}$, the latter being the indecomposable objects of **Tilt**, see e.g. [Wil17, Section 1] for a concise summary of the main definitions and properties regarding **Tilt**.

Let us now elaborate a bit further on the representation-theoretic implications of Corollary A. Almost all of these are, of course, well-understood. However, the reader might find it helpful to see how they can be derived from our results in the previous sections.

It is well-known that

$$\mathbf{Tilt} = \bigoplus_{e\in\mathrm{Eve}} \mathbf{Tilt}_{e-1}, \quad \mathbf{Tilt}_{e-1} = \{\mathtt{T}(v-1) \mid v-1 \in (e)_{\mathsf{p}}\},$$

whose direct summands are called *blocks*, which are equivalent as additive, $\mathbb{K}$-linear categories. From our discussion we immediately get the following.

Proposition 5.3. *There is an equivalence of additive,* $\mathbb{K}$*-linear categories*

$$\mathscr{F}'_{e-1} \colon \mathbf{Tilt}_{e-1} \xrightarrow{\cong} p\mathbf{Mod}\text{-}\mathtt{Z}_{e-1},$$

sending indecomposable tilting modules to indecomposable projectives. Moreover, $\mathrm{Hom}_{\mathbf{Tilt}}(\mathtt{X},\mathtt{Y}) = 0$ *if* $\mathtt{X} \in \mathbf{Tilt}_{e-1}$ *and* $\mathtt{Y} \in \mathbf{Tilt}_{e'-1}$ *for* $e, e' \in \mathrm{Eve}, e \neq e'$*. Finally, there is an isomorphism of algebras* $\mathtt{Z}_{e-1} \cong \mathtt{Z}_{e'-1}$ *for all* $e, e' \in \mathrm{Eve}$ *with equal non-zero digits.*

Proof. Directly from Theorem A and Corollary A, combined with Theorem 3.2 and the above. □

In fact $\mathtt{Z}_{e-1} \cong \mathtt{Z}_{e'-1}$ for all $e, e' \in \mathrm{Eve}$, but such isomorphisms involve non-trivial rescalings in our presentation.

Another consequence we get are the tilting–dual Weyl multiplicities.

Proposition 5.4. *We have*

$$\big(\mathtt{T}(v-1) : \nabla(w-1)\big) = \begin{cases} 1 & \text{if } w \in \mathrm{supp}(v), \\ 0 & \text{else.} \end{cases}$$

Proof. Note that the basis Theorem 3.2.(Basis) is part of the family of bases constructed in [AST18], and the proposition follows from the construction of these bases in *loc. cit.*, and our main statements Theorem A and Corollary A. □

Hence, we get the tilting characters $\chi^{\mathtt{T}}_{v-1} = \sum_{w\in\mathbb{N}} \big(\mathtt{T}(v-1) : \nabla(w-1)\big)\chi^{\nabla}_{w-1}$, where the characters $\chi^{\nabla}_{w-1} = \chi^{\Delta}_{w-1}$ are the characteristic zero characters well-known e.g. by Weyl's character formula. By reciprocity, cf. [RW18, Proposition 1.14], we also get the Weyl–simple multiplicities. To state these explicitly, let us recursively define a set $X(v)$ as follows. For $v \leq \mathsf{p}-1$ let $X(v) = \{0\}$. For $v > \mathsf{p}-1$ let

$$X(v) = \begin{cases} \mathsf{p}X\big((v-a_0)/\mathsf{p}\big) \cup \Big(a_0 + 1 + \mathsf{p}X\big((v-a_0-\mathsf{p})/\mathsf{p}\big)\Big) & \text{if } a_0 \neq \mathsf{p}-1, \\ \mathsf{p}X\big((v-a_0)/\mathsf{p}\big) & \text{if } a_0 = \mathsf{p}-1, \end{cases}$$

where we again meet losp. Then we get

$$\big[\Delta(w-1) : \mathtt{L}(v-1)\big] = \begin{cases} 1 & \text{if } v \in w - 2X(w), \\ 0 & \text{else.} \end{cases}$$

Thus, we get $\chi^{\Delta}_{w-1} = \sum_{v\in\mathbb{N}} \big[\Delta(w-1) : \mathtt{L}(v-1)\big]\chi^{\mathtt{L}}_{v-1} = \sum_{v\in w-X(w)} \chi^{\mathtt{L}}_{v-1}$, which determines the simple characters by inverting the change of basis matrix.

Example 5.5. For $\mathsf{p}=3$ we have $\chi^{\mathtt{T}}_{22} = \chi^{\nabla}_{22} + \chi^{\nabla}_{18} + \chi^{\nabla}_{16} + \chi^{\nabla}_{12}$, cf. [JW17, Figure 1]. Moreover, we also get $\chi^{\Delta}_{22} = \chi^{\mathtt{L}}_{22} + \chi^{\mathtt{L}}_{18} + \chi^{\mathtt{L}}_{12} + \chi^{\mathtt{L}}_{10}$.

The final consequence we would like to derive in this paper is the following.

Proposition 5.6. *Let* $\mathbf{I}_v = \{\mathtt{T}(w-1) \mid w \geq v\}$. *For any thick* $\otimes$*-ideal* $\mathbf{I} \neq 0$ *in* **Tilt** *there exists* $k \in \mathbb{N}_0$ *such that*

$$\mathbf{I} = \mathbf{I}_{\mathsf{p}^k} \overset{\mathsf{p}\neq 2}{=} \big\{\mathtt{T}(v-1) \mid \nu_{\mathsf{p}}\big(\dim_{\mathbb{F}_{\mathsf{p}}}\big(\mathtt{T}(v-1)\big)\big) \leq k\big\},$$

with the latter equality holding in case $\mathsf{p} \neq 2$.

Proof. We will use that $\mathtt{T}(1)$ is a $\otimes$-generator of **Tilt** and Weyl and dual Weyl modules have classical characters.

Assume that $\mathtt{T}(v-1) \in \mathbf{I}$ for v minimal. Then it is clear by Proposition 5.4 that $\mathbf{I} = \mathbf{I}_v$. Thus, it remains to determine the possible minimal v. To this end, note that the decomposition of $\mathtt{T}(1)\otimes\mathtt{T}(w-1) \cong \mathtt{T}(w-1)\otimes\mathtt{T}(1)$ into its indecomposable summands is completely determined by Proposition 5.4 and the classical $SL_2(\mathbb{C})$ tensor product combinatorics. Analyzing now how tensoring with $\mathtt{T}(1)$ affects the support, Proposition 5.4 then also implies that v needs to be a prime power, and conversely, that a prime power works as a minimal v.

Finally, the last statement follows from Proposition 3.14 since $\mathsf{e}_{e-1} = \overline{\mathsf{e}}_{e-1} = \tilde{\mathsf{e}}_{e-1}$ for $e \in$ Eve. ☐

Thus, the thick $\otimes$-ideals in **Tilt** are $\mathbf{Tilt} = \mathbf{I}_{\mathsf{p}^0} \supset \mathbf{I}_{\mathsf{p}^1} \supset \mathbf{I}_{\mathsf{p}^2} \supset \mathbf{I}_{\mathsf{p}^3} \supset \mathbf{I}_{\mathsf{p}^4} \supset \cdots$.

Example 5.7. The elements of $\mathbf{I}_{\mathsf{p}^1}$ are the so-called *negligible* modules.

Note that the above implies that **Tilt** has no projective modules since these would form the minimal thick $\otimes$-ideal.

Table of notation and central concepts

In general we use a tilde, e.g. $\tilde{f}$, to indicate that we work over $\mathbb{Q}$, an overline, e.g. $\overline{f}$, to indicate that we have something that reduces mod p but we want to consider it over $\mathbb{Q}$, and no extra decoration if we work in $\mathbb{F}_{\mathsf{p}}$.

Name	Symbol	Description
Ringel dual of SL_2	$\mathtt{Z}$	the path algebra of a quiver with relations presented in Theorem 3.2.
—	$\mathtt{f}, \mathtt{g}, \mathtt{h}$	functions $\mathbb{F}_{\mathsf{p}} \to \mathbb{F}_{\mathsf{p}}$ used in the presentation of $\mathtt{Z}$, Subsection 3A.
JW projector	$\tilde{\mathsf{e}}_{v-1}$	the Jones–Wenzl projectors, corresponding to projections to $\Delta(v-1)$ in $\mathtt{T}(1)^{\otimes(v-1)}$; defined over $\mathbb{Q}$, Definition 2.1.
pJW projector	e_{v-1}	the pJones–Wenzl projectors, corresponding to projections to $\mathtt{T}(v-1)$ in $\mathtt{T}(1)^{\otimes(v-1)}\mathrm{k}$; defined over $\mathbb{F}_{\mathsf{p}}$, Definition 2.26.
rational pJW proj.	$\overline{\mathsf{e}}_{v-1}$	the pJones–Wenzl projectors, corresponding to projections to $\mathtt{T}(v-1)$ in $\mathtt{T}(1)^{\otimes(v-1)}$, but considered over $\mathbb{Q}$, Definition 2.22.
—	$\lambda_{v,S}$	scalars in the definition of projectors (2-8).
integral morphisms	$\mathrm{d}_S\mathbb{1}_{v-1}$	down or up morphisms given by cups or caps; these work integrally, Definition 2.15.
standard morphisms	$\tilde{\mathrm{d}}_S\mathbb{1}_{v-1}$	down or up morphisms given by cups or caps together with JW projectors; these over $\mathbb{Q}$, Definition 2.18.
p morphisms	$\mathrm{D}_S\mathsf{e}_{v-1}$	down or up morphisms given by cups or caps together with pJW projectors; these over $\mathbb{F}_{\mathsf{p}}$, Definition 3.1.
standard loops	$\tilde{\mathrm{L}}_{v-1}^S$	compositions of down and up morphisms; form a basis of endomorphism spaces over $\mathbb{Q}$, Definition 2.18.
ploops	L_{v-1}^S	compositions of down and up morphisms; form a basis of endomorphism spaces over $\mathbb{F}_{\mathsf{p}}$, Definition 3.1.
eve	e	a number with a single non-zero p-adic digit, Definition 2.5.
mother of v	$\mathtt{m}_v$	defined (unless v is an eve) by setting the last non-zero p-adic digit of v to zero, Definition 2.5.
ancestors of v	$\mathtt{m}_v, \mathtt{m}_v^2 \cdots$	positive numbers obtained by setting last p-adic digits of v to zero, Definition 2.5.
generation of v	$\mathtt{g}_v$	the number of ancestors of v, Definition 2.5.
stretches	—	sets of consecutive digits in the p-adic expansion of a number, Definition 2.8.
admissibility	—	whether a set of digits of a p-adic expansion is suitable for reflecting up or down, Definition 2.8.
admissible hull	$\overline{S}$	an admissible set containing S, Definition 2.8.

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