

LOCAL LANGLANDS CORRESPONDENCE FOR UNITARY GROUPS VIA THETA LIFTS

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ABSTRACT. Using the theta correspondence, we extend the classification of irreducible representations of quasi-split unitary groups (the so-called local Langlands correspondence, which is due to Mok) to non quasi-split unitary groups. We also prove that our classification satisfies some good properties, which characterize it uniquely. In particular, this paper provides an alternative approach to the works of Kaletha-Mínguez-Shin-White and Mœglin-Renard.

1. INTRODUCTION

In his monumental book [Art13], Arthur gave a complete description of the automorphic discrete spectra of quasi-split orthogonal groups and symplectic groups, by using the stable trace formula and the theory of endoscopy. One of the main local theorems in that book is the local Langlands correspondence (“LLC” for short), which gives a classification of irreducible tempered representations of quasi-split classical groups. Following Arthur’s method, Mok established the same results for quasi-split unitary groups [Mok15]. To extend these results to non quasi-split classical groups, one can use the stable trace formula à la Arthur. This was partially carried out by Kaletha-Mínguez-Shin-White in [KMSW14] for unitary groups. In particular, they established the LLC for all unitary groups, in the enhanced version of Vogan. Mœglin-Renard also have some related results [MR18]. Both these two papers use very difficult techniques.

However, the theta correspondence provides us a rather cheap way to establish, or “transfer”, results from one group to another group. Indeed, this idea has been used in many papers, for example, [GT11], [GS12], [GI18], and a recent paper [Ish19]. This paper is another exploitation of this idea. The main goal of this paper is to construct a (Vogan version) LLC for unitary groups over a p -adic field, based on the LLC for quasi-split unitary groups. We will also prove that this LLC satisfies several desired properties; these properties will uniquely determine the LLC (see Theorem 2.5.1). Among these properties, the most important one is so-called “local intertwining relations” (“LIR” for short), which allows us to distinguish representations in a tempered L -packet by using (normalized) intertwining operators. We would like to remark here that the LIR we used here is the same as in [GI16], which is a little bit different from the LIR formulated by Arthur/ Mok/ KMSW (see Remark 2.5.4). As in other instances where the LLC was shown using the theta correspondence (such as [GT11] and [GS12]), we do not show the (twisted) endoscopic character identities for the L -packets we constructed. To show that our L -packets satisfy the endoscopic character identities, one would need to appeal to

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the stable trace formula (or a simple form of it), as was done in [CG15] and [Luo20]. Although essentially there is no new result in this paper, it provides an alternative approach to the works of [KMSW14] and [MR18].

We would like to mention some related works. In [GI16], Gan-Ichino proved the so-called Prasad conjecture, which describes the almost equal rank theta lifts in terms of the LLC; similarly, in [AG17], Atobe-Gan described the theta lifts of tempered representations in terms of the LLC. In this paper, we “turn the table around”, namely, imitating the prediction of Prasad conjecture, we construct a Vogan version LLC for unitary groups. We also write a parallel paper [CZ21], in which we use the same method to deal with the even orthogonal groups (we write it separately to avoid making notations too complicated). In a sequel to this paper, we carry out the global counterpart of this paper and establish the Arthur’s multiplicity formula for the tempered part of automorphic discrete spectra of even orthogonal groups/ unitary groups.

We now give a summary of the layout of this paper. We formulate the main theorem (i.e. the desired LLC, Theorem 2.5.1) in Section 2, taking the chance to recall some results from [Mok15] that we are using. After recalling some basics of theta correspondence in Section 3, we give our construction in Section 4, and prove several properties of the desired LLC along the way. Then in Section 5 we recall some results from [GI16], which will be the ingredients in the proof of the LIR in Section 6. Finally in Section 7, with the help of the LIR, we are able to finish the proof of the main theorem. To keep the paper in a reasonable length, we omit many repeated details. Readers can refer to the arXiv version of this paper if they would like the full details.

NOTATION

We set up some notations at the beginning of this paper. Let F be a non-Archimedean local field of characteristic 0 and residue characteristic p . Let E be a quadratic field extension of F and let $\omega_{E/F}$ be the quadratic character of F^\times associated to E/F by local class field theory. We denote by c the non-trivial Galois automorphism of E over F . Let $\text{Tr}_{E/F}$ and $\text{Nm}_{E/F}$ be the trace and norm maps from E to F . We denote by E^1 the subgroup of E^\times consisting of norm 1 elements. We choose an element $\delta \in E^\times$ such that $\text{Tr}_{E/F}(\delta) = 0$. We write $|\cdot| = |\cdot|_E$ for the normalized absolute value on E . If ψ is an additive character of F , we shall use ψ_E to denote the additive character of E defined by $\psi_E = \psi \circ \text{Tr}_{E/F}$. If π is a representation of some group G , we shall use π^\vee to denote the contragredient of π .

2. LOCAL LANGLANDS CORRESPONDENCE

In this section, we formulate the desired LLC for unitary groups.

2.1. Hermitian and skew-Hermitian spaces. Fix $\varepsilon = \pm 1$. Let V be a finite dimensional vector space over E equipped with a non-degenerate ε -Hermitian form

$$\langle \cdot, \cdot \rangle_V : V \times V \longrightarrow E.$$

Put $n = \dim V$ and $\text{disc } V = (-1)^{(n-1)n/2} \cdot \det V$, so that

$$\text{disc } V \in \begin{cases} F^\times / \text{Nm}_{E/F}(E^\times) & \text{if } \varepsilon = +1; \\ \delta^n \cdot F^\times / \text{Nm}_{E/F}(E^\times) & \text{if } \varepsilon = -1. \end{cases}$$

We define $\epsilon(V) = \pm 1$ by

$$\epsilon(V) = \begin{cases} \omega_{E/F}(\text{disc } V) & \text{if } \epsilon = +1; \\ \omega_{E/F}(\delta^{-n} \cdot \text{disc } V) & \text{if } \epsilon = -1. \end{cases}$$

Given a positive integer n , there are precisely two isometry classes of n -dimensional ϵ -Hermitian spaces V , which are distinguished from each other by their signs $\epsilon(V)$. Note that

- $\epsilon(V)$ depends on the choice of δ if $\epsilon = -1$ and n is odd;
- V^+ always has the maximal possible Witt index $[\dim V^+ / 2]$.

Let $U(V)$ be the unitary group of V . If $n = 0$, we interpret $U(V)$ as the trivial group $\{1\}$.

Sometimes we also need to consider a tower of ϵ -Hermitian spaces. Let V_{an} be an anisotropic space over E , and for $r \geq 0$, let

$$V_{an,r} = V_{an} \oplus \mathcal{H}^r,$$

where \mathcal{H} is the (ϵ -Hermitian) hyperbolic plane. Let $U(V_{an,r})$ be the unitary group associated to $V_{an,r}$. The collection

$$\{V_{an,r} \mid r \geq 0\}$$

is called a Witt tower of spaces. We note that any given ϵ -Hermitian space V is a member of a unique Witt tower of spaces \mathcal{V} .

2.2. Langlands parameters and component groups. Let W_E be the Weil group of E and $WD_E = W_E \times SL_2(\mathbb{C})$ the Weil-Deligne group of E . Recall that an L -parameter for the unitary group $U(V)$ is an n -dimensional conjugate self-dual representation of WD_E

$$\phi : WD_E \longrightarrow GL_n(\mathbb{C})$$

with sign $(-1)^{n-1}$. Let $\Phi(n)$ be the set of equivalence classes of L -parameters for unitary groups of n variables. Given $\phi \in \Phi(n)$, we may decompose it into a direct sum

$$\phi = \bigoplus_i m_i \phi_i$$

with pairwise inequivalent irreducible representations ϕ_i of WD_E and multiplicities m_i . We say that ϕ is square-integrable if it is multiplicity-free and ϕ_i is conjugate self-dual with sign $(-1)^{n-1}$ for all i , and we say that ϕ is tempered if the image of W_E is bounded.

For an L -parameter ϕ for $U(V)$, we can define the component group \mathcal{S}_ϕ associated to ϕ following [GGP12] Section 8. If we write $\phi = \bigoplus_i m_i \phi_i$, then \mathcal{S}_ϕ has an explicit description of the form

$$\mathcal{S}_\phi = \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j$$

with a canonical basis $\{a_j\}$, where the product ranges over all j such that ϕ_j is conjugate self-dual with sign $(-1)^{n-1}$. For $a = a_{j_1} + \dots + a_{j_r} \in \mathcal{S}_\phi$, we put

$$\phi^a = \phi_{j_1} \oplus \dots \oplus \phi_{j_r}.$$

We shall let z_ϕ denote the image of $-1 \in GL_n(\mathbb{C})$ in \mathcal{S}_ϕ . More explicitly, we have

$$z_\phi = (m_j a_j) \in \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j.$$

Let $\overline{\mathcal{S}_\phi} = \mathcal{S}_\phi / \langle z_\phi \rangle$. Then the canonical epimorphism $\mathcal{S}_\phi \twoheadrightarrow \overline{\mathcal{S}_\phi}$ induces an inclusion

$$\widehat{\overline{\mathcal{S}_\phi}} \hookrightarrow \widehat{\mathcal{S}_\phi}.$$

Here, we denote by \widehat{A} the Pontryagin dual of an abelian group A .

2.3. Whittaker data. To describe our main result, we need to choose a Whittaker datum of $U(V^+)$, which is a conjugacy class of pairs (N, ξ) , where

- N is the unipotent radical of a Borel subgroup of the quasi-split unitary group $U(V^+)$,
- ξ is a generic character of N .

When n is odd, such a datum is canonical. When $n = 2m$ is even, as explained in [GGP12] Section 12, it is determined by the choice of an $\text{Nm}_{E/F}(E^\times)$ -orbit of non-trivial additive characters

$$\begin{cases} \psi^E : E/F \longrightarrow \mathbb{C}^\times & \text{if } \varepsilon = +1; \\ \psi : F \longrightarrow \mathbb{C}^\times & \text{if } \varepsilon = -1. \end{cases}$$

According to this choice, we write

$$\begin{cases} \mathcal{W}_{\psi^E} & \text{if } \varepsilon = +1; \\ \mathcal{W}_\psi & \text{if } \varepsilon = -1, \end{cases}$$

for the corresponding Whittaker datum.

Assume that $\varepsilon = +1$ for a while. So that V^+ is a Hermitian space. By choosing a non-zero trace zero element $\delta \in E$, we can define a skew-Hermitian space $W^+ = \delta \cdot V^+$, which is the space V^+ equipped with the skew-Hermitian form $\delta \cdot \langle \cdot, \cdot \rangle_{V^+}$. Then $U(V^+)$ and $U(W^+)$ are physically equal as subgroups of $GL(V^+)$. Let ψ be a non-trivial additive character of F , and

$$\psi^E = \psi \left(\frac{1}{2} \text{Tr}_{E/F}(\delta \cdot \) \right)$$

be a character of E trivial on F . Then, there is a Whittaker datum \mathcal{W}_{ψ^E} of $U(V^+)$, and a Whittaker datum \mathcal{W}_ψ of $U(W^+)$. We have

$$\mathcal{W}_{\psi^E} = \mathcal{W}_\psi.$$

Now we return to the general case. Sometimes we need to consider the LLC for two (or more) unitary groups associated to spaces in a same Witt tower simultaneously, hence we must choose a Whittaker datum of each group in a compatible way. Let \mathcal{W} be a Whittaker datum of the unitary group $U(V^+)$. Then, for each space \widetilde{V}^+ in the Witt tower containing V^+ , we may choose a Whittaker datum of $U(\widetilde{V}^+)$ as follows. Let ψ (or ψ^E) be a non-trivial character of F (or E/F), such that

$$\mathcal{W} = \mathcal{W}_\psi \quad (\text{or } \mathcal{W} = \mathcal{W}_{\psi^E}).$$

Then there is an obvious choice of the Whittaker datum of $U(\widetilde{V}^+)$, namely, the Whittaker datum associated to ψ (or ψ^E). By abuse of notation, we shall also denote this Whittaker datum of $U(\widetilde{V}^+)$ by \mathcal{W} .

2.4. Local factors. To characterize the correspondence that will be established later, we need to introduce two representation-theoretic local factors.

The first one is the standard γ -factor. Let V be an n -dimensional vector space over E equipped with a non-degenerate ε -Hermitian form, π be an irreducible smooth representation of $U(V)$, and χ be a character of E^\times . Following [LR05], [GI14], one can define the standard γ -factor

$$\gamma(s, \pi, \chi, \psi)$$

by using the doubling zeta integral. We remark here that in this paper we shall use the definition in [GI14] Section 10, which is slightly different from the definition in [LR05] (see [GI14] page 546 or [Kak20] Remark 5.4 for the modification of the definition and explanations). The standard γ -factors satisfy many good properties, for example, “Ten Commandments”.

The second one we want to introduce is the Plancherel measure. Let ψ be a non-trivial additive character of F . Again let V be an n -dimensional vector space over E equipped with a non-degenerate ε -Hermitian form, π be an irreducible smooth representation of $U(V)$, and τ be an irreducible smooth representation of $GL_k(E)$. For any $s \in \mathbb{C}$, we put $\tau_s := \tau \otimes |\det|^s$. Let \tilde{V} be the $(n + 2k)$ -dimensional ε -Hermitian space in the Witt tower containing V , and $P = M_P U_P$ be a maximal parabolic subgroup of $U(\tilde{V})$ with Levi component M_P and unipotent radical U_P , such that

$$M_P \simeq GL_k(E) \times U(V).$$

Consider the (normalized parabolic) induced representation

$$\text{Ind}_P^{U(\tilde{V})}(\tau_s \boxtimes \pi).$$

Let $\bar{P} = M_P U_{\bar{P}}$ be the parabolic subgroup of $U(\tilde{V})$ opposite to P , and $U_{\bar{P}}$ be the unipotent radical of \bar{P} . Fix a Haar measure $du \times d\bar{u}$ on $U_P \times U_{\bar{P}}$ as in [GI14] Appendix B (this Haar measure depends on the choice of the additive character ψ). We define an intertwining operator

$$\mathcal{M}_{\bar{P}|P}(\tau_s \boxtimes \pi) : \text{Ind}_P^{U(\tilde{V})}(\tau_s \boxtimes \pi) \longrightarrow \text{Ind}_{\bar{P}}^{U(\tilde{V})}(\tau_s \boxtimes \pi)$$

by (the meromorphic continuation of) the integral

$$\mathcal{M}_{\bar{P}|P}(\tau_s \boxtimes \pi)\Phi_s(g) = \int_{U_{\bar{P}}} \Phi_s(ug)du.$$

Then there exists a meromorphic function $\mu_\psi(\tau_s \boxtimes \pi)$ of s such that

$$\mathcal{M}_{P|\bar{P}}(\tau_s \boxtimes \pi) \circ \mathcal{M}_{\bar{P}|P}(\tau_s \boxtimes \pi) = \mu_\psi(\tau_s \boxtimes \pi)^{-1}.$$

In this paper, by “Plancherel measures”, we mean the functions of the form $\mu_\psi(\tau_s \boxtimes \pi)$.

Given a representation ρ of WD_E , one can define the Galois-theoretic γ -factor

$$\gamma(s, \rho, \psi_E)$$

as usual. We denote by As^+ the Asai representation of the L -group of $\text{Res}_{E/F} GL_k$ and $As^- = As^+ \otimes \omega_{E/F}$ its twist. Readers may refer to [GGP12] Section 7 for these representations.

2.5. Main theorem. Now we can formulate our desired LLC for unitary groups.

Theorem 2.5.1. *There is a canonical finite-to-one surjection*

$$\mathcal{L} : \text{Irr } U(V^+) \sqcup \text{Irr } U(V^-) \longrightarrow \Phi(n),$$

where V^+ and V^- are the n -dimensional ε -Hermitian spaces with $\epsilon(V^+) = +1$ and $\epsilon(V^-) = -1$. For each L -parameter ϕ , we denote the inverse image of ϕ by Π_ϕ , and call Π_ϕ the L -packet associated to ϕ . For each L -packet Π_ϕ , there is a bijection (depends on the choice of a Whittaker datum \mathcal{W} of $U(V^+)$)

$$\mathcal{J}_\mathcal{W} : \Pi_\phi \longrightarrow \widehat{\mathcal{S}}_\phi.$$

We shall use $\pi(\phi, \eta)$ to denote the element in Π_ϕ corresponding to η (with respect to \mathcal{W}).

This assignment $\pi \mapsto (\phi = \mathcal{L}(\pi), \eta = \mathcal{J}_\mathcal{W}(\pi))$ satisfies following properties:

- (1) The map \mathcal{L} preserves square-integrability.
- (2) The map \mathcal{L} preserves temperedness.
- (3) The map \mathcal{L} respects standard γ -factors, in the sense that

$$\gamma(s, \pi, \chi, \psi) = \gamma(s, \phi\chi, \psi_E)$$

for any $\pi \in \text{Irr } U(V^\epsilon)$ whose parameter is ϕ , and any character χ of E^\times .

- (4) The map \mathcal{L} respects Plancherel measures, in the sense that

$$\begin{aligned} \mu_\psi(\tau_s \boxtimes \pi) &= \gamma(s, \phi_\tau \otimes \phi^\vee, \psi_E) \cdot \gamma(-s, \phi_\tau^\vee \otimes \phi, \psi_E^{-1}) \\ &\quad \times \gamma(2s, \text{As}^{(-1)^n} \circ \phi_\tau, \psi) \cdot \gamma(-2s, \text{As}^{(-1)^n} \circ \phi_\tau^\vee, \psi^{-1}) \end{aligned}$$

for any $\pi \in \text{Irr } U(V^\epsilon)$ whose parameter is ϕ , and any irreducible square-integrable representation τ of $GL_k(E)$ with L -parameter ϕ_τ . In particular, the Plancherel measures are invariants of an L -packet. Namely, if π_1, π_2 has the same L -parameter ϕ , then we have

$$\mu_\psi(\tau_s \boxtimes \pi_1) = \mu_\psi(\tau_s \boxtimes \pi_2)$$

for any irreducible square-integrable representation τ of $GL_k(E)$.

- (5) $\pi = \pi(\phi, \eta)$ is a representation of $U(V^\epsilon)$ if and only if $\eta(z_\phi) = \epsilon$.
- (6) Assume that ϕ is a tempered L -parameter, then there is an unique \mathcal{W} -generic representation of $U(V^+)$ in Π_ϕ , which corresponds to the trivial character of \mathcal{S}_ϕ .
- (7) (**Local intertwining relation**) Assume that

$$\phi = \phi_\tau \oplus \phi_0 \oplus (\phi_\tau^c)^\vee,$$

where ϕ_τ is an irreducible tempered representation of WD_E which corresponds to an irreducible (unitary) discrete series representation τ of $GL_k(E)$ and ϕ_0 is a tempered element in $\Phi(n - 2k)$. So there is a natural embedding $\mathcal{S}_{\phi_0} \hookrightarrow \mathcal{S}_\phi$. Let $\pi_0 = \pi(\phi_0, \eta_0)$ be an irreducible tempered representation of $U(V_0^\epsilon)$, where V_0^ϵ is the $(n - 2k)$ -dimensional ε -Hermitian space with sign ϵ . There is a maximal parabolic subgroup of $U(V^\epsilon)$, say P , with Levi component M , so that

$$M \simeq GL_k(E) \times U(V_0^\epsilon).$$

Then the induced representation $\text{Ind}_P^{U(V^\epsilon)}(\tau \boxtimes \pi_0)$ has a decomposition

$$\text{Ind}_P^{U(V^\epsilon)}(\tau \boxtimes \pi_0) = \bigoplus_{\eta} \pi(\phi, \eta),$$

where the sum ranges over all $\eta \in \widehat{\mathcal{S}}_\phi$ such that $\eta|_{\mathcal{S}_{\phi_0}} = \eta_0$. Moreover, if ϕ_τ is conjugate self-dual, let

$$R(w, \tau \boxtimes \pi_0) \in \text{End}_{U(V^\epsilon)} \left(\text{Ind}_P^{U(V^\epsilon)} (\tau \boxtimes \pi_0) \right)$$

be the normalized intertwining operator to be defined later in Section 5.2, where w is the unique non-trivial element in the relative Weyl group for M . Then the restriction of $R(w, \tau \boxtimes \pi_0)$ to $\pi(\phi, \eta)$ is the scalar multiplication by

$$\begin{cases} \epsilon^k \cdot \eta(a_\tau) & \text{if } \phi_\tau \text{ has sign } (-1)^{n-1}; \\ \epsilon^k & \text{if } \phi_\tau \text{ has sign } (-1)^n, \end{cases}$$

where a_τ is the element in \mathcal{S}_ϕ corresponding to ϕ_τ .

(8) **(Compatibility with Langlands quotients)** Assume that

$$\phi = (\phi_{\tau_1} | \cdot |^{s_1} \oplus \cdots \oplus \phi_{\tau_r} | \cdot |^{s_r}) \oplus \phi_0 \oplus ((\phi_{\tau_1} | \cdot |^{s_1} \oplus \cdots \oplus \phi_{\tau_r} | \cdot |^{s_r})^c)^\vee,$$

where for $i = 1, \dots, r$, ϕ_{τ_i} is an irreducible tempered representation of WD_E which corresponds to an irreducible (unitary) discrete series representation τ_i of $GL_{k_i}(E)$, and s_i is a real number such that

$$s_1 \geq \cdots \geq s_r > 0;$$

ϕ_0 is a tempered element in $\Phi(n - 2k)$, where $k = k_1 + \cdots + k_r$. So there is a natural isomorphism $\mathcal{S}_{\phi_0} \simeq \mathcal{S}_\phi$. Let $\eta \in \widehat{\mathcal{S}}_\phi$, and $\eta_0 := \eta|_{\mathcal{S}_{\phi_0}}$. Let $\pi_0 = \pi(\phi_0, \eta_0)$ be an irreducible tempered representation of $U(V_0^\epsilon)$, where V_0^ϵ is the $(n - 2k)$ -dimensional ϵ -Hermitian space with sign ϵ . Then there is a parabolic subgroup of $U(V^\epsilon)$, say P , with Levi component M , so that

$$M \simeq GL_{k_1}(E) \times \cdots \times GL_{k_r}(E) \times U(V_0^\epsilon),$$

and $\pi(\phi, \eta)$ is the unique irreducible quotient of the standard module

$$\text{Ind}_P^{U(V^\epsilon)} (\tau_1 | \det |^{s_1} \boxtimes \cdots \boxtimes \tau_r | \det |^{s_r} \boxtimes \pi_0).$$

(9) If $\pi = \pi(\phi, \eta)$, and χ is a character of E^1 , then the representation $\pi\chi := \pi \otimes (\chi \circ \det)$ has L -parameter $\phi \cdot \tilde{\chi}$ and the associated character $\eta_{\pi\chi} = \eta$, where $\tilde{\chi}$ is the base change of χ , i.e. the pull-back of χ along

$$\begin{aligned} E^\times &\rightarrow E^1 \\ x &\mapsto x/c(x), \end{aligned}$$

and we use the obvious isomorphism between \mathcal{S}_ϕ and $\mathcal{S}_{\phi \cdot \tilde{\chi}}$ to identify them.

(10) If $\pi = \pi(\phi, \eta)$, then the contragredient representation π^\vee of π has L -parameter ϕ^\vee and associated character $\eta_{\pi^\vee} = \eta \cdot \nu$, where ν is a character of \mathcal{S}_ϕ given by

$$\nu(a) = \begin{cases} \omega_{E/F}(-1)^{\dim \phi^a} & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

for $a \in \mathcal{S}_\phi$. Here we use the obvious isomorphism between \mathcal{S}_ϕ and \mathcal{S}_{ϕ^\vee} to identify them.

Remark 2.5.2. Here, the formulations of some properties involve the LLC for a smaller unitary group $U(V_0^\epsilon)$. Hence we need to specify which Whittaker datum of $U(V_0^+)$ we are using. Notice that V_0^+ is in the Witt tower containing V^+ , as explicated in Section 2.3, the Whittaker datum \mathscr{W} of $U(V^+)$ uniquely determines a Whittaker datum of $U(V_0^+)$, which we shall also denote by \mathscr{W} . The LLC for $U(V_0^\epsilon)$ we are using is with respect to this Whittaker datum \mathscr{W} .

Following the method of Arthur, Mok established the LLC for quasi-split unitary groups in [Mok15] (supplemented by some results of many others):

Theorem 2.5.3. *There is a canonical finite-to-one surjection*

$$\mathcal{L}^+ : \text{Irr}(U(V^+)) \longrightarrow \Phi(n),$$

where V^+ is the n -dimensional ε -Hermitian space with $\epsilon(V^+) = 1$. For an L -parameter ϕ , let Π_ϕ^+ be the inverse image of ϕ under \mathcal{L}^+ . For each Π_ϕ^+ , we have a bijection (depends on the choice of a Whittaker datum \mathscr{W} of $U(V^+)$)

$$\mathcal{J}_\mathscr{W}^+ : \Pi_\phi^+ \longrightarrow \widehat{\mathcal{S}}_\phi.$$

This assignment $\pi \mapsto (\phi = \mathcal{L}^+(\pi), \eta = \mathcal{J}_\mathscr{W}^+(\pi))$ satisfies all properties listed in Theorem 2.5.1.

Remark 2.5.4.

- (1) There are also some existed results on the LLC for (non quasi-split) unitary groups, see [KMSW14], and [MR18]. Their methods are based on trace formulas and endoscopic character identities. But we are not sure if all properties listed in Theorem 2.5.1 were verified in their works. For example, it seems to the authors that properties (3), (4), (9), and (10) are verified only for quasi-split groups, though it is expected that these properties can be verified through the endoscopic character identities. The approach in this paper is independent with these works.
- (2) Indeed, the LIR we formulated in Theorem 2.5.1 is the same as that in [GI16], but is different from the LIR formulated by Mok in [Mok15] Proposition 3.4.4, or KMSW’s version in [KMSW14] Chapter 2. There are several different points between their LIR and the LIR we formulated here:
 - their LIR is formulated for A -packets, rather than individual representations;
 - their LIR is formulated in terms of distributions;
 - the normalizing fator we used in the definition of the normalized intertwining operator is slightly different from theirs (however our normalized intertwining operator is still the same as theirs, see Remark 5.2.2).

In [Ato17], Atope proved that for tempered L -packets, the LIR we used here is a consequence of the LIR formulated by Mok/ KMSW. So we shall take it as given for quasi-split unitary groups.

We emphasize that our proof of Theorem 2.5.1 relies on Theorem 2.5.3. Firstly we shall extend Mok’s result to all odd unitary groups. Observe that when n is odd, we may take $V^- = a \cdot V^+$, where $a \in F^\times \setminus \text{Nm}_{E/F}(E^\times)$. Then $U(V^+)$ and $U(V^-)$ are physically equal as subgroups of $GL(V^+)$, and the identity map between them induces a bijection

$$\text{id}^* : \text{Irr } U(V^-) \longrightarrow \text{Irr } U(V^+).$$

Under this identification, we can extend the map \mathcal{L}^+ to a map

$$\mathcal{L} : \text{Irr } U(V^+) \sqcup \text{Irr } U(V^-) \longrightarrow \Phi(n)$$

as follows:

$$\mathcal{L}(\pi) = \begin{cases} \mathcal{L}^+(\pi) & \text{if } \pi \in \text{Irr } U(V^+); \\ \mathcal{L}^+(\text{id}^* \pi) & \text{if } \pi \in \text{Irr } U(V^-). \end{cases}$$

Then for each parameter ϕ , we have

$$\Pi_\phi = \Pi_\phi^+ \sqcup \Pi_\phi^-,$$

where $\Pi_\phi := \mathcal{L}^{-1}(\phi)$, $\Pi_\phi^+ := (\mathcal{L}^+)^{-1}(\phi)$ and $\Pi_\phi^- := (\text{id}^*)^{-1}(\Pi_\phi^+)$. We can also extend the bijection $\mathcal{J}_{\mathcal{W}}^+$ to a bijection

$$\mathcal{J}_{\mathcal{W}} : \Pi_\phi \longrightarrow \widehat{\mathcal{S}}_\phi$$

by letting

$$\mathcal{J}_{\mathcal{W}}(\pi) = \begin{cases} \mathcal{J}_{\mathcal{W}}^+(\pi) & \text{if } \pi \in \Pi_\phi^+; \\ \mathcal{J}_{\mathcal{W}}^+(\text{id}^* \pi) \cdot \eta_- & \text{if } \pi \in \Pi_\phi^-, \end{cases}$$

where η_- is a character of \mathcal{S}_ϕ given by

$$\eta_-(a) = (-1)^{\dim \phi^a}$$

for $a \in \mathcal{S}_\phi$. One can easily check that \mathcal{L} and $\mathcal{J}_{\mathcal{W}}$ give us what we want:

Theorem 2.5.5. *Theorem 2.5.1 holds for n odd.*

Hence in the rest of this paper, we will focus on proving Theorem 2.5.1 for n even.

3. THETA CORRESPONDENCE

In this section, we recall the notion of the Weil representation and local theta correspondence.

3.1. Weil representations. Let V be a Hermitian space and W a skew-Hermitian space. To consider the theta correspondence for the reductive dual pair $U(V) \times U(W)$, one requires some additional data:

- a non-trivial additive character ψ of F ;
- a pair of characters χ_V and χ_W of E^\times such that

$$\chi_V|_{F^\times} = \omega_{E/F}^{\dim V} \quad \text{and} \quad \chi_W|_{F^\times} = \omega_{E/F}^{\dim W};$$

- a trace zero element $\delta \in E^\times$.

To elaborate, the tensor product $V \otimes W$ has a natural symplectic form defined by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \text{Tr}_{E/F}(\langle v_1, v_2 \rangle_V \cdot \langle w_1, w_2 \rangle_W).$$

Then there is a natural map

$$U(V) \times U(W) \longrightarrow Sp(V \otimes W).$$

One has the metaplectic S^1 -cover $Mp(V \otimes W)$ of $Sp(V \otimes W)$, and the character ψ (together with the form $\langle \cdot, \cdot \rangle$ on $V \otimes W$) determines a Weil representation ω_ψ of $Mp(V \otimes W)$. The datum $\underline{\psi} := (\psi, \chi_V, \chi_W, \delta)$ then allows one to specify a splitting of the metaplectic cover over $U(V) \times U(W)$. In [Kud94], [HKS96], it is showed that this splitting in fact does not depend on the choice of δ . Hence, we have a Weil

representation $\omega_{\underline{\psi},V,W}$ of $U(V) \times U(W)$. The Weil representation $\omega_{\underline{\psi},V,W}$ depends only on the orbit of ψ under $\text{Nm}_{E/F} E^\times$.

3.2. Local theta correspondence. Given an irreducible representation π of $U(W)$, the maximal π -isotypic quotient of $\omega_{\underline{\psi},V,W}$ is of the form

$$\Theta_{\underline{\psi},V,W}(\pi) \boxtimes \pi$$

for some smooth representation $\Theta_{\underline{\psi},V,W}(\pi)$ of $U(V)$ of finite length. By the Howe duality, which was proved by Waldspurger [Wal90] for $p \neq 2$ and by Gan-Takeda [GT16a], [GT16b] for any p , we have

- The maximal semi-simple quotient $\theta_{\underline{\psi},V,W}(\pi)$ of $\Theta_{\underline{\psi},V,W}(\pi)$ is irreducible if $\Theta_{\underline{\psi},V,W}(\pi)$ is non-zero;
- If π_1 and π_2 are irreducible smooth representations of $U(W)$, such that both $\theta_{\underline{\psi},V,W}(\pi_1)$ and $\theta_{\underline{\psi},V,W}(\pi_2)$ are non-zero. Assume that $\pi_1 \not\cong \pi_2$. Then $\theta_{\underline{\psi},V,W}(\pi_1) \not\cong \theta_{\underline{\psi},V,W}(\pi_2)$.

In this paper, we use the theta correspondence for $U(V) \times U(W)$ with

$$|\dim V - \dim W| \leq 1$$

to construct the LLC for even unitary groups. In our proofs, we shall use some results in the context of the theta correspondence from [GI14]. We emphasize that the proofs of those results are independent of the LLC for unitary groups.

Here we give a generalization of [GI14] Proposition C.4, which will be frequently used in later proofs.

Lemma 3.2.1. *Let $l = \dim W - \dim V$. Assume that $l = -1$. Let π be an irreducible tempered representation of $U(W)$ such that*

$$\pi \subset \text{Ind}_Q^{U(W)}(\tau\chi_V \boxtimes \pi_0),$$

where Q is a maximal parabolic subgroup of $U(W)$ with Levi component $GL_k(E) \times U(W_0)$, τ is an irreducible (unitary) discrete series representation of $GL_k(E)$ and π_0 is an irreducible tempered representation of $U(W_0)$. Let

$$m_Q(\pi) = \dim \text{Hom}_{U(W)}\left(\pi, \text{Ind}_Q^{U(W)}(\tau\chi_V \boxtimes \pi_0)\right),$$

and

$$m_P\left(\theta_{\underline{\psi},V,W}(\pi)\right) = \dim \text{Hom}_{U(V)}\left(\theta_{\underline{\psi},V,W}(\pi), \text{Ind}_P^{U(V)}(\tau\chi_W \boxtimes \theta_{\underline{\psi},V_0,W_0}(\pi_0))\right).$$

Then we have

$$m_Q(\pi) \leq m_P\left(\theta_{\underline{\psi},V,W}(\pi)\right).$$

Proof. The proof is almost the same as that of [GI14] Proposition C.1. The only difference is that we count the multiplicities. Readers may also refer to [CZ21] Lemma 3.5 and Lemma 8.2. □

4. CONSTRUCTIONS

In this section, we will construct an LLC for even unitary groups. We will first construct such a correspondence for tempered representations, and then extend the construction to non-tempered representations based on the tempered case. Several properties listed in Theorem 2.5.1 will be proved along the way.

Before we start, we set up some notations here. For $\epsilon = \pm 1$, let \mathcal{V}^ϵ be the Witt tower of Hermitian spaces which consists of all V_{2n+1}^ϵ , where V_{2n+1}^ϵ is the $(2n + 1)$ -dimensional Hermitian space over E with sign ϵ . Similarly, let \mathcal{W}^ϵ be the Witt tower of skew-Hermitian spaces which consists of all W_{2n}^ϵ , where W_{2n}^ϵ is the $2n$ -dimensional skew-Hermitian space over E with sign ϵ . Let

$$\underline{\psi} = (\psi, \chi_V, \chi_W, \delta)$$

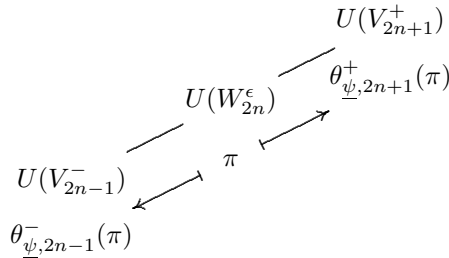
be a tuple of data described in Section 3.1. Let W be an even dimensional skew-Hermitian space. For an irreducible smooth representation π of $U(W)$, we will use $\theta_{\underline{\psi}, 2n+1}^\epsilon(\pi)$ to denote the theta lift of π to V_{2n+1}^ϵ , with respect to the datum $\underline{\psi}$. Similarly, let V be an odd dimensional Hermitian space. For an irreducible smooth representation σ of $U(V)$, we will use $\theta_{\underline{\psi}, 2n}^\epsilon(\sigma)$ to denote the theta lift of σ to W_{2n}^ϵ , with respect to the datum $\underline{\psi}$.

4.1. Construction of \mathcal{L} . First of all, we attach L -parameters to irreducible tempered representations of even unitary groups. We shall use two steps to achieve this purpose. In the first step, for each tuple of data $\underline{\psi} = (\psi, \chi_V, \chi_W, \delta)$, we construct a map

$$\mathcal{L}_{\underline{\psi}} : \text{Irr}_{\text{temp}} U(W_{2n}^+) \sqcup \text{Irr}_{\text{temp}} U(W_{2n}^-) \longrightarrow \Phi_{\text{temp}}(2n).$$

Then in the second step, we show that indeed $\mathcal{L}_{\underline{\psi}}$ is independent of the choice of $\underline{\psi}$, so we get the desired map \mathcal{L} .

In this subsection we do the first step. Fix a tuple of data $\underline{\psi} = (\psi, \chi_V, \chi_W, \delta)$. Given $\pi \in \text{Irr}_{\text{temp}} U(W_{2n}^\epsilon)$, consider its theta lifts to $U(V_{2n+1}^+)$ and $U(V_{2n-1}^-)$:



By the conservation relation (see [M12], [SZ15]), we know that exactly one of these two representations is non-zero. Also, by [GI14] Proposition C.4, this non-zero representation is also tempered.

Case 1. If $\sigma := \theta_{\underline{\psi}, 2n+1}^+(\pi) \neq 0$, then we have:

Lemma 4.1.1. *Let ϕ_σ be the L -parameter of σ . Then we have $\chi_W \subset \phi_\sigma$.*

Proof. By the Howe duality, $\theta_{\underline{\psi}, 2n}^\epsilon(\sigma) = \pi$ is non-zero. Hence by [GI14] Proposition 11.2, $\gamma(s, \sigma, \chi_W^{-1}, \psi)$ has a pole at $s = 1$. Applying the LLC for odd unitary groups, we get

$$\gamma(s, \sigma, \chi_W^{-1}, \psi) = \gamma(s, \phi_\sigma \chi_W^{-1}, \psi_E).$$

Since σ is tempered, ϕ_σ is also tempered. This implies that $\phi_\sigma \chi_W^{-1}$ contains a trivial representation. Hence we conclude $\chi_W \subset \phi_\sigma$ as desired. \square

In this case, we define $\mathcal{L}_{\underline{\psi}}(\pi)$ to be

$$\phi := (\phi_\sigma - \chi_W) \chi_W^{-1} \chi_V.$$

Case 2. If $\sigma := \theta_{\underline{\psi}, 2n-1}^-(\pi) \neq 0$. Let ϕ_σ be the L -parameter of σ . In this case, we simply define $\mathcal{L}_{\underline{\psi}}(\pi)$ to be

$$\phi := \phi_\sigma \chi_W^{-1} \chi_V \oplus \chi_V.$$

Notice that in either case, ϕ is a tempered parameter. Thus we get a map

$$\mathcal{L}_{\underline{\psi}} : \text{Irr}_{\text{temp}} U(W_{2n}^+) \sqcup \text{Irr}_{\text{temp}} U(W_{2n}^-) \longrightarrow \Phi_{\text{temp}}(2n).$$

Lemma 4.1.2. *Let π be an irreducible tempered representation of $U(W_{2n}^+)$. Let ϕ be the L -parameter of π in the sense of Mok’s LLC for quasi-split unitary groups, i.e. $\phi = \mathcal{L}^+(\pi)$. Then we have*

$$\mathcal{L}_{\underline{\psi}}(\pi) = \phi.$$

Proof. See [GI14] Page 652. (If we only consider the case $\pi \in \text{Irr}_{\text{temp}} U(W_{2n}^+)$, the proof in [GI14] Page 652 will only involve Mok’s LLC for quasi-split unitary groups, without referring any non quasi-split unitary groups.) \square

4.2. Independency. In the previous subsection we have constructed the map $\mathcal{L}_{\underline{\psi}}$. Now we do the second step.

Lemma 4.2.1. *Let $\pi \in \text{Irr}_{\text{temp}} U(W_{2n}^\epsilon)$ and $\phi = \mathcal{L}_{\underline{\psi}}(\pi)$.*

(1) *For any character χ of E^\times , we have*

$$\gamma(s, \pi, \chi, \psi) = \gamma(s, \phi\chi, \psi_E).$$

(2) *For any irreducible square-integrable representation τ of $GL_k(E)$ with L -parameter ϕ_τ , we have*

$$\begin{aligned} \mu_\psi(\tau_s \boxtimes \pi) &= \gamma(s, \phi_\tau \otimes \phi^\vee, \psi_E) \cdot \gamma(-s, \phi_\tau^\vee \otimes \phi, \psi_E^{-1}) \\ &\quad \times \gamma(2s, As^{(-1)^n} \circ \phi_\tau, \psi) \cdot \gamma(-2s, As^{(-1)^n} \circ \phi_\tau^\vee, \psi^{-1}). \end{aligned}$$

Proof. We only prove the first statement here. The proof of the second is similar. According to our construction, we need to consider two cases.

Case 1. Suppose that $\sigma := \theta_{\underline{\psi}, 2n+1}^+(\pi) \neq 0$. Then for any character χ of E^\times , by [GI14] Theorem 11.5, we have

$$\frac{\gamma(s, \sigma, \chi \chi_W^{-1} \chi_V, \psi)}{\gamma(s, \pi, \chi, \psi)} = \gamma(s, \chi \chi_V, \psi_E).$$

Let ϕ_σ be the L -parameter of σ . It follows from our construction that

$$\phi_\sigma = \phi \chi_V^{-1} \chi_W \oplus \chi_W.$$

By the LLC for odd unitary groups, we have

$$\begin{aligned} \gamma(s, \sigma, \chi \chi_W^{-1} \chi_V, \psi) &= \gamma(s, \phi_\sigma \chi \chi_W^{-1} \chi_V, \psi_E) \\ &= \gamma(s, \phi\chi, \psi_E) \cdot \gamma(s, \chi \chi_V, \psi_E). \end{aligned}$$

Combining these equalities, we get

$$\gamma(s, \pi, \chi, \psi) = \gamma(s, \phi\chi, \psi_E).$$

Hence the first statement holds in this case.

Case 2. Suppose that $\sigma := \theta_{\psi, 2n-1}^-(\pi) \neq 0$. In this case the desired formula also follows from a similar computation. We omit the details here.

□

Corollary 4.2.2. *The map $\mathcal{L}_{\underline{\psi}}$ is independent of the choice of $\underline{\psi}$.*

Proof. Assume that $\underline{\psi}' = (\psi', \chi'_V, \chi'_W, \delta)$ is another tuple of data. We define the map

$$\mathcal{L}_{\underline{\psi}'} : \text{Irr}_{\text{temp}} U(W_{2n}^+) \sqcup \text{Irr}_{\text{temp}} U(W_{2n}^-) \longrightarrow \Phi_{\text{temp}}(2n)$$

in a similar procedure. By Lemma 4.1.2, the restrictions of both $\mathcal{L}_{\underline{\psi}}$ and $\mathcal{L}_{\underline{\psi}'}$ to $\text{Irr}_{\text{temp}} U(W_{2n}^+)$ coincide with \mathcal{L}^+ , i.e.

$$\mathcal{L}_{\underline{\psi}} \Big|_{\text{Irr}_{\text{temp}} U(W_{2n}^+)} = \mathcal{L}^+ = \mathcal{L}_{\underline{\psi}'} \Big|_{\text{Irr}_{\text{temp}} U(W_{2n}^+)}.$$

Now given any irreducible tempered representation π of $U(W_{2n}^\epsilon)$, we can find a representation $\pi' \in \text{Irr}_{\text{temp}} U(W_{2n}^+)$, such that

$$\mathcal{L}^+(\pi') = \mathcal{L}_{\underline{\psi}}(\pi') = \mathcal{L}_{\underline{\psi}}(\pi).$$

Hence by Lemma 4.2.1, for all $k \geq 1$, and all irreducible square-integrable representation τ of $GL_k(E)$, we have

$$\mu_{\psi'}(\tau_s \boxtimes \pi') = C_{\psi', \psi, 2n, k} \cdot \mu_{\psi}(\tau_s \boxtimes \pi') = C_{\psi', \psi, 2n, k} \cdot \mu_{\psi}(\tau_s \boxtimes \pi) = \mu_{\psi'}(\tau_s \boxtimes \pi),$$

where $C_{\psi', \psi, 2n, k}$ is a constant only depends on $\psi', \psi, 2n$ and k . This equality, together with Lemma 4.2.1 and [GI16] Lemma A.6, implies that

$$\mathcal{L}^+(\pi') = \mathcal{L}_{\underline{\psi}'}(\pi') = \mathcal{L}_{\underline{\psi}'}(\pi).$$

Hence $\mathcal{L}_{\underline{\psi}}(\pi) = \mathcal{L}_{\underline{\psi}'}(\pi)$. In other words, $\mathcal{L}_{\underline{\psi}}$ is independent of the choice of $\underline{\psi}$. □

After proving that the map $\mathcal{L}_{\underline{\psi}}$ is indeed independent of the choice of $\underline{\psi}$, we will denote the map abstractly by \mathcal{L} . For an irreducible tempered representation π of $U(W_{2n}^\epsilon)$, we call $\mathcal{L}(\pi)$ the L -parameter of π . For a tempered L -parameter ϕ , we let Π_ϕ be the fiber $\mathcal{L}^{-1}(\phi)$, and call it the L -packet of ϕ . For $\epsilon = \pm 1$, we also let $\Pi_\phi^\epsilon = \Pi_\phi \cap \text{Irr} U(W_{2n}^\epsilon)$. Combining Lemma 4.2.1 and Corollary 4.2.2, we get

Corollary 4.2.3. *For tempered representations, the map \mathcal{L} respects standard γ -factors and Plancherel measures.*

4.3. Counting sizes of packets. Our next goal is to attach a character of component group to each irreducible tempered representation of even unitary groups. To do this, we need some preparations. In this subsection we consider the behaviour of L -parameters under the local theta correspondence and count the sizes of L -packets for even unitary groups. In this subsection when we talk about representations of odd unitary groups, the L -parameter of a representation is in the sense of Theorem 2.5.5; whereas when we talk about representations of even unitary groups, the L -parameter of a tempered representation is in the sense of \mathcal{L} .

To define the map $\mathcal{J}_{\mathcal{W}}$, we need to fix a Whittaker datum \mathcal{W} of $U(W_{2n}^+)$. As explained in Section 2.3, once we fix the Whittaker datum \mathcal{W} , we may pick up a non-trivial additive character ψ of F , such that

$$\mathcal{W} = \mathcal{W}_\psi.$$

We fix a pair of characters (χ_V, χ_W) of E^\times and a trace zero element $\delta \in E^\times$ as in Section 3.1. If there is no further explanation, the theta lifts used in the rest of this section will be with respect to the datum

$$\underline{\psi} = (\psi, \chi_V, \chi_W, \delta).$$

We shall use $\mathcal{L}_{\underline{\psi}}$ to ‘realize’ the map \mathcal{L} . For simplicity, we shall drop the subscript “ $\underline{\psi}$ ” and just denote $\theta_{\underline{\psi},*}^\pm$ by θ_*^\pm . In the rest of this section, if ρ is an irreducible smooth representation of some group G , we shall use the symbol ϕ_ρ to denote the L -parameter of ρ . If G is an odd unitary group, we shall also use the symbol η_ρ to denote the character of \mathcal{S}_{ϕ_ρ} associated to ρ .

Lemma 4.3.1.

(1) *If $\pi \in \text{Irr}_{\text{temp}} U(W_{2n}^\epsilon)$, such that $\sigma := \theta_{2n+1}^{\epsilon'}(\pi) \neq 0$. Then*

$$\phi_\sigma = \phi_\pi \chi_V^{-1} \chi_W \oplus \chi_W.$$

(2) *Similarly, if $\sigma \in \text{Irr}_{\text{temp}} U(V_{2m-1}^\epsilon)$, such that $\pi := \theta_{2m}^{\epsilon'}(\sigma) \neq 0$. Then*

$$\phi_\pi = \phi_\sigma \chi_W^{-1} \chi_V \oplus \chi_V.$$

Proof. With Lemma 4.2.1 at hand, we can appeal to the same argument of [GI14] Page 652 to prove this lemma. We omit the details here. □

As a consequence, we deduce

Corollary 4.3.2.

(1) *Let $\pi \in \text{Irr}_{\text{temp}} U(W_{2n}^\epsilon)$. If $\chi_V \not\subset \phi_\pi$, then $\theta_{2n-1}^\pm(\pi) = 0$. Hence by the conservation relation, both $\theta_{2n+1}^+(\pi)$ and $\theta_{2n+1}^-(\pi)$ are non-zero.*

(2) *Similarly, let $\sigma \in \text{Irr}_{\text{temp}} U(V_{2m-1}^\epsilon)$. If $\chi_W \not\subset \phi_\sigma$, then $\theta_{2m-2}^\pm(\sigma) = 0$. Hence by the conservation relation, both $\theta_{2m}^+(\sigma)$ and $\theta_{2m}^-(\sigma)$ are non-zero.*

Lemma 4.3.3.

(1) *Let $\pi \in \text{Irr}_{\text{temp}} U(W_{2n}^\epsilon)$. If $\chi_V \subset \phi_\pi$, then exactly one of $\theta_{2n-1}^+(\pi)$ and $\theta_{2n-1}^-(\pi)$ is non-zero.*

(2) *Similarly, let $\sigma \in \text{Irr}_{\text{temp}} U(V_{2m-1}^\epsilon)$. If $\chi_W \subset \phi_\sigma$, then exactly one of $\theta_{2m-2}^+(\sigma)$ and $\theta_{2m-2}^-(\sigma)$ is non-zero.*

Proof. We only prove the first statement here. The proof of the second statement is similar. We shall prove this by counting fibers of the map $\mathcal{L} = \mathcal{L}_{\underline{\psi}}$. We define a map

$$\theta_{2n+1} : \text{Irr } U(W_{2n}^+) \sqcup \text{Irr } U(W_{2n}^-) \longrightarrow \text{Irr } U(V_{2n+1}^+) \sqcup \text{Irr } U(V_{2n+1}^-)$$

as follows:

$$\pi' \mapsto \begin{cases} \theta_{2n+1}^+(\pi') & \text{if } \theta_{2n+1}^+(\pi') \neq 0; \\ \theta_{2n+1}^-(\pi') & \text{otherwise.} \end{cases}$$

By the Howe duality and the conservation relation, this map is well-defined and injective. For each tempered L -parameter ϕ , by Lemma 4.3.1, the restriction of this map to the L -packet Π_ϕ gives an injection

$$\theta_{2n+1} : \Pi_\phi \hookrightarrow \Pi_{\phi^+},$$

where $\phi^+ := \phi \chi_V^{-1} \chi_W \oplus \chi_W$.

Now we let $\phi = \phi_\pi$. By our assumption, $\chi_V \subset \phi$. Let $\phi^- := (\phi - \chi_V)\chi_V^{-1}\chi_W$.

Case 1. If $2\chi_V \subset \phi$, then $\chi_W \subset \phi^-$. Similarly we can define another map

$$\theta_{2n} : \text{Irr } U(V_{2n-1}^+) \sqcup \text{Irr } U(V_{2n-1}^-) \longrightarrow \text{Irr } U(W_{2n}^+) \sqcup \text{Irr } U(W_{2n}^-)$$

in the same way as θ_{2n+1} . Again by Lemma 4.3.1, the restriction of this map to the packet Π_{ϕ^-} gives an injection

$$\theta_{2n} : \Pi_{\phi^-} \hookrightarrow \Pi_\phi.$$

Hence we have

$$|\Pi_{\phi^-}| \leq |\Pi_\phi| \leq |\Pi_{\phi^+}|.$$

But in this case, $\mathcal{S}_{\phi^-} \simeq \mathcal{S}_\phi \simeq \mathcal{S}_{\phi^+}$, by using the LLC for odd unitary groups, we get

$$|\Pi_{\phi^-}| = |\widehat{\mathcal{S}_{\phi^-}}| = |\widehat{\mathcal{S}_{\phi^+}}| = |\Pi_{\phi^+}|.$$

This implies that θ_{2n} is surjective. Hence in this case the lemma holds.

Case 2. If $2\chi_V \not\subset \phi$, then $\chi_W \not\subset \phi^-$. We can define a map

$$\theta_{2n}^+ \sqcup \theta_{2n}^- : \Pi_{\phi^-} \sqcup \Pi_{\phi^-} \longrightarrow \Pi_\phi$$

by

$$\begin{cases} \sigma \mapsto \theta_{2n}^+(\sigma) & \text{for } \sigma \text{ in the first copy of } \Pi_{\phi^-}; \\ \sigma \mapsto \theta_{2n}^-(\sigma) & \text{for } \sigma \text{ in the second copy of } \Pi_{\phi^-}. \end{cases}$$

Again, by the Howe duality, the conservation relation, and Corollary 4.3.2, it's easy to see that this map is well-defined and injective. Thus we have

$$2|\Pi_{\phi^-}| \leq |\Pi_\phi| \leq |\Pi_{\phi^+}|.$$

Also, in this case,

$$\mathcal{S}_{\phi^+} \simeq \mathcal{S}_\phi \simeq \mathcal{S}_{\phi^-} \oplus (\mathbb{Z}/2\mathbb{Z})e,$$

where e is the element in \mathcal{S}_ϕ corresponding to $\chi_V \subset \phi$. By using the LLC for odd unitary groups, we get

$$2|\Pi_{\phi^-}| = 2|\widehat{\mathcal{S}_{\phi^-}}| = |\widehat{\mathcal{S}_{\phi^+}}| = |\Pi_{\phi^+}|.$$

This implies that $\theta_{2n}^+ \sqcup \theta_{2n}^-$ is surjective. Hence in this case the lemma also holds. □

As a consequence of this lemma, we can compute the sizes of L -packets.

Corollary 4.3.4. *Let $\phi \in \Phi_{\text{temp}}(2n)$. Then the size of the L -packet Π_ϕ is exactly the same as the size of $\widehat{\mathcal{S}_\phi}$. In particular, the packet is non-empty.*

Proof. The case when $\chi_V \subset \phi$ follows directly from the proof of Lemma 4.3.3. So it is sufficient to prove the case when $\chi_V \not\subset \phi$. Similar to the proof of Lemma 4.3.3, the theta lift gives us injections

$$\theta_{2n+1}^\epsilon : \Pi_\phi \hookrightarrow \Pi_{\phi^+}^\epsilon$$

for $\epsilon = \pm 1$. Lemma 4.3.3 tells us these injections are also surjective. Notice that in this case, we have

$$\mathcal{S}_{\phi^+} \simeq \mathcal{S}_\phi \oplus (\mathbb{Z}/2\mathbb{Z})e,$$

where e is the element in \mathcal{S}_{ϕ^+} corresponding to $\chi_W \subset \phi^+$. This induces an isomorphism

$$\mathcal{S}_\phi \hookrightarrow \mathcal{S}_{\phi^+} \xrightarrow{\overline{}} \overline{\mathcal{S}_{\phi^+}}.$$

Hence by the LLC for odd unitary groups, we conclude that

$$|\Pi_\phi| = |\Pi_{\phi^+}^\epsilon| = |\widehat{\mathcal{S}}_{\phi^+}| = |\widehat{\mathcal{S}}_\phi|$$

as desired. □

4.4. Construction of $\mathcal{J}_\mathcal{W}$. Now given a tempered parameter $\phi \in \Phi_{\text{temp}}(2n)$, we have shown the size of the L -packet Π_ϕ is the same as $\widehat{\mathcal{S}}_\phi$. Next, we are going to define the bijection

$$\mathcal{J}_\mathcal{W} : \Pi_\phi \longrightarrow \widehat{\mathcal{S}}_\phi.$$

We separate the construction into two cases.

Case 1. If $\chi_V \not\subset \phi$, then by Corollary 4.3.2, we have $\sigma := \theta_{2n+1}^+(\pi) \neq 0$. And by our construction, $\phi_\sigma = \phi\chi_V^{-1}\chi_W \oplus \chi_W$. Therefore

$$\mathcal{S}_{\phi_\sigma} \simeq \mathcal{S}_\phi \oplus (\mathbb{Z}/2\mathbb{Z})e,$$

where e is the element in $\mathcal{S}_{\phi_\sigma}$ corresponding to $\chi_W \subset \phi_\sigma$. This induces an isomorphism

$$\iota : \mathcal{S}_\phi \hookrightarrow \mathcal{S}_{\phi_\sigma} \twoheadrightarrow \overline{\mathcal{S}_{\phi_\sigma}}.$$

In this case we define the character $\eta \in \widehat{\mathcal{S}}_\phi$ associated to π to be

$$\eta := \eta_\sigma|_{\mathcal{S}_\phi}.$$

Case 2. If $\chi_V \subset \phi$, then by the Lemma 4.3.3, there exists a unique ϵ' , such that $\theta_{2n-1}^{\epsilon'}(\pi)$ is non-zero, hence $\sigma := \theta_{2n+1}^{\epsilon'}(\pi)$ is also non-zero by the persistence of theta lifts. According to Lemma 4.3.1, $\phi_\sigma = \phi\chi_V^{-1}\chi_W \oplus \chi_W$. Thus

$$\mathcal{S}_\phi \simeq \mathcal{S}_{\phi_\sigma}.$$

In this case we define the character $\eta \in \widehat{\mathcal{S}}_\phi$ associated to π to be

$$\eta := \eta_\sigma|_{\mathcal{S}_\phi}.$$

By the LLC for odd unitary groups, the Howe duality, and Corollary 4.3.4, it is easy to check that the assignment constructed here gives a bijection between Π_ϕ and $\widehat{\mathcal{S}}_\phi$.

4.5. From tempered to non-tempered. So far, we have attached L -parameters and characters of component groups for all irreducible tempered representations of $U(W_{2n}^\epsilon)$. Next, for an irreducible non-tempered representation π of $U(W_{2n}^\epsilon)$, we shall attach an L -parameter and a character of component group to it. Readers may also refer to [ABPS14].

Let π be an irreducible smooth representation of $U(W_{2n}^\epsilon)$. By Langlands' classification for p -adic groups [Sil78], [Kon03], we know that π is the unique irreducible quotient of a standard module

$$\text{Ind}_P^{U(W_{2n}^\epsilon)} (\tau_1 | \det |^{s_1} \boxtimes \cdots \boxtimes \tau_r | \det |^{s_r} \boxtimes \pi_0),$$

where P is a parabolic subgroup of $U(W_{2n}^\epsilon)$, with a Levi component

$$M \simeq GL_{k_1}(E) \times \cdots \times GL_{k_r}(E) \times U(W_{2n}^\epsilon), \quad k = k_1 + \cdots + k_r;$$

τ_i is an irreducible (unitary) square-integrable representation of $GL_{k_i}(E)$, s_i is a real number such that

$$s_1 \geq \cdots \geq s_r > 0;$$

and π_0 is an irreducible tempered representation of $U(W_{2n-2k}^\epsilon)$. Let ϕ_{τ_i} be the L -parameter of τ_i , and $\pi_0 = \pi(\phi_0, \eta_0)$. We define the L -parameter of π to be

$$\phi = (\phi_{\tau_1} | \cdot |^{s_1} \oplus \cdots \oplus \phi_{\tau_r} | \cdot |^{s_r}) \oplus \phi_0 \oplus ((\phi_{\tau_1} | \cdot |^{s_1} \oplus \cdots \oplus \phi_{\tau_r} | \cdot |^{s_r})^c)^\vee.$$

Notice that $\mathcal{S}_\phi \simeq \mathcal{S}_{\phi_0}$. Via this natural identification, we define the character in $\widehat{\mathcal{S}}_\phi$ associated to π to be

$$\eta = \eta_0.$$

Since the datum (P, τ_i, s_i, π_0) is uniquely determined by π up to Weyl group conjugate, ϕ and η are well-defined.

From now on, we shall use $\pi(\phi, \eta)$ to denote the element in Π_ϕ corresponding to η . It follows directly from our construction that

Proposition 4.5.1. *The LLC we constructed for even unitary groups is compatible with Langlands quotients.*

An easy computation shows that

Proposition 4.5.2. *The LLC we constructed for even unitary groups respects standard γ -factors and Plancherel measures.*

Proof. We have proved this proposition for tempered representations. The general case follows from Lemma 4.2.3 and multiplicativity of standard γ -factors and Plancherel measures (see [GI14] Section 10.2, and Appendix B.5). □

4.6. Preservation. In this section we prove two further properties of the map \mathcal{L} .

Proposition 4.6.1. *The map \mathcal{L} preserves square-integrability.*

Proof. Let π be an irreducible smooth representation of $U(W_{2n}^\epsilon)$, and ϕ be the L -parameter of π . We first prove that if π is square-integrable, then ϕ is square-integrable. We divide this into two cases.

Case 1. If $\chi_V \not\subset \phi$, then by the Corollary 4.3.2, $\theta_{2n-1}^+(\pi) = 0$ and $\sigma := \theta_{2n+1}^+(\pi) \neq 0$. Hence by [GI14] Corollary C.3, σ is also square-integrable. The LLC for odd unitary groups then implies that

$$\phi_\sigma = \phi \chi_V^{-1} \chi_W \oplus \chi_W$$

is square-integrable. Thus ϕ is also square-integrable.

Case 2. If $\chi_V \subset \phi$, then by Lemma 4.3.3, there exists $\epsilon' \in \{\pm 1\}$, such that $\sigma := \theta_{2n-1}^{\epsilon'}(\pi) \neq 0$. Hence by [GI14] Corollary C.3, σ is also square-integrable. The LLC for odd unitary groups then implies that ϕ_σ is square-integrable. We claim that $\chi_W \not\subset \phi_\sigma$. Indeed, suppose on the contrary that $\chi_W \subset \phi_\sigma$, by Lemma 4.3.3 and the conservation relation, we must have

$$\theta_{2n-2}^\epsilon(\sigma) \neq 0.$$

Again by [GI14] Corollary C.3, $\pi = \theta_{2n}^\epsilon(\sigma)$ cannot be square-integrable. This contradicts with our assumption. It follows that

$$\phi = \phi_\sigma \chi_W^{-1} \chi_V \oplus \chi_V$$

is also square-integrable.

Now it remains to prove that if ϕ is square-integrable, then π is square-integrable. Indeed the proof follows from the same idea of the proof of the first part. We omit the details here. □

Proposition 4.6.2. *The map \mathcal{L} preserves temperedness.*

Proof. This automatically follows from our construction. □

5. PREPARATIONS FOR THE PROOF OF LOCAL INTERTWINING RELATION

Our next step is to prove that \mathcal{L} and $\mathcal{J}_{\mathcal{H}}$ satisfy the LIR. In this section, we first briefly recall the definition of normalized intertwining operators, following [GI16] Section 7; and then recall a result in [GI16], which is the ingredient of our later proof. Fix $\varepsilon = \pm 1$. In this section, we let V and W be an ε -Hermitian space and an $(-\varepsilon)$ -Hermitian space respectively. Put

$$m = \dim V \quad \text{and} \quad n = \dim W.$$

5.1. Parabolic subgroups. Let r be the Witt index of V and V_{an} an anisotropic kernel of V . Choose a basis $\{v_i, v_i^* \mid i = 1, \dots, r\}$ of the orthogonal complement of V_{an} such that

$$\langle v_i, v_j \rangle_V = \langle v_i^*, v_j^* \rangle_V = 0, \quad \langle v_i, v_j^* \rangle_V = \delta_{i,j}$$

for $1 \leq i, j \leq r$. Let k be a positive integer with $k \leq r$ and set

$$X = Ev_1 \oplus \dots \oplus Ev_k, \quad X^* = Ev_1^* \oplus \dots \oplus Ev_k^*.$$

Let V_0 be the orthogonal complement of $X \oplus X^*$ in V , so that V_0 is a ε -Hermitian space of dimension $m_0 = m - 2k$ over E . We shall write an element in the unitary group $U(V)$ as a block matrix relative to the decomposition $V = X \oplus V_0 \oplus X^*$. Let $P = M_P U_P$ be the maximal parabolic subgroup of $U(V)$ stabilizing X , where M_P is the Levi component of P stabilizing X^* and U_P is the unipotent radical of P . We have

$$M_P = \{m_P(a) \cdot h_0 \mid a \in GL(X), h_0 \in U(V_0)\},$$

$$U_P = \{u_P(b) \cdot u_P(c) \mid b \in \text{Hom}(V_0, X), c \in \text{Herm}(X^*, X)\},$$

where

$$m_P(a) = \begin{pmatrix} a & & \\ & 1_{V_0} & \\ & & (a^*)^{-1} \end{pmatrix},$$

$$u_P(b) = \begin{pmatrix} 1_X & b & -\frac{1}{2}bb^* \\ & 1_{V_0} & -b^* \\ & & 1_{X^*} \end{pmatrix},$$

$$u_P(c) = \begin{pmatrix} 1_X & & c \\ & 1_{V_0} & \\ & & 1_{X^*} \end{pmatrix},$$

and

$$\text{Herm}(X^*, X) = \{c \in \text{Hom}(X^*, X) \mid c^* = -c\}.$$

Here, the elements $a^* \in GL(X^*)$, $b^* \in \text{Hom}(X^*, V_0)$, and $c^* \in \text{Hom}(X^*, X)$ are the adjoints of a , b , and c respectively. In particular, $M_P \simeq GL(X) \times U(V_0)$ and we have an exact sequence

$$1 \longrightarrow \text{Herm}(X^*, X) \longrightarrow U_P \longrightarrow \text{Hom}(V_0, X) \longrightarrow 1.$$

Put

$$\rho_P = \frac{m_0 + k}{2}, \quad w_P = \begin{pmatrix} & & -I_X \\ & 1_{V_0} & \\ -\varepsilon I_X^{-1} & & \end{pmatrix},$$

where $I_X \in \text{Isom}(X^*, X)$ is defined by $I_X v_i^* = v_i$ for $1 \leq i \leq k$.

Similarly, let r' be the Witt index of W and choose a basis $\{w_i, w_i^* \mid i = 1, \dots, r'\}$ of the orthogonal complement of an anisotropic kernel of W such that

$$\langle w_i, w_j \rangle_W = \langle w_i^*, w_j^* \rangle_W = 0, \quad \langle w_i, w_j^* \rangle_W = \delta_{i,j}$$

for $1 \leq i, j \leq r'$. We assume that $k \leq r'$ and set

$$Y = Ew_1 \oplus \dots \oplus Ew_k, \quad Y^* = Ew_1^* \oplus \dots \oplus Ew_k^*.$$

Let W_0 be the orthogonal complement of $Y \oplus Y^*$ in W , so that W_0 is a $(-\varepsilon)$ -Hermitian space of dimension $n_0 = n - 2k$ over E . Let $Q = M_Q U_Q$ be the maximal parabolic subgroup of $U(W)$ stabilizing Y , where M_Q is the Levi component of Q stabilizing Y^* and U_Q is the unipotent radical of Q . For $a \in GL(Y)$, $b \in \text{Hom}(W_0, Y)$ and $c \in \text{Herm}(Y^*, Y)$, we define elements $m_Q(a) \in M_Q$ and $u_Q(b), u_Q(c) \in U_Q$ as above. We have $M_Q \simeq GL(Y) \times U(W_0)$ and

$$1 \longrightarrow \text{Herm}(Y^*, Y) \longrightarrow U_Q \longrightarrow \text{Hom}(W_0, Y) \longrightarrow 1.$$

Put

$$\rho_Q = \frac{n_0 + k}{2}, \quad w_Q = \begin{pmatrix} & & -I_Y \\ & 1_{W_0} & \\ \varepsilon I_Y^{-1} & & \end{pmatrix},$$

where $I_Y \in \text{Isom}(Y^*, Y)$ is defined by $I_Y w_i^* = w_i$ for $1 \leq i \leq k$.

5.2. Intertwining operators. To define the local intertwining operators, firstly we need to choose Haar measures on various groups. For this part, readers may refer to [GI16], Section 7.2. We follow their conventions on Haar measures.

Let τ be an irreducible (unitary) square-integrable representation of $GL(X)$ on a space \mathcal{V}_τ with central character ω_τ . For any $s \in \mathbb{C}$, we realize the representation $\tau_s := \tau \otimes |\det|^s$ on \mathcal{V}_τ by setting $\tau_s(a)v := |\det a|^s \tau(a)v$ for $a \in GL(X)$ and $v \in \mathcal{V}_\tau$. Let σ_0 be an irreducible tempered representation of $U(V_0)$ on a space \mathcal{V}_{σ_0} . We consider the induced representation

$$\text{Ind}_P^{U(V)}(\tau_s \boxtimes \sigma_0)$$

of $U(V)$, which is realized on the space of smooth functions $\Phi_s : U(V) \rightarrow \mathcal{V}_\tau \otimes \mathcal{V}_{\sigma_0}$ such that

$$\Phi_s(um_P(a)h_0h) = |\det a|^{s+\rho_P} \tau(a)\sigma_0(h_0)\Phi_s(h)$$

for all $u \in U_P$, $a \in GL(X)$, $h_0 \in U(V_0)$, and $h \in U(V)$. Let A_P be the split component of the center of M_P and $W(M_P) = N_{U(V)}(A_P)/M_P$ be the relative Weyl group for M_P . Noting that $W(M_P) \simeq \mathbb{Z}/2\mathbb{Z}$, we denote by w the non-trivial element in $W(M_P)$. For any representative $\tilde{w} \in U(V)$ of w , we define an unnormalized intertwining operator

$$\mathcal{M}(\tilde{w}, \tau_s \boxtimes \sigma_0) : \text{Ind}_P^{U(V)}(\tau_s \boxtimes \sigma_0) \longrightarrow \text{Ind}_P^{U(V)}(w(\tau_s \boxtimes \sigma_0))$$

by (the meromorphic continuation of) the integral

$$\mathcal{M}(\tilde{w}, \tau_s \boxtimes \sigma_0)\Phi_s(h) = \int_{U_P} \Phi_s(\tilde{w}^{-1}uh)du,$$

where $w(\tau_s \boxtimes \sigma_0)$ is the representation of M_P on $\mathcal{V}_\tau \otimes \mathcal{V}_{\sigma_0}$ given by

$$(w(\tau_s \boxtimes \sigma_0))(m) = (\tau_s \boxtimes \sigma_0)(\tilde{w}^{-1}m\tilde{w})$$

for $m \in M_P$.

Next we shall normalize the intertwining operator $\mathcal{M}(\tilde{w}, \tau_s \boxtimes \sigma_0)$, depending on the choice of a Whittaker datum. Having fixed the additive character ψ and the trace zero element δ , we define the sign $\epsilon(V)$ and use the Whittaker datum

$$\begin{cases} \mathcal{W}_{\psi^E} & \text{if } \epsilon = +1, \text{ where } \psi^E = \psi(\frac{1}{2} \text{Tr}_{E/F}(\delta \cdot)); \\ \mathcal{W}_\psi & \text{if } \epsilon = -1. \end{cases}$$

Also, we need to choose the following data appropriately:

- a representative \tilde{w} ;
- a normalizing factor $r(w, \tau_s \boxtimes \sigma_0)$;
- an intertwining isomorphism \mathcal{A}_w .

For the representative, we take $\tilde{w} \in U(V)$ defined by

$$\tilde{w} = w_P \cdot m_P \left((-1)^{m'} \cdot \kappa_V \cdot J \right) \cdot (-1_{V_0})^k,$$

where w_P is as in the previous subsections, $m' = [\frac{m}{2}]$,

$$\kappa_V = \begin{cases} -\delta & \text{if } m \text{ is even and } \epsilon = +1; \\ 1 & \text{if } m \text{ is even and } \epsilon = -1; \\ -1 & \text{if } m \text{ is odd and } \epsilon = +1; \\ -\delta & \text{if } m \text{ is odd and } \epsilon = -1, \end{cases}$$

and

$$J = \begin{pmatrix} & & & (-1)^{k-1} \\ & & & \\ & & \ddots & \\ & & -1 & \\ 1 & & & \end{pmatrix} \in GL_k(E).$$

Here, we have identified $GL(X)$ with $GL_k(E)$ using the basis v_1, \dots, v_k . In [GI16] Section 7.3, it is showed that the representative defined above coincides with the representative defined in [Mok15] when $\epsilon(V) = 1$.

Next we define the normalizing factor $r(w, \tau_s \boxtimes \sigma_0)$. Let $\lambda(E/F, \psi)$ be the Langlands λ -factor and put

$$\lambda(w, \psi) = \begin{cases} \lambda(E/F, \psi)^{(k-1)k/2} & \text{if } m \text{ is even;} \\ \lambda(E/F, \psi)^{(k+1)k/2} & \text{if } m \text{ is odd.} \end{cases}$$

Let ϕ_τ and ϕ_0 be the L -parameters of τ and σ_0 respectively. We set

$$r(w, \tau_s \boxtimes \sigma_0) = \lambda(w, \psi) \cdot \gamma(s, \phi_\tau \otimes \phi_0^\vee, \psi_E)^{-1} \cdot \gamma(2s, As^{(-1)^m} \circ \phi_\tau, \psi)^{-1},$$

and the normalized intertwining operator

$$\mathcal{R}(w, \tau_s \boxtimes \sigma_0) := |\kappa_V|^{k\rho_P} \cdot r(w, \tau_s \boxtimes \sigma_0)^{-1} \cdot \mathcal{M}(\tilde{w}, \tau_s \boxtimes \sigma_0).$$

Lemma 5.2.1. *The normalized intertwining operators satisfy the multiplicative property*

$$\mathcal{R}(w, w(\tau_s \boxtimes \sigma_0)) \circ \mathcal{R}(w, \tau_s \boxtimes \sigma_0) = 1,$$

as well as the adjoint property

$$\mathcal{R}(w, w(\tau_s \boxtimes \sigma_0))^* = \mathcal{R}(w, \tau_{-\bar{s}} \boxtimes \sigma_0).$$

In particular, when s is purely imaginary, $\mathcal{R}(w, \tau_s \boxtimes \sigma_0)$ is unitary. Hence the normalized intertwining operator $\mathcal{R}(w, \tau_s \boxtimes \sigma_0)$ is holomorphic at $s = 0$.

Proof. An easy computation shows that

$$\mathcal{M}(\tilde{w}, \tau_s \boxtimes \sigma_0) = |\kappa_V|^{-k\rho_P} \ell(\tilde{w}) \circ \mathcal{M}_{\overline{P}|P}(\tau_s \boxtimes \sigma_0),$$

where

$$\ell(\tilde{w}) : \text{Ind}_{\overline{P}}^{U(V)}(\tau_s \boxtimes \sigma_0) \longrightarrow \text{Ind}_P^{U(V)}(w(\tau_s \boxtimes \sigma_0))$$

is defined by

$$\ell(\tilde{w})\Psi_s(h) = \Psi_s(\tilde{w}^{-1}h)$$

for $\Psi \in \text{Ind}_{\overline{P}}^{U(V)}(\tau_s \boxtimes \sigma_0)$. Here the factor $|\kappa_V|^{k\rho_P}$ arises because of our choices of the Haar measures on U_P and $U_P \times U_{\overline{P}}$ in the definition of $\mathcal{M}(\tilde{w}, \tau_s \boxtimes \sigma_0)$ and $\mathcal{M}_{\overline{P}|P}(\tau_s \boxtimes \sigma_0)$. Hence

$$\begin{aligned} &\mathcal{R}(w, w(\tau_s \boxtimes \sigma_0)) \circ \mathcal{R}(w, \tau_s \boxtimes \sigma_0) \\ &= r(w, \tau_s \boxtimes \sigma_0)^{-1} \cdot r(w, w(\tau_s \boxtimes \sigma_0))^{-1} \cdot \ell(\tilde{w}^2) \circ \mathcal{M}_{P|\overline{P}}(\tau_s \boxtimes \sigma_0) \circ \mathcal{M}_{\overline{P}|P}(\tau_s \boxtimes \sigma_0) \\ &= \lambda(w, \psi)^{-2} \cdot \frac{\gamma(s, \phi_\tau \otimes \phi_0^\vee, \psi_E) \cdot \gamma(2s, As^{(-1)^m} \circ \phi_\tau, \psi)}{\gamma(s, \phi_\tau \otimes \phi_0^\vee, \psi_E) \cdot \gamma(2s, As^{(-1)^m} \circ \phi_\tau, \psi)} \\ &\quad \times \frac{\gamma(-s, \phi_\tau^\vee \otimes \phi_0, \psi_E) \cdot \gamma(-2s, As^{(-1)^m} \circ \phi_\tau^\vee, \psi)}{\gamma(-s, \phi_\tau^\vee \otimes \phi_0, \psi_E^{-1}) \cdot \gamma(-2s, As^{(-1)^m} \circ \phi_\tau^\vee, \psi^{-1})} \cdot \ell(\tilde{w}^2) \\ &= \lambda(w, \psi)^{-2} \cdot \det(\phi_\tau^\vee \otimes \phi_0)(-1) \cdot \det(As^{(-1)^m} \circ \phi_\tau^\vee)(-1) \cdot (\tau_s \boxtimes \sigma_0)(\tilde{w}^2) \\ &= \lambda(w, \psi)^{-2} \cdot \omega_\tau(-1)^m \omega_{E/F}(-1)^{(m-1)mk} \\ &\quad \times \omega_\tau(-1)^k \omega_{E/F}(-1)^{\dim R\phi_\tau} \cdot \omega_\tau \left(\varepsilon \cdot \kappa_V (\kappa_V^c)^{-1} \cdot (-1)^{k-1} \right), \end{aligned}$$

where

$$R = \begin{cases} \Lambda^2 & \text{if } m \text{ is even;} \\ \text{Sym}^2 & \text{if } m \text{ is odd.} \end{cases}$$

It follows that

$$\mathcal{R}(w, w(\tau_s \boxtimes \sigma_0)) \circ \mathcal{R}(w, \tau_s \boxtimes \sigma_0) = 1$$

as desired. The adjoint property can be proved exactly the same as [Art13] Proposition 2.3.1. \square

Finally we define the intertwining isomorphism. Assume that $w(\tau \boxtimes \sigma_0) \simeq \tau \boxtimes \sigma_0$, which is equivalent to $(\tau^c)^\vee \simeq \tau$. We may take the unique isomorphism

$$\mathcal{A}_w : \mathcal{V}_\tau \otimes \mathcal{V}_{\sigma_0} \longrightarrow \mathcal{V}_\tau \otimes \mathcal{V}_{\sigma_0}$$

such that:

- $\mathcal{A}_w \circ (w(\tau \boxtimes \sigma_0))(m) = (\tau \boxtimes \sigma_0)(m) \circ \mathcal{A}_w$ for all $m \in M_P$;
- $\mathcal{A}_w = \mathcal{A}'_w \otimes 1_{\mathcal{V}_{\sigma_0}}$ with an isomorphism

$$\mathcal{A}'_w : \mathcal{V}_\tau \longrightarrow \mathcal{V}_\tau$$

such that $\Lambda \circ \mathcal{A}'_w = \Lambda$. Here, $\Lambda : \mathcal{V}_\tau \rightarrow \mathbb{C}$ is the unique (up to a scalar) Whittaker functional with respect to the Whittaker datum (N_k, ψ_{N_k}) , where N_k is the group of unipotent upper triangular matrices in $GL_k(E)$ and ψ_{N_k} is the generic character of N_k given by $\psi_{N_k}(x) = \psi_E(x_{1,2} + \dots + x_{k-1,k})$.

Note that $\mathcal{A}_w^2 = 1_{\mathcal{V}_\tau \otimes \mathcal{V}_{\sigma_0}}$. We define a self-intertwining operator

$$R(w, \tau \boxtimes \sigma_0) : \text{Ind}_P^{U(V)}(\tau \boxtimes \sigma_0) \longrightarrow \text{Ind}_P^{U(V)}(\tau \boxtimes \sigma_0)$$

by

$$R(w, \tau \boxtimes \sigma_0)\Phi(h) = \mathcal{A}_w(\mathcal{R}(w, \tau \boxtimes \sigma_0)\Phi(h))$$

for $\Phi \in \text{Ind}_P^{U(V)}(\tau \boxtimes \sigma_0)$, and $h \in U(V)$. By construction,

$$R(w, \tau \boxtimes \sigma_0)^2 = 1.$$

We shall also use the notation $R(w, \tau \boxtimes \sigma_0, \psi)$ if we want to emphasize the dependence of $R(w, \tau \boxtimes \sigma_0)$ on the additive character ψ .

Remark 5.2.2.

- (1) The normalizing factor we defined here is the same as in [GI16] Section 7. It is not exactly the same as the normalizing factor defined in [Mok15] or [KMSW14]; but they have the same analytic behavior near $s = 0$. So the final self-intertwining operator $R(w, \tau \boxtimes \sigma_0)$ we defined here coincides with Mok's when $U(V)$ is quasi-split.
- (2) In the definition of the self-intertwining operator $R(w, \tau \boxtimes \sigma_0)$, if we replace the additive character ψ by ψ_a , where $a \in F^\times$, then it follows from an easy computation that

$$R(w, \tau \boxtimes \sigma_0, \psi_a) = \begin{cases} R(w, \tau \boxtimes \sigma_0, \psi) \cdot \omega_\tau(a) & \text{if } m \text{ is even;} \\ R(w, \tau \boxtimes \sigma_0, \psi) & \text{if } m \text{ is odd.} \end{cases}$$

In particular, the self-intertwining operator $R(w, \tau \boxtimes \sigma_0)$ only depends on the choice of the Whittaker datum.

Similarly, we can define the normalized intertwining operator for $U(W)$. We put

$$\tilde{w} = w_Q \cdot m_Q \left((-1)^{n'} \cdot \kappa_W \cdot J \right) \cdot (-1_{W_0})^k,$$

where w_Q is as in the previous subsection, and $n' = \lfloor \frac{n}{2} \rfloor$. Let π_0 be an irreducible tempered representation of $U(W_0)$. We denote the L -parameters of τ and π_0 by ϕ_τ and ϕ_0 respectively. We set

$$r(w, \tau_s \boxtimes \pi_0) = \lambda(w, \psi) \cdot \gamma(s, \phi_\tau \otimes \phi_0^\vee, \psi_E)^{-1} \cdot \gamma(2s, As^{(-1)^n} \circ \phi_\tau, \psi)^{-1},$$

and the normalized intertwining operator

$$\mathcal{R}(w, \tau_s \boxtimes \pi_0) := |\kappa_W|^{k\rho_Q} \cdot r(w, \tau_s \boxtimes \pi_0)^{-1} \cdot \mathcal{M}(\tilde{w}, \tau_s \boxtimes \pi_0).$$

Assume that $w(\tau \boxtimes \pi_0) \simeq \tau \boxtimes \pi_0$, we take an isomorphism \mathcal{A}_w similarly, and define the self-intertwining operator $R(w, \tau \boxtimes \pi_0)$ by

$$R(w, \tau \boxtimes \pi_0)\Phi(g) = \mathcal{A}_w(\mathcal{R}(w, \tau_s \boxtimes \pi_0)\Phi(g))$$

for $\Phi \in \text{Ind}_Q^{U(W)}(\tau \boxtimes \pi_0)$, and $g \in U(W)$. We have

$$R(w, \tau \boxtimes \pi_0)^2 = 1.$$

5.3. An equivariant map. In [GI16] Section 8, Gan-Ichino constructed an equivariant map. We will apply this map to do some computations in later sections. Now we briefly recall some related results.

Let τ be an irreducible square-integrable representation of $GL_k(E)$, π_0 be an irreducible tempered representation of $U(W_0)$, and $\sigma_0 = \theta_{\psi, V_0, W_0}(\pi_0)$ be the theta lift of π_0 to $U(V_0)$.

Proposition 5.3.1.

(1) *There is a family of $U(V) \times U(W)$ -equivariant maps*

$$\mathcal{T}_s : \omega \otimes \text{Ind}_P^{U(V)}(\tau_s^c \chi_W^c \boxtimes \sigma_0^\vee) \longrightarrow \text{Ind}_Q^{U(W)}(\tau_s \chi_V \boxtimes \pi_0)$$

parametrized by $s \in \mathbb{C}$. This family of maps \mathcal{T}_s is holomorphic in s .

(2) *Assume that $m \geq n$. Let $\Phi \in \text{Ind}_P^{U(V)}(\tau^c \chi_W^c \boxtimes \sigma_0^\vee)$. If $\Phi \neq 0$, then there exists $\varphi \in \mathcal{S}$ such that*

$$\mathcal{T}_0(\varphi \otimes \Phi) \neq 0.$$

Proof. See [GI16] Lemma 8.1 and Lemma 8.3. □

Let ϕ_τ, ϕ_0 , and ϕ'_0 be the L -parameters of τ, π_0 , and σ_0 respectively. We denote by \tilde{w}' and \tilde{w} the representatives of the non-trivial element in $W(M_P)$ and $W(M_Q)$ respectively, as described in the previous subsection.

Proposition 5.3.2. *The diagram*

$$\begin{array}{ccc} \omega \otimes \text{Ind}_P^{U(V)}(\tau_s^c \chi_W^c \boxtimes \sigma_0^\vee) & \xrightarrow{\mathcal{T}_s} & \text{Ind}_Q^{U(W)}(\tau_s \chi_V \boxtimes \pi_0) \\ 1 \otimes \mathcal{R}(\tilde{w}', s) \downarrow & & \downarrow \mathcal{R}(\tilde{w}, s) \\ \omega \otimes \text{Ind}_P^{U(V)}(w'(\tau_s^c \chi_W^c \boxtimes \sigma_0^\vee)) & \xrightarrow{\mathcal{T}_{-s}} & \text{Ind}_Q^{U(W)}(w(\tau_s \chi_V \boxtimes \pi_0)) \end{array}$$

commutes up to a scalar. Indeed, for $\varphi \in \mathcal{S}$ and $\Phi_s \in \text{Ind}_P^{U(V)}(\tau_s^c \chi_W^c \boxtimes \sigma_0^\vee)$, we have

$$\mathcal{R}(\tilde{w}, \tau_s \chi_V \boxtimes \pi_0) \mathcal{T}_s(\varphi \otimes \Phi_s) = \alpha \cdot \beta(s) \cdot \mathcal{T}_{-s}(\varphi \otimes \mathcal{R}(\tilde{w}', \tau_s^c \chi_W^c \boxtimes \sigma_0^\vee) \Phi_s),$$

where

$$\begin{aligned} \alpha &= \left[\gamma_V^{-1} \cdot \gamma_W \cdot \chi_V \left((-1)^{n'} \cdot \varepsilon \cdot \kappa_W^{-1} \right) \cdot \chi_W \left((-1)^{m'-1} \cdot \kappa_V^{-1} \right) \cdot (\chi_V^{-n} \chi_W^m)(\delta) \right]^k \\ &\quad \times \omega_\tau \left((-1)^{m'+n'-1} \cdot \kappa_V^c \kappa_W^{-1} \right) \cdot \lambda(w', \psi) \cdot \lambda(w, \psi)^{-1} \end{aligned}$$

and

$$\begin{aligned} \beta(s) &= L \left(s - s_0 + \frac{1}{2}, \phi_\tau \right)^{-1} \cdot L \left(-s - s_0 + \frac{1}{2}, (\phi_\tau^c)^\vee \right) \cdot \gamma \left(-s - s_0 + \frac{1}{2}, (\phi_\tau^c)^\vee, \psi_E \right) \\ &\quad \times |\kappa_V \kappa_W^{-1}|^{-ks} \cdot \gamma(s, \phi_\tau^c \otimes \phi'_0 \otimes \chi_W^c, \psi_E)^{-1} \cdot \gamma(s, \phi_\tau \otimes \phi_0^\vee \otimes \chi_V, \psi_E). \end{aligned}$$

Proof. See [GI16] Corollary 8.5. □

6. LOCAL INTERTWINING RELATION

In this section, we prove the LLC we constructed for even unitary groups (i.e. \mathcal{L} and $\mathcal{J}_{\mathcal{W}}$) satisfy the LIR. We retain notations in Section 4.4.

Assume that $\phi \in \Phi_{\text{temp}}(2n)$ is a tempered L -parameter, such that

$$\phi = \phi_{\tau} \oplus \phi_0 \oplus (\phi_{\tau}^{\epsilon})^{\vee},$$

where ϕ_{τ} is an irreducible tempered representation of WD_E which corresponds to an irreducible (unitary) discrete series representation τ of $GL_k(E)$, and $\phi_0 \in \Phi_{\text{temp}}(2n_0)$, where $n_0 = n - k$. So there is a natural embedding $\mathcal{S}_{\phi_0} \hookrightarrow \mathcal{S}_{\phi}$. Let $\pi_0 = \pi(\phi_0, \eta_0)$ be an irreducible tempered representation of $U(W_{2n_0}^{\epsilon})$. We can write

$$W_{2n}^{\epsilon} = Y \oplus W_{2n_0}^{\epsilon} \oplus Y^*,$$

where Y and Y^* are k -dimensional totally isotropic subspaces of W_{2n}^{ϵ} such that $Y \oplus Y^*$ is non-degenerate and orthogonal to $W_{2n_0}^{\epsilon}$. Let Q be the maximal parabolic subgroup of $U(W_{2n}^{\epsilon})$ stabilizing Y , and L be the Levi component of Q stabilizing Y^* , so that

$$L \simeq GL(Y) \times U(W_{2n_0}^{\epsilon}).$$

Our goal is to completely analyze the induced representation $\text{Ind}_Q^{U(W_{2n}^{\epsilon})}(\tau \boxtimes \pi_0)$.

We divide our proof into three parts. In the first part, we analyze the L -parameter for each irreducible constituent π of $\text{Ind}_Q^{U(W_{2n}^{\epsilon})}(\tau \boxtimes \pi_0)$; and as a corollary, we get some information on the reducibility of $\text{Ind}_Q^{U(W_{2n}^{\epsilon})}(\tau \boxtimes \pi_0)$. In the second part, we analyze the action of the normalized local intertwining operator $R(w, \tau \boxtimes \pi_0)$ on $\text{Ind}_Q^{U(W_{2n}^{\epsilon})}(\tau \boxtimes \pi_0)$. In the last part, we relate the character $\eta = \mathcal{J}_{\mathcal{W}}(\pi)$ with η_0 .

6.1. L -parameters and reducibilities. We first prove that

Proposition 6.1.1. *Let π be an irreducible subrepresentation of $\text{Ind}_Q^{U(W_{2n}^{\epsilon})}(\tau \boxtimes \pi_0)$. Then the L -parameter of π is ϕ .*

Proof. We pick up $\epsilon' \in \{\pm\}$ appropriately such that $\sigma := \theta_{2n+1}^{\epsilon'}(\pi)$ is non-zero. By Lemma 3.2.1, we have

$$\sigma \subset \text{Ind}_P^{U(V_{2n+1}^{\epsilon'})}(\tau \chi_V^{-1} \chi_W \boxtimes \sigma_0),$$

where P is a maximal parabolic subgroup of $U(V_{2n+1}^{\epsilon'})$ with Levi component $GL_k(E) \times U(V_{2n_0+1}^{\epsilon'})$, and $\sigma_0 := \theta_{2n_0+1}^{\epsilon'}(\pi_0)$. By using the LLC for odd unitary groups, it is easy to see that

$$\phi_{\sigma} = \phi_{\tau} \chi_V^{-1} \chi_W \oplus \phi_{\sigma_0} \oplus (\phi_{\tau}^{\epsilon})^{\vee} \chi_V^{-1} \chi_W.$$

On the other hand, by Lemma 4.3.1, we have

$$\begin{aligned} \phi_{\sigma} &= \phi_{\pi} \chi_V^{-1} \chi_W \oplus \chi_W, \\ \phi_{\sigma_0} &= \phi_0 \chi_V^{-1} \chi_W \oplus \chi_W. \end{aligned}$$

From these equalities, we get $\phi_{\pi} = \phi$. □

Recall that there is a natural embedding $\mathcal{S}_{\phi_0} \hookrightarrow \mathcal{S}_{\phi}$ of component groups. We identify \mathcal{S}_{ϕ_0} with a subgroup of \mathcal{S}_{ϕ} via this embedding.

Corollary 6.1.2. *The induced representation $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is semi-simple and multiplicity free. Moreover, we have*

- (1) *If $\mathcal{S}_{\phi_0} = \mathcal{S}_\phi$, then $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is irreducible.*
- (2) *If \mathcal{S}_{ϕ_0} is a proper subgroup of \mathcal{S}_ϕ , then $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is reducible, and has two inequivalent constituents.*

Proof. Since $\tau \boxtimes \pi_0$ is an irreducible unitary representation of L , the parabolic induction $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is unitary and of finite length, hence semi-simple. Let π be an irreducible constituent of $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$, and

$$m_Q(\pi) = \dim \text{Hom}_{U(W_{2n}^\epsilon)} \left(\pi, \text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0) \right).$$

As in the proof of Proposition 6.1.1, there exists $\epsilon' \in \{\pm 1\}$, such that $\sigma := \theta_{2n+1}^{\epsilon'}(\pi)$ is non-zero and

$$\sigma \subset \text{Ind}_P^{U(V_{2n+1}^{\epsilon'})}(\tau \chi_V^{-1} \chi_W \boxtimes \sigma_0),$$

where P is a maximal parabolic subgroup of $U(V_{2n+1}^{\epsilon'})$ with Levi component $GL_k(E) \times U(V_{2n_0+1}^{\epsilon'})$, and $\sigma_0 := \theta_{2n_0+1}^{\epsilon'}(\pi_0)$. By the LLC for odd unitary groups,

$$\text{Ind}_P^{U(V_{2n+1}^{\epsilon'})}(\tau \chi_V^{-1} \chi_W \boxtimes \sigma_0)$$

is multiplicity free. It then follows from Lemma 3.2.1 that

$$m_Q(\pi) \leq 1.$$

Hence $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is also multiplicity free. We denote by

$$JH \left(\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0) \right)$$

the set of irreducible constituents of $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$. Consider the set

$$\bigsqcup_{\pi_0} JH \left(\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0) \right),$$

where the disjoint union runs over all $\pi_0 \in \Pi_{\phi_0}$. By the Howe duality, Lemma 3.2.1, and Proposition 6.1.1, this set is indeed a subset of Π_ϕ .

Now suppose that $\mathcal{S}_{\phi_0} = \mathcal{S}_\phi$. Then we have

$$|\Pi_\phi| \geq \left| \bigsqcup_{\pi_0} JH \left(\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0) \right) \right| \geq |\Pi_{\phi_0}|.$$

But in this case, it follows from Corollary 4.3.4 that

$$|\Pi_\phi| = |\widehat{\mathcal{S}}_\phi| = |\widehat{\mathcal{S}}_{\phi_0}| = |\Pi_{\phi_0}|.$$

Therefore we must have

$$\left| JH \left(\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0) \right) \right| = 1$$

for all $\pi_0 \in \Pi_{\phi_0}$. In other words, $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is irreducible.

Next suppose that \mathcal{S}_{ϕ_0} is a proper subgroup of \mathcal{S}_ϕ . In this case, \mathcal{S}_{ϕ_0} is an index two subgroup of \mathcal{S}_ϕ . We first show that for all $\pi_0 \in \Pi_{\phi_0}$, we have

$$\left| JH \left(\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0) \right) \right| \geq 2.$$

In other words, $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is reducible. Let

$$\begin{aligned} \phi_0^+ &:= \phi_0 \chi_V^{-1} \chi_W \oplus \chi_W, \\ \phi^+ &:= \phi \chi_V^{-1} \chi_W \oplus \chi_W. \end{aligned}$$

Depending on the relative size of $\mathcal{S}_{\phi_0^+}$ and \mathcal{S}_{ϕ^+} , there are two subcases:

Subcase 1. If $\phi_\tau \neq \chi_V$, then $\mathcal{S}_{\phi_0^+}$ is also a proper subgroup of \mathcal{S}_{ϕ^+} . Pick up any $\epsilon' \in \{\pm 1\}$ such that $\sigma_0 := \theta_{2n_0+1}^{\epsilon'}(\pi_0)$ is non-zero. Then σ_0 has L -parameter ϕ_0^+ , and by the LLC for odd unitary groups,

$$\text{Ind}_P^{U(V_{2n+1}^{\epsilon'})} \left((\tau \chi_V^{-1} \chi_W)^c \boxtimes \sigma_0^\vee \right) \simeq \left(\text{Ind}_P^{U(V_{2n+1}^{\epsilon'})} (\tau \chi_V^{-1} \chi_W \boxtimes \sigma_0) \right)^\vee$$

is reducible. By Proposition 5.3.1, there is a $U(V) \times U(W)$ -equivariant map

$$\mathcal{T}_0 : \omega \otimes \text{Ind}_P^{U(V_{2n+1}^{\epsilon'})} \left((\tau \chi_V^{-1} \chi_W)^c \boxtimes \sigma_0^\vee \right) \longrightarrow \text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0),$$

such that for any irreducible constituent σ of $\text{Ind}_P^{U(V_{2n+1}^{\epsilon'})}(\tau \chi_V^{-1} \chi_W \boxtimes \sigma_0)$, the restriction $\mathcal{T}_0|_{\omega \otimes \sigma^\vee}$ is non-vanishing, and its image is just $\theta_{2n}^{\epsilon'}(\sigma)$. Hence $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ at least contains all these $\theta_{2n}^{\epsilon'}(\sigma)$ as subrepresentations. In particular, $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is also reducible.

Subcase 2. If $\phi_\tau = \chi_V$, then the natural embedding $\mathcal{S}_{\phi_0^+} \hookrightarrow \mathcal{S}_{\phi^+}$ is an isomorphism. Our assumptions in this subcase imply that $\chi_V \not\subset \phi_0$. By Corollary 4.3.2, both $\sigma_0^+ := \theta_{2n_0+1}^+(\pi_0)$ and $\sigma_0^- := \theta_{2n_0+1}^-(\pi_0)$ are non-zero. Moreover, for $\epsilon' \in \{\pm 1\}$, $\sigma_0^{\epsilon'}$ has L -parameter ϕ_0^+ , and by the LLC for odd unitary groups,

$$\text{Ind}_P^{U(V_{2n+1}^{\epsilon'})} \left((\tau \chi_V^{-1} \chi_W)^c \boxtimes (\sigma_0^{\epsilon'})^\vee \right) \simeq \left(\text{Ind}_P^{U(V_{2n+1}^{\epsilon'})} (\tau \chi_V^{-1} \chi_W \boxtimes \sigma_0^{\epsilon'}) \right)^\vee$$

is irreducible. Similar to Subcase 1, there are non-vanishing $U(V) \times U(W)$ -equivariant maps

$$\mathcal{T}_0^{\epsilon'} : \omega \otimes \text{Ind}_P^{U(V_{2n+1}^{\epsilon'})} \left((\tau \chi_V^{-1} \chi_W)^c \boxtimes (\sigma_0^{\epsilon'})^\vee \right) \longrightarrow \text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0).$$

Let

$$\pi^{\epsilon'} := \text{Im}(\mathcal{T}_0^{\epsilon'}) \subset \text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0).$$

Then by the Howe duality, we know that $\pi^{\epsilon'}$ is irreducible and is just the theta lift of $\text{Ind}_P^{U(V_{2n+1}^{\epsilon'})}(\tau \chi_V^{-1} \chi_W \boxtimes \sigma_0^{\epsilon'})$. Since $\pi^{\epsilon'}$ has L -parameter ϕ and $\chi_V \subset \phi$, it follows from Lemma 4.3.3 that $\pi^+ \not\subset \pi^-$,

which implies that $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is reducible.

Now similar to the previous case, we have

$$|\Pi_\phi| \geq \left| \bigsqcup_{\pi_0} JH \left(\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0) \right) \right| \geq 2|\Pi_{\phi_0}|.$$

Again in this case, Corollary 4.3.4 forces these inequalities to be equalities. Therefore we conclude that

$$\left| JH \left(\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0) \right) \right| = 2$$

for all $\pi_0 \in \Pi_{\phi_0}$. This completes the proof. □

6.2. Actions of intertwining operators. In the previous subsection, we showed that $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is semi-simple and multiplicity free. In this subsection, we prove the following:

Proposition 6.2.1. *Assume that ϕ_τ is conjugate self-dual. Let $\pi = \pi(\phi, \eta)$ be an irreducible constituent of $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$. Then the restriction of the normalized intertwining operator $R(w, \tau \boxtimes \pi_0)$ to π is the scalar multiplication by*

$$\begin{cases} \epsilon^k \cdot \eta(a_\tau) & \text{if } \phi_\tau \text{ is conjugate symplectic;} \\ \epsilon^k & \text{if } \phi_\tau \text{ is conjugate orthogonal,} \end{cases}$$

where a_τ is the element in \mathcal{S}_ϕ corresponding to ϕ_τ .

Proof. Since $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$ is multiplicity free, the restriction of $R(w, \tau \boxtimes \pi_0)$ to π gives a self intertwining operator of π . Hence by Schur’s lemma, $R(w, \tau \boxtimes \pi_0)$ acts on π by a scalar. Let’s denote this scalar by $\mathcal{R}(\pi)$. We want to relate the scalar $\mathcal{R}(\pi)$ with the character η .

Let

$$\epsilon' = \begin{cases} + & \text{if } \theta_{2n+1}^+(\pi) \neq 0; \\ - & \text{otherwise,} \end{cases}$$

and let $\sigma := \theta_{2n+1}^{\epsilon'}(\pi)$ (which is non-zero by the conservation relation). Recall that there is a natural embedding of component groups $\mathcal{S}_\phi \hookrightarrow \mathcal{S}_{\phi_\sigma}$, and it follows from our construction that $\eta = \eta_\sigma|_{\mathcal{S}_\phi}$. According to Lemma 3.2.1, we have

$$\sigma \subset \text{Ind}_P^{U(V_{2n+1}^{\epsilon'})}(\tau\chi_V^{-1}\chi_W \boxtimes \sigma_0),$$

where P is a maximal parabolic subgroup of $U(V_{2n+1}^{\epsilon'})$ with Levi component $GL_k(E) \times U(V_{2n_0+1}^{\epsilon'})$, and $\sigma_0 := \theta_{2n_0+1}^{\epsilon'}(\pi_0)$. Hence

$$\sigma^\vee \subset \text{Ind}_P^{U(V_{2n+1}^{\epsilon'})} \left((\tau\chi_V^{-1}\chi_W)^c \boxtimes \sigma_0^\vee \right).$$

By Lemma 5.3.1, there exists a $U(V_{2n+1}^{\epsilon'}) \times U(W_{2n}^\epsilon$)-equivariant map

$$\mathcal{T}_0 : \omega \otimes \text{Ind}_P^{U(V_{2n+1}^{\epsilon'})} \left((\tau\chi_V^{-1}\chi_W)^c \boxtimes \sigma_0^\vee \right) \longrightarrow \text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0),$$

whose restriction to $\omega \otimes \sigma^\vee$ gives an epimorphism

$$\mathcal{T}_0 : \omega \otimes \sigma^\vee \longrightarrow \pi.$$

Applying Proposition 5.3.2, we get

$$\mathcal{R}(\pi) = \alpha \cdot \beta(0) \cdot \mathcal{R}(\sigma^\vee),$$

where $\mathcal{R}(\sigma^\vee)$ is the scalar defined by the action of the normalized intertwining operator $R(w', (\tau\chi_V^{-1}\chi_W)^c \boxtimes \sigma_0^\vee)$ on σ^\vee . Following the calculation in [GI16] Section 8.4, we have

$$\epsilon^k \cdot (\epsilon')^k \cdot \alpha \cdot \beta(0) = 1.$$

Then one can easily deduce the desired formula for $\mathcal{R}(\pi)$ from these two equalities and the LLC for odd unitary groups. □

6.3. Matching characters of component groups. Let π be an irreducible constituent of $\text{Ind}_Q^{U(W_{2n}^\epsilon)}(\tau \boxtimes \pi_0)$. We showed in Proposition 6.1.1 that the L -parameter of π is ϕ . In this subsection, we are going to relate the character $\eta = \mathcal{J}_{\mathcal{W}}(\pi)$ of \mathcal{S}_ϕ with η_0 .

We first consider a special case.

Lemma 6.3.1. *Assume that the natural embedding $\mathcal{S}_{\phi_0} \hookrightarrow \mathcal{S}_\phi$ is an isomorphism. Then*

$$\eta|_{\mathcal{S}_{\phi_0}} = \eta_0.$$

Proof. Similar to the proof of Proposition 6.2.1, we can pick up $\epsilon'_0, \epsilon' \in \{\pm\}$ appropriately such that $\sigma_0 := \theta_{2n_0+1}^{\epsilon'_0}(\pi_0)$ and $\sigma := \theta_{2n+1}^{\epsilon'}(\pi)$ are non-zero, and by our construction,

$$\begin{aligned} \eta_0 &= \eta_{\sigma_0}|_{\mathcal{S}_{\phi_0}}, \\ \eta &= \eta_\sigma|_{\mathcal{S}_\phi}. \end{aligned}$$

One can easily check case-by-case that under our assumption,

$$(\dagger) \quad \epsilon'_0 = \epsilon'.$$

On the other hand, by Lemma 3.2.1, we have

$$\sigma \subset \text{Ind}_P^{U(V_{2n+1}^{\epsilon'})}(\tau \chi_V^{-1} \chi_W \boxtimes \sigma_0),$$

where P is a maximal parabolic subgroup of $U(V_{2n+1}^{\epsilon'})$ with Levi component $GL_k(E) \times U(V_{2n_0+1}^{\epsilon'})$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_{\phi_0} & \longrightarrow & \mathcal{S}_\phi \\ \downarrow & & \downarrow \\ \mathcal{S}_{\phi_{\sigma_0}} & \longrightarrow & \mathcal{S}_{\phi_\sigma} \end{array}$$

Here every arrow in this diagram is the natural one. Hence we get

$$\begin{aligned} \eta|_{\mathcal{S}_{\phi_0}} &= \left(\eta_\sigma|_{\mathcal{S}_\phi} \right) \Big|_{\mathcal{S}_{\phi_0}} && \text{(by our construction of } \eta) \\ &= \left(\eta_\sigma|_{\mathcal{S}_{\phi_{\sigma_0}}} \right) \Big|_{\mathcal{S}_{\phi_0}} && \text{(by the commutative diagram)} \\ &= \eta_{\sigma_0}|_{\mathcal{S}_{\phi_0}} && \text{(by the LLC for odd unitary groups)} \\ &= \eta_0. && \text{(by our construction of } \eta_0) \end{aligned}$$

□

Here in this lemma, the assumption is only used to guarantee that the equality (\dagger) holds, which may fail in the general case. With this special case at hand, we can show that

Corollary 6.3.2. *Let $\epsilon' \in \{\pm 1\}$. Assume that $\sigma_0 := \theta_{2n_0+1}^{\epsilon'}(\pi_0)$ is non-zero. Then*

$$\eta_0 = \eta_{\sigma_0}|_{\mathcal{S}_{\phi_0}}.$$

Here we use the natural embedding $\mathcal{S}_{\phi_0} \hookrightarrow \mathcal{S}_{\phi_{\sigma_0}}$ to identify \mathcal{S}_{ϕ_0} with a subgroup of $\mathcal{S}_{\phi_{\sigma_0}}$.

Proof. We use an argument similar to that of [Ato18] Section 7.3. Let ϕ_ρ be any irreducible conjugate symplectic subrepresentation of ϕ_0 , which corresponds to a square-integrable representation ρ of $GL_d(E)$, for some $d \leq 2n_0$. We can write

$$W_{2n_0+2d}^\epsilon = Y_\rho \oplus W_{2n_0}^\epsilon \oplus Y_\rho^*,$$

where Y_ρ and Y_ρ^* are d -dimensional totally isotropic subspaces of $W_{2n_0+2d}^\epsilon$ such that $Y_\rho \oplus Y_\rho^*$ is non-degenerate and orthogonal to $W_{2n_0}^\epsilon$. Let \tilde{Q} be the maximal parabolic subgroup of $U(W_{2n_0+2d}^\epsilon)$ stabilizing Y_ρ and \tilde{L} be its Levi component stabilizing Y_ρ^* , so that

$$\tilde{L} \simeq GL(Y_\rho) \times U(W_{2n_0}^\epsilon).$$

We consider the induced representation $\tilde{\pi}_0 := \text{Ind}_{\tilde{Q}}^{U(W_{2n_0+2d}^\epsilon)}(\rho \boxtimes \pi_0)$. By Corollary 6.1.2, $\tilde{\pi}_0$ is irreducible. Moreover, it follows from Proposition 6.1.1 and Lemma 6.3.1 that

$$\tilde{\pi}_0 = \pi(\tilde{\phi}_0, \eta_0)$$

is the element in $\Pi_{\tilde{\phi}_0}$ corresponding to η_0 , where

$$\tilde{\phi}_0 = \phi_\rho \oplus \phi_0 \oplus (\phi_\rho^c)^\vee,$$

and we use the natural isomorphism $\mathcal{S}_{\phi_0} \simeq \mathcal{S}_{\tilde{\phi}_0}$ to identify \mathcal{S}_{ϕ_0} and $\mathcal{S}_{\tilde{\phi}_0}$.

Similarly, we can write

$$V_{2n_0+2d+1}^{\epsilon'} = X_\rho \oplus V_{2n_0+1}^{\epsilon'} \oplus X_\rho^*,$$

where X_ρ and X_ρ^* are d -dimensional totally isotropic subspaces of $V_{2n_0+2d+1}^{\epsilon'}$ such that $X_\rho \oplus X_\rho^*$ is non-degenerate and orthogonal to $V_{2n_0+1}^{\epsilon'}$. Let \tilde{P} be the maximal parabolic subgroup of $U(V_{2n_0+2d+1}^{\epsilon'})$ stabilizing X_ρ and \tilde{M} be its Levi component stabilizing X_ρ^* , so that

$$\tilde{M} \simeq GL(X_\rho) \times U(V_{2n_0+1}^{\epsilon'}).$$

Set $\tilde{\sigma}_0 := \text{Ind}_{\tilde{P}}^{U(V_{2n_0+2d+1}^{\epsilon'})}(\rho \chi_V^{-1} \chi_W \boxtimes \sigma_0)$. By the LLC for odd unitary groups, $\tilde{\sigma}_0$ is irreducible. Moreover, we have

$$\tilde{\sigma}_0 = \pi(\tilde{\phi}_{\sigma_0}, \eta_{\sigma_0})$$

is the element in $\Pi_{\tilde{\phi}_{\sigma_0}}$ corresponding to η_{σ_0} , where

$$\tilde{\phi}_{\sigma_0} = \phi_\rho \chi_V^{-1} \chi_W \oplus \phi_{\sigma_0} \oplus ((\phi_\rho \chi_V^{-1} \chi_W)^c)^\vee,$$

and we use the natural isomorphism $\mathcal{S}_{\phi_{\sigma_0}} \simeq \mathcal{S}_{\tilde{\phi}_{\sigma_0}}$ to identify $\mathcal{S}_{\phi_{\sigma_0}}$ and $\mathcal{S}_{\tilde{\phi}_{\sigma_0}}$.

Recall by Proposition 5.3.1, there exists a non-zero $U(V_{2n_0+2d+1}^{\epsilon'}) \times U(W_{2n_0+2d}^\epsilon)$ -equivariant epimorphism

$$\tilde{\mathcal{T}}_0 : \omega \otimes \tilde{\sigma}_0^\vee \longrightarrow \tilde{\pi}_0.$$

Applying Proposition 5.3.2, we get

$$\mathcal{R}(\tilde{\pi}_0) = \alpha \cdot \beta(0) \cdot \mathcal{R}(\tilde{\sigma}_0^\vee),$$

where $\mathcal{R}(\tilde{\pi}_0)$ is the scalar defined by the action of the normalized intertwining operator $R(w, \rho \boxtimes \pi_0)$ on $\tilde{\pi}_0$, and $\mathcal{R}(\tilde{\sigma}_0^\vee)$ is defined similarly. Following the calculation in [GI16] Section 8.4, we have

$$\epsilon^k \cdot (\epsilon')^k \cdot \alpha \cdot \beta(0) = 1.$$

Combining these two equalities, the LLC for odd unitary groups, and Proposition 6.2.1, we get

$$\eta_0(a_\rho) = \eta_{\sigma_0}(a'_\rho),$$

where a_ρ is the element in \mathcal{S}_{ϕ_0} corresponding to ϕ_ρ , and a'_ρ is the element in $\mathcal{S}_{\phi_{\sigma_0}}$ corresponding to $\phi_\rho \chi_V^{-1} \chi_W$. Since ϕ_ρ is arbitrary, we deduce that

$$\eta_0 = \eta_{\sigma_0}|_{\mathcal{S}_{\phi_0}}.$$

This completes the proof. □

Finally we can prove the general case:

Proposition 6.3.3. *We have*

$$\eta|_{\mathcal{S}_{\phi_0}} = \eta_0.$$

Proof. The proof is almost the same as that of Lemma 6.3.1. Although the equality (†) may not hold anymore without the assumption in Lemma 6.3.1, we can still appeal to Corollary 6.3.2 to complete the comparison of $\eta|_{\mathcal{S}_{\phi_0}}$ and η_0 . □

Combining Proposition 6.1.1, Corollary 6.1.2, Proposition 6.2.1, and Proposition 6.3.3, we get

Proposition 6.3.4. *The LIR holds for the LLC we constructed for even unitary groups.*

7. COMPLETION OF THE PROOF

Now we are equipped with enough powerful arms and able to complete the proof of our main result Theorem 2.5.1. In this section, to simplify notations, we let V^ϵ be the $(2n + 1)$ -dimensional Hermitian space over E with sign ϵ , and $U(V^\epsilon)$ be the unitary group associated to V^ϵ . Similarly, we let W^ϵ be the $2n$ -dimensional skew-Hermitian space over E with sign ϵ , and $U(W^\epsilon)$ be the unitary group associated to W^ϵ . The idea of many proofs in this section comes from [Ato18] Section 7.3.

7.1. Comparison with LLC à la Mok. In this subsection, we compare the LLC for even unitary groups constructed in Section 4 with the LLC for quasi-split unitary groups constructed by Mok in [Mok15].

Fix a Whittaker datum \mathcal{W} of $U(W^+)$. Let π be an irreducible smooth representation of $U(W^+)$. Recall that in Section 4, we associated a pair

$$(\phi = \mathcal{L}(\pi), \eta = \mathcal{J}_{\mathcal{W}}(\pi))$$

to π . Also, in [Mok15], Mok associated a pair

$$(\phi^M = \mathcal{L}^+(\pi), \eta^M = \mathcal{J}_{\mathcal{W}}^+(\pi))$$

to π . Moreover, the LLC for quasi-split unitary groups constructed by Mok satisfies all properties listed in Theorem 2.5.1.

Theorem 7.1.1. *We have*

$$\phi = \phi^M \quad \text{and} \quad \eta = \eta^M.$$

Proof. Since both two LLC are compatible with Langlands quotients, without loss of generality, we may assume that π is tempered. Then the desired conclusion follows from Proposition 4.1.2 and the same argument as that of Corollary 6.3.2. □

Remark 7.1.2. Similarly, one can easily show that the bijection $\mathcal{J}_{\mathcal{W}}$ constructed in Section 4.4 is independent of the choice of the datum $\underline{\psi} = (\psi, \chi_V, \chi_W, \delta)$, but only depends on the choice of the Whittaker datum \mathcal{W} .

As a consequence of this comparison, we deduce

Proposition 7.1.3. *The LLC we constructed for even unitary groups satisfies following properties:*

- (1) *Let $\pi = \pi(\phi, \eta)$ be the element in Π_ϕ corresponding to η . Then π is a representation of $U(W^\epsilon)$ if and only if $\eta(z_\phi) = \epsilon$.*
- (2) *Assume that ϕ is a tempered L -parameter, then there is an unique \mathcal{W} -generic representation of $U(W^+)$ in Π_ϕ corresponding to the trivial character of \mathcal{S}_ϕ .*

7.2. Twisting by characters. In this subsection, we prove a formula which concerns the behavior of the LLC we constructed with respect to twisting by characters.

Let $\pi = \pi(\phi, \eta)$ be the representation of $U(W^\epsilon)$ in Π_ϕ corresponding to η , where $\epsilon = \eta(z_\phi)$. Let χ be a character of E^1 , and let $\tilde{\chi}$ to be the base change of χ , i.e. the pull-back of χ along

$$E^\times \rightarrow E^1$$

$$x \mapsto x/c(x).$$

Let $\pi\chi := \pi \otimes (\chi \circ \det)$. Denote by ϕ_χ the L -parameter of $\pi\chi$.

Lemma 7.2.1. *We have $\phi_\chi = \phi \cdot \tilde{\chi}$.*

Proof. We first assume that π is square-integrable. Then $\pi\chi$ is also square-integrable. By Proposition 4.6.1, we can write

$$\phi = \sum_i \phi_i$$

with pairwise inequivalent irreducible conjugate symplectic representation ϕ_i of WD_E . For each i , we may regard ϕ_i as an L -parameter of $GL_{k_i}(E)$, where $k_i = \dim \phi_i$. We denote by ρ_i the irreducible square-integrable representation of $GL_{k_i}(E)$ corresponding to ϕ_i . Let $\widetilde{W}_{\phi_i} = W^\epsilon \oplus \mathcal{H}^{k_i}$, where \mathcal{H} is the (skew-Hermitian) hyperbolic plane. We can decompose \widetilde{W}_{ϕ_i} as follows

$$\widetilde{W}_{\phi_i} = Y_{\phi_i} \oplus W^\epsilon \oplus Y_{\phi_i}^*,$$

where Y_{ϕ_i} and $Y_{\phi_i}^*$ are k_i -dimensional totally isotropic subspaces of \widetilde{W}_{ϕ_i} such that $Y_{\phi_i} \oplus Y_{\phi_i}^* \simeq \mathcal{H}^{k_i}$ and orthogonal to W^ϵ . Let \widetilde{Q}_{ϕ_i} be the maximal parabolic subgroup of $U(\widetilde{W}_{\phi_i})$ stabilizing Y_{ϕ_i} and \widetilde{L}_{ϕ_i} be its Levi component stabilizing $Y_{\phi_i}^*$, so that

$$\widetilde{L}_{\phi_i} \simeq GL(Y_{\phi_i}) \times U(W^\epsilon).$$

Consider the induced representation

$$\text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \tilde{\chi} \boxtimes \pi\chi) \simeq \text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \boxtimes \pi) \otimes (\chi \circ \det).$$

By Proposition 6.3.4, $\text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \boxtimes \pi)$ is irreducible. Hence the induced representation $\text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \tilde{\chi} \boxtimes \pi\chi)$ is also irreducible. Again by Proposition 6.3.4, it

follows that

$$\phi_i \cdot \tilde{\chi} \subset \phi_\chi.$$

This containment holds for all i . Therefore we must have

$$\phi_\chi = \sum \phi_i \cdot \tilde{\chi} = \phi \cdot \tilde{\chi}.$$

When π is tempered but not square-integrable, the lemma follows from Proposition 6.3.4 and induction in stages. In the general case, the lemma follows from the compatibility of the LLC with Langlands quotients. \square

Next we consider the character $\eta_{\pi\chi}$ of \mathcal{S}_{ϕ_χ} associated to $\pi\chi$.

Lemma 7.2.2. *If we use the natural isomorphism $\mathcal{S}_\phi \simeq \mathcal{S}_{\phi_\chi}$ to identify them, then we have*

$$\eta_{\pi\chi} = \eta.$$

Proof. Since the LLC we constructed for even unitary groups is compatible with Langlands quotients, without loss of generality, we may assume that π is tempered.

Similar to the proof of Lemma 7.2.1, given any irreducible conjugate symplectic representation ϕ_i of WD_E , we define \widetilde{W}_{ϕ_i} , \widetilde{Q}_{ϕ_i} and \widetilde{L}_{ϕ_i} . Consider two induced representations

$$\text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \boxtimes \pi) \quad \text{and} \quad \text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \tilde{\chi} \boxtimes \pi\chi).$$

Then the LIR asserts that it is sufficient to prove the equality

$$R(w, \rho_i \boxtimes \pi) = R(w, \rho_i \tilde{\chi} \boxtimes \pi\chi),$$

where $R(w, \rho_i \boxtimes \pi)$ and $R(w, \rho_i \tilde{\chi} \boxtimes \pi\chi)$ are intertwining operators defined in Section 5.2. Consider the following commutative diagram

$$\begin{array}{ccc} \text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \boxtimes \pi) \otimes (\chi \circ \det) & \xrightarrow{\mathcal{F}} & \text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \tilde{\chi} \boxtimes \pi\chi) \\ \downarrow R(w, \rho_i \boxtimes \pi) \otimes 1 & & \downarrow R(w, \rho_i \tilde{\chi} \boxtimes \pi\chi) \\ \text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \boxtimes \pi) \otimes (\chi \circ \det) & \xrightarrow{\mathcal{F}} & \text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \tilde{\chi} \boxtimes \pi\chi) \end{array}$$

where the horizontal arrow

$$\mathcal{F} : \text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \boxtimes \pi) \otimes (\chi \circ \det) \longrightarrow \text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \tilde{\chi} \boxtimes \pi\chi)$$

is given by

$$\mathcal{F}(\Phi)(g) = \chi(\det g)\Phi(g)$$

for $\Phi \in \text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \boxtimes \pi)$. Here we realize $\text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \boxtimes \pi) \otimes (\chi \circ \det)$ on the same space as $\text{Ind}_{\widetilde{Q}_{\phi_i}}^{U(\widetilde{W}_{\phi_i})} (\rho_i \boxtimes \pi)$, but with the action twisted by χ . The desired equality follows from this commutative diagram easily. \square

Combining these two lemmas, we get

Proposition 7.2.3. *Let $\pi = \pi(\phi, \eta)$ be the representation of $U(W^\epsilon)$, where $\epsilon = \eta(z_\phi)$. Let χ be a character of E^1 . Then*

$$\pi\chi = \pi(\phi \cdot \tilde{\chi}, \eta).$$

Here we use the obvious isomorphism between \mathcal{S}_ϕ and \mathcal{S}_{ϕ_χ} to identify them.

7.3. Changes of Whittaker data. In this subsection, we prove a formula which concerns the behavior of the LLC we constructed for even unitary groups with respect to changes of the Whittaker data.

Let $\phi \in \Phi(2n)$, and π be an irreducible smooth representation of $U(W^\epsilon)$ with L -parameter ϕ . Let \mathscr{W} and \mathscr{W}' be the two Whittaker data of $U(W^+)$. Recall that in Section 4, we have constructed two bijections

$$\mathcal{J}_{\mathscr{W}} : \Pi_\phi \longrightarrow \widehat{\mathcal{S}}_\phi \quad \text{and} \quad \mathcal{J}_{\mathscr{W}'} : \Pi_\phi \longrightarrow \widehat{\mathcal{S}}_\phi.$$

Proposition 7.3.1. *Let $\eta = \mathcal{J}_{\mathscr{W}}(\pi)$ and $\eta' = \mathcal{J}_{\mathscr{W}'}(\pi)$. Then we have*

$$\eta' = \eta \cdot \eta_-,$$

where η_- is a character of \mathcal{S}_ϕ given by

$$\eta_-(a) = (-1)^{\dim \phi^a}$$

for $a \in \mathcal{S}_\phi$.

Proof. As described in Section 2.3, we may choose a non-trivial additive character ψ of F , such that

$$\mathscr{W} = \mathscr{W}_\psi \quad \text{and} \quad \mathscr{W}' = \mathscr{W}_{\psi_{a^w}},$$

where $a^w \in F^\times \setminus \text{Nm}_{E/F}(E^\times)$. Then by applying an argument similar to that of Corollary 6.3.2, one can see immediately that the desired formula follows from the LIR and Remark 5.2.2. \square

Using this formula, we are able to prove the last property listed in our main result Theorem 2.5.1, which concerns the behavior of the LLC we constructed with respect to taking contragredient.

Proposition 7.3.2. *Let $\pi = \pi(\phi, \eta)$ be the representation of $U(W^\epsilon)$, where $\epsilon = \eta(z_\phi)$ (with respect to the Whittaker datum \mathscr{W}). Then*

$$\pi^\vee = \pi(\phi^\vee, \eta \cdot \nu),$$

where ν is a character of \mathcal{S}_ϕ given by

$$\nu(a) = \omega_{E/F}(-1)^{\dim \phi^a}$$

for $a \in \mathcal{S}_\phi$. Here we use the obvious isomorphism between \mathcal{S}_ϕ and \mathcal{S}_{ϕ^\vee} to identify them.

Remark 7.3.3. In [Kal13], Kaletha proved such a formula using endoscopic character identities for quasi-split groups. Here, based on Kaletha’s results for odd unitary groups, we use an elementary argument to establish the desired formula for all even unitary groups.

Proof of Proposition 7.3.2. Since the LLC we constructed for even unitary groups is compatible with Langlands quotients, without loss of generality, we may assume that π is tempered.

Pick up a non-trivial additive character ψ of F , such that $\mathscr{W} = \mathscr{W}_\psi$. Let $\underline{\psi} = (\psi, \chi_V, \chi_W, \delta)$ be a tuple of data as described in Section 3.1, and $\epsilon' \in \{\pm 1\}$ such that

$$\Theta_{\underline{\psi}, V^{\epsilon'}, W^\epsilon}(\pi) = \theta_{\underline{\psi}, V^{\epsilon'}, W^\epsilon}(\pi) \neq 0.$$

Then we have

$$\theta_{\underline{\psi}', V^{\epsilon'}, W^\epsilon}(\pi^\vee \chi_V) \simeq \theta_{\underline{\psi}, V^{\epsilon'}, W^\epsilon}(\pi)^{MVW} \chi_W,$$

where $\underline{\psi}' = (\psi^{-1}, \chi_V, \chi_W, \delta)$ (see also [GI14] Section 6.1). By applying Lemma 4.3.1, Proposition 7.2.3, and Theorem 2.5.5 to this equality, we get

$$\mathcal{L}(\pi^\vee) = \phi^\vee.$$

Moreover, by applying Corollary 6.3.2 and Theorem 2.5.5 to the same equality, we get

$$\mathcal{J}_{\mathcal{W}_{\psi^{-1}}}(\pi^\vee \chi_V) = \mathcal{J}_{\mathcal{W}_\psi}(\pi).$$

It then follows from Proposition 7.2.3 and Proposition 7.3.1 that

$$\eta_{\pi^\vee} = \eta \cdot \nu$$

as desired. \square

So now, we have finished proving all properties listed in our main result Theorem 2.5.1.

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REFERENCES

- [ABPS14] Anne-Marie Aubert, Paul Baum, Roger Plymen, and Maarten Solleveld, *On the local Langlands correspondence for non-tempered representations*, Münster J. Math. **7** (2014), no. 1, 27–50. MR3271238
- [AG17] Hiraku Atobe and Wee Teck Gan, *Local theta correspondence of tempered representations and Langlands parameters*, Invent. Math. **210** (2017), no. 2, 341–415, DOI 10.1007/s00222-017-0730-8. MR3714507
- [Art13] James Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups, DOI 10.1090/coll/061. MR3135650
- [Ato17] Hiraku Atobe, *On the uniqueness of generic representations in an L -packet*, Int. Math. Res. Not. IMRN **23** (2017), 7051–7068, DOI 10.1093/imrn/rnw220. MR3801418
- [Ato18] Hiraku Atobe, *The local theta correspondence and the local Gan-Gross-Prasad conjecture for the symplectic-metaplectic case*, Math. Ann. **371** (2018), no. 1-2, 225–295, DOI 10.1007/s00208-017-1620-5. MR3788848
- [CG15] Ping-Shun Chan and Wee Teck Gan, *The local Langlands conjecture for $\mathrm{GSp}(4)$ III: Stability and twisted endoscopy*, J. Number Theory **146** (2015), 69–133, DOI 10.1016/j.jnt.2013.07.009. MR3267112
- [CZ21] Rui Chen and Jialiang Zou, *Local Langlands correspondence for even orthogonal groups via theta lifts*, Selecta Math. (N.S.) **27** (2021), no. 5, Paper No. 88, DOI 10.1007/s00029-021-00704-8. MR4308934
- [GGP12] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups* (English, with English and French summaries), Astérisque **346** (2012), 1–109. Sur les conjectures de Gross et Prasad. I. MR3202556
- [GI14] Wee Teck Gan and Atsushi Ichino, *Formal degrees and local theta correspondence*, Invent. Math. **195** (2014), no. 3, 509–672, DOI 10.1007/s00222-013-0460-5. MR3166215

- [GI16] Wee Teck Gan and Atsushi Ichino, *The Gross-Prasad conjecture and local theta correspondence*, *Invent. Math.* **206** (2016), no. 3, 705–799, DOI 10.1007/s00222-016-0662-8. MR3573972
- [GI18] Wee Teck Gan and Atsushi Ichino, *The Shimura-Waldspurger correspondence for Mp_{2n}* , *Ann. of Math.* (2) **188** (2018), no. 3, 965–1016, DOI 10.4007/annals.2018.188.3.5. MR3866889
- [GS12] Wee Teck Gan and Gordan Savin, *Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence*, *Compos. Math.* **148** (2012), no. 6, 1655–1694, DOI 10.1112/S0010437X12000486. MR2999299
- [GT11] Wee Teck Gan and Shuichiro Takeda, *The local Langlands conjecture for $\mathrm{GSp}(4)$* , *Ann. of Math.* (2) **173** (2011), no. 3, 1841–1882, DOI 10.4007/annals.2011.173.3.12. MR2800725
- [GT16a] Wee Teck Gan and Shuichiro Takeda, *On the Howe duality conjecture in classical theta correspondence*, *Advances in the theory of automorphic forms and their L -functions*, *Contemp. Math.*, vol. 664, Amer. Math. Soc., Providence, RI, 2016, pp. 105–117, DOI 10.1090/conm/664/13063. MR3502978
- [GT16b] Wee Teck Gan and Shuichiro Takeda, *A proof of the Howe duality conjecture*, *J. Amer. Math. Soc.* **29** (2016), no. 2, 473–493, DOI 10.1090/jams/839. MR3454380
- [HKS96] Michael Harris, Stephen S. Kudla, and William J. Sweet, *Theta dichotomy for unitary groups*, *J. Amer. Math. Soc.* **9** (1996), no. 4, 941–1004, DOI 10.1090/S0894-0347-96-00198-1. MR1327161
- [Ish19] Hiroshi Ishimoto, *Local intertwining relation for metaplectic groups*, *Compos. Math.* **156** (2020), no. 8, 1560–1594, DOI 10.1112/s0010437x20007253. MR4157428
- [Kak20] Hirotaka Kakuhamada, *On the local factors of irreducible representations of quaternionic unitary groups*, *Manuscripta Math.* **163** (2020), no. 1-2, 57–86, DOI 10.1007/s00229-019-01153-6. MR4131991
- [Kal13] Tasho Kaletha, *Genericity and contragredience in the local Langlands correspondence*, *Algebra Number Theory* **7** (2013), no. 10, 2447–2474, DOI 10.2140/ant.2013.7.2447. MR3194648
- [KMSW14] Tasho Kaletha, Alberto Minguez, Sug Woo Shin, and Paul-James White. *Endoscopic classification of representations: Inner forms of unitary groups*, preprint, [arXiv:1409.3731](https://arxiv.org/abs/1409.3731), 2014.
- [Kon03] Takuya Konno, *A note on the Langlands classification and irreducibility of induced representations of p -adic groups*, *Kyushu J. Math.* **57** (2003), no. 2, 383–409, DOI 10.2206/kyushujm.57.383. MR2050093
- [Kud86] Stephen S. Kudla, *On the local theta-correspondence*, *Invent. Math.* **83** (1986), no. 2, 229–255, DOI 10.1007/BF01388961. MR818351
- [Kud94] Stephen S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, *Israel J. Math.* **87** (1994), no. 1-3, 361–401, DOI 10.1007/BF02773003. MR1286835
- [LR05] Erez M. Lapid and Stephen Rallis, *On the local factors of representations of classical groups*, *Automorphic representations, L -functions and applications: progress and prospects*, *Ohio State Univ. Math. Res. Inst. Publ.*, vol. 11, de Gruyter, Berlin, 2005, pp. 309–359, DOI 10.1515/9783110892703.309. MR2192828
- [Luo20] Caihua Luo, *Endoscopic character identities for metaplectic groups*, *J. Reine Angew. Math.* **768** (2020), 1–37, DOI 10.1515/crelle-2019-0042. MR4168685
- [M12] Alberto Mínguez, *The conservation relation for cuspidal representations*, *Math. Ann.* **352** (2012), no. 1, 179–188, DOI 10.1007/s00208-011-0636-5. MR2885581
- [Mok15] Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups*, *Mem. Amer. Math. Soc.* **235** (2015), no. 1108, vi+248, DOI 10.1090/memo/1108. MR3338302
- [MR18] Colette Mœglin and David Renard, *Sur les paquets d’Arthur des groupes classiques et unitaires non quasi-déployés* (French), *Relative aspects in representation theory, Langlands functoriality and automorphic forms*, *Lecture Notes in Math.*, vol. 2221, Springer, Cham, 2018, pp. 341–361. MR3839702
- [MVW87] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger, *Correspondances de Howe sur un corps p -adique* (French), *Lecture Notes in Mathematics*, vol. 1291, Springer-Verlag, Berlin, 1987, DOI 10.1007/BFb0082712. MR1041060

- [Sha90] Freydoon Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p -adic groups*, Ann. of Math. (2) **132** (1990), no. 2, 273–330, DOI 10.2307/1971524. MR1070599
- [Sil78] Allan J. Silberger, *The Langlands quotient theorem for p -adic groups*, Math. Ann. **236** (1978), no. 2, 95–104, DOI 10.1007/BF01351383. MR507262
- [SZ15] Binyong Sun and Chen-Bo Zhu, *Conservation relations for local theta correspondence*, J. Amer. Math. Soc. **28** (2015), no. 4, 939–983, DOI 10.1090/S0894-0347-2014-00817-1. MR3369906
- [Wal90] J.-L. Waldspurger, *Démonstration d'une conjecture de dualité de Howe dans le cas p -adique, $p \neq 2$* (French), Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. 2, Weizmann, Jerusalem, 1990, pp. 267–324. MR1159105
- [Zel80] A. V. Zelevinsky, *Induced representations of reductive p -adic groups. II. On irreducible representations of $GL(n)$* , Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210. MR584084

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