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MIRKOVIĆ–VILONEN BASIS IN TYPE A_1

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Abstract. Let G be a connected reductive algebraic group over $\mathbb{C}$. Through the geometric Satake equivalence, the fundamental classes of the Mirković–Vilonen cycles define a basis in each tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$ of irreducible representations of G. We compute this basis in the case $G = \mathrm{SL}_2(\mathbb{C})$ and conclude that in this case it coincides with the dual canonical basis at $q = 1$.

1. Introduction

Let G be a connected reductive algebraic group over $\mathbb{C}$, endowed with a Borel subgroup B and a maximal torus $T \subset B$. Irreducible rational representations of G are classified by their highest weight: to the dominant integral weight λ corresponds the irreducible representation $V(\lambda)$.

Several constructions provide bases of $V(\lambda)$, for instance:

- Exploiting the geometry of moduli spaces of quiver representations, Lusztig defines in [16] the so-called canonical basis of the quantum deformation $V_q(\lambda)$. We are in fact interested in the dual of this basis, aka Kashiwara's upper global basis [14]. Taking the classical limit $q = 1$ provides a basis of $V(\lambda)$.
- The geometric Satake correspondence [15, 18] realizes $V(\lambda)$ as the intersection cohomology of a certain Schubert variety $\overline{\mathrm{Gr}^\lambda}$ in the affine Grassmannian of the Langlands dual of G. The fundamental classes of the Mirković–Vilonen cycles form a basis of this cohomology space, hence of $V(\lambda)$.

These two bases share nice properties; they can both be endowed with a Kashiwara crystal structure which controls the action of the Chevalley generators of the Lie algebra of G, and they are both bases over $\mathbb{Z}$ of the costandard integral form of $V(\lambda)$. They coincide in small rank but usually differ: counterexamples were found in [2] with $G = \mathrm{SO}_8(\mathbb{C})$ and $\mathrm{SL}_6(\mathbb{C})$. This disparity seems related to the theory of cluster algebras. Namely, the algebra $\mathbb{C}[N]$ of regular functions on the unipotent radical N of B has a cluster structure, which is of infinite type if $G = \mathrm{SO}_8(\mathbb{C})$ or $\mathrm{SL}_6(\mathbb{C})$. Each representation $V(\lambda)$ can be embedded into $\mathbb{C}[N]$, and the counterexamples in [2] are located at points where the dual canonical basis elements are not cluster monomials (see e.g. [10], sect. 19).

A tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$ of irreducible representations also admits a dual canonical basis (see chapter 27 in [17]) and a Mirković–Vilonen basis (see sect. 2.4 in [12]). In each case, one modifies the tensor product of the bases of the factors in a specific way to produce a new basis, that still enjoys the pleasant

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properties mentioned earlier, and is moreover compatible with the isotypic filtration. In this more general setup, the dual canonical basis and the Mirković–Vilonen basis already differ for $G = \mathrm{SL}_3(\mathbb{C})$ [6].

It is possible to carry out the complete calculations in the case $G = \mathrm{SL}_2(\mathbb{C})$. This task was performed by Frenkel and Khovanov [7] for the dual canonical basis, and the purpose of this paper is to do the same for the Mirković–Vilonen basis. Recall that each dominant weight of $\mathrm{SL}_2(\mathbb{C})$ is a nonnegative multiple of the fundamental weight ϖ. The following result is a corollary of our work.

Theorem. *For $G=\mathrm{SL}_2(\mathbb{C})$, the Mirković–Vilonen basis of a tensor product $V(n_1\varpi) \otimes \cdots \otimes V(n_r\varpi)$ coincides with the dual canonical basis of this space specialized at $q = 1$.*

In truth, this theorem holds only after reversal of the order of the tensor factors, but this defect is merely caused by a discrepancy in the conventions.

The theorem is trivial in the case $r = 1$ and can be expeditiously deduced from general properties shared by the two bases in the case $r = 2$. When $r \geq 3$, the presence of multiplicities in the tensor product renders the result less obvious. Perhaps one can deduce the theorem from a compatibility of both bases with an appropriate rigid structure. (One option here would be to use the symmetric Howe duality relative to the dual pair $(\mathrm{GL}_2, \mathrm{GL}_r)$ in conjunction with the cluster algebras $\mathbb{C}[B_K^-\backslash G]$ studied in [11].) However, we shall not pursue this avenue. Instead, we regard $V(n_1\varpi) \otimes \cdots \otimes V(n_r\varpi)$ as a quotient of $V(\varpi)^{\otimes(n_1+\cdots+n_r)}$ and note that both the dual canonical basis and the Mirković–Vilonen basis behave well under this operation. We can therefore reduce the general statement to the particular case of the tensor power $V(\varpi)^{\otimes n}$. We then deal with the latter by direct though complicated calculations.

The paper is organized in the following way. In sect. 2, we define a basis of $V(\varpi)^{\otimes n}$ by a simple recursive formula and argue that it matches Frenkel and Khovanov's characterization of the dual canonical basis. In sect. 3, we recall the definition of the Mirković–Vilonen basis in tensor products of irreducible representations and prove its good behavior under the quotient operation of the previous paragraph. In sect. 4, we show that the Mirković–Vilonen basis of $V(\varpi)^{\otimes n}$ satisfies the recursive formula from sect. 2 (this is the difficult part in the paper).

While preparing this paper, we learned that independently Pak-Hin Li computed the Mirković–Vilonen basis for the tensor product of two irreducible representations of $\mathrm{SL}_2(\mathbb{C})$.

This work is based on the PhD thesis of the second author [5]. We however rewrote the proof to render it more accessible and remove ambiguities.

2. Combinatorics and linear algebra

Let $\mathbb{K}$ be a field and let V be the vector space $\mathbb{K}^2$. In this section, we define in an elementary manner an explicit basis in each tensor power $V^{\otimes n}$ that has nice properties with respect to the natural action of $\mathrm{SL}_2(\mathbb{K})$.

2.1. **Words.** Given a nonnegative integer n, we set $\mathscr{C}_n = \{+, -\}^n$. We regard an element in $\mathscr{C}_n$ as a word of length n on the alphabet $\{+, -\}$. Concatenation of words endows $\mathscr{C} = \bigcup_{n\geq 0} \mathscr{C}_n$ with the structure of a monoid. The word of length zero is denoted by $\varnothing$.

The weight of a word $w \in \mathscr{C}$, denoted by $\operatorname{wt}(w)$, is the number of letters $+$ minus the number of letters $-$ that w contains. A word $w = w(1)w(2)\cdots w(n)$ is said to be semistable if its weight is 0 and if each initial segment $w(1)\cdots w(j)$ has nonpositive weight.

Words are best understood through a representation as planar paths, where letters $+$ and $-$ are depicted by upward and downward segments, respectively. A word is semistable if the endpoints of its graphical representation are on the same horizontal line and if the whole path lies below this line.

Any word w can be uniquely factorized as a concatenation

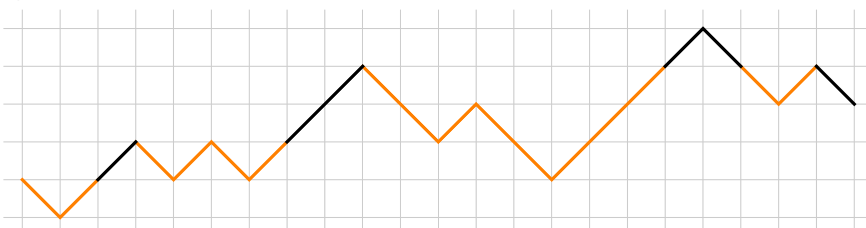
$$w_{-r}{+}\cdots{+}w_{-1}{+}w_0{-}w_1{-}\cdots{-}w_s$$

where r and s are nonnegative integers and where the words $w_{-r}, \ldots, w_s$ are semistable. The r letters $+$ and the s letters $-$ that do not occur in the semistable words are called significant. Informally, a letter $+$ is significant if it records the first time an altitude is reached, and a letter $-$ is significant if it marks a descent from a height that is never attained again. A word is semistable if and only if it does not contain any significant letter.

Example. The following picture illustrates the factorization of the word

$$w = {-}{+}{+}{-}{+}{-}{+}{+}{+}{-}{-}{+}{-}{-}{+}{+}{+}{+}{-}{-}{+}{-}.$$

This word has length 22 and weight 2. Here $(r, s) = (4, 2)$ and the words w_{-2}, w_0 and w_2 are empty. Significant letters are written in black, non-significant ones in orange.



Given a word w, we denote by $\mathscr{P}(w)$ the set of words obtained from w by changing a single significant letter $+$ into a $-$. With our previous notation, $\mathscr{P}(w)$ has r elements.

2.2. **Bases.** Let (x_+, x_-) be the standard basis of the vector space V. Each word $w = w(1)w(2)\cdots w(n)$ in $\mathscr{C}^n$ defines an element $x_w = x_{w(1)} \otimes \cdots \otimes x_{w(n)}$ in the n-th tensor power of V. The family $(x_w)_{w\in\mathscr{C}_n}$ is a basis of $V^{\otimes n}$.

We define another family of elements $(y_w)_{w\in\mathscr{C}}$ in the tensor algebra of V by the convention $y_\varnothing = 1$ and the recursive formulas

$$y_{+w} = x_+ \otimes y_w \quad \text{and} \quad y_{-w} = x_- \otimes y_w - \sum_{v\in\mathscr{P}(w)} x_+ \otimes y_v.$$

Rewriting the latter as

$$x_+ \otimes y_w = y_{+w} \quad \text{and} \quad x_- \otimes y_w = y_{-w} + \sum_{v\in\mathscr{P}(w)} y_{+v} \tag{1}$$

one easily shows by induction on the length of words that each element x_w can be expressed as a linear combination of elements y_v, using only words v that have the same length and weight as w. It follows that for each nonnegative integer n, the family $(y_w)_{w\in\mathscr{C}_n}$ spans $V^{\otimes n}$, hence is a basis of this space.

Proposition 1. *The family $(y_w)_{w\in\mathscr{C}}$ is characterized by the following conditions:*

(i) *If w is of the form $+\cdots+-\cdots-$ (a collection of $+$ followed by a collection of $-$), then $y_w = x_w$.*

(ii) $y_{-+} = x_{-+} - x_{+-}$.

(iii) *Let u be a semistable word and let $(w', w'') \in \mathscr{C}_{n'} \times \mathscr{C}_{n''}$. Write $y_{w'w''} = \sum_i z_i' \otimes z_i''$ with $(z_i', z_i'') \in V^{\otimes n'} \times V^{\otimes n''}$. Then $y_{w'uw''} = \sum_i z_i' \otimes y_u \otimes z_i''$.*

Proof. Statements (i) and (ii) follow straightforwardly from the definition of the elements y_w. We prove (iii) by induction on the length of $w'uw''$. Discarding a trivial case, we assume that u is not the empty word.

Suppose first that w' is the empty word. Let us write u as a concatenation $-u'+u''$ where u' and u'' are (possibly empty) semistable words. Equation (1) gives

$$x_- \otimes y_{w''} = y_{-w''} + \sum_{v\in\mathscr{P}(w'')} y_{+v}.$$

Applying the induction hypothesis to the semistable word u'' and the pairs $(-, w'')$ and $(+, v)$, for each $v \in \mathscr{P}(w'')$, we obtain

$$x_- \otimes y_{u''} \otimes y_{w''} = y_{-u''w''} + \sum_{v\in\mathscr{P}(w'')} y_{+u''v}.$$

Since $x_- \otimes y_{u''} = y_{-u''}$, we get

$$y_{-u''} \otimes y_{w''} = y_{-u''w''} + \sum_{v\in\mathscr{P}(w'')} y_{+u''v}$$

and applying once more the induction hypothesis, this time to the semistable word u' and the pairs $(\varnothing, -u'')$, $(\varnothing, -u''w'')$ and $(\varnothing, +u''v)$, we arrive at

$$y_{u'-u''} \otimes y_{w''} = y_{u'-u''w''} + \sum_{v\in\mathscr{P}(w'')} y_{u'+u''v}. \tag{2}$$

Starting now with

$$x_+ \otimes y_{w''} = y_{+w''}$$

we arrive by similar transformations at

$$y_{u'+u''} \otimes y_{w''} = y_{u'+u''w''}. \tag{3}$$

Since $\mathscr{P}(u'+u'') = \{u'-u''\}$, we have by definition

$$y_u = x_- \otimes y_{u'+u''} - x_+ \otimes y_{u'-u''}. \tag{4}$$

Likewise, $\mathscr{P}(u'+u''w'') = \{u'-u''w''\} \cup \{u'+u''v \mid v \in \mathscr{P}(w'')\}$ leads to

$$y_{uw''} = x_- \otimes y_{u'+u''w''} - x_+ \otimes y_{u'-u''w''} - \sum_{v\in\mathscr{P}(w'')} x_+ \otimes y_{u'+u''v}. \tag{5}$$

Combining (2)–(5), we obtain the desired equation

$$y_{uw''} = y_u \otimes y_{w''}.$$

We now address the case where w' is not empty. Suppose that the first letter of w' is a $+$ and write $w' = +\widetilde{w}'$. Then

$$y_{w'w''} = x_+ \otimes y_{\widetilde{w}'w''} \quad \text{and} \quad y_{w'uw''} = x_+ \otimes y_{\widetilde{w}'uw''}$$

and the result follows from the induction hypothesis applied to the semistable word u and the pair $(\widetilde{w}', w'')$.

If on the contrary the first letter of w' is a $-$, then we write $w' = -\widetilde{w}'$. Since u is semistable, its insertion in the middle of a word does not add or remove any significant letter; in particular, the set of significant letters in $\widetilde{w}'w''$ is in natural bijection with the set of significant letters in $\widetilde{w}'uw''$. This observation leads to a bijection from $\mathscr{P}(\widetilde{w}'w'')$ onto $\mathscr{P}(\widetilde{w}'uw'')$, which splits a word v in two subwords $v' \in \mathscr{C}_{n'-1}$ and $v'' \in \mathscr{C}_{n''}$ and then returns $v'uv''$. With this notation,

$$y_{w'w''} = x_- \otimes y_{\widetilde{w}'w''} - \sum_{v \in \mathscr{P}(\widetilde{w}'w'')} x_+ \otimes y_{v'v''}$$

and

$$y_{w'uw''} = x_- \otimes y_{\widetilde{w}'uw''} - \sum_{v \in \mathscr{P}(\widetilde{w}'w'')} x_+ \otimes y_{v'uv''}.$$

Again the desired equation follows from the induction hypothesis applied to the semistable word u and the pairs $(\widetilde{w}', w'')$ and (v', v''), for each $v \in \mathscr{P}(\widetilde{w}'w'')$.

Condition (iii) computes $y_{w'uw''}$ from the datum of $y_{w'w''}$ and y_u whenever u is semistable; condition (i) provide the value of y_w when w is of the form $+\cdots+-\cdots-$; and condition (ii) provides the value of y_{-+}. Noting that any word in $\mathscr{C}$ can be obtained from a word of the form $+\cdots+-\cdots-$ by repetitively inserting the semistable word $-+$ (possibly at non disjoint positions), we conclude that conditions (i)–(iii) fully characterize the family $(y_w)_{w\in\mathscr{C}}$. $\square$

As a consequence of this proposition, we see that if

$$w_{-k}+\cdots+w_{-1}+w_0-w_1-\cdots-w_\ell$$

is the factorization of a word w, as in section 2.1, then

$$y_w = y_{w_{-k}} \otimes x_+ \otimes \cdots \otimes x_+ \otimes y_{w_{-1}} \otimes x_+ \otimes y_{w_0} \otimes x_- \otimes y_{w_1} \otimes x_- \otimes \cdots \otimes x_- \otimes y_{w_\ell}. \tag{6}$$

Remark. The transition matrix between the two bases $(x_w)_{w\in\mathscr{C}_n}$ and $(y_w)_{w\in\mathscr{C}_n}$ of $V^{\otimes n}$ is unitriangular: if we write

$$x_w = \sum_{v\in\mathscr{C}_n} n_{w,v}\, y_v$$

then the diagonal coefficient $n_{w,w}$ is equal to one and the entry $n_{w,v}$ is zero except when the path representing v lies above the path representing w. In addition, all the coefficients $n_{w,v}$ are nonnegative integers. The proof of these facts is left to the reader.

2.3. **Representations.** In this section, we regard V as the defining representation of $\mathrm{SL}_2(\mathbb{K})$. From now on, we assume that $\mathbb{K}$ has characteristic zero. We denote by (e, h, f) the usual basis of $\mathfrak{sl}_2(\mathbb{K})$.

Fix a nonnegative integer n. Given a word $w \in \mathscr{C}_n$, we denote by $\varepsilon(w)$ (respectively, $\varphi(w)$) the number of significant letters $-$ (respectively, $+$) that w contains. Thus, in the notation of section 2.1, $\varepsilon(w) = s$ and $\varphi(w) = r$. If $\varepsilon(w) > 0$, we can change in w the leftmost significant letter $-$ into a $+$; the resulting word is denoted by $\tilde{e}(w)$. Likewise, if $\varphi(w) > 0$, we can change in w the rightmost significant letter $+$ into a $-$; the resulting word is denoted by $\tilde{f}(w)$. If these operations are not feasible, then $\tilde{e}(w)$ or $\tilde{f}(w)$ is defined to be 0. Endowed with the maps wt, ε, φ, $\tilde{e}$, $\tilde{f}$, the set $\mathscr{C}_n$ identifies with the crystal* of the $\mathfrak{sl}_2(\mathbb{K})$-module $V^{\otimes n}$.

We denote by $\ell(w) = \varepsilon(w) + \varphi(w)$ the number of significant letters in a word $w \in \mathscr{C}_n$; thus w is semistable if and only if $\ell(w) = 0$. For each $p \in \{0, \dots, n\}$, we denote by $(V^{\otimes n})_{\leq p}$ the subspace of $V^{\otimes n}$ spanned by the elements y_w such that $\ell(w) \leq p$. We agree that $(V^{\otimes n})_{\leq -1} = \{0\}$.

Proposition 2. *The basis $(y_w)_{w \in \mathscr{C}_n}$ of $V^{\otimes n}$ enjoys the following properties.*

(i) *For each $w \in \mathscr{C}_n$, we have*

$$e \cdot y_w \equiv \varepsilon(w)\, y_{\tilde{e}(w)} \quad \text{and} \quad f \cdot y_w \equiv \varphi(w)\, y_{\tilde{f}(w)}$$

modulo terms in $(V^{\otimes n})_{\leq \ell(w)-1}$.

(ii) *For each $p \in \{0, \dots, n\}$, the subspace $(V^{\otimes n})_{\leq p}$ is a subrepresentation of $V^{\otimes n}$, and the quotient $(V^{\otimes n})_{\leq p}/(V^{\otimes n})_{\leq p-1}$ is an isotypic representation, specifically the sum of simple $\mathfrak{sl}_2(\mathbb{K})$-modules of dimension $p+1$.*

(iii) *The elements y_w with w semistable form a basis of the space of invariants $(V^{\otimes n})^{\mathrm{SL}_2(\mathbb{K})}$.*

Sketch of proof. We first note that y_{-+} is invariant under the action of $\mathrm{SL}_2(\mathbb{K})$ on $V^{\otimes 2}$ and that any semistable word is the result of repetitive insertions of the word $-+$ inside the empty word (possibly at non disjoint positions). From Proposition 1(iii), it then follows that any element y_w with w semistable is $\mathrm{SL}_2(\mathbb{K})$-invariant. Using now (6), we reduce the proof of statement (i) to the case where w is of the form $+\cdots+-\cdots-$ (perhaps for a smaller n), which is easily dealt with.

Statement (ii) is a direct consequence of statement (i) and implies that $(V^{\otimes n})_{\leq 0}$ is the subspace of invariants $(V^{\otimes n})^{\mathrm{SL}_2(\mathbb{K})}$, an assertion equivalent to statement (iii). $\square$

The basis $(y_w)_{w \in \mathscr{C}_n}$ of $V^{\otimes n}$ is even more remarkable than what Proposition 2 claims. In fact, let V_q be the vector space with basis (x_+, x_-) over the field $\mathbb{C}(q)$. On the one hand, we can recover V (in the case $\mathbb{K} = \mathbb{C}$) as the specialization of V_q at $q = 1$; on the other hand, we can regard V_q as the defining representation of the quantum group $U_q(\mathfrak{sl}_2)$. Frenkel and Khovanov showed ([7], Theorem 1.9) that the elements in the dual canonical basis of the $U_q(\mathfrak{sl}_2)$-module $V_q^{\otimes n}$ are produced by inserting repetitively the element $x_+ \otimes x_- - q^{-1} x_- \otimes x_+$ inside an element of the form $x_- \otimes \cdots \otimes x_- \otimes x_+ \otimes \cdots \otimes x_+$. Comparing with Proposition 1, we deduce:

Theorem 3. *Up to the reversal of the order of the tensor factors, $(y_w)_{w \in \mathcal{C}_n}$ is the dual canonical basis of $V_q^{\otimes n}$ specialized at $q = 1$.*

3. The Mirković–Vilonen basis

In this section, we consider a connected reductive group G over $\mathbb{C}$ and explain the definition of the Mirković–Vilonen basis (from now on: MV basis) in a tensor product $V(\boldsymbol{\lambda}) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$ of irreducible representations of G. References for the material presented here are [18] and sect. 2.4 in [12]. We recall the recipe from [1] to compute the transition matrix between the MV basis of $V(\boldsymbol{\lambda})$ and the tensor product of the MV bases of the factors $V(\lambda_1)$, ..., $V(\lambda_n)$. We state and prove a compatibility property of the MV bases with tensor products of projections onto Cartan components.

*In fact, here we use the opposite of the usual tensor product of crystals.

3.1. **Definition of the basis.** We choose a maximal torus T and a Borel subgroup B of G such that $T \subset B$. The Langlands dual $G^\vee$ of G comes with a maximal torus $T^\vee$ and a Borel subgroup $B^\vee$. We denote by $N^{-,\vee}$ the unipotent radical of the Borel subgroup of $G^\vee$ opposite to $B^\vee$ with respect to $T^\vee$. We denote by Λ the weight lattice of T and by $\Lambda^+ \subset \Lambda$ the set of dominant weights. Let $\leq$ be the dominance order on Λ: positive elements with respect to $\leq$ are sums of positive roots. We view the half-sum of all positive coroots as a linear form $\rho : \Lambda \to \mathbb{Q}$.

The affine Grassmannian of $G^\vee$ is the homogeneous space

$$\mathrm{Gr} = G^\vee\big(\mathbb{C}\big[z, z^{-1}\big]\big)/G^\vee(\mathbb{C}[z]),$$

where z is an indeterminate. It is endowed with the structure of an ind-variety.

A weight $\lambda \in \Lambda$ is a cocharacter of $T^\vee$. Its value at z is a point $z^\lambda \in T^\vee\big(\mathbb{C}\big[z, z^{-1}\big]\big)$, whose image in Gr is denoted by L_λ.

Assume that λ is dominant. Then the $G^\vee(\mathbb{C}[z])$-orbit through L_λ in Gr, denoted by Gr^λ, is a smooth connected variety of dimension $2\rho(\lambda)$. The Cartan decomposition implies that

$$\mathrm{Gr} = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}^\lambda; \quad \text{moreover} \quad \overline{\mathrm{Gr}^\lambda} = \bigsqcup_{\substack{\mu \in \Lambda^+ \\ \mu \leq \lambda}} \mathrm{Gr}^\mu.$$

The geometric Satake correspondence identifies the irreducible representation of G of highest weight λ with the intersection cohomology of $\overline{\mathrm{Gr}^\lambda}$ with trivial local system of coefficients:

$$V(\lambda) = IH\Big(\overline{\mathrm{Gr}^\lambda}, \underline{\mathbb{C}}\Big).$$

Let n be a positive integer. The group $G^\vee(\mathbb{C}[z])^n$ acts on the space $G^\vee\big(\mathbb{C}\big[z, z^{-1}\big]\big)^n$ by

$$(h_1, \ldots, h_n) \cdot (g_1, \ldots, g_n) = (g_1 h_1^{-1}, h_1 g_2 h_2^{-1}, \ldots, h_{n-1} g_n h_n^{-1})$$

where $(h_1, \ldots, h_n) \in G^\vee(\mathbb{C}[z])^n$ and $(g_1, \ldots, g_n) \in G^\vee\big(\mathbb{C}\big[z, z^{-1}\big]\big)^n$. The quotient is called the n-fold convolution variety and is denoted by Gr_n. We will use the customary notation

$$\mathrm{Gr}_n = G^\vee\big(\mathbb{C}\big[z, z^{-1}\big]\big) \times^{G^\vee(\mathbb{C}[z])} \cdots \times^{G^\vee(\mathbb{C}[z])} G^\vee\big(\mathbb{C}\big[z, z^{-1}\big]\big) \,/\, G^\vee(\mathbb{C}[z])$$

to indicate this construction and denote the image in Gr_n of a tuple $(g_1, \ldots, g_n)$ by $[g_1, \ldots, g_n]$. Then Gr_n is endowed with the structure of an ind-variety. One notes that Gr_1 is just the affine Grassmannian Gr. We define a map $m_n : \mathrm{Gr}_n \to \mathrm{Gr}$ by $m_n([g_1, \ldots, g_n]) = [g_1 \cdots g_n]$.

For each tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in Λ^n, we set

$$|\boldsymbol{\lambda}| = \lambda_1 + \cdots + \lambda_n.$$

Given $\lambda \in \Lambda^+$, we set $\widehat{\mathrm{Gr}^\lambda} = G^\vee(\mathbb{C}[z])\, z^\lambda\, G^\vee(\mathbb{C}[z])$; this is the preimage of Gr^λ under the quotient map $G^\vee\big(\mathbb{C}\big[z, z^{-1}\big]\big) \to \mathrm{Gr}$. Given $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$, we define

$$\mathrm{Gr}_n^{\boldsymbol{\lambda}} = \widehat{\mathrm{Gr}^{\lambda_1}} \times^{G^\vee(\mathbb{C}[z])} \cdots \times^{G^\vee(\mathbb{C}[z])} \widehat{\mathrm{Gr}^{\lambda_n}} \,/\, G^\vee(\mathbb{C}[z]),$$

a subset of Gr_n. The geometric Satake correspondence identifies the tensor product

$$V(\boldsymbol{\lambda}) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$$

with the intersection cohomology of $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}$.

Given $\mu \in \Lambda$, the $N^{-,\vee}\big(\mathbb{C}\big[z, z^{-1}\big]\big)$-orbit through L_μ is denoted by T_μ; this is a locally closed sub-ind-variety of Gr. The Iwasawa decomposition implies that

$$\mathrm{Gr} = \bigsqcup_{\mu \in \Lambda} T_\mu; \quad \text{moreover} \quad \overline{T_\mu} = \bigsqcup_{\substack{\nu \in \Lambda \\ \nu \geq \mu}} T_\nu.$$

For each $(\lambda, \mu) \in \Lambda^+ \times \Lambda$, the intersection $\overline{\mathrm{Gr}^\lambda} \cap T_\mu$ (if non-empty) has pure dimension $\rho(\lambda - \mu)$. Using this fact, Mirković and Vilonen set up the geometric Satake correspondence so that the μ-weight subspace of $V(\lambda)$ identifies with the top-dimensional Borel–Moore homology of $\mathrm{Gr}^\lambda \cap T_\mu$ ([18], Corollary 7.4):

$$V(\lambda)_\mu = H^{\mathrm{BM}}_{2\rho(\lambda-\mu)}\Big(\mathrm{Gr}^\lambda \cap T_\mu\Big).$$

We denote by $\mathscr{Z}(\lambda)_\mu$ the set of irreducible components of $\overline{\mathrm{Gr}^\lambda} \cap T_\mu$. If $Z \in \mathscr{Z}(\lambda)_\mu$, then $Z \cap \mathrm{Gr}^\lambda$ is an irreducible component of $\mathrm{Gr}^\lambda \cap T_\mu$, whose fundamental class in Borel–Moore homology is denoted by $\langle Z \rangle$. The classes $\langle Z \rangle$, for $Z \in \mathscr{Z}(\lambda)_\mu$, form a basis of $V(\lambda)_\mu$.

Likewise, for each $(\boldsymbol{\lambda}, \mu) \in (\Lambda^+)^n \times \Lambda$, the intersection $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\mu)$ has pure dimension $\rho(|\boldsymbol{\lambda}| - \mu)$, and we can identify

$$V(\boldsymbol{\lambda})_\mu = H^{\mathrm{BM}}_{2\rho(|\boldsymbol{\lambda}|-\mu)}\Big(\mathrm{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\mu)\Big).$$

We denote by $\mathscr{Z}(\boldsymbol{\lambda})_\mu$ the set of irreducible components of $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\mu)$. If $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu$, then $\mathbf{Z} \cap \mathrm{Gr}_n^{\boldsymbol{\lambda}}$ is an irreducible component of $\mathrm{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\mu)$, whose fundamental class in Borel–Moore homology is denoted by $\langle \mathbf{Z} \rangle$. The classes $\langle \mathbf{Z} \rangle$, for $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu$, form a basis of $V(\boldsymbol{\lambda})_\mu$.

We set

$$\mathscr{Z}(\lambda) = \bigsqcup_{\mu \in \Lambda} \mathscr{Z}(\lambda)_\mu \quad \text{and} \quad \mathscr{Z}(\boldsymbol{\lambda}) = \bigsqcup_{\mu \in \Lambda} \mathscr{Z}(\boldsymbol{\lambda})_\mu.$$

Elements in these sets are called Mirković–Vilonen (MV) cycles, and the bases of $V(\lambda)$ and $V(\boldsymbol{\lambda})$ obtained above are called MV bases.

3.2. **Indexation of the Mirković–Vilonen cycles.** In this short section, we explain that there is a natural bijection

$$\mathscr{Z}(\boldsymbol{\lambda}) \cong \mathscr{Z}(\lambda_1) \times \cdots \times \mathscr{Z}(\lambda_n) \tag{7}$$

for any $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$. The construction goes back to Braverman and Gaitsgory [4]; details can be found in [1], Proposition 2.2 and Corollary 4.10.

For $\mu \in \Lambda$, we define

$$\widetilde{T_\mu} = N^{-,\vee}\big(\mathbb{C}\big[z, z^{-1}\big]\big)\, z^\mu$$

and note that the natural map

$$\widetilde{T_\mu} \,/\, N^{-,\vee}(\mathbb{C}[z]) \to T_\mu$$

is bijective. Given a $N^{-,\vee}(\mathbb{C}[z])$-invariant subset $Z \subset T_\mu$, we denote by $\widetilde{Z}$ the preimage of Z by the quotient map $\widetilde{T_\mu} \to T_\mu$. In particular, the notation $\widetilde{Z}$ is defined for any MV cycle Z.

Pick $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ in Λ^n and $\mathbf{Z} = (Z_1, \ldots, Z_n)$ in $\mathscr{Z}(\lambda_1)_{\mu_1} \times \cdots \times \mathscr{Z}(\lambda_n)_{\mu_n}$. Then the closure of

$$\Big\{ [g_1, \ldots, g_n] \;\Big|\; (g_1, \ldots, g_n) \in \widetilde{Z_1} \times \cdots \times \widetilde{Z_n} \Big\}$$

in $(m_n)^{-1}\big(T_{|\boldsymbol{\mu}|}\big)$ is an MV cycle, and actually belongs to $\mathscr{Z}(\boldsymbol{\lambda})_{|\boldsymbol{\mu}|}$. Each MV cycle in $\mathscr{Z}(\boldsymbol{\lambda})$ can be uniquely obtained in this manner, which defines the bijection (7).

Because of this, we will allow ourselves to write elements in $\mathscr{Z}(\boldsymbol{\lambda})$ as tuples $\mathbf{Z}$ as above.

3.3. **Transition matrix.** We continue with our tuple of dominant weights $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$. To compute the MV basis of $V(\boldsymbol{\lambda})$, we compare it with the tensor product of the MV bases of the factors $V(\lambda_1)$, …, $V(\lambda_n)$. This requires the introduction of a nice geometric object.

Let n be a positive integer. We define the n-fold Beilinson–Drinfeld convolution variety $\mathcal{G}r_n$ as the set of pairs $(x_1, \dots, x_n; [g_1, \dots, g_n])$, where $(x_1, \dots, x_n) \in \mathbb{C}^n$ and $[g_1, \dots, g_n]$ belongs to

$$G^\vee\big(\mathbb{C}\big[z,(z-x_1)^{-1}\big]\big) \times^{G^\vee(\mathbb{C}[z])} \cdots \times^{G^\vee(\mathbb{C}[z])} G^\vee\big(\mathbb{C}\big[z,(z-x_n)^{-1}\big]\big) \,/\, G^\vee\big(\mathbb{C}[z]\big).$$

We denote by $\pi : \mathcal{G}r_n \to \mathbb{C}^n$ the morphism which forgets $[g_1, \dots, g_n]$. It is known that $\mathcal{G}r_n$ is endowed with the structure of an ind-variety and that π is ind-proper.

To each composition $\mathbf{n} = (n_1, \dots, n_r)$ of n in r parts corresponds a partial diagonal $\Delta_{\mathbf{n}}$ in $\mathbb{C}^n$, defined as the set of all elements of the form

$$\mathbf{x} = (\underbrace{x_1, \dots, x_1}_{n_1 \text{ times}}, \dots, \underbrace{x_r, \dots, x_r}_{n_r \text{ times}}) \tag{8}$$

for $(x_1, \dots, x_r) \in \mathbb{C}^r$. The small diagonal is the particular case $\mathbf{n} = (n)$; we denote it simply by Δ. We define $\mathcal{G}r_n\big|_{\Delta_{\mathbf{n}}}$ to be $\pi^{-1}(\Delta_{\mathbf{n}})$.

Given $g \in G^\vee\big(\mathbb{C}\big[z, z^{-1}\big]\big)$ and $x \in \mathbb{C}$, we denote by $g_{|x}$ the result of substituting $z - x$ for z in g. We define $\mathcal{G}r_n^{\boldsymbol{\lambda}}$ to be the set of all pairs $\big(x_1, \dots, x_n; \big[g_{1|x_1}, \dots, g_{n|x_n}\big]\big)$ with $(x_1, \dots, x_n) \in \mathbb{C}^n$ and $g_j \in \widehat{\mathrm{Gr}^{\lambda_j}}$ for each $j \in \{1, \dots, n\}$. Similarly, given $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ in Λ^n, we define $\mathcal{T}_{\boldsymbol{\mu}}$ to be the set of all pairs $\big(x_1, \dots, x_n; \big[g_{1|x_1}, \dots, g_{n|x_n}\big]\big)$ with $(x_1, \dots, x_n) \in \mathbb{C}^n$ and $g_j \in \widetilde{T_{\mu_j}}$ for each $j \in \{1, \dots, n\}$. For $\mu \in \Lambda$, we set (leaving n out of the notation)

$$\dot{T}_\mu = \bigcup_{\substack{\boldsymbol{\mu} \in \Lambda^n \\ |\boldsymbol{\mu}| = \mu}} \mathcal{T}_{\boldsymbol{\mu}}.$$

Given $(\mu_1, \dots, \mu_n) \in \Lambda^n$ and $\mathbf{Z} = (Z_1, \dots, Z_n)$ in $\mathscr{Z}(\lambda_1)_{\mu_1} \times \cdots \times \mathscr{Z}(\lambda_n)_{\mu_n}$, we define $\dot{\mathcal{X}}(\mathbf{Z})$ to be the set of all pairs $\big(x_1, \dots, x_n; \big[g_{1|x_1}, \dots, g_{n|x_n}\big]\big)$ with $(x_1, \dots, x_n) \in \mathbb{C}^n$ and $g_j \in \widetilde{Z_j}$ for each $j \in \{1, \dots, n\}$. Given in addition a composition $\mathbf{n}$ of n, we define

$$\mathcal{X}(\mathbf{Z}, \mathbf{n}) = \overline{\dot{\mathcal{X}}(\mathbf{Z})\big|_{\Delta_{\mathbf{n}}}} \cap \mathcal{G}r_n^{\boldsymbol{\lambda}}.$$

(In [1], $\dot{\mathcal{X}}(\mathbf{Z})$ is denoted by $\Psi(Z_1 \propto \cdots \propto Z_n)$ and $\mathcal{X}(\mathbf{Z}, \mathbf{n})$ is defined as $\overline{\dot{\mathcal{X}}(\mathbf{Z})\big|_{\Delta_{\mathbf{n}}}} \cap \mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu$.)

For given $\boldsymbol{\lambda}$, μ and $\mathbf{n}$, the subsets $\mathcal{X}(\mathbf{Z}, \mathbf{n}) \cap \dot{T}_\mu$ for $\mathbf{Z}$ in

$$\mathscr{Z}(\boldsymbol{\lambda})_\mu = \bigsqcup_{\substack{(\mu_1, \dots, \mu_n) \in \Lambda^n \\ \mu_1 + \cdots + \mu_n = \mu}} \mathscr{Z}(\lambda_1)_{\mu_1} \times \cdots \times \mathscr{Z}(\lambda_n)_{\mu_n}$$

are the irreducible components of $\big(\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu\big)\big|_{\Delta_{\mathbf{n}}}$ (see [1], proof of Proposition 5.4). We adopt a special notation for the small diagonal and set $\mathcal{Y}(\mathbf{Z}) = \mathcal{X}(\mathbf{Z}, (n))$.

Now fix n, the tuple $\boldsymbol{\lambda} \in (\Lambda^+)^n$, the weight $\mu \in \Lambda$, and the composition $\mathbf{n}$ of n. We write $\boldsymbol{\lambda}$ as a concatenation $\big(\boldsymbol{\lambda}_{(1)}, \dots, \boldsymbol{\lambda}_{(r)}\big)$, where each $\boldsymbol{\lambda}_{(j)}$ belongs to $(\Lambda^+)^{n_j}$, and similarly we write each tuple $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu$ as $\big(\mathbf{Z}_{(1)}, \dots, \mathbf{Z}_{(r)}\big)$ with $\mathbf{Z}_{(j)} \in \mathscr{Z}(\boldsymbol{\lambda}_{(j)})$. Then

$$V(\boldsymbol{\lambda}) = V\big(\boldsymbol{\lambda}_{(1)}\big) \otimes \cdots \otimes V\big(\boldsymbol{\lambda}_{(r)}\big) \quad \text{and} \quad \big\langle \mathbf{Z}_{(j)} \big\rangle \in V\big(\boldsymbol{\lambda}_{(j)}\big).$$

With this notation ([1], Proposition 5.10):

Proposition 4. *Let $(\mathbf{Z}', \mathbf{Z}'') \in (\mathscr{Z}(\boldsymbol{\lambda})_\mu)^2$. The coefficient $b_{\mathbf{Z}',\mathbf{Z}''}$ in the expansion*

$$\big\langle \mathbf{Z}''_{(1)} \big\rangle \otimes \cdots \otimes \big\langle \mathbf{Z}''_{(r)} \big\rangle = \sum_{\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu} b_{\mathbf{Z},\mathbf{Z}''} \, \langle \mathbf{Z} \rangle$$

is the multiplicity of $\mathcal{Y}(\mathbf{Z}')$ in the intersection product $\mathcal{X}(\mathbf{Z}'', \mathbf{n}) \cdot \mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_\Delta$ computed in the ambient space $\mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_{\Delta_\mathbf{n}}$.

3.4. **Projecting onto Cartan components.** To begin with, let n be a positive integer and let $\boldsymbol{\lambda} \in (\Lambda^+)^n$. We denote by $p : V(\boldsymbol{\lambda}) \to V(|\boldsymbol{\lambda}|)$ the projection onto the Cartan component of $V(\boldsymbol{\lambda})$, i.e. the top step in the isotypic filtration. The map $m_n : \mathrm{Gr}_n \to \mathrm{Gr}$ restricts to an isomorphism $\mathrm{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}\big(\mathrm{Gr}^{|\boldsymbol{\lambda}|}\big) \to \mathrm{Gr}^{|\boldsymbol{\lambda}|}$ (see [13], p. 2110). Given $\mu \in \Lambda$ and $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu$, we define $|\mathbf{Z}|$ to be the closure in T_μ of $m_n(\mathbf{Z}) \cap \mathrm{Gr}^{|\boldsymbol{\lambda}|}$.

Proposition 5 is a direct consequence of Theorem 3.4 in [1] and its proof.

Proposition 5.

(i) *The map $\mathbf{Z} \mapsto |\mathbf{Z}|$ defines a bijection $\big\{\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda}) \bigm| |\mathbf{Z}| \neq \varnothing\big\} \to \mathscr{Z}(|\boldsymbol{\lambda}|)$.*

(ii) *Let $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})$. If $|\mathbf{Z}| \neq \varnothing$, then $p(\langle \mathbf{Z} \rangle) = \big\langle |\mathbf{Z}| \big\rangle$; otherwise $p(\langle \mathbf{Z} \rangle) = 0$.*

By Corollary 4.10 in [1], the condition $|\mathbf{Z}| \neq \varnothing$ concretely means that under the bijection (7), $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})$ belongs to the connected component of highest weight $|\boldsymbol{\lambda}|$ of the tensor product of crystals $\mathscr{Z}(\lambda_1) \times \cdots \times \mathscr{Z}(\lambda_n)$.

Now let $\mathbf{n} = (n_1, \dots, n_r)$ be a composition of n in r parts. We again write $\boldsymbol{\lambda}$ as a concatenation $\big(\boldsymbol{\lambda}_{(1)}, \dots, \boldsymbol{\lambda}_{(r)}\big)$, where each $\boldsymbol{\lambda}_{(j)}$ belongs to $(\Lambda^+)^{n_j}$, and set $\|\boldsymbol{\lambda}\| = \big(\big|\boldsymbol{\lambda}_{(1)}\big|, \dots, \big|\boldsymbol{\lambda}_{(r)}\big|\big)$; then

$$V(\|\boldsymbol{\lambda}\|) = V\big(\big|\boldsymbol{\lambda}_{(1)}\big|\big) \otimes \cdots \otimes V\big(\big|\boldsymbol{\lambda}_{(r)}\big|\big).$$

For each $j \in \{1, \dots, r\}$, we denote by $p_{(j)} : V\big(\boldsymbol{\lambda}_{(j)}\big) \to V\big(\big|\boldsymbol{\lambda}_{(j)}\big|\big)$ the projection onto the Cartan component and define

$$\mathbf{p} = p_{(1)} \otimes \cdots \otimes p_{(r)};$$

thus $\mathbf{p} : V(\boldsymbol{\lambda}) \to V(\|\boldsymbol{\lambda}\|)$.

Likewise, we again write each tuple $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})$ as a concatenation $\big(\mathbf{Z}_{(1)}, \dots, \mathbf{Z}_{(r)}\big)$ with $\mathbf{Z}_{(j)} \in \mathscr{Z}(\boldsymbol{\lambda}_{(j)})$ and set $\|\mathbf{Z}\| = \big(\big|\mathbf{Z}_{(1)}\big|, \dots, \big|\mathbf{Z}_{(r)}\big|\big)$.

Proposition 6. *Let $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})$. If $\big|\mathbf{Z}_{(j)}\big| \neq \varnothing$ for all $j \in \{1, \dots, r\}$, then $\mathbf{p}(\langle \mathbf{Z} \rangle) = \big\langle \|\mathbf{Z}\| \big\rangle$; otherwise $p(\langle \mathbf{Z} \rangle) = 0$.*

Proof. Let $\mathring{\mathscr{Z}}(\boldsymbol{\lambda})$ be the set of all $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})$ such that $\big|\mathbf{Z}_{(j)}\big| \neq \varnothing$ for all $j \in \{1, \dots, r\}$; then the map $\mathbf{Z} \mapsto \|\mathbf{Z}\|$ realizes a bijection from $\mathring{\mathscr{Z}}(\boldsymbol{\lambda})$ onto $\mathscr{Z}(\|\boldsymbol{\lambda}\|)$.

We fix a weight $\mu \in \Lambda$ and introduce the transition matrices $(b_{\mathbf{Z}',\mathbf{Z}''})$ and $(a_{\mathbf{Y}',\mathbf{Y}''})$, where $(\mathbf{Z}',\mathbf{Z}'') \in (\mathscr{Z}(\boldsymbol{\lambda})_\mu)^2$ and $(\mathbf{Y}',\mathbf{Y}'') \in (\mathscr{Z}(\|\boldsymbol{\lambda}\|)_\mu)^2$, that encode the expansions

$$\left\langle \mathbf{Z}''_{(1)} \right\rangle \otimes \cdots \otimes \left\langle \mathbf{Z}''_{(r)} \right\rangle = \sum_{\mathbf{Z}' \in \mathscr{Z}(\boldsymbol{\lambda})_\mu} b_{\mathbf{Z}',\mathbf{Z}''} \left\langle \mathbf{Z}' \right\rangle$$

and

$$\left\langle Y''_1 \right\rangle \otimes \cdots \otimes \left\langle Y''_r \right\rangle = \sum_{\mathbf{Y}' \in \mathscr{Z}(\|\boldsymbol{\lambda}\|)_\mu} a_{\mathbf{Y}',\mathbf{Y}''} \left\langle \mathbf{Y}' \right\rangle$$

in the MV bases of $V(\boldsymbol{\lambda})$ and $V(\|\boldsymbol{\lambda}\|)$. We claim that if $\mathbf{Z}' \in \mathring{\mathscr{Z}}(\boldsymbol{\lambda})$, then

$$b_{\mathbf{Z}',\mathbf{Z}''} = \begin{cases} a_{\|\mathbf{Z}'\|,\|\mathbf{Z}''\|} & \text{if } \mathbf{Z}'' \in \mathring{\mathscr{Z}}(\boldsymbol{\lambda}), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Assuming (9), we conclude the proof as follows. Let $\widetilde{\mathbf{p}} : V(\boldsymbol{\lambda}) \to V(\|\boldsymbol{\lambda}\|)$ be the linear map defined by the requirement that for all $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})$,

$$\widetilde{\mathbf{p}}(\langle \mathbf{Z} \rangle) = \begin{cases} \left\langle \|\mathbf{Z}\| \right\rangle & \text{if } \mathbf{Z} \in \mathring{\mathscr{Z}}(\boldsymbol{\lambda}), \\ 0 & \text{otherwise.} \end{cases}$$

Then (9) gives

$$\widetilde{\mathbf{p}}\big(\left\langle \mathbf{Z}_{(1)} \right\rangle \otimes \cdots \otimes \left\langle \mathbf{Z}_{(r)} \right\rangle\big) = \begin{cases} \left\langle |\mathbf{Z}_{(1)}| \right\rangle \otimes \cdots \otimes \left\langle |\mathbf{Z}_{(r)}| \right\rangle & \text{if } \mathbf{Z} \in \mathring{\mathscr{Z}}(\boldsymbol{\lambda}), \\ 0 & \text{otherwise,} \end{cases}$$

and from Proposition 5, we conclude that $\widetilde{\mathbf{p}} = \mathbf{p}$.

We are thus reduced to prove (9). We define a map $\mathbf{m_n} : \mathcal{G}r_n\big|_{\Delta_\mathbf{n}} \to \mathcal{G}r_r$ by

$$\begin{aligned} &\mathbf{m_n}(\mathbf{x}; [g_1, \ldots, g_n]) \\ &\quad = (x_1, \ldots, x_r; [g_1 \cdots g_{n_1},\ g_{n_1+1} \cdots g_{n_1+n_2},\ \ldots,\ g_{n_1+\ldots+n_{r-1}+1} \cdots g_n]) \end{aligned}$$

for $\mathbf{x}$ as in (8). Then $\mathcal{U} = \mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_{\Delta_\mathbf{n}} \cap (\mathbf{m_n})^{-1}\left(\mathcal{G}r_r^{\|\boldsymbol{\lambda}\|}\right)$ is an open subset of $\mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_{\Delta_\mathbf{n}}$ and $\mathbf{m_n}$ restricts to an isomorphism $\mathcal{U} \to \mathcal{G}r_r^{\|\boldsymbol{\lambda}\|}$.

Let $(\mathbf{Z}',\mathbf{Z}'') \in (\mathscr{Z}(\boldsymbol{\lambda})_\mu)^2$. By Proposition 4, the coefficient $b_{\mathbf{Z}',\mathbf{Z}''}$ is the multiplicity of $\mathcal{Y}(\mathbf{Z}')$ in the intersection product $\mathcal{X}(\mathbf{Z}'',\mathbf{n}) \cdot \big(\mathcal{G}r_n^{\boldsymbol{\lambda}}\big)\big|_\Delta$ computed in the ambient space $\mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_{\Delta_\mathbf{n}}$.

Assume first that both $\mathbf{Z}'$ and $\mathbf{Z}''$ lie in $\mathring{\mathscr{Z}}(\boldsymbol{\lambda})$. Then the open subset $\mathcal{U}$ meets $\mathcal{Y}(\mathbf{Z}')$ and $\mathcal{X}(\mathbf{Z}'',\mathbf{n})$. Since intersection multiplicities are of local nature, $b_{\mathbf{Z}',\mathbf{Z}''}$ is the multiplicity of $\mathcal{Y}(\mathbf{Z}') \cap \mathcal{U}$ in the intersection product $\big(\mathcal{X}(\mathbf{Z}'',\mathbf{n}) \cap \mathcal{U}\big) \cdot \mathcal{U}\big|_\Delta$ computed in the ambient space $\mathcal{U}\big|_{\Delta_\mathbf{n}}$. On the other hand, Proposition 4 for the composition $(1^r) = (1, \ldots, 1)$ of r gives that $a_{\|\mathbf{Z}'\|,\|\mathbf{Z}''\|}$ is the multiplicity of $\mathcal{Y}(\|\mathbf{Z}'\|)$ in the intersection product $\mathcal{X}(\|\mathbf{Z}''\|,(1^r)) \cdot \big(\mathcal{G}r_r^{\|\boldsymbol{\lambda}\|}\big)\big|_\Delta$ computed in the ambient space $\mathcal{G}r_r^{\|\boldsymbol{\lambda}\|}$. Observing that

$$\mathbf{m_n}\big(\mathcal{Y}(\mathbf{Z}') \cap \mathcal{U}\big) = \mathcal{Y}(\|\mathbf{Z}'\|) \quad \text{and} \quad \mathbf{m_n}\big(\mathcal{X}(\mathbf{Z}'',\mathbf{n}) \cap \mathcal{U}\big) = \mathcal{X}(\|\mathbf{Z}''\|,(1^r)),$$

we conclude that $b_{\mathbf{Z}',\mathbf{Z}''} = a_{\|\mathbf{Z}'\|,\|\mathbf{Z}''\|}$ in this case.

Now assume that $\mathbf{Z}'$ is in $\mathring{\mathscr{Z}}(\boldsymbol{\lambda})$ but not $\mathbf{Z}''$. Then there exists $j \in \{1, \ldots, r\}$ such that $\mathbf{Z}''_{(j)}$ is contained in $F = \overline{\mathrm{Gr}_{n_j}^{\boldsymbol{\lambda}_{(j)}}} \setminus (m_{n_j})^{-1}\big(\mathrm{Gr}^{|\boldsymbol{\lambda}_{(j)}|}\big)$. For $x \in \mathbb{C}$, denote by $\widehat{F}_{|x}$ the set of all tuples $\bigl(g_{1|x},\dots,g_{n_j|x}\bigr)$ where

$$(g_1,\dots,g_{n_j})\in\bigl(G^\vee\bigl(\mathbb{C}\bigl[z,z^{-1}\bigr]\bigr)\bigr)^{n_j}\quad\text{and}\quad[g_1,\dots,g_{n_j}]\in F,$$

and denote by $\mathcal{F}$ the subset of $\mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_{\Delta_{\mathbf{n}}}$ consisting of all pairs $(\mathbf{x};[g_1,\dots,g_n])$ such that

$$(g_{n_1+\cdots+n_{j-1}+1},\dots,g_{n_1+\cdots+n_j})\in\widehat{F}_{|x_j}$$

where $\mathbf{x}$ is written as in (8). Then F is closed in $\overline{\mathrm{Gr}_{n_j}^{\boldsymbol{\lambda}_{(j)}}}$ and $\mathcal{X}(\mathbf{Z}'',\mathbf{n})$ is contained in $\mathcal{F}$. As $\mathcal{Y}(\mathbf{Z}')$ is not contained in $\mathcal{F}$, it is not contained in $\mathcal{X}(\mathbf{Z}'',\mathbf{n})$, so here $b_{\mathbf{Z}',\mathbf{Z}''}=0$. □

3.5. **Truncation.** In this section, we come back to the setup of sect. 3.3 and record a property which will simplify our analysis.

We fix nonnegative integers n_1, n_2, n_3 and tuples $\boldsymbol{\lambda}_{(1)}\in(\Lambda^+)^{n_1}$, $\boldsymbol{\lambda}_{(2)}\in(\Lambda^+)^{n_2}$, $\boldsymbol{\lambda}_{(3)}\in(\Lambda^+)^{n_3}$. We define $\boldsymbol{\lambda}$ to be the concatenation $\bigl(\boldsymbol{\lambda}_{(1)},\boldsymbol{\lambda}_{(2)},\boldsymbol{\lambda}_{(3)}\bigr)$ and we regard elements $\mathbf{Z}\in\mathscr{Z}(\boldsymbol{\lambda})$ as concatenations $\bigl(\mathbf{Z}_{(1)},\mathbf{Z}_{(2)},\mathbf{Z}_{(3)}\bigr)$ where each $\mathbf{Z}_{(j)}$ belongs to $\mathscr{Z}(\boldsymbol{\lambda}_{(j)})$. If $\nu\in\Lambda$ and $\mathbf{Z}_{(3)}\in\mathscr{Z}(\boldsymbol{\lambda}_{(3)})_\nu$, then we set $\operatorname{wt}\mathbf{Z}_{(3)}=\nu$.

We fix a weight $\mu\in\Lambda$ and introduce the transition matrix $(a_{\mathbf{Z}',\mathbf{Z}''})$, where $(\mathbf{Z}',\mathbf{Z}'')\in(\mathscr{Z}(\boldsymbol{\lambda})_\mu)^2$, that encodes the expansions

$$\bigl\langle\mathbf{Z}''_{(1)}\bigr\rangle\otimes\bigl\langle\bigl(\mathbf{Z}''_{(2)},\mathbf{Z}''_{(3)}\bigr)\bigr\rangle=\sum_{\mathbf{Z}'\in\mathscr{Z}(\boldsymbol{\lambda})_\mu}a_{\mathbf{Z}',\mathbf{Z}''}\,\bigl\langle\bigl(\mathbf{Z}'_{(1)},\mathbf{Z}'_{(2)},\mathbf{Z}'_{(3)}\bigr)\bigr\rangle$$

in the MV basis of $V(\boldsymbol{\lambda})$.

Proposition 7.

(i) *Let* $(\mathbf{Z}',\mathbf{Z}'')\in(\mathscr{Z}(\boldsymbol{\lambda})_\mu)^2$. *If* $a_{\mathbf{Z}',\mathbf{Z}''}\neq0$, *then either* $\operatorname{wt}\mathbf{Z}'_{(3)}<\operatorname{wt}\mathbf{Z}''_{(3)}$ *or* $\mathbf{Z}'_{(3)}=\mathbf{Z}''_{(3)}$.

(ii) *Let* $\mathbf{Z}''\in\mathscr{Z}(\boldsymbol{\lambda})_\mu$. *Then*

$$\bigl\langle\mathbf{Z}''_{(1)}\bigr\rangle\otimes\bigl\langle\mathbf{Z}''_{(2)}\bigr\rangle=\sum_{\substack{\mathbf{Z}'\in\mathscr{Z}(\boldsymbol{\lambda})_\mu\\ \mathbf{Z}'_{(3)}=\mathbf{Z}''_{(3)}}}a_{\mathbf{Z}',\mathbf{Z}''}\,\bigl\langle\bigl(\mathbf{Z}'_{(1)},\mathbf{Z}'_{(2)}\bigr)\bigr\rangle$$

in $V\bigl(\boldsymbol{\lambda}_{(1)}\bigr)\otimes V\bigl(\boldsymbol{\lambda}_{(2)}\bigr)$.

Proof. Let $\mathbf{Z}''\in\mathscr{Z}(\boldsymbol{\lambda})_\mu$ and set $\nu=\operatorname{wt}\mathbf{Z}''_{(3)}$. Expanding $\bigl\langle\mathbf{Z}''_{(1)}\bigr\rangle\otimes\bigl\langle\mathbf{Z}''_{(2)}\bigr\rangle$ in the MV basis of $V\bigl(\boldsymbol{\lambda}_{(1)}\bigr)\otimes V\bigl(\boldsymbol{\lambda}_{(2)}\bigr)$, we write

$$\bigl\langle\mathbf{Z}''_{(1)}\bigr\rangle\otimes\bigl\langle\mathbf{Z}''_{(2)}\bigr\rangle=\sum_{\mathbf{Z}\in\mathscr{Z}(\boldsymbol{\lambda}_{(1)},\boldsymbol{\lambda}_{(2)})_{\mu-\nu}}c_{\mathbf{Z}}\,\langle\mathbf{Z}\rangle$$

for some complex numbers $c_{\mathbf{Z}}$.

We denote by $V\bigl(\boldsymbol{\lambda}_{(3)}\bigr)_{<\nu}$ the sum of the ξ-weight subspaces of $V\bigl(\boldsymbol{\lambda}_{(3)}\bigr)$ with $\xi<\nu$. By Theorem 5.13 in [1],

$$\bigl\langle\mathbf{Z}''_{(2)}\bigr\rangle\otimes\bigl\langle\mathbf{Z}''_{(3)}\bigr\rangle\equiv\bigl\langle\bigl(\mathbf{Z}''_{(2)},\mathbf{Z}''_{(3)}\bigr)\bigr\rangle\pmod{V\bigl(\boldsymbol{\lambda}_{(2)}\bigr)\otimes V\bigl(\boldsymbol{\lambda}_{(3)}\bigr)_{<\nu}}$$

and for each $\mathbf{Z}\in\mathscr{Z}(\boldsymbol{\lambda}_{(1)},\boldsymbol{\lambda}_{(2)})$,

$$\langle\mathbf{Z}\rangle\otimes\bigl\langle\mathbf{Z}''_{(3)}\bigr\rangle\equiv\bigl\langle\bigl(\mathbf{Z},\mathbf{Z}''_{(3)}\bigr)\bigr\rangle\pmod{V\bigl(\boldsymbol{\lambda}_{(1)}\bigr)\otimes V\bigl(\boldsymbol{\lambda}_{(2)}\bigr)\otimes V\bigl(\boldsymbol{\lambda}_{(3)}\bigr)_{<\nu}}.$$

Consequently,

$$\sum_{\mathbf{Z}'\in\mathscr{Z}(\boldsymbol{\lambda})_\mu} a_{\mathbf{Z}',\mathbf{Z}''}\,\big\langle\big(\mathbf{Z}'_{(1)},\mathbf{Z}'_{(2)},\mathbf{Z}'_{(3)}\big)\big\rangle \equiv \sum_{\mathbf{Z}\in\mathscr{Z}(\boldsymbol{\lambda}_{(1)},\boldsymbol{\lambda}_{(2)})_{\mu-\nu}} c_{\mathbf{Z}}\,\big\langle\big(\mathbf{Z},\mathbf{Z}''_{(3)}\big)\big\rangle$$

modulo $V\big(\boldsymbol{\lambda}_{(1)}\big)\otimes V\big(\boldsymbol{\lambda}_{(2)}\big)\otimes V\big(\boldsymbol{\lambda}_{(3)}\big)_{<\nu}$.

We conclude by noticing that the subspace $V\big(\boldsymbol{\lambda}_{(1)}\big)\otimes V\big(\boldsymbol{\lambda}_{(2)}\big)\otimes V\big(\boldsymbol{\lambda}_{(3)}\big)_{<\nu}$ is spanned by the basis vectors $\langle\mathbf{Z}'\rangle$ such that $\operatorname{wt}\mathbf{Z}'_{(3)}<\nu$; see Corollary 5.12 in [1]. $\square$

4. Geometry

In this section, we prove that the MV basis of the tensor powers of the natural representation of $G=\mathrm{SL}_2(\mathbb{C})$ is the basis (y_w) from sect. 2. As a matter of fact, by Theorem 5.13 in [1], the MV basis satisfies the first equation in (1), so we only have to prove that it satisfies the second one too.

4.1. **Notation.** We endow G with its usual maximal torus and Borel subgroup. The weight lattice is represented as usual as the quotient $(\mathbb{Z}\varepsilon_1\oplus\mathbb{Z}\varepsilon_2)/\mathbb{Z}(\varepsilon_1+\varepsilon_2)$. The fundamental weight ϖ is the image of ε_1 in this quotient. The notation Gr indicates the affine Grassmannian of $G^\vee=\mathrm{PGL}_2(\mathbb{C})$.

In this section, $\boldsymbol{\lambda}$ will always be of the form $(\varpi,\dots,\varpi)$; the number n of times ϖ is repeated will usually appears as a subscript in notation like $\mathrm{Gr}_n^{\boldsymbol{\lambda}}$ or $\mathcal{G}r_n^{\boldsymbol{\lambda}}$.

The cell Gr^ϖ is isomorphic to the projective line, hence is closed. The two MV cycles in $\mathscr{Z}(\varpi)$ are

$$Z_+=\mathrm{Gr}^\varpi\cap T_\varpi=\left\{\left[\begin{pmatrix} z & 0\\ 0 & 1\end{pmatrix}\right]\right\}\quad\text{and}\quad Z_-=\mathrm{Gr}^\varpi\cap T_{-\varpi}=\left\{\left[\begin{pmatrix} 1 & 0\\ a & z\end{pmatrix}\right]\;\middle|\; a\in\mathbb{C}\right\}$$

(the matrices above should actually be viewed in $\mathrm{PGL}_2\big(\mathbb{C}\big[z,z^{-1}\big]\big)$). The standard basis of $V(\varpi)=\mathbb{C}^2$ is then $(x_+,x_-)=(\langle Z_+\rangle,\langle Z_-\rangle)$.

Given a word $v\in\mathscr{C}_n$, we set

$$P(v)=\big\{\ell\in\{1,\dots,n\}\;\big|\;v(\ell)=+\big\}\quad\text{and}\quad \mathbf{Z}_v=\big(Z_{v(1)},\dots,Z_{v(n)}\big).$$

Thanks to the bijection (7), we regard $\mathbf{Z}_v$ as an element in $\mathscr{Z}(\boldsymbol{\lambda})$.

For $(x,a)\in\mathbb{C}^2$, we set

$$\varphi_+(x,a)=\begin{pmatrix} z-x & a\\ 0 & 1\end{pmatrix}\quad\text{and}\quad \varphi_-(x,a)=\begin{pmatrix} 1 & 0\\ a & z-x\end{pmatrix}.$$

Recall the notation introduced in sect. 3.3. For each word $v\in\mathscr{C}_n$, we define an embedding $\phi_v:\mathbb{C}^{2n}\to\mathcal{G}r_n^{\boldsymbol{\lambda}}$ by

$$\phi_v(\mathbf{x};\mathbf{a})=\big(\mathbf{x};\big[\varphi_{v(1)}(x_1,a_1),\dots,\varphi_{v(n)}(x_n,a_n)\big]\big)$$

where $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{a}=(a_1,\dots,a_n)$. The image of ϕ_v is an open subset U_v and ϕ_v can be regarded as a chart on the manifold $\mathcal{G}r_n^{\boldsymbol{\lambda}}$. This chart is designed so that $\dot{\mathcal{X}}(\mathbf{Z}_v)$ is the algebraic subset of U_v defined by the equations $a_\ell=0$ for $\ell\in P(v)$ (compare with the construction presented in [9]).

4.2. **The simplest example.** In this section, we consider the case $n = 2$; the variety $\mathcal{G}r_2^{\boldsymbol{\lambda}}$ has dimension 4. The words $v = +-$ and $w = -+$ give rise to charts ϕ_v and ϕ_w on $\mathcal{G}r_2^{\boldsymbol{\lambda}}$ defined by

$$\phi_v(x_1, x_2; a_1, a_2) = \left(x_1, x_2; \left[\begin{pmatrix} z - x_1 & a_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_2 & z - x_2 \end{pmatrix}\right]\right),$$

$$\phi_w(x_1, x_2; b_1, b_2) = \left(x_1, x_2; \left[\begin{pmatrix} 1 & 0 \\ b_1 & z - x_1 \end{pmatrix}, \begin{pmatrix} z - x_2 & b_2 \\ 0 & 1 \end{pmatrix}\right]\right).$$

The transition map $(\phi_w)^{-1} \circ \phi_v$ is given by

$$b_1 = 1/a_1, \qquad b_2 = -a_1(x_2 - x_1 + a_1 a_2)$$

on the domain

$$(\phi_v)^{-1}(U_v \cap U_w) = \left\{(x_1, x_2, a_1, a_2) \in \mathbb{C}^4 \;\middle|\; a_1 \neq 0\right\}.$$

We set $A = \mathbb{C}[x_1, x_2, a_1, a_2]$; this is the coordinate ring of $(\phi_v)^{-1}(U_v)$. We let $B = \mathscr{S}^{-1}A$ be the localization of A with respect to the multiplicative subset $\mathscr{S}$ generated by a_1; this is the coordinate ring of $(\phi_v)^{-1}(U_v \cap U_w)$.

In the chart ϕ_v, the cycle $\mathcal{Y}(\mathbf{Z}_v)$ is defined by the equations $a_1 = x_1 - x_2 = 0$, so the ideal in A of the subvariety

$$V = (\phi_v)^{-1}(U_v \cap \mathcal{Y}(\mathbf{Z}_v))$$

is

$$\mathfrak{p} = (a_1, x_1 - x_2).$$

In the chart ϕ_w, the cycle $\dot{\mathcal{X}}(\mathbf{Z}_w)$ is defined by the equation $b_2 = 0$, and the closure in U_v of $U_v \cap \dot{\mathcal{X}}(\mathbf{Z}_w)$ is $U_v \cap \mathcal{X}(\mathbf{Z}_w, (1,1))$. Therefore the ideal in B of $(\phi_v)^{-1}\big(U_v \cap \dot{\mathcal{X}}(\mathbf{Z}_w)\big)$ is $\mathring{\mathfrak{q}} = (-a_1(x_2 - x_1 + a_1 a_2))$ and the ideal in A of the subvariety

$$X = (\phi_v)^{-1}(U_v \cap \mathcal{X}(\mathbf{Z}_w, (1,1)))$$

is the preimage

$$\mathfrak{q} = (x_2 - x_1 + a_1 a_2)$$

of $\mathring{\mathfrak{q}}$ under the canonical map $A \to B$.

Plainly $\mathfrak{q} \subset \mathfrak{p}$, which shows that $V \subset X$. The local ring $\mathscr{O}_{V,X}$ of X along V is the localization of $\overline{A} = A/\mathfrak{q}$ at the ideal $\overline{\mathfrak{p}} = \mathfrak{p}/\mathfrak{q}$. Since a_2 is not in $\mathfrak{p}$, its image in $\overline{A}_{\overline{\mathfrak{p}}}$ is invertible, and then we see that $x_1 - x_2$ generates the maximal ideal of $\overline{A}_{\overline{\mathfrak{p}}}$. As a consequence, the order of vanishing of $x_1 - x_2$ along V (see [8], sect. 1.2) is equal to one. By definition, this is the multiplicity of $\mathcal{Y}(\mathbf{Z}_v)$ in the intersection product $\mathcal{X}(\mathbf{Z}_w, (1,1)) \cdot \mathcal{G}r_2^{\boldsymbol{\lambda}}\big|_\Delta$.

Proposition 4 then asserts that $y_{+-} = \langle \mathbf{Z}_v \rangle$ occurs with coefficient one in the expansion of $x_w = \langle Z_- \rangle \otimes \langle Z_+ \rangle$ on the MV basis of $V(\varpi)^{\otimes 2}$, in agreement with the equation

$$x_{-+} = y_{-+} + y_{+-}.$$

The proof of the general case follows the same pattern, but more elaborate combinatorics is needed to manage the equations.

4.3. **Transition maps.** Pick v, w in $\mathscr{C}_n$. Set $P_0 = S_0 = 1$ and $Q_0 = R_0 = 0$. For $\ell \in \{1, \ldots, n\}$, let $K_\ell = \mathbb{C}(x_1, \ldots, x_\ell, a_1, \ldots, a_\ell)$ be the field of rational functions and define by induction an element $b_\ell \in K_\ell$ and a matrix

$$\begin{pmatrix} P_\ell & Q_\ell \\ R_\ell & S_\ell \end{pmatrix}$$

with coefficients in $K_\ell[z]$ and determinant one as follows:

- If $(v(\ell), w(\ell)) = (+, +)$, then

$$b_\ell = \frac{\big(a_\ell P_{\ell-1} + Q_{\ell-1}\big)(x_\ell)}{\big(a_\ell R_{\ell-1} + S_{\ell-1}\big)(x_\ell)}, \quad \begin{cases} P_\ell = P_{\ell-1} - b_\ell R_{\ell-1}, & Q_\ell = \dfrac{a_\ell P_{\ell-1} + Q_{\ell-1} - b_\ell S_\ell}{z - x_\ell}, \\ R_\ell = (z - x_\ell) R_{\ell-1}, & S_\ell = a_\ell R_{\ell-1} + S_{\ell-1}. \end{cases}$$

- If $(v(\ell), w(\ell)) = (-, +)$, then

$$b_\ell = \frac{\big(P_{\ell-1} + a_\ell Q_{\ell-1}\big)(x_\ell)}{\big(R_{\ell-1} + a_\ell S_{\ell-1}\big)(x_\ell)}, \quad \begin{cases} P_\ell = \dfrac{P_{\ell-1} + a_\ell Q_{\ell-1} - b_\ell R_\ell}{z - x_\ell}, & Q_\ell = Q_{\ell-1} - b_\ell S_{\ell-1}, \\ R_\ell = R_{\ell-1} + a_\ell S_{\ell-1}, & S_\ell = (z - x_\ell) S_{\ell-1}. \end{cases}$$

- If $(v(\ell), w(\ell)) = (+, -)$, then

$$b_\ell = \frac{\big(a_\ell R_{\ell-1} + S_{\ell-1}\big)(x_\ell)}{\big(a_\ell P_{\ell-1} + Q_{\ell-1}\big)(x_\ell)}, \quad \begin{cases} P_\ell = (z - x_\ell) P_{\ell-1}, & Q_\ell = a_\ell P_{\ell-1} + Q_{\ell-1}, \\ R_\ell = R_{\ell-1} - b_\ell P_{\ell-1}, & S_\ell = \dfrac{a_\ell R_{\ell-1} + S_{\ell-1} - b_\ell Q_\ell}{z - x_\ell}. \end{cases}$$

- If $(v(\ell), w(\ell)) = (-, -)$, then

$$b_\ell = \frac{\big(R_{\ell-1} + a_\ell S_{\ell-1}\big)(x_\ell)}{\big(P_{\ell-1} + a_\ell Q_{\ell-1}\big)(x_\ell)}, \quad \begin{cases} P_\ell = P_{\ell-1} + a_\ell Q_{\ell-1}, & Q_\ell = (z - x_\ell) Q_{\ell-1}, \\ R_\ell = \dfrac{R_{\ell-1} + a_\ell S_{\ell-1} - b_\ell P_\ell}{z - x_\ell}, & S_\ell = S_{\ell-1} - b_\ell Q_{\ell-1}. \end{cases}$$

Since the matrix $\begin{pmatrix} P_{\ell-1} & Q_{\ell-1} \\ R_{\ell-1} & S_{\ell-1} \end{pmatrix}$ has determinant one, the denominator in the fraction that defines b_ℓ is not the zero polynomial and everything is well-defined.

Proposition 8. *The transition map*

$$(\phi_w)^{-1} \circ \phi_v : (\phi_v)^{-1}(U_v \cap U_w) \to (\phi_w)^{-1}(U_v \cap U_w)$$

is given by the rational map

$$(x_1, \ldots, x_n; a_1, \ldots, a_n) \mapsto (x_1, \ldots, x_n; b_1, \ldots, b_n)$$

where b_1, ..., b_n are defined above.

Proof. The definitions are set up so that

$$\varphi_{w(\ell)}(x_\ell, b_\ell) \begin{pmatrix} P_\ell & Q_\ell \\ R_\ell & S_\ell \end{pmatrix} = \begin{pmatrix} P_{\ell-1} & Q_{\ell-1} \\ R_{\ell-1} & S_{\ell-1} \end{pmatrix} \varphi_{v(\ell)}(x_\ell, a_\ell)$$

and therefore

$$\left(\prod_{j=1}^{\ell} \varphi_{w(j)}(x_j, b_j) \right) \begin{pmatrix} P_\ell & Q_\ell \\ R_\ell & S_\ell \end{pmatrix} = \left(\prod_{j=1}^{\ell} \varphi_{v(j)}(x_j, a_j) \right)$$

for each $\ell \in \{1, \dots, n\}$. Thus, when complex values are assigned to the indeterminates $x_1, \dots, x_n, a_1, \dots, a_n$, we get

$$\left[\prod_{j=1}^{\ell} \varphi_{v(j)}(x_j, a_j)\right] = \left[\prod_{j=1}^{\ell} \varphi_{w(j)}(x_j, b_j)\right]$$

in $\mathrm{PGL}_2\big(\mathbb{C}\big[z, (z-x_1)^{-1}, \dots, (z-x_\ell)^{-1}\big]\big) \,/\, \mathrm{PGL}_2(\mathbb{C}[z])$. This implies the equality

$$\phi_v(x_1, \dots, x_n; a_1, \dots, a_n) = \phi_w(x_1, \dots, x_n; b_1, \dots, b_n)$$

in $\mathcal{G}r_n$. $\square$

The parameters b_ℓ and the coefficients of the polynomials P_ℓ, Q_ℓ, R_ℓ, S_ℓ were defined as elements in K_ℓ. We can however be more precise and define recursively a subring $B_\ell \subset K_\ell$ to which they belong: we start with $B_0 = \mathbb{C}$, and for $\ell \in \{1, \dots, n\}$, we set $B_\ell = B_{\ell-1}\big[x_\ell, a_\ell, f_\ell^{-1}\big]$, where $f_\ell \in B_{\ell-1}[x_\ell, a_\ell]$ is the denominator in the fraction that defines b_ℓ.

Let $A_\ell = \mathbb{C}[x_1, \dots, x_\ell, a_1, \dots, a_\ell]$ be the polynomial algebra. One can easily build by induction a finitely generated multiplicative set $\mathscr{S}_\ell \subset A_\ell$ such that B_ℓ is the localization $\mathscr{S}_\ell^{-1} A_\ell$. While A_n is the coordinate ring of $(\phi_v)^{-1}(U_v)$, we see that B_n is the coordinate ring of the open subset $(\phi_v)^{-1}(U_v \cap U_w)$. In fact, since the matrix $\begin{pmatrix} P_\ell & Q_\ell \\ R_\ell & S_\ell \end{pmatrix}$ has determinant one, the numerator and the denominator of b_ℓ cannot both vanish at the same time. As a consequence, $(\phi_w)^{-1} \circ \phi_v$ cannot be defined at a point where a function in $\mathscr{S}_n$ vanishes.

4.4. **Finding the equations.** To prove that the MV basis satisfies the equation (1), we need intersection multiplicities in the ambient space $\mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_{\Delta_{(1,n-1)}}$. In practice, we make the base change $\Delta_{(1,n-1)} \to \mathbb{C}^n$ by letting $x_2 = \cdots = x_n$ in the definition of the charts and by agreeing that **from now on,** U_v **actually means** $U_v\big|_{\Delta_{(1,n-1)}}$**.** Then, in view of the invariance of the whole system under translation along the small diagonal Δ, all our equations will only involve the difference $x = x_1 - x_2$.

We will consider words v and w in $\mathscr{C}_n$ such that $(v(1), w(1)) = (+, -)$ and $\mathrm{wt}\,(v) = \mathrm{wt}\,(w)$. The planar paths that represent v and w have then the same endpoints. We write w as a concatenation $-w'$ where $w' \in \mathscr{C}_{n-1}$. Proposition 4 asserts that the basis element y_v occurs in the expansion of $x_- \otimes y_{w'}$ in the MV basis of $V(\varpi)^{\otimes n}$ only if $\mathcal{Y}(\mathbf{Z}_v) \subset \mathcal{X}(\mathbf{Z}_w, (1, n-1))$, and when this condition is fulfilled, its coefficient is the multiplicity of $\mathcal{Y}(\mathbf{Z}_v)$ in the intersection product $\mathcal{X}(\mathbf{Z}_w, (1, n-1)) \cdot \mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_{\Delta}$.

The next sections are devoted to the determination of these inclusions and intersection multiplicities. The actual calculations require the ideals in A_n of the subvarieties $(\phi_v)^{-1}(U_v \cap \mathcal{Y}(\mathbf{Z}_v))$ and $(\phi_v)^{-1}(U_v \cap \mathcal{X}(\mathbf{Z}_w, (1, n-1)))$ of $(\phi_v)^{-1}(U_v)$: the first one, denoted by $\mathfrak{p}$, is generated by x and the elements a_ℓ for $\ell \in P(v)$; the second one, denoted by $\mathfrak{q}$, is less easily determined.

Taking into account our notational convention regarding the base change $\Delta_{(1,n-1)} \to \mathbb{C}^n$, we observe that $U_v \cap \mathcal{X}(\mathbf{Z}_w, (1, n-1))$ is the closure in U_v of $U_v \cap \dot{\mathcal{X}}(\mathbf{Z}_w)$. Let $\mathring{\mathfrak{q}}_n$ be the ideal in B_n of the closed subset $(\phi_v)^{-1}\big(U_v \cap \dot{\mathcal{X}}(\mathbf{Z}_w)\big)$ of $(\phi_v)^{-1}(U_v \cap U_w)$. Then $\mathring{\mathfrak{q}}_n$ is generated by the elements b_ℓ for $\ell \in P(w)$ and $\mathfrak{q}$ is the preimage of $\mathring{\mathfrak{q}}_n$ under the canonical map $A_n \to B_n$. In other words, $\mathfrak{q}$ is the saturation with respect to $\mathscr{S}_n$ of the ideal of A_n generated by the numerators of the elements b_ℓ for $\ell \in P(w)$. Though algorithmically doable in any concrete example, finding the saturation is a demanding calculation, which we will bypass by replacing $\mathfrak{q}$ by an approximation $\widetilde{\mathfrak{q}}_n$.

4.5. **Inclusion and multiplicity, I.** This section is devoted to the situation where the paths representing v and w stay parallel to each other at distance two; specifically, we assume that $v(\ell) = w(\ell)$ for each $\ell \in \{2, \dots, n-1\}$ and $(v(n), w(n)) = (-, +)$.

Proposition 9. *Under these assumptions:*

(i) *The inclusion* $\mathcal{Y}(\mathbf{Z}_v) \subset \mathcal{X}(\mathbf{Z}_w, (1, n-1))$ *holds if and only if the last letter of* w' *is significant.*

(ii) *If the condition in* (i) *is fulfilled, then the multiplicity of* $\mathcal{Y}(\mathbf{Z}_v)$ *in the intersection product* $\mathcal{X}(\mathbf{Z}_w, (1, n-1)) \cdot \mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_\Delta$ *is equal to one.*

The proof of Proposition 9 fills the remainder of this section.

Let us denote by $S(v)$ the set of all positions $\ell \in \{1, \dots, n\}$ such that the letter $v(\ell)$ is significant in v.

In agreement with the convention set forth in section 4.4, we define $A_\ell = \mathbb{C}[x_2][x, a_1, \dots, a_\ell]$ for each $\ell \in \{1, \dots, n\}$, where $x = x_1 - x_2$. We rewrite the indeterminate z as $\tilde{z} + x_2$. We set $\widetilde{P}_1 = \tilde{z} - x$ and $\widetilde{Q}_1 = a_1$. For $\ell \in \{2, \dots, n-1\}$, we define by induction two polynomials $\widetilde{P}_\ell$, $\widetilde{Q}_\ell$ in $A_\ell[\tilde{z}]$ as follows:

- If $v(\ell) = w(\ell) = +$ and $\ell \in S(v)$, then

$$\widetilde{P}_\ell = \widetilde{P}_{\ell-1} \quad \text{and} \quad \widetilde{Q}_\ell = \frac{a_\ell \widetilde{P}_{\ell-1} + \widetilde{Q}_{\ell-1} - \big(a_\ell \widetilde{P}_{\ell-1} + \widetilde{Q}_{\ell-1}\big)(0)}{\tilde{z}}.$$

- If $v(\ell) = w(\ell) = +$ and $\ell \notin S(v)$, then $\widetilde{P}_\ell = \widetilde{P}_{\ell-1}$ and $\widetilde{Q}_\ell = \big(\widetilde{Q}_{\ell-1} - \widetilde{Q}_{\ell-1}(0)\big)/\tilde{z}$.
- If $v(\ell) = w(\ell) = -$, then $\widetilde{P}_\ell = \widetilde{P}_{\ell-1} + a_\ell \widetilde{Q}_{\ell-1}$ and $\widetilde{Q}_\ell = \tilde{z}\, \widetilde{Q}_{\ell-1}$.

Moreover, in the case where $v(\ell) = w(\ell) = +$, set

$$\widetilde{c}_\ell = \begin{cases} \big(a_\ell \widetilde{P}_{\ell-1} + \widetilde{Q}_{\ell-1}\big)(0) & \text{if } \ell \in S(v), \\ a_\ell & \text{otherwise,} \end{cases}$$

and set

$$\widetilde{c}_n = \big(\widetilde{P}_{n-1} + a_n \widetilde{Q}_{n-1}\big)(0).$$

Remark 10. The polynomials $\widetilde{P}_\ell$ and $\widetilde{Q}_\ell$ do not depend on the variables a_j with $j \in P(v) \setminus S(v)$. The elements $\widetilde{c}_\ell$ for $\ell \in \{2, \dots, n-1\} \cap P(v) \cap S(v)$ and $\widetilde{c}_n$ enjoy the same property.

For $\ell \in \{1, \dots, n\}$:

- let $\mathring{\mathfrak{q}}_\ell$ be the ideal of B_ℓ generated by $\{b_j \mid j \in P(w),\ j \le \ell\}$;
- let $\widetilde{\mathfrak{q}}_\ell$ be the ideal of A_ℓ generated by $\{\widetilde{c}_j \mid j \in P(w),\ j \le \ell\}$;
- let d_ℓ be the weight of the word $v(1)v(2)\cdots v(\ell)$ and set $D_\ell = \max(d_1, d_2, \dots, d_\ell)$.

As noticed before, a $+$ letter at position ℓ in v is significant if and only if ℓ marks the first time that the path representing v reaches a new height; agreeing that $D_0 = 0$, this translates to

$$\ell \in P(v) \cap S(v) \iff d_\ell > D_{\ell-1}.$$

For the record, we also note that the last letter of w' is significant if and only if $d_{n-1} = D_{n-1}$.

Lemma 11. *For $\ell \in \{1, \dots, n-1\}$, we have*

(i)$_\ell$ $\mathscr{S}_\ell^{-1}\widetilde{\mathfrak{q}}_\ell = \mathring{\mathfrak{q}}_\ell$,

(ii)$_\ell$ $\widetilde{P}_\ell(\tilde{z}) \equiv P_\ell(z) \pmod{\mathring{\mathfrak{q}}_\ell[z]}$ *and* $\widetilde{Q}_\ell(\tilde{z}) \equiv Q_\ell(z) \pmod{\mathring{\mathfrak{q}}_\ell[z]}$,

(iii)$_\ell$ $\tilde{z}^{D_\ell - d_\ell}$ *divides* $\widetilde{Q}_\ell$.

Proof. We proceed by induction on ℓ. The statements are banal for $\ell = 1$. Suppose that $2 \le \ell \le n-1$ and that statements (i)$_{\ell-1}$, (ii)$_{\ell-1}$ and (iii)$_{\ell-1}$ hold.

Suppose first that $(v(\ell), w(\ell)) = (+,+)$. Then by construction

$$b_\ell = \bigl(a_\ell P_{\ell-1} + Q_{\ell-1}\bigr)\bigl(x_2\bigr) \times f_\ell^{-1}, \tag{10}$$

$$P_\ell = P_{\ell-1} - b_\ell R_{\ell-1}, \qquad Q_\ell = \frac{a_\ell P_{\ell-1} + Q_{\ell-1} - b_\ell S_\ell}{z - x_2}. \tag{11}$$

If $\ell \notin S(v)$, then $d_{\ell-1} + 1 = d_\ell \le D_{\ell-1}$, and we see by (iii)$_{\ell-1}$ that $\widetilde{Q}_{\ell-1}(0) = 0$. Using (ii)$_{\ell-1}$, we deduce that $Q_{\ell-1}(x_2) \in \mathring{\mathfrak{q}}_{\ell-1}$. On the other hand, the matrix $\begin{pmatrix} P_{\ell-1}(x_2) & Q_{\ell-1}(x_2) \\ R_{\ell-1}(x_2) & S_{\ell-1}(x_2) \end{pmatrix}$ with coefficients in $B_{\ell-1}$ has determinant one. After reduction modulo $\mathring{\mathfrak{q}}_{\ell-1}$, the coefficient in the top right corner becomes zero; it follows that $P_{\ell-1}(x_2)$ is invertible in the quotient ring $B_{\ell-1}/\mathring{\mathfrak{q}}_{\ell-1}$. Reducing (10) modulo $\mathring{\mathfrak{q}}_{\ell-1}B_\ell$ and noting that here $\widetilde{c}_\ell = a_\ell$, we deduce that b_ℓ and $\widetilde{c}_\ell$ generate the same ideal in $B_\ell/\mathring{\mathfrak{q}}_{\ell-1}B_\ell$. This piece of information allows to deduce (i)$_\ell$ from (i)$_{\ell-1}$. From (11) and the fact that $a_\ell \in \mathring{\mathfrak{q}}_\ell$, we get

$$P_\ell \equiv P_{\ell-1} \pmod{\mathring{\mathfrak{q}}_\ell[z]}, \qquad Q_\ell \equiv \frac{Q_{\ell-1} - Q_{\ell-1}(x_2)}{z - x_2} \pmod{\mathring{\mathfrak{q}}_\ell[z]}.$$

Then (ii)$_\ell$ and (iii)$_\ell$ follow from (ii)$_{\ell-1}$ and (iii)$_{\ell-1}$ and from the definition of $\widetilde{P}_\ell$ and $\widetilde{Q}_\ell$.

If $\ell \in S(v)$, then (10) and (ii)$_{\ell-1}$ lead to $b_\ell \equiv \widetilde{c}_\ell/f_\ell$ modulo $\mathring{\mathfrak{q}}_{\ell-1}B_\ell$. Again, b_ℓ and $\widetilde{c}_\ell$ generate the same ideal in $B_\ell/\mathring{\mathfrak{q}}_{\ell-1}B_\ell$, so we can deduce (i)$_\ell$ from (i)$_{\ell-1}$. Then (ii)$_\ell$ follows from (ii)$_{\ell-1}$ and (11). Also, (iii)$_{\ell-1}$ holds trivially since $D_\ell = d_\ell$.

It remains to tackle the case $(v(\ell), w(\ell)) = (-,-)$. Here (i)$_\ell$, (ii)$_\ell$ and (iii)$_\ell$ can be deduced from (i)$_{\ell-1}$, (ii)$_{\ell-1}$ and (iii)$_{\ell-1}$ without ado. $\square$

Lemma 12. *With the notation above,*

$$\mathscr{S}_n^{-1}\widetilde{\mathfrak{q}}_n = \mathring{\mathfrak{q}}_n \quad \textit{and} \quad \mathfrak{q} = \bigl\{ g \in A_n \bigm| \exists f \in \mathscr{S}_n,\ fg \in \widetilde{\mathfrak{q}}_n \bigr\}.$$

Proof. From $(v(n), w(n)) = (-,+)$, we deduce

$$b_n = \bigl(P_{n-1} + a_n Q_{n-1}\bigr)\bigl(x_2\bigr) \times f_n^{-1}.$$

From the assertion (ii)$_{n-1}$ in Lemma 11, we deduce that $b_n \equiv \widetilde{c}_n/f_n$ modulo $\mathring{\mathfrak{q}}_{n-1}B_n$. Thus, b_n and $\widetilde{c}_n$ generate the same ideal in $B_n/\mathring{\mathfrak{q}}_{n-1}B_n$, and from the assertion $(\mathrm{i})_{n-1}$ in Lemma 11, we conclude that $\mathscr{S}_n^{-1}\widetilde{\mathfrak{q}}_n = \mathring{\mathfrak{q}}_n$. The second announced equality then follows from the definition of $\mathfrak{q}$ as the preimage of $\mathring{\mathfrak{q}}_n$ under the canonical map $A_n \to B_n$, with $B_n = \mathscr{S}_n^{-1}A_n$. $\square$

Lemma 13. *If the last letter of w' is not significant, then $\mathring{\mathfrak{q}}_n = B_n$.*

Proof. Assume that the last letter of w' is not significant. Then $D_{n-1} - d_{n-1} \geq 1$, and by assertion $(\mathrm{iii})_{n-1}$ in Lemma 11, we get $\widetilde{Q}_{n-1}(0) = 0$. Using assertion $(\mathrm{ii})_{n-1}$ in that lemma, we deduce that $Q_{n-1}(x_2) \in \mathring{\mathfrak{q}}_{n-1}$. Since the matrix $\begin{pmatrix} P_{n-1}(x_2) & Q_{n-1}(x_2) \\ R_{n-1}(x_2) & S_{n-1}(x_2) \end{pmatrix}$ has determinant 1, we see that $P_{n-1}(x_2)$ is invertible in the ring $B_{n-1}/\mathring{\mathfrak{q}}_{n-1}$. Then $b_n = \big(P_{n-1} + a_n Q_{n-1}\big)\big(x_2\big) \times f_n^{-1}$ is invertible in $B_n/\mathring{\mathfrak{q}}_{n-1}B_n$, and we conclude that $\mathring{\mathfrak{q}}_n = B_n$. $\square$

Lemma 13 asserts that if the last letter of w' is not significant, then $U_v \cap \dot{\mathcal{X}}(\mathbf{Z}_w) = \varnothing$, and thus $U_v \cap \mathcal{X}(\mathbf{Z}_w, (1, n-1)) = \varnothing$. Since U_v contains a dense subset of $\mathcal{Y}(\mathbf{Z}_v)$, this proves half of Proposition 9(i).

For the rest of this section, we assume that the last letter of w' is significant. We want to show that $\mathcal{Y}(\mathbf{Z}_v)$ is contained in $\mathcal{X}(\mathbf{Z}_w, (1, n-1))$. It would be rather easy to prove the inclusion $\widetilde{\mathfrak{q}}_n \subset \mathfrak{p}$, but this would not be quite enough, since we do not know that $\widetilde{\mathfrak{q}}_n = \mathfrak{q}$. (We believe that this equality is correct but we are not able to prove it.) Instead we will look explicitly at the zero set of $\widetilde{\mathfrak{q}}_n$ in the neighborhood of $(\phi_v)^{-1}(U_v \cap \mathcal{Y}(\mathbf{Z}_v))$. This zero set is the algebraic subset of $(\phi_v)^{-1}(U_v)$ defined by the equations $\widetilde{c}_\ell$ for $\ell \in P(w)$.

Our analysis is pedestrian. We observe that there are two kinds of equations $\widetilde{c}_\ell$. When $\ell \in P(v) \setminus S(v)$, the equation $\widetilde{c}_\ell$ reduces to the variable a_ℓ; this equation and variable can simply be discarded because a_ℓ is an equation for $\mathcal{Y}(\mathbf{Z}_v)$ as well. The other equations involve the other variables.

Set $D = D_n$. The map $\ell \mapsto d_\ell$ is an increasing bijection from $P(v) \cap S(v)$ onto $\{1, \ldots, D\}$. We define L as the largest element in $P(v) \cap S(v)$; then L is the smallest element in $\{\ell \mid d_\ell = D\}$. For $\ell \in \{1, \ldots, n\}$, we denote by ℓ^- the largest element in $\{1, \ldots, \ell\} \cap P(v) \cap S(v)$. In particular, $\ell^- = \ell$ if $\ell \in P(v) \cap S(v)$ and $\ell^- = L$ if $\ell \geq L$; also $d_{\ell^-} = D_\ell$.

Given $\ell \in \{1, \ldots, n\}$, let σ_ℓ be the sum of the variables a_j for $j \in \{2, \ldots, \ell\}$ such that $v(j) = -$ and $d_{j-1} = D$; thus $\sigma_\ell = 0$ if $\ell \leq L$.

We define a grading on A_n by setting $\deg x = 1$, $\deg a_\ell = D + 1 - d_\ell$ for $\ell \in P(v) \cap S(v)$, and $\deg a_\ell = 0$ for the other variables. For $d \geq 1$, we denote by J_d the ideal of A_n spanned by monomials of degree at least d.

Lemma 14. *Let $\ell \in \{1, \ldots, n-1\}$.*

- $(\mathrm{i})_\ell$ *If $\ell \leq L$, then $\widetilde{P}_\ell(\tilde{z}) \equiv \tilde{z} - x \pmod{J_2[\tilde{z}]}$; if $\ell \geq L$, then $\widetilde{P}_\ell(0) \equiv a_L\sigma_\ell - x \pmod{J_2}$.*
- $(\mathrm{ii})_\ell$ $\widetilde{Q}_\ell(\tilde{z}) \equiv \tilde{z}^{D_\ell - d_\ell}\, a_{\ell^-} \pmod{J_{D+2-d_{\ell^-}}[\tilde{z}]}$.

Proof. The proof starts with a banal verification for $\ell = 1$ and then proceeds by induction on ℓ. Suppose that $2 \leq \ell \leq n-1$ and that statements $(\mathrm{i})_{\ell-1}$ and $(\mathrm{ii})_{\ell-1}$ hold.

Assume first that $v(\ell) = w(\ell) = -$. Here $(\mathrm{ii})_\ell$ is an immediate consequence of $(\mathrm{ii})_{\ell-1}$. If $\ell - 1 < L$, then $d_{(\ell-1)^-} < D$, so $\deg a_{(\ell-1)^-} \geq 2$, and $\widetilde{Q}_{\ell-1} \in J_2[\tilde{z}]$ by statement $(\mathrm{ii})_{\ell-1}$. As a result, $\widetilde{P}_\ell \equiv \widetilde{P}_{\ell-1} \pmod{J_2[\tilde{z}]}$, so $(\mathrm{i})_\ell$ follows from $(\mathrm{i})_{\ell-1}$.

If $\ell-1\geq L$, then either $d_{\ell-1}=D$, in which case $\widetilde{Q}_{\ell-1}(0)\equiv a_L \pmod{J_2}$ and $\sigma_\ell=\sigma_{\ell-1}+a_\ell$, or $d_{\ell-1}<D$, in which case $\widetilde{Q}_{\ell-1}(0)\equiv 0 \pmod{J_2}$ and $\sigma_\ell=\sigma_{\ell-1}$. In both cases, $\widetilde{P}_\ell(0)-(a_L\sigma_\ell)\equiv\widetilde{P}_{\ell-1}(0)-(a_L\sigma_{\ell-1}) \pmod{J_2}$, and again (i)$_\ell$ follows from (i)$_{\ell-1}$.

Assume now that $v(\ell)=w(\ell)=+$ and that $\ell\in S(v)$. Certainly then (i)$_\ell$ follows from (i)$_{\ell-1}$. Further, $d_{(\ell-1)^-}=d_{\ell^-}-1$, so $\deg a_{(\ell-1)^-}=D+2-d_{\ell^-}$, hence $\widetilde{Q}_{\ell-1}$ is zero modulo $J_{D+2-d_{\ell^-}}[\tilde z]$ by (ii)$_{\ell-1}$. Using (i)$_{\ell-1}$, we conclude that $\widetilde{Q}_\ell\equiv a_\ell$ $\pmod{J_{D+2-d_{\ell^-}}[\tilde z]}$, so (ii)$_\ell$ holds.

The third situation, namely $v(\ell)=w(\ell)=+$ and $\ell\notin S(v)$, presents no difficulties. □

Lemma 15.

(i) *For* $\ell\in\{2,\dots,n-1\}\cap P(v)\cap S(v)$, *we have* $\widetilde{c}_\ell\equiv -a_\ell\, x+a_{(\ell-1)^-} \pmod{J_{D+3-d_\ell}}$.

(ii) *We have* $\widetilde{c}_n\equiv a_L\sigma_n-x \pmod{J_2}$.

Proof. Let $\ell\in\{2,\dots,n-1\}\cap P(v)\cap S(v)$. Then $D_{\ell-1}=d_{\ell-1}$ and $d_{(\ell-1)^-}=d_\ell-1$. By Lemma 14, $\widetilde{P}_{\ell-1}(0)\equiv -x \pmod{J_2}$ and $\widetilde{Q}_{\ell-1}(0)\equiv a_{(\ell-1)^-} \pmod{J_{D+3-d_\ell}}$. This gives (i).

Since the last letter of w' is assumed to be significant, we have $d_{n-1}=D_{n-1}=D$, so $\sigma_n=\sigma_{n-1}+a_n$. From Lemma 14, we get $\widetilde{P}_{n-1}(0)\equiv a_L\sigma_{n-1}-x \pmod{J_2}$ and $\widetilde{Q}_{n-1}(0)\equiv a_L \pmod{J_2}$. This gives (ii). □

Lemma 16. *There exists an element* $\widetilde{g}\in A_n$, *which depends only on the variables* x, a_1, *and* a_j *with* $v(j)=-$, *such that*

$$\widetilde{g}\equiv\widetilde{c}_n\,x^{D-1}\times\prod_{\substack{\ell\in P(v)\cap S(v)\\ \ell\geq 2}}\Big(-\widetilde{P}_{\ell-1}(0)\Big)^{p_\ell}\pmod{\widetilde{\mathfrak{q}}_L}\tag{12}$$

$$\widetilde{g}\equiv x^q\left(a_1\sigma_n-x^D\right)\pmod{J_{q+D+1}}\tag{13}$$

where each p_ℓ *and* q *are nonnegative integers.*

Proof. Consider

$$\widetilde{g}_L=\widetilde{c}_n\,x^{D-1}+\sum_{\substack{\ell\in P(v)\cap S(v)\\ \ell\geq 2}}\widetilde{c}_\ell\,\sigma_n\,x^{d_\ell-2}.$$

An immediate calculation based on Lemma 15 yields

$$\widetilde{g}_L\equiv a_1\sigma_n-x^D\pmod{J_{D+1}}.$$

This $\widetilde{g}_L$ meets the specifications for $\widetilde{g}$ (with p_ℓ and q all equal to zero) except that it may involve other variables than those prescribed.

We are not bothered by the variables a_j for $j\in P(v)\setminus S(v)$ because $\widetilde{g}_L$ do not depend on them (see Remark 10). The variables x and a_j with $v(j)=-$ are allowed. The only trouble comes then from the variables a_j with $j\in\{2,\dots,n-1\}\cap P(v)\cap S(v)$. We will eliminate them in turn.

Assume that $L\geq 2$. Let $\ell\in\{2,\dots,n-1\}\cap P(v)\cap S(v)$ and assume that we succeeded in constructing an element $\widetilde{g}_\ell\in\widetilde{\mathfrak{q}}_n$ which satisfies (12) and (13) and depends only on the variables x and a_j with $v(j) = -$ or $j \le \ell$. Expand $\widetilde{g}_\ell$ as a polynomial in a_ℓ

$$\widetilde{g}_\ell = \sum_{s=0}^{r} h_s\, a_\ell^s$$

where the coefficients h_s only depend on x and on the variables a_j such that $v(j) = -$ or $j < \ell$. Then define

$$\widetilde{g}_{(\ell-1)^-} = \sum_{s=0}^{r} h_s \left(-\widetilde{P}_{\ell-1}(0)\right)^{r-s} \left(\widetilde{Q}_{\ell-1}(0)\right)^{s}.$$

This $\widetilde{g}_{(\ell-1)^-}$ only involves the variables x and a_j with $v(j) = -$ or $j \le \ell - 1$. In fact, we can strengthen the latter inequality to $j \le (\ell-1)^-$ because $\widetilde{g}_{(\ell-1)^-}$ does not depend on the variables a_j with $j \in P(v) \setminus S(v)$. Moreover, $\widetilde{g}_{(\ell-1)^-}$ also satisfies (12) and (13), but for different integers than $\widetilde{g}_\ell$: one has to increase p_ℓ and q by r. (To verify that $\widetilde{g}_{(\ell-1)^-}$ satisfies (13) with $q + r$ instead of q, one observes that

$$h_0 \equiv x^q \left(a_1\sigma_n - x^D\right) \qquad (\mathrm{mod}\ J_{q+D+1})$$

$$h_s \in J_{q+D+1-s(D+1-d_\ell)} \quad \text{for each } s \in \{1, \dots, r\}$$

and uses Lemma 14.)

At the end of the process, we obtain an element $\widetilde{g} = \widetilde{g}_1$ which enjoys the desired properties. $\square$

Let us recall a few important points:

- $A_n = \mathbb{C}[x_2][x, a_1, \dots, a_n]$ is the coordinate ring of $(\phi_v)^{-1}(U_v)$. The variable x_2 is dummy (no equations depend on it); we get rid of it by specializing it to an arbitrary value.
- The ring B_1 is $\mathbb{C}[x_2]\left[x, a_1, f_1^{-1}\right]$ with $f_1 = a_1$. For $\ell \ge 2$, we produce an explicit function $f_\ell \in B_{\ell-1}[a_\ell]$ and we set $B_\ell = B_{\ell-1}\left[a_\ell, f_\ell^{-1}\right]$. The ring B_n is the coordinate ring of $(\phi_v)^{-1}(U_v \cap U_w)$.
- $\mathscr{S}_n$ is a finitely generated multiplicative subset of A_n such that $B_n = \mathscr{S}_n^{-1} A_n$.
- Polynomials $\widetilde{c}_\ell \in A_\ell$ are defined for each $\ell \in P(w)$. The ideal of A_n generated by these elements is denoted by $\widetilde{\mathfrak{q}}_n$.
- The ideal $\mathfrak{p} \subset A_n$ of $(\phi_v)^{-1}(U_v \cap \mathcal{Y}(\mathbf{Z}_v))$ is generated by the variables x and a_ℓ for $\ell \in P(v)$.
- The ideal $\mathfrak{q} \subset A_n$ of $(\phi_v)^{-1}(U_v \cap \mathcal{X}(\mathbf{Z}_w, (1, n-1)))$ is the saturation of $\widetilde{\mathfrak{q}}_n$ with respect to $\mathscr{S}_n$.
- $\sigma_1, \dots, \sigma_n$ are certain sums of variables a_ℓ with $v(\ell) = -$; these linear forms are not pairwise distinct, but σ_n differs from all the other ones, for only it involves a_n.

Lemma 17. *Fix $\alpha_\ell \in \mathbb{C}$ for each $\ell \in \{1, \dots, n\} \setminus P(v)$ such that, when a_ℓ is assigned the value α_ℓ, the linear form σ_n takes a value different from all the other σ_j. Consider these numbers α_ℓ as constant functions of the variable ξ. Set also $\alpha_\ell = 0$ for $\ell \in P(v) \setminus S(v)$. Then there exists a neighborhood Ω of 0 in $\mathbb{C}$ and analytic functions $\alpha_\ell : \Omega \to \mathbb{C}$ for $\ell \in P(v) \cap S(v)$ such that*

(i) *If $\ell \in P(v) \cap S(v)$, then $\alpha_\ell(\xi) \sim \xi^{D+1-d_\ell}/\sigma_n$.*
(ii) *The point $(\xi, \alpha_1(\xi), \dots, \alpha_n(\xi))$ belongs to the zero locus of $\widetilde{\mathfrak{q}}_n$ for each $\xi \in \Omega$.*
(iii) *The point $(\xi, \alpha_1(\xi), \dots, \alpha_n(\xi))$ belongs to $(\phi_v)^{-1}(U_v \cap U_w)$ for each $\xi \neq 0$ in Ω.*

Proof. Let $\widetilde{g}$ be as in Lemma 16. We consider that the variables a_ℓ with $\ell > 1$ occurring in $\widetilde{g}$ are assigned the values α_ℓ fixed in the statement of the lemma. We can then regard $\widetilde{g}$ as a polynomial in the indeterminates x and a_1 with complex coefficients, or as a polynomial in the indeterminate a_1 with coefficients in the valued field $\mathbb{C}(\!(x)\!)$. Equation (13) shows that the points $(0, D+q)$ and $(1, q)$ are vertices of the Newton polygon of $\widetilde{g}$. Therefore $\widetilde{g}$ admits a unique root of valuation D in $\mathbb{C}(\!(x)\!)$, which we denote by α_1, and the power series α_1 has a positive radius of convergence. Proceeding by induction on $\ell \in \{2, \dots, n-1\} \cap P(v) \cap S(v)$, and solving the equation $\widetilde{c}_\ell = 0$, we define

$$\alpha_\ell(\xi) = -\widetilde{Q}_{\ell-1}(0)/\widetilde{P}_{\ell-1}(0), \tag{14}$$

where the right-hand side is evaluated at $(\xi, \alpha_1(\xi), \dots, \alpha_{\ell-1}(\xi))$; this is a well-defined process and $\alpha_\ell(\xi)$ satisfies the equivalent given in the statement, because Lemma 14 guarantees that after evaluation

$$\widetilde{P}_{\ell-1}(0) = -\xi + O\big(\xi^2\big) \quad \text{and} \quad \widetilde{Q}_{\ell-1}(0) = \alpha_{(\ell-1)^-}(\xi) + O\Big(\xi^{D+2-d_{(\ell-1)^-}}\Big),$$

so the denominator in (14) does not vanish if $\xi \neq 0$. Moreover, (12) ensures that the equation $\widetilde{c}_n = 0$ is enforced too. Therefore this construction gives (i) and (ii).

We will prove (iii) by showing that none of the functions f_ℓ vanish when evaluated at the point $(\xi, \alpha_1(\xi), \dots, \alpha_n(\xi))$ with $\xi \neq 0$ in Ω. Further, to achieve this result, we have the latitude to shrink Ω as needed.

Since $f_1 = a_1$, we have $f_1(\xi, \alpha_1(\xi), \dots, \alpha_n(\xi)) = \alpha_1(\xi)$, and this quantity does not vanish for $\xi \neq 0$ small enough, because $\alpha_1(\xi) \sim \xi^D/\sigma_n$. Proceeding by induction, we assume known that $f_1, \dots, f_{\ell-1}$ do not vanish at our point.

- In the case $(v(\ell), w(\ell)) = (+,+)$, we have

  $$f_\ell = \big(a_\ell R_{\ell-1} + S_{\ell-1}\big)(x_2).$$

  The congruences in Lemma 11 allow to rewrite the equation $\widetilde{c}_\ell = 0$ in the form

  $$\big(a_\ell P_{\ell-1} + Q_{\ell-1}\big)(x_2) = 0;$$

  this is satisfied after evaluation at the point $(\xi, \alpha_1(\xi), \dots, \alpha_n(\xi))$. Using then the relation $\big(P_{\ell-1}S_{\ell-1} - Q_{\ell-1}R_{\ell-1}\big)(x_2) = 1$, we obtain

$$P_{\ell-1}(x_2) \times f_\ell = P_{\ell-1}(x_2)\big(a_\ell R_{\ell-1} + S_{\ell-1}\big)(x_2) = 1 + R_{\ell-1}(x_2)\big(a_\ell P_{\ell-1} + Q_{\ell-1}\big)(x_2) = 1.$$

  Thus, f_ℓ does not vanish at $(\xi, \alpha_1(\xi), \dots, \alpha_n(\xi))$.
- The case $(v(\ell), w(\ell)) = (-,+)$, that is $\ell = n$, is amenable to a similar treatment.
- The remaining case is $(v(\ell), w(\ell)) = (-,-)$. Here by Lemma 11 we have after substitution

  $$f_\ell = \big(P_{\ell-1} + a_\ell Q_{\ell-1}\big)(x_2) = \big(\widetilde{P}_{\ell-1} + a_\ell \widetilde{Q}_{\ell-1}\big)(0),$$

  and by Lemma 14 and the equivalence in (i)

  $$\widetilde{P}_{\ell-1}(0) = (\sigma_{\ell-1}/\sigma_n - 1)\,\xi + O\big(\xi^2\big)$$

  and

  $$\widetilde{Q}_{\ell-1}(0) = \begin{cases} \xi/\sigma_n + O(\xi^2) & \text{if } d_{\ell-1} = D_{\ell-1} = D, \\ O(\xi^2) & \text{otherwise.} \end{cases}$$

Therefore $f_\ell(\xi, \alpha_1(\xi), \dots, \alpha_n(\xi))$ is equivalent to $(\sigma_\ell/\sigma_n - 1)\,\xi$, hence does not vanish if $\xi \neq 0$ is small enough.

This concludes the induction and establishes (iii). $\square$

To sum up, the lemma constructs a germ of a (parameterized) smooth algebraic curve $\xi \mapsto (\xi, \alpha_1(\xi), \dots, \alpha_n(\xi))$ contained in the zero locus of $\widetilde{\mathfrak{q}}_n$. The ideal of this curve is a prime ideal of A_n which contains $\widetilde{\mathfrak{q}}_n$ and is disjoint from $\mathscr{S}_n$, hence it contains $\mathfrak{q}$. As a result, our curve is contained in $(\phi_v)^{-1}(U_v \cap \mathcal{X}(\mathbf{Z}_w, (1, n-1)))$. The point $\xi = 0$ of this curve has for coordinates $x = 0$, $a_\ell = 0$ if $\ell \in P(v)$, and $a_\ell = \alpha_\ell$ if $\ell \in \{1, \dots, n\} \setminus P(v)$. Now points of this form fill an open dense subset of $(\phi_v)^{-1}(U_v \cap \mathcal{Y}(\mathbf{Z}_v))$, because the values α_ℓ were chosen arbitrarily, subject to the sole requirement that $\sigma_n \neq \sigma_j$ for $j \in \{1, \dots, n-1\}$. We can then conclude that $\mathcal{Y}(\mathbf{Z}_v) \subset \mathcal{X}(\mathbf{Z}_w, (1, n-1))$. This proves the missing half of Proposition 9(i) (the first half was obtained just after Lemma 13).

As a consequence, $\mathfrak{q} \subset \mathfrak{p}$. To ease the reading of the sequel, we will omit the subscripts n in the notation A_n and $\widetilde{\mathfrak{q}}_n$. For $\ell \in \{1, \dots, n\}$, we set $R(\ell) = \big\{j \in \{2, \dots, \ell\} \bigm| v(j) = -,\ d_{j-1} = D_{j-1}\big\}$.

Lemma 18.

(i) *For each $\ell \in \{1, \dots, n-1\}$, we have*

$$\widetilde{P}_\ell \equiv \tilde{z} \pmod{\mathfrak{p}[\tilde{z}]}, \qquad \widetilde{Q}_\ell \equiv \tilde{z}^{D_\ell - d_\ell}\, a_{\ell^-} \pmod{\mathfrak{p}^2[\tilde{z}]},$$

$$\widetilde{P}_\ell(0) \equiv -x + \sum_{j \in R(\ell)} a_{(j-1)^-} a_j \pmod{\mathfrak{p}^2}.$$

(ii) *In the local ring $A_\mathfrak{p}$, we have $\mathfrak{p}A_\mathfrak{p} = xA_\mathfrak{p} + \mathfrak{q}A_\mathfrak{p} + \mathfrak{p}^2 A_\mathfrak{p}$.*

Proof. Statement (i) is proved by a banal induction. Let us tackle (ii).

If $\ell \in P(v) \setminus S(v)$, then $a_\ell = \widetilde{c}_\ell$ belongs to $\widetilde{\mathfrak{q}}$.

If $\ell \in (P(v) \cap S(v)) \setminus \{L\}$, then there exists $m \in P(v) \cap S(v)$ such that $d_\ell = d_m - 1$. Then $\ell = (m-1)^-$ and $D_{m-1} = d_{m-1}$, whence by statement (i)

$$a_\ell \equiv \widetilde{Q}_{m-1}(0) = \widetilde{c}_m - a_m \widetilde{P}_{m-1}(0) \equiv \widetilde{c}_m \pmod{\mathfrak{p}^2},$$

and therefore $a_\ell \in \widetilde{\mathfrak{q}} + \mathfrak{p}^2$.

Surely $D_{n-1} = d_{n-1} = D$ and $L = (n-1)^-$, so again by statement (i), we have

$$\widetilde{c}_n = \widetilde{P}_{n-1}(0) + a_n \widetilde{Q}_{n-1}(0) \equiv \widetilde{P}_{n-1}(0) + a_L a_n \equiv -x + \sum_{j \in R(n)} a_{(j-1)^-} a_j \pmod{\mathfrak{p}^2}.$$

In the last sum, we gather the terms with the same value ℓ for $(j-1)^-$: denoting by τ_ℓ the sum of the variables a_j for $j \in \{2, \dots, n\}$ such that $v(j) = -$ and $d_{j-1} = D_{j-1} = d_\ell$, we obtain

$$\widetilde{c}_n \equiv -x + \sum_{\ell \in P(v) \cap S(v)} a_\ell\, \tau_\ell \pmod{\mathfrak{p}^2}.$$

Noting that $a_\ell \in \widetilde{\mathfrak{q}} + \mathfrak{p}^2$ for $\ell \in P(v) \cap S(v) \setminus \{L\}$ and that $\tau_L = \sigma_n$, we get $a_L \sigma_n \in (x) + \widetilde{\mathfrak{q}} + \mathfrak{p}^2$. Since σ_n is invertible in $A_\mathfrak{p}$, we conclude that $a_L \in xA_\mathfrak{p} + \widetilde{\mathfrak{q}}A_\mathfrak{p} + \mathfrak{p}^2 A_\mathfrak{p}$.

Altogether the remarks above show the inclusion

$$\mathfrak{p}A_\mathfrak{p} \subset xA_\mathfrak{p} + \widetilde{\mathfrak{q}}A_\mathfrak{p} + \mathfrak{p}^2 A_\mathfrak{p}.$$

Joint with $\widetilde{\mathfrak{q}} \subset \mathfrak{q} \subset \mathfrak{p}$, this gives statement (ii). $\square$

The ideal in A of the subvarieties

$$V = (\phi_v)^{-1}(U_v \cap \mathcal{Y}(\mathbf{Z}_v)) \quad \text{and} \quad X = (\phi_v)^{-1}(U_v \cap \mathcal{X}(\mathbf{Z}_w, (1, n-1)))$$

are $\mathfrak{p}$ and $\mathfrak{q}$, respectively. The local ring $\mathscr{O}_{V,X}$ of X along V is the localization of $\overline{A} = A/\mathfrak{q}$ at the ideal $\overline{\mathfrak{p}} = \mathfrak{p}/\mathfrak{q}$. Lemma 18(ii) combined with Nakayama's lemma shows that the image of $x = x_1 - x_2$ in $\overline{A}$ generates the ideal $\overline{\mathfrak{p}}\,\overline{A}_{\overline{\mathfrak{p}}}$. As a consequence, the order of vanishing of $x_1 - x_2$ along V is equal to one, and by definition, this is the multiplicity of $\mathcal{Y}(\mathbf{Z}_v)$ in the intersection product $\mathcal{X}(\mathbf{Z}_w, (1, n-1)) \cdot \mathcal{G}r_n^{\boldsymbol{\lambda}}\big|_\Delta$. This proves Proposition 9(ii).

4.6. **Inclusion, II.** In this section, we again consider words v and w such that $(v(1), w(1)) = (+,-)$ and $\mathrm{wt}\,(v) = \mathrm{wt}\,(w)$ and explore the situation where the path representing v lies strictly above the one representing w (except of course at the two endpoints) but does not stay parallel to it. We thus assume that there exists $k \in \{2, \ldots, n-1\}$ such that $(v(k), w(k)) = (+,-)$.

Proposition 19. *Under these assumptions,* $\mathcal{Y}(\mathbf{Z}_v) \not\subset \mathcal{X}(\mathbf{Z}_w, (1, n-1))$.

The proof of Proposition 19 fills the remainder of this section. Our argument is similar to our proof in Proposition 9(i).

For each $\ell \in \{1, \ldots, n\}$, we define $A_\ell = \mathbb{C}[x_2][x, a_1, \ldots, a_\ell]$, where $x = x_1 - x_2$. We introduce $\tilde{z} = z - x_2$.

In addition:

- let K be the largest integer $k \in \{2, \ldots, n-1\}$ such that $(v(k), w(k)) = (+,-)$;
- for $\ell \in \{K, \ldots, n\}$, let d_ℓ be the weight of the word $v(K+1)v(K+2)\cdots v(\ell)$, with the convention $d_K = 0$;
- let L be the smallest position $\ell > K$ such that $(v(\ell), w(\ell)) = (-,+)$ or $d_\ell > 0$.

Set $\widetilde{P}_1 = \tilde{z} - x$ and $\widetilde{Q}_1 = a_1$. For $\ell \in \{2, \ldots, L-1\}$, define by induction two polynomials $\widetilde{P}_\ell$, $\widetilde{Q}_\ell$ in $A_\ell[z]$ as follows:

- If $(v(\ell), w(\ell)) = (+,+)$, then

$$\widetilde{P}_\ell = \widetilde{P}_{\ell-1} \quad \text{and} \quad \widetilde{Q}_\ell = \begin{cases} \dfrac{a_\ell \widetilde{P}_{\ell-1} + \widetilde{Q}_{\ell-1} - \big(a_\ell \widetilde{P}_{\ell-1} + \widetilde{Q}_{\ell-1}\big)(0)}{\tilde{z}} & \text{if } \ell < K, \\[2ex] \dfrac{\widetilde{Q}_{\ell-1} - \widetilde{Q}_{\ell-1}(0)}{\tilde{z}} & \text{if } \ell > K. \end{cases}$$

- If $(v(\ell), w(\ell)) = (-,+)$, then

$$\widetilde{P}_\ell = \frac{\widetilde{P}_{\ell-1} + a_\ell \widetilde{Q}_{\ell-1} - \big(\widetilde{P}_{\ell-1} + a_\ell \widetilde{Q}_{\ell-1}\big)(0)}{\tilde{z}} \quad \text{and} \quad \widetilde{Q}_\ell = \widetilde{Q}_{\ell-1}.$$

- If $(v(\ell), w(\ell)) = (+,-)$, then $\widetilde{P}_\ell = \tilde{z}\,\widetilde{P}_{\ell-1}$ and $\widetilde{Q}_\ell = a_\ell \widetilde{P}_{\ell-1} + \widetilde{Q}_{\ell-1}$.
- If $(v(\ell), w(\ell)) = (-,-)$, then $\widetilde{P}_\ell = \widetilde{P}_{\ell-1} + a_\ell \widetilde{Q}_{\ell-1}$ and $\widetilde{Q}_\ell = \tilde{z}\,\widetilde{Q}_{\ell-1}$.

For $\ell \in \{1, \ldots, L\}$:

- let $\mathring{\mathfrak{q}}_\ell$ be the ideal of B_ℓ generated by $\{b_j \mid j \in P(w),\ j \leq \ell\}$;
- if $\ell \geq K$, let σ_ℓ be the sum of the a_j for $j \in \{K+1, \ldots, \ell\}$ such that $v(j) = -$ and $d_{j-1} = 0$, with the convention $\sigma_K = 0$.

Lemma 20. *For* $\ell \in \{1, \ldots, L-1\}$, *we have*

(i)$_\ell$ $\widetilde{P}_\ell(\tilde{z}) \equiv P_\ell(z) \pmod{\mathring{\mathfrak{q}}_\ell[z]}$ *and* $\widetilde{Q}_\ell(\tilde{z}) \equiv Q_\ell(z) \pmod{\mathring{\mathfrak{q}}_\ell[z]}$,

(ii)$_\ell$ *if* $\ell \geq K$, *then* $\widetilde{P}_\ell(0) = \widetilde{Q}_K(0)\sigma_\ell$ *and* $\widetilde{Q}_\ell = \tilde{z}^{-d_\ell}\widetilde{Q}_K$.

Proof. One again proceeds by induction. The details are straightforward, except in the case where $(v(\ell), w(\ell)) = (+,+)$ and $\ell > K$, where one can follow the arguments offered in the proof of Lemma 11 to get $a_\ell \in \mathring{\mathfrak{q}}_\ell$. ☐

We now distinguish three cases:

- Assume that $d_{L-1} < 0$. Then necessarily $(v(L), w(L)) = (-,+)$. By assertion $(\mathrm{ii})_{L-1}$ in Lemma 20, we get $\widetilde{Q}_{L-1}(0) = 0$. Using assertion $(\mathrm{i})_{L-1}$ in that lemma, we deduce that $Q_{L-1}(x_2) \in \mathring{\mathfrak{q}}_{L-1}$. Then, by the identity $P_{L-1}S_{L-1} - Q_{L-1}R_{L-1} = 1$, we see that $P_{L-1}(x_2)$ is invertible in the ring $B_{L-1}/\mathring{\mathfrak{q}}_{L-1}$. Thus, $b_L = \big(P_{L-1} + a_L Q_{L-1}\big)\big(x_2\big) \times f_L^{-1}$ is invertible in $B_L/\mathring{\mathfrak{q}}_{L-1}B_L$. We conclude that $\mathring{\mathfrak{q}}_L = B_L$, and therefore $\mathring{\mathfrak{q}}_n = B_n$. Thus, $U_v \cap \dot{\mathcal{X}}(\mathbf{Z}_w) = \varnothing$, so $\mathcal{X}(\mathbf{Z}_w, (1, n-1))$ does not meet U_v and cannot contain $\mathcal{Y}(\mathbf{Z}_v)$.
- Assume that $d_{L-1} = 0$ and $(v(L), w(L)) = (-,+)$. We note that $P_K(x_2) = 0$ by construction. The identity $P_K S_K - Q_K R_K = 1$ then implies that $Q_K(x_2)$ is invertible in B_K, and by assertion $(\mathrm{i})_K$ in Lemma 20, $\widetilde{Q}_K(0)$ is invertible in $B_K/\mathring{\mathfrak{q}}_K$. Moreover, $f_L b_L = \big(P_{L-1} + a_L Q_{L-1}\big)\big(x_2\big)$ belongs to $\mathring{\mathfrak{q}}_L$. Using assertion $(\mathrm{ii})_{L-1}$ in Lemma 20, we deduce that
$$\big(\widetilde{P}_{L-1} + a_L \widetilde{Q}_{L-1}\big)\big(0\big) = \widetilde{Q}_K(0)(\sigma_{L-1} + a_L) = \widetilde{Q}_K(0)\sigma_L$$
belongs to $\mathring{\mathfrak{q}}_L$ too. Therefore σ_L belongs to $\mathring{\mathfrak{q}}_L$, hence to $\mathfrak{q}$. However $\sigma_L \notin \mathfrak{p}$, because a_L is a summand in the sum that defines σ_L whereas $L \notin P(v)$. We must then conclude that $\mathfrak{q} \not\subset \mathfrak{p}$, in other words that $\mathcal{Y}(\mathbf{Z}_v) \not\subset \mathcal{X}(\mathbf{Z}_w, (1, n-1))$.
- Assume that $d_{L-1} = 0$ and $(v(L), w(L)) = (+,+)$. As in the previous case, we note that $\widetilde{Q}_K(0)$ is invertible in $B_K/\mathring{\mathfrak{q}}_K$. But now we have $f_L b_L = \big(a_L P_{L-1} + Q_{L-1}\big)\big(x_2\big)$, so we get
$$\widetilde{Q}_K(0)(a_L\sigma_{L-1} + 1) \in \mathring{\mathfrak{q}}_L$$
and then $a_L\sigma_{L-1} + 1 \in \mathfrak{q}$. Here however $a_L \in \mathfrak{p}$, so $a_L\sigma_{L-1} + 1 \notin \mathfrak{p}$. Again we must conclude that $\mathfrak{q} \not\subset \mathfrak{p}$ and $\mathcal{Y}(\mathbf{Z}_v) \not\subset \mathcal{X}(\mathbf{Z}_w, (1, n-1))$.

Proposition 19 is then proved.

4.7. **Loose ends.** We can now prove that the MV basis of $V(\varpi)^{\otimes n}$ satisfies the second formula in (1). We consider two words v and w in $\mathscr{C}_n$ with $w(1) = -$ and $\mathrm{wt}\,(v) = \mathrm{wt}\,(w)$ and we look for the coefficient of y_v in the expansion of $x_- \otimes y_{w'}$ in the MV basis, where w' is the word w stripped from its first letter.

If $v(1) = -$, then this coefficient is zero except for $v = w$, in which case the coefficient is one. This follows from Theorem 5.13 in [1].

If $v(1) = +$, then the path representing v starts above the path representing w. We distinguish two cases.

In the case where v stays strictly above w until the very end, we can refer to Propositions 9 and 19: the coefficient of y_v is non-zero only if v stays parallel to w at distance two and the last letter of w' is significant. If this condition is fulfilled, then the coefficient is one.

In the case where v and w rejoin before the end, after m letters, then we write v and w as concatenations $+v_{(2)}v_{(3)}$ and $-w_{(2)}w_{(3)}$, respectively, with $v_{(2)}$ and $w_{(2)}$ of length $m-1$ and $v_{(3)}$ and $w_{(3)}$ of length $n-m$. By assumption, $\mathrm{wt}\,v_{(3)} = \mathrm{wt}\,w_{(3)}$. We can then apply Proposition 7 with $n_1 = 1$, $n_2 = m-1$ and $n_3 = n-m$: if $v_{(3)} \neq w_{(3)}$, then the coefficient of y_v in the expansion of $x_- \otimes y_{w'}$ is zero; otherwise, it is equal to the coefficient of $y_{+v(2)}$ in the expansion of $x_- \otimes y_{w(2)}$ in the MV basis of $V(\varpi)^{\otimes m}$.

Thus, Proposition 7 reduces the second case to the first one, but for words of length m. The coefficient is then non-zero only if $+v_{(2)}$ stays parallel to $-w_{(2)}$ at distance two and the last letter of $w_{(2)}$ is significant, in which case the coefficient is one.

To sum up: if $(v(1), w(1)) = (+, -)$, then the coefficient of y_v in the expansion of $x_- \otimes y_{w'}$ is either zero or one; it is one if and only if v is obtained by flipping the first letter $-$ of w into a $+$ and flipping a significant letter $+$ in w' into a $-$. This shows that the MV basis satisfies the second formula in (1). We have proved:

Theorem 21. *$(y_w)_{w\in\mathcal{C}_n}$ is the MV basis of $V(\varpi)^{\otimes n}$.*

Putting Theorem 21 alongside Theorem 3, Proposition 6, and Theorem 1.11 in [7], we obtain the result stated in section 1.

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