REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 25, Pages 807–843 (October 5, 2021)
https://doi.org/10.1090/ert/585

PARABOLIC INDUCTION VIA THE PARABOLIC PRO-p IWAHORI–HECKE ALGEBRA

CLAUDIUS HEYER

Abstract. Let $\mathbf{G}$ be a connected reductive group defined over a locally compact non-archimedean field F, let $\mathbf{P}$ be a parabolic subgroup with Levi $\mathbf{M}$ and compatible with a pro-p Iwahori subgroup of $G := \mathbf{G}(F)$. Let R be a commutative unital ring.

We introduce the parabolic pro-p Iwahori–Hecke R-algebra $\mathcal{H}_R(P)$ of $P := \mathbf{P}(F)$ and construct two R-algebra morphisms $\Theta_M^P \colon \mathcal{H}_R(P) \to \mathcal{H}_R(M)$ and $\Xi_G^P \colon \mathcal{H}_R(P) \to \mathcal{H}_R(G)$ into the pro-p Iwahori–Hecke R-algebra of $M := \mathbf{M}(F)$ and G, respectively. We prove that the resulting functor $\text{Mod-}\mathcal{H}_R(M) \to \text{Mod-}\mathcal{H}_R(G)$ from the category of right $\mathcal{H}_R(M)$-modules to the category of right $\mathcal{H}_R(G)$-modules (obtained by pulling back via Θ_M^P and extension of scalars along Ξ_G^P) coincides with the parabolic induction due to Ollivier–Vignéras.

The maps Θ_M^P and Ξ_G^P factor through a common subalgebra $\mathcal{H}_R(M,G)$ of $\mathcal{H}_R(G)$ which is very similar to $\mathcal{H}_R(M)$. Studying these algebras $\mathcal{H}_R(M,G)$ for varying (M,G) we prove a transitivity property for tensor products. As an application we give a new proof of the transitivity of parabolic induction.

Contents

1. Introduction

1.1. **Motivation.** Let $\mathbf{G}$ be a connected reductive group over a locally compact non-archimedean field F with residue field of characteristic $p > 0$. In this introduction all representations will be on vector spaces over a fixed field k. Given a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Levi $\mathbf{M}$, parabolic induction is a process to obtain smooth representations of $G := \mathbf{G}(F)$ from smooth representations of $M := \mathbf{M}(F)$.

Received by the editors October 29, 2020, and, in revised form, March 31, 2021.
2020 *Mathematics Subject Classification.* Primary 11E95, 20C08, 20G25.
The author was funded by the University of Münster and Germany's Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics-Geometry-Structure.

More precisely, it is obtained as the composite

$$\begin{aligned} \mathrm{Rep}_k(M) &\longrightarrow \mathrm{Rep}_k(P) \longrightarrow \mathrm{Rep}_k(G), \\ V &\longmapsto V \longmapsto \mathrm{Ind}_P^G V. \end{aligned} \tag{1}$$

Here, V is first viewed as a smooth representation of $P \coloneqq \mathbf{P}(F)$ by letting the unipotent radical $U_P \coloneqq \mathbf{U_P}(F)$ of P act trivially. Then we form the space $\mathrm{Ind}_P^G V$ of locally constant functions $f\colon G \to V$ satisfying $f(\gamma g) = \gamma \cdot f(g)$ for all $\gamma \in P$ and $g \in G$. Parabolic induction plays a fundamental role in classifying the smooth admissible representations of G.

A second important method to study smooth representations of G is via the functor of K-invariants $V \mapsto V^K$, for K any compact open subgroup of G. The ring of G-equivariant endomorphisms

$$H_k(K, G) \coloneqq \mathrm{End}_{k[G]}\big(k[K\backslash G]\big)$$

of the space $k[K\backslash G]$ of maps $K\backslash G \to k$ with finite support acts on

$$V^K \cong \mathrm{Hom}_{k[G]}(k[K\backslash G], V)$$

by Frobenius reciprocity. In this way one obtains a functor

$$\mathrm{Rep}_k(G) \longrightarrow \mathrm{Mod}\text{-}H_k(K, G), \quad V \longmapsto V^K \tag{2}$$

from $\mathrm{Rep}_k(G)$ into the category of right modules over the k-algebra $H_k(K, G)$. If k has characteristic p, it suffices to restrict attention to a pro-p Iwahori subgroup I_1 of G; this is because for $K = I_1$ the functor (2) sends non-zero representations to non-zero modules. One is therefore led to study the modules over the pro-p Iwahori–Hecke algebra $\mathcal{H}_k(G) \coloneqq H_k(I_1, G)$. A systematic study of $\mathcal{H}_k(G)$ was carried out by Vignéras in [Vig05] (for F-split $\mathbf{G}$) and [Vig16] (for arbitrary $\mathbf{G}$).

Again, classifying the simple modules of $\mathcal{H}_k(G)$ is accomplished by studying induction functors

$$\mathrm{Mod}\text{-}\mathcal{H}_k(M) \longrightarrow \mathrm{Mod}\text{-}\mathcal{H}_k(G),$$

for parabolic subgroups $\mathbf{P} = \mathbf{M}\mathbf{U_P}$ of $\mathbf{G}$ that are compatible with I_1, which is compatible with parabolic induction via (2). Such a functor was defined and studied by Ollivier [Oll10] for $\mathbf{G} = \mathrm{GL}_n$, and for general $\mathbf{G}$ by Ollivier–Vignéras [OV18], Vignéras [Vig15], and Abe [Abe16, Abe19]. Explicitly, this functor is given by

$$\mathfrak{m} \longmapsto \mathfrak{m} \otimes_{\mathcal{H}_k(M^+)} \mathcal{H}_k(G), \tag{3}$$

where $\mathcal{H}_k(M^+)$ is a certain subalgebra of $\mathcal{H}_k(M)$ embedding naturally into $\mathcal{H}_k(G)$. This algebra $\mathcal{H}_k(M^+)$ depends on $I_1 \cap \mathbf{U_P}(F)$, hence also on P. However, the analogy between (1) and (3) is not as strong as one might hope for. Let us recall the reason why the parabolic P shows up in (1). One could directly define an induction functor $\mathrm{Rep}_k(M) \to \mathrm{Rep}_k(G)$ by $V \mapsto \mathrm{Ind}_M^G V$. But this functor is very difficult to handle, because the coset space G/M and hence the representation $\mathrm{Ind}_M^G V$ is "too large". On the other hand, the quotient G/P is compact which makes it possible to effectively study $\mathrm{Ind}_P^G V$. In this light it is surprising that (3) is defined using the small algebra $\mathcal{H}_k(M^+)$. Instead, one would expect to use as big an algebra as possible to define (3); the parabolic Hecke algebra $\mathcal{H}_k(P) \coloneqq H_k(I_1 \cap P, P)$ is a natural candidate although it is not at all obvious how this can be achieved.

To summarize, this article is motivated by the following questions:

(Q1) Can we replace $\mathcal{H}_k(M^+)$ in (3) by the algebra $\mathcal{H}_k(P)$?

(Q2) Assume the answer to (Q1) is affirmative. Is the functor

$$\mathrm{Mod}\text{-}\mathcal{H}_k(M) \longrightarrow \mathrm{Mod}\text{-}\mathcal{H}_k(G), \quad \mathfrak{m} \longmapsto \mathfrak{m} \otimes_{\mathcal{H}_k(P)} \mathcal{H}_k(G)$$

naturally isomorphic to (3)?

1.2. **Main results.** We answer question (Q1) positively even for k an arbitrary commutative unital ring. This amounts to constructing two k-algebra homomorphisms $\Theta_M^P\colon \mathcal{H}_k(P) \to \mathcal{H}_k(M)$ and $\Xi_G^P\colon \mathcal{H}_k(P) \to \mathcal{H}_k(G)$.

The map Θ_M^P (see Proposition 4.3) exists quite generally and is induced by the canonical projection map $k[I_1 \cap P \backslash P] \to k[I_1 \cap M \backslash M]$. Its origins can be traced back to the works of Andrianov; see, *e.g.*, [And77, (3.3)] or [AZ95, Definition of Φ before Chapter 3, Proposition 3.28], who seems to have been the first to study "parabolic Hecke algebras", albeit in a different context.

The main contribution of this work lies in the construction of Ξ_G^P. Observing that the image of Θ_M^P contains $\mathcal{H}_k(M^+)$, the idea is to try to extend the embedding $\xi^+\colon \mathcal{H}_k(M^+) \to \mathcal{H}_k(G)$ to $\mathrm{Im}(\Theta_M^P)$ and then to define Ξ_G^P as the composite $\mathcal{H}_k(P) \xrightarrow{\Theta_M^P} \mathrm{Im}(\Theta_M^P) \to \mathcal{H}_k(G)$. However, since the goal for $\Xi_G^P\colon \mathcal{H}_k(P) \to \mathcal{H}_k(G)$ is to have as large image as possible, this approach will not always yield optimal results. Since in this approach Ξ_G^P factors through a subalgebra of $\mathcal{H}_k(M)$, the best we can expect is for the image of Ξ_G^P to be canonically isomorphic to $\mathcal{H}_k(M)$ as a k-module. If k happens to be p-torsionfree, this is indeed the case. We prove:

Theorem (Proposition 4.7)**.** *Assume that k is p-torsionfree. The embedding*

$$\xi^+\colon \mathcal{H}_k(M^+) \to \mathcal{H}_k(G)$$

extends to an injective k-algebra morphism $\mathrm{Im}(\Theta_M^P) \to \mathcal{H}_k(G)$*. Moreover, this extension is unique and* $\mathrm{Im}(\Theta_M^P)$ *is the maximal subalgebra of $\mathcal{H}_k(M)$ with this property.*

Further, Corollary 4.4 shows that $\mathrm{Im}(\Theta_M^P)$ identifies canonically with $\mathcal{H}_k(M)$ as a k-module. If, on the other hand, k is not p-torsionfree, then $\mathrm{Im}(\Theta_M^P)$ is much smaller than $\mathcal{H}_k(M)$. (For example, if k is a field of characteristic p, then $\mathrm{Im}(\Theta_M^P) = \mathcal{H}_k(M^+)$.) Hence, even if the assertion of Proposition 4.7 could be proved without requiring that k be p-torsionfree, defining Ξ_G^P as the composite $\mathcal{H}_k(P) \to \mathrm{Im}(\Theta_M^P) \to \mathcal{H}_k(G)$ would yield a morphism with small image. Instead, we make the following crucial definition:

Definition. Let k be arbitrary. We define

$$\Xi_G^P = \mathrm{id}_k \otimes \Xi_{G,\mathbb{Z}}^P\colon \mathcal{H}_k(P) \to \mathcal{H}_k(G).$$

Here, $\Xi_{G,\mathbb{Z}}^P$ is the composite $\mathcal{H}_{\mathbb{Z}}(P) \xrightarrow{\Theta_M^P} \mathrm{Im}(\Theta_M^P) \to \mathcal{H}_{\mathbb{Z}}(G)$, where the second arrow is the map in Proposition 4.7 (for $k = \mathbb{Z}$).

In this way the image of Ξ_G^P will always be large, that is, it can be canonically identified with $\mathcal{H}_k(M)$ as a k-module. The assumption on k in Proposition 4.7 is therefore inconsequential for the rest of the paper.

Question (Q2) is a consequence of the construction of Θ_M^P and Ξ_G^P, that is, we prove:

Theorem (Theorem 4.9)**.** *The functor* $\mathrm{Mod}\text{-}\mathcal{H}_k(M) \to \mathrm{Mod}\text{-}\mathcal{H}_k(G)$*,* $\mathfrak{m} \mapsto \mathfrak{m} \otimes_{\mathcal{H}_k(P)} \mathcal{H}_k(G)$ *is naturally isomorphic to* (3)*.*

As an application we will give a new proof (see Corollary 5.8) of the transitivity of parabolic induction originally due to Vignéras [Vig15, Proposition 4.3].

1.3. **Structure of the paper.** Section 2 is devoted to setting up the notation and reviewing some parts of Bruhat–Tits theory. This part draws heavily from the original paper [BT72] by Bruhat–Tits and from the comprehensive summary in [Vig16].

In section 3 we will study the group index $\mu_{U_P}(g) := [I_{U_P} : I_{U_P} \cap g^{-1} I_{U_P} g]$ for $g \in P$, where $I_{U_P} = I_1 \cap U_P$. This index appears in the double coset formula (Proposition 3.1) which is used to give an explicit description of the map Θ_M^P (Proposition 4.3). The results of subsection 3.2 are used to prove the estimate $\mu_{U_P}(g) \geq \mu_{U_P}(g_M)$, where g_M is the image of the projection of g in M; see Proposition 3.4. It allows us to give a concrete description of a basis of $\mathrm{Im}(\Theta_M^P)$ which will be necessary for the construction of Ξ_G^P. In subsection 3.4 we explain how μ_{U_P} naturally gives rise to a function defined on the Iwahori–Weyl group W_M of M, again denoted μ_{U_P}. Proposition 3.14 shows that μ_{U_P} measures how far the length function on W_M deviates from the length function on the Iwahori–Weyl group W of G, and that μ_{U_P} is compatible with the Bruhat order on W_M. These properties will become useful in the construction of Ξ_G^P (Proposition 4.7) and in the study of the algebras $\mathcal{H}_k(M, G)$ (section 5). We also obtain new and short proofs of two lemmas due to Abe in Corollaries 3.16 and 3.17.

Section 4 contains the main results. Subsection 4.1 gives a short introduction to abstract Hecke algebras as double coset algebras following [AZ95, Chapter 3]. In subsection 4.2 we describe the map Θ_M^P explicitly; see Proposition 4.3. Together with the inequality $\mu_{U_P}(g) \geq \mu_{U_P}(g_M)$ this yields a minimal generating system of $\mathrm{Im}(\Theta_M^P)$ which is even a basis provided the coefficient ring R is p-torsionfree; see Corollary 4.4. The main results on the construction of Ξ_G^P and on parabolic induction are proved in Proposition 4.7 and Theorem 4.9, respectively.

Finally, section 5 is devoted to applying our previous results to give a new proof of the transitivity of parabolic induction. The algebras $\mathcal{H}_k(M, G)$ are introduced and studied in subsection 5.1. In Proposition 5.3 we show that these algebras, for varying G, are localizations of each other. The main result is the general Theorem 5.7 from which we deduce that parabolic induction is transitive. To finish, we construct in subsection 5.3 a natural filtration on $\mathcal{H}_k(M, G)$ (Proposition 5.10) which might be of independent interest.

2. Notations and preliminaries

Throughout the article we fix a locally compact non-archimedean field F with residue field $\mathbb{F}_q$ of characteristic p and normalized valuation $\mathrm{val}_F \colon F \to \mathbb{Z} \cup \{\infty\}$.

Given a group G, a subgroup $H \subseteq G$, and elements $g, h \in G$, we write

$$h^g := g^{-1} h g, \qquad H^g := \{h^g \mid h \in H\}, \qquad H_{(g)} := H \cap H^g.$$

We also write $[g, h] := g h g^{-1} h^{-1}$ for the commutator of g and h.

The symbol $\bigsqcup$ denotes "disjoint union".

Given an algebraic group $\mathbf{H}$ over F, we denote by the corresponding lightface letter $H := \mathbf{H}(F)$ its group of F-rational points. The topology of F makes H into a topological group. We denote

- $\mathbf{H}^\circ$ the identity component of $\mathbf{H}$;

– $\mathrm{X}^*(H)$ (resp. $\mathrm{X}_*(H)$) the group of algebraic F-characters (resp. F-cocharacters) of $\mathbf{H}$.

Let $\mathbf{G}$ be a connected reductive group defined over F. Fix a maximal F-split torus $\mathbf{T}$ and denote by $\mathbf{Z} := \mathbf{Z}_{\mathbf{G}}(\mathbf{T})$ (resp. $\mathbf{N} := \mathbf{N}_{\mathbf{G}}(\mathbf{T})$) the centralizer (resp. normalizer) of $\mathbf{T}$ in $\mathbf{G}$. The finite Weyl group $W_0 := N/Z$ acts on the (relative) root system $\Phi := \Phi(\mathbf{G}, \mathbf{T}) \subseteq \mathrm{X}^*(T)$ associated with $\mathbf{T}$; it identifies with the Weyl group of Φ.

Notation. Given a subset $\Psi \subseteq \Phi$, we denote $\Psi_{\mathrm{red}} := \left\{\alpha \in \Psi \,\middle|\, \frac{1}{2}\alpha \notin \Psi\right\}$ the set of *reduced roots* in Ψ.

Let $\mathbf{U}_\alpha$ be the root group associated with $\alpha \in \Phi$. Then $\mathbf{U}_{2\alpha} \subseteq \mathbf{U}_\alpha$ whenever $\alpha, 2\alpha \in \Phi$. Fix a minimal parabolic F-subgroup $\mathbf{B}$ with Levi decomposition $\mathbf{B} = \mathbf{ZU}$. This corresponds to a choice Φ^+ of positive roots in Φ, and we have $\mathbf{U} = \prod_{\alpha \in \Phi^+_{\mathrm{red}}} \mathbf{U}_\alpha$. In this article parabolic subgroups are always standard and defined over F. By a Levi subgroup we mean the unique Levi F-subgroup $\mathbf{M}$ of $\mathbf{P}$ containing $\mathbf{Z}$; this is expressed by writing $\mathbf{P} = \mathbf{MU}_{\mathbf{P}}$, where $\mathbf{U}_{\mathbf{P}}$ denotes the unipotent radical of $\mathbf{P}$. Conversely, a Levi subgroup $\mathbf{M}$ in $\mathbf{G}$ determines a unique parabolic group $\mathbf{P}_{\mathbf{M}}$ with Levi $\mathbf{M}$ and unipotent radical $\mathbf{U}_{\mathbf{P}_{\mathbf{M}}} = \prod_{\alpha \in (\Phi^+ \smallsetminus \Phi_M)_{\mathrm{red}}} \mathbf{U}_\alpha$.

2.1. **The standard apartment.** Consider the finite-dimensional $\mathbb{R}$-vector space

$$V := \mathbb{R} \otimes \mathrm{X}_*(T)/\mathrm{X}_*(C),$$

where $\mathbf{C}$ denotes the connected center of $\mathbf{G}$. The finite Weyl group W_0 acts on V via the conjugation action on T, and the natural pairing $\langle\,\cdot\,,\cdot\,\rangle\colon V^* \times V \to \mathbb{R}$ is W_0-invariant, where V^* denotes the $\mathbb{R}$-linear dual of V. Fix a W_0-invariant scalar product $(\,\cdot\,,\cdot\,)$ on V. The root system Φ embeds into V^*. For each $\alpha \in \Phi$ there exists a unique coroot $\alpha^\vee \in V$ with $\langle \alpha, \alpha^\vee \rangle = 2$ such that the reflection

$$s_\alpha\colon V^* \longrightarrow V^*, \quad x \longmapsto x - \langle x, \alpha^\vee \rangle \cdot \alpha$$

leaves Φ invariant.

2.1.1. *Valuations.* A valuation $\varphi = (\varphi_\alpha)_{\alpha \in \Phi}$ on the root group datum $(Z, (U_\alpha)_{\alpha \in \Phi})$ consists of a family of functions $\varphi_\alpha\colon U_\alpha \to \mathbb{R} \cup \{\infty\}$ satisfying a list of axioms [BT72, (6.2.1)]. Then φ is called *discrete* if the set of values $\Gamma_\alpha := \varphi_\alpha(U_\alpha \smallsetminus \{1\}) \subseteq \mathbb{R}$ is discrete. It is called *special* if $0 \in \Gamma_\alpha$ for all $\alpha \in \Phi_{\mathrm{red}}$. We write

$$\Gamma'_\alpha := \{\varphi_\alpha(u) \,|\, u \in U_\alpha \smallsetminus \{1\} \text{ and } \varphi_\alpha(u) = \sup \varphi_\alpha(uU_{2\alpha})\}\,.$$

Then $\Gamma'_\alpha = \Gamma_\alpha$ if $2\alpha \notin \Phi$ and $\Gamma_\alpha = \Gamma'_\alpha \cup (\frac{1}{2}\Gamma_{2\alpha})$ for all $\alpha \in \Phi$ [BT72, (6.2.2)].

The group N acts naturally on the set of valuations via

$$(n.\varphi)_\alpha(u) = \varphi_{w^{-1}(\alpha)}(n^{-1}un), \qquad \text{for } u \in U_\alpha,\ n \in N,$$

where $w := {}^{\mathrm{v}}\nu(n) \in W_0 = N/Z$ and ${}^{\mathrm{v}}\nu\colon N \to N/Z = W_0$ denotes the projection map.

The vector space V acts faithfully on the set of valuations via

$$(\varphi + v)_\alpha(u) = \varphi_\alpha(u) + \langle \alpha, v \rangle, \qquad \text{for } u \in U_\alpha,\ v \in V.$$

One easily verifies $n.(\varphi + v) = n.\varphi + {}^{\mathrm{v}}\nu(n)(v)$ for $n \in N$, $v \in V$ [BT72, (6.2.5)].

Given $z \in Z$, there is a unique vector $\nu(z) \in V$ satisfying $z.\varphi = \varphi + \nu(z)$ [BT72, Proposition (6.2.10), Proof of (i)]. In this way we obtain a group homomorphism

$$\nu\colon Z \longrightarrow V, \quad z \longmapsto \nu(z). \tag{4}$$

Restriction of characters realizes $\mathrm{X}^*(Z)$ as a subgroup of finite index in $\mathrm{X}^*(T)$ [Ren10, V.2.6. Lemme]. Therefore, given $\chi \in \mathrm{X}^*(T)$, there exists $n \in \mathbb{Z}_{>0}$ with $n\chi \in \mathrm{X}^*(Z)$, and one easily verifies that the definition

$$(\mathrm{val}_F \circ \chi)(z) := \frac{1}{n} \cdot \mathrm{val}_F\big((n\chi)(z)\big) \in \mathbb{R}, \qquad \text{for } z \in Z,$$

is independent of n. We say that φ is *compatible with* val_F if

$$\big\langle \alpha, \nu(z) \big\rangle = -(\mathrm{val}_F \circ \alpha)(z), \qquad \text{for all } z \in Z,\ \alpha \in \Phi.$$

2.1.2. *The apartment, hyperplanes, and affine roots.* From now on we fix a discrete, special valuation $\varphi_0 = (\varphi_{0,\alpha})_{\alpha\in\Phi}$ on $(Z, (U_\alpha)_{\alpha\in\Phi})$ which is compatible with val_F; it exists by [BT84, 5.1.20. Theorème and 5.1.23. Proposition]. By [Vig16, (37)] the subgroups

$$U_{\alpha,r} := \{u \in U_\alpha \mid \varphi_{0,\alpha}(u) \geq r\}, \qquad \text{for } r \in \mathbb{R},$$

form a basis of compact open neighborhoods of the neutral element in U_α, $\alpha \in \Phi$.

The *apartment* of G is defined as the affine space

$$\mathscr{A} := \{\varphi_0 + v \mid v \in V\}$$

under V. It follows from [BT72, Proposition (6.2.10)] that the action of N on valuations induces a group homomorphism

$$\nu \colon N \longrightarrow \operatorname{Aut} \mathscr{A}$$

extending (4). The fixed W_0-invariant scalar product on V endows $\mathscr{A}$ with a Euclidean metric.

Given $\alpha \in V^*$ and $r \in \mathbb{R}$, we put

$$\begin{aligned} a_{\alpha,r} &:= \{\varphi_0 + v \in \mathscr{A} \mid \langle \alpha, v\rangle + r \geq 0\} \quad \text{and} \\ H_{\alpha,r} &:= \{\varphi_0 + v \in \mathscr{A} \mid \langle \alpha, v\rangle + r = 0\}. \end{aligned}$$

We call $\Phi^{\mathrm{aff}} := \{a_{\alpha,r} \mid \alpha \in \Phi \text{ and } r \in \Gamma'_\alpha\}$ the set of *affine roots* of $\mathscr{A}$ and denote

$$\mathfrak{H} := \{H_{\alpha,r} \mid \alpha \in \Phi \text{ and } r \in \Gamma'_\alpha\} = \{H_{\alpha,r} \mid \alpha \in \Phi_{\mathrm{red}} \text{ and } r \in \Gamma_\alpha\}$$

the set of hyperplanes in $\mathscr{A}$. A connected component of $\mathscr{A} \smallsetminus \bigcup \mathfrak{H}$ is called a *chamber*. Associated with φ_0 and $\mathbf{B}$ there is a unique chamber $\mathfrak{C}$ determined by $\varphi_0 \in \overline{\mathfrak{C}}$ (topological closure) and

$$\mathfrak{C} \subseteq \big\{\varphi_0 + v \in \mathscr{A} \,\big|\, \langle \alpha, v\rangle > 0, \quad \text{for all } \alpha \in \Phi^+\big\}.$$

We call $\mathfrak{C}$ the *fundamental chamber*.

The action of N on $\mathscr{A}$ induces natural actions on Φ^{aff} and on $\mathfrak{H}$. Explicitly, we have

$$n.a_{\alpha,r} = a_{w(\alpha), r - \langle w(\alpha), n.\varphi_0 - \varphi_0\rangle},$$

for all $a_{\alpha,r} \in \Phi^{\mathrm{aff}}$ and $n \in N$, where $w := {}^{\mathrm{v}}\nu(n) \in W_0$. A similar formula holds for $H_{\alpha,r}$. Likewise,

$$nU_{\alpha,r}n^{-1} = U_{w(\alpha), r - \langle w(\alpha), n.\varphi_0 - \varphi_0\rangle}, \tag{5}$$

for all $\alpha \in \Phi$, $r \in \mathbb{R}$, $n \in N$.

2.1.3. *The affine Weyl group.* Denote $s_H \in \operatorname{Aut} \mathscr{A}$ the orthogonal reflection through $H \in \mathfrak{H}$ and put

$$S(\mathfrak{H}) \coloneqq \{s_H \mid H \in \mathfrak{H}\}\,.$$

Conversely, we denote $H_s \in \mathfrak{H}$ the hyperplane fixed by $s \in S(\mathfrak{H})$. The *affine Weyl group* W^{aff} is the subgroup of $\widetilde{W} \coloneqq \nu(N) \subseteq \operatorname{Aut} \mathscr{A}$ generated by $S(\mathfrak{H})$. Notice that $ws_Hw^{-1} = s_{w(H)}$ and $wH_s = H_{wsw^{-1}}$ for all $w \in \widetilde{W}$, $s \in S(\mathfrak{H})$, and $H \in \mathfrak{H}$. We denote by S^{aff} the set of reflections in the walls of the fundamental chamber $\mathfrak{C}$. It generates W^{aff} as a group. Moreover, W^{aff} acts simply transitively on the set of chambers of $\mathscr{A}$.

The stabilizer W_{φ_0} of φ_0 in $\widetilde{W}$ identifies with W_0, because φ_0 is special. We obtain semidirect product decompositions

$$\widetilde{W} = W_0 \ltimes (\widetilde{W} \cap V) \qquad \text{and} \qquad W^{\text{aff}} = W_0 \ltimes (W^{\text{aff}} \cap V),$$

and $W^{\text{aff}} \cap V$ is generated by the translations $r\alpha^\vee$, for $\alpha \in \Phi_{\text{red}}$ and $r \in \Gamma_\alpha$ [BT72, Proposition (6.2.19)].

By [BT72, Proposition (6.2.22)] there exists a unique reduced root system Σ in V^* such that W^{aff} is the affine Weyl group of Σ, *i.e.*, it is the subgroup of $\operatorname{Aut} \mathscr{A}$ generated by the reflections

$$s_{\alpha,k} \colon \operatorname{Aut} \mathscr{A} \longrightarrow \operatorname{Aut} \mathscr{A}, \quad \varphi_0 + v \longmapsto \varphi_0 + v - \big(\langle \alpha, v\rangle + k\big) \cdot \alpha^\vee,$$

for $(\alpha, k) \in \Sigma^{\text{aff}} \coloneqq \Sigma \times \mathbb{Z}$. By a suitable scaling we obtain a surjective map $\Phi \to \Sigma$, $\alpha \mapsto \varepsilon_\alpha \alpha$ between root systems which induces a bijection $\Phi_{\text{red}} \cong \Sigma$. By [Vig16, (39) ff] we have $\varepsilon_\alpha = \varepsilon_{-\alpha} \in \mathbb{Z}_{>0}$ and $\Gamma_\alpha = \varepsilon_\alpha^{-1}\mathbb{Z}$, for $\alpha \in \Phi_{\text{red}}$, is a group.

Notation. In order to avoid confusion when working with the two root systems Φ_{red} and Σ we will write $H_{(\alpha,k)}$ and $U_{(\alpha,k)}$ instead of $H_{\alpha,k}$ and $U_{\beta,\varepsilon_\beta^{-1}k}$ whenever $\beta \in \Phi_{\text{red}}$, $\alpha = \varepsilon_\beta \beta$, and $k \in \mathbb{Z}$.

Given $n \in N$ with image w in $\widetilde{W}$ and $(\alpha, k) \in \Sigma^{\text{aff}}$, we have $wH_{(\alpha,k)} = H_{w\cdot(\alpha,k)}$ and

$$nU_{(\alpha,k)}n^{-1} = U_{w\cdot(\alpha,k)}. \tag{6}$$

2.2. **Parahoric subgroups.** The pointwise stabilizer of φ_0, resp. $\mathfrak{C}$, in the kernel of the Kottwitz homomorphism κ_G [Kot97, 7.1 to 7.4] is denoted by K, resp. I. We call I the *Iwahori subgroup*; its pro-p Sylow subgroup I_1 is called the *pro-p Iwahori subgroup.* Both K and I are examples of *parahoric subgroups* [HR08]; they are compact open subgroups of G.

We remark that K is a special parahoric subgroup containing I, and it satisfies [Vig16, (51)]

$$K \cap U_\alpha = U_{(\alpha,0)}, \qquad \text{for all } \alpha \in \Phi.$$

Put $Z_0 \coloneqq Z \cap K = Z \cap I$ [HR09, Lemma 4.2.1] with pro-p radical $Z_1 = Z \cap I_1$. Then Z_0 is the unique parahoric subgroup of Z, and N normalizes both Z_0 and Z_1. The multiplication map

$$\prod_{\alpha \in -\Sigma^+} U_{(\alpha,1)} \times Z_1 \times \prod_{\alpha \in \Sigma^+} U_{(\alpha,0)} \xrightarrow{\cong} I_1 \tag{7}$$

is a homeomorphism [Vig16, Corollary 3.20] with respect to any ordering of the factors.

2.3. **The Iwahori–Weyl group.** We call

$$W := N/Z_0, \qquad \text{resp.} \qquad W(1) := N/Z_1$$

the *Iwahori–Weyl group*, resp. *pro-p Iwahori–Weyl group.* There are exact sequences

$$1 \longrightarrow Z_0/Z_1 \longrightarrow W(1) \longrightarrow W \longrightarrow 1$$

and

$$0 \longrightarrow \Lambda \longrightarrow W \longrightarrow W_0 \longrightarrow 1, \tag{8}$$

where $\Lambda := Z/Z_0$ is a finitely generated abelian group with finite torsion and the same rank as $\mathrm{X}_*(T)$ [HR09, Theorem 1.0.1]. It is thus written additively. When viewed as an element of W we use the exponential notation e^λ rather than $\lambda \in \Lambda$ in order to avoid confusion.

Given a subset $X \subseteq W$, we denote $X(1)$ the preimage of X under the projection $W(1) \to W$.

We remark that the sequence (8) splits, providing a semidirect product decomposition

$$W = \Lambda \rtimes W_0.$$

In particular, W_0 acts on Λ via $w(\lambda) = we^\lambda w^{-1}$. The group $\Lambda(1)$ is not abelian in general. Notice that (4) factors through Λ (and hence $\Lambda(1)$).

The inclusion $N \hookrightarrow G$ induces bijections

$$W_0 \cong B\backslash G/B, \qquad W \cong I\backslash G/I, \qquad W(1) \cong I_1\backslash G/I_1. \tag{9}$$

The group $\Omega = \{u \in W \mid u\mathfrak{C} = \mathfrak{C}\}$ is abelian and acts on S^{aff} by conjugation, and we have a decomposition

$$W = W^{\mathrm{aff}} \rtimes \Omega.$$

The length function ℓ on the Coxeter group $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ extends to a length function ℓ on W if we define $\ell(wu) = \ell(w)$ for $w \in W^{\mathrm{aff}}$, $u \in \Omega$. By inflation we also obtain a length function ℓ on $W(1)$.

We denote by $\leq$ the Bruhat order on W^{aff}. It extends to the Bruhat order $\leq$ on W if we put

$$wu \leq w'u' \iff w \leq w' \text{ and } u = u', \qquad \text{for } w, w' \in W^{\mathrm{aff}},\ u, u' \in \Omega.$$

We define $v < w$ as $v \leq w$ and $v \neq w$. By inflation, we obtain the Bruhat order $<$ on $W(1)$: given $\widetilde{v}, \widetilde{w} \in W(1)$ with image $v, w \in W$, respectively, we define

$$\widetilde{v} < \widetilde{w} \iff v < w.$$

2.4. **The integers q_w.** Given $n \in N$ with image w in W (or $W(1)$), one defines

$$q_w := \left|InI/I\right| = \left|I_1 n I_1/I_1\right|.$$

An application of [Vig16, Proposition 3.38] shows

$$q_w = q_{s_1} \cdots q_{s_{\ell(w)}}$$

whenever $w = s_1 \cdots s_{\ell(w)} u$ with $s_i \in S^{\mathrm{aff}}$ and $u \in \Omega$ (or $s_i \in S^{\mathrm{aff}}(1)$ and $u \in \Omega(1)$). Given $s \in S^{\mathrm{aff}}$, write $H_s = H_{\beta,r} = H_{(\alpha,k)}$ with $\beta \in \Phi^+_{\mathrm{red}}$, $r \in \Gamma_\beta$ and $\alpha = \varepsilon_\beta \beta \in \Sigma^+$, $k = \varepsilon_\beta r \in \mathbb{Z}$. Then

$$q_s = \left|U_{\beta,r}/U_{\beta,r+}\right| = \left|U_{(\alpha,k)}/U_{(\alpha,k+1)}\right|, \tag{10}$$

where $U_{\beta,r+} \coloneqq \bigcap_{r'>r} U_{\beta,r'}$. Indeed, if $r \in \Gamma_\beta \setminus \Gamma'_\beta$, then $2\beta \in \Phi^+$ and $2r \in \Gamma'_{2\beta}$. In this case we have $U_{\beta,r+} \cdot U_{2\beta,2r} = U_{\beta,r}$, and [Vig16, Lemma 3.8] yields an isomorphism

$$U_{\beta,r}/U_{\beta,r+} \cong U_{2\beta,2r}/U_{2\beta,2r+}.$$

Now, the first equality in (10) follows from [Vig16, Corollary 3.31] while the second equality is clear.

Notice that $q_s = q_{s'}$ whenever $s, s' \in S^{\mathrm{aff}}$ are conjugate in W [Vig16, (67)]. As every hyperplane in $\mathfrak{H}$ is of the form wH_s for some $w \in W$, $s \in S^{\mathrm{aff}}$, we obtain a well-defined function

$$\mathfrak{H} \longrightarrow q^{\mathbb{Z}_{>0}}, \quad q(wH_s) \coloneqq q_s. \tag{11}$$

For every $w \in W$ (or $W(1)$) we then have [Vig16, Definition 4.14]

$$q_w = \prod_{H \in \mathfrak{H}_w} q(H), \tag{12}$$

where $\mathfrak{H}_w$ denotes the set of hyperplanes in $\mathfrak{H}$ separating $\mathfrak{C}$ and $w\mathfrak{C}$.

For $v, w \in W$ (or $W(1)$) there exists a unique integer $q_{v,w} \in q^{\mathbb{Z}_{\geq 0}}$ satisfying [Vig16, Definition 4.14]

$$q_v q_w = q_{vw} q_{v,w}^2.$$

Notice that

$$q_{v,w} = \prod_{H \in \mathfrak{H}_v \cap v\mathfrak{H}_w} q(H).$$

(In [Vig16, Lemma 4.19] this is only stated for $v, w \in W^{\mathrm{aff}}$. But this follows in general from the facts that $\mathfrak{H}_{wu} = \mathfrak{H}_w$ and $u\mathfrak{H}_w = \mathfrak{H}_{uwu^{-1}}$ whenever $w \in W^{\mathrm{aff}}$ and $u \in \Omega$.)

Remark. (a) If $\mathbf{G}$ is F-split, then $q_w = q^{\ell(w)}$ for all $w \in W$. For this reason the function $w \mapsto q_w$ may be viewed as a generalized length function on W.
(b) In general, $q_{v,w} = 1$ if and only if $\ell(vw) = \ell(v) + \ell(w)$.

2.5. **Levi subgroups.** Let $\mathbf{P} = \mathbf{M}\mathbf{U}_\mathbf{P}$ be a (standard) parabolic subgroup. Then $\mathbf{Z} \subseteq \mathbf{M}$ and $\mathbf{N}_\mathbf{M} \coloneqq \mathbf{N}_\mathbf{M}(\mathbf{T}) = \mathbf{N} \cap \mathbf{M}$. All the objects we have defined for $\mathbf{G}$ have an analogue for $\mathbf{M}$ and we denote them by attaching the index M; for example, we write Φ_M, $W_{0,M}$, $\mathscr{A}_M$, $\mathfrak{H}_M$, $q_{M,w}$, etc.

Notice that $W_{0,M}$ is contained in W_0, W_M identifies with $\Lambda \rtimes W_{0,M}$, and $W_M(1)$ is the preimage of W_M under $W(1) \to W$. The restriction $\varphi_{0,M}$ of φ_0 to $(Z, (U_\alpha)_{\alpha\in\Phi_M})$ is again discrete, special, and compatible with val_F. The $\mathbb{R}$-vector space

$$V_M \coloneqq \mathbb{R} \otimes X_*(T)/X_*(C_M),$$

where $\mathbf{C}_\mathbf{M} = \left(\bigcap_{\alpha\in\Phi_M^+} \operatorname{Ker}\alpha\right)^\circ$ is the connected center of $\mathbf{M}$, is a quotient of V. Then $\Phi_M = V_M^* \cap \Phi$ and $\Sigma_M = V_M^* \cap \Sigma$. The projection $V \twoheadrightarrow V_M$ induces a natural N_M-equivariant projection

$$p_M \colon \mathscr{A} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{A}_M$$

sending $\varphi_0 \mapsto \varphi_{0,M}$. Taking inverse images we obtain N_M-equivariant inclusions $\Phi_M^{\mathrm{aff}} \subseteq \Phi^{\mathrm{aff}}$ and $\mathfrak{H}_M \subseteq \mathfrak{H}$. Therefore, we have also $S(\mathfrak{H}_M) \subseteq S(\mathfrak{H})$ and hence $W_M^{\mathrm{aff}} \subseteq W^{\mathrm{aff}}$.

Remark. In general we have only $p_M(\mathfrak{C}) \subseteq \mathfrak{C}_M$ and not an equality. In this case we have $S_M^{\mathrm{aff}} \nsubseteq S^{\mathrm{aff}}$. Therefore, the length and Bruhat order on W_M^{aff} are not obtained by restricting the length and Bruhat order of W^{aff}.

3. Groups, double cosets, and indices

We fix a (standard) parabolic F-subgroup $\mathbf{P} = \mathbf{MU_P}$ of $\mathbf{G}$.

3.1. **A double coset formula.** Let $\Gamma \subseteq P$ be a compact open subgroup satisfying $\Gamma = \Gamma_M \Gamma_{U_P}$, where $\Gamma_M := \Gamma \cap M$ and $\Gamma_{U_P} := \Gamma \cap U_P$. This means that every $g \in \Gamma$ can be uniquely written as

$$g = g_M \cdot g_{U_P}, \qquad \text{for some } g_M \in \Gamma_M,\, g_{U_P} \in \Gamma_{U_P}.$$

Notice that for all $g \in P$ and $m \in M$ the indices

$$\mu(g) := [\Gamma : \Gamma_{(g)}], \qquad \mu_{U_P}(g) := [\Gamma_{U_P} : (\Gamma_{U_P})_{(g)}], \qquad \mu_M(m) := [\Gamma_M : (\Gamma_M)_{(m)}]$$

are finite, because $\Gamma \subseteq P$ is compact open. We also remark that the projection map $\mathrm{pr}_M \colon P \twoheadrightarrow M$ is continuous and open, so that $\nu_M(g) := [(\Gamma_M)_{(g_M)} : \mathrm{pr}_M(\Gamma_{(g)})]$ is finite.

Proposition 3.1 generalizes [Gri88, Lemma 2].

Proposition 3.1. *Given $g \in P$, consider the coset decompositions*

$$\Gamma_M = \bigsqcup_{i=1}^{\mu_M(g_M)} (\Gamma_M)_{(g_M)} m_i, \quad (\Gamma_M)_{(g_M)} = \bigsqcup_{j=1}^{\nu_M(g)} \mathrm{pr}_M(\Gamma_{(g)}) h_j,$$

$$\Gamma_{U_P} = \bigsqcup_{s=1}^{\mu_{U_P}(g)} (\Gamma_{U_P})_{(g)} u_s.$$

Then one has a decomposition of the double coset

$$\Gamma g \Gamma = \bigcup_{i=1}^{\mu_M(g_M)} \bigcup_{j=1}^{\nu_M(g)} \bigcup_{s=1}^{\mu_{U_P}(g)} \Gamma g u_s h_j m_i. \tag{13}$$

Moreover, the union is disjoint. In particular, $\mu(g) = \mu_M(g_M) \cdot \nu_M(g) \cdot \mu_{U_P}(g)$.

Proof. It is clear that the right hand side of (13) is contained in $\Gamma g \Gamma$. For the converse inclusion let $\gamma = \gamma_M \gamma_{U_P} \in \Gamma$. Write

$$\begin{aligned} \gamma_M &= m m_i, && \text{for some } m \in (\Gamma_M)_{(g_M)} \text{ and } 1 \le i \le \mu_M(g_M),\\ m &= h h_j, && \text{for some } h \in \mathrm{pr}_M(\Gamma_{(g)}) \text{ and } 1 \le j \le \nu_M(g). \end{aligned}$$

By definition of h there exists $u \in \Gamma_{U_P}$ with $hu^{-1} \in \Gamma_{(g)}$, *i.e.*, with $ghu^{-1} = yg$ for some $y \in \Gamma$. Thus,

$$\gamma = \gamma_M \gamma_{U_P} = h h_j m_i \gamma_{U_P} = hu^{-1} \cdot u \gamma_{U_P}^{(h_j m_i)^{-1}} \cdot h_j m_i.$$

As Γ_M normalizes Γ_{U_P} we may write $v u_s = u \gamma_{U_P}^{(h_j m_i)^{-1}} \in \Gamma_{U_P}$ for some $v \in (\Gamma_{U_P})_{(g)}$ and some integer $1 \le s \le \mu_{U_P}(g)$. Then also $v^{g^{-1}} \in \Gamma_{U_P}$, and therefore

$$\begin{aligned} g\gamma &= ghu^{-1} \cdot u \gamma_{U_P}^{(h_j m_i)^{-1}} \cdot h_j m_i\\ &= yg \cdot v u_s \cdot h_j m_i\\ &= y v^{g^{-1}} \cdot g \cdot u_s h_j m_i \in \Gamma g u_s h_j m_i. \end{aligned}$$

This proves equality in (13). To see disjointness in (13), assume $g u_s h_j m_i = \gamma g u_t h_a m_b$ for some $\gamma \in \Gamma$. Rearranging gives

$$\gamma^g = u_s h_j m_i m_b^{-1} h_a^{-1} u_t^{-1} \in \Gamma_{(g)}. \tag{14}$$

Applying pr_M to (14) yields

$$\gamma_M^{g_M} = h_j m_i m_b^{-1} h_a^{-1} \in \mathrm{pr}_M\big(\Gamma_{(g)}\big). \tag{15}$$

In particular, $m_i m_b^{-1} = h_j^{-1} \gamma_M^{g_M} h_a \in (\Gamma_M)_{(g_M)}$, whence $i = b$. Therefore, equation (15) reads $\gamma_M^{g_M} = h_j h_a^{-1} \in \mathrm{pr}_M\big(\Gamma_{(g)}\big)$, and we deduce $j = a$. Going back to (14) gives $\gamma^g = u_s u_t^{-1} \in \Gamma_{(g)} \cap U_P$. But notice that $\Gamma_{(g)} \cap U_P = (\Gamma_{U_P})_{(g)}$, because g normalizes U_P. Consequently, $u_s u_t^{-1} \in (\Gamma_{U_P})_{(g)}$, whence $s = t$. This concludes the proof of the disjointness assertion.

The map $\Gamma_{(g)}\backslash\Gamma \to \Gamma\backslash\Gamma g\Gamma$, sending $\Gamma_{(g)}\gamma \mapsto \Gamma g\gamma$, is well-defined and bijective. Hence, we have $\mu(g) = [\Gamma : \Gamma_{(g)}] = |\Gamma\backslash\Gamma g\Gamma|$ and the last assertion follows. □

Remark. In general, we have $\nu_M(g) \neq 1$, *i.e.*, $\mathrm{pr}_M(\Gamma_{(g)}) \subsetneqq (\Gamma_M)_{(g_M)}$. As a concrete example consider the group P of upper triangular matrices inside $\mathrm{GL}_2(\mathbb{Q}_p)$. It contains the subgroup M of diagonal matrices and the subgroup U_P of upper triangular unipotent matrices. Let $\Gamma = \left(\begin{smallmatrix} (1+p\mathbb{Z}_p)^\times & \mathbb{Z}_p \\ 0 & (1+p\mathbb{Z}_p)^\times \end{smallmatrix}\right)$ and $g := \left(\begin{smallmatrix} 1 & p^{-n-1} \\ 0 & 1 \end{smallmatrix}\right)$ for some integer $n \in \mathbb{Z}_{\geq 0}$. Then $g_M = 1$, whence $(\Gamma_M)_{(g_M)} = \Gamma_M$. Given any $\gamma = \left(\begin{smallmatrix} 1+pa & b \\ 0 & 1+pc \end{smallmatrix}\right)$ in Γ, we compute $g\gamma g^{-1} = \left(\begin{smallmatrix} 1+pa & b+p^{-n}(c-a) \\ 0 & 1+pc \end{smallmatrix}\right)$. Therefore, $g\gamma g^{-1} \in \Gamma$ if and only if $c - a \in p^n\mathbb{Z}_p$. Thus, $\Gamma_{(g)} = \left\{ \left(\begin{smallmatrix} 1+pa & b \\ 0 & 1+pa+p^{n+1}c \end{smallmatrix}\right) \,\middle|\, a, b, c \in \mathbb{Z}_p \right\}$, so that

$$\mathrm{pr}_M\big(\Gamma_{(g)}\big) = \left\{ \begin{pmatrix} 1+pa & 0 \\ 0 & 1+pa+p^{n+1}c \end{pmatrix} \,\middle|\, a, c \in \mathbb{Z}_p \right\}.$$

From this description it is already clear that $\nu_M(g) \neq 1$ in general. As an exercise, and in order to illustrate the methods employed in section 3.3, we explicitly compute $\nu_M(g)$. Consider the reduction modulo p^{n+1} map

$$\psi\colon \Gamma_M = \begin{pmatrix} (1+p\mathbb{Z}_p)^\times & 0 \\ 0 & (1+p\mathbb{Z}_p)^\times \end{pmatrix} \longrightarrow \begin{pmatrix} (\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^\times & 0 \\ 0 & (\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^\times \end{pmatrix}.$$

Its kernel is contained in $\mathrm{pr}_M\big(\Gamma_{(g)}\big)$, and we have

$$\psi\big(\Gamma_M\big) = \left\{ \begin{pmatrix} 1+pa+p^{n+1}\mathbb{Z}_p & 0 \\ 0 & 1+pc+p^{n+1}\mathbb{Z}_p \end{pmatrix} \,\middle|\, a, c \in \mathbb{Z}_p \right\} \quad \text{and}$$

$$\psi\big(\mathrm{pr}_M(\Gamma_{(g)})\big) = \left\{ \begin{pmatrix} 1+pa+p^{n+1}\mathbb{Z}_p & 0 \\ 0 & 1+pa+p^{n+1}\mathbb{Z}_p \end{pmatrix} \,\middle|\, a \in \mathbb{Z}_p \right\},$$

whence $|\psi(\Gamma_M)| = p^{2n}$ and $|\psi\big(\mathrm{pr}_M(\Gamma_{(g)})\big)| = p^n$. Therefore,

$$\nu_M(g) = [\Gamma_M : \mathrm{pr}_M(\Gamma_{(g)})] = |\psi(\Gamma_M)|/|\psi\big(\mathrm{pr}_M(\Gamma_{(g)})\big)| = p^n.$$

3.2. **Two technical lemmas.** In this subsection we prove two technical lemmas which will be needed for the proof of the fundamental Proposition 3.4.

Recall the finite-dimensional $\mathbb{R}$-vector space V, the root system Φ inside the dual vector space V^*, and the set of positive roots Φ^+. Fix a subset $\Psi \subseteq \Phi^+$. We consider the partial ordering on Ψ defined by

$$\alpha \leq \beta$$

if there exist $\gamma_1, \ldots, \gamma_n \in \Psi$ and $r, s_1, \ldots, s_n \in \mathbb{Z}_{\geq 0}$, with $r > 0$, such that $\beta = r\alpha + \sum_{i=1}^n s_i\gamma_i$. The relation is clearly reflexive and transitive. It is also antisymmetric, since there exists $v \in V$ with $\langle \alpha, v \rangle > 0$ for all $\alpha \in \Phi^+$. We write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

Let X be a group. For each $\alpha \in \Psi$ let Y_α be a subgroup of X. Define

$$X_\alpha := \begin{cases} Y_\alpha, & \text{if } 2\alpha \notin \Psi, \\ Y_\alpha Y_{2\alpha}, & \text{if } 2\alpha \in \Psi. \end{cases}$$

Put $X_{2\alpha} := \{1\}$ if $2\alpha \notin \Psi$. We impose the following conditions:

(i) X is generated by the Y_α, for $\alpha \in \Psi$.

(ii) For all $\alpha, \beta \in \Psi$ the commutator subgroup $[Y_\alpha, Y_\beta]$ is contained in the subgroup of X generated by the $Y_{r\alpha+s\beta}$, where $r, s \in \mathbb{Z}_{>0}$ with $r\alpha+s\beta \in \Psi$.

(iii) The intersection of the groups generated by $\bigcup_{\substack{\alpha\in\Psi,\\ \langle\alpha,v\rangle\leq 0}} Y_\alpha$ and $\bigcup_{\substack{\alpha\in\Psi,\\ \langle\alpha,v\rangle> 0}} Y_\alpha$, respectively, is trivial for each $v \in V$.

Notice that X_α is a group thanks to (ii). Fix a bijection $o\colon\Psi_{\mathrm{red}} \to \{1, 2, \ldots, |\Psi_{\mathrm{red}}|\}$ and define

$$\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha := x_{o^{-1}(1)} \cdot x_{o^{-1}(2)} \cdots x_{o^{-1}(|\Psi_{\mathrm{red}}|)}, \qquad \text{for } x_\alpha \in X_\alpha.$$

We call o an *ordering of the factors.* It follows from [BT72, Lemme (6.1.7)] that the multiplication map

$$\prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha \longrightarrow X$$

is bijective.

Lemma 3.2. *Let $f\colon X \to X$ be a group homomorphism such that*

$$f(x_\alpha)x_\alpha^{-1} \in \langle X_\beta \mid \beta > \alpha\rangle, \qquad \textit{for all } x_\alpha \in X_\alpha, \textit{ all } \alpha \in \Psi_{\mathrm{red}}.$$

For all $\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha \in X$, with $x_\alpha \in X_\alpha$, one has

$$f\Big(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha\Big) = \prod_{\alpha\in\Psi_{\mathrm{red}}} z_\alpha \widetilde{z}_\alpha(x_\alpha) x_\alpha, \tag{16}$$

where $z_\alpha = z_\alpha((x_\beta)_{\beta<\alpha}) \in X_\alpha$ depends only on $(x_\beta)_{\beta<\alpha}$, and where $\widetilde{z}_\alpha\colon X_\alpha \to X_{2\alpha}$ is a group homomorphism factoring through $X_\alpha/X_{2\alpha}$. Moreover,

- *z_α and $\widetilde{z}_\alpha$ are uniquely determined by* (16).
- *$\widetilde{z}_\alpha$ depends only on those x_β with $\beta < \alpha$* and *$o(\beta) > o(\alpha)$.*

In particular, if o is such that $\beta_1 < \beta_2$ implies $o(\beta_1) < o(\beta_2)$, then $\widetilde{z}_\alpha$ does not depend on $(x_\beta)_{\beta<\alpha}$. In this case, $\widetilde{z}_\alpha(x_\alpha)$ is the image of $f(x_\alpha)x_\alpha^{-1}$ under the projection $\prod_{\beta\in\Psi_{\mathrm{red}}} X_\beta \to X_\alpha$.

Remark. (a) The homomorphism $f\colon X \to X$ in Lemma 3.2 is necessarily an automorphism.

(b) The main example to keep in mind is the case where $f\colon X \to X$ is conjugation by some element of X. (See Lemma 3.6 for a proof of why such f satisfies the hypothesis of Lemma 3.2.) This is also the only morphism to which we apply Lemma 3.2.

Proof of Lemma 3.2. The uniqueness assertions are immediate (for example, the uniqueness of z_α follows by letting $x_\alpha = 1$). Notice that (ii) above implies that $X_{2\alpha}$ is central in X_α and that the commutator subgroup $[X_\alpha, X_\alpha]$ is contained in $X_{2\alpha}$.

We prove (16) by induction on $|\Psi_{\mathrm{red}}|$. Suppose $\Psi_{\mathrm{red}} = \{\alpha\}$. By the hypothesis on f the map

$$\widetilde{z}_\alpha\colon X_\alpha \longrightarrow X_{2\alpha}, \qquad x \longmapsto f(x)x^{-1}$$

is well-defined and satisfies $\widetilde{z}_\alpha(X_{2\alpha}) = \{1\}$. Given $x, y \in X_\alpha$, we compute

$$\begin{aligned}\widetilde{z}_\alpha(xy) &= f(xy)\cdot(xy)^{-1} = f(x)f(y)y^{-1}x^{-1}\\ &= f(x)\widetilde{z}_\alpha(y)x^{-1} = f(x)x^{-1}\cdot\widetilde{z}_\alpha(y) = \widetilde{z}_\alpha(x)\cdot\widetilde{z}_\alpha(y).\end{aligned}$$

Hence, $\widetilde{z}_\alpha$ is a group homomorphism, which proves the base case (with $z_\alpha := 1$).

Suppose now $|\Psi_{\mathrm{red}}| > 1$ and choose a root $\alpha_0 \in \Psi_{\mathrm{red}}$ maximal with respect to the partial order. We start by proving the following useful claim:

Claim. Suppose $f\big(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha\big) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha$, where $x_\alpha, y_\alpha \in X_\alpha$. Then y_{α_0} depends only on the x_β with $\beta \leq \alpha_0$.

Proof of the claim. Let Ψ' be the largest subset of Ψ with $\Psi'_{\mathrm{red}} = \{\beta\in\Psi_{\mathrm{red}} \mid \beta \not\leq \alpha_0\}$, and let Z_{α_0} be the subgroup of X generated by the groups X_β, for $\beta \in \Psi'_{\mathrm{red}}$. Notice that Ψ' is upwards closed in Ψ, that is, $\gamma \geq \beta$ with $\gamma \in \Psi$ and $\beta \in \Psi'$ implies $\gamma \in \Psi'$. Therefore, Z_{α_0} is a normal subgroup of X. The hypotheses (i)–(iii) remain satisfied if we replace X, Ψ, and $(Y_\alpha)_{\alpha\in\Psi}$ by Z_{α_0}, Ψ', and $(Y_\alpha)_{\alpha\in\Psi'}$, respectively. Hence, the multiplication map $\prod_{\alpha\in\Psi'_{\mathrm{red}}} X_\alpha \to Z_{\alpha_0}$ is bijective, and the canonical projection

$$\mathrm{pr}_{\leq\alpha_0}\colon X \cong \prod_{\beta\in\Psi_{\mathrm{red}}} X_\beta \longrightarrow \prod_{\beta\leq\alpha_0} X_\beta \cong X/Z_{\alpha_0},$$

is a group homomorphism with kernel Z_{α_0}. The hypothesis on f implies $f(Z_{\alpha_0}) \subseteq Z_{\alpha_0}$, again since Ψ' is upwards closed. We obtain an induced group homomorphism $\overline{f}\colon X/Z_{\alpha_0} \to X/Z_{\alpha_0}$ such that, after identifying $X \cong \prod_{\beta\in\Psi_{\mathrm{red}}} X_\beta$ and $X/Z_{\alpha_0} \cong \prod_{\beta\leq\alpha_0} X_\beta$, the diagram

$$\begin{tikzcd}\prod_{\beta\in\Psi_{\mathrm{red}}} X_\beta \arrow[r, "f"] \arrow[d, "\mathrm{pr}_{\leq\alpha_0}"'] & \prod_{\beta\in\Psi_{\mathrm{red}}} X_\beta \arrow[d, "\mathrm{pr}_{\leq\alpha_0}"] \\ \prod_{\beta\leq\alpha_0} X_\beta \arrow[r, "\overline{f}"'] & \prod_{\beta\leq\alpha_0} X_\beta\end{tikzcd}$$

is commutative. As the restriction of $\mathrm{pr}_{\leq\alpha_0}$ to X_{α_0} is injective, it follows immediately that y_{α_0} only depends on the x_β with $\beta \leq \alpha_0$. The claim is proved. □

Let $(x_\alpha)_\alpha \in \prod_{\alpha\in\Psi_{\mathrm{red}}}$ and write $f\big(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha\big) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha$ as in the claim. We prove (16) in two steps.

Step 1. We have $y_\alpha = z_\alpha \widetilde{z}_\alpha(x_\alpha)x_\alpha$, for $\alpha \neq \alpha_0$, with z_α and $\widetilde{z}_\alpha$ as in the statement of the lemma.

This follows from the induction hypothesis as follows: as X_{α_0} is normal in X, the quotient $X' := X/X_{\alpha_0}$ is a group. Put $\Psi' := \Psi \smallsetminus \{\alpha_0, 2\alpha_0\}$. Under the projection map $X \twoheadrightarrow X'$ the subgroups Y_α, X_α of X embed into X' for $\alpha \in \Psi'$. Denote $f'\colon X' \to X'$ the homomorphism induced by f. The hypotheses of the lemma remain satisfied if we replace X, Ψ, $(Y_\alpha)_{\alpha\in\Psi}$, f by X', Ψ', $(Y_\alpha)_{\alpha\in\Psi'}$, f', respectively. The diagram

$$\begin{tikzcd}X \cong \prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha \arrow[r, "f"] \arrow[d, "\mathrm{pr}"'] & \prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha \cong X \arrow[d, "\mathrm{pr}"] \\ X' \cong \prod_{\alpha\in\Psi'_{\mathrm{red}}} X_\alpha \arrow[r, "f'"'] & \prod_{\alpha\in\Psi'_{\mathrm{red}}} X_\alpha \cong X'\end{tikzcd}$$

is commutative. Therefore, $f'\bigl(\prod_{\alpha\in\Psi'_{\mathrm{red}}} x_\alpha\bigr) = \prod_{\alpha\in\Psi'_{\mathrm{red}}} y_\alpha$, and the induction hypothesis implies $y_\alpha = z_\alpha\widetilde{z}_\alpha(x_\alpha)x_\alpha$ for certain elements $z_\alpha \in X_\alpha$ and group homomorphisms $\widetilde{z}_\alpha\colon X_\alpha \to X_{2\alpha}$ factoring through $X_\alpha/X_{2\alpha}$ and only depending on the x_β with $\beta < \alpha$, $\alpha \in \Psi'_{\mathrm{red}} = \Psi_{\mathrm{red}} \smallsetminus \{\alpha_0\}$.

Step 2. We have $y_{\alpha_0} = z_{\alpha_0}\widetilde{z}_{\alpha_0}(x_{\alpha_0})x_{\alpha_0}$ with z_{α_0} and $\widetilde{z}_{\alpha_0}$ as in the statement of the lemma.

We introduce the following notation:

$$x^< := \prod_{\substack{\alpha\in\Psi_{\mathrm{red}}\\ o(\alpha)<o(\alpha_0)}} x_\alpha, \quad x^> := \prod_{\substack{\alpha\in\Psi_{\mathrm{red}}\\ o(\alpha)>o(\alpha_0)}} x_\alpha, \quad x := x^< \cdot x_{\alpha_0} \cdot x^>, \quad x' := x^< \cdot x^>,$$

$$f(x^<) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y^<_\alpha, \quad f(x^>) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y^>_\alpha, \quad f(x) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha, \quad f(x') = \prod_{\alpha\in\Psi_{\mathrm{red}}} y'_\alpha.$$

The claim implies that $y^<_{\alpha_0}$ (resp. $y^>_{\alpha_0}$, resp. y'_{α_0}) depends only on the x_β with $\beta < \alpha_0$ and $o(\beta) < o(\alpha_0)$ (resp. $o(\beta) > o(\alpha_0)$, resp. $o(\beta) \neq o(\alpha_0)$). Using that $X_{2\alpha_0}$ is central in X and that X_{α_0} is centralized by all X_β with $\beta \neq \alpha_0$ we compute

$$\begin{aligned}\prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha &= f(x) = f(x^<)\cdot f(x_{\alpha_0})\cdot f(x^>) = f(x^<)\cdot f(x_{\alpha_0})\cdot \prod_{\alpha\in\Psi_{\mathrm{red}}} y^>_\alpha\\ &= f(x^<)\cdot\Bigl(\prod_{\alpha\in\Psi_{\mathrm{red}}} y^>_\alpha\Bigr)\cdot [f(x_{\alpha_0}), y^>_{\alpha_0}]\cdot f(x_{\alpha_0})\\ &= f(x')\cdot [f(x_{\alpha_0}), y^>_{\alpha_0}]\cdot f(x_{\alpha_0}) = \Bigl(\prod_{\alpha\in\Psi_{\mathrm{red}}} y'_\alpha\Bigr)\cdot [f(x_{\alpha_0}), y^>_{\alpha_0}]\cdot f(x_{\alpha_0}).\end{aligned}$$

We obtain $y_{\alpha_0} = y'_{\alpha_0}\cdot [f(x_{\alpha_0}), y^>_{\alpha_0}]\cdot f(x_{\alpha_0})$. The claim implies that the element $z_{\alpha_0} := y'_{\alpha_0} \in X_\alpha$ only depends on the x_β with $\beta < \alpha_0$. Moreover, define

$$\widetilde{z}_{\alpha_0}(x_{\alpha_0}) := [f(x_{\alpha_0}), y^>_{\alpha_0}]\cdot f(x_{\alpha_0})x^{-1}_{\alpha_0} \in X_{2\alpha_0}. \tag{17}$$

Now, $X_{2\alpha_0}$ is central in X_{α_0} and f is the identity on $X_{2\alpha_0}$, whence $\widetilde{z}_{\alpha_0}(X_{2\alpha_0}) = \{1\}$. It remains to show that $\widetilde{z}_{\alpha_0}\colon X_{\alpha_0} \to X_{2\alpha_0}$ is a group homomorphism. The base case shows that $X_{\alpha_0} \to X_{2\alpha_0}$, $x \mapsto f(x)x^{-1}$ is a homomorphism. As $X_{2\alpha_0}$ is abelian it suffices to show that $X_{\alpha_0} \to X_{2\alpha_0}$, $x \mapsto [f(x), y^>_{\alpha_0}]$ is a homomorphism. But this is immediate from the general identity $[uv, w] = u[v, w]u^{-1}\cdot[u, w]$ for all $u, v, w \in X_{\alpha_0}$. Hence, $y_{\alpha_0} = z_{\alpha_0}\widetilde{z}_{\alpha_0}(x_{\alpha_0})x_{\alpha_0}$, with z_{α_0} and $\widetilde{z}_{\alpha_0}$ depending only on the x_β with $\beta < \alpha_0$.

Putting together Steps 1 and 2 finishes the proof of (16). For the last statement we may assume that $\alpha = \alpha_0$ is maximal. Then the claim follows from (17), because $y^>_{\alpha_0}$ depends only on those x_β with $\beta < \alpha_0$ and $o(\beta) > o(\alpha_0)$. □

For Lemma 3.3 we choose the ordering o of the factors such that $\alpha < \beta$ implies $o(\alpha) < o(\beta)$.

Lemma 3.3. *Assume that* X *is finite. Let* $Y \subseteq X$ *and* $Z_\alpha \subseteq X_\alpha$*, for each* $\alpha \in \Psi_{\mathrm{red}}$*, be subgroups. Assume further that the following condition is satisfied:*

For all $1 \leq i \leq |\Psi_{\mathrm{red}}|$ *and all* $(x_1, \ldots, x_{i-1}) \in \prod_{j=1}^{i-1} X_{o^{-1}(j)}$*, there exists* $z \in X_{o^{-1}(i)}$ *depending only on* $(x_1, \ldots, x_{i-1})$ *and satisfying the following property: whenever* $\prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha \in Y$ *is such that* $y_{o^{-1}(j)} = x_j$*, for all* $1 \leq j \leq i-1$*, we have* $y_{o^{-1}(i)} \in zZ_{o^{-1}(i)}$*.*

Then $|Y| \leq \prod_{\alpha\in\Psi_{\mathrm{red}}} |Z_\alpha|$*.*

Proof. Given $0 \leq i \leq |\Psi_{\mathrm{red}}|$ and $(x_1, \ldots, x_i) \in \prod_{j=1}^{i} X_{o^{-1}(j)}$, we put

$$Y(x_1, \ldots, x_i) := \left\{ \prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha \in Y \,\middle|\, y_{o^{-1}(j)} = x_j \text{ for all } 1 \leq j \leq i \right\}.$$

For $i = 0$ we define $(x_1, \ldots, x_0)$ to be the empty tuple $()$. In this case we have $Y() = Y$. Whenever $i < |\Psi_{\mathrm{red}}|$ we say that $x_{i+1} \in X_{o^{-1}(i+1)}$ *extends* $(x_1, \ldots, x_i)$ if $Y(x_1, \ldots, x_{i+1})$ is not the empty set.

With i as above we will prove

$$|Y(x_1, \ldots, x_i)| \leq \prod_{j=i+1}^{|\Psi_{\mathrm{red}}|} |Z_{o^{-1}(j)}|, \qquad \text{for all } (x_1, \ldots, x_i) \in \prod_{j=1}^{i} X_{o^{-1}(j)} \tag{18}$$

by descending induction on the length i of the tuple $(x_1, \ldots, x_i)$. Then (18) for $i = 0$ is precisely the assertion of the lemma.

The base case $i = |\Psi_{\mathrm{red}}|$ is satisfied, because the right hand side equals 1 (being the empty product) and since $|Y(x_1, \ldots, x_{|\Psi_{\mathrm{red}}|})|$ is 1 or 0 depending on whether $x_1 \cdots x_{|\Psi_{\mathrm{red}}|}$ lies in Y or not. Assume now that (18) is satisfied for some $1 \leq i \leq |\Psi_{\mathrm{red}}|$. Take $(x_1, \ldots, x_{i-1}) \in \prod_{j=1}^{i-1} X_{o^{-1}(j)}$ and put

$$J(x_1, \ldots, x_{i-1}) := \left\{ y \in X_{o^{-1}(i)} \,\middle|\, y \text{ extends } (x_1, \ldots, x_{i-1}) \right\}.$$

We may assume that $Y(x_1, \ldots, x_{i-1})$ is non-empty. In this case $J(x_1, \ldots, x_{i-1})$ is also non-empty and condition (3.3) implies $|J(x_1, \ldots, x_{i-1})| = |Z_{o^{-1}(i)}|$. Observe that

$$Y(x_1, \ldots, x_{i-1}) = \bigcup_{y\in J(x_1,\ldots,x_{i-1})} Y(x_1, \ldots, x_{i-1}, y).$$

We now compute

$$\begin{aligned} |Y(x_1, \ldots, x_{i-1})| &\leq \sum_{y\in J(x_1,\ldots,x_{i-1})} |Y(x_1, \ldots, x_{i-1}, y)| \\ &\leq \sum_{y\in J(x_1,\ldots,x_{i-1})} \prod_{j=i+1}^{|\Psi_{\mathrm{red}}|} |Z_{o^{-1}(j)}| \\ &= |Z_{o^{-1}(i)}| \cdot \prod_{j=i+1}^{|\Psi_{\mathrm{red}}|} |Z_{o^{-1}(j)}| = \prod_{j=i}^{|\Psi_{\mathrm{red}}|} |Z_{o^{-1}(j)}|, \end{aligned}$$

where the second estimate uses the induction hypothesis. This finishes the induction step and proves the lemma. □

3.3. **An inequality of indices.** Let $\Gamma \subseteq P$ be an open compact subgroup with $\Gamma = \Gamma_M I_{U_P}$, where $\Gamma_M = \Gamma \cap M$ and $I_{U_P} = I \cap U_P$. Here I is the (fixed) Iwahori subgroup of G. For example, Γ could be $K \cap P$ or $I \cap P$ or even $I_1 \cap P$.

Remark. Since the function μ_{U_P} takes values in $q^{\mathbb{Z}_{\geq 0}}$, there is an equivalence

$$\mu_{U_P}(g) \leq \mu_{U_P}(g') \iff \mu_{U_P}(g) \text{ divides } \mu_{U_P}(g').$$

Therefore, we use "$\leq$" in this context to implicitly mean "divides".

The main goal of this section is to prove the following fundamental result.

Proposition 3.4. *Each $g \in P$ with image g_M in M satisfies*

$$\mu_{U_P}(g) \geq \mu_{U_P}(g_M).$$

Example. The ensuing proof is long and technical notwithstanding the lemmas in subsection 3.2. We will therefore discuss first an example in order to fix ideas. Let $G = \mathrm{GL}_3(\mathbb{Q}_p)$, P the Borel subgroup of upper triangular matrices, and M the torus of diagonal matrices. Let I be the standard Iwahori determined by P. Let $g = g_U g_M$ with $g_U = \left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right) \in U_P$ and $g_M = \mathrm{diag}(p^{m+n}, p^n, 1) \in M$ with $n, m \in \mathbb{Z}$. The inequality in Proposition 3.4 is equivalent to

$$\big|(I_{U_P})_{(g)}/H\big| \leq \big|(I_{U_P})_{(g_M)}/H\big|, \tag{19}$$

where $H \subseteq I_{U_P}$ is any sufficiently small open normal subgroup. For a general element $\left(\begin{smallmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{smallmatrix}\right)$ of I_{U_P} we compute

$$g_M^{-1} \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} g_M = \begin{pmatrix} 1 & p^{-m}u & p^{-m-n}w \\ 0 & 1 & p^{-n}v \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

For (20) to lie in $(I_{U_P})_{(g_M)}$ it is necessary and sufficient that u, v, w satisfy the following conditions:

(i) $\mathrm{val}_{\mathbb{Q}_p}(p^{-m}u) \geq 0$; (ii) $\mathrm{val}_{\mathbb{Q}_p}(p^{-n}v) \geq 0$; (iii) $\mathrm{val}_{\mathbb{Q}_p}(p^{-m-n}w) \geq 0$.

We deduce

$$(I_{U_P})_{(g_M)} = \begin{pmatrix} 1 & p^{\max\{0,-m\}}\mathbb{Z}_p & p^{\max\{0,-m-n\}}\mathbb{Z}_p \\ 0 & 1 & p^{\max\{0,-n\}}\mathbb{Z}_p \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, compute that $g^{-1}\left(\begin{smallmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{smallmatrix}\right)g$ equals

$$\begin{pmatrix} 1 & p^{-m}u & y \cdot p^{-m}u - x \cdot p^{-n}v + p^{-m-n}w \\ 0 & 1 & p^{-n}v \\ 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

Observe that (21) and the right hand side of (20) differ only in the upper right entry, and their difference is $z(u,v) := y \cdot p^{-m}u - x \cdot p^{-n}v$ which depends only on the terms coming from root groups of smaller roots. Lemma 3.2 shows that this is the general behavior. For (21) to lie in $(I_{U_P})_{(g)}$ it is necessary that u and v satisfy (i), (ii), and

(iii') $\mathrm{val}_{\mathbb{Q}_p}(z(u,v)) \geq \min\{0, -m-n\}$.

The important observation to make here is that (iii') may fail even if u and v satisfy (i) and (ii). This further restriction on u and v is the main reason why there is an inequality in (19). Now, we assume that u and v do satisfy (iii') and we determine the possible upper right entries in (21). If $\mathrm{val}_{\mathbb{Q}_p}(z(u,v)) \geq -m-n$, then the possible entries lie exactly in $p^{\max\{0,-m-n\}}\mathbb{Z}_p$. If, on the other hand, we have $-m-n > \operatorname{val}_{\mathbb{Q}_p}(z(u,v)) \geq 0$, then they lie exactly in the proper coset $z(u,v)+p^{\max\{0,-m-n\}}\mathbb{Z}_p$. Note that in both cases $p^{\max\{0,-m-n\}}\mathbb{Z}_p$ does not depend on u and v. If $H = \left(\begin{smallmatrix} 1 & p^r\mathbb{Z}_p & p^r\mathbb{Z}_p \\ 0 & 1 & p^r\mathbb{Z}_p \\ 0 & 0 & 1 \end{smallmatrix}\right)$, for $r \gg 0$ large enough, this discussion shows

$$\begin{aligned} \left|(I_{U_P})_{(g)}/H\right| &\leq \left|\frac{p^{\max\{0,-m\}}\mathbb{Z}_p}{p^r\mathbb{Z}_p}\right| \cdot \left|\frac{p^{\max\{0,-n\}}\mathbb{Z}_p}{p^r\mathbb{Z}_p}\right| \cdot \left|\frac{p^{\max\{0,-m-n\}}\mathbb{Z}_p}{p^r\mathbb{Z}_p}\right| \\ &= \left|(I_{U_P})_{(g_M)}/H\right|. \end{aligned}$$

The role of Lemma 3.3 is to show that we can make this estimate in general.

We now turn to the proof of Proposition 3.4. As the example above illustrates, it will be necessary to analyse $(I_{U_P})_{(g)} = I_{U_P} \cap g^{-1}I_{U_P}g$. We will make extensive use of the identification

$$I_{U_P} \cong \prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} U_{(\alpha,0)},$$

which follows from (7). As $\mathbf{G}$ is not assumed F-split, the root system Φ need not be reduced. If $\alpha, 2\alpha \in \Phi$, the root group U_α is not abelian in general; it contains the non-trivial abelian subgroup $U_{2\alpha}$, and the quotient $U_\alpha/U_{2\alpha}$ is abelian. This phenomenon motivates the next definition.

Definition. Let **FGrp** be the category whose

- objects are pairs (X_0, X) of groups such that X_0 is a normal subgroup of X;
- morphisms $(X_0, X) \to (X_0', X')$ are group homomorphisms $f\colon X \to X'$ satisfying $f(X_0) \subseteq f(X_0')$.

We write $(X_0, X) \subseteq (X_0', X')$ if $X \subseteq X'$ and $X_0 = X_0' \cap X$.

There is a canonical functor $\operatorname{gr}\colon \mathbf{FGrp} \to \mathbf{Grp}$ into the category of graded groups given by

$$\operatorname{gr}(X_0, X) := \operatorname{gr}_0(X_0, X) \times \operatorname{gr}_1(X_0, X),$$

where $\operatorname{gr}_0(X_0, X) := X_0$ and $\operatorname{gr}_1(X_0, X) := X/X_0$. We will need the following elementary lemma, the proof of which will be left as an exercise for the reader.

Lemma 3.5. *Let* $(X_0, X) \subseteq (X_0', X')$ *be objects in* **FGrp***.*

(a) *One has* $\operatorname{gr}(X_0, X) \subseteq \operatorname{gr}(X_0', X')$*.*
(b) *Assume* $[X' : X] < \infty$*. Then* $[X' : X] = [\operatorname{gr}(X_0', X') : \operatorname{gr}(X_0, X)]$*.*

3.3.1. *Proof of Proposition* 3.4. Let $g \in P$. Recall that g_M denotes the image of g in M and that $g_U := g_M^{-1}g \in U_P$. The group $K_M = K \cap M$ normalizes I_{U_P} and hence the function $\mu_{U_P}\colon P \to q^{\mathbb{Z}_{\geq 0}}$ is constant on $K_M g K_M$. By [HR09, Lemma 4.1.1] the group K_M is a maximal parahoric subgroup of M. Hence, the Cartan decomposition [HR09, Theorem 1.0.3] implies that the intersection $K_M g_M K_M \cap Z$ is non-empty. Thus, we may assume $g_M \in Z$.

For each $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$ we have $g_M^{-1}U_{(\alpha,0)}g_M = U_{(\alpha,\langle\alpha,\nu(g_M)\rangle)}$ by (5) and hence

$$(U_{(\alpha,0)})_{(g_M)} = \begin{cases} U_{(\alpha,0)}, & \text{if } \langle\alpha,\nu(g_M)\rangle \leq 0, \\ U_{(\alpha,0)}^{g_M}, & \text{otherwise.} \end{cases} \tag{22}$$

Write $\Psi := \Phi^+ \smallsetminus \Phi_M$ and choose an ordering o of the factors of $\prod_{\alpha\in\Psi_{\mathrm{red}}} U_\alpha$ in such a way that $\beta < \alpha$ implies $o(\beta) < o(\alpha)$.

Consider the automorphism $f\colon U_P \to U_P$, $x \mapsto g_U^{-1} x g_U$. We proceed with a series of lemmas.

Lemma 3.6. *For each $x_\alpha \in U_\alpha$, $\alpha \in \Psi_{\mathrm{red}}$, the element $f(x_\alpha)x_\alpha^{-1}$ is contained in the subgroup $\langle U_\beta \mid \beta > \alpha\rangle$ of U_P generated by the U_β for $\beta \in \Psi_{\mathrm{red}}$ with $\beta > \alpha$.*

Proof. Write $g_U = u_{\alpha_1} \cdots u_{\alpha_r}$ for certain $u_{\alpha_i} \in U_{\alpha_i}$ and $\alpha_1, \ldots, \alpha_r \in \Psi_{\mathrm{red}}$. We prove the assertion by induction on r.

As $(Z, (U_\alpha)_{\alpha\in\Phi})$ is a root group datum, it satisfies [BT72, (6.1.1), (DR 2)]:

(DR 2) For each $\alpha, \beta \in \Phi$ the commutator subgroup $[U_\alpha, U_\beta]$ is contained in the group generated by the $U_{r\alpha+s\beta}$, for $r, s \in \mathbb{Z}_{>0}$ with $r\alpha + s\beta \in \Phi$.

The base case $r = 1$ is clear from (DR 2). Now, assume $r > 1$ and

$$y := (u_{\alpha_1} \cdots u_{\alpha_{r-1}})^{-1} \cdot x_\alpha \cdot (u_{\alpha_1} \cdots u_{\alpha_{r-1}}) \cdot x_\alpha^{-1} \in \langle U_\beta \mid \beta > \alpha\rangle.$$

Again from (DR 2) we know $y^{u_{\alpha_r}} \in \langle U_\beta \mid \beta > \alpha\rangle$. Therefore, $g_U^{-1} x_\alpha g_U x_\alpha^{-1} = y^{u_{\alpha_r}} \cdot [u_{\alpha_r}^{-1}, x_\alpha]$ is contained in $\langle U_\beta \mid \beta > \alpha\rangle$, proving the assertion. ☐

Lemma 3.6 allows us to apply Lemma 3.2: for each $(x_\alpha)_\alpha \in \prod_{\alpha\in\Psi_{\mathrm{red}}} U_\alpha$ there exist an element $z_\alpha(x_\beta)_{\beta<\alpha} \in U_\alpha$ (depending only on $(x_\beta)_{\beta<\alpha}$) and a group homomorphism $\widetilde{z}_\alpha\colon U_\alpha \to U_{2\alpha}$, factoring through $U_\alpha/U_{2\alpha}$ and independent of $(x_\beta)_{\beta<\alpha}$, such that

$$f\Big(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha\Big) = \prod_{\alpha\in\Psi_{\mathrm{red}}} z_\alpha(x_\beta)_{\beta<\alpha} \cdot \widetilde{z}_\alpha(x_\alpha) \cdot x_\alpha. \tag{23}$$

Identify Ψ_{red} with $\Sigma^+ \smallsetminus \Sigma_M$. For $\alpha \in \Psi_{\mathrm{red}}$ we consider the homomorphism

$$f_\alpha\colon U_{(\alpha,0)}^{g_M} \longrightarrow U_\alpha, \quad x \longmapsto \widetilde{z}_\alpha(x) \cdot x.$$

Remark. Observe that f_α is the identity if the root system Φ is reduced (*e.g.*, if $\mathbf{G}$ is F-split). In this case Lemmas 3.7 and 3.8 are trivial.

Lemma 3.7. *The image of f_α is open in U_α.*

Proof. Put $\Psi' := \{\beta \in \Psi_{\mathrm{red}} \mid \beta \not< \alpha\}$ and let $Z_\alpha \subseteq U_P$ be the subgroup generated by the U_β with $\beta \in \Psi'$. Then Z_α is normal in U_P by (DR 2), and the multiplication map induces a homeomorphism $\prod_{\beta\in\Psi'} U_\beta \cong Z_\alpha$. The projection map $\mathrm{pr}_\alpha\colon Z_\alpha \to U_\alpha$ and the automorphism $f' := f\big|_{Z_\alpha}$, induced by the inner automorphism f, are open. Hence, the subset

$$f_\alpha\big(U_{(\alpha,0)}^{g_M}\big) = (\mathrm{pr}_\alpha \circ f')\Big(U_{(\alpha,0)}^{g_M} \times \prod_{\beta\in\Psi'\smallsetminus\{\alpha\}} U_{(\beta,0)}\Big) \subseteq U_\alpha$$

is open. ☐

Lemma 3.8. $\big[U_{(\alpha,0)} : f_\alpha\big(U_{(\alpha,0)}^{g_M}\big) \cap U_{(\alpha,0)}\big] \geq \big[U_{(\alpha,0)} : (U_{(\alpha,0)})_{(g_M)}\big]$

Proof. For each subgroup X of U_α we have $(U_{2\alpha} \cap X, X) \subseteq (U_{2\alpha}, U_\alpha)$ in **FGrp**. In order to simplify the notation we write X instead of $(U_{2\alpha} \cap X, X)$.

Step 1. We show $\mathrm{gr}_0\big(f_\alpha\big(U_{(\alpha,0)}^{g_M}\big) \cap U_{(\alpha,0)}\big) = \mathrm{gr}_0\big((U_{(\alpha,0)})_{(g_M)}\big)$, *i.e.*,

$$f_\alpha\big(U_{(\alpha,0)}^{g_M}\big) \cap U_{(\alpha,0)} \cap U_{2\alpha} = (U_{(\alpha,0)})_{(g_M)} \cap U_{2\alpha}. \tag{24}$$

Take $x \in U_{(\alpha,0)}$ with $f_\alpha(x^{g_M}) = \widetilde{z}_\alpha(x^{g_M}) \cdot x^{g_M} \in U_{(\alpha,0)} \cap U_{2\alpha}$. As $\widetilde{z}_\alpha$ takes values in $U_{2\alpha}$, we must have $x^{g_M} \in U_{2\alpha}$. As $\widetilde{z}_\alpha$ vanishes on $U_{2\alpha}$ we deduce $f_\alpha(x^{g_M}) = x^{g_M}$ which is contained in the right hand side of (24). Conversely, given $x \in U_{(\alpha,0)}$ with $x^{g_M} \in U_{(\alpha,0)} \cap U_{2\alpha}$, we have $x^{g_M} = f_\alpha(x^{g_M})$, which is contained in the left hand side of (24).

Step 2. We prove $\mathrm{gr}_1\big(f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}\big) \subseteq \mathrm{gr}_1\big((U_{(\alpha,0)})_{(g_M)}\big)$. We first show

$$\big(f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}\big) \cdot U_{2\alpha} \subseteq U^{g_M}_{(\alpha,0)} U_{2\alpha} \cap U_{(\alpha,0)} U_{2\alpha} = (U_{(\alpha,0)})_{(g_M)} \cdot U_{2\alpha}. \tag{25}$$

The inclusion is a consequence of $f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) U_{2\alpha} = U^{g_M}_{(\alpha,0)} U_{2\alpha}$ and the equality follows from (22). We compute

$$\begin{aligned}
&\mathrm{gr}_1\big(f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}\big)\\
&\quad = \frac{f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}}{f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)} \cap U_{2\alpha}} = \frac{\big(f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}\big) \cdot U_{2\alpha}}{U_{2\alpha}}\\
&\quad \overset{(25)}{\subseteq} \frac{(U_{(\alpha,0)})_{(g_M)} \cdot U_{2\alpha}}{U_{2\alpha}} = \frac{(U_{(\alpha,0)})_{(g_M)}}{(U_{(\alpha,0)})_{(g_M)} \cap U_{2\alpha}} = \mathrm{gr}_1\big((U_{(\alpha,0)})_{(g_M)}\big).
\end{aligned}$$

Step 3. Proof of the assertion. Steps 1 and 2 imply $\mathrm{gr}\big(f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}\big) \subseteq \mathrm{gr}\big((U_{(\alpha,0)})_{(g_M)}\big)$. The index of $f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}$ in $U_{(\alpha,0)}$ is finite by Lemma 3.7. Applying Lemma 3.5.(b) twice finally shows

$$\begin{aligned}
\big[U_{(\alpha,0)} : f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}\big] &= \big[\mathrm{gr}(U_{(\alpha,0)}) : \mathrm{gr}\big(f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}\big)\big]\\
&\geq \big[\mathrm{gr}(U_{(\alpha,0)}) : \mathrm{gr}\big((U_{(\alpha,0)})_{(g_M)}\big)\big]\\
&= \big[U_{(\alpha,0)} : (U_{(\alpha,0)})_{(g_M)}\big]. \qquad \square
\end{aligned}$$

Given $\alpha \in \Psi_{\mathrm{red}}$ and $(x_\beta)_{\beta<\alpha} \in \prod_{\substack{\beta \in \Psi_{\mathrm{red}} \\ \beta < \alpha}} U_\beta$, we consider the subset

$$X_{(x_\beta)_{\beta<\alpha}} := U_{(\alpha,0)} \cap \Big\{ z_\alpha\big(x^{g_M}_\beta\big)_{\beta<\alpha} \cdot \widetilde{z}_\alpha\big(x^{g_M}_\alpha\big) \cdot x^{g_M}_\alpha \;\Big|\; x_\alpha \in U_{(\alpha,0)} \Big\} \subseteq U_{(\alpha,0)}.$$

As $(I_{U_P})_{(g)} = I_{U_P} \cap f\big(I^{g_M}_{U_P}\big)$ it is immediate from (23) that

$$(I_{U_P})_{(g)} = \left\{ \prod_{\alpha \in \Psi_{\mathrm{red}}} y_\alpha = f\Big(\prod_{\alpha \in \Psi_{\mathrm{red}}} x^{g_M}_\alpha\Big) \in \prod_{\alpha \in \Psi_{\mathrm{red}}} U_{(\alpha,0)} \;\middle|\; \begin{array}{l} x_\alpha \in U_{(\alpha,0)} \text{ and} \\ y_\alpha \in X_{(x_\beta)_{\beta<\alpha}} \\ \text{for all } \alpha \in \Psi_{\mathrm{red}} \end{array} \right\}. \tag{26}$$

Lemma 3.9. *The set $X_{(x_\beta)_{\beta<\alpha}}$ is either empty or a left coset of $f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}$.*

Proof. Clearly, $X_{(x_\beta)_{\beta<\alpha}}$ is stable under right multiplication by elements of $f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}$. Now, take two elements

$$\gamma_i := z_\alpha\big(x^{g_M}_\beta\big)_{\beta<\alpha} \cdot \widetilde{z}_\alpha\big(x^{g_M}_i\big) \cdot x^{g_M}_i \in X_{(x_\beta)_{\beta<\alpha}}, \quad \text{with } x_i \in U_{(\alpha,0)},\ i = 1,2.$$

Then $\gamma_2^{-1}\gamma_1 = f_\alpha\big((x_2^{-1}x_1)^{g_M}\big) \in f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)}$ proving the claim. $\square$

Choose $r \in \mathbb{Z}_{>0}$ big enough so that $U_{(\alpha,r)}$ is contained in $f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) \cap U_{(\alpha,0)} \cap U^{g_M}_{(\alpha,0)}$ for all $\alpha \in \Psi_{\mathrm{red}}$ and such that the subgroup $H := \prod_{\alpha \in \Psi_{\mathrm{red}}} U_{(\alpha,r)}$ of I_{U_P} is contained in $(I_{U_P})_{(g)} \cap (I_{U_P})_{(g_M)}$. Observe that H is normal, since the valuation φ_0 satisfies [BT72, (6.2.1), (V 3)].

We will apply Lemma 3.3 next with X, Y, Z_α equal to I_{U_P}/H and $(I_{U_P})_{(g)}/H$ and $f_\alpha(U_{(\alpha,0)}^{g_M}) \cap U_{(\alpha,0)}/U_{(\alpha,r)}$, respectively. If $\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha \in \prod_{\alpha\in\Psi_{\mathrm{red}}} U_{(\alpha,0)}$ and $\prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha = f(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha^{g_M})$, then each x_α, and hence $(x_\beta)_{\beta<\alpha}$, depends only on $(y_\beta)_{\beta<\alpha}$ (apply the Lemma 3.2 to f^{-1}). Therefore, $X_{(x_\beta)_{\beta<\alpha}}$ depends only on $(y_\beta)_{\beta<\alpha}$ and, in particular, on $(y_1,\dots,y_{o(\alpha)-1})$ by our choice of o. In fact, $X_{(x_\beta)_{\beta<\alpha}}$ depends only on the cosets $y_\beta U_{(\beta,r)}$, for $\beta<\alpha$, since we assumed $U_{(\beta,r)} \subseteq f_\beta\big(U_{(\beta,0)}^{g_M}\big) \cap U_{(\beta,0)}$. Now, (26) and Lemma 3.9 show that condition (3.3) is satisfied. Lemma 3.3 implies

$$\big|(I_{U_P})_{(g)}/H\big| \le \prod_{\alpha\in\Psi_{\mathrm{red}}} \big|f_\alpha(U_{(\alpha,0)}^{g_M}) \cap U_{(\alpha,0)}/U_{(\alpha,r)}\big|. \tag{27}$$

Further, using Lemma 3.8 we estimate

$$\begin{aligned}\big|f_\alpha\big(U_{(\alpha,0)}^{g_M}\big) \cap U_{(\alpha,0)}/U_{(\alpha,r)}\big| &= \frac{|U_{(\alpha,0)}/U_{(\alpha,r)}|}{[U_{(\alpha,0)} : f_\alpha\big(U_{(\alpha,0)}^{g_M}\big) \cap U_{(\alpha,0)}]} \\ &\le \frac{|U_{(\alpha,0)}/U_{(\alpha,r)}|}{[U_{(\alpha,0)} : (U_{(\alpha,0)})_{(g_M)}]} = \big|(U_{(\alpha,0)})_{(g_M)}/U_{(\alpha,r)}\big|.\end{aligned} \tag{28}$$

Putting (27) and (28) together yields

$$\big|(I_{U_P})_{(g)}/H\big| \le \prod_{\alpha\in\Psi_{\mathrm{red}}} \big|(U_{(\alpha,0)})_{(g_M)}/U_{(\alpha,r)}\big| = \big|(I_{U_P})_{(g_M)}/H\big|.$$

Finally, we conclude $\mu_{U_P}(g) = \frac{|I_{U_P}/H|}{|(I_{U_P})_{(g)}/H|} \ge \frac{|I_{U_P}/H|}{|(I_{U_P})_{(g_M)}/H|} = \mu_{U_P}(g_M)$, finishing the proof of Proposition 3.4.

3.4. Properties of $\mu_{U_P}(w)$.

Notation. As $\mu_{U_P}\colon M \to q^{\mathbb{Z}_{\ge 0}}$ is constant on double cosets with respect to K_M (hence also I_M), we obtain from the Bruhat decomposition (9) an induced function

$$\mu_{U_P}\colon W_M \longrightarrow q^{\mathbb{Z}_{\ge 0}}. \tag{29}$$

As K_M contains representatives of $W_{0,M}$, it follows that μ_{U_P} is constant on the double cosets with respect to $W_{0,M}$.

The analogous map $\mu_{U_P}\colon W_M(1) \to q^{\mathbb{Z}_{\ge 0}}$ is obtained from (29) by inflation. It is clear that all results in this section are still true if we replace W_M by $W_M(1)$.

Our goal in this section will be to study the properties of the function μ_{U_P}.

Lemma 3.10. *Let $\lambda\in\Lambda$ and $w_0\in W_{0,M}$. Then*

$$\mu_{U_P}(e^\lambda w_0) = \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(\lambda)\rangle>0}} \big|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(\lambda)\rangle)}\big|.$$

Proof. Let $m\in Z$ be a representative of $\lambda\in\Lambda$. Then the multiplication map induces an m-equivariant bijection $I_{U_P} \cong \prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} U_{(\alpha,0)}$, and we compute

$$\begin{aligned}(I_{U_P})_{(m)} = I_{U_P} \cap m^{-1} I_{U_P} m &\cong \prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} U_{(\alpha,0)} \cap m^{-1} U_{(\alpha,0)} m \\ &= \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(\lambda)\rangle\le 0}} U_{(\alpha,0)} \times \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(\lambda)\rangle>0}} U_{(\alpha,\langle\alpha,\nu(\lambda)\rangle)}.\end{aligned}$$

But then

$$\mu_{U_P}(e^\lambda w_0) = \mu_{U_P}(e^\lambda) = [I_{U_P} : (I_{U_P})_{(m)}] = \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(\lambda)\rangle>0}} \left|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(\lambda)\rangle)}\right|. \qquad \square$$

Definition. Given $(\alpha, k) \in \Sigma^{\mathrm{aff}} = \Sigma \times \mathbb{Z}$, we put (cf. (11))

$$q(\alpha, k) := q(H_{(\alpha,k)}) = \left|U_{(\alpha,k)}/U_{(\alpha,k+1)}\right| \in q^{\mathbb{Z}_{>0}}.$$

Notice that $q(\alpha, k) = q\big(w \cdot (\alpha, k)\big)$ by (6). In particular, $q(\alpha, k) = q(-\alpha, k)$ (take $w = s_\alpha$) and $q(\alpha, k) = q(\alpha, k + \langle\alpha, \nu(\lambda)\rangle)$ (take $w = e^{-\lambda} \in \Lambda$).

3.4.1. *The opposite parabolic.* Let $\mathbf{P}^{\mathrm{op}}$ be the parabolic subgroup opposite $\mathbf{P}$ with Levi $\mathbf{M}$ and unipotent radical $\mathbf{U}_{\mathbf{P}^{\mathrm{op}}}$. Writing $I_{U_{P^{\mathrm{op}}}} = I \cap U_{P^{\mathrm{op}}}$, we define

$$\mu_{U_{P^{\mathrm{op}}}}(g) := [I_{U_{P^{\mathrm{op}}}} : (I_{U_{P^{\mathrm{op}}}})_{(g)}], \qquad \text{for } g \in P.$$

It is constant on double cosets with respect to K_M, hence restricts to a function $\mu_{U_{P^{\mathrm{op}}}} : W_M \to q^{\mathbb{Z}_{\geq 0}}$. Proposition 3.11 explains how $\mu_{U_{P^{\mathrm{op}}}}$ is related with μ_{U_P}.

Proposition 3.11. *$\mu_{U_{P^{\mathrm{op}}}}(w) = \mu_{U_P}(w^{-1})$ for all $w \in W_M$.*

Proof. We may assume $w = e^\lambda \in \Lambda$. Lemma 3.10 implies

$$\mu_{U_P}(-\lambda) = \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(-\lambda)\rangle>0}} \prod_{k=0}^{\langle\alpha,\nu(-\lambda)\rangle-1} q(\alpha, k).$$

A similar argument shows

$$\mu_{U_{P^{\mathrm{op}}}}(\lambda) = \prod_{\substack{\alpha\in(-\Sigma^+)\smallsetminus\Sigma_M \\ \langle\alpha,\nu(\lambda)\rangle>0}} \left|U_{(\alpha,1)}/U_{(\alpha,\langle\alpha,\nu(\lambda)\rangle+1)}\right| = \prod_{\substack{\alpha\in(-\Sigma^+)\smallsetminus\Sigma_M \\ \langle\alpha,\nu(\lambda)\rangle>0}} \prod_{k=1}^{\langle\alpha,\nu(\lambda)\rangle} q(\alpha, k).$$

Using $\langle-\alpha, \nu(-\lambda)\rangle = \langle\alpha, \nu(\lambda)\rangle$, $q(-\alpha, k) = q(\alpha, k)$, and $q(\alpha, 0) = q(\alpha, \langle\alpha, \nu(\lambda)\rangle)$, we obtain $\mu_{U_P}(-\lambda) = \mu_{U_{P^{\mathrm{op}}}}(\lambda)$. $\square$

3.4.2. *Changing the parabolic subgroup.* Let $\mathbf{Q} = \mathbf{L}\mathbf{U}_{\mathbf{Q}}$ be a parabolic subgroup of $\mathbf{G}$ with $\mathbf{M} \subseteq \mathbf{L}$. Then $\mathbf{P} \cap \mathbf{L}$ is a parabolic subgroup of $\mathbf{L}$ with Levi $\mathbf{M}$ and unipotent radical $\mathbf{U}_{\mathbf{P}} \cap \mathbf{L}$.

Proposition 3.12. *$\mu_{U_P}(w) = \mu_{U_P\cap L}(w) \cdot \mu_{U_Q}(w)$ for all $w \in W_M$.*

Proof. It suffices to prove the assertion for $w \in \Lambda$. Using Lemma 3.10 we obtain

$$\begin{aligned} \mu_{U_P}(w) &= \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(w)\rangle>0}} \left|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(w)\rangle)}\right| \\ &= \prod_{\substack{\alpha\in\Sigma_L^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(w)\rangle>0}} \left|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(w)\rangle)}\right| \cdot \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_L \\ \langle\alpha,\nu(w)\rangle>0}} \left|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(w)\rangle)}\right| \\ &= \mu_{U_P\cap L}(w) \cdot \mu_{U_Q}(w). \end{aligned} \qquad \square$$

3.4.3. *Relating μ_{U_P} with the structure of W_M.* The fact that μ_{U_P} can be defined on W_M may seem rather coincidental. There is, however, a strong relationship between the two. To begin with, the function

$$\delta_M \colon M \longrightarrow \mathbb{Q}^\times, \qquad m \longmapsto \mu_{U_P}(m)/\mu_{U_P}(m^{-1}) \tag{30}$$

is easily seen to be a group homomorphism by showing $\delta_M(m) = [mI_{U_P}m^{-1} : I_{U_P}]$ (generalized index; see [Vig96, I.2.7] for various properties), and this clearly induces a group homomorphism $\delta_M \colon W_M \to \mathbb{Q}^\times$. One can say more:

Lemma 3.13. *δ_M is trivial on W^{aff}, hence factors through a character $\Omega_M \to \mathbb{Q}^\times$.*

Proof. Given $s \in S^{\mathrm{aff}}$, we have $s = s^{-1}$. As δ_M is a group homomorphism taking only positive values, this implies $\delta_M(s) = 1$. As W^{aff} is generated by S^{aff}, the assertion follows. ☐

Thus, μ_{U_P} carries (at least some) information about the group structure of W_M. Our main result in this section shows that μ_{U_P} measures the deviation between the length functions ℓ on W and ℓ_M on W_M, and that it is monotone with respect to the Bruhat order $\leq_M$ on W_M.

Proposition 3.14. *Let $v, w \in W_M$. Then:*

(a) $q_w = \mu_{U_P}(w)\mu_{U_P}(w^{-1}) \cdot q_{M,w}$*;*

(b) $\mu_{U_P}(v) \leq \mu_{U_P}(w)$ *whenever* $v \leq_M w$*.*

Before we give the proof, we deduce a simple corollary:

Corollary 3.15. *$\mu_{U_P}(vw) \leq \mu_{U_P}(v)\mu_{U_P}(w)$ and $q_{v,w} = \frac{\mu_{U_P}(v)\mu_{U_P}(w)}{\mu_{U_P}(vw)} \cdot q_{M,v,w}$ for all $v, w \in W_M$.*

Proof. As $\delta_M \colon W_M \to \mathbb{Q}^\times$ is a group homomorphism, we have $\frac{\mu_{U_P}(v)\mu_{U_P}(w)}{\mu_{U_P}(vw)} = \frac{\mu_{U_P}(v^{-1})\mu_{U_P}(w^{-1})}{\mu_{U_P}(w^{-1}v^{-1})}$. Hence, Proposition 3.14.(a) implies

$$\begin{aligned} q_{v,w}^2 = \frac{q_v q_w}{q_{vw}} &= \frac{\mu_{U_P}(v)\mu_{U_P}(v^{-1}) \cdot \mu_{U_P}(w)\mu_{U_P}(w^{-1})}{\mu_{U_P}(vw)\mu_{U_P}(w^{-1}v^{-1})} \cdot \frac{q_{M,v} \cdot q_{M,w}}{q_{M,vw}} \\ &= \left(\frac{\mu_{U_P}(v) \cdot \mu_{U_P}(w)}{\mu_{U_P}(vw)} \cdot q_{M,v,w} \right)^2 . \end{aligned}$$

Taking the (positive) square root implies the second assertion. Since $\mathfrak{H}_{M,w} = \mathfrak{H}_w \cap \mathfrak{H}_M$ for all $w \in W$ (see (12) for the definition of $\mathfrak{H}_w$), we have $\mathfrak{H}_{M,v} \cap v\mathfrak{H}_{M,w} \subseteq \mathfrak{H}_v \cap v\mathfrak{H}_w$. Therefore $q_{M,v,w}$ divides $q_{v,w}$, and the first assertion follows from the second. ☐

Proof of Proposition 3.14.

(a) Under the inclusion $\mathfrak{H}_M \subseteq \mathfrak{H}$ we have

$$\mathfrak{H}_M = \left\{ H_{(\alpha,k)} \in \mathfrak{H} \,\middle|\, (\alpha, k) \in \Sigma_M^{\mathrm{aff}} \right\}.$$

Take $w = e^\lambda w_0$, with $\lambda \in \Lambda$ and $w_0 \in W_{0,M}$. By [Vig16, Lemma 5.6 and Proposition 5.9, 2)] (using $w_0(\Sigma^+ \smallsetminus \Sigma_M) = \Sigma^+ \smallsetminus \Sigma_M$) there is a decomposition

$$\mathfrak{H}_w = \mathfrak{H}_w^+ \sqcup \mathfrak{H}_w^- \sqcup \mathfrak{H}_{M,w},$$

where

$$\mathfrak{H}_w^+ := \left\{ H_{(\alpha,k)} \;\middle|\; \begin{array}{l} \alpha \in \Sigma^+ \smallsetminus \Sigma_M \text{ with } \langle \alpha, \nu(\lambda) \rangle > 0 \\ \text{and } k \in \{-\langle \alpha, \nu(\lambda)\rangle, \dots, -1\} \end{array} \right\} \qquad \text{and}$$

$$\mathfrak{H}_w^- := \left\{ H_{(\alpha,k)} \;\middle|\; \begin{array}{l} \alpha \in \Sigma^+ \smallsetminus \Sigma_M \text{ with } \langle \alpha, \nu(\lambda) \rangle < 0 \\ \text{and } k \in \{0, \dots, -\langle \alpha, \nu(\lambda)\rangle - 1\} \end{array} \right\}.$$

Thanks to Lemma 3.10 we compute

$$\prod_{H \in \mathfrak{H}_w^-} q(H) = \prod_{\substack{\alpha \in \Sigma^+ \smallsetminus \Sigma_M \\ \langle \alpha, \nu(\lambda)\rangle < 0}} \prod_{k=0}^{-\langle \alpha, \nu(\lambda)\rangle - 1} q(\alpha, k) = \mu_{U_P}(-\lambda) = \mu_{U_P}(w^{-1}).$$

Likewise, a short computation reveals $\prod_{H \in \mathfrak{H}_w^+} q(H) = \mu_{U_P}(w)$ keeping in mind $q(\alpha, k) = q(\alpha, k + \langle \alpha, \nu(\lambda)\rangle)$. Taking everything together, the assertion follows from (12) applied to both q_w and $q_{M,w}$.

(b) Let $v, w \in W_M$ with $v \leq_M w$. By definition of the Bruhat order, *e.g.*, [BB06, Definition 2.1.1], there exists a chain $v = v_0, v_1, \dots, v_k = w$ in W_M with $v_i v_{i-1}^{-1} \in S(\mathfrak{H}_M)$ ($\iff v_{i-1}^{-1} v_i \in S(\mathfrak{H}_M)$) and $\ell_M(v_{i-1}) < \ell_M(v_i)$ for all $1 \leq i \leq k$.

We are therefore reduced to the case $w = tv$ and $\ell_M(v) < \ell_M(w)$ for some $t \in S(\mathfrak{H}_M)$. In fact, by [BB06, Theorem 2.2.6], we may even assume $\ell_M(w) = \ell_M(v) + 1$.

Step 1. We show $\mu_{U_P}(tw)\mu_{U_P}\big((tw)^{-1}\big) \leq \mu_{U_P}(w)\mu_{U_P}(w^{-1})$. The hypothesis $\ell_M(tw) = \ell_M(w) - 1$ implies $H_t \in \mathfrak{H}_{M,w} \smallsetminus \mathfrak{H}_{M,tw}$. But since $\mathfrak{H}_{M,w'} = \mathfrak{H}_{w'} \cap \mathfrak{H}_M$, for all $w' \in W_M$, we also have $H_t \in \mathfrak{H}_w \smallsetminus \mathfrak{H}_{tw}$. Let Γ_w be a minimal gallery in $\mathscr{A}$ connecting $\mathfrak{C}$ and $w\mathfrak{C}$. Denote $(H_1, H_2, \dots, H_{\ell(w)})$ the sequence of hyperplanes crossed by Γ_w. Then $H_t = H_i$ for some $1 \leq i \leq \ell(w)$. By folding Γ_w along H_t and deleting the repeated chamber we obtain a gallery Γ_{tw} of length $\ell(w) - 1$ in $\mathscr{A}$ connecting $\mathfrak{C}$ and $tw\mathfrak{C}$, given by crossing the hyperplanes

$$(H_1, \dots, H_{i-1}, tH_{i+1}, \dots, tH_{\ell(w)}).$$

Of course, Γ_{tw} crosses all hyperplanes in $\mathfrak{H}_{tw}$ at least once. Applying (a) twice, we compute

$$\begin{aligned} \mu_{U_P}(w)\mu_{U_P}(w^{-1})q_{M,w} = q_w &= \prod_{j=1}^{\ell(w)} q(H_j) \\ &= \prod_{j=1}^{i-1} q(H_j) \cdot \prod_{j=i+1}^{\ell(w)} q(tH_j) \cdot q(H_t) \\ &\geq q_{tw} \cdot q(H_t) \\ &= \mu_{U_P}(tw)\mu_{U_P}\big((tw)^{-1}\big) \cdot q_{M,tw} \cdot q(H_t). \end{aligned}$$

Since $\ell_M(w) = \ell_M(tw) + 1$, we have $q_{M,w} = q_{M,tw} \cdot q(H_t)$, finishing the proof of Step 1.

Step 2. We show $\mu_{U_P}(tw) \leq \mu_{U_P}(w)$. By Lemma 3.13 we have $\delta_M(tw) = \delta_M(w)$. Hence,

$$\begin{aligned}\mu_{U_P}(tw)^2 &= \mu_{U_P}(tw)\mu_{U_P}\big((tw)^{-1}\big) \cdot \delta_M(tw)\\ &\leq \mu_{U_P}(w)\mu_{U_P}(w^{-1}) \cdot \delta_M(w) = \mu_{U_P}(w)^2\end{aligned}$$

using Step 1. Taking square roots, the assertion follows. ☐

3.5. **Positive elements.** We collect here some results on positive elements in the Levi subgroup M which will become important for the definition of parabolic induction. The results in this section are not new, but we will give new proofs of two results obtained by Abe in [Abe16]

Definition. An element $m \in M$ is called *M-positive* (or just *positive* if no confusion arises) if $\mu_{U_P}(m) = 1$. The monoid of positive elements is denoted M^+ (or even $M^{+,G}$ if we want to stress that M is considered as a Levi subgroup in G). Notice that $K_M \subseteq M^+$.

The elements in the monoid $M^- := (M^+)^{-1}$ are called *M-negative* (or *negative*).

Remark. In view of $\mu_{U_{P^{\mathrm{op}}}}(m^{-1}) = \mu_{U_P}(m)$ (Proposition 3.11), m being positive is equivalent to

$$mI_{U_P}m^{-1} \subseteq I_{U_P} \quad \text{and} \quad I_{U_{P^{\mathrm{op}}}} \subseteq mI_{U_{P^{\mathrm{op}}}}m^{-1},$$

where $I_{U_{P^{\mathrm{op}}}} = I \cap U_{P^{\mathrm{op}}}$. This recovers the classical definition, cf. [BK98, (6.5)] or [Vig98, II.4].

Definition.

(a) An element $w \in W_M$ (or in $W_M(1)$) is called *M-positive* (or *positive*) if $\mu_{U_P}(w) = 1$. The monoid of positive elements in W_M is denoted W_{M^+}.

The elements in $W_{M^-} := (W_{M^+})^{-1}$ are called *M-negative* (or *negative*).

(b) An element $\lambda \in \Lambda$ (or in $\Lambda(1)$) is called *strictly M-positive* (or *strictly positive*) if there exists a central element $a \in M$ with $\lambda = aZ_0$ (or $\lambda = aZ_1$) and if

$$\langle \alpha, \nu(\lambda) \rangle < 0 \quad \text{for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_M.$$

(c) A central element $a \in M$ is called *strictly M-positive* (or *strictly positive*) if $aZ_0 \in \Lambda$ is.

Remark.

(a) If $\lambda \in \Lambda$ is strictly positive, then $\langle \alpha, \nu(\lambda) \rangle = 0$ for all $\alpha \in \Sigma_M$, because $\lambda = aZ_0$ for some element a in the center of M. The strictly positive elements are contained in $\Lambda_{M^+} := \Lambda \cap W_{M^+}$ (*e.g.*, by Lemma 3.10).

(b) An element $\lambda \in \Lambda$ is strictly positive if and only if it lies in the image of a strongly positive element a in the sense of [BK98, (6.16)]. In particular, [BK98, (6.14)], strictly positive elements exist. Moreover, given any $m \in M$ we have $a^n m \in M^+$ for $n \gg 0$.

(c) The Bruhat decomposition of M (9) induces bijections $W_{M^+} \cong I_M \backslash M^+ / I_M$ and $W_{M^+}(1) \cong I_{1,M} \backslash M^+ / I_{1,M}$ [OV18, Remark 2.11(2)].

(d) We have $W_{M^+} \cong \Lambda_{M^+} \rtimes W_{0,M}$ and [Vig15, Lemma 2.2]

$$\Lambda_{M^+} = \left\{ \lambda \in \Lambda \,\middle|\, \langle \alpha, \nu(\lambda) \rangle \leq 0 \quad \text{for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_M \right\}.$$

It follows that if $\lambda \in \Lambda$ (or in $\Lambda(1)$) is strictly positive, then for all $w \in W_M$ (or in $W_M(1)$) there exists $n \gg 0$ with $e^{n\lambda} w \in W_{M^+}$ (or in $W_{M^+}(1)$).

Finally, we give new proofs of two useful lemmas due to Abe. They will not be needed in the rest of the paper.

Corollary 3.16 ([Abe16, Lemma 4.5]). *Let either* $v, w \in W_{M^+}$ *or* $v, w \in W_{M^-}$. *Then we have* $q_{v,w} = q_{M,v,w}$ *and, in particular,*

$$\ell_M(v) + \ell_M(w) - \ell_M(vw) = \ell(v) + \ell(w) - \ell(vw).$$

Proof. If $v, w \in W_{M^+}$ then Corollary 3.15 yields $q_{v,w} = q_{M,v,w}$ (and also $vw \in W_{M^+}$). If, on the other hand, $v, w \in W_{M^-}$ then $v^{-1}, w^{-1}, (vw)^{-1} \in W_{M^+}$ and $\mu_{U_P}(vw) = \mu_{U_P}(v) \cdot \mu_{U_P}(w)$, because (30) is a group homomorphism. Again, Corollary 3.15 implies $q_{v,w} = q_{M,v,w}$.

Notice that $q_{v,w} = q_{M,v,w}$ implies $\mathfrak{H}_v \cap v\mathfrak{H}_w = \mathfrak{H}_{M,v} \cap v\mathfrak{H}_{M,w}$. Hence, the second assertion follows from $\ell(vw) = \ell(v) + \ell(w) - 2 \cdot |\mathfrak{H}_v \cap v\mathfrak{H}_w|$, [Vig16, Remark 4.18], and a similar formula for $\ell_M(vw)$. ☐

Corollary 3.17 ([Abe16, Lemma 4.1]). *Let* $w \in W_{M^+}$ *(resp.* $w \in W_{M^-}$*) and* $v \in W_M$ *with* $v \leq_M w$. *Then* $v \in W_{M^+}$ *(resp.* $v \in W_{M^-}$*).*

Proof. Replacing w by w^{-1} if necessary, we may assume $w \in W_{M^+}$. Then Proposition 3.14.(b) implies $v \in W_{M^+}$. ☐

4. Parabolic induction

4.1. **Reminder on abstract Hecke algebras.**

We recall briefly the basics of Hecke algebras. A thorough introduction with complete proofs can be found in [AZ95, Chapter 3, §1.2].

Let X be a topological group and $S \subseteq X$ a submonoid containing a compact open subgroup Γ. Then S acts on the right on the free $\mathbb{Z}$-module on the set of right cosets $\Gamma\backslash S$

$$\mathbb{Z}[\Gamma\backslash S] := \bigoplus_{\Gamma g \in \Gamma\backslash S} \mathbb{Z}.(\Gamma g).$$

The $\mathbb{Z}$-module of Γ-invariants

$$H(\Gamma, S) = \mathbb{Z}[\Gamma\backslash S]^\Gamma = \{t \in \mathbb{Z}[\Gamma\backslash S] \,|\, t\gamma = t \quad \text{for all } \gamma \in \Gamma\}$$

is free. The element

$$T_g := T_g^S := \sum_{\Gamma h} (\Gamma h) \in H(\Gamma, S),$$

where the sum ranges over all right cosets contained in the double coset $\Gamma g \Gamma$, depends only on $\Gamma g \Gamma$. Moreover, $(T_g)_{\Gamma g \Gamma \in \Gamma\backslash S/\Gamma}$ forms a $\mathbb{Z}$-basis of $H(\Gamma, S)$.

The isomorphism

$$\mathrm{End}_S\big(\mathbb{Z}[\Gamma\backslash S]\big) \xrightarrow{\cong} H(\Gamma, S), \quad T \longmapsto T\big((\Gamma)\big)$$

endows $H(\Gamma, S)$ with the structure of an associative ring with unit T_1. Concretely, given elements $t = \sum_i a_i \cdot (\Gamma g_i)$ and $t' = \sum_j a'_j \cdot (\Gamma g'_j)$ in $H(\Gamma, S)$, one has

$$t \cdot t' = \sum_{i,j} a_i a'_j \cdot (\Gamma g_i g'_j) \in H(\Gamma, S). \tag{31}$$

Given a commutative unital ring R, the R-algebra

$$H_R(\Gamma, S) := R \otimes H(\Gamma, S)$$

is called the *Hecke algebra* over R associated with (Γ, S).

4.2. **Parabolic induction.** Let $\mathbf{P} = \mathbf{M}\mathbf{U}_{\mathbf{P}}$ be a parabolic F-subgroup of $\mathbf{G}$. Recall the pro-p Iwahori subgroup I_1 of G. Then $I_{1,M} = I_1 \cap M$ is a pro-p Iwahori subgroup of M [HR09, Lemma 4.1.1]. Fix a commutative unital ring R. The algebras

$$\mathcal{H}_R(G) \coloneqq H_R(I_1, G) \qquad \text{and} \qquad \mathcal{H}_R(M) \coloneqq H_R(I_{1,M}, M)$$

are the usual pro-p Iwahori–Hecke R-algebras [Vig16, (1)] of G and M, respectively. We write $T_w \coloneqq T_g$ whenever $w \in W(1)$ corresponds to $I_1 g I_1$ under (9); a similar convention applies to $\mathcal{H}_R(M)$. Then $(T_w)_{w\in W(1)}$ and $(T^M_w)_{w\in W_M(1)}$ are R-bases of $\mathcal{H}_R(G)$ and $\mathcal{H}_R(M)$, respectively.

Definition. The R-algebra $\mathcal{H}_R(P) \coloneqq H_R(I_1 \cap P, P)$ is called the *parabolic pro-p Iwahori–Hecke R-algebra.*

The main goal of this section will be to construct two R-algebra morphisms

$$\begin{tikzcd} & \mathcal{H}_R(P) \arrow[dl, "\Theta^P_M"'] \arrow[dr, "\Xi^P_G"] & \\ \mathcal{H}_R(M) & & \mathcal{H}_R(G). \end{tikzcd}$$

Pulling back along Θ^P_M and extending scalars along Ξ^P_G then defines a functor

$$\text{Mod-}\mathcal{H}_R(M) \longrightarrow \text{Mod-}\mathcal{H}_R(G), \qquad \mathfrak{m} \longmapsto \mathfrak{m} \otimes_{\mathcal{H}_R(P)} \mathcal{H}_R(G) \tag{32}$$

from the category of right $\mathcal{H}_R(M)$-modules to the category of right $\mathcal{H}_R(G)$-modules. We then go on to prove that (32) is naturally isomorphic to the parabolic induction functor studied in [OV18, (4.2)] and [Vig15].

4.2.1. *The positive subalgebra.* Recall the monoid M^+ of M-positive elements. The algebra

$$\mathcal{H}_R(M^+) \coloneqq H_R(I_{1,M}, M^+)$$

is called the *positive subalgebra* of $\mathcal{H}_R(M)$. (The fact that it is indeed a subalgebra of $\mathcal{H}_R(M)$ is clear from the explicit definition of the multiplication.)

We collect two well-known and fundamental properties of $\mathcal{H}_R(M^+)$.

Proposition 4.1 ([Vig98, II.5])**.** *Consider the injective R-linear map*

$$\xi \colon \mathcal{H}_R(M) \longrightarrow \mathcal{H}_R(G), \quad T^M_m \longmapsto T_m.$$

The restriction $\xi^+ \coloneqq \xi\big|_{\mathcal{H}_R(M^+)}$ respects the product.

Remark. Proposition 4.1 relies solely on the Iwahori decomposition of I_1, *i.e.*, we could replace I_1 by any compact open subgroup Γ satisfying $\Gamma = (\Gamma \cap U_{P^{\mathrm{op}}}) \cdot (\Gamma \cap M) \cdot (\Gamma \cap U_P)$. However, Proposition 4.1 fails for groups like K that do not admit an Iwahori decomposition.

Proposition 4.2 ([Vig98, II.6])**.** *The following assertions are equivalent:*

(a) *$\xi^+ \colon \mathcal{H}_R(M^+) \to \mathcal{H}_R(G)$ extends to a morphism $\widetilde{\xi^+} \colon \mathcal{H}_R(M) \to \mathcal{H}_R(G)$ of R-algebras.*

(b) *There exists a strictly positive element $a \in M$ such that T_a is invertible in $\mathcal{H}_R(G)$.*

If one of the assertions holds then T_a is invertible for all strictly positive elements $a \in M$ and $\widetilde{\xi^+}$ is unique.

Proof. If $a \in M$ is strictly positive then T_a^M is central and invertible in $\mathcal{H}_R(M)$. Since $(T_a^M)^n \cdot T_m^M = T_{a^n m}^M \in \mathcal{H}_R(M^+)$, for any $m \in M$ and $n \gg 0$, it follows that $\mathcal{H}_R(M)$ is the localization of $\mathcal{H}_R(M^+)$ at T_a^M. The proposition is a formal consequence of this fact. $\square$

4.2.2. *The morphism Θ_M^P.* Let $\Gamma \subseteq P$ be a compact open subgroup satisfying $\Gamma = \Gamma_M \Gamma_{U_P}$, where $\Gamma_M = \Gamma \cap M$ and $\Gamma_{U_P} = \Gamma \cap U_P$. For example, Γ could be the intersection of P with one of the groups K, I, I_1. Recall the notation $g_M \coloneqq \mathrm{pr}_M(g)$, where $\mathrm{pr}_M \colon P \to M$ is the projection.

Proposition 4.3. *The R-linear map*

$$\begin{aligned} \Theta_M^P \coloneqq \Theta_{M,R}^P \colon H_R(\Gamma, P) &\longrightarrow H_R(\Gamma_M, M), \\ T_g^P &\longmapsto \nu_M(g)\mu_{U_P}(g) \cdot T_{g_M}^M \end{aligned}$$

is a homomorphism of R-algebras.

Proof. The projection $\mathrm{pr}_M \colon P \to M$ induces an R-linear map $\vartheta \colon R[\Gamma\backslash P] \to R[\Gamma_M\backslash M]$, given explicitly by $(\Gamma g) \mapsto (\Gamma_M g_M)$. Since $\mathrm{pr}_M(\Gamma) \subseteq \Gamma_M$ and ϑ is right P-linear if we let P act by inflation on $R[\Gamma_M\backslash M]$, it follows that ϑ maps $H_R(\Gamma, P) = R[\Gamma\backslash P]^\Gamma$ into $H_R(\Gamma_M, M) = R[\Gamma_M\backslash M]^{\Gamma_M}$. By restriction we obtain an R-linear map $\theta \colon H_R(\Gamma, P) \to H_R(\Gamma_M, M)$. It is obvious from the explicit description of multiplication (31) that θ respects the product. Therefore, it remains to prove $\theta = \Theta_M^P$. Let $g \in P$. By Proposition 3.1 we have

$$T_g^P = \sum_{i=1}^{\mu_M(g_M)} \sum_{j=1}^{\nu_M(g)} \sum_{s=1}^{\mu_{U_P}(g)} (\Gamma g u_s h_j m_i)$$

in $R[\Gamma\backslash P]$, for certain $u_s \in \Gamma_{U_P}$, $h_j \in (\Gamma_M)_{(g_M)}$, and $m_i \in \Gamma_M$ with $\Gamma_M = \bigsqcup_{i=1}^{\mu_M(g_M)} (\Gamma_M)_{(g_M)} m_i$. Since $g_M h_j \in \Gamma_M g_M$, by definition of h_j, we obtain

$$\theta(T_g^P) = \sum_{i,j,s} (\Gamma_M g_M h_j m_i) = \sum_{i,j,s} (\Gamma_M g_M m_i) = \nu_M(g)\mu_{U_P}(g) \cdot T_{g_M}^M,$$

where the last equality uses $\Gamma_M g_M \Gamma_M = \bigsqcup_{i=1}^{\mu_M(g_M)} \Gamma_M g_M m_i$, cf. [AZ95, Chapter 3, Lemma 1.2]. Hence, θ and Θ_M^P coincide. $\square$

For the next two consequences we assume $\Gamma_{U_P} = I_{U_P}$.

Corollary 4.4. *The system $(\mu_{U_P}(m) T_m^M)_m$, where m runs through a system of representatives of $\Gamma_M\backslash M/\Gamma_M$ with $\mu_{U_P}(m) \neq 0$ in R, generates $\mathrm{Im}(\Theta_{M,R}^P)$ over R. If R is p-torsionfree, then it is an R-basis.*

Proof. Notice that $\nu_M(m) = 1$ whenever $m \in M$. Moreover, we have $\mu_{U_P}(g) \geq \mu_{U_P}(g_M)$, for all $g \in P$, by Proposition 3.4. The assertion now follows from Proposition 4.3. $\square$

Corollary 4.5. *The algebra $H_R(\Gamma_M, M^+)$ is contained in $\mathrm{Im}(\Theta_M^P)$ with equality whenever $qR = 0$.*

Proof. Since Γ_M normalizes $\Gamma_{U_P} = I_{U_P}$, we have $\Gamma_M \subseteq M^+$ and $H_R(\Gamma_M, M^+)$ is defined. Keeping in mind that μ_{U_P} takes values in $q^{\mathbb{Z}_{\geq 0}}$, the assertion follows from Corollary 4.4. $\square$

4.2.3. *The morphism* Ξ_G^P. Lacking a good reference, we formulate Vignéras' "fundamental lemma", a proof of which appears in [Vig05, Lemma 13] (or [Vig06, 1.2]) for F-split $\mathbf{G}$, and which for general $\mathbf{G}$ is known to the experts.

Lemma 4.6 (Fundamental Lemma). *Let* $v, w \in W(1)$. *Then*

$$q_{v,w} \cdot T_v^{-1} T_{vw} - T_w \in \bigoplus_{w' < w} \mathbb{Z}.T_{w'} \qquad \text{in } \mathcal{H}_{\mathbb{Z}}(G),$$

where $<$ *denotes the Bruhat order in* $W(1)$.

Proof. Let o be the orientation of $(\mathscr{A}, \mathfrak{H})$ (cf. [Vig16, 5.2]) which places $\mathfrak{C}$ in the negative half-space of each hyperplane $H \in \mathfrak{H}$. Then $E_o(w) = T_w$ in the notation of [Vig16, Definition 5.22]. Computing inside $\mathcal{H}_{\mathbb{Z}[p^{-1}]}(G)$, we have

$$q_{v,w} T_v^{-1} T_{vw} = q_{v,w} E_o(v)^{-1} E_o(vw) = E_{o \bullet v}(w) \in \mathcal{H}_{\mathbb{Z}}(G),$$

by [Vig16, Theorem 5.25], and the assertion follows from [Vig16, Corollary 5.26]. □

Proposition 4.7. *Assume that R is p-torsionfree. Then* $\xi^+ \colon \mathcal{H}_R(M^+) \to \mathcal{H}_R(G)$ *(see Proposition* 4.1*) extends uniquely to an injective R-algebra morphism*

$$\widetilde{\xi}^+ \colon \operatorname{Im}(\Theta_M^P) \longrightarrow \mathcal{H}_R(G),$$

and $\operatorname{Im}(\Theta_M^P)$ *is the maximal subalgebra of* $\mathcal{H}_R(M)$ *with this property.*

Definition. For arbitrary R we obtain an R-algebra morphism

$$\Xi_G^P \coloneqq \Xi_{G,R}^P \coloneqq \mathrm{id}_R \otimes \big(\widetilde{\xi}^+ \circ \Theta_{M,\mathbb{Z}}^P\big) \colon \mathcal{H}_R(P) \longrightarrow \mathcal{H}_R(G).$$

Proof of Proposition 4.7. We view $\mathcal{H}_R(M)$ (resp. $\mathcal{H}_R(G)$) as a subalgebra of $\mathcal{H}_{R[p^{-1}]}(M)$ (resp. $\mathcal{H}_{R[p^{-1}]}(G)$), which is possible by our assumption on R. Let $a \in M^+$ be a strictly positive element. Then T_a is invertible in $\mathcal{H}_{R[p^{-1}]}(G)$ by [Vig16, Proposition 4.13 1)]. By Proposition 4.2 the map ξ^+ extends uniquely to an $R[p^{-1}]$-algebra morphism

$$\widetilde{\xi}^+ \colon \mathcal{H}_{R[p^{-1}]}(M) \longrightarrow \mathcal{H}_{R[p^{-1}]}(G).$$

Explicitly, it is given as follows: let $m \in M$, and choose $n \in \mathbb{Z}_{>0}$ such that $a^n m$ is M-positive. Then

$$\widetilde{\xi}^+\big(T_m^M\big) = T_a^{-n} \cdot \xi^+\big(T_{a^n m}^M\big) = T_a^{-n} T_{a^n m}. \tag{33}$$

It suffices to prove the following:

Claim. The preimage of $\mathcal{H}_R(G)$ under $\widetilde{\xi}^+$ coincides with $\operatorname{Im}(\Theta_M^P)$.

The claim implies the assertion of the proposition: given any extension $\xi' \colon \mathcal{A} \to \mathcal{H}_R(G)$ of $\xi^+ \colon \mathcal{H}_R(M^+) \to \mathcal{H}_R(G)$, we have $\xi' = \widetilde{\xi}^+\big|_{\mathcal{A}}$ by the uniqueness of $\widetilde{\xi}^+$, and then $\mathcal{A} \subseteq \operatorname{Im}(\Theta_M^P)$ by the claim.

We now prove the claim. By Corollary 4.4 the family $(\mu_{U_P}(w) T_w^M)_{w \in W_M(1)}$ is an R-basis of $\operatorname{Im}(\Theta_M^P)$. As a is strictly positive, so is the element $\lambda \coloneqq a Z_1 \in \Lambda(1)$. Given any $w \in W_M(1)$, there exists $n \in \mathbb{Z}_{>0}$ such that $e^{n\lambda} w \in W_{M^+}(1)$. Now, Corollary 3.15 shows

$$q_{n\lambda,w} = \frac{\mu_{U_P}(n\lambda)\mu_{U_P}(w)}{\mu_{U_P}(e^{n\lambda} w)} \cdot q_{M,n\lambda,w} = \mu_{U_P}(w).$$

The second equality uses $q_{M,n\lambda} \cdot q_{M,w} = q_{M,e^{n\lambda}w}$, that means, $q_{M,n\lambda,w} = 1$, which holds because a lies in the center of M. By Lemma 4.6 we have

$$\begin{aligned}\widetilde{\xi}^{+}\big(\mu_{U_P}(w)T_w^M\big) &= \mu_{U_P}(w) \cdot T_\lambda^{-n} T_{e^{n\lambda}w} = q_{n\lambda,w} \cdot T_{n\lambda}^{-1} \cdot T_{e^{n\lambda}w} \\ &= T_w + \sum_{w'<w} c_{w'} T_{w'} \in \mathcal{H}_R(G),\end{aligned} \tag{34}$$

for certain $c_{w'} \in \mathbb{Z}$, viewed as elements of R. This shows that $\widetilde{\xi}^{+}$ is injective and that $\mathrm{Im}(\Theta_M^P)$ is contained in $(\widetilde{\xi}^{+})^{-1}(\mathcal{H}_R(G))$.

Conversely, let $T = \sum_{i=1}^k x_i \cdot T_{w_i}^M \in \mathcal{H}_{R[p^{-1}]}(M)$ with $x_i \in R[p^{-1}] \smallsetminus \{0\}$ and $\widetilde{\xi}^{+}(T) \in \mathcal{H}_R(G)$. We prove $T \in \mathrm{Im}(\Theta_M^P)$ by induction on k. The case $k = 0$ is trivial. Assume $k > 0$. Rearranging if necessary, we may assume that $w_k \in W_M(1)$ is maximal among $\{w_1, \dots, w_k\}$ with respect to the Bruhat order in $W(1)$. Let $n \in \mathbb{Z}_{>0}$ with $e^{n\lambda}w_i \in W_{M^+}(1)$ for all i. Then $T_{n\lambda}^M \cdot T = \sum_{i=1}^k x_i \cdot T_{e^{n\lambda}w_i}^M$ lies in $\mathcal{H}_{R[p^{-1}]}(M^+)$, and hence

$$\widetilde{\xi}^{+}(T) = \sum_{i=1}^k x_i \cdot T_{n\lambda}^{-1} T_{e^{n\lambda}w_i} = \sum_{i=1}^k x_i q_{n\lambda,w_i}^{-1} \cdot \Big(T_{w_i} + \sum_{w_i'<w_i} c_{w_i'} T_{w_i'}\Big) \in \mathcal{H}_R(G).$$

Again, we have $q_{n\lambda,w_i} = \mu_{U_P}(w_i)$. Hence, maximality of w_k implies $x_k \in R.\mu_{U_P}(w_k)$, whence $x_k T_{w_k} \in \mathrm{Im}(\Theta_M^P)$. By the induction hypothesis we have $T - x_k T_{w_k}^M \in \mathrm{Im}(\Theta_M^P)$. We conclude $T \in \mathrm{Im}(\Theta_M^P)$, finishing the proof. □

We note the following useful consequence of the proof of Proposition 4.7.

Corollary 4.8. *Let $a \in M$ be strictly positive and $g \in P$ arbitrary. Then*

$$T_{a^n} \cdot \Xi_G^P(T_g^P) = \nu_M(g)\mu_{U_P}(g) \cdot T_{a^n g_M}, \qquad \text{in } \mathcal{H}_R(G)$$

whenever $n \in \mathbb{Z}_{>0}$ is such that $a^n g_M \in M^+$.

Proof. The assertion follows by extension of scalars from the case $R = \mathbb{Z}$. Thus, it suffices to prove

$$T_{a^n} \cdot \widetilde{\xi}^{+}\big(\Theta_{M,\mathbb{Z}}^P(T_g^P)\big) = \nu_M(g)\mu_{U_P}(g) \cdot T_{a^n g_M}, \qquad \text{in } \mathcal{H}_{\mathbb{Z}}(G),$$

where the computation takes place in $\mathcal{H}_{\mathbb{Z}[p^{-1}]}(G)$. But this is clear from (33) and the fact that $\Theta_{M,\mathbb{Z}}^P(T_g^P) = \nu_M(g)\mu_{U_P}(g) \cdot T_{g_M}^M$. □

4.2.4. *Equivalence of parabolic inductions.* Having constructed two morphisms $\Theta_M^P \colon \mathcal{H}_R(P) \to \mathcal{H}_R(M)$ and $\Xi_G^P \colon \mathcal{H}_R(P) \to \mathcal{H}_R(G)$, we obtain a functor

$$\text{Mod-}\mathcal{H}_R(M) \longrightarrow \text{Mod-}\mathcal{H}_R(G), \quad \mathfrak{m} \longmapsto \mathfrak{m} \otimes_{\mathcal{H}_R(P)} \mathcal{H}_R(G) \tag{35}$$

from the category of right $\mathcal{H}_R(M)$-modules to the category of right $\mathcal{H}_R(G)$-modules by viewing $\mathfrak{m}$ via Θ_M^P as a right $\mathcal{H}_R(P)$-module and then extending scalars along Ξ_G^P. There is also the parabolic induction, due to [OV18, (4.2)],

$$\text{Mod-}\mathcal{H}_R(M) \longrightarrow \text{Mod-}\mathcal{H}_R(G), \quad \mathfrak{m} \longmapsto \mathfrak{m} \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G), \tag{36}$$

given by viewing $\mathfrak{m}$ as a right $\mathcal{H}_R(M^+)$-module and extending scalars along the R-algebra morphism $\xi^+ \colon \mathcal{H}_R(M^+) \to \mathcal{H}_R(G)$ (Proposition 4.1). Theorem 4.9 is an easy consequence of the construction of Ξ_G^P.

Theorem 4.9. *The functors* (35) *and* (36) *are canonically isomorphic.*

Proof. Let $\mathfrak{n}$ be a right $\mathcal{H}_R(G)$-module and let

$$\rho\colon \mathfrak{m} \times \mathcal{H}_R(G) \longrightarrow \mathfrak{n}$$

be an R-bilinear map satisfying $\rho(m, TT') = \rho(m, T)T'$ for all $m \in \mathfrak{m}$ and $T, T' \in \mathcal{H}_R(G)$. The assertion of the theorem is then tantamount with the equivalence of the following two properties:

(i) $\rho(mT^M, T) = \rho\big(m, \xi^+(T^M)T\big)$ for all $m \in \mathfrak{m}$, $T^M \in \mathcal{H}_R(M^+)$, and $T \in \mathcal{H}_R(G)$.

(ii) $\rho\big(m\Theta_M^P(T^P), T\big) = \rho\big(m, \Xi_G^P(T^P)T\big)$ for all $m \in \mathfrak{m}$, $T^P \in \mathcal{H}_R(P)$, and $T \in \mathcal{H}_R(G)$.

Given $m \in M^+$, we have $\Theta_M^P(T_m^P) = T_m^M$ and $\Xi_G^P(T_m^P) = \xi^+(T_m^M)$. Thus, (ii) implies (i).

Conversely, assume (i) and fix a strictly positive element $a \in M$. Let $g \in P$ and choose $n \in \mathbb{Z}_{>0}$ such that $a^n g_M \in M^+$. Then

$$\begin{aligned}
\rho\big(m \cdot \Theta_M^P(T_g^P), T\big) &= \rho\big(m \cdot \nu_M(g)\mu_{U_P}(g) \cdot T_{g_M}^M, T\big) \\
&= \rho\big(m \cdot (T_{a^n}^M)^{-1} \cdot \nu_M(g)\mu_{U_P}(g) T_{a^n g_M}^M, T\big) \\
&= \rho\big(m \cdot (T_{a^n}^M)^{-1}, \nu_M(g)\mu_{U_P}(g) T_{a^n g_M} \cdot T\big) && \text{(by (i))} \\
&= \rho\big(m \cdot (T_{a^n}^M)^{-1}, T_{a^n} \cdot \Xi_G^P(T_g^P) \cdot T\big) && \text{(by Corollary 4.8)} \\
&= \rho\big(m, \Xi_G^P(T_g^P) \cdot T\big) && \text{(by (i))}
\end{aligned}$$

keeping in mind $\xi^+(T_m^M) = T_m$, for all $m \in M^+$. Hence, (i) implies (ii). □

5. Transitivity of parabolic induction

We observe that only a proper quotient of the parabolic pro-p Iwahori–Hecke algebra $\mathcal{H}_R(P)$ affects the parabolic induction functor: both morphisms Θ_M^P and Ξ_G^P factor through

$$R \otimes \operatorname{Im}(\Theta_{M,\mathbb{Z}}^P).$$

This suggests to study this algebra.

5.1. Definitions and compatibilities.

Definition. We put $\mathcal{H}_R(M, G) \coloneqq R \otimes \operatorname{Im}(\Theta_{M,\mathbb{Z}}^P)$. Given $w \in W_M(1)$, we define

$$\tau_w^{M,G} \coloneqq 1 \otimes \mu_{U_P}(w) T_w^M \in \mathcal{H}_R(M, G).$$

From Corollary 4.4 it follows that $\big(\tau_w^{M,G}\big)_{w \in W_M(1)}$ is an R-basis of $\mathcal{H}_R(M, G)$. Finally, write

$$\begin{aligned}
\theta_M^{M,G}&\colon \mathcal{H}_R(M, G) \longrightarrow \mathcal{H}_R(M), \quad \text{and} \\
\xi_{G,M}^G&\colon \mathcal{H}_R(M, G) \longrightarrow \mathcal{H}_R(G)
\end{aligned}$$

for the maps induced by Θ_M^P and Ξ_G^P, respectively.

Remark. (a) Although not explicit in the notation, the algebra $\mathcal{H}_R(M, G)$ depends on $\mathbf{P}$. However, in our context $\mathbf{M}$ and $\mathbf{P}$ determine each other so that no confusion will arise.

(b) Notice that $\mathcal{H}_R(G, G) = \mathcal{H}_R(G)$.

(c) The computation (34) actually shows

$$\xi^G_{G,M}\big(\tau^{M,G}_w\big) = T_w + \sum_{w'<w} c_{w'}T_{w'} \in \mathcal{H}_R(G), \tag{37}$$

for all $w \in W_M(1)$. In particular, $\xi^G_{G,M}$ is injective.

Lemma 5.1. *Let $v, w \in W_M(1)$ with $q_{v,w} = 1$. Then $\tau^{M,G}_v \cdot \tau^{M,G}_w = \tau^{M,G}_{vw}$.*

Proof. We may assume $R = \mathbb{Z}$. Corollary 3.15 shows $\mu_{U_P}(v) \cdot \mu_{U_P}(w) = \mu_{U_P}(vw)$ and $q_{M,v,w} = 1$. Hence, $\tau^{M,G}_v \cdot \tau^{M,G}_w = \mu_{U_P}(v)\mu_{U_P}(w) \cdot T^M_v T^M_w = \mu_{U_P}(vw) \cdot T^M_{vw} = \tau^{M,G}_{vw}$. □

5.1.1. *The morphisms $\theta^{L,G}_M$.*

Lemma 5.2. *Let $\mathbf{M} \subseteq \mathbf{L}$ be Levi subgroups in $\mathbf{G}$. The map*

$$\theta^{L,G}_M \colon \mathcal{H}_R(M,G) \longrightarrow \mathcal{H}_R(M,L), \quad \tau^{M,G}_w \longmapsto \mu_{U_{P_L}}(w) \cdot \tau^{M,L}_w$$

is a morphism of R-algebras. Given another Levi subgroup $\mathbf{L}'$ containing $\mathbf{L}$, the diagram

$$\begin{tikzcd} \mathcal{H}_R(M,G) \arrow[r, "\theta^{L',G}_M"] \arrow[dr, "\theta^{L,G}_M"'] & \mathcal{H}_R(M,L') \arrow[d, "\theta^{L,L'}_M"] \\ & \mathcal{H}_R(M,L) \end{tikzcd}$$

commutes, i.e., $\theta^{L,G}_M = \theta^{L,L'}_M \circ \theta^{L',G}_M$.

Proof. We may assume $R = \mathbb{Z}$. By Proposition 3.12 we compute

$$\tau^{M,G}_w = \mu_{U_P}(w)T^M_w = \mu_{U_{P_L}}(w) \cdot \mu_{U_P \cap L}(w)T^M_w = \mu_{U_{P_L}}(w) \cdot \tau^{M,L}_w,$$

for all $w \in W_M(1)$. Hence, $\theta^{L,G}_M$ is the inclusion map and, in particular, a morphism of $\mathbb{Z}$-algebras. If $\mathbf{L}'$ is another Levi subgroup containing $\mathbf{L}$, then

$$\begin{aligned}\big(\theta^{L,L'}_M \circ \theta^{L',G}_M\big)\big(\tau^{M,G}_w\big) &= \theta^{L,L'}_M\big(\mu_{U_{P_{L'}}}(w)\tau^{M,L'}_w\big) = \mu_{U_{P_{L'}}}(w)\mu_{U_{P_L}\cap L'}(w) \cdot \tau^{M,L}_w \\ &= \mu_{U_{P_L}}(w) \cdot \tau^{M,L}_w = \theta^{L,G}_M\big(\tau^{M,G}_w\big),\end{aligned}$$

for all $w \in W_M(1)$, again by Proposition 3.12. □

Proposition 5.3. *Let $\mathbf{M} \subseteq \mathbf{L}$ be Levi subgroups in $\mathbf{G}$. Let $\lambda \in \Lambda(1)$ be a strictly L-positive element. Then*

$$\mathcal{H}_R(M,L) \cong \mathcal{H}_R(M,G)\big[(\tau^{M,G}_\lambda)^{-1}\big]$$

and $\theta^{L,G}_M \colon \mathcal{H}_R(M,G) \to \mathcal{H}_R(M,L)$ is the localization morphism.

Proof. Notice that $\tau^{M,G}_\lambda$ lies in the center of $\mathcal{H}_R(M,G)$, since λ is lifted by a central element in L (and hence in M). Also $\theta^{L,G}_M(\tau^{M,G}_{n\lambda}) = \tau^{M,L}_{n\lambda}$ is central and invertible in $\mathcal{H}_R(M,L)$ for each $n \in \mathbb{Z}_{>0}$. Hence, $\theta^{L,G}_M$ induces a well-defined R-algebra morphism

$$\widetilde{\theta}^{L,G}_M \colon \mathcal{H}_R(M,G)\big[(\tau^{M,G}_\lambda)^{-1}\big] \longrightarrow \mathcal{H}_R(M,L).$$

It suffices to construct an R-linear inverse. Let $w \in W_M(1)$. Choose $n \in \mathbb{Z}_{>0}$ such that $e^{n\lambda}w \in W_{L^+}(1)$. As $n\lambda$ is lifted by a central element of L, we have $q_{L,n\lambda,w} = 1$. Hence, Lemma 5.1 shows

$$\tau^{M,L}_w = \tau^{M,L}_{-n\lambda} \cdot \tau^{M,L}_{n\lambda} \cdot \tau^{M,L}_w = \tau^{M,L}_{-n\lambda} \cdot \tau^{M,L}_{e^{n\lambda}w} = \tau^{M,L}_{-n\lambda} \cdot \theta^{L,G}_M\big(\tau^{M,G}_{e^{n\lambda}w}\big). \tag{38}$$

Hence, we obtain an R-linear map

$$\gamma\colon \mathcal{H}_R(M,L) \longrightarrow \mathcal{H}_R(M,G)\big[(\tau_\lambda^{M,G})^{-1}\big], \quad \tau_w^{M,L} \longmapsto \frac{\tau_{e^{n\lambda}w}^{M,G}}{\tau_{n\lambda}^{M,G}},$$

which does not depend on the choice of n. By (38) we have $\widetilde{\theta}_M^{L,G} \circ \gamma = \mathrm{id}_{\mathcal{H}_R(M,L)}$. Conversely, let $w \in W_M(1)$ and $n \in \mathbb{Z}_{>0}$. Take $m \in \mathbb{Z}_{>0}$ with $e^{m\lambda}w \in W_{L^+}(1)$. As $m\lambda$ is lifted by a central element in L, we have $\mu_{U_P \cap L}(e^{m\lambda}w) = \mu_{U_P\cap L}(w)$. Applying Proposition 3.12 twice, we compute

$$\begin{aligned}\mu_{U_{P_L}}(w)\cdot\mu_{U_P}(e^{m\lambda}w) &= \mu_{U_{P_L}}(w)\cdot\mu_{U_P\cap L}(e^{m\lambda}w)\cdot\mu_{U_{P_L}}(e^{m\lambda}w)\\ &= \mu_{U_{P_L}}(w)\cdot\mu_{U_P\cap L}(w) = \mu_{U_P}(w).\end{aligned}$$

This shows $\tau_{m\lambda}^{M,G}\cdot\tau_w^{M,G} = \mu_{U_{P_L}}(w)\cdot\tau_{e^{m\lambda}w}^{M,G}$. Now,

$$\begin{aligned}&(\gamma\circ\widetilde{\theta}_M^{L,G})\left(\frac{\tau_w^{M,G}}{\tau_{n\lambda}^{M,G}}\right)\\ &\quad= \gamma\big(\tau_{-n\lambda}^{M,L}\cdot\theta_M^{L,G}(\tau_w^{M,G})\big) = \mu_{U_{P_L}}(w)\cdot\gamma\big(\tau_{-n\lambda}^{M,L}\cdot\tau_w^{M,L}\big)\\ &\quad= \mu_{U_{P_L}}(w)\cdot\gamma\big(\tau_{e^{-n\lambda}w}^{M,L}\big) = \frac{\mu_{U_{P_L}}(w)\cdot\tau_{e^{m\lambda}w}^{M,G}}{\tau_{(n+m)\lambda}^{M,G}} = \frac{\tau_{m\lambda}^{M,G}\cdot\tau_w^{M,G}}{\tau_{m\lambda}^{M,G}\cdot\tau_{n\lambda}^{M,G}} = \frac{\tau_w^{M,G}}{\tau_{n\lambda}^{M,G}}.\end{aligned}$$

Hence, $\gamma\circ\widetilde{\theta}_M^{L,G} = \mathrm{id}_{\mathcal{H}_R(M,G)[(\tau_\lambda^{M,G})^{-1}]}$ finishing the proof. □

5.1.2. *The morphisms $\xi_{L,M}^G$.*

Lemma 5.4. *Let* $\mathbf{M}\subseteq\mathbf{L}$ *be Levi subgroups in* $\mathbf{G}$. *There exists a unique R-algebra morphism*

$$\xi_{L,M}^G\colon \mathcal{H}_R(M,G)\longrightarrow\mathcal{H}_R(L,G)$$

which is natural in R and satisfies the following property: for all Levi subgroups $\mathbf{M}\subseteq\mathbf{L}\subseteq\mathbf{L}'$ *in* $\mathbf{G}$, *the diagram*

$$\begin{tikzcd} \mathcal{H}_R(M,G) \arrow[r, "\xi_{L,M}^G"] \arrow[d, "\theta_M^{L',G}"'] & \mathcal{H}_R(L,G) \arrow[d, "\theta_L^{L',G}"] \\ \mathcal{H}_R(M,L') \arrow[r, "\xi_{L,M}^{L'}"'] & \mathcal{H}_R(L,L') \end{tikzcd} \tag{39}$$

commutes, i.e., $\theta_L^{L',G}\circ\xi_{L,M}^G = \xi_{L,M}^{L'}\circ\theta_M^{L',G}$. *Moreover,* $\xi_{L,M}^G$ *is injective.*

Proof. We first construct a unique morphism $\xi_{L,M}^G$, natural in R, making the diagram

$$\begin{tikzcd} \mathcal{H}_R(M,G) \arrow[r, "\xi_{L,M}^G"] \arrow[d, "\theta_M^{L,G}"'] & \mathcal{H}_R(L,G) \arrow[d, "\theta_L^{L,G}"] \\ \mathcal{H}_R(M,L) \arrow[r, "\xi_{L,M}^L"'] & \mathcal{H}_R(L) \end{tikzcd} \tag{40}$$

commutative. Afterwards, we check injectivity and that (39) commutes.

Step 1. We prove unique existence provided R is p-torsionfree. In this case, $\theta_M^{L,G}$ and $\theta_L^{L,G}$ are the canonical inclusions, hence uniqueness is clear. We have to show that $\xi^L_{L,M}$ maps $\mathcal{H}_R(M,G)$ into $\mathcal{H}_R(L,G)$. Let $w \in W_M(1)$. Recall that by (37) we have

$$\xi^L_{L,M}\big(\tau_w^{M,L}\big) = T_w^L + \sum_{w' <_L w} c_{w'} T^L_{w'} \quad \text{in } \mathcal{H}_R(L),$$

where $<_L$ denotes the Bruhat order in $W_L(1)$. By Proposition 3.14.(b) we have $\mu_{U_{P_L}}(w') \leq \mu_{U_{P_L}}(w)$ for all $w' \in W_L(1)$ with $w' <_L w$. We deduce that

$$\begin{aligned}\xi^L_{L,M}\big(\tau_w^{M,G}\big) = \mu_{U_{P_L}}(w) \cdot \xi^L_{L,M}\big(\tau_w^{M,L}\big) &= \mu_{U_{P_L}}(w) T_w^L + \sum_{w' <_L w} c_{w'} \mu_{U_{P_L}}(w) T^L_{w'} \\ &= \tau_w^{L,G} + \sum_{w' <_L w} c'_{w'} \tau^{L,G}_{w'}\end{aligned}$$

lies in $\mathcal{H}_R(L,G)$. This proves existence of an embedding $\xi^G_{L,M} \colon \mathcal{H}_R(M,G) \to \mathcal{H}_R(L,G)$ making (40) commutative.

Step 2. We prove unique existence for general R. Existence follows from Step 1 by extension of scalars from $\mathbb{Z}$ to R. (If R is p-torsionfree this construction coincides with the one in Step 1 by the uniqueness assertion.) We have to prove uniqueness for general R. Take a surjection $f \colon R' \twoheadrightarrow R$ for some p-torsionfree ring R' (*e.g.*, the large polynomial ring $\mathbb{Z}[X_r \mid r \in R]$). By naturality of the diagram

$$\begin{tikzcd}
\mathcal{H}_{R'}(M,G) \arrow[r, "\xi^G_{L,M}"] \arrow[d, two heads, "f\otimes\mathrm{id}"'] & \mathcal{H}_{R'}(L,G) \arrow[d, two heads, "f\otimes\mathrm{id}"] \\
\mathcal{H}_R(M,G) \arrow[r, "\xi^G_{L,M}"'] & \mathcal{H}_R(L,G)
\end{tikzcd}$$

it follows that $\xi^G_{L,M}$ is uniquely determined by the naturality requirement.

Step 3. Injectivity of $\xi^G_{L,M}$. By construction we have

$$\xi^G_{L,M}\big(\tau_w^{M,G}\big) = \tau_w^{L,G} + \sum_{w' <_L w} c'_{w'} \tau^{L,G}_{w'}, \qquad \text{for all } w \in W_M(1),$$

for certain $c'_{w'} \in R$. In particular, $\xi^G_{L,M}$ is injective.

Step 4. Commutativity of (39). By naturality we may assume $R = \mathbb{Z}$. The outer and lower square in

$$\begin{tikzcd}
\mathcal{H}_{\mathbb{Z}}(M,G) \arrow[r, "\xi^G_{L,M}"] \arrow[d, "\theta^{L',G}_M"'] \arrow[dd, bend right=60, "\theta^{L,G}_M"'] & \mathcal{H}_{\mathbb{Z}}(L,G) \arrow[d, "\theta^{L',G}_L"] \arrow[dd, bend left=60, "\theta^{L,G}_L"] \\
\mathcal{H}_{\mathbb{Z}}(M,L') \arrow[r, "\xi^{L'}_{L,M}"] \arrow[d, "\theta^{L,L'}_M"'] & \mathcal{H}_{\mathbb{Z}}(L,L') \arrow[d, "\theta^{L,L'}_L"] \\
\mathcal{H}_{\mathbb{Z}}(M,L) \arrow[r, "\xi^L_{L,M}"'] & \mathcal{H}_{\mathbb{Z}}(L)
\end{tikzcd}$$

commute by construction. By Lemma 5.2, and since $\theta_L^{L,L'}$ is injective, the upper square commutes.

□

Lemma 5.5. *Let* $\mathbf{M} \subseteq \mathbf{L} \subseteq \mathbf{L}'$ *be Levi subgroups in* $\mathbf{G}$*. Then the diagram*

$$\begin{tikzcd}
\mathcal{H}_R(M,G) \arrow[r, "\xi^G_{L,M}"] \arrow[dr, "\xi^G_{L',M}"'] & \mathcal{H}_R(L,G) \arrow[d, "\xi^G_{L',L}"] \\
 & \mathcal{H}_R(L',G)
\end{tikzcd}$$

commutes, i.e., $\xi^G_{L',M} = \xi^G_{L',L} \circ \xi^G_{L,M}$.

Proof. By naturality it suffices to prove the assertion for $R = \mathbb{Z}$. By naturality, and since $\mathcal{H}_{\mathbb{Z}}(M',G) \subseteq \mathcal{H}_{\mathbb{Z}[p^{-1}]}(M',G) = \mathcal{H}_{\mathbb{Z}[p^{-1}]}(M')$ for all Levi subgroups $\mathbf{M}'$ in $\mathbf{G}$, we may even assume $R = \mathbb{Z}[p^{-1}]$. Hence, we need to prove commutativity of

$$\begin{tikzcd}
\mathcal{H}_{\mathbb{Z}[p^{-1}]}(M) \arrow[r, "\xi^L_{L,M}"] \arrow[dr, "\xi^{L'}_{L',M}"'] & \mathcal{H}_{\mathbb{Z}[p^{-1}]}(L) \arrow[d, "\xi^{L'}_{L',L}"] \\
 & \mathcal{H}_{\mathbb{Z}[p^{-1}]}(L').
\end{tikzcd}$$

(Notice that $\xi^G_{L,M} = \xi^L_{L,M}$ for all Levi subgroups $\mathbf{M} \subseteq \mathbf{L}$ in $\mathbf{G}$ whenever p is invertible, because then $\theta^{L,G}_M$ and $\theta^{L,G}_L$ are the identity morphisms in (40).)

Let $w \in W_M(1)$ and take a strictly M-positive element $\lambda \in \Lambda(1)$. Let $n \in \mathbb{Z}_{>0}$ with $e^{n\lambda}w \in W_{M^+}(1)$. Then both $n\lambda$ and $e^{n\lambda}w$ are $M^{+,L}$-, $M^{+,L'}$-, and $L^{+,L'}$-positive. Hence,

$$\xi^L_{L,M}(T^M_w) = (T^L_{n\lambda})^{-1} \cdot T^L_{e^{n\lambda}w}, \qquad \xi^{L'}_{L',M}(T^M_w) = (T^{L'}_{n\lambda})^{-1} \cdot T^{L'}_{e^{n\lambda}w},$$
$$\xi^{L'}_{L',L}(T^L_{n\lambda}) = T^{L'}_{n\lambda}, \qquad \xi^{L'}_{L',L}(T^L_{e^{n\lambda}w}) = T^{L'}_{e^{n\lambda}w}.$$

Therefore, we compute

$$\big(\xi^{L'}_{L',L} \circ \xi^L_{L,M}\big)\big(T^M_w\big) = \xi^{L'}_{L',L}\big((T^L_{n\lambda})^{-1} \cdot T^L_{e^{n\lambda}w}\big) = (T^{L'}_{n\lambda})^{-1} \cdot T^{L'}_{e^{n\lambda}w} = \xi^{L'}_{L',M}(T^M_w). \quad \square$$

5.2. **Transitivity of parabolic induction.**

Proposition 5.6. *Let* $\mathbf{M} \subseteq \mathbf{L} \subseteq \mathbf{L}'$ *be Levi subgroups in* $\mathbf{G}$*. The canonical map*

$$\begin{aligned} \mathcal{H}_R(M,L') \otimes_{\mathcal{H}_R(M,G)} \mathcal{H}_R(L,G) &\longrightarrow \mathcal{H}_R(L,L'), \\ x \otimes y &\longmapsto \xi^{L'}_{L,M}(x) \cdot \theta^{L',G}_L(y) \end{aligned} \tag{41}$$

is an isomorphism of $\mathcal{H}_R(M,L')$*-*$\mathcal{H}_R(L,G)$*-bimodules.*[1]

Proof. The map is well-defined, since (39) commutes, and preserves the bimodule structure by definition. Let $\lambda \in \Lambda(1)$ be a strictly L'-positive element. By Proposition 5.3 the map (41) identifies with the canonical map

$$\mathcal{H}_R(M,G)\big[(\tau^{M,G}_\lambda)^{-1}\big] \otimes_{\mathcal{H}_R(M,G)} \mathcal{H}_R(L,G) \longrightarrow \mathcal{H}_R(L,G)\big[(\tau^{L,G}_\lambda)^{-1}\big],$$

which is clearly an R-linear isomorphism. $\square$

---

[1] The maps $\mathcal{H}_R(M,G) \to \mathcal{H}_R(M,L')$ and $\mathcal{H}_R(M,G) \to \mathcal{H}_R(L,G)$ are the obvious ones, namely $\theta^{L',G}_M$ and $\xi^G_{L,M}$, respectively. Likewise, the bimodule structure is the obvious one.

Theorem 5.7. *Let* $\mathbf{L} \subseteq \mathbf{L}'$ *and* $\mathbf{M} \subseteq \mathbf{M}' \subseteq \mathbf{M}''$ *be Levi subgroups in* $\mathbf{G}$ *with* $\mathbf{M} \subseteq \mathbf{L}$ *and* $\mathbf{M}' \subseteq \mathbf{L}'$. *Then the canonical map*

$$\mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,G)}{\otimes} \mathcal{H}_R(M'',G) \longrightarrow \mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,L')}{\otimes} \mathcal{H}_R(M',L') \underset{\mathcal{H}_R(M',G)}{\otimes} \mathcal{H}_R(M'',G),$$
$$x \otimes y \longmapsto x \otimes 1 \otimes y$$

is an isomorphism of $\mathcal{H}_R(M,L)$-$\mathcal{H}_R(M'',G)$*-bimodules.*

Proof. There are natural isomorphisms

$$\begin{aligned}
&\mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,G)}{\otimes} \mathcal{H}_R(M'',G)\\
&\quad\cong \mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,L')}{\otimes} \mathcal{H}_R(M,L') \underset{\mathcal{H}_R(M,G)}{\otimes} \mathcal{H}_R(M',G) \underset{\mathcal{H}_R(M',G)}{\otimes} \mathcal{H}_R(M'',G)\\
&\quad\cong \mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,L')}{\otimes} \mathcal{H}_R(M',L') \underset{\mathcal{H}_R(M',G)}{\otimes} \mathcal{H}_R(M'',G),
\end{aligned}$$

the second isomorphism being given by Proposition 5.6. The composite sends $x \otimes y \mapsto x \otimes 1 \otimes y$. ☐

As an application we give another proof of the transitivity of parabolic induction, which is originally due to Vignéras [Vig15, Proposition 4.3].

Corollary 5.8. *Let* $\mathbf{M} \subseteq \mathbf{L}$ *be Levi subgroups in* $\mathbf{G}$. *Let* $\mathfrak{m}$ *be a right* $\mathcal{H}_R(M)$*-module. Then there is a natural isomorphism of right* $\mathcal{H}_R(G)$*-modules*

$$\mathfrak{m} \otimes_{\mathcal{H}_R(M^{+,L})} \mathcal{H}_R(L) \otimes_{\mathcal{H}_R(L^+)} \mathcal{H}_R(G) \cong \mathfrak{m} \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G).$$

Proof. Since there is a natural right $\mathcal{H}_R(G)$-linear isomorphism

$$\mathfrak{m} \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G) \cong \mathfrak{m} \otimes_{\mathcal{H}_R(M)} \mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G)$$

(and similarly with (M^+, G) replaced by $(M^{+,L}, L)$), we are reduced to proving the assertion for $\mathfrak{m} = \mathcal{H}_R(M)$. Now, by Theorems 4.9 and 5.7 there are $\mathcal{H}_R(M)$-$\mathcal{H}_R(G)$-bimodule isomorphisms

$$\begin{aligned}
\mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G) &\cong \mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M,G)} \mathcal{H}_R(G)\\
&\cong \mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M,L)} \mathcal{H}_R(L) \otimes_{\mathcal{H}_R(L,G)} \mathcal{H}_R(G)\\
&\cong \mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M^{+,L})} \mathcal{H}_R(L) \otimes_{\mathcal{H}_R(L^+)} \mathcal{H}_R(G). \qquad \square
\end{aligned}$$

5.3. **Alcove walk bases and a filtration.** We finish by describing a natural $\mathbb{Z}_{\geq 0}$-filtration on the R-algebra $\mathcal{H}_R(M,G)$ coming from $\mu_{U_P} : W_M(1) \to \mathbb{Z}_{\geq 0}$. To do this we need to describe alcove walk bases for $\mathcal{H}_R(M,G)$.

Definition. Let o be an orientation of $(\mathscr{A}_M, \mathfrak{H}_M)$ [Vig16, 5.2]. Let $(E_o(w))_{w \in W_M(1)}$ be the associated alcove walk basis in $\mathcal{H}_{\mathbb{Z}}(M)$ [Vig16, Definition 5.22]. We define

$$E_o^{M,G}(w) := 1 \otimes \mu_{U_P}(w) \cdot E_o(w) \in \mathcal{H}_R(M,G), \qquad \text{for all } w \in W_M(1).$$

Remark. The element $E_o^{M,G}(w)$ is indeed well-defined: since $E_o(w) = T_w^M + \sum_{w' <_M w} c_{w'} T_{w'}^M$, for certain $c_{w'} \in \mathbb{Z}$, [Vig16, Corollary 5.26] and by Proposition 3.14.(b), we even have

$$E_o^{M,G}(w) = \tau_w^{M,G} + \sum_{w' <_M w} c'_{w'} \cdot \tau_{w'}^{M,G} \in \mathcal{H}_R(M,G), \tag{42}$$

where $c'_{w'}$ is the image of $\frac{\mu_{U_P}(w)}{\mu_{U_P}(w')} \cdot c_{w'}$ in R. Hence, $(E_o^{M,G}(w))_{w \in W_M(1)}$ is an R-basis of $\mathcal{H}_R(M,G)$.

Lemma 5.9. *Let o be an orientation of $(\mathscr{A}_M, \mathfrak{H}_M)$. Let $v, w \in W_M(1)$. Then*

$$E_o^{M,G}(v) \cdot E_{o\bullet v}^{M,G}(w) = q_{v,w} \cdot E_o^{M,G}(vw).$$

Proof. We may assume $R = \mathbb{Z}$. Then $E_o(v) \cdot E_{o\bullet v}(w) = q_{M,v,w} \cdot E_o(vw)$ by [Vig16, Theorem 5.25]. Corollary 3.15 shows

$$E_o^{M,G}(v) \cdot E_{o\bullet v}^{M,G}(w) = \frac{\mu_{U_P}(v)\mu_{U_P}(w)}{\mu_{U_P}(vw)} \cdot q_{M,v,w} E_o^{M,G}(vw) = q_{v,w} \cdot E_o^{M,G}(vw). \quad \square$$

Definition. A $\mathbb{Z}_{\geq 0}$-*filtration* of an R-algebra A is a family $(\mathcal{F}_i A)_{i \in \mathbb{Z}_{\geq 0}}$ of R-submodules satisfying

- $\mathcal{F}_i A \subseteq \mathcal{F}_{i+1} A$ for all $i \geq 0$;
- $\mathcal{F}_i A \cdot \mathcal{F}_j A \subseteq \mathcal{F}_{i+j} A$ for all $i, j \geq 0$;
- $1 \in \mathcal{F}_0 A$;
- $A = \cup_{i \geq 0} \mathcal{F}_i A$.

Proposition 5.10. *The free R-submodules $\mathcal{F}_n^{M,G}$ of $\mathcal{H}_R(M,G)$ generated by $\{\tau_w^{M,G}\}_{\substack{w \in W_M(1), \\ \mu_{U_P}(w) \leq q^n}}$ define a $\mathbb{Z}_{\geq 0}$-filtration on $\mathcal{H}_R(M,G)$.*

Moreover, $\mathcal{F}_0^{M,G} \cong \mathcal{H}_R(M^+)$ via $\theta_M^{M,G}$.

Proof. The only thing that is not immediately clear is $\mathcal{F}_i^{M,G} \cdot \mathcal{F}_j^{M,G} \subseteq \mathcal{F}_{i+j}^{M,G}$, for $i, j \geq 0$. Given any orientation o of $(\mathscr{A}_M, \mathfrak{H}_M)$, the set $\left\{E_o^{M,G}(w)\right\}_{\substack{w \in W_M(1) \\ \mu_{U_P}(w) \leq q^n}}$ is an R-basis of $\mathcal{F}_n^{M,G}$ by (42) and Proposition 3.14.(b). Hence, the claim follows from Lemma 5.9 and Corollary 3.15. $\square$

ACKNOWLEDGMENTS

This article constitutes a part of my doctoral dissertation [Hey19] at the Humboldt University in Berlin. It is a pleasure to thank my advisor Elmar Große-Klönne for suggesting the topic and supporting me throughout the years. My thanks extends to Peter Schneider for his interest in my work. I am especially grateful to the referee for carefully reading this article, for pointing out a gap, and for providing several helpful comments and suggestions.

REFERENCES

[Abe16] Noriyuki Abe, *Modulo p parabolic induction of pro-p-Iwahori Hecke algebra*, J. Reine Angew. Math. **749** (2019), 1–64, DOI 10.1515/crelle-2016-0043. MR3935898

[Abe19] Noriyuki Abe, *Parabolic inductions for pro-p-Iwahori Hecke algebras*, Adv. Math. **355** (2019), 106776, 63, DOI 10.1016/j.aim.2019.106776. MR3996728

[And77] A. N. Andrianov, *The expansion of Hecke polynomials for the symplectic group of genus n* (Russian), Mat. Sb. (N.S.) **104(146)** (1977), no. 3, 390–427, 511. MR0562510

[AZ95] A. N. Andrianov and V. G. Zhuravlëv, *Modular forms and Hecke operators*, Translations of Mathematical Monographs, vol. 145, American Mathematical Society, Providence, RI, 1995. Translated from the 1990 Russian original by Neal Koblitz, DOI 10.1090/mmono/145. MR1349824

[BB06] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR2133266

[BK98] Colin J. Bushnell and Philip C. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 582–634, DOI 10.1112/S0024611598000574. MR1643417

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local* (French), Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251. MR327923

[BT84] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée* (French), Inst. Hautes Études Sci. Publ. Math. **60** (1984), 197–376. MR756316

[Gri88] V. A. Gritsenko, *Parabolic extensions of the Hecke ring of the general linear group* (Russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **154** (1986), no. Anal. Teor. Chisel i Teor. Funktsiĭ. 7, 36–45, 176, DOI 10.1007/BF01374983; English transl., J. Soviet Math. **43** (1988), no. 4, 2533–2540. MR859521

[Hey19] Claudius Heyer. *Applications of parabolic Hecke algebras: parabolic induction and Hecke polynomials*. PhD thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät, 2019. `doi:http://dx.doi.org/10.18452/20137`.

[HR08] Thomas Haines and Michael Rapoport. On parahoric subgroups. *Advances in Mathematics*, 219(1):188–198, 2008. `doi:10.1016/j.aim.2008.04.020`.

[HR09] Thomas J. Haines and Sean Rostami, *The Satake isomorphism for special maximal parahoric Hecke algebras*, Represent. Theory **14** (2010), 264–284, DOI 10.1090/S1088-4165-10-00370-5. MR2602034

[Kot97] Robert E. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339, DOI 10.1023/A:1000102604688. MR1485921

[Oll10] Rachel Ollivier, *Parabolic induction and Hecke modules in characteristic p for p-adic GL_n*, Algebra Number Theory **4** (2010), no. 6, 701–742, DOI 10.2140/ant.2010.4.701. MR2728487

[OV18] Rachel Ollivier and Marie-France Vignéras, *Parabolic induction in characteristic p*, Selecta Math. (N.S.) **24** (2018), no. 5, 3973–4039, DOI 10.1007/s00029-018-0440-0. MR3874689

[Ren10] David Renard, *Représentations des groupes réductifs p-adiques* (French), Cours Spécialisés [Specialized Courses], vol. 17, Société Mathématique de France, Paris, 2010. MR2567785

[Vig96] Marie-France Vignéras, *Représentations l-modulaires d'un groupe réductif p-adique avec $l \neq p$* (French, with English summary), Progress in Mathematics, vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996. MR1395151

[Vig98] Marie-France Vignéras, *Induced R-representations of p-adic reductive groups*, Selecta Math. (N.S.) **4** (1998), no. 4, 549–623, DOI 10.1007/s000290050040. MR1668044

[Vig05] Marie-France Vignéras, *Pro-p-Iwahori Hecke ring and supersingular $\overline{\mathbf{F}}_p$-representations*, Math. Ann. **331** (2005), no. 3, 523–556, DOI 10.1007/s00208-004-0592-4. MR2122539

[Vig06] Marie-France Vignéras, *Algèbres de Hecke affines génériques* (French, with French summary), Represent. Theory **10** (2006), 1–20, DOI 10.1090/S1088-4165-06-00185-3. MR2192484

[Vig15] Marie-France Vignéras, *The pro-p Iwahori Hecke algebra of a reductive p-adic group, V (parabolic induction)*, Pacific J. Math. **279** (2015), no. 1-2, 499–529, DOI 10.2140/pjm.2015.279.499. MR3437789

[Vig16] Marie-France Vigneras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group I*, Compos. Math. **152** (2016), no. 4, 693–753, DOI 10.1112/S0010437X15007666. MR3484112

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstrasse 62, D-48149 Münster, Germany

Email address: `cheyer@uni-muenster.de`