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INTERTWINING MAPS BETWEEN p-ADIC PRINCIPAL SERIES OF p-ADIC GROUPS

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Abstract. In this paper we study p-adic principal series representation of a p-adic group G as a module over the maximal compact subgroup G_0. We show that there are no non-trivial G_0-intertwining maps between principal series representations attached to characters whose restrictions to the torus of G_0 are distinct, and there are no non-scalar endomorphisms of a fixed principal series representation. This is surprising when compared with another result which we prove: that a principal series representation may contain infinitely many closed G_0-invariant subspaces. As for the proof, we work mainly in the setting of Iwasawa modules, and deduce results about G_0-representations by duality.

1. Introduction

In this paper we study intertwining maps between p-adic principal series representations of compact p-adic groups.

We take L a p-adic field and K a finite extension of L, and denote by o_L and o_K their rings of integers. We take G to be the L points of a split connected reductive $\mathbb{Z}$-group $\mathbf{G}$. Let $G_0 = \mathbf{G}(o_L)$. We equip $\mathbf{G}$ with a choice of Borel $\mathbf{P}$, having unipotent radical $\mathbf{U}$ and split maximal torus $\mathbf{T} \subset \mathbf{P}$. Let $P = \mathbf{P}(L)$ and $P_0 = \mathbf{P}(o_L)$, and $U_0 = \mathbf{U}(o_L)$. Let B be the standard Iwahori subgroup of G_0. If $\chi_0 : P_0 \to o_K^\times$ is a continuous character, trivial on U_0, we let

$$\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1}) = \{f : G_0 \to K \text{ continuous } \mid f(gp) = \chi_0(p) f(g)\, \forall p \in P_0, g \in G_0\},$$

where G_0 acts on the left by $g \cdot f(h) = f(g^{-1}h)$. These are the principal series representations which we study.

Our approach is based on the duality theory of Schneider and Teitelbaum [11]. Let $K[[G_0]]$ be the Iwasawa algebra of G_0 (see section 3 for the definition of $K[[G_0]]$). The character χ_0 extends uniquely to a continuous character of $K[[P_0]]$. Let $K^{(\chi_0)}$ denote the corresponding one dimensional $K[[P_0]]$-module, and let $M^{(\chi_0)} = K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi_0)}$.

The space $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ is a Banach space, with continuous G-action. Its continuous dual is isomorphic to $M^{(\chi_0)}$. Since $M^{(\chi_0)}$ is generated as a $K[[G_0]]$-module by a single element $1 \otimes 1$, it follows that $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ is an admissible Banach space representation [11, Lemma 3.4].

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Given two continuous characters χ_1 and χ_2 of P_0, we want to describe the space of continuous G_0-intertwining operators $\mathrm{Hom}_{G_0}(\mathrm{Ind}_{P_0}^{G_0}(\chi_1^{-1}), \mathrm{Ind}_{P_0}^{G_0}(\chi_2^{-1}))$. By duality [11, Theorem 3.5], this is equivalent to describing the space of $K[[G_0]]$-linear maps $\mathrm{Hom}_{K[[G_0]]}(M^{(\chi_1)}, M^{(\chi_2)})$.

Our main result is the following (Corollary 10.2):

Theorem 1.1. *For any two continuous characters χ_1 and χ_2 of P_0, we have*

$$\mathrm{Hom}_{K[[G_0]]}(M^{(\chi_1)}, M^{(\chi_2)}) = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2, \\ K \cdot \mathrm{id} & \text{if } \chi_1 = \chi_2. \end{cases}$$

This was partially known in the case $G_0 = GL_2(\mathbb{Z}_p)$: Proposition 4.5 from [11] states that $\mathrm{Hom}_{K[[GL_2(\mathbb{Z}_p)]]}(M^{(\chi_1)}, M^{(\chi_2)}) = 0$ if $\chi_1 \neq \chi_2$. The first step in our proof generalizes the argument in section 4 of [11]. The Bruhat decomposition of G gives rise to a decomposition of $M^{(\chi)}$ as a direct sum of $K[[B]]$-modules indexed by the elements of the Weyl group.

By duality, Theorem 1.1 implies the following (Corollary 10.3):

Corollary 1.2. *For any two continuous characters χ_1 and χ_2 of P_0, we have*

$$\mathrm{Hom}_{G_0}(\mathrm{Ind}_{P_0}^{G_0}(\chi_1), \mathrm{Ind}_{P_0}^{G_0}(\chi_2)) = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2, \\ K \cdot \mathrm{id} & \text{if } \chi_1 = \chi_2. \end{cases}$$

An analogous result for principal series representations of G was proved by Peter Schneider in an unpublished note. Suppose $\chi : P \to K^\times$ is a continuous character, and set $\chi_0 = \chi|_{P_0}$. Note that restriction to G_0 gives an isomorphism $\mathrm{Ind}_P^G(\chi) \to \mathrm{Ind}_{P_0}^{G_0}(\chi_0)$. The G-representation $\mathrm{Ind}_P^G(\chi)$, however, differs significantly from the G_0-representation $\mathrm{Ind}_{P_0}^{G_0}(\chi_0)$. For example, we know from [14] that in the case of $\mathbf{G} = GL_2$, the $GL_2(\mathbb{Z}_p)$-representation $\mathrm{Ind}_{P_0}^{G_0}(\chi_0)$ can have infinitely many finite dimensional subrepresentations, while the $GL_2(\mathbb{Q}_p)$-representation $\mathrm{Ind}_P^G(\chi)$, if reducible, has a unique irreducible subrepresentation. With this in mind, the result of Corollary 1.2 for G_0 seems surprising. Examples of $\mathrm{Ind}_{P_0}^{G_0}(\chi_0)$ with infinitely many finite dimensional subrepresentations for a general group $\mathbf{G}$ are constructed in the Appendix.

The structure of the paper is as follows. In section 2, we introduce some notation. In section 3, we give a projective limit realization of $M_0^{(\chi_0)} = o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi_0)}$. In section 4, we introduce a decomposition $M^{(\chi_0)} = \bigoplus_{w \in W} M_w^{(\chi_0)}$ into components $M_w^{(\chi_0)}$ indexed by the Weyl group W of $\mathbf{G}$. In section 5 we describe $M_w^{(\chi_0)}$ as a tensor product, thus obtaining a $K[[B]]$-module decomposition

$$M^{(\chi_0)} \cong \bigoplus_{w \in W} K[[B]] \otimes_{K[[P_{\frac{1}{2}}^{w,\pm}]]} K^{(w\chi_0)},$$

where $P_{\frac{1}{2}}^{w,\pm} = B \cap w P_0 w^{-1}$ (Corollary 5.3). This decomposition generalizes the decomposition $M_\chi \cong N_\chi \oplus N_{w\chi}^-$ for $G_0 = GL_2(\mathbb{Z}_p)$ which appears on p. 376 of [11]. The next step is to prove that

$$\mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, M_w^{(\chi_2)}) = 0$$

for all w other than the identity. In fact, what we prove in Proposition 8.1 is a more general statement which allows us, in corollary 8.2, to show that $\mathrm{Hom}_{K[[B]]}(M_w^{(\chi_1)},$

$M_v^{(\chi_2)}) \neq 0$ implies $w = v$– a result which seems interesting in its own right. Sections 6 and 7 are devoted to technical preliminaries that are required to prove the results in section 8.

Sections 9 and 10 contain the proof of the main result.

2. Notation

Let L be a finite extension of $\mathbb{Q}_p$. Let G be the group of L-points of a split connected reductive L-group $\mathbf{G}_L$. The group $\mathbf{G}_L$ is determined, up to an L-isomorphism, by its root datum [13, Theorems 16.3.2 and 16.3.3]. On the other hand, there exists a split connected reductive $\mathbb{Z}$-group $\mathbf{G}_\mathbb{Z}$ with the same root datum [5, Théorème 1.1, Exposé XXV]. Denote by $(\mathbf{G}_\mathbb{Z})_L$ the corresponding L-group. Then $(\mathbf{G}_\mathbb{Z})_L$ is isomorphic to $\mathbf{G}_L$. Hence, we may assume that $G = \mathbf{G}(L)$, where $\mathbf{G}$ is a split connected reductive $\mathbb{Z}$-group.

We take $\mathbf{P}$ a Borel subgroup and $\mathbf{T} \subset \mathbf{P}$ a maximal torus, and denote the unipotent radical of $\mathbf{P}$ by $\mathbf{U}$. The unipotent radical of the opposite parabolic is denoted $\mathbf{U}^-$. We write Φ for the roots of $\mathbf{T}$ in $\mathbf{G}$ and Φ^+ (resp. Φ^-) for the set of positive (resp. negative) roots determined by the choice of $\mathbf{P}$. For each $\alpha \in \Phi$ the root subgroup attached to α is denoted $\mathbf{U}_\alpha$. For each root α of $\mathfrak{g}$ one defines a morphism x_α from the additive $\mathbb{Z}$-group $\mathbb{G}_a$ to $\mathbf{U}_\alpha$.

We denote by o_L the ring of integers of L and by $\mathfrak{p}_L$ its unique maximal ideal. Let q_L be the cardinality of the residue field of L.

For each algebraic subgroup $\mathbf{H}$ of $\mathbf{G}$ we let $H = \mathbf{H}(L)$ and $H_0 = \mathbf{H}(o_L)$. We write pr_n for the canonical map $o_L \to o_L/\mathfrak{p}_L^n$ and also for the induced map $H_0 \to \mathbf{H}(o_L/\mathfrak{p}_L^n)$ for any $\mathbf{H}$. The kernel of pr_n in H_0 is denoted H_n. Finally, $H(o_L/\mathfrak{p}_L)$ is denoted $\bar{H}$. Let $B = \mathrm{pr}_1^{-1}(\bar{P})$ be the standard Iwahori subgroup.

We denote the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$ by W. For each $w \in W$ we select a representative $\dot{w} \in \mathbf{G}(\mathbb{Z})$.

We work with p-adic representations; their coefficient field is a finite extension K of L. Then we have o_K, $\mathfrak{p}_K$, and q_K defined similarly as above. Let $|\ | = |\ |_K$ be the absolute value on K given by $|\varpi_K| = q_K^{-1}$, where ϖ_K is a uniformizer of K.

If X is a set, $\mathbf{1}_X$ denotes the characteristic function of X.

2.1. Some unipotent subgroups of G_0.

For $w \in W$, let $V_w^{\pm} = \dot{w}U^-\dot{w}^{-1}$. Note that $V_w^{\pm}$ is the product of all the root subgroups U_α attached to roots α such that $w\alpha < 0$. We define

$$\begin{aligned}
U^-_{w,\frac{1}{2}} &= \dot{w}^{-1}B\dot{w} \cap U_0^- = (\dot{w}^{-1}U_0\dot{w} \cap U_0^-)(\dot{w}^{-1}U_1^-\dot{w} \cap U_0^-),\\
V^{\pm}_{w,\frac{1}{2}} &= \dot{w}U^-_{w,\frac{1}{2}}\dot{w}^{-1} = (U_0 \cap \dot{w}U^-\dot{w}^{-1})(U_1^- \cap \dot{w}U^-\dot{w}^{-1}),\\
V^{w,\pm}_{\frac{1}{2}} &= B \cap \dot{w}U_0\dot{w}^{-1} = (U_0 \cap \dot{w}U_0\dot{w}^{-1})(U_1^- \cap \dot{w}U_0\dot{w}^{-1}).
\end{aligned}$$

Then $V^{\pm}_{w,1} \subset V^{\pm}_{w,\frac{1}{2}} \subset V^{\pm}_{w,0}$. The subscript $\frac{1}{2}$ indicates that $V^{\pm}_{w,\frac{1}{2}}$ is a mixture of $U_{\alpha,1}$'s and $U_{\alpha,0}$'s, while the superscript ± 1 indicates that some roots α are positive and some are negative.

From [1, Section 4.1], $\coprod_{w\in W} \dot{w}U^-_{w,\frac{1}{2}}$ is a set of coset representatives of G_0/P_0. In particular, $B\dot{w}B = \dot{w}U^-_{w,\frac{1}{2}}P_0 = V^{\pm}_{w,\frac{1}{2}}\dot{w}P_0$ and we have the disjoint union decomposition

$$G_0 = \coprod_{w\in W} \dot{w}U^-_{w,\frac{1}{2}}P_0 = \coprod_{w\in W} V^{\pm}_{w,\frac{1}{2}}\dot{w}P_0.$$

3. Projective limit realization of $M_0^{(\chi)}$

3.1. **Iwasawa algebra.** Define

$$o_K[[G_0]] = \varprojlim_N o_K[G_0/N] \quad \text{and} \quad K[[G_0]] = K \otimes_{o_K} o_K[[G_0]],$$

where N runs through all open normal subgroups of G_0. We equip $o_K[[G_0]]$ with the projective limit topology and $K[[G_0]]$ with the corresponding locally convex topology [11]. As a projective limit of compact rings, $o_K[[G_0]]$ is compact.

Since $\{G_n \mid n \in \mathbb{N}\}$ is a neighborhood basis of the identity consisting of open normal subgroups of G_0, we have $o_K[[G_0]] = \varprojlim_{n\in\mathbb{N}} o_K[G_0/G_n]$. The projective limit $\varprojlim_{n\in\mathbb{N}} o_K[G_0/G_n]$ can be realized as a subspace of the topological space $\prod_{n\in\mathbb{N}} o_K[G_0/G_n]$ and we have natural projections $\varphi_n : o_K[[G_0]] \to o_K[G_0/G_n]$. For $\mu \in o_K[[G_0]]$, set $\mu_n = \varphi_n(\mu)$. Then we identify

$$\mu = (\mu_n)_{n=1}^{\infty} \in \prod_{n\in\mathbb{N}} o_K[G_0/G_n].$$

The surjections $o_K[G_0] \to o_K[G_0/G_n]$ induce in the limit the injective ring homomorphism

$$o_K[G_0] \to o_K[[G_0]]$$

[11, Section 2]. We use this homomorphism to identify $o_K[G_0]$ with its image in $o_K[[G_0]]$.

3.2. **Canonical pairing.** Let $C(G_0, K)$ be the space of continuous K-valued functions on G_0. We equip $C(G_0, K)$ with the Banach space topology induced by the sup norm. We denote by $C^\infty(G_0, K)$ the subspace of $C(G_0, K)$ consisting of smooth (i.e., locally constant) functions. Then $C^\infty(G_0, K)$ is dense in $C(G_0, K)$. This follows from Example 3.D on page 47 in [15], noticing that, by compactness of G_0, the continuous functions on G_0 are bounded.

Let $D^c(G_0, K)$ be the continuous dual of $C(G_0, K)$. We have the canonical pairing $\langle\ ,\ \rangle : D^c(G_0, K) \times C(G_0, K) \to K$ given by

$$\langle \mu, h\rangle = \mu(h).$$

The Iwasawa algebra $K[[G_0]]$ can be identified with $D^c(G_0, K)$ by identifying $g \in G_0$ with the Dirac distribution δ_g [12, Section 2]. This gives us the canonical pairing $\langle\ ,\ \rangle : K[[G_0]] \times C(G_0, K) \to K$.

We can describe the pairing explicitly (see Section 12 in [10]). Let $\mu \in o_K[[G_0]]$ and $h \in C(G_0, K)$. Write $\mu = (\mu_n)_{n=1}^{\infty}$, where $\mu_n \in o_K[G_0/G_n]$. On the other hand, h can be uniformly approximated by a sequence $\{h_n\}_{n=1}^{\infty}$ of smooth functions such that h_n is right G_n-invariant. If $g_1 G_n = g_2 G_n$, then $\delta_{g_1}(h_n) = \delta_{g_2}(h_n)$. It follows that we have a well-defined pairing $\langle \mu_n, h_n\rangle$. More specifically, if $\{g_1, \ldots, g_s\}$ is a set of representatives of G_0/G_n, we can write

$$\mu_n = a_1 g_1 G_n + \cdots + a_s g_s G_n \quad \text{and} \quad h_n = b_1 1_{g_1 G_n} + \cdots + b_s 1_{g_s G_n},$$

where $a_i \in o_K$ and $b_i \in K$ for all i. Then

$$\langle \mu_n, h_n\rangle = a_1 b_1 + \cdots + a_s b_s.$$

It can be shown that $\{\langle \mu_n, h_n\rangle\}_{n=1}^{\infty}$ is a Cauchy sequence whose limit is independent of the choice of $\{h_n\}_{n=1}^{\infty}$. Then

$$\langle \mu, h\rangle = \lim_{n\to\infty} \langle \mu_n, h_n\rangle.$$

Observe that $h_n \in C(G_0, K)$, so we can apply the above formula to evaluate $\langle \mu, h_n \rangle$. It is easy to show that $\langle \mu, h_n \rangle = \langle \mu_n, h_n \rangle$.

3.3. **Extending characters of P_0 to $o_K[[P_0]]$.** Let $\chi : P_0 \to o_K^\times$ be a continuous character. By Lemma 2.1 and Corollary 2.2 of [11], it extends uniquely to a continuous homomorphism of o_K-modules $\chi : o_K[[P_0]] \to o_K$ and a continuous homomorphism of K-algebras $\chi : K[[P_0]] \to K$. The extension is achieved by $\langle \nu, \chi \rangle$, where $\langle\ ,\ \rangle : K[[P_0]] \times C(P_0, K) \to K$ is the canonical pairing described in section 3.2. Hence, for $\nu \in K[[P_0]]$ we have

$$\chi(\nu) = \langle \nu, \chi \rangle.$$

We denote by $o_K^{(\chi)}$ (respectively, $K^{(\chi)}$) the corresponding one dimensional $o_K[[P_0]]$-module (respectively, $K[[P_0]]$-module).

3.4. **Module $M_0^{(\chi)}$.** From now on, χ is a continuous character $\chi : P_0 \to o_K^\times$ which is trivial on U_0. Equivalently, χ is a continuous character $\chi : T_0 \to o_K^\times$ which we extend trivially to U_0. Define

$$M_0^{(\chi)} = o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)}.$$

As a quotient of the compact ring $o_K[[G_0]]$, $M_0^{(\chi)}$ is a compact $o_K[[G_0]]$-module.

In Proposition 3.3, we give a realization of $M_0^{(\chi)}$ as the projective limit over $n \in \mathbb{N}$ of tensor products $o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$. We start by proving two technical lemmas about those tensor products. We follow the convention that $\mathfrak{p}_K^0 = o_K$ and $\mathfrak{p}_K^\infty = 0$.

Lemma 3.1. *Let $\chi : P_0 \to o_K^\times$ be a continuous character and let $n \in \mathbb{N}$. Define $m(\chi, n) = \sup\{m \in \mathbb{N} \cup \{0\} \mid \chi(p) \in 1 + \mathfrak{p}_K^m$ for all $p \in P_n\}$.*

(i) *In $o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$, for any $\xi \in o_K[G_0/G_n]$ and any $b \in \mathfrak{p}_K^{m(\chi,n)}$ we have*

$$\xi \otimes b = 0.$$

(ii) *The o_K-module $o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$ is isomorphic to*

$$\bigoplus_{w \in W} o_K/\mathfrak{p}_K^{m(\chi,n)}[U^-_{w,\frac{1}{2}}/U^-_n],$$

where $U^-_{w,\frac{1}{2}}$ is as in section 2.1.

Note that $m(\chi, n) = \infty$ if and only if $\chi|_{P_n} = 1$. In any case, $\lim_{n\to\infty} m(\chi, n) = \infty$ by continuity of χ.

Proof. (i) If $m(\chi, n) = \infty$, then there is nothing to prove.

Assume $m(\chi, n) < \infty$. For any $p \in P_n$ and any $\xi \in o_K[G_0/G_n]$, we have $\xi = \xi p$, and hence

$$\xi \otimes (1 - \chi(p)) = (\xi \otimes 1) - (\xi \otimes \chi(p)) = (\xi \otimes 1) - (\xi p \otimes 1) = 0.$$

Now, take $p_0 \in P_n$ such that $ord_K(\chi(p_0) - 1) = m(\chi, n)$. Then any $b \in \mathfrak{p}_K^{m(\chi,n)}$ can be written as $b = b_0(1 - \chi(p_0))$ for some $b_0 \in o_K$. It follows

$$\xi \otimes b = \xi \otimes b_0(1 - \chi(p_0)) = b_0(\xi \otimes (1 - \chi(p_0))) = 0.$$

(ii) We first recall the disjoint union decomposition $G_0 = \coprod_{w\in W} \dot{w}U^-_{w,\frac{1}{2}}P_0$. Define

$$h_w : o_K[U^-_{w,\frac{1}{2}}/U^-_n] \to o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$$
$$\mu \mapsto \dot{w}\mu \otimes 1.$$

Then $\bigoplus_w h_w : \bigoplus_w o_K[U^-_{w,\frac{1}{2}}/U^-_n] \to o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$ is easily seen to be surjective.

Next, we want to realize $o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$ as the dual of a suitable space of functions. We consider the o_K-module

$$i(\chi, n) := \{f : G_0/G_n \to o_K/\mathfrak{p}_K^{m(\chi,n)} \mid f(gp) = \mathrm{pr}_{m(\chi,n)}\, \chi(p)f(g), \quad \text{for } g \in G_0/G_n \text{ and } p \in P_0/P_n\},$$

where $\mathrm{pr}_{m(\chi,n)}$ is the canonical projection $o_K \to o_K/\mathfrak{p}_K^{m(\chi,n)}$. The mapping

$$(g, a) \mapsto \lambda_{g,a}, \quad \text{where } \lambda_{g,a}(f) = af(g), \quad a \in o_K,\ g \in G_0/G_n,\ f \in i(\chi, n)$$

extends to a surjective middle linear map from $o_K[G_0/G_n] \times o_K$ to the o_K-module

$$i(\chi, n)^* := \mathrm{Hom}_{o_K}(i(\chi, n), o_K/\mathfrak{p}_K^{m(\chi,n)}).$$

This middle linear map then induces a linear map $o_K[G_0/G_n] \otimes_{o_K[P_0/P_n]} o_K^{(\chi)} \to i(\chi, n)^*$. It is then easy to see that the kernel of the map from $\bigoplus_w w o_K[U^-_{w,\frac{1}{2}}/U^-_n]$ into the $i(\chi, n)^*$ is $\bigoplus_w w\mathfrak{p}_K^{m(\chi,n)}[U^-_{w,\frac{1}{2}}/U^-_n]$. □

Lemma 3.2. *Let $\mu \in o_K[[G_0]]$ and $\nu \in o_K[[P_0]]$. Write $\mu = (\mu_n)_{n=1}^\infty$ and $\nu = (\nu_n)_{n=1}^\infty$ as in section* 3.1. *Then $\mu_n\nu_n \otimes a = \mu_n \otimes \chi(\nu)a$ in $o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$.*

Proof. Let $\{c_n\}_{n=1}^\infty$ be a sequence of functions as in section 3.2: each $c_n : P_0 \to o_K$ is right P_n-invariant and $\chi = \lim_{n\to\infty} c_n$.

Let us make a reasonable and explicit choice of $\{c_n\}_{n=1}^\infty$. For each n, we select $c_n : P_0 \to o_K$ which is constant on cosets of P_n, such that inside each coset there is at least one point p_0, where $c_n(p_0) = \chi(p_0)$.

Now, let $m(\chi, n)$ be the maximal integer such that $\chi(P_n) \subset 1 + \mathfrak{p}_K^{m(\chi,n)}$. If $p_1P_n = p_2P_n$, then $\chi(p_1) - \chi(p_2) \in \mathfrak{p}_K^{m(\chi,n)}$. It follows

$$c_n(p) - \chi(p) \in \mathfrak{p}_K^{m(\chi,n)}, \quad \text{for all } p \in P_0.$$

Consequently, $\langle \xi, c_n - \chi\rangle \in \mathfrak{p}_K^{m(\chi,n)}$ for all $\xi \in o_K[[P_0]]$.

Let $\{p_1, \dots, p_s\}$ be a set of coset representatives of P_0/P_n consisting of points satisfying $c_n(p_i) = \chi(p_i)$ for all i. (By our construction of c_n, such points exist.) Then we can write $\nu_n = a_1p_1P_n + \cdots + a_sp_sP_n$, where $a_i \in o_K$. Define

$$\eta = a_1p_1 + \cdots + a_sp_s.$$

This is an element of $o_K[P_0] \subset o_K[[P_0]]$ such that $\eta_n = \nu_n$. Since

$$\chi(\eta) = a_1\chi(p_1) + \cdots + a_s\chi(p_s) = a_1c_n(p_1) + \cdots + a_sc_n(p_s) = \langle \eta, c_n\rangle,$$

it follows

$$\chi(\eta) = \langle \eta, c_n\rangle = \langle \eta_n, c_n\rangle = \langle \nu_n, c_n\rangle = \langle \nu, c_n\rangle \in \chi(\nu) + \mathfrak{p}_K^{m(\chi,n)}.$$

Now, in $o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$, we have

$$\mu_n\nu_n \otimes a = \mu_n\eta_n \otimes a = \mu_n \otimes \chi(\eta)a.$$

To show that the above expression is equal to $\mu_n \otimes \chi(\nu)a$, we observe that $\chi(\eta) - \chi(\nu) \in \mathfrak{p}_K^{m(\chi,n)}$, and apply Lemma 3.1(i). □

Proposition 3.3. *Let $\chi : P_0 \to o_K^\times$ be a continuous character trivial on U_0. Then*

$$M_0^{(\chi)} \cong \varprojlim_{n\in\mathbb{N}} \left(o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}\right).$$

Proof. As explained in section 3.1, any $\mu \in o_K[[G_0]]$ can be written as $\mu = (\mu_n)_{n=1}^{\infty}$, where $\mu_n \in o_K[G_0/G_n]$. For each $n \in \mathbb{N}$, we define a map

$$\psi_n : o_K[[G_0]] \times o_K^{(\chi)} \to o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$$
$$(\mu, a) \mapsto \mu_n \otimes a.$$

It follows from Lemma 3.2 that ψ_n is $o_K[[P_0]]$-middle linear. Hence, it gives rise to a linear map

$$\Psi_n : o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)} \to o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}.$$

Now, $\{\Psi_n\}_{n\in\mathbb{N}}$ is a family of compatible continuous linear maps from

$$o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)}$$

to the inverse system $\{o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}\}_{n\in\mathbb{N}}$. By the universal property of projective limits, there exists a continuous linear map

$$\Psi : M_0^{(\chi)} = o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)} \to \varprojlim_{n\in\mathbb{N}} \left(o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}\right).$$

This map is surjective because $M_0^{(\chi)} = o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)}$ is compact and Ψ_n are surjective [8, Corollary 1.1.6].

For injectivity, we first recall from [1, Corollary 6.3] that $\bigoplus_w K[[U^-_{w,\frac{1}{2}}]]$ maps isomorphically onto $K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi)}$. From the embedding $o_K[[G_0]] \hookrightarrow K[[G_0]]$ we obtain an isomorphism

$$f : \bigoplus_{w\in W} o_K[[U^-_{w,\frac{1}{2}}]] \xrightarrow{\sim} o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)},$$

where the restriction f_w of f to $o_K[[U^-_{w,\frac{1}{2}}]]$ is given by $f_w : \mu \mapsto \dot{w}\mu \otimes 1$. Note that $f = \bigoplus_w f_w$.

For every $w \in W$, we have the following commutative diagram

$$\begin{array}{ccc} o_K[[U^-_{w,\frac{1}{2}}]] & \xrightarrow{h_w} & \varprojlim_{n\in\mathbb{N}} \left(o_K/\mathfrak{p}_K^{m(\chi,n)}[U^-_{w,\frac{1}{2}}/U^-_n]\right) \\ {\scriptstyle f_w}\downarrow & & \downarrow{\scriptstyle g_w} \\ o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)} & \xrightarrow{\Psi} & \varprojlim_{n\in\mathbb{N}} \left(o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}\right). \end{array}$$

The map h_w is built from the natural projections $o_K[[U^-_{w,\frac{1}{2}}]] \to o_K/\mathfrak{p}_K^{m(\chi,n)}[U^-_{w,\frac{1}{2}}/U^-_n]$, using the universal property of projective limits. The map g_w is defined as follows. We know from the proof of Lemma 3.1(ii) that the maps $g_{n,w} : o_K/\mathfrak{p}_K^{m(\chi,n)}[U^-_{w,\frac{1}{2}}/U^-_n] \to o_K[G_0/G_n] \otimes_{o_K[P_0]} o_K^{(\chi)}$, given by $g_{n,w} : \mu \mapsto \dot{w}\mu \otimes 1$, are injective, and that $g_n = \bigoplus_w g_{n,w}$ is an isomorphism of o_K-modules. Define $g_w = \varprojlim_n g_{n,w}$. Then

g_w is injective. Thus, we reduce our proof to proving the injectivity of h_w, for all $w \in W$.

Suppose η is a nonzero element of $o_K[[U^-_{w,\frac{1}{2}}]]$ and write $\eta = (\eta_n)_{n=1}^\infty$, where $\eta_n \in o_K[U^-_{w,\frac{1}{2}}/U^-_n]$. Then for each n we have

$$\eta_n = \sum_{\overline{u} \in U^-_{w,\frac{1}{2}}/U^-_n} c_{\overline{u}} \overline{u}$$

and for some $n_0, \overline{u}_0$ $c_{\overline{u}_0} \neq 0$. Then for all $n \geq n_0$ there exists $\overline{u} \in U^-_{w,\frac{1}{2}}/U^-_n$ such that $|c_{\overline{u}}| \geq |c_{\overline{u}_0}|$. Then for all n sufficiently large we will have $c_{\overline{u}_0} \notin \mathfrak{p}_K^{m(n,\chi)}$, and hence the image of η_n in $o_K/\mathfrak{p}_K^{m(\chi,n)}[U^-_{w,\frac{1}{2}}/U^-_n]$ is nonzero. □

4. The space $M^{(\chi)}$ and its decomposition

The continuous principal series representation

$$\mathrm{Ind}_{P_0}^{G_0}(\chi^{-1}) = \{f \in C(G_0, K) \mid f(gp) = \chi(p)f(g) \text{ for all } p \in P_0, g \in G_0\}$$

is a closed subspace of the Banach space $C(G_0, K)$, so it is itself a Banach space. Its continuous dual is isomorphic to

$$M^{(\chi)} = K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi)}.$$

We have the canonical pairing $\langle\ ,\ \rangle : M^{(\chi)} \times \mathrm{Ind}_{P_0}^{G_0}(\chi^{-1}) \to K$. There is no confusion in using the same notation as for the pairing $\langle\ ,\ \rangle : K[[G_0]] \times C(G_0, K) \to K$ for the following reason. For $\mu \in K[[G_0]]$, we denote its image in $M^{(\chi)}$ by $[\mu]$. As explained in [1, Section 6], if $f \in \mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})$ and $\mu \in K[[G_0]]$, then $\langle \mu, f\rangle$ depends only on $[\mu]$ and it is equal to $\langle [\mu], f\rangle$.

The principal series representation $\mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})$ decomposes as

$$\mathrm{Ind}_{P_0}^{G_0}(\chi^{-1}) = \bigoplus_{w \in W} \mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})_w,$$

where $\mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})_w := \{f \in \mathrm{Ind}_{P_0}^{G_0}(\chi^{-1}) : \mathrm{supp}(f) \subset B\dot{w}B = V^{\pm}_{w,\frac{1}{2}}\dot{w}P_0\}$ [1, Section 6.1]. Define $M_w^{(\chi)} = \{[\mu] \in M^{(\chi)} : \langle [\mu], f\rangle = 0,\ f \in \mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})_{w'},\ w' \neq w\}$. This is a closed subspace of $M^{(\chi)}$. Since each subspace $\mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})_{w'}$ is B-invariant, $M_w^{(\chi)}$ is also B-invariant, and therefore it is a $K[[B]]$-module. Then, as in [1, Section 6.1], we have the $K[[B]]$-module decomposition

$$M^{(\chi)} = \bigoplus_{w \in W} M_w^{(\chi)}.$$

Lemma 4.1 is Corollary 6.3 from [1] (as already mentioned in the proof of Proposition 3.3). We briefly review its proof, to introduce notation needed in the rest of the paper. Let $U^-_{w,\frac{1}{2}}$ and $V^{\pm}_{w,\frac{1}{2}}$ be as in section 2.1.

Lemma 4.1. *As $K[[V^{\pm}_{w,\frac{1}{2}}]]$-modules, $M_w^{(\chi)} \cong K[[V^{\pm}_{w,\frac{1}{2}}]]$.*

Proof. As shown in [1, Corollary 6.3] the subspace

$$\dot{w}K[[U^-_{w,\frac{1}{2}}]] = K[[V^{\pm}_{w,\frac{1}{2}}]]\dot{w}$$

maps isomorphically onto $M_w^{(\chi)}$. Clearly, the map sending $\eta \in K[[V^{\pm}_{w,\frac{1}{2}}]]$ to $\eta\dot{w} \in K[[G_0]]$ is a $K[[V^{\pm}_{w,\frac{1}{2}}]]$-intertwining map.

Explicitly, $M_w^{(\chi)}$ is identified with the dual of $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})_w$. Each element of this space is of the form

$$f_h(v\dot{w}p) = h(v)\chi(p)$$

for some unique $h \in C(V^{\pm}_{w,\frac{1}{2}}, K)$. Clearly, the map $h \to f_h$ commutes with left inverse translation by elements of $V^{\pm}_{w,\frac{1}{2}}$. □

The space $M_w^{(\chi)}$ is a $K[[B]]$-module, and so the isomorphism from Lemma 4.1 induces a $K[[B]]$-module structure on $K[[V^{\pm}_{w,\frac{1}{2}}]]$. The action of T_0 can be described explicitly. Let $A : T_0 \to \mathrm{Aut}(C(V^{\pm}_{w,\frac{1}{2}}, K))$ be the action by conjugation: $[A(t).h](v) = h(t^{-1}vt)$. Then

$$t \cdot f_h = w\chi(t)^{-1} f_{A(t).h},$$

where the action $t \cdot f_h$ is by left inverse translation, and $w\chi(t) = \chi(w^{-1}tw)$. In particular, the induced action of T_0 on $\mathrm{Aut}(C(V^{\pm}_{w,\frac{1}{2}}, K))$ is given by

$$[A_\chi(t).h](v) = w\chi(t)^{-1}h(t^{-1}vt).$$

The induced action on $K[[V^{\pm}_{w,\frac{1}{2}}]]$ is given by

$$\langle A_\chi(t).\mu, h\rangle = \langle \mu, A_\chi(t^{-1}).h\rangle, \qquad \mu \in K[[V^{\pm}_{w,\frac{1}{2}}]],\ h \in C(V^{\pm}_{w,\frac{1}{2}}, K),\ t \in T_0.$$

Combined with the action of $K[[V^{\pm}_{w,\frac{1}{2}}]]$ on itself by left translation, this action of T_0 makes $K[[V^{\pm}_{w,\frac{1}{2}}]]$ into a $K[[Q^{\pm}_{w,\frac{1}{2}}]]$-module, where

$$Q^{\pm}_{w,\frac{1}{2}} = T_0 V^{\pm}_{w,\frac{1}{2}} = B \cap wP_0^- w^{-1}.$$

Write $K[[V^{\pm}_{w,\frac{1}{2}}]]^{(w\chi)}$ for this $K[[Q^{\pm}_{w,\frac{1}{2}}]]$-module structure on $K[[V^{\pm}_{w,\frac{1}{2}}]]$. Then we have proved Lemma 4.2:

Lemma 4.2. *As $K[[Q^{\pm}_{w,\frac{1}{2}}]]$-modules, $M_w^{(\chi)} \cong K[[V^{\pm}_{w,\frac{1}{2}}]]^{(w\chi)}$.*

5. An alternate description of $M_w^{(\chi)}$

Recall the space $\mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})_w = \{f \in \mathrm{Ind}_{P_0}^{G_0}(\chi^{-1}) \mid \mathrm{supp}(f) \subset B\dot{w}B\}$, and its dual $M_w^{(\chi)}$. The purpose of this section is to give a realization of $M_w^{(\chi)}$ as a tensor product, analogous to the realization of $M^{(\chi)}$ itself as $K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi_0)}$.

We will prove that $\mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})_w$ is isomorphic as a B-module to a representation induced from $B \cap \dot{w}P_0\dot{w}^{-1}$, and obtain the corresponding tensor product expression for $M_w^{(\chi)}$. Both results depend on the fact that multiplication is a homeomorphism $V^{\pm}_{w,\frac{1}{2}} \times (B \cap \dot{w}P_0\dot{w}^{-1}) \to B$.

To prepare for the proof, we introduce the following technical result.

Lemma 5.1. *Let F be any field. Let $\Phi^+ = S_1 \coprod S_2$ be any partition of the positive roots into two disjoint sets. Take any numbering of S_1 as $\{\beta_1, \ldots, \beta_n\}$ and any numbering of S_2 as $\{\gamma_1, \ldots, \gamma_m\}$. Then*

$$\Big((b_1, \ldots, b_n),\ t,\ (c_1, \ldots, c_m)\Big) \mapsto x_{\beta_1}(b_1) \ldots x_{\beta_n}(b_n) \cdot t \cdot x_{\gamma_1}(c_1) \ldots x_{\gamma_m}(c_m)$$

is a bijection $F^n \times T(F) \times F^m \to P(F)$.

Proof. By §14.4 of [2] multiplying root subgroups gives an isomorphism of varieties $\prod_\alpha U_\alpha \to U$, for any ordering of the roots. That is

$$((b_1, \dots, b_n), (c_1, \dots, c_n)) \to x_{\beta_1}(b_1) \dots x_{\beta_n}(b_n) x_{\gamma_1}(c_1) \dots x_{\gamma_m}(c_m)$$

is a bijection $F^n \times F^m \to U(F)$. On the other hand, $P = TU$, and we can conjugate $t \in T(F)$ to the middle. □

Define

$$P^{w,\pm}_{\frac{1}{2}} = B \cap \dot{w} P_0 \dot{w}^{-1} = T_0 (U_0 \cap \dot{w} U_0 \dot{w}^{-1})(U_1^- \cap \dot{w} U_0 \dot{w}^{-1}).$$

Lemma 5.2.

(1) *Multiplication induces a homeomorphism* $V^\pm_{w,\frac{1}{2}} \times P^{w,\pm}_{\frac{1}{2}} \to B$.

(2) *As representations of* B,

$$\mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})_w \cong \mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}} w\chi^{-1}.$$

(3) *As a* $K[[B]]$*-module,*

$$M_w^{(\chi)} \cong K[[B]] \otimes_{K[[P^{w,\pm}_{\frac{1}{2}}]]} K^{(w\chi)}.$$

Proof. (1) Write $\bar{P}$ for $P_0/P_1 = \mathbf{P}(o_L/\mathfrak{p}_L) = B/G_1$. Define $\bar{T}, \bar{U}$ and $\bar{U}_\alpha$ for each root α similarly. Given $b \in B$ let $\bar{b}$ be the image in $\bar{P}$. We factor it as $\bar{u}_1 \bar{t} \bar{u}_2$, where $\bar{u}_1 \in \prod_{\alpha>0,\ w^{-1}\alpha<0} \mathbf{U}_\alpha(o_L/\mathfrak{p}_L)$ and $\bar{u}_2 \in \prod_{\alpha>0,\ w^{-1}\alpha>0} \mathbf{U}_\alpha(o_L/\mathfrak{p}_L)$. Choosing representatives in $\mathbf{U}_\alpha(o_L)$, and $T(o_L)$ we obtain u_1, u_2 and t such that $g_1 := u_1^{-1} b u_2^{-1} t^{-1} \in G_1 \subset B$. Now using the Iwahori factorization of B we can write $\dot{w}^{-1} g_1 \dot{w}$ as $v_1 v_2$ with $v_1 \in U_1^-$ and $v_2 \in P_1 = T_1 U_1$. Put $v_i' = \dot{w} v_i \dot{w}^{-1}$. Then $g_1 = v_1' v_2'$ with $v_1' \in \dot{w} U_1^- \dot{w}^{-1}$ and $v_2' \in T_1 \dot{w} U_1 \dot{w}^{-1}$. Now $b = u_1 v_1' v_2' t u_2$, and $u_1 v_1' \in V^\pm_{w,\frac{1}{2}}$ and $v_2' t u_2 \in P^{w,\pm}_{\frac{1}{2}}$.

(2) For each f in $(\mathrm{Ind}_{P_0}^{G_0} \chi^{-1})_w$, define $T.f : B \to K$ by $T.f(b) = f(b\dot{w})$. Then T is clearly injective. Moreover, if $f \in \mathrm{Ind}_{P_0}^{G_0} \chi^{-1}, b \in B$ and $p \in P^{w,\pm}_{\frac{1}{2}}$, then $T.f(bp) = f(bp\dot{w}) = f(b\dot{w}\dot{w}^{-1}p\dot{w}) = \chi(\dot{w}^{-1}p\dot{w}) f(b\dot{w})$, because $\dot{w}^{-1}p\dot{w} \in P_0$ by definition of $P^{w,\pm}_{\frac{1}{2}}$. But this is equal to $w\chi(p) T.f(b)$, which proves that $T.f \in \mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}}(w\chi^{-1})$. From the homeomorphism $V^\pm_{w,\frac{1}{2}} \times P^{w,\pm}_{\frac{1}{2}} \to B$, we deduce that $\mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}}(w\chi^{-1})$ is isomorphic to $C(V^\pm_{w,\frac{1}{2}}, K)$ as a vector space (and even a $V^\pm_{w,\frac{1}{2}}$-module). Concretely, every element of $\mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}}(w\chi^{-1})$ is given by

$$f(vp) = h(v)\chi(p), \qquad (v \in V^\pm_{w,\frac{1}{2}}, p \in P^{w,\pm}_{\frac{1}{2}}),$$

for some $h \in C(V^\pm_{w,\frac{1}{2}}, K)$. From this, it easily follows that T is surjective.

(3) It follows readily from the definitions that $\langle \mu\pi, f \rangle = w\chi(\pi)\langle \mu, f \rangle$ for all $\mu \in K[[B]]$, $\pi \in K[[P^{w,\pm}_{\frac{1}{2}}]]$, and $f \in \mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}}(w\chi^{-1})$. It follows that the map $(\mu, a) \to a\mu\Big|_{\mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}}(w\chi^{-1})}$ is a middle-linear map from $K[[B]] \times K^{(w\chi)}$ to the dual of $\mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}}(w\chi^{-1})$, and hence gives rise to a map from $K[[B]] \otimes_{K[[P^{w,\pm}_{\frac{1}{2}}]]} K^{(w\chi)}$ to this dual.

Since $\mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}}(w\chi^{-1}) \cong C(V^{\pm}_{w,\frac{1}{2}}, K)$, it follows that $K[[V^{\pm}_{w,\frac{1}{2}}]]$ maps isomorphically onto the dual, and from this it follows that the map from $K[[B]] \otimes_{K[[P^{w,\pm}_{\frac{1}{2}}]]} K^{(w\chi)}$ is surjective.

The same reasoning used in [1, Corollary 6.3] to show that $\dot{w}K[[U^{-}_{w,\frac{1}{2}}]]$ surjects onto $M^{(\chi)}_w$ may be used here to prove that $K[[V^{\pm}_{w,\frac{1}{2}}]]$ surjects onto $K[[B]] \otimes_{K[[P^{w,\pm}_{\frac{1}{2}}]]} K^{(w\chi)}$. Since the map from $K[[V^{\pm}_{w,\frac{1}{2}}]]$ to the dual of $\mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}}(w\chi^{-1})$ is an isomorphism, the map from $K[[B]] \otimes_{K[[P^{w,\pm}_{\frac{1}{2}}]]} K^{(w\chi)}$ onto the dual of $\mathrm{Ind}^B_{P^{w,\pm}_{\frac{1}{2}}}(w\chi^{-1})$ must be injective, which completes the proof. □

Corollary 5.3. *As a $K[[B]]$-module,*

$$M^{(\chi)} \cong \bigoplus_{w\in W} K[[B]] \otimes_{K[[P^{w,\pm}_{\frac{1}{2}}]]} K^{(w\chi)}.$$

6. Invariant distributions on vector groups

Let $\chi_1, \chi_2 : T_0 \to o_L^\times$ be continuous characters. In this section we establish some technical results which will be used in the proof that

$$\mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, M^{(\chi_2)}_w) \neq 0 \implies w = e,$$

and

$$\mathrm{Hom}_{K[[B]]}(M^{(\chi_1)}_w, M^{(\chi_2)}_v) \neq 0 \implies w = v.$$

The key point is that the only invariant distribution on a group isomorphic to several copies of $\mathbb{Z}_p$ is the trivial one. If $\mathbf{N}$ is an abelian unipotent group, this applies to the groups $\mathbf{N}_k, k \in \mathbb{N}$. These are the “vector groups” of the title.

Lemma 6.1. *Suppose $V \cong o_L^r$ for some positive integer r and that $\mu \in K[[V]]$ satisfies*

$$v \cdot \mu = \mu, \qquad \forall v \in V.$$

Then $\mu = 0$. That is, the space $K[[V]]^V$ of V-invariant distributions on V is 0.

Proof. Let $V_0 = V$ and for $n = 1, 2, 3\dots$ let V_n be the image of the r-copies of $\mathfrak{p}_L^n$ under an isomorphism $o_L^r \to V$. So, $[V_m : V_n] = q_L^{r(n-m)}$ for any nonnegative integers m, n with $m < n$. Now, there is a constant c such that $|\mu(f)| \leq c$ for all $f \in C(V, o_K)$. This follows from the fact that $\mu = a\mu_0$ for some $a \in K$ and $\mu_0 \in o_K[[G_0]]$, and $|\mu_0(f)| \leq 1$ for all $f \in C(V, o_K)$. Then

$$|\mu(\mathbf{1}_{V_m})| = |q_L^{r(n-m)}\mu(\mathbf{1}_{V_n})| \leq c|q_L^{r(n-m)}|$$

for all n, m. Since $c|q_L^{r(n-m)}| \to 0$ as $n \to \infty$ for each fixed m, we deduce that $\mu(\mathbf{1}_{V_m}) = 0$ for all m. But the space spanned by translates of these functions is $C^\infty(V, K)$ and, as observed in section 3.2, it is dense in $C(V, K)$. □

Corollary 6.2. *Take V as in Lemma 6.1. Regard K as a $K[[V]]$ module with trivial action. Then*

$$\mathrm{Hom}_{K[[V]]}(K, K[[V]]) = 0.$$

Proof. The image of any element of $\mathrm{Hom}_{K[[V]]}(K, K[[V]])$ is an element of $K[[V]]^V$. □

7. "Partially invariant" distributions on unipotent groups

Lemma 7.1. *Let $V_0 \subset G_0$ be a subgroup. Let V_1 be a closed subgroup of V_0 which is isomorphic to o_L^r for some r, and V_2 a closed subset of V_0 such that multiplication is a homeomorphism $V_1 \times V_2 \to V_0$. Then*

$$K[[V_0]]^{V_1} = 0, \qquad \text{and hence} \qquad \mathrm{Hom}_{K[[V_1]]}(K, K[[V_0]]) = 0.$$

Proof. Since multiplication is a homeomorphism $V_1 \times V_2 \to V_0$, we have an injective map $C(V_1, K) \times C(V_2, K) \hookrightarrow C(V_0, K)$. For each fixed nonzero $h \in C(V_2, K)$ we get an injective map $i_h : C(V_1, K) \hookrightarrow C(V_0, K)$. Explicitly

$$i_h.f(uv) = f(u)h(v), \qquad (u \in V_1,\ v \in V_2).$$

Assume that $\mu \in K[[V_0]]$ is a V_1-invariant element. Then $\mu \circ i_h$ is an invariant element of $K[[V_1]]$. Thus $\mu \circ i_h = 0$, by Lemma 6.1. But this means that μ vanishes on $i_h.f$ for all f and all h. That is μ vanishes on the image of $C(V_1, K) \times C(V_2, K)$ in $C(V_0, K)$. But the span of this image is dense, so μ must vanish identically. □

8. Key application of theorem on partially invariant distributions

Proposition 8.1. *Suppose $\chi, \xi : T_0 \to o_K^\times$ are continuous characters (we allow $\chi = \xi$). If $w, v \in W$, $w \neq v$, then*

$$\mathrm{Hom}_{K[[P_{\frac{1}{2}}^{w,\pm}]]}(K^{(w\chi)}, M_v^{(\xi)}) = 0.$$

Proof. With $V_{\frac{1}{2}}^{w,\pm}$ as in section 2.1, we have $P_{\frac{1}{2}}^{w,\pm} = T_0 V_{\frac{1}{2}}^{w,\pm}$. We select a root γ such that $w^{-1}\gamma > 0$ and $v^{-1}\gamma < 0$, and define

$$\epsilon = \begin{cases} 0, & \text{if } \gamma > 0, \\ 1, & \text{if } \gamma < 0. \end{cases}$$

Then $U_{\gamma,\epsilon} \subset V_{\frac{1}{2}}^{w,\pm} \cap V_{v,\frac{1}{2}}^{\pm}$. Clearly $\mathrm{Hom}_{K[[P_{\frac{1}{2}}^{w,\pm}]]}(K^{(w\chi)}, M_v^{(\xi)}) \subset \mathrm{Hom}_{K[[U_{\gamma,\epsilon}]]}(K, M_v^{(\xi)})$. But $M_v^{(\xi)} \cong K[[V_{v,\frac{1}{2}}^{\pm}]]$ as a $K[[V_{v,\frac{1}{2}}^{\pm}]]$ and hence as a $K[[U_{\gamma,\epsilon}]]$-module, and it follows from Lemma 7.1 that $\mathrm{Hom}_{K[[U_{\gamma,\epsilon}]]}(K, K[[V_{v,\frac{1}{2}}^{\pm}]]) = 0$. □

Corollary 8.2. *Suppose $\chi, \xi : T_0 \to o_K^\times$ are continuous characters (we allow $\chi = \xi$). If $w, v \in W$, $w \neq v$, then*

$$\mathrm{Hom}_{K[[B]]}(M_w^{(\chi)}, M_v^{(\xi)}) = 0.$$

Proof. Since $M_w^{(\chi)} \cong K[[B]] \otimes_{K[[P_{\frac{1}{2}}^{w,\pm}]]} K^{(w\chi)}$,

$$\mathrm{Hom}_{K[[B]]}(M_w^{(\chi)}, M_v^{(\xi)}) \cong \mathrm{Hom}_{K[[B]]}(K[[B]] \otimes_{K[[P_{\frac{1}{2}}^{w,\pm}]]} K^{(w\chi)}, M_v^{(\xi)}).$$

We can regard $K[[B]]$ as a bimodule with $K[[B]]$ acting on the left and $K[[P_{\frac{1}{2}}^{w,\pm}]]$ acting on the right, and apply adjoint associativity (see [9, Theorem 2.11]). It follows that

$$\begin{aligned} &\mathrm{Hom}_{K[[B]]}(K[[B]] \otimes_{K[[P_{\frac{1}{2}}^{w,\pm}]]} K^{(w\chi)}, M_v^{(\xi)}) \\ &\qquad \cong \mathrm{Hom}_{K[[P_{\frac{1}{2}}^{w,\pm}]]}(K^{(w\chi)}, \mathrm{Hom}_{K[[B]]}(K[[B]], M_v^{(\xi)})). \end{aligned}$$

And $\mathrm{Hom}_{K[[B]]}(K[[B]], M_v^{(\xi)})$ has the structure of a left $K[[P_{\frac{1}{2}}^{w,\pm}]]$-module isomorphic to $M_v^{(\xi)}$ by theorems 1.15 and 1.16 of [9], so

$$\mathrm{Hom}_{K[[P_{\frac{1}{2}}^{w,\pm}]]}(K^{(w\chi)}, \mathrm{Hom}_{K[[B]]}(K[[B]], M_v^{(\xi)})) \cong \mathrm{Hom}_{K[[P_{\frac{1}{2}}^{w,\pm}]]}(K^{(w\chi)}, M_v^{(\xi)}),$$

which is zero by Proposition 8.1. □

9. T_0-equivariant distributions on unipotent groups

Let $\mathbf{V}$ be a $\mathbf{T}$-stable unipotent $\mathbb{Z}$-subgroup of $\mathbf{G}$. The natural action of T on V by conjugation induces actions of T_0 on V_n/V_m for all positive integers m, n with $m > n$.

To describe T_0-equivariant distributions on V, we will consider their canonical pairing with characteristic functions of the form $1_{u_0 V_n}$. Fix $n > 0$ and $u_0 \in V_0$ such that $u_0 \notin V_n$. We will further decompose $1_{u_0 V_n}$ as the sum of characteristic functions of the form $1_{u V_m}$, for $m > n$. To simplify notation, we define, for $m > n$,

$$\mathcal{Q}_m = \mathcal{Q}_m^{u_0,n} = \{u_0 v V_m \mid v \in V_n\}.$$

This is a subset of the quotient group V_0/V_m.

Lemma 9.1. *The set $u_0 V_n = \{u_0 v \mid v \in V_n\}$ is preserved under the action of T_n. Consequently, the set $\mathcal{Q}_m$ is also preserved under the action of T_n.*

Proof. From the hypothesis that $\mathbf{V}$ is $\mathbf{T}$-stable, we deduce that there is a set S of roots such that taking any fixed order on the elements of S and multiplying in that order gives an isomorphism of $\mathbb{Z}$-schemes $\prod_{\alpha\in S} \mathbf{U}_\alpha \to \mathbf{V}$ (cf. [6, §1.7, p. 159]). In particular, we get a homeomorpishm $\prod_{\alpha\in S} U_\alpha(o_L) \to V(o_L)$, which induces a bijection $\prod_{\alpha\in S} U_\alpha(o_L/\mathfrak{p}_L^n) \to V(o_L/\mathfrak{p}_L^n)$, for each n, from which we deduce that the preimage of V_n in $\prod_{\alpha\in S} U_\alpha$ is $\prod_{\alpha\in S} U_{\alpha,n}$.

Hence, we can write u_0 as

$$u_0 = \prod_{\alpha\in S} x_\alpha(r_\alpha),$$

where $r_\alpha \in o_L$. Let $t \in T_n$. For each root $\alpha \in S$

$$t x_\alpha(r_\alpha) t^{-1} = x_\alpha(r_\alpha t^\alpha) = x_\alpha(r_\alpha) x_\alpha(r_\alpha(t^\alpha - 1)).$$

But $t^\alpha \equiv 1 \pmod{\mathfrak{p}_L^n}$, so $x_\alpha(r_\alpha(t^\alpha - 1)) \in V_n$. □

Given $\bar{u} \in \mathcal{Q}_m$, we denote by $\mathrm{Orb}_{T_n}(\bar{u})$ its orbit under the action of T_n. Then $\mathrm{Orb}_{T_n}(\bar{u}) \subset \mathcal{Q}_m \subset V_0/V_m$ is a finite set and we denote its cardinality by $|\mathrm{Orb}_{T_n}(\bar{u})|$.

Lemma 9.2. *Let $\mathbf{V}$ be a $\mathbf{T}$-stable unipotent $\mathbb{Z}$-subgroup of $\mathbf{G}$. Take $u_0 \in V_0$ and $n > 0$ such that $u_0 \notin V_n$. Then*

$$\lim_{m\to\infty} \min_{\bar{u}\in\mathcal{Q}_m} ord_p(|\mathrm{Orb}_{T_n}(\bar{u})|) = \infty,$$

where ord_p is the p-adic valuation on $\mathbb{Z}$.

Proof. Let $\bar{u} = u_0 v V_m \in \mathcal{Q}_m$ and write $u_0 v = \prod_{\alpha\in S} x_\alpha(u_\alpha)$. Denote by $\mathrm{Stab}_{T_n}(\bar{u})$ the stabilizer of $\bar{u}$ in T_n. Let $t \in T_n$. We may then note that

$$t \cdot \bar{u} = t\left(\prod_{\alpha\in S} x_\alpha(u_\alpha)\right) t^{-1} V_m = \prod_{\alpha\in S} t x_\alpha(u_\alpha) t^{-1} V_m = \prod_{\alpha\in S} x_\alpha(t^\alpha u_\alpha) V_m,$$

and hence

$$t\cdot\bar u=\bar u \iff \prod_{\alpha\in S} x_\alpha(t^\alpha u_\alpha)V_m=\prod_{\alpha\in S} x_\alpha(u_\alpha)V_m$$
$$\iff (t^\alpha-1)u_\alpha\in\mathfrak{p}_L^m,\ \text{ for all } \alpha\in S.$$

It follows

$$\mathrm{Stab}_{T_n}(\bar u)=\{t\in T_n \mid ord_L(t^\alpha-1)\geq m-ord_L(u_\alpha), \text{ for all } \alpha\in S\}.$$

Of course, u_α's can be zero for some roots α. In this case $ord_L(u_\alpha)=\infty$ and the condition becomes vacuous. However, the requirement that $u_0\notin V_n$ implies that for any $u\in u_0V_n$ there will be at least one root α_0 such that $ord_L(u_{\alpha_0})<n$.

For $m>n+ord_L(u_{\alpha_0})$, the condition $ord_L(t^{\alpha_0}-1)\geq m-ord_L(u_{\alpha_0})$ determines a subgroup of T_n of index $q_L^{m-ord_L(u_{\alpha_0})-n}$. Indeed α_0 is a homomorphism $T_n\to 1+\mathfrak{p}_L^n$. The condition $ord_L(t^{\alpha_0}-1)\geq m-ord_L(u_{\alpha_0})$ is equivalent to $t^{\alpha_0}\in 1+\mathfrak{p}_L^{m-ord_L(u_{\alpha_0})}$. If $m-ord_L(u_{\alpha_0})>n$, then $1+\mathfrak{p}_L^{m-ord_L(u_{\alpha_0})}$ is a subgroup of index $q_L^{m-ord_L(u_{\alpha_0})-n}$ in $1+\mathfrak{p}_L^n$ and $\{t\in T_n : t^{\alpha_0}\in 1+\mathfrak{p}_L^{m-ord_L(u_{\alpha_0})}\}$ is a subgroup of the same index in T_n. The actual stabilizer $\mathrm{Stab}_{T_n}(\bar u)$ is then a subgroup of this subgroup, and its index is a multiple of $q_L^{m-ord_L(u_{\alpha_0})-n}$. Hence,

$$ord_p(|\mathrm{Orb}_{T_n}(\bar u)|)\geq (m-ord_L(u_{\alpha_0})-n)\, ord_p(q_L),$$

which tends to ∞ with m. □

The action of T_0 on V_0 (by conjugation) induces the action of T_0 on $C(V_0,K)$ given by $t\cdot h(u)=h(t^{-1}ut)$. Then we define an action of T_0 on $K[[V_0]]$ by

$$\langle t\cdot\mu,h\rangle=\langle\mu,t^{-1}\cdot h\rangle$$

for $\mu\in K[[V_0]]$, $t\in T_0$, and $h\in C(V_0,K)$.

Lemma 9.3. *Let μ be a nonzero element of $K[[V_0]]$. Suppose there exists a character ξ of T_0 such that $t\cdot\mu=\xi(t)\mu$ for all $t\in T_0$. Then ξ is trivial and $\mu=c\cdot 1$ for some scalar c.*

Proof. Take a nonzero $\mu\in K[[V_0]]$ and assume that $\langle\mu,t^{-1}\cdot h\rangle=\xi(t)\langle\mu,h\rangle$ for all $t\in T_0$.

Suppose first that ξ is not smooth and consider the characteristic function 1_{vV_n}, for some $v\in V_0$ and some $n\in\mathbb{N}$. Then there exists $t\in T_n$ such that $\xi(t)\neq 1$. Notice that $t^{-1}\cdot 1_{vV_n}=1_{vV_n}$. This follows from Lemma 9.1 if $v\notin V_n$ and it holds trivially if $v\in V_n$. Then

$$\langle\mu,t^{-1}\cdot 1_{vV_n}\rangle=\langle\mu,1_{vV_n}\rangle=\xi(t)\langle\mu,1_{vV_n}\rangle$$

implies $\langle\mu,1_{vV_n}\rangle=0$. This condition forces $\langle\mu,h\rangle=0$ for all smooth h, and then for all h. Thus we are reduced to the case when ξ is smooth.

To treat the case when ξ is smooth, we will show that $\langle\mu,1_{u_0V_n}\rangle=0$ for any $u_0\in V_0$ and positive integer n such that u_0V_n does not contain the identity.

There exists n_0 such that the restriction of ξ to T_{n_0} is trivial. Fix u_0 and n as above and assume $n\geq n_0$. For $m>n$,

$$1_{u_0V_n}=\sum_{\bar u\in\mathcal{Q}_m}1_{\bar u}.$$

Hence

$$\langle \mu, 1_{u_0 V_n} \rangle = \sum_{\bar{u} \in \mathcal{Q}_m} \langle \mu, 1_{\bar{u}} \rangle.$$

Now, let T_n act on V_0/V_m. We know from Lemma 9.1 that $\mathcal{Q}_m$ is preserved under this action. Write $[T_n \setminus \mathcal{Q}_m]$ for a set of representatives of the distinct orbits in $\mathcal{Q}_m$. Then

$$\begin{aligned} \langle \mu, 1_{u_0 V_n} \rangle &= \sum_{\bar{u} \in [T_n \setminus \mathcal{Q}_m]} \sum_{t \cdot \bar{u} \in \mathrm{Orb}_{T_n}(\bar{u})} \langle \mu, t \cdot 1_{\bar{u}} \rangle \\ &= \sum_{\bar{u} \in [T_n \setminus \mathcal{Q}_m]} |\mathrm{Orb}_{T_n}(\bar{u})| \langle \mu, 1_{\bar{u}} \rangle \end{aligned}$$

because $\langle \mu, t \cdot 1_{\bar{u}} \rangle = \xi(t^{-1}) \langle \mu, 1_{\bar{u}} \rangle = \langle \mu, 1_{\bar{u}} \rangle$ for any $t \in T_n$. But $\min_{\bar{u} \in \mathcal{Q}_m} \langle \mu, 1_{\bar{u}} \rangle$ is bounded independently of m by $\|\mu\|$, and

$$\lim_{m \to \infty} \min_{\bar{u} \in \mathcal{Q}_m} ord_p(|\mathrm{Orb}_{T_n}(\bar{u})|) = \infty,$$

by Lemma 9.2. It follows that $\langle \mu, 1_{u_0 V_n} \rangle = 0$, for any $u_0 \in V_0$ and positive integer n such that $u_0 V_n$ does not contain the identity.

Thus μ is supported at the identity, so it is $c \cdot 1$ for some c. It now follows easily that ξ is trivial. □

Corollary 9.4. *Let $\xi : T_0 \to o_K^\times$ be a continuous character. Let $\varphi : K \to K[[V_0]]$ be the map defined by $\varphi(a) = a \cdot 1$. Then*

$$\mathrm{Hom}_{K[[T_0]]}(K^{(\xi)}, K[[V_0]]) = \begin{cases} K \cdot \varphi, & \xi \text{ is trivial}, \\ 0, & \text{otherwise}. \end{cases}$$

(Here $K[[T_0]]$ acts on $K[[V_0]]$ as in Lemma 9.3.)

Proof. Take $\psi \in \mathrm{Hom}_{K[[T_0]]}(K^{(\xi)}, K[[V_0]])$ and set $\mu = \psi(1)$. Then for any $t \in T_0$,

$$t \cdot \mu = \psi(t \cdot 1) = \xi(t)\psi(1) = \xi(t)\mu.$$

By Lemma 9.3, if $\xi \neq 1$, then $\mu = 0$, while for $\xi = 1$ we have $\mu = c \cdot 1$ for some scalar c. □

Corollary 9.5. *Suppose $\chi_1, \chi_2 : T_0 \to o_K^\times$ are continuous characters. Let $\varphi : K \to K[[V_0]]$ be the map defined by $\varphi(a) = a \cdot 1$, and let $K[[V_0]]^{(\chi_2)}$ denote the space $K[[V_0]]$ equipped with an action of $K[[T_0]]$ such that*

$$\langle t \cdot \mu, h \rangle = \chi_2(t) \langle \mu, t^{-1} h \rangle \qquad t \in T_0, \mu \in K[[V_0]],\ h \in C(V_0, K).$$

Then

$$\mathrm{Hom}_{K[[T_0]]}(K^{(\chi_1)}, K[[V_0]]^{(\chi_2)}) = \begin{cases} K \cdot \varphi, & \chi_1 = \chi_2, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Write A for the action of T_0 on $K[[V_0]]$ in Lemma 9.3 and A_{χ_2} for the action of T_0 on $K[[V_0]]^{(\chi_2)}$. Then $A_{\chi_2}(t).\mu = \chi_2(t) A(t).\mu$. Hence, if

$$\psi \in \mathrm{Hom}_{K[[T_0]]}(K^{(\chi_1 \chi_2^{-1})}, K[[V_0]])$$

then

$$\chi_1 \chi_2^{-1}(t) \psi(x) = \psi(\chi_1 \chi_2^{-1}(t).x) = A(t).\psi(x),$$

whence

$$\chi_1(t)\psi(x) = \chi_2(t) A(t).\psi(x) = A_{\chi_2}(t).\psi(x).$$

So, ψ is also in $\mathrm{Hom}_{K[[T_0]]}(K^{(\chi_1)}, K[[V_0]]^{(\chi_2)})$. A similar argument shows containment in the other direction. $\square$

10. Main result

Theorem 10.1. *For any two continuous characters χ_1 and χ_2 of T_0, we have*

$$\mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, M^{(\chi_2)}) = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2, \\ K \cdot \varphi & \text{if } \chi_1 = \chi_2, \end{cases}$$

where $\varphi : K \to M^{(\chi_2)}$ sends $a \in K$ to $a \cdot 1$ in $M^{(\chi_2)}$.

Proof. Since $M^{(\chi_2)} = \bigoplus_{v \in W} M_v^{(\chi_2)}$, as a $K[[B]]$-module and hence as a $K[[P_0]]$ module, we obtain

$$\mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, M^{(\chi_2)}) = \bigoplus_{v \in W} \mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, M_v^{(\chi_2)}).$$

We apply Proposition 8.1, taking w to be the identity element of W, which we denote e. Then $P_{\frac{1}{2}}^{w,\pm} = P_0$, and from Proposition 8.1 we deduce that

$$\mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, M_v^{(\chi_2)}) = 0, \qquad \forall v \neq e.$$

Thus $\mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, M^{(\chi_2)}) = \mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, M_e^{(\chi_2)})$. Now, by Lemma 4.2, $M_e^{(\chi_2)}$ is isomorphic to $K[[U_1^-]]$, as a $K[[Q_{e,\frac{1}{2}}^{\pm}]]$-module and in particular as a $K[[T_0]]$-module. And, by Corollary 9.5,

$$\mathrm{Hom}_{K[[T_0]]}(K^{(\chi_1)}, K[[U_1^-]]^{\chi_2}) = \begin{cases} K \cdot \varphi, & \chi_1 = \chi_2, \\ 0, & \text{otherwise.} \end{cases}$$

One readily confirms that φ is a $K[[P_0]]$-module map, and this completes the proof. $\square$

Corollary 10.2. *For any two continuous characters χ_1 and χ_2 of P_0, we have*

$$\mathrm{Hom}_{K[[G_0]]}(M^{(\chi_1)}, M^{(\chi_2)}) = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2, \\ K \cdot \mathrm{id} & \text{if } \chi_1 = \chi_2. \end{cases}$$

Proof. Similarly to the proof of Corollary 8.2, using adjoint associativity from [9, Theorem 2.11], we have

$$\begin{aligned} \mathrm{Hom}_{K[[G_0]]}(M^{(\chi_1)}, M^{(\chi_2)}) &= \mathrm{Hom}_{K[[G_0]]}(K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi_1)}, M^{(\chi_2)}) \\ &\cong \mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, \mathrm{Hom}_{K[[G_0]]}(K[[G_0]], M^{(\chi_2)})) \\ &\cong \mathrm{Hom}_{K[[P_0]]}(K^{(\chi_1)}, M^{(\chi_2)}). \end{aligned}$$

The statement now follows from Theorem 10.1. $\square$

Since $M^{(\chi)}$ is the dual of $\mathrm{Ind}_{P_0}^{G_0}(\chi^{-1})$, we obtain the following result.

Corollary 10.3. *For any two continuous characters χ_1 and χ_2 of P_0, we have*

$$\mathrm{Hom}_{G_0}(\mathrm{Ind}_{P_0}^{G_0}(\chi_1), \mathrm{Ind}_{P_0}^{G_0}(\chi_2)) = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2, \\ K \cdot \mathrm{id} & \text{if } \chi_1 = \chi_2. \end{cases}$$

Appendix: Finite dimensional G_0-invariant subspaces

In this section we discuss finite dimensional G_0-invariant subspaces of the representations $V = \mathrm{Ind}_P^G(\chi^{-1})$.

We begin by recalling the notion of a $\mathbb{Q}_p$-rational representation from [3]. Such a representation is obtained by viewing G as a subgroup of $\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{G}(K)$, and restricting a K-rational representation of $\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{G}$ to G. Note that the group $\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{G}$ splits over K and $\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{G}(K)$ is isomorphic to the product of several copies of $\mathbf{G}(K)$, indexed by the embeddings $\sigma : L \to K$. The induced embedding of G into $\prod_\sigma \mathbf{G}(K)$ maps g to the element whose σ component is $\sigma(g)$ for each σ.

This construction works for any reductive group, and by applying it to our torus, we obtain a notion of $\mathbb{Q}_p$-rational character. We say that $a \in X(\mathbf{T})$ (the rational characters of $\mathbf{T}$) is dominant if $\langle a, \alpha_i^\vee\rangle \geq 0$ for each simple root α_i. Here $\langle\ ,\ \rangle$ is the canonical pairing $X(\mathbf{T}) \times X^\vee(\mathbf{T}) \to \mathbb{Z}$. Our choice of Borel for $\mathbf{G}$ determines a Borel for $\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{G}$ and hence a notion of "dominant" for K-rational characters of the K-split maximal torus $\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{T}$ of $\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{G}$. We say that a $\mathbb{Q}_p$-rational character of T is dominant if the underlying K-rational character of $\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{T}$ is dominant. Explicitly, we may think of a K-rational character of $\widetilde{\mathbf{T}} := \mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{T}$ as a tuple $(a_\sigma)_{\sigma:L\to K}$ of rational characters of $\mathbf{T}$ indexed by $\sigma : L \to K$, and it is dominant if each component a_σ is so, and induces a map $T \to K^\times$ that maps $t \in T$ to $\prod_{\sigma:L\to K}\sigma(a_\sigma(t))$.

For a dominant K-rational character a of $\widetilde{\mathbf{T}}$ we have the corresponding finite dimensional algebraically induced representation of $\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{G}(K)$. Restricting to G we obtain a $\mathbb{Q}_p$-rational representation of G. It is realized explicitly as follows. We have noted that a may be identified with a tuple $(a_\sigma)_{\sigma:L\to K}$, where $a_\sigma \in X(\mathbf{T})$ is dominant for each σ. Then for each σ we obtain the algebraic induced space $AI_{\mathbf{P}}^{\mathbf{G}} a_\sigma^{-1}$ and our $\mathbb{Q}_p$-rational representation is the span of all functions of the form $f(g) = \prod_{\sigma:L\to K}\sigma(f_\sigma(g))$, where $f_\sigma \in AI_{\mathbf{P}}^{\mathbf{G}} a_\sigma$ for each σ. If χ_{alg} is the $\mathbb{Q}_p$-rational character of T induced by a then the $\mathbb{Q}_p$-rational representation thus obtained is a finite dimensional invariant subspace of $\mathrm{Ind}_P^G(\chi_{\mathrm{alg}}^{-1})$ and we denote it $\mathrm{Ind}_P^G(\chi_{\mathrm{alg}}^{-1})_{\mathrm{alg}}$.

Suppose $\chi = \chi_{\mathrm{alg}}\chi_{\mathrm{sm}}$, where χ_{sm} is smooth and χ_{alg} is $\mathbb{Q}_p$-rational. Suppose in addition that χ_{alg} is dominant. Let $U = \mathrm{Ind}_P^G(\chi_{\mathrm{sm}}^{-1})_{\mathrm{sm}}$ be the subspace of smooth elements in $\mathrm{Ind}_P^G(\chi_{\mathrm{sm}}^{-1})$. Any element of $\mathrm{Ind}_P^G(\chi_{\mathrm{sm}}^{-1})$ is a sum over $w \in W$ of elements f_h, where $h \in C(V^{\pm}_{w,\frac{1}{2}}, K)$ and f_h is defined as in the proof of Lemma 4.1. If h is smooth, then f_h is also smooth, by smoothness of χ_{sm}. Since $C^\infty(V^{\pm}_{w,\frac{1}{2}}, K)$ is dense in $C(V^{\pm}_{w,\frac{1}{2}}, K)$, it follows that U is dense in $\mathrm{Ind}_P^G(\chi_{\mathrm{sm}}^{-1})$.[1]

Let $W = \mathrm{Ind}_P^G(\chi_{\mathrm{alg}}^{-1})_{\mathrm{alg}}$. The representation W is finite dimensional and irreducible. We consider the locally algebraic representation $U \otimes_K W$. There is a natural map

$$U \otimes_K W \to V,$$

given by pointwise multiplication of functions. We claim that this map is injective. In the case when χ_{alg} is L-algebraic, this follows from [7], using exactness of the functor $\mathcal{F}_P^G$ for the split group $\mathbf{G}$. We prove it in general for χ_{alg} $\mathbb{Q}_p$-rational. We

[1] There is an intermediate space of locally analytic vectors $U \subset \mathrm{Ind}_P^G(\chi_{\mathrm{sm}}^{-1})_{an} \subset \mathrm{Ind}_P^G(\chi_{\mathrm{sm}}^{-1})$. Then U is closed in $\mathrm{Ind}_P^G(\chi_{\mathrm{sm}}^{-1})_{an}$, under an appropriate topology, and both spaces are dense in $\mathrm{Ind}_P^G(\chi_{\mathrm{sm}}^{-1})$ with respect to the Banach space topology [12, Section 3].

may identify G with $\mathrm{Res}_{L/\mathbb{Q}_p} \mathbf{G}(\mathbb{Q}_p)$. Then the elements of W are polynomials with coefficients in K and the elements of U are locally constant. Given a finite linear combination

$$\sum_{i=1}^{n} c_i u_i w_i, \qquad c_i \in K,\ u_i \in U,\ w_i \in W,$$

we may choose n such that $u_i \in U^{G_n}$ for all n. Since each element of U^{G_n} may be expressed as a K-linear combination of elements supported on a single P_0, G_n-double coset, we may assume each u_i is such an element and then after collecting like terms we may assume that they are all distinct. But then if

$$\sum_{i=1}^{n} c_i u_i w_i = 0$$

we may deduce that each of the polynomial-functions w_i vanishes identically on the double coset supporting u_i. Since each of these double cosets is an open subset of G we deduce that each w_i is the zero polynomial, so that $\sum_{i=1}^{n} c_i u_i \otimes w_i$ is zero in $U \otimes_K W$. Hence, we can identify $U \otimes_K W$ with a subspace of V.

Now, we consider the corresponding G_0-representations. The algebraic representation W remains irreducible when restricted to G_0. Then U decomposes as a countable direct sum of finite dimensional representations ρ with finite multiplicities

$$U \cong \bigoplus_{\rho} m(\rho)\rho.$$

Then V contains

$$U \otimes_K W \cong \bigoplus_{\rho} m(\rho)(\rho \otimes_K W).$$

Note that every subspace $\rho \otimes_K W$ is finite-dimensional, and hence closed in V. Alternatively, we can use Corollary 4.2.9 of [4] to show that $U \otimes_K W$ decomposes as a direct sum of irreducible finite-dimensional representations. In conclusion, the G_0-representation V contains countably many finite-dimensional topologically irreducible subrepresentations. Still, by Lemma 10.3, $\mathrm{Hom}_{G_0}(V, V) = K \cdot \mathrm{id}$.

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