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A PROOF OF CASSELMAN'S COMPARISON THEOREM

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ABSTRACT. Let G be a real linear reductive group and K be a maximal compact subgroup. Let P be a minimal parabolic subgroup of G with complexified Lie algebra $\mathfrak{p}$, and $\mathfrak{n}$ be its nilradical. In this paper we show that: for any admissible finitely generated moderate growth smooth Fréchet representation V of G, the inclusion $V_K \subset V$ induces isomorphisms $H_i(\mathfrak{n}, V_K) \cong H_i(\mathfrak{n}, V)$ $(i \geq 0)$, where V_K denotes the $(\mathfrak{g}, K)$ module of K finite vectors in V. This is called Casselman's comparison theorem (see Henryk Hecht and Joseph L. Taylor [*A remark on Casselman's comparison theorem*, Birkhäuser Boston, Boston, Ma, 1998, pp. 139–146]). As a consequence, we show that: for any $k \geq 1$, $\mathfrak{n}^k V$ is a closed subspace of V and the inclusion $V_K \subset V$ induces an isomorphism $V_K/\mathfrak{n}^k V_K = V/\mathfrak{n}^k V$. This strengthens Casselman's automatic continuity theorem (see W. Casselman [Canad. J. Math. 41 (1989), pp. 385–438] and Nolan R. Wallach [*Real reductive groups*, Academic Press, Boston, MA, 1992]).

CONTENTS

INTRODUCTION

Let G be a real linear reductive group and K be a maximal compact subgroup. Let W be a finitely generated admissible $(\mathfrak{g}, K)$-module and W^* be its dual $(\mathfrak{g}, K)$-module. Schmid constructed a minimal globalization $W_{\min}$ which consists of analytic vectors, and the topological dual of $(W^*)_{\min}$ is called a maximal globalization of W ([18]). Casselman and Wallach constructed a smooth globalization W_∞ (also called a canonical globalization or a Casselman-Wallach globalization, [7], [23])

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which is a moderate growth smooth Fréchet representation of G, and one has a globalization $W_{-\infty}$ as the topological dual of $(W^*)_\infty$. They fit into a sequence

$$W \subset W_{\min} \subset W_\infty \subset W_{-\infty} \subset W_{\max}.$$

Each kind of globalization has important applications. For example, the smooth globalization W_∞ is very useful in the theory of automorphic forms, and the maximal globalization $W_{\max}$ is used to realize standard derived functor modules ([19]).

Let's recall the Casselman-Wallach globalization ([7], [23], [2]). The presentation in this paragraph mainly follows [24]. A $(\mathfrak{g}, K)$-module W is said to be admissible if $\dim \operatorname{Hom}_K(\tau, W) < \infty$ for any irreducible complex linear representation τ of K; it is said to be finitely generated if there exists a finite dimensional subspace $F \subset W$ such that $W = \mathcal{U}(\mathfrak{g})F$. For a continuous representation (π, V) of G on a topological vector space V, let V_K be the subspace of K-finite vectors, i.e.,

$$V_K = \{v \in V : \dim \operatorname{span}_{\mathbb{C}}\{\pi(k)v : k \in K\} < \infty\}.$$

If $\dim \operatorname{Hom}_K(\tau, V) < \infty$ for any irreducible complex linear representation τ of K, then we call V admissible. In this case V_K consists of smooth vectors and it is an admissible $(\mathfrak{g}, K)$-module ([16, Proposition 8.5]). An admissible continuous representation (π, V) is said to be finitely generated if V_K is finitely generated; it is said to be a smooth Fréchet representation if V is a Fréchet space and for any $v \in V$, the map $g \mapsto \pi(g)v$ is of class C^∞ on G. For a complex square matrix $X = \{a_{i,j}\}_{1\le i,j\le n}$, the Hilbert-Schmidt norm of X is defined by

$$\|X\| = \sqrt{\sum_{1\le i,j\le n} |a_{i,j}|^2}.$$

Fix a linear group imbedding $p : G \to \mathrm{GL}(n, \mathbb{C})$. For any $g \in G$, we define a norm p on G by

$$\|g\| = \max\{\|p(g)\|, \|p(g)^{-1}\|\},$$

which satisfies the following properties:

(i) $\|g\| \ge 1$ ($\forall g \in G$);
(ii) $\|xy\| \le \|x\|\,\|y\|$ ($\forall x, y \in G$);
(iii) For any $t \in \mathbb{R}$, $\{g \in G : \|g\| \le t\}$ is a compact set.

A smooth Fréchet representation (π, V) is said to be moderate growth if for every continuous seminorm λ on V, there exists a continuous seminorm μ on V and a constant $r \in \mathbb{R}$ such that

$$\lambda(\pi(g)v) \le \|g\|^r \mu(v), \ \forall g \in G, \forall v \in V.$$

Write $\mathcal{H}(\mathfrak{g}, K)$ for the category of $(\mathfrak{g}, K)$-modules that are admissible and finitely generated; write $\mathcal{HF}_{\mathrm{mod}}(G)$ for the category of admissible finitely generated moderate growth smooth Fréchet representations of G. The celebrated Casselman-Wallach theorem asserts that the functor

$$\mathcal{HF}_{\mathrm{mod}}(G) \longrightarrow \mathcal{H}(\mathfrak{g}, K), \quad V \mapsto V_K$$

is an equivalence of categories.

Let P be a minimal parabolic subgroup of G with complexfied Lie algebra $\mathfrak{p}$. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{p}$. In this paper we show that: for any admissible finitely generated moderate growth smooth Fréchet representation V of G, the inclusion $V_K \subset V$ induces isomorphisms $H_i(\mathfrak{n}, V_K) \cong H_i(\mathfrak{n}, V)$ ($i \ge 0$). This is called Casselman's comparison theorem ([13]). As a consequence, we show that: for any $k \ge 1$,

$\mathfrak{n}^k V$ is a closed subspace of V and the inclusion $V_K \subset V$ induces an isomorphism $V_K/\mathfrak{n}^k V_K = V/\mathfrak{n}^k V$. This implies Casselman's automatic continuity theorem saying that $V_K/\mathfrak{n}^k V_K = V/\overline{\mathfrak{n}^k V}$ for each $k \geq 0$ ([7, p. 416]). Let V' be the strong dual of V ([17]), which is a dual nuclear Fréchet space. Our proof uses the *Casselman-Jacquet module*

$$V'^{[\mathfrak{n}]} = \{u \in V' : \exists k \in \mathbb{Z}_{>0}, \mathfrak{n}'^k \cdot u = 0\}$$

of V' in an essential way.

Let's describe the strategy of our proof briefly. It is well-known that $H_i(\mathfrak{n}, V_K)$ and $H^i(\mathfrak{n}, V_K^*)$ are finite-dimensional ([9, Corollary 2.4]). By a duality argument ([8, Lemma 5.11]), the homological comparison theorem $H_i(\mathfrak{n}, V_K) \cong H_i(\mathfrak{n}, V)$ $(i \geq 0)$ and the cohomological comparison theorem $H^i(\mathfrak{n}, V_K^*) \cong H^i(\mathfrak{n}, V')$ $(i \geq 0)$ are equivalent, where V_K^* denotes the dual space of V_K. Moreover, a reduction argument in [13, Proposition 3] reduces the homological comparison theorem to the principal series case. Let $V = I(\sigma) := \mathrm{Ind}_P^G(\sigma)$ (the un-normalized smooth parabolic induction) be a principal series, where (σ, V_σ) is a finite-dimensional complex linear algebraic representation of P. By a theorem of Hecht-Schmid ([10, Lemma 2.37]), we have $H^i(\mathfrak{n}, V_K^*) = H^i(\mathfrak{n}, (V_K^*)^{[\mathfrak{n}]})$. By Casselman's automatic continuity theorem ([7, p. 416],[2, Theorem 11.4], [23, p.77]), we have $V'^{[\mathfrak{n}]} = (V_K^*)^{[\mathfrak{n}]}$. Then, it reduces to the following assertion: the inclusion $I(\sigma)'^{[\mathfrak{n}]} \subset I(\sigma)'$ induces isomorphisms

$$H^i(\mathfrak{n}, I(\sigma)'^{[\mathfrak{n}]}) = H^i(\mathfrak{n}, I(\sigma)'),\ \forall i \in \mathbb{Z}. \tag{1}$$

Choose a maximal split torus A contained in P. Let $W = N_G(A)/Z_G(A)$ be the Weyl group of G. For each element $w \in W$, choose an element $\dot{w} \in K$ representing w and put $C(w) = N\dot{w}P \subset G/P$, which is called the Bruhat cell attached to w. Write $W = \{w_1, \ldots, w_r\}$ so that $\dim C(w_i) \leq \dim C(w_{i+1})$ $(1 \leq i \leq r-1)$. For each k $(0 \leq k \leq r)$, write

$$Z_k = \bigcup_{1 \leq i \leq k} C(w_i).$$

Then, by the Bruhat decomposition we have a stratification

$$\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_r = X$$

of the flag variety $X := G/P$. Put

$$E(\sigma) = G \times_P V_\sigma$$

for a G equivariant bundle on X induced from a finite-dimensional complex linear algebraic representation V_σ of P. Then, $C^\infty(E(\sigma)) = I(\sigma)$ and $C^\infty(E(\sigma))' = I(\sigma)'$. For each k, let $I_{k,\sigma}$ be the space of $E(\sigma)$-distributions with support contained in Z_k. For each $w \in W$, in [8] the authors defined a space $\mathcal{T}(C(w), E(\sigma))$ of *tempered* $E(\sigma)$*-distributions on an open subanalytic neighborhood of* $C(w)$ with support in $C(w)$. Then,

$$I_{k,\sigma}/I_{k-1,\sigma} = \mathcal{T}(C(w_k), E(\sigma))$$

for each k. For each integer $p \geq 0$, in [8] the authors defined a space $F_p\mathcal{T}(C(w), E(\sigma))$ of distributions in $\mathcal{T}(C(w), E(\sigma))$ of *transversal degree* $\leq p$. Put

$$\mathrm{Gr}^p\mathcal{T}(C(w), E(\sigma)) = F_p\mathcal{T}(C(w), E(\sigma))/F_{p-1}\mathcal{T}(C(w), E(\sigma)).$$

We show that

$$\mathcal{T}(C(w_k), E(\sigma))^{[\mathfrak{n}]} = I_{k,\sigma}^{[\mathfrak{n}]}/I_{k-1,\sigma}^{[\mathfrak{n}]},\ \forall k \geq 1$$

and

$$(\mathrm{Gr}^p\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]}=(F_p\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]}/(F_{p-1}\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]},\ \forall p\geq 0.$$

When N acts trivially on V_σ, we show that each $\mathrm{Gr}^p\,\mathcal{T}(C(w),E(\sigma))$ admits a finite increasing filtration such that each graded piece is isomorphic to $\mathcal{S}(C(w))'$ (the space of Schwarz distributions on $C(w)$ as a real Euclidean space) and $\mathcal{S}(C(w))'^{[\mathfrak{n}]} = R(C(w))\delta_{C(w)}$, where $R(C(w))$ denotes the space of regular functions on $C(w)$ and $\delta_{C(w)}$ is an N invariant volume measure on $C(w)$. Taking induction on k and p, (1) is reduced to the following assertion: for each $w\in W$, the inclusion $R(C(w))\delta_{C(w)}\subset\mathcal{S}(C(w))'$ induces isomorphisms

$$H^i(\mathfrak{n},R(C(w))\delta_{C(w)})=H^i(\mathfrak{n},\mathcal{S}(C(w))'),\ \forall i\in\mathbb{Z}. \tag{2}$$

We prove a more general statement than this (Proposition 4.3) by a method inspired by ideas in [8, §5]. For a general finite-dimensional algebraic representation (σ,V_σ) of P, we prove (1) by induction on the length of the composition series of V_σ.

For each of the four kinds of globalization above, there is a comparison conjecture for $\mathfrak{n}$-homology (or $\mathfrak{n}$-cohomology) for $\mathfrak{n}$ the nilradical of certain parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$ ([21, Conj. 10.3]). Casselman claimed a proof of the comparison theorem when $\mathfrak{p}$ is any standard parabolic subalgebra, but the proof remains unpublished. For minimal globalization, the comparison conjecture is shown in [12], [5] and [4]. All of these proofs use sophisticated analytic results about D-modules (e.g., [11], [14]). In [13] the above comparison theorem for smooth globalization and for $\mathfrak{p}$ a standard minimal parabolic subalgebra is deduced from the corresponding comparison theorem for minimal globalization. Compared to these developments, our proof of the comparison theorem is more elementary and much easier. The closedness of $\mathfrak{n}^kV$ and the isomorphism $V_K/\mathfrak{n}^kV_K=V/\mathfrak{n}^kV$ ($\forall k\geq 1$) shown in this paper are new.

Notation and conventions. Let G be a real Lie group. Write $\mathfrak{g}_0$ for the Lie algebra of G, and write $\mathfrak{g}=\mathfrak{g}_0\otimes_{\mathbb{R}}\mathbb{C}$ for the complexified Lie algebra of G. Similarly, we have Lie algebras (resp. complexified Lie algebras) $\mathfrak{p}_0$, $\mathfrak{n}_0$, $\mathfrak{a}_0$ (resp. $\mathfrak{p}$, $\mathfrak{n}$, $\mathfrak{a}$) of Lie groups P, N, A appearing in this paper.

For a complex Lie algebra $\mathfrak{h}$, write $\mathcal{U}(\mathfrak{h})$ for the enveloping algebra of $\mathfrak{h}$. For an integer $p\geq 0$, write $\mathcal{U}_p(\mathfrak{h})$ for the subspace of $\mathcal{U}(\mathfrak{h})$ generated by elements $X_1\cdots X_q\in\mathcal{U}(\mathfrak{h})$ where $X_1,\ldots,X_q\in\mathfrak{h}$ and $q\leq p$.

For an affine algebraic variety Y over $\mathbb{R}$, write $R(Y)$ for the ring of complex coefficient regular functions on Y. For each integer $l\geq 0$, write $R_l(Y)$ for the space of complex coefficient regular functions on Y of degree at most l.

1. Tempered E-distributions supported in a subanalytic submanifold

In this section we review the theory of tempered E-distributions supported in a subanalytic submanifold and its transversal degree filtration defined in [8, §2]. The exposition in [8] is excellent and the paper itself is well known to experts. For most results we only give a sketch of proof. Interested readers are encouraged to consult the original paper [8] for more complete account and proof.

Let X be a compact analytic manifold and E be an analytic vector bundle on X of finite rank. Let $C^\infty(X,E)$ denote the space of smooth sections of E. It is a *nuclear Fréchet space* (NF space for short) with seminorms

$$p_{D,f}: s\mapsto \max_{x\in X}|f(x)((Ds)(x))|,$$

where $f \in C^\infty(X, E^*)$ is a smooth section of the dual vector bundle E^* of E and $D \in \mathrm{Diff}(X, E)$ is an E-valued smooth differential operator on X. Let $C^\infty(X, E)'$ be the *strong dual* of $C^\infty(X, E)$ ([17]), which is a *dual nuclear Fréchet space* (DNF space for short). For an open subset U of X, let $C_0^\infty(U, E)$ be the space of compactly supported E-sections on U, which is again a nuclear Fréchet space. The strong dual $C_0^\infty(U, E)'$ of $C_0^\infty(U, E)$ is called the space of *E-distributions on U*. There is a natural restriction map

$$\mathrm{res}_U : C^\infty(X, E)' \to C_0^\infty(U, E)'.$$

Elements in the image of res_U are called *tempered E-distributions on U with respect to X*. When E is the trivial line bundle, we will omit the symbol E from notations like $C^\infty(X, E)$, $C^\infty(X, E)'$, etc. For example, $C_0^\infty(U)'$ stands for the space of distributions on an open subset U.

There is a short exact sequence

$$0 \to \ker \mathrm{res}_U \to C^\infty(X, E)' \to \mathrm{im}\, \mathrm{res}_U \to 0.$$

Put

$$Z := X - U.$$

Then, elements in $\ker \mathrm{res}_U$ are E-distributions on X supported in the closed subset Z. By dualizing, there is a dual exact sequence

$$0 \to (\mathrm{im}\, \mathrm{res}_U)' \to C^\infty(X, E) \to (\ker \mathrm{res}_U)' \to 0.$$

Put

$$\mathcal{S}(U, E) = (\mathrm{im}\, \mathrm{res}_U)'$$

and call it the *(relative) Schwartz space of sections of E on U with respect to X*. Then, $\mathcal{S}(U, E)' = \mathrm{im}\, \mathrm{res}_U$ by duality. Lemma 1.1 gives a characterization of sections in the space $\mathcal{S}(U, E)' = (\mathrm{im}\, \mathrm{res}_U)'$.

Lemma 1.1 ([8, Lemma 2.2]). *The subspace* $(\mathrm{im}\, \mathrm{res}_U)'$ *of* $C^\infty(X, E)$ *consists of all global sections in* $C^\infty(X, E)$ *vanishing with all of their derivatives on* Z.

Proof. Let $s \in (\mathrm{im}\, \mathrm{res}_U)'$. Any distribution supported in a point $x \in Z$ is a finite linear combination of derivatives of delta like distributions $s \mapsto \alpha(s(x))$ ($\alpha \in E_x^*$). Evaluating these distributions at $s \in (\mathrm{im}\, \mathrm{res}_U)'$, it follows that s vanishes with all derivatives at x. Then, all $s \in (\mathrm{im}\, \mathrm{res}_U)'$ vanish with all derivatives on Z. The converse follows from [8, Lemma 2.1], which asserts that any order $\leq m$ E-distribution T on X vanishes on $s \in C^\infty(X, E)$ such that $Ds|_{\mathrm{supp}(T)} = 0$ for all order $\leq m$ smooth differential operators D on E. □

Let Y be a submanifold of X which is also a *subanalytic set* ([3]) in X (note that being a submanifold implies that Y is an open subset of $\bar{Y}$). An open subset U containing Y is called a *subanalytic neighborhood* of Y if U is subanalytic in X and Y is a closed subset of U, i.e, $\bar{Y} \cap U = Y$. Let $\mathcal{T}(Y, E)$ denote the space of *tempered E-distributions on U with support in Y*. By Lemma 1.2, the space $\mathcal{T}(Y, E)$ is independent of the choice of the subanalytic open neighborhood U.

Lemma 1.2 ([8, Lemma 2.6]). *If $U' \subset U$ are two subanalytic open neighborhoods of Y, then the restriction map $\mathrm{res}_{U,U'}$ induces an isomorphism of the space of tempered E-distributions on U with support in Y onto the space of tempered E-distributions on U' with support in Y.*

Proof. Injectivity of the restriction map $\mathrm{res}_{U,U'}$ is due to the supporting set condition. The inverse of the restriction map $\mathrm{res}_{U,U'}$ is given by "the extension by zero" map. □

For a closed subset K of X, let $C^\infty(X,E)'_K$ denote the space of E-distributions on X with support contained in K. Since Y is a closed subset of U, we have $\bar{Y} \cap U = Y$. Then, $\bar{Y} - Y \subset X - U = Z$. Thus, $Y \cup Z$ is a closed subset of X. Lemma 1.3 is clear.

Lemma 1.3. *The space $\mathcal{T}(Y,E)$ is equal to $C^\infty(E)'_{Z\cup Y}/C^\infty(E)'_Z$.*

For an integer p, let M_p denote the closed subspace of $\mathcal{S}(U,E)$ consisting of sections which vanish with all derivatives of order $\leq p$ along Y. Put $F_p\mathcal{T}(Y,E) = M_p^\perp$ and call it the *space of tempered E-distributions with support in Y and of transversal degree* $\leq p$. Then,

$$F\mathcal{T}(Y,E) = \{F_p\mathcal{T}(Y,E)\}_{p=0}^\infty$$

is an exhaustive increasing filtration of $\mathcal{T}(Y,E)$.

For each $p \in \mathbb{Z}$, let M_{0p} denote the subspace of M_p consisting of sections with compact support contained in U. Lemma 1.4 gives a useful characterization of $F_p\mathcal{T}(Y,E)$ in practice. The proof of Lemma 1.4 is a bit complicated. We refer interested readers to consult the original paper [8].

Lemma 1.4 ([8, Lemma 2.9])**.** *For any integer p, we have*

$$M_{0p}^\perp = M_p^\perp.$$

The following is [8, Example 2.3].

Example 1.5 ([8, Example 2.3])**.** Let $X = S^n$ be an n-dimensional sphere, $p \in S^n$ and $\pi : S^n - \{p\} \to \mathbb{R}^n$ be the stereographical projection of $S^n - \{p\}$ onto $\mathbb{R}^n$ with respect to the pole p. The closed subspace of $C^\infty(S^n)$ consisting of smooth functions vanishing with all derivatives at p can be identified with the "classical" Schwartz space $\mathcal{S}(\mathbb{R}^n)$ via the map π. Dually, this identifies the space of tempered distributions on $S^n - \{p\}$ with respect to S^n with the space $\mathcal{S}(\mathbb{R}^n)'$ of "classical" tempered distributions on $\mathbb{R}^n$.

The following is a generalization of [8, Example 2.11] to a general linear subspace.

Example 1.6 ([8, Example 2.11])**.** Let S^k be a k-dimensional sphere in $X = S^n$ passing through p such that the stereographical projection image of $Y = S^k - \{p\}$ is equal to the linear subspace $\mathbb{R}^k$ of $\mathbb{R}^n$ defined by $x_{k+1} = \cdots = x_n = 0$. Using Lemma 1.4 one can show that

$$F_p\mathcal{T}(Y) = \{T \in \mathcal{S}(\mathbb{R}^n)' : (x_{k+1}^{a_{k+1}} \cdots x_n^{a_n})T = 0, a_j \geq 0, \sum_{k+1\leq j\leq n} a_j = p+1\}.$$

Then,

$$F_0\mathcal{T}(Y) = \mathcal{S}(\mathbb{R}^k)', \tag{3}$$

where $\mathcal{S}(\mathbb{R}^k)'$ means the image of the inclusion of $\mathcal{S}(\mathbb{R}^k)'$ into $\mathcal{S}(\mathbb{R}^n)'$. Moreover (cf. [20, Ch. III, §10]),

$$\mathcal{T}(Y) = \bigoplus_{a_j\geq 0} \partial_{k+1}^{a_{k+1}} \cdots \partial_n^{a_n} F_0\mathcal{T}(Y) \tag{4}$$

and

$$F_p\mathcal{T}(Y) = \bigoplus_{a_j\geq 0,\ \sum a_j\leq p} \partial_{k+1}^{a_{k+1}}\cdots\partial_n^{a_n}F_0\mathcal{T}(Y) \tag{5}$$

for each $p\geq 0$.

2. Bruhat filtration

In this section we review the Bruhat filtration defined in [8, §3-4]. The same as in §1, for most results we only give a sketch of proof. Interested readers are encouraged to consult the excellent paper [8] for more complete account and proof.

Let G be a real linear reductive group with Lie algebra $\mathfrak{g}_0$ and K be a maximal compact subgroup with Lie algebra $\mathfrak{k}_0$. Let $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{q}_0$ be the corresponding Cartan decomposition. Choose a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{q}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{q}$, $\mathfrak{a}$ be the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{q}_0$, $\mathfrak{a}_0$ respectively. Write $\Phi=\Phi(\mathfrak{g},\mathfrak{a})$ for the restricted root system from the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$. Choose a positive system $\Phi^+=\Phi^+(\mathfrak{g},\mathfrak{a})$ of Φ. Let $\mathfrak{n}$ be the subalgebra of $\mathfrak{g}$ spanned by root spaces of positive roots in Φ^+, and $\mathfrak{n}_0=\mathfrak{n}\cap\mathfrak{g}_0$. There is an Iwasawa decomposition $G=KAN$ where A and N are the closed Lie subgroups of G corresponding to the Lie subalgebras $\mathfrak{a}_0,\mathfrak{n}_0$ of $\mathfrak{g}_0$ respectively. Put $M=Z_K(\mathfrak{a}_0)$ and $P=MAN$. Then, P is a minimal parabolic subgroup of G.

Let $X=G/P$ be the flag variety associated to the minimal parabolic subgroup P. Write $W=N_G(A)/Z_G(A)$ and put $r=|W|$. It is well known that the conjugaction action of W on $\mathfrak{a}$ identifies W with the Weyl group of the restricted root system Φ. For each $w\in W$, choose an element $\dot{w}\in N_K(A)$ representing w and write $x_w=\dot{w}P\in C(w)$. Set

$$C(w)=N\cdot x_w=N\dot{w}P/P\subset X=G/P,$$

which is called the *Bruhat cell attached to* w ([8, line -1]). Each $C(w)$ is a smooth submanifold and is a subanalytic set in X. The Bruhat decomposition of G relative to P asserts that

$$G/P=\bigsqcup_{w\in W}C(w).$$

Let w_0 be the longest element in W, i.e., the unique element in W which maps Φ^+ to $-\Phi^+$. For each $w\in W$, put $N^w=wNw^{-1}$. Then, the map

$$(N\cap N^w)\times(N\cap N^{ww_0})\to N,\quad n_1\times n_2\mapsto n_1n_2$$

is an isomorphism of algebraic varieties. Thus, the map $n\mapsto n\dot{w}P$ ($n\in N\cap N^{ww_0}$) gives an isomorphism $N\cap N^{ww_0}\cong C(w)$. Moreover, we know that the complexified Lie algebra $\bar{\mathfrak{n}}=\mathfrak{n}^{w_0}$ of $\bar{N}:=N^{w_0}$ is spanned by root spaces of negative roots in $\Phi^-=-\Phi^+$.

Write $W=\{w_1,\dots,w_r\}$ so that $\dim C(w_i)\leq\dim C(w_{i+1})$ ($1\leq i\leq r-1$). For each k ($0\leq k\leq r$), write

$$Z_k=\bigcup_{1\leq i\leq k}C(w_i).$$

Here, we remark that the stratification

$$\emptyset=Z_0\subset Z_1\subset\cdots\subset Z_r=X$$

is a refinement of the stratification defined in [8]. Then, the Buhat filtration defined below is a refinement of that in [8].

Let (σ, V_σ) be a finite-dimensional complex linear algebraic representation of P. Here, being algebraic means *matrix coefficients*

$$c_{\alpha,v}(x) = \langle \alpha, x \cdot v \rangle \ (x \in P)$$

of σ are regular functions on P (as an algebraic variety) for all $(\alpha, v) \in V_\sigma^* \times V_\sigma$. Different from that in [8], we don't assume that the action of N on V_σ is trivial. All statements in [8] extend to our setting. Write

$$E(\sigma) := G \times_P V_\sigma$$

for a smooth equivariant vector bundle with fibre space V_σ at $P \in G/P$. Write $I(\sigma) = \mathrm{Ind}_P^G(\sigma)$ for the *un-normalized smooth parabolic induction* from the representation σ of P. Then,

$$C^\infty(E(\sigma)) = I(\sigma) \text{ and } C^\infty(E(\sigma))' = I(\sigma)'.$$

For each k $(0 \leq k \leq r)$, let $C_k^\infty(\sigma)$ be the space of smooth sections of $E(\sigma)$ which vanish with all derivatives along Z_k. We call the following finite decreasing filtration of $\mathcal{U}(\mathfrak{g})$ submodules of $I(\sigma)$:

$$I(\sigma) = C_0^\infty(\sigma) \supset C_1^\infty(\sigma) \supset \cdots \supset C_{r-1}^\infty(\sigma) \supset C_r^\infty(\sigma) = \{0\},$$

the *Bruhat filtration of* $I(\sigma)$. Write

$$\mathrm{gr} I(\sigma) = \sum_{k \geq 0} C_k^\infty(\sigma)/C_{k+1}^\infty(\sigma)$$

for the graded module corresponding to the Bruhat filtration of $I(\sigma)$. Let

$$I_{k,\sigma} = C_k^\infty(\sigma)^\perp$$

be the space of $E(\sigma)$-distributions vanishing on all sections in $C_k^\infty(\sigma)$. Then, $I_{0,\sigma} = 0$ and $I_{r,\sigma} = I(\sigma)'$.

For each $w \in W$, choose a subanalytic open neighborhood U of $C(w)$ in X. Let $\mathcal{T}(C(w), E(\sigma))$ be the space of *tempered* $E(\sigma)$*-distributions on* U *with support in* $C(w)$ and $J(w, \sigma)$ be the strong dual of $\mathcal{T}(C(w), E(\sigma))$. By Lemma 1.2, $\mathcal{T}(C(w), E(\sigma))$ and $J(w, \sigma)$ do not depend on the choice of the subanalytic open neighborhood U. Put

$$V^w = N^{ww_0} \dot{w} P/P.$$

As shown in [8, p. 166] and [1, Lemma 2.4], V^w is a subanalytic open neighborhood of $C(w)$ in X. Hence, we could take $U = V^w$ while defining $\mathcal{T}(C(w), E(\sigma))$ and $J(w, \sigma)$.

Lemma 2.1. *For each* k *(*$1 \leq k \leq r$*), the space* $I_{k,\sigma} = C_k^\infty(\sigma)^\perp$ *is equal to the space of* $E(\sigma)$*-distributions supported in* Z_k*. There is an exact sequence*

$$0 \to C_{k-1}^\infty(\sigma)^\perp \to C_k^\infty(\sigma)^\perp \to \mathcal{T}(C(w_k), E(\sigma)) \to 0.$$

Proof. The first statement follows from Lemma 1.1 by dualizing. The second statement is a special case of Lemma 1.3. □

By Lemma 2.1, we have

$$I_{k,\sigma}/I_{k-1,\sigma} = \mathcal{T}(C(w_k), E(\sigma)), \ \forall k, 1 \leq k \leq r.$$

Theorem 2.2 ([8, Theorem 4.1]). *For each k $(1 \leq k \leq r)$, there is a short exact sequence*

$$0 \longrightarrow C_k^\infty(\sigma) \longrightarrow C_{k-1}^\infty(\sigma) \longrightarrow J(w_k, \sigma) \longrightarrow 0.$$

There is an isomorphism of $\mathcal{U}(\mathfrak{g})$-modules:

$$\mathrm{gr} I(\sigma) \cong \bigoplus_{k \in \mathbb{Z}_{1 \leq k \leq r}} J(w_k, \sigma).$$

Proof. By dualizing, the first statement follows from Lemma 2.1. The second statement is a consequence of the first one. □

As in §1, let $M_{p,w,\sigma}$ denote the subspace of $\mathcal{S}(V^w, E(\sigma))$ consisting of sections which vanish with all derivatives of order $\leq p$ along $C(w)$. Then,

$$F_p\mathcal{T}(C(w), E(\sigma)) = M_{p,w,\sigma}^\perp \ (\forall p \in \mathbb{Z}).$$

Put

$$\mathrm{Gr}^p\, \mathcal{T}(C(w), E(\sigma)) = F_p\mathcal{T}(C(w), E(\sigma))/F_{p-1}\mathcal{T}(C(w), E(\sigma)).$$

Lemma 2.3 ([8, Lemma 3.5]). *For any $p \in \mathbb{Z}$, the subspace $F_p\mathcal{T}(C(w), E(\sigma))$ of $\mathcal{T}(C(w), E(\sigma))$ is $\mathcal{U}(\mathfrak{p})$-invariant.*

For $Y \in \mathfrak{g}_0$, define a differential operator $\tilde{L}_Y$ on $C^\infty(X, E(\sigma))$ by setting

$$(\tilde{L}_Y\varphi)(g) = \frac{d}{dt}\varphi(\exp(-tY)g)|_{t=0}$$

for any $\varphi \in C^\infty(X, E(\sigma))$ and any $g \in G$, where φ is regarded as a V_σ-valued function on G. We have

$$\tilde{L}_{[Y_1,Y_2]} = \tilde{L}_{Y_1}\tilde{L}_{Y_2} - \tilde{L}_{Y_2}\tilde{L}_{Y_1}, \ \forall Y_1, Y_2 \in \mathfrak{g}_0.$$

Then, $\tilde{L}.$ extends to an action of $\mathcal{U}(\mathfrak{g})$ on $C^\infty(X, E(\sigma))$ by setting

$$\tilde{L}(\lambda Y_1 \cdots Y_k)\varphi = \lambda \tilde{L}_{Y_1}(\cdots(\tilde{L}_{Y_k}\varphi)\cdots)$$

for any $Y_1, \ldots, Y_k \in \mathfrak{g}_0$, $\lambda \in \mathbb{C}$ and $\varphi \in C^\infty(X, E(\sigma))$. For any $Y_1, \ldots, Y_k \in \mathfrak{g}$, we define

$$(Y_1 \cdots Y_k)^t = (-1)^k Y_k \cdots Y_1$$

and extends it linearly to all elements in $\mathcal{U}(\mathfrak{g})$. Then, $Y \mapsto Y^t$ $(Y \in \mathcal{U}(\mathfrak{g}))$ is an anti-automorphism of $\mathcal{U}(\mathfrak{g})$. For an element $Y \in \mathcal{U}(\mathfrak{g})$ and a distribution $T \in C^\infty(X, E(\sigma))'$, define a distribution $\tilde{L}_Y(T) \in C^\infty(X, E(\sigma))'$ by setting

$$\langle \tilde{L}_Y(T), \varphi \rangle = \langle T, \tilde{L}_{Y^t}(\varphi) \rangle, \ \forall \varphi \in C^\infty(X, E(\sigma)).$$

Lemma 2.4. *We have the following assertions:*

(i) *For each $w \in W$, we have*

$$F_0\mathcal{T}(C(w), E(\sigma)) = \mathcal{S}(C(w), E(\sigma)|_{C(w)})'.$$

(ii) *For each $p \geq 0$, the map $(Y, T) \mapsto \tilde{L}_Y(T)$ gives an isomorphism*

$$\mathcal{U}_p(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}}) \otimes_{\mathbb{C}} F_0\mathcal{T}(C(w), E(\sigma)) \cong F_p\mathcal{T}(C(w), E(\sigma)).$$

(iii) *The map $(Y, T) \mapsto \tilde{L}_Y(T)$ gives an isomorphism*

$$\mathcal{U}(\mathfrak{n}^{ww_0}) \otimes_{\mathcal{U}(\mathfrak{n}^{ww_0} \cap \mathfrak{n})} F_0\mathcal{T}(C(w), E(\sigma)) \cong \mathcal{T}(C(w), E(\sigma)).$$

Proof. The equality in (i) follows from (3).

Choose an element $H \in \mathfrak{a}_0$ such that $-w\alpha(H) \in \mathbb{Z}_{>0}$ for each positive root $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$. Write $s = \dim(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}})$ and $s' = \dim(\mathfrak{n}^{ww_0} \cap \mathfrak{n})$. Choose a basis $\{X_i : 1 \le i \le s + s'\}$ of $\mathfrak{n}_0^{ww_0}$ such that each X_i is an $\mathfrak{a}_0$-weight vector, $\{X_i : 1 \le i \le s\}$ is a basis of $\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}}$ and $\{X_i : s+1 \le i \le s+s'\}$ is a basis of $\mathfrak{n}^{ww_0} \cap \mathfrak{n}$. Let $\nu_i \in \mathfrak{a}^*$ be the $\mathfrak{a}_0$-weight of X_i $(1 \le i \le s+s')$. Let $\phi : \mathfrak{n}_0^{ww_0} \to V^w$ be defined by

$$\phi(Y) = \exp(Y)\dot{w}P, \ \forall Y \in \mathfrak{n}_0^{ww_0}.$$

Then, ϕ is an isomorphism and $\phi^{-1}(C(w)) = \mathfrak{n}_0^{ww_0} \cap \mathfrak{n}_0$. For each $Y \in \mathfrak{n}_0^{ww_0}$, define vector fields $D(Y)$ and $D'(Y)$ on $\mathfrak{n}_0^{ww_0}$ by

$$D(Y)_{Y'} = \frac{\mathrm{d}\exp^{-1}(\exp(tY)\exp(Y'))}{\mathrm{d}t}|_{t=0}, \ \forall Y' \in \mathfrak{n}_0^{ww_0}$$

and

$$D'(Y)_{Y'} = Y, \ \forall Y' \in \mathfrak{n}_0^{ww_0}.$$

Let $\phi^*(D(Y))$ (and $\phi^*(D'(Y))$) be vector fields on V^w corresponding to $D(Y)$ (and $D'(Y)$) via the isomorphism ϕ. For each integer $q \ge 0$, put

$$D'_q = \mathrm{span}_{\mathbb{C}}\{\phi^*(D'(X_1))^{k_1} \cdots \phi^*(D'(X_s))^{k_s} : k_i \in \mathbb{Z}_{\ge 0}, k_1 + \cdots + k_s \le q\}.$$

For each $q \ge 0$ and each $k \in \mathbb{Z}_{\ge 0}$, put

$$\begin{aligned} D'_{q,k} \ &= \ \mathrm{span}_{\mathbb{C}}\{\phi^*(D'(X_1))^{k_1} \cdots \phi^*(D'(X_s))^{k_s} : \\ &\qquad k_i \in \mathbb{Z}_{\ge 0}, k_1 + \cdots + k_s = q, \sum_{1 \le i \le s} k_i \nu_i(H) \ge k\}. \end{aligned}$$

By the Campbell-Baker-Hausdorff formula ([15, Theorem B.22]) and the nilpotence of $\mathfrak{n}_0^{ww_0}$, we have

$$\exp^{-1}(\exp Y \exp Y') = Y + Y' + \frac{1}{2}[Y, Y'] + \sum_{3 \le k \le s+s'} H_k \tag{6}$$

for any $Y, Y' \in \mathfrak{n}_0^{ww_0}$, where H_k $(3 \le k \le s+s')$ is a finite linear combination of terms of the form

$$(\mathrm{ad}\, Y_1) \cdots (\mathrm{ad}\, Y_{k-1}) Y_k$$

with each $Y_i = Y$ or Y'. By (6), we have

$$D(Y)|_{Y'} = Y + \frac{1}{2}[Y, Y'] + \sum_{3 \le k \le s+s'} c_k (\mathrm{ad}\, Y')^{k-1} Y \tag{7}$$

for some constants $c_3, \ldots, c_{s+s'} \in \mathbb{R}$. By (5) we have

$$F_q\mathcal{T}(C(w), E(\sigma)) = D'_q F_0 \mathcal{T}(C(w), E(\sigma)) \cong D'_q \otimes_{\mathbb{C}} F_0\mathcal{T}(C(w), E(\sigma)). \tag{8}$$

Let $k_i \in \mathbb{Z}_{\ge 0}$ $(1 \le i \le s)$. Put $q = \sum_{1 \le i \le s} k_i$ and $k = \sum_{1 \le i \le s} k_i \nu_i(H)$. By (7), we have

$$\begin{aligned} &(\phi^*(D(X_1))^{k_1} \cdots \phi^*(D(X_s))^{k_s} - \phi^*(D'(X_1))^{k_1} \cdots \phi^*(D'(X_s))^{k_s})T \\ \in \ &(D'_{q-1} + D'_{q,k+1}) F_0 \mathcal{T}(C(w), E(\sigma)) \end{aligned} \tag{9}$$

for any $T \in F_0\mathcal{T}(C(w), E(\sigma))$. Note that $F_0\mathcal{T}(C(w), E(\sigma))$ is invariant by multiplying functions in $R(V^w)$ and under the action of

$$\phi^*(D'(X_{s+1})), \ldots, \phi^*(D'(X_{s+s'})).$$

Taking double induction on (p,k), using (9) one can show that

$$\begin{aligned} & F_p\mathcal{T}(C(w),E(\sigma)) \\ = \ & \mathcal{U}_p(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})F_0\mathcal{T}(C(w),E(\sigma)) \\ \cong \ & \mathcal{U}_p(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})\otimes_{\mathbb{C}} F_0\mathcal{T}(C(w),E(\sigma)) \end{aligned}$$

by (8).

Due to the isomorphism

$$\mathcal{U}(\mathfrak{n}^{ww_0})\otimes_{\mathcal{U}(\mathfrak{n}^{ww_0}\cap\mathfrak{n})} F_0\mathcal{T}(C(w),E(\sigma))\cong\mathcal{U}(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})\otimes_{\mathbb{C}} F_0\mathcal{T}(C(w),E(\sigma)),$$

(iii) follows from (ii). □

Let J be the ideal of polynomials in $R(V^w)$ vanishing along $C(w)$. From the multiplication map

$$J\otimes_{R(V^w)} F_p\mathcal{T}(C(w),E(\sigma))\to F_{p-1}\mathcal{T}(C(w),E(\sigma)),$$

we get

$$J^p/J^{p+1}\otimes_{R(C(w))}\mathrm{Gr}^p\,\mathcal{T}(C(w),E(\sigma))\to F_0\mathcal{T}(C(w),E(\sigma)).$$

Put

$$L_p(w):=\mathrm{Hom}_{R(C(w))}(J^p/J^{p+1},R(C(w))).$$

Notice that J^p/J^{p+1} is a free $R(C(w))$-module of finite rank, and so is $L_p(w)$. Then, we get a $\mathcal{U}(\mathfrak{p})$-module homomorphism

$$\alpha_p:\mathrm{Gr}^p\,\mathcal{T}(C(w),E(\sigma))\to L_p(w)\otimes_{R(C(w))}F_0\mathcal{T}(C(w),E(\sigma)),$$

which is clearly an isomorphism. Then, we get Lemma 2.5.

Lemma 2.5 ([8, Lemma 4.3])**.** *As $\mathcal{U}(\mathfrak{p})$ modules, we have*

$$\begin{aligned} & \mathrm{Gr}^p\,\mathcal{T}(C(w),E(\sigma)) \\ \cong \ & L_p(w)\otimes_{R(C(w))}F_0\mathcal{T}(C(w),E(\sigma)) \\ \cong \ & L_p(w)\otimes_{R(C(w))}\mathcal{S}(C(w),E(\sigma))'. \end{aligned}$$

Lemma 2.6 ([8, Lemma 4.4])**.** *The $R(C(w))$ module $L_p(w)$ admits a finite increasing filtration by N-equivariant $R(C(w))$-submodules such that each graded piece is isomorphic to $R(C(w))$ as an N-equivariant $R(C(w))$ module.*

Proof. The inclusion $C(w)\subset V^w$ can be identified with $N\cap N^{ww_0}\subset N^{ww_0}$. Taking exponential map this is equivalent to $\mathfrak{n}\cap\mathfrak{n}^{ww_0}\subset\mathfrak{n}^{ww_0}$, which is a linear subspace in a real linear space. Hence, $L_p(w)$ is the space of global regular sections of a finite rank free $\mathcal{O}_{C(w)}$-module $\mathcal{V}_p$. This $\mathcal{O}_{C(w)}$-module is a sheaf $\mathcal{V}_p$ of local sections of an N-homogeneous algebraic vector bundle V_p on $C(w)$. Notice that N is a unipotent algebraic group. Then, so is $\mathrm{Stab}_N(x_w)=N\cap wNw^{-1}$. Then, the geometric fibre of V_p at x_w as a representation of $\mathrm{Stab}_N(x_w)$ admits a finite increasing filtration with each graded piece a trivial representation of $\mathrm{Stab}_N(x_w)$. Then, the N-homogeneous algebraic vector bundle V_p admits a finite increasing filtration with each graded piece isomorphic to the trivial bundle on $C(w)$ and with trivial N action. Since $C(w)$ is affine, by Serre's theorem the global section functor is exact. Therefore, $L_p(w)$ admits a finite increasing filtration with each graded piece being isomorphic to $R(C(w))$ as an N-equivariant $R(C(w))$ module. □

Corollary 2.7 is a direct consequence of Lemmas 2.5 and 2.6.

Corollary 2.7. *For each $p \geq 0$, $\mathrm{Gr}^p\,\mathcal{T}(C(w), E(\sigma))$ admits a finite increasing filtration such that each graded piece is isomorphic to $\mathcal{S}(C(w), E(\sigma))'$ as a $\mathcal{U}(\mathfrak{n})$ module.*

3. Casselman-Jacquet modules

Let $\mathfrak{n}'$ be a nilpotent complex Lie algebra and U be a complex linear representation of $\mathcal{U}(\mathfrak{n}')$. We define

$$U^{[\mathfrak{n}']} = \{v \in U : \exists k \in \mathbb{Z}_{>0}, {\mathfrak{n}'}^k \cdot v = 0\}$$

and call it *the space of nilpotnet elements* in U.

Write $\eta : \mathcal{U}(\mathfrak{n}') \to \mathbb{C}$ for the augmented homomorphism of $\mathcal{U}(\mathfrak{n}')$ defined by $\eta(X_1 \cdots X_q) = 0$ for any $X_1, \ldots, X_q \in \mathfrak{n}'$ whenever $q > 0$. Then,

$$\ker \eta = \mathfrak{n}'\mathcal{U}(\mathfrak{n}') = \mathcal{U}(\mathfrak{n}')\mathfrak{n}'$$

and it is the augmented ideal of $\mathcal{U}(\mathfrak{n}')$.

Lemma 3.1. *For any $v \in V$, the following conditions are equivalent:*

(1) *For some $k \geq 1$, $(\ker \eta)^k v = 0$.*
(2) *For some $k \geq 1$, ${\mathfrak{n}'}^k \cdot v = 0$.*

Proof. (2) $\Rightarrow$ (1). Suppose ${\mathfrak{n}'}^k \cdot v = 0$. Then, $(\mathcal{U}(\mathfrak{n}'){\mathfrak{n}'}^k) \cdot v = 0$. Since $\mathcal{U}(\mathfrak{n}') = \mathcal{U}_k(\mathfrak{n}') + \mathcal{U}(\mathfrak{n}'){\mathfrak{n}'}^k$ and $\dim \mathcal{U}_k(\mathfrak{n}') < \infty$, we get $\dim \mathcal{U}(\mathfrak{n}')v < \infty$. Put $U := \mathcal{U}(\mathfrak{n}')v$. Then, U is a finite-dimensional representation of $\mathcal{U}(\mathfrak{n}')$ and each $X \in \mathfrak{n}'$ acts on U as a nilpotent endomorphism. By Engel's theorem, there is a filtration $U = U_0 \supset U_1 \supset \cdots \supset U_k = 0$ of U such that $\mathfrak{n}'U_i \subset U_{i-1}$ $(1 \leq i \leq k)$. Then, $(\ker \eta)U_i \subset U_{i-1}$ $(1 \leq i \leq k)$. Thus, $(\ker \eta)^k v = 0$.

(1) $\Rightarrow$ (2). It follows from $\mathfrak{n}' \subset \ker \eta$. □

Alternatively, from the fact $\mathfrak{n}'\mathcal{U}(\mathfrak{n}') = \mathcal{U}(\mathfrak{n}')\mathfrak{n}'$ we get $(\ker \eta)^k = (\mathcal{U}(\mathfrak{n}')\mathfrak{n}')^k = \mathcal{U}(\mathfrak{n}'){\mathfrak{n}'}^k$. Then, if follows that (1) $\Leftrightarrow$ (2) in Lemma 3.1.

Lemma 3.2. *Let V, U be two complex linear representations of $\mathcal{U}(\mathfrak{n}')$. Then we have the following assertions:*

(i) $V^{[\mathfrak{n}']} \otimes U^{[\mathfrak{n}']} \subset (V \otimes U)^{[\mathfrak{n}']}$.
(ii) *If $V = V^{[\mathfrak{n}']}$, then $(V \otimes U)^{[\mathfrak{n}']} = V \otimes U^{[\mathfrak{n}']}$.*
(iii) *If $U = U^{[\mathfrak{n}']}$, then $(V \otimes U)^{[\mathfrak{n}']} = V^{[\mathfrak{n}']} \otimes U$.*

Proof. (i) is obvious. (ii) and (iii) are similar. We show (ii) below.

Suppose that $V = V^{[\mathfrak{n}']}$. For each $k \geq 1$, put

$$V_k = \{v \in V : (\ker \eta)^k v = 0\}.$$

Then, $\mathfrak{n}'V_k \subset V_{k-1}$ for each $k \geq 1$. By Lemma 3.1, we have

$$V_k = \{v \in V : {\mathfrak{n}'}^k v = 0\}.$$

Then,

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k \subset \cdots$$

is an exhaustive ascending filtration of V by the assumption $V = V^{[\mathfrak{n}']}$. Let x be an element in $(V \otimes U)^{[\mathfrak{n}']}$. From the above, we can find a set of linearly independent elements $\{v_i : 1 \leq i \leq k\}$ of V such that

$$(\ker \eta)v_i \subset \mathrm{span}\{v_j : 1 \leq j \leq i-1\}$$

for each i ($1 \leq i \leq k$) and $x \in \operatorname{span}\{v_j : 1 \leq j \leq k\} \otimes U$. Write

$$x = \sum_{1 \leq j \leq k} v_j \otimes u_j,$$

where $u_j \in U$ ($1 \leq j \leq k$). Projecting to

$$(\operatorname{span}\{v_j : 1 \leq j \leq k\} / \operatorname{span}\{v_j : 1 \leq j \leq k-1\}) \otimes U,$$

we get that $v_k \otimes u_k$ is annihilated by a power of $\ker \eta$ in this quotient module of $\operatorname{span}\{v_j : 1 \leq j \leq k\} \otimes U$. Since $\operatorname{span}\{v_j : 1 \leq j \leq k\} / \operatorname{span}\{v_j : 1 \leq j \leq k-1\}$ is isomorphic to the trivial representation of $\mathcal{U}(\mathfrak{n}')$, we get $u_k \in U^{[\mathfrak{n}']}$. Then by (i), we have $v_k \otimes u_k \in (V \otimes U)^{[\mathfrak{n}']}$. Thus,

$$\sum_{1 \leq j \leq k-1} v_j \otimes u_j \in (V \otimes U)^{[\mathfrak{n}']}.$$

Inductively, we show that $u_{k-1}, \ldots, u_1$ are all in $U^{[\mathfrak{n}']}$. Hence, $x \in V \otimes U^{[\mathfrak{n}']}$. $\square$

For a $\mathcal{U}(\mathfrak{p})$-module V, we call $V^{[\mathfrak{n}]}$ the *Casselman-Jacquet module* of V. Then, $V^{[\mathfrak{n}]}$ is still a $\mathcal{U}(\mathfrak{p})$-module, and $\mathfrak{n}$ acts on it locally nilpotently in the sense that: for any $v \in V^{[\mathfrak{n}]}$, $\mathcal{U}(\mathfrak{n}) \cdot v$ is a finite-dimensional space. Moreover, if V is a $\mathcal{U}(\mathfrak{g})$-module, then so is $V^{[\mathfrak{n}]}$ ([6]). It is clear that the functor $V \to V^{[\mathfrak{n}]}$ is left exact. However, it is not right exact in general. The goal of this section is to show that the functor $V \to V^{[\mathfrak{n}]}$ preserves exactness of some short exact sequences arising in §2.

Let U be a Zariski closed subgroup of N. In [8, p. 169] there is a definition of polynomials on the affine space N/U: a smooth function f on N/U is called a *polynomial* if its annihilator in $\mathcal{U}(\mathfrak{n})$ contains $\mathfrak{n}^k$ for some $k \geq 1$. The notions of polynomial and regular function coincide in this case, as Lemma 3.3 shows. Lemma 3.3 is an adjustment of [1, Proposition A.4]. The same proof as that in [1] works. Define a distribution $\delta_{N/U}$ on N/U by

$$\delta_{N/U}(f) = \int_{N/U} f(x)\, \mathrm{d}x$$

for any $f \in C_c^\infty(N/U)$, where $\mathrm{d}x$ is a fixed N invariant measure on N/U.

Lemma 3.3 ([1, Proposition A.4])**.** *We have the following assertions.*

(i) *For any $k \geq 1$, there exists $l \geq 1$ such that if a distribution T on N/U satisfies $\mathfrak{n}^k T = 0$, then $T \in R_l(N/U)\delta_{N/U}$.*

(ii) *For any $l \geq 1$, there exists $k \geq 1$ such that $\mathfrak{n}^k(R_l(N/U)\delta_{N/U}) = 0$.*

Let $w \in W$. For $f \in R(C(w))$ and $u' \in V_\sigma^*$, define a distribution $(f \otimes u')\delta_{C(w)} \in \mathcal{S}(C(w), E(\sigma))'$ by setting

$$\langle (f \otimes u')\delta_{C(w)}, g \rangle = \int_{N \cap N^{ww_0}} f(y\dot{w}P) u'(g(y))\, \mathrm{d}y$$

for all $g \in \mathcal{S}(C(w), E(\sigma))$, where $\mathrm{d}y$ is a volume form on $N \cap N^{ww_0}$ which induces an N invariant measure $\mathrm{d}x$ on $C(w) = N/N \cap N^w \cong N \cap N^{ww_0}$ and g is regarded as a function on N. Choose a basis $\{v_1, \ldots, v_m\}$ of V_σ and let $\{\alpha_1, \ldots, \alpha_m\}$ be the dual basis of V_σ^*. For any $v \in V_\sigma$ and $\alpha \in V_\sigma^*$, define the *matrix coefficient*

$$\tilde{c}_{\alpha,v}(x) = \langle \alpha, x^{-1} \cdot v \rangle = c_{\alpha,v}(x^{-1}),\ \forall x \in P.$$

Lemma 3.4. *For each $w \in W$, we have*

$$\mathcal{S}(C(w), E(\sigma))'^{[\mathfrak{n}]} = (R(C(w)) \otimes V_\sigma^*)\delta_{C(w)}.$$

Proof. Due to the fact that N is a unipotent group and V_σ is finite-dimensional algebraic representation of P, we have $V_\sigma^{[\mathfrak{n}\cap\mathfrak{n}^{ww_0}]} = V_\sigma$. Note that $C(w) \cong N \cap N^{ww_0}$ and $E(\sigma)|_{C(w)}$ is a trivial $N \cap N^{ww_0}$-equivariant bundle. Then,

$$\mathcal{S}(C(w), E(\sigma))'^{[\mathfrak{n}]} \subset \mathcal{S}(C(w), E(\sigma))'^{[\mathfrak{n}\cap\mathfrak{n}^{ww_0}]} = \mathcal{S}(C(w))'^{[\mathfrak{n}\cap\mathfrak{n}^{ww_0}]} \otimes V_\sigma^*$$

by Lemma 3.2 (iii) and

$$\mathcal{S}(C(w))'^{[\mathfrak{n}\cap\mathfrak{n}^{ww_0}]} = R(C(w))\delta_{C(w)}$$

by Lemma 3.3. Thus,

$$\mathcal{S}(C(w), E(\sigma))'^{[\mathfrak{n}]} \subset (R(C(w))\delta_{C(w)}) \otimes V_\sigma^* = (R(C(w)) \otimes V_\sigma^*)\delta_{C(w)}. \tag{10}$$

Define a representation $(\dot{w}\sigma, V_{\dot{w}\sigma})$ of $N \cap N^w$ whose underlying space is V_σ and the group action is given by

$$(\dot{w}\sigma)(x) = \sigma(\dot{w}^{-1}x\dot{w}), \forall x \in N \cap N^w.$$

Then, we have

$$E(\sigma)|_{C(w)} \cong N \times_{N\cap N^w} V_{\dot{w}\sigma}$$

via the identification $C(w) \cong N/N \cap N^w$. Recall that a section of the dual bundle $N \times_{N\cap N^w} V_{\dot{w}\sigma}^*$ may be identified with a function $h : N \to V_{\dot{w}\sigma}^*$ such that

$$h(nn') = (\dot{w}\sigma^*(n'^{-1}))h(n),\ \forall (n, n') \in N \times (N \cap N^w).$$

Write $R_{\dot{w},\sigma}$ for the space of such sections $h : N \to V_{\dot{w}\sigma}^*$ such that $\langle h(x), v\rangle$ is a regular function on N for any fixed vector $v \in V_\sigma = V_{\dot{w}\sigma}$. We may identify a section $g \in \mathcal{S}(C(w), E(\sigma)|_{C(w)})$ with a function $g : N \to V_{\dot{w}\sigma}$ such that

$$g(nn') = (\dot{w}\sigma(n'^{-1}))g(n),\ \forall (n, n') \in N \times (N \cap N^w)$$

and $\langle \alpha, g(x)\rangle$ is a Schwartz function on $N \cap N^{ww_0}$ for any fixed vector $\alpha \in V_\sigma^* = V_{\dot{w}\sigma}^*$. For any section $h \in R_{\dot{w},\sigma}$ and any section $g \in \mathcal{S}(C(w), E(\sigma)|_{C(w)})$, define

$$(h \otimes g)(x) = \langle h(x), g(x)\rangle, \forall x \in N,$$

where $\langle\cdot,\cdot\rangle$ is the pairing $V_\sigma^* \times V_\sigma \to \mathbb{C}$. Then,

$$(h \otimes g)(nn') = (h \otimes g)(n), \forall (n, n') \in N \times (N \cap N^w).$$

Thus, $h \otimes g$ represents a function on $C(w) \cong N/(N \cap N^w)$ and it is a Schwartz function. Hence, we can define

$$\langle h, g\rangle = \int_{C(w)} (h \otimes g)(x)\,\mathrm{d}x = \int_{N\cap N^{ww_0}} (h \otimes g)(y)\,\mathrm{d}y.$$

For any $f \in R(C(w))$ and $u' \in V_\sigma^*$, define $f \otimes u' : N \to V_{\dot{w}\sigma}$ by setting

$$(f \otimes u')(nn') = f(n\dot{w}P)(\dot{w}\sigma^*(n'^{-1})u'),\ \forall (n, n') \in (N \cap N^{ww_0}) \times (N \cap N^w).$$

Then, we have $f \otimes u' \in R_{\dot{w},\sigma}$. Due to the fact that $C(w) \cong N/(N \cap N^w) \cong N \cap N^{ww_0}$, the map

$$(f \otimes u')\delta_{C(w)} \mapsto f \otimes u',\ f \in R(C(w)), u' \in V_\sigma^*$$

gives a bijection

$$(R(C(w)) \otimes V_\sigma^*)\delta_{C(w)} \cong R_{\dot{w},\sigma}.$$

Moreover, we have

$$\langle (f\otimes u')\delta_{C(w)}, g\rangle = \langle f\otimes u', g\rangle$$

for any $f\in R(C(w))$, any $u'\in V_\sigma^*$ and any $g\in \mathcal{S}(C(w),E(\sigma)|_{C(w)})$. Therefore, the $\mathcal{U}(\mathfrak{n})$ representations $(R(C(w))\otimes V_\sigma^*)\delta_{C(w)}$ and $R_{\dot{w},\sigma}$ are isomorphic.

When N acts trivially on V_σ, $R_{\dot{w},\sigma}$ is isomorphic to the direct sum of $\dim V_\sigma$ copies of $R(C(w))$. By Lemma 3.3 (ii), each section $h\in R_{\dot{w},\sigma}$ is annihilated by $\mathfrak{n}^k$ for some $k\geq 1$. Then,

$$(R(C(w))\otimes V_\sigma^*)\delta_{C(w)} \subset \mathcal{S}(C(w),E(\sigma))'^{[\mathfrak{n}]}.$$

In general, take a composition series

$$0=V_0\subset V_1\subset\cdots\subset V_s=V_\sigma$$

of V_σ and write σ_i for the representation of P on V_i/V_{i-1} $(1\leq i\leq s)$. Accordingly, we have a filtration

$$0=R_{\dot{w},\sigma_0}\subset R_{\dot{w},\sigma_1}\subset\cdots\subset R_{\dot{w},\sigma_s}=R_{\dot{w},\sigma}.$$

For each i $(1\leq i\leq s)$, since V_i/V_{i-1} is an irreducible algebraic representation of P, then N acts trivially on it. Hence, any element of $R_{\dot{w},\sigma_i}/R_{\dot{w},\sigma_{i-1}}\cong R_{\dot{w},\sigma_i/\sigma_{i-1}}$ is annihilated by $\mathfrak{n}^k$ for some $k\geq 1$. Then, any element of $R_{\dot{w},\sigma}$ is annihilated by $\mathfrak{n}^k$ for some $k\geq 1$. Thus,

$$(R(C(w))\otimes V_\sigma^*)\delta_{C(w)} \subset \mathcal{S}(C(w),E(\sigma))'^{[\mathfrak{n}]}. \tag{11}$$

Combining (10) and (11), we get

$$\mathcal{S}(C(w),E(\sigma))'^{[\mathfrak{n}]} = (R(C(w))\otimes V_\sigma^*)\delta_{C(w)}.$$

□

Set

$$J_{w,\sigma}=\mathrm{span}\{\tilde{L}_Y((f\otimes u')\delta_{C(w)}) : Y\in\mathcal{U}(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}}), f\in R(C(w)), u'\in V_\sigma^*\}. \tag{12}$$

Lemma 3.5 follows from results in [1, §2 and §3]. Since Lemma 3.7 does not directly follow from results in [1], we give a complete proof for Lemma 3.5 which can be adjusted to show Lemma 3.7.

Lemma 3.5. *For each $w\in W$, we have*

$$\mathcal{T}(C(w),E(\sigma))^{[\mathfrak{n}]} = J_{w,\sigma}.$$

Proof. Put

$$J'_{w,\sigma}=\mathrm{span}\{\tilde{L}_Y((f\otimes u')\delta_{C(w)}) : Y\in\mathcal{U}(\mathfrak{g}), f\in R(C(w)), u'\in V_\sigma^*\}. \tag{13}$$

Then, it is clear that

$$J_{w,\sigma}\subset J'_{w,\sigma}. \tag{14}$$

By Lemma 3.4, $J'_{w,\sigma}$ is a quotient of $\mathcal{U}(\mathfrak{g})\otimes_{\mathbb{C}}(F_0\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]}$ as a representation of $\mathcal{U}(\mathfrak{n})$. Since each element of $\mathcal{U}(\mathfrak{g})$ (or $(F_0\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]}$) is annihilated by a power of $\mathfrak{n}$, by Lemma 3.2 (i) we have

$$J'_{w,\sigma}\subset \mathcal{T}(C(w),E(\sigma))^{[\mathfrak{n}]}. \tag{15}$$

By Lemma 2.4 (ii), we have

$$\mathcal{T}(C(w),E(\sigma))\cong\mathcal{U}(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})\otimes_{\mathbb{C}} F_0\mathcal{T}(C(w),E(\sigma)).$$

Take an element $H \in \mathfrak{a}_0$ such that $\alpha(H) \in \mathbb{Z}_{>0}$ for each root $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$. The conjugation action of H on $\mathcal{U}(\mathfrak{g})$ gives a grading

$$\mathcal{U}(\mathfrak{g}) = \sum_{k\in\mathbb{Z}} \mathcal{U}(\mathfrak{g})[k]$$

defined by

$$\mathcal{U}(\mathfrak{g})[k] = \{Y \in \mathcal{U}(\mathfrak{g}) : HY - YH = kY\}.$$

Put

$$\mathcal{U}(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}})[k] = \mathcal{U}(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}}) \cap \mathcal{U}(\mathfrak{g})[k].$$

Since H normalizes $\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}}$ and $\bar{\mathfrak{n}}$ is the sum of H-eigenspaces of negative eigenvalues, we have a grading

$$\mathcal{U}(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}}) = \sum_{k\in\mathbb{Z}_{\geq 0}} \mathcal{U}(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}})[-k].$$

For each $k \in \mathbb{Z}_{\geq 0}$, put

$$J_{w,\sigma,k} = \mathrm{span}\{\tilde{L}_Y T : Y \in \mathcal{U}(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}})[-k'], k' \leq k, T \in F_0\mathcal{T}(C(w), E(\sigma))\}. \tag{16}$$

Then,

$$\mathbb{C} = J_{w,\sigma,0} \subset \cdots J_{w,\sigma,k} \subset \cdots$$

form an exhaustive ascending filtration of $\mathcal{T}(C(w), E(\sigma))$.

We show that: for each $k \geq 0$,

$$Y(\tilde{L}_{Y'} Y) \in \tilde{L}_{Y'}(YT) + J_{w,\sigma,k-1} \tag{17}$$

for any $Y \in \mathfrak{n}^{ww_0} \cap \mathfrak{n}$, $Y' \in \mathcal{U}(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}})[k]$ and $T \in F_0\mathcal{T}(C(w), E(\sigma))$. Prove by induction on k. When $k = 0$, this is clear. Let $k_0 \geq 1$ and suppose (17) holds when $k < k_0$. When $k = k_0$, we have

$$Y(\tilde{L}_{Y_1\cdots Y_l} T) = \sum_{1\leq i\leq l} (Y_1 \cdots Y_{i-1}[Y, Y_i])\tilde{L}_{Y_{i+1}\cdots Y_l} T + (Y_1 \cdots Y_l Y)T \tag{18}$$

for any $Y_1, \ldots, Y_l \in \mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}}$, $Y \in \mathfrak{n}^{ww_0} \cap \mathfrak{n}$ and $T \in F_0\mathcal{T}(C(w), E(\sigma))$. Without loss of generality we assume that $Y, Y_1, \ldots, Y_l$ are all $\mathfrak{a}$-weight vectors. For each i $(1 \leq i \leq k)$, $[Y, Y_i]$ is an $\mathfrak{a}$-weight vector contained in $\mathfrak{n}^{ww_0}$. Thus, $[Y, Y_i] \in \mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}}$ or $\mathfrak{n}^{ww_0} \cap \mathfrak{n}$. When $[Y, Y_i] \in \mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}}$, the H-weight of $Y_1 \cdots Y_{i-1}[Y, Y_i]Y_{i+1} \cdots Y_l$ is bigger than $-k_0$. Then,

$$(Y_1 \cdots Y_{i-1}[Y, Y_i])\tilde{L}_{Y_{i+1}\cdots Y_l} T = \tilde{L}_{Y_1\cdots Y_{i-1}[Y,Y_i]Y_{i+1}\cdots Y_l} T \in J_{w,\sigma,k_0-1}.$$

When $[Y, Y_i] \in \mathfrak{n}^{ww_0} \cap \mathfrak{n}$, let $-k_1$ be the H-weight of $Y_1 \cdots Y_{i-1}$ and let $-k_2$ be the H-weight of $Y_{i+1} \cdots Y_l$. Then, $k_1 + k_2 < k_0$. By induction we have

$$[Y, Y_i]\tilde{L}_{Y_{i+1}\cdots Y_l} T \in J_{w,\sigma,k_2}.$$

Then,

$$(Y_1 \cdots Y_{i-1}[Y, Y_i])\tilde{L}_{Y_{i+1}\cdots Y_l} T \in J_{w,\sigma,k_0-1}.$$

For the last term, we have

$$(Y_1 \cdots Y_l Y)T = \tilde{L}_{Y_1\cdots Y_l}(YT).$$

This finishes the proof of (17).

From (17), we get

$$J_{w,\sigma,k}^{[\mathfrak{n}^{ww_0}\cap\mathfrak{n}]} \subset (\mathcal{U}(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}})[k])(F_0\mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}^{ww_0}\cap\mathfrak{n}]} + J_{w,\sigma,k-1}.$$

By the proof of Lemma 3.4, we have

$$(F_0\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}^{ww_0}\cap\mathfrak{n}]}=(F_0\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]}=(R(C(w))\otimes V_\sigma^*)\delta_{C(w)}.$$

Then,

$$J_{w,\sigma,k}^{[\mathfrak{n}^{ww_0}\cap\mathfrak{n}]}\subset J_{w,\sigma}+J_{w,\sigma,k-1}.$$

By (14) and (15), each element of $J_{w,\sigma}$ is annihilated by a power of $\mathfrak{n}$. Then,

$$J_{w,\sigma,k}^{[\mathfrak{n}^{ww_0}\cap\mathfrak{n}]}\subset J_{w,\sigma}+J_{w,\sigma,k-1}^{[\mathfrak{n}^{ww_0}\cap\mathfrak{n}]}.$$

Proving by induction on k, we get

$$\mathcal{T}(C(w),E(\sigma))^{[\mathfrak{n}^{ww_0}\cap\mathfrak{n}]}\subset J_{w,\sigma}.$$

Thus,

$$\mathcal{T}(C(w),E(\sigma))^{[\mathfrak{n}]}\subset J_{w,\sigma}. \tag{19}$$

Combining (14), (15) and (19), we get

$$\mathcal{T}(C(w),E(\sigma))^{[\mathfrak{n}]}=J_{w,\sigma}=J'_{w,\sigma}.$$

□

For each p, set

$$J_{p,w,\sigma}=\operatorname{span}\{\tilde{L}_Y((f\otimes u')\delta_{C(w)}):Y\in\mathcal{U}_p(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}}),f\in R(C(w)),u'\in V_\sigma^*\}. \tag{20}$$

Lemma 3.6. *For each $w\in W$ and each $p\in\mathbb{Z}$, we have*

$$(F_p\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]}=J_{p,w,\sigma}.$$

Proof. By Lemma 3.5, we have

$$\begin{aligned}&(F_p\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]}\\=\ &F_p\mathcal{T}(C(w),E(\sigma))\cap\mathcal{T}(C(w),E(\sigma))^{[\mathfrak{n}]}\\=\ &F_p\mathcal{T}(C(w),E(\sigma))\cap J_{w,\sigma}.\end{aligned}$$

By Lemma 2.4 (ii), (12) and (20), the latter is equal to $J_{p,w,\sigma}$. □

Lemma 3.7. *For any $w\in W$ and each $p\in\mathbb{Z}$, we have*

$$(\operatorname{Gr}^p\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]}=J_{p,w,\sigma}/J_{p-1,w,\sigma}.$$

Proof. There is a short exact sequence

$$0\to F_{p-1}\mathcal{T}(C(w),E(\sigma))\to F_p\mathcal{T}(C(w),E(\sigma))\to\operatorname{Gr}^p\mathcal{T}(C(w),E(\sigma))\to 0.$$

By the left exactness of the Casselman-Jacquet functor and Lemma 3.6, it suffices to show the following assertion: each element in $(\operatorname{Gr}^p\mathcal{T}(C(w),E(\sigma)))^{[\mathfrak{n}]}$ is the image of some element of $J_{p,w,\sigma}$ in $\operatorname{Gr}^p\mathcal{T}(C(w),E(\sigma))$.

By Lemma 2.4, we have

$$\begin{aligned}&\operatorname{Gr}^p\mathcal{T}(C(w),E(\sigma))\\=\ &F_p\mathcal{T}(C(w),E(\sigma))/F_{p-1}\mathcal{T}(C(w),E(\sigma))\\\cong\ &\mathcal{U}_p(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})F_0\mathcal{T}(C(w),E(\sigma))/\mathcal{U}_{p-1}(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})F_0\mathcal{T}(C(w),E(\sigma))\\\cong\ &(\mathcal{U}_p(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})/\mathcal{U}_{p-1}(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}}))\otimes_{\mathbb{C}}F_0\mathcal{T}(C(w),E(\sigma)).\end{aligned} \tag{21}$$

Following the proof of Lemma 3.5, we write

$$\mathcal{U}_p(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})[k]=\mathcal{U}_p(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})\cap\mathcal{U}(\mathfrak{n}^{ww_0}\cap\bar{\mathfrak{n}})[k]$$

and put

$$J_{p,w,\sigma,k} = \operatorname{span}\{\tilde{L}_Y T : Y \in \mathcal{U}_p(\mathfrak{n}^{ww_0} \cap \bar{\mathfrak{n}})[-k'], k' \leq k, T \in F_0\mathcal{T}(C(w), E(\sigma))\}. \tag{22}$$

Then,

$$0 \subset J_{p,w,\sigma,0} \subset \cdots J_{p,w,\sigma,k} \subset \cdots$$

form an exhaustive ascending filtration of $F_p\mathcal{T}(C(w), E(\sigma))$. Moreover, this filtration is compatible with the corresponding filtration of $F_{p-1}\mathcal{T}(C(w), E(\sigma))$ in the sense that

$$J_{p,w,\sigma,k} \cap F_{p-1}\mathcal{T}(C(w), E(\sigma)) = J_{p-1,w,\sigma,k}.$$

Then,

$$J_{p,w,\sigma,k} + F_{p-1}\mathcal{T}(C(w), E(\sigma))/F_{p-1}\mathcal{T}(C(w), E(\sigma)) \cong J_{p,w,\sigma,k}/J_{p-1,w,\sigma,k}$$

form an exhaustive filtration of $\operatorname{Gr}^p \mathcal{T}(C(w), E(\sigma))$. As in the proof of Lemma 3.5, using (17) one can show that

$$\begin{aligned} &((J_{p,w,\sigma,k} + F_{p-1}\mathcal{T}(C(w), E(\sigma)))/F_{p-1}\mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}^{ww_0} \cap \mathfrak{n}]} \\ \subset\ &(J_{p,w,\sigma} + J_{p,w,\sigma,k-1} + F_{p-1}\mathcal{T}(C(w), E(\sigma)))/F_{p-1}\mathcal{T}(C(w), E(\sigma)) \end{aligned}$$

for each $k \geq 0$. Taking induction on k, it follows that

$$\begin{aligned} &(F_p\mathcal{T}(C(w), E(\sigma))/F_{p-1}\mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}^{ww_0} \cap \mathfrak{n}]} \\ \subset\ &(J_{p,w,\sigma} + F_{p-1}\mathcal{T}(C(w), E(\sigma)))/F_{p-1}\mathcal{T}(C(w), E(\sigma)). \end{aligned}$$

Thus, any element in $(\operatorname{Gr}^p \mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}]}$ is the image of some element of $J_{p,w,\sigma}$ in $\operatorname{Gr}^p \mathcal{T}(C(w), E(\sigma))$. ☐

Lemma 3.8. *For each $w \in W$ and each $p \in \mathbb{Z}$, there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow\ &(F_{p-1}\mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}]} \longrightarrow (F_p\mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}]} \\ \longrightarrow\ &(\operatorname{Gr}^p \mathcal{T}(C(w), E(\sigma)))^{[\mathfrak{n}]} \longrightarrow 0. \end{aligned}$$

Proof. This follows from Lemmas 3.6 and 3.7 directly. ☐

Lemma 3.9. *For each k, there is an exact sequence*

$$0 \to I_{k-1,\sigma}^{[\mathfrak{n}]} \to I_{k,\sigma}^{[\mathfrak{n}]} \to (I_{k,\sigma}/I_{k-1,\sigma})^{[\mathfrak{n}]} \to 0.$$

Proof. We only need to show the surjectivity of the map $I_{k,\sigma}^{[\mathfrak{n}]} \to (I_{k,\sigma}/I_{k-1,\sigma})^{[\mathfrak{n}]}$. By Lemma 3.5, we have

$$(I_{k,\sigma}/I_{k-1,\sigma})^{[\mathfrak{n}]} = \mathcal{T}(C(w_k), E(\sigma))^{[\mathfrak{n}]} = J_{w_k,\sigma}.$$

Then, it suffices to find a distribution in $I_{k,\sigma}^{[\mathfrak{n}]}$ with restriction $\tilde{L}_Y((f \otimes u')\delta_{C(w_k)})$ on V^{w_k} for any $Y \in \mathcal{U}(\mathfrak{g})$, $f \in R(C(w_k))$ and $u' \in V_\sigma^*$. This is shown in [1, Lemma 4.6]. ☐

Lemma 3.10. *Let $0 \to \sigma_1 \to \sigma_2 \to \sigma_3 \to 0$ be a short exact sequence of finite-dimensional complex linear algebraic representations of P. Then, the sequence*

$$0 \to I(\sigma_1)'^{[\mathfrak{n}]} \to I(\sigma_2)'^{[\mathfrak{n}]} \to I(\sigma_3)'^{[\mathfrak{n}]} \to 0$$

is exact.

Proof. We show that: for each $k \geq 0$, the following sequence

$$0 \to I_{k,\sigma_1}^{[\mathfrak{n}]} \to I_{k,\sigma_2}^{[\mathfrak{n}]} \to I_{k,\sigma_3}^{[\mathfrak{n}]} \to 0$$

is exact. By the left exactness of the Casselman-Jacquet functor, we only need to show that the map $I_{k,\sigma_2}^{[\mathfrak{n}]} \to I_{k,\sigma_3}^{[\mathfrak{n}]}$ is surjective. When $k=0$, we have $I_{k,\sigma_2}^{[\mathfrak{n}]} = I_{k,\sigma_3}^{[\mathfrak{n}]} = 0$. Hence, the assertion is trivial. Let $l \geq 1$ and suppose the assertion holds true whenever $k < l$. When $k = l$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & I_{l-1,\sigma_2}^{[\mathfrak{n}]} \arrow[r] \arrow[d] & I_{l,\sigma_2}^{[\mathfrak{n}]} \arrow[r] \arrow[d] & \mathcal{T}(C(w_l), E(\sigma_2))^{[\mathfrak{n}]} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{l-1,\sigma_3}^{[\mathfrak{n}]} \arrow[r] & I_{l,\sigma_3}^{[\mathfrak{n}]} \arrow[r] & \mathcal{T}(C(w_l), E(\sigma_3))^{[\mathfrak{n}]} \arrow[r] & 0.
\end{tikzcd}$$

By Lemma 3.9, two horizontal lines of this commutative diagram are short exact sequences. By Lemma 3.5, the map

$$\mathcal{T}(C(w_l), E(\sigma_2))^{[\mathfrak{n}]} \to \mathcal{T}(C(w_l), E(\sigma_3))^{[\mathfrak{n}]}$$

is surjective. By hypothesis, the map $I_{l-1,\sigma_2}^{[\mathfrak{n}]} \to I_{l-1,\sigma_3}^{[\mathfrak{n}]}$ is surjective. By the five lemma, the surjectivity of the map $I_{l,\sigma_2}^{[\mathfrak{n}]} \to I_{l,\sigma_3}^{[\mathfrak{n}]}$ follows. Taking $k = r$, we get the conclusion of this lemma. □

Among results in this section, Lemmas 3.3, 3.4, 3.5 and 3.9 follow from results in [1]. Let's remark on the difference between our proof and the proof in [1]. The hard part in the proof of Lemma 3.4 is showing that each distribution $(f \otimes u')\delta_{C(w)}$ ($f \in R(C(w))$, $u' \in V_\sigma^*$) is annihilated by a power of $\mathfrak{n}$, which is the content of [1, §3]. Different from Abe's proof which defined and used a right action of $\mathcal{U}(\mathfrak{g})$ on $E(\sigma)$-distributions, we only use the left action $\tilde{L}.$ by differential operators. The hard part in the proof of Lemma 3.5 is showing that

$$\mathcal{T}(C(w), E(\sigma))^{[\mathfrak{n}]} \subset J_{w,\sigma},$$

which is the content of [1, §2]. We avoid the backward induction used in [1, Lemma 2.9], and use induction on the H-grading degree instead. In another aspect, we consider only principal series rather than representations induced from a general real parabolic subgroup, and only the Casselman-Jacquet functor with respect to the nilradical of a minimal real parabolic subalgebra and the trivial character $\eta = 1$. For this reason we avoid complicated computation of differential operator action in [1] and use only elementary representation theory.

4. A comparison result on nilpotent homogeneous affine variety

Let U be a closed linear subgroup of N. The same as in Lemma 3.4 we have $\mathcal{S}(N/U)'^{[\mathfrak{n}]} = R(N/U)\delta_{N/U}$. In this section we show that the inclusion $R(N/U)\delta_{N/U} \subset \mathcal{S}(N/U)'$ induces isomorphisms

$$H^i(\mathfrak{n}, R(N/U)\delta_{N/U}) = H^i(\mathfrak{n}, \mathcal{S}(N/U)'), \ \forall i \geq 0.$$

Our proof is inspired by ideas in [8, §5]. Note that all closed subgroups of a unipotent algebraic group are connected.

Lemma 4.1. *Let U be a closed linear subgroup of N, and C be a one-dimensional central closed subgroup of N which is not contained in U. Put $V = CU \subset N$. Then*

$$H^i(\mathfrak{c}, \mathcal{S}(N/U)') = \begin{cases} \mathcal{S}(N/V)' & \textit{for } i = 0 \\ 0 & \textit{for } i \neq 0 \end{cases} \tag{23}$$

and

$$H^i(\mathfrak{c}, R(N/U)\delta_{N/U}) = \begin{cases} R(N/V)\delta_{N/V} & \textit{for } i = 0 \\ 0 & \textit{for } i \neq 0. \end{cases} \tag{24}$$

Proof. Let $p : N/U \longrightarrow N/V$ be the natural projection. There is a natural short exact sequence:

$$0 \longrightarrow \mathcal{S}(N/U) \xrightarrow{\xi} \mathcal{S}(N/U) \xrightarrow{\pi} \mathcal{S}(N/V) \longrightarrow 0,$$

where ξ is an N invariant differential operator on N/U associated to a nonzero element $\xi \in \mathfrak{z}_0$ and the map π is defined by

$$\pi(h)(p(x)) = \int_C h(xz)\,\mathrm{d}z \ (\forall h \in \mathcal{S}(N/U)), \forall x \in N/U.$$

Then, it induces the following exact sequences:

$$0 \longrightarrow \mathcal{S}(N/V)' \xrightarrow{\pi^*} \mathcal{S}(N/U)' \xrightarrow{\xi^*} \mathcal{S}(N/U)' \longrightarrow 0$$

and

$$0 \longrightarrow R(N/V)\delta_{N/V} \xrightarrow{\pi^*} R(N/U)\delta_{N/U} \xrightarrow{\xi^*} R(N/U)\delta_{N/U} \longrightarrow 0\ .$$

Note that for each $f \in R(N/V)$ and each $h \in \mathcal{S}(N/U)$, we have

$$\begin{aligned}
&\langle \pi^*(f\delta_{N/V}), h\rangle \\
&= \langle f\delta_{N/V}, \pi(h)\rangle \\
&= \int_{N/V} f(y)\pi(h)(y)\,\mathrm{d}y \\
&= \int_{N/V} f(y) \int_{V/U} h(yz)\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_{N/U} \tilde{f}(x)h(x)\,\mathrm{d}x \\
&= \langle \tilde{f}\delta_{N/U}, h\rangle,
\end{aligned}$$

where $\tilde{f} \in R(N/U)$ is given by $\tilde{f}(x) = f(p(x))$ $(\forall x \in N/U)$. Then, $\pi^*(f\delta_{N/V}) = \tilde{f}\delta_{N/U}$. Consequently,

$$H^i(\mathfrak{c}, \mathcal{S}(N/U)') = \begin{cases} \mathcal{S}(N/V)' & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

and

$$H^i(\mathfrak{c}, R(N/U)\delta_{N/U}) = \begin{cases} R(N/V)\delta_{N/V} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$

□

Lemma 4.2. *Let D be an abelian real linear group. Then we have*

$$H^j(\mathfrak{d}, \mathcal{S}(D)') = \begin{cases} \mathbb{C}\delta_D & \text{for } j = 0 \\ 0 & \text{for } j \neq 0 \end{cases} \tag{25}$$

and

$$H^j(\mathfrak{d}, R(D)\delta_D) = \begin{cases} \mathbb{C}\delta_D & \text{for } j = 0 \\ 0 & \text{for } j \neq 0. \end{cases} \tag{26}$$

Proof. We show (26). The equality (25) can be proved in the same way. Prove by induction on $\dim D$. When $\dim D = 0$, (26) is trivial. Suppose (26) holds whenever $\dim D < k$. Now let $\dim D = k \geq 1$. Choose a one-dimensional closed linear subgroup Z of D. By (24) the Hochschild-Serre spectral sequence

$$H^p(\mathfrak{d}/\mathfrak{z}, H^q(\mathfrak{z}, R(D)\delta_D)) \Longrightarrow H^{p+q}(\mathfrak{d}, R(D)\delta_D)$$

degenerates and

$$H^p(\mathfrak{d}, R(D)\delta_D) = H^p(\mathfrak{d}/\mathfrak{z}, R(D/Z)\delta_{D/Z}).$$

Then, the conclusion follows from the induction hypothesis. □

Proposition 4.3. *Let U be a closed linear subgroup N. Then the inclusion $R(N/U)\delta_{N/U} \subset \mathcal{S}(N/U)'$ induces isomorphisms*

$$H^i(\mathfrak{n}, R(N/U)\delta_{N/U}) = H^i(\mathfrak{n}, \mathcal{S}(N/U)'),\ \forall i \geq 0.$$

Proof. We prove the proposition by induction on $\dim N$.

Case I (N is abelian). In this case, $N = U \times D$ for some complementary abelian Lie subgroup D. The subgroup U acts on $\mathcal{S}(N/U)'$ and $R(N/U)\delta_{N/U}$ trivially. Then, the differentials in the Koszul complexes that compute $H^*(\mathfrak{u}, \mathcal{S}(N/U)')$ and $H^*(\mathfrak{u}, \mathcal{S}(N/U)')$ are all 0. Thus,

$$H^j(\mathfrak{u}, \mathcal{S}(N/U)') = \bigwedge^j \mathfrak{u}^* \otimes \mathcal{S}(N/U)',\ j \in \mathbb{Z} \tag{27}$$

and

$$H^j(\mathfrak{u}, R(N/U)\delta_{N/U}) = \bigwedge^j \mathfrak{u}^* \otimes R(N/U)\delta_{N/U},\ j \in \mathbb{Z}. \tag{28}$$

By (25) and (26), $H^p(\mathfrak{n}/\mathfrak{u}, H^q(\mathfrak{u}, \cdot)) \neq 0$ only when $p = 0$ while $\cdot = \mathcal{S}(N/U)'$ or $R(N/U)\delta_{N/U}$. Then, both Hochschild-Serre spectral sequences

$$H^p(\mathfrak{n}/\mathfrak{u}, H^q(\mathfrak{u}, \cdot)) \Rightarrow H^{p+q}(\mathfrak{n}, \cdot)$$

for $\cdot = \mathcal{S}(N/U)'$ and $R(N/U)\delta_{N/U}$ degenerate. Thus, we have

$$H^i(\mathfrak{n}, \mathcal{S}(N/U)') = H^0(\mathfrak{n}/\mathfrak{u}, \bigwedge^i \mathfrak{u}^* \otimes \mathcal{S}(N/U)') = \bigwedge^i \mathfrak{u}^*,\ \forall i \geq 0$$

and

$$H^i(\mathfrak{n}, R(N/U)\delta_{N/U}) = H^0(\mathfrak{n}/\mathfrak{u}, \bigwedge^i \mathfrak{u}^* \otimes R(N/U)\delta'_{N/U}) = \bigwedge^i \mathfrak{u}^*,\ \forall i \geq 0.$$

Hence, we get isomorphisms $H^i(\mathfrak{n}, R(N/U)\delta_{N/U}) = H^i(\mathfrak{n}, \mathcal{S}(N/U)')$ $(i \geq 0)$.

Case II (N is non-abelian). Choose a one-dimensional closed linear central subgroup C of N. First, assume that $C \subset U$. Put $N' = N/C$ and $U' = U/C$. Then, $N/U = N'/U'$. Note that the action of C on $R(N/U)\delta_{N/U}$ is trivial. Hence,

$$H^i(\mathfrak{c}, \mathcal{S}(N/U)') = \begin{cases} \mathcal{S}(N/U)' & i = 0, 1 \\ 0 & i \neq 0, 1 \end{cases} \tag{29}$$

and

$$H^i(\mathfrak{c}, R(N/U)\delta_{N/U}) = \begin{cases} R(N/U)\delta_{N/U} & \text{for } i = 0, 1 \\ 0 & \text{for } i \neq 0, 1. \end{cases} \tag{30}$$

Then, the Hochschild-Serre spectral sequences

$$H^p(\mathfrak{n}', H^q(\mathfrak{c}, R(N/U)\delta_{N/U})) \Longrightarrow H^{p+q}(\mathfrak{n}, R(N/U)\delta_{N/U})$$

and

$$H^p(\mathfrak{n}', H^q(\mathfrak{c}, \mathcal{S}(N/U)')) \Longrightarrow H^{p+q}(\mathfrak{n}, \mathcal{S}(N/U)')$$

degenerate into short exact sequences and we are led to the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^i(\mathfrak{n}', R(N'/U')\delta_{N'/U'}) \arrow[r] \arrow[d] & H^i(\mathfrak{n}, R(N/U)\delta_{N/U}) \arrow[d] \\
0 \arrow[r] & H^i(\mathfrak{n}', \mathcal{S}(N'/U')') \arrow[r] & H^i(\mathfrak{n}, \mathcal{S}(N/U)')
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^{i-1}(\mathfrak{n}', R(N'/U')\delta_{N'/U'}) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & H^{i-1}(\mathfrak{n}', \mathcal{S}(N'/U')) \arrow[r] & 0.
\end{tikzcd}$$

By induction hypothesis, the first and the third vertical arrows in the above commutative diagram are isomorphisms. By the five lemma, we get isomorphisms $H^i(\mathfrak{n}, R(N/U)\delta_{N/U}) = H^i(\mathfrak{n}, \mathcal{S}(N/U)')$ ($\forall i \geq 0$).

Second, assume that $C \not\subseteq U$. Put $N' = N/C$ and $U' = UC/C$. Let $V = UC$. By (23) and (24), the Hochschild-Serre spectral sequences

$$H^p(\mathfrak{n}/\mathfrak{c}, H^q(\mathfrak{c}, *)) \Rightarrow H^{p+q}(\mathfrak{n}, *)$$

for $* = \mathcal{S}(N/U)'$ and $R(N/U)\delta_{N/U}$ degenerate. Then, we have

$$H^i(\mathfrak{n}, \mathcal{S}(N/U)') = H^i(\mathfrak{n}', \mathcal{S}(N'/U')'), \ \forall i \geq 0$$

and

$$H^i(\mathfrak{n}, R(N/U)\delta_{N/U}) = H^i(\mathfrak{n}', R(N'/U')\delta_{N'/U'}), \ \forall i \geq 0.$$

Since $\dim N' = \dim N - 1 < \dim N$, by induction hypothesis we get isomorphisms $H^i(\mathfrak{n}, R(N/U)\delta_{N/U}) = H^i(\mathfrak{n}, \mathcal{S}(N/U)')$ ($\forall i \geq 0$). $\square$

5. Casselman's comparison theorem

5.1. Comparison theorems.

Theorem 5.1. *Let (σ, V_σ) be a finite-dimensional complex linear algebraic representation of P. Then the inclusion $I(\sigma)' \to I(\sigma)^*_K$ induces isomorphisms*

$$H^i(\mathfrak{n}, I(\sigma)') = H^i(\mathfrak{n}, I(\sigma)^*_K), \ \forall i \geq 0.$$

Proof. By [10, Lemma 2.37], we have

$$H^i(\mathfrak{n}, I(\sigma)^*_K) = H^i(\mathfrak{n}, (I(\sigma)^*_K)^{[\mathfrak{n}]}), \ \forall i \geq 0. \tag{31}$$

By Casselman's automatic continuity theorem ([7, p. 416],[2, Theorem 11.4], [23, p.77]), we have

$$I(\sigma)'^{[\mathfrak{n}]} = (I(\sigma)^*_K)^{[\mathfrak{n}]}. \tag{32}$$

Then, it reduces to show the following assertion: the inclusion $I(\sigma)'^{[\mathfrak{n}]} \subset I(\sigma)$ induces isomorphisms

$$H^i(\mathfrak{n}, I(\sigma)'^{[\mathfrak{n}]}) = H^i(\mathfrak{n}, I(\sigma)'), \ \forall i \geq 0. \tag{33}$$

First, suppose that the action of N on V_σ is trivial. Since $I(\sigma) = C^\infty(E(\sigma))$, then $I(\sigma)' = C^\infty(E(\sigma))'$ and $I(\sigma)'^{[\mathfrak{n}]} = C^\infty(E(\sigma))'^{[\mathfrak{n}]}$. Take the Bruhat filtration of $C^\infty(E(\sigma))'$. By Lemma 3.9, it suffices to prove that: for each $k \in \mathbb{Z}$, the inclusion $(I_k/I_{k-1})^{[\mathfrak{n}]} \subset I_k/I_{k-1}$ induces isomorphisms for cohomology on all degrees. Since $I_k/I_{k-1} = \mathcal{T}(C(w_k), E(\sigma))$, it suffices to prove the cohomological isomorphism for the inclusions $\mathcal{T}(C(w), E(\sigma))^{[\mathfrak{n}]} \subset \mathcal{T}(C(w), E(\sigma))$ $(w \in W)$. Consider the filtration through transversal degree. Since Lie algebra cohomology is compatible with direct limit, it suffices to show the cohomological isomorphism for each inclusion $(F_p\mathcal{T}(C(w), E))^{[\mathfrak{n}]} \subset F_p\mathcal{T}(C(w), E)$ $(p \in \mathbb{Z}_{\geq 0})$. By Lemma 3.8 and taking induction on the degree p, it suffices to show that the inclusions $(\mathrm{Gr}^p\,\mathcal{T}(C(w), E))^{[\mathfrak{n}]} \subset \mathrm{Gr}^p\,\mathcal{T}(C(w), E)$ induce isomorphisms for cohomology on all degrees. By Corollary 2.7, it suffices to show so for the inclusions $\mathcal{S}(C(w), E(\sigma))'^{[\mathfrak{n}]} \subset \mathcal{S}(C(w), E(\sigma))'$ $(w \in W)$, which follows from Lemma 3.4 and Proposition 4.3.

In general, take a composition series of V_σ as a P representation:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = V_\sigma.$$

Then, the action of N on each graded piece V_j/V_{j-1} $(1 \leq j \leq k)$ is trivial. Thus, we have isomorphisms

$$H^i(\mathfrak{n}, I(V_j/V_{j-1})'^{[\mathfrak{n}]}) = H^i(\mathfrak{n}, I(V_j/V_{j-1})'), \ \forall i \in \mathbb{Z}.$$

By Lemma 3.10, we have short exact sequences

$$0 \to I(V_{j-1})'^{[\mathfrak{n}]} \to I(V_j)'^{[\mathfrak{n}]} \to I(V_j/V_{j-1})'^{[\mathfrak{n}]} \to 0.$$

By the cohomological long exact sequences associated to short exact sequences

$$0 \to I(V_{j-1})' \to I(V_j)' \to I(V_j/V_{j-1})' \to 0$$

and

$$0 \to I(V_{j-1})'^{[\mathfrak{n}]} \to I(V_j)'^{[\mathfrak{n}]} \to I(V_j/V_{j-1})'^{[\mathfrak{n}]} \to 0$$

and taking induction on j, one shows that each inclusion $I(V_j)'^{[\mathfrak{n}]} \to I(V_j)'$ induces isomorphisms

$$H^i(\mathfrak{n}, I(V_j)'^{[\mathfrak{n}]}) = H^i(\mathfrak{n}, I(V_j)'), \ \forall i \in \mathbb{Z}.$$

Taking $j = k$, we get the conclusion of the theorem. $\square$

Theorem 5.2. *Let V be an admissible finitely generated moderate growth smooth Fréchet representation of G. Then the inclusion $V_K \subset V$ induces isomorphisms*

$$H_i(\mathfrak{n}, V_K) = H_i(\mathfrak{n}, V),\ \forall i \in \mathbb{Z} \tag{34}$$

and the inclusion $V' \to V_K^*$ induces isomorphisms

$$H^i(\mathfrak{n}, V') = H^i(\mathfrak{n}, V_K^*),\ \forall i \in \mathbb{Z}. \tag{35}$$

Proof. First, we show the homological comparison theorem (34) and the cohomological comparison theorem (35) are equivalent. Suppose we have the cohomological comparison theorem. Then, $H^i(\mathfrak{n}, V)$ are all finite-dimensional as $H^i(\mathfrak{n}, V_K)$ are ([9, Corollary 2.4]). By a duality argument in [8, Lemma 5.11], we get the homological comparison theorem. The proof for the converse direction is similar. We show the homological comparison theorem (34) below.

Second, when $V = I(\sigma)$ is a principal series with σ a finite-dimensional complex linear algebraic representation of P, (35) is shown in Theorem 5.1. By the duality argument in [8, Lemma 5.11], we get (34) for $V = I(\sigma)$.

Third, by [13, Proposition 3], one can reduce the homological comparison theorem for a general representation $V \in \mathcal{HF}_{\mathrm{mod}}(G)$ to the principal series case. For readers' convenience, we recall Hecht-Taylor's proof in [13]. By Casselman's subrepresentation theorem there is a finite-dimensional complex linear algebraic representation σ_0 of P such that there is an injection $V \hookrightarrow I(\sigma_0)$ ([22, Proposition 4.2.3]). Let V_1 be the cokernel, which is still in the category $\mathcal{HF}_{\mathrm{mod}}(G)$. By Casselman's subrepresentation theorem again, there is a finite-dimensional complex linear algebraic representation σ_1 of P such that there is an injection $V_1 \hookrightarrow I(\sigma_1)$. Continuing in this way, we obtain a resolution of V by principal series:

$$V \to I(\sigma_0) \to I(\sigma_1) \to \cdots .$$

This double complex and the corresponding $(\mathfrak{g}, K)$-module double complex have bounded p parameter. Thus, the spectral sequences associated to two standard filtrations are convergent. The natural map of double complexes

$$\wedge^p \mathfrak{n} \otimes I(\sigma_q)_K \to \wedge^p \mathfrak{n} \otimes I(\sigma_q)$$

can be analyzed by means of the convergent spectral sequences associated to the two standard filtrations. The spectral sequence corresponding to the second filtration degenerates at $E_2 : E_2^{q,-p}$-terms are zero except when $q = 0$, and the resulting map on the $(-p)$-cohomology of the total complex is nothing but

$$H_p(\mathfrak{n}, V_K) \to H_p(\mathfrak{n}, V). \tag{36}$$

On the other hand, the map between the $E_1^{-p,q}$ terms of the spectral sequence associated to the first filtration is

$$H_p(\mathfrak{n}, I(\sigma_q)_K) \to H_p(\mathfrak{n}, I(\sigma_q)). \tag{37}$$

As shown above, maps in (37) are isomorphisms. Then, the maps in (36) are isomorphisms. □

5.2. **Closedness.** Theorem 5.2 has the following immediate consequence.

Corollary 5.3. *Let V be an admissible finitely generated moderate growth smooth Fréchet representation of G. For the Koszul complex associated to V:*

$$0 \longrightarrow \wedge^d \mathfrak{n} \otimes_{\mathbb{C}} V \xrightarrow{\partial_d} \cdots \xrightarrow{\partial_2} \mathfrak{n} \otimes_{\mathbb{C}} V \xrightarrow{\partial_1} V \longrightarrow 0\ ,$$

where $d = \dim \mathfrak{n}$, each $\mathrm{Im}(\partial_i)$ is an NF space and it is a closed subspace of $\wedge^{i-1}\mathfrak{n} \otimes_{\mathbb{C}} V$ $(1 \leq i \leq d)$. Similarly, we have DNF property and closedness for images of boundary operators in the Koszul complex defining $\mathfrak{n}$-cohomology of V'.

Proof. By Theorem 5.2, $\mathrm{Im}(\partial_i)$ is a finite co-dimensional subspace of $\ker(\partial_{i-1})$. Since ∂_{i-1} is a linear continuous map, then $\ker(\partial_{i-1})$ is a closed subspace of the NF space $\wedge^{i-1}\mathfrak{n} \otimes_{\mathbb{C}} V$. Thus, $\ker(\partial_{i-1})$ is also an NF space. By Theorem 5.1, the linear continuous map $\partial_i : \wedge^i \mathfrak{n} \otimes_{\mathbb{C}} V \to \ker(\partial_{i-1})$ has finite co-dimensional image for each i. By [8, Lemma A.1], it follows that $\mathrm{Im}(\partial_i)$ is an NF space. Hence, it is a closed subspace of $\wedge^{i-1}\mathfrak{n} \otimes_{\mathbb{C}} V$. The proof for the DNF property and closedness of images of boundary operators in the Koszul complex defining $\mathfrak{n}$-cohomology of V' is similar. □

Let V be an admissible finitely generated moderate growth smooth Fréchet representation of G. Recall that Casselman's automatic continuity theorem says that

$$V_K/\mathfrak{n}^k V_K = V/\overline{\mathfrak{n}^k V},\ \forall k \geq 0.$$

In Theorem 5.2, we show that each $\mathfrak{n}^k V$ is a closed subspace of V. Hence, $V_K/\mathfrak{n}^k V_K = V/\mathfrak{n}^k V$.

Theorem 5.4. *Let V be an admissible finitely generated moderate growth smooth Fréchet representation of G. Then for every $k \geq 0$, $\mathfrak{n}^k V$ is a closed subspace of V and the inclusion $V_K \subset V$ induces an isomorphism*

$$V_K/\mathfrak{n}^k V_K = V/\mathfrak{n}^k V.$$

Proof. By the homological comparison theorem in Theorem 5.2, $\mathfrak{n}V$ is a finite co-dimensional subspace of V. Thus, there is a finite-dimension subspace U of V such that $V = \mathfrak{n}V + U$. Hence, for each $k \geq 0$,

$$V = \mathfrak{n}^k V + \Big(\sum_{0 \leq j \leq k-1} \mathfrak{n}^j U\Big).$$

By this, $\mathfrak{n}^k V$ is a finite co-dimensional subspace of V. Then, $\mathfrak{n}^k \otimes V \to V$ is a linear continuous map having a finite co-dimensional image. By [8, Lemma A.1], it follows that $\mathfrak{n}^k V$ is a closed subspace of V. By Casselman's automatic continuity theorem we have $V_K/\mathfrak{n}^k V_K = V/\overline{\mathfrak{n}^k V}$. Then, it follows that $V_K/\mathfrak{n}^k V_K = V/\mathfrak{n}^k V$. □

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References

[1] Noriyuki Abe, *Generalized Jacquet modules of parabolically induced representations*, Publ. Res. Inst. Math. Sci. **48** (2012), no. 2, 419–473, DOI 10.2977/PRIMS/75. MR2928148

[2] Joseph Bernstein and Bernhard Krötz, *Smooth Fréchet globalizations of Harish-Chandra modules*, Israel J. Math. **199** (2014), no. 1, 45–111, DOI 10.1007/s11856-013-0056-1. MR3219530

[3] Edward Bierstone and Pierre D. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. **67** (1988), 5–42. MR972342

[4] Tim Bratten, *A comparison theorem for Lie algebra homology groups*, Pacific J. Math. **182** (1998), no. 1, 23–36, DOI 10.2140/pjm.1998.182.23. MR1610614

[5] Ulrich Bunke and Martin Olbrich, *Cohomological properties of the canonical globalizations of Harish-Chandra modules. Consequences of theorems of Kashiwara-Schmid, Casselman, and Schneider-Stuhler*, Ann. Global Anal. Geom. **15** (1997), no. 5, 401–418, DOI 10.1023/A:1006538211648. MR1482895

[6] W. Casselman, *Jacquet modules for real reductive groups*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, pp. 557–563. MR562655

[7] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Canad. J. Math. **41** (1989), no. 3, 385–438, DOI 10.4153/CJM-1989-019-5. MR1013462

[8] William Casselman, Henryk Hecht, and Dragan Miličić, *Bruhat filtrations and Whittaker vectors for real groups*, The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), Proc. Sympos. Pure Math., vol. 68, Amer. Math. Soc., Providence, RI, 2000, pp. 151–190, DOI 10.1090/pspum/068/1767896. MR1767896

[9] William Casselman and M. Scott Osborne, *The restriction of admissible representations to* $\mathfrak{n}$, Math. Ann. **233** (1978), no. 3, 193–198, DOI 10.1007/BF01405350. MR480884

[10] Henryk Hecht and Wilfried Schmid, *Characters, asymptotics and* $\mathfrak{n}$*-homology of Harish-Chandra modules*, Acta Math. **151** (1983), no. 1-2, 49–151, DOI 10.1007/BF02393204. MR716371

[11] Henryk Hecht and Joseph L. Taylor, *Analytic localization of group representations*, Adv. Math. **79** (1990), no. 2, 139–212, DOI 10.1016/0001-8708(90)90062-R. MR1033077

[12] Henryk Hecht and Joseph L. Taylor, *A comparison theorem for* $\mathfrak{n}$*-homology*, Compositio Math. **86** (1993), no. 2, 189–207. MR1214457

[13] Henryk Hecht and Joseph L. Taylor, *A remark on Casselman's comparison theorem*, Geometry and representation theory of real and p-adic groups (Córdoba, 1995), Progr. Math., vol. 158, Birkhäuser Boston, Boston, MA, 1998, pp. 139–146. MR1486138

[14] Masaki Kashiwara and Wilfried Schmid, *Quasi-equivariant $\mathcal{D}$-modules, equivariant derived category, and representations of reductive Lie groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 457–488, DOI 10.1007/978-1-4612-0261-5_16. MR1327544

[15] Anthony W. Knapp, *Lie groups beyond an introduction*, 2nd ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002. MR1920389

[16] Anthony W. Knapp, *Representation theory of semisimple groups*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001. An overview based on examples; Reprint of the 1986 original. MR1880691

[17] Helmut H. Schaefer, *Topological vector spaces*, The Macmillan Company, New York; Collier-Macmillan Ltd., London, 1966. MR0193469

[18] Wilfried Schmid, *Boundary value problems for group invariant differential equations*, Astérisque **Numéro Hors Série** (1985), 311–321. The mathematical heritage of Élie Cartan (Lyon, 1984). MR837206

[19] Wilfried Schmid and Joseph A. Wolf, *Geometric quantization and derived functor modules for semisimple Lie groups*, J. Funct. Anal. **90** (1990), no. 1, 48–112, DOI 10.1016/0022-1236(90)90080-5. MR1047577

[20] Laurent Schwartz, *Théorie des distributions* (French), Publications de l'Institut de Mathématique de l'Université de Strasbourg, IX-X, Hermann, Paris, 1966. Nouvelle édition, entiérement corrigée, refondue et augmentée. MR0209834

[21] David A. Vogan Jr., *Unitary representations and complex analysis*, Representation theory and complex analysis, Lecture Notes in Math., vol. 1931, Springer, Berlin, 2008, pp. 259–344, DOI 10.1007/978-3-540-76892-0_5. MR2409701

[22] Nolan R. Wallach, *Real reductive groups. I*, Pure and Applied Mathematics, vol. 132, Academic Press, Inc., Boston, MA, 1988. MR929683

[23] Nolan R. Wallach, *Real reductive groups. II*, Pure and Applied Mathematics, vol. 132, Academic Press, Inc., Boston, MA, 1992. MR1170566

[24] N. Wallach, *Fréchet completions of moderate growth old and (somewhat) new results*, `arXiv:1403.3658`, 2014.

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