

AFFINE HIGHEST WEIGHT CATEGORIES AND QUANTUM AFFINE SCHUR-WEYL DUALITY OF DYNKIN QUIVER TYPES

RYO FUJITA

ABSTRACT. For a Dynkin quiver Q (of type ADE), we consider a central completion of the convolution algebra of the equivariant K -group of a certain Steinberg type graded quiver variety. We observe that it is affine quasi-hereditary and prove that its category of finite-dimensional modules is identified with a block of Hernandez-Leclerc's monoidal category \mathcal{C}_Q of modules over the quantum loop algebra $U_q(L\mathfrak{g})$ via Nakajima's homomorphism. As an application, we show that Kang-Kashiwara-Kim's generalized quantum affine Schur-Weyl duality functor gives an equivalence between the category of finite-dimensional modules over the quiver Hecke algebra associated with Q and Hernandez-Leclerc's category \mathcal{C}_Q , assuming the simpleness of some poles of normalized R -matrices for type E.

INTRODUCTION

0.1. For a Dynkin quiver Q (of type ADE), one can associate the following two interesting monoidal categories \mathcal{C}_Q and \mathcal{M}_Q .

The first one \mathcal{C}_Q is a certain monoidal subcategory of the category of finite-dimensional modules over the quantum loop algebra $U_q(L\mathfrak{g})$, where \mathfrak{g} is the complex simple Lie algebra whose Dynkin diagram is the underlying graph of the quiver Q . This category was introduced by Hernandez-Leclerc [21]. The definition of the category \mathcal{C}_Q involves the Auslander-Reiten quiver of the path algebra of Q . The complexified Grothendieck ring of \mathcal{C}_Q is known to be isomorphic to the coordinate algebra $\mathbb{C}[N]$ of the unipotent group N associated with the positive part of \mathfrak{g} , under which the classes of simple modules correspond to the dual canonical basis elements. Note that we have a decomposition $\mathcal{C}_Q = \bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{C}_{Q,\beta}$, which corresponds to the weight decomposition $\mathbb{C}[N] = \bigoplus_{\beta \in \mathbb{Q}^+} \mathbb{C}[N]_{\beta}$.

The second one \mathcal{M}_Q is the direct sum of the categories $\mathcal{M}_{Q,\beta} = H_Q(\beta)\text{-mod}_{\text{fd}}^0$ of finite-dimensional modules over the quiver Hecke algebra $H_Q(\beta)$ on which the center acts nilpotently. The monoidal structure of \mathcal{M}_Q is given by an analog of parabolic inductions. The quiver Hecke algebra (which is also known as the Khovanov-Lauda-Rouquier algebra) was introduced by Khovanov-Lauda [29] and by Rouquier [41] as an algebraic object which generalizes the affine Hecke algebra of type A in the sense that it gives a categorification of the dual of the integral form $U_q(\mathfrak{g})_{\mathbb{Z}}^+$ of the

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positive half of the quantized enveloping algebra $U_q(\mathfrak{g})$. More precisely, the quiver Hecke algebra $H_Q(\beta)$ is equipped with a \mathbb{Z} -grading and hence the direct sum of the Grothendieck groups of the categories of finite-dimensional graded modules over $H_Q(\beta)$ for various β becomes a $\mathbb{Z}[q^{\pm 1}]$ -algebra, where q corresponds to the grading shift. It is isomorphic to the dual of $U_q(\mathfrak{g})_{\mathbb{Z}}^+$ with the classes of self-dual simple modules corresponding to the dual canonical basis elements. Our category \mathcal{M}_Q is obtained by forgetting the gradings, which corresponds to specializing q to 1 at the level of the Grothendieck ring. Therefore the complexified Grothendieck ring of the monoidal category \mathcal{M}_Q is also isomorphic to $\mathbb{C}[N]$.

Thus, we encounter a natural question, originally asked by Hernandez-Leclerc [21], whether there is any functorial relationship between these two monoidal categories \mathcal{C}_Q and \mathcal{M}_Q . Kang-Kashiwara-Kim [22] gave an elegant answer to this question by constructing the *generalized quantum affine Schur-Weyl duality functor* $\mathcal{F}_Q: \mathcal{M}_Q \rightarrow \mathcal{C}_Q$ for any quiver Q of type AD. The functor \mathcal{F}_Q is a direct sum $\bigoplus_{\beta \in Q^+} \mathcal{F}_{Q,\beta}$ of functors $\mathcal{F}_{Q,\beta}: \mathcal{M}_{Q,\beta} \rightarrow \mathcal{C}_{Q,\beta}$, where $\mathcal{F}_{Q,\beta}$ is given by a certain $(U_q(L\mathfrak{g}), H_Q(\beta))$ -bimodule constructed by using the normalized R -matrices for ℓ -fundamental modules of $U_q(L\mathfrak{g})$. Here the ℓ -fundamental modules are quantum loop analogs of the fundamental modules of \mathfrak{g} , and the normalized R -matrices are certain intertwining operators between tensor product modules. A normalized R -matrix can be seen as a matrix-valued rational function, whose singularity is strongly related to the structure of tensor product modules. Kang-Kashiwara-Kim's construction also works for type E if we assume the simpleness of some specific poles of normalized R -matrices.¹ Moreover, under this assumption, it was proved in [22] that the functor \mathcal{F}_Q is an exact monoidal functor which induces an isomorphism between the Grothendieck rings for any quiver Q of type ADE.

When the quiver Q is an equioriented quiver of type A, Kang-Kashiwara-Kim's construction coincides with that of the usual quantum affine Schur-Weyl duality between $U_q(L\mathfrak{sl}_n)$ and the affine Hecke algebra of type A. In particular, the functor \mathcal{F}_Q gives an equivalence of categories in this special case thanks to Chari-Pressley [9]. Thus it is natural to expect that the functor \mathcal{F}_Q also gives an equivalence of categories for general Q of type ADE. The goal of this paper is to verify this expectation.

0.2. We briefly explain our strategy. An isomorphism between Grothendieck rings does not imply an actual equivalence of categories because by passing to Grothendieck rings we forget some homological information such as extensions among modules. Note that, in our case, both categories \mathcal{C}_Q and \mathcal{M}_Q are far from semi-simple. Therefore, in order to verify that the functor \mathcal{F}_Q gives an equivalence, one should ask whether it respects the homological properties. However, the categories \mathcal{C}_Q and \mathcal{M}_Q are not suitable for a homological study because they have no projective modules.² Thus we want to discuss some suitable “completions” $\widehat{\mathcal{C}}_Q$ and $\widehat{\mathcal{M}}_Q$ of categories \mathcal{C}_Q and \mathcal{M}_Q respectively to get enough projective modules.

For the category \mathcal{M}_Q , the desired completion is easily obtained. Namely, we just take the central completion $\widehat{H}_Q(\beta)$ of the quiver Hecke algebra along the trivial

¹After the initial submission of this paper, it was also proved that this assumption is always satisfied for type E by Oh-Scrimshaw [40] and by the author [18] independently.

²In fact, for any simple module L in \mathcal{C}_Q (or \mathcal{M}_Q), we can construct a module of arbitrary length with a simple head $\simeq L$ as a finite-dimensional quotient of the affinization of L (see Remark 2.14), which implies that we do not have a projective cover of L in the category \mathcal{C}_Q (nor \mathcal{M}_Q).

central character and define the category $\widehat{\mathcal{M}}_{Q,\beta}$ as the category of finitely generated $\widehat{H}_Q(\beta)$ -modules. The homological properties of the category $\widehat{\mathcal{M}}_{Q,\beta}$ are well-understood, due to Kato [26] and Brundan-Kleshchev-McNamara [5]. More precisely, the category $\widehat{\mathcal{M}}_{Q,\beta}$ has a structure of *affine highest weight category*, which is equivalent to saying that the algebra $\widehat{H}_Q(\beta)$ is an *affine quasi-hereditary algebra*. The notion of affine highest weight category was axiomatized by Kleshchev [30] as a generalization of the notion of highest weight category introduced by Cline-Parshall-Scott [12]. In particular, an affine highest weight category has *standard modules*, which filter projective modules.

On the other hand, it is not obvious how to define a suitable completion $\widehat{\mathcal{C}}_{Q,\beta}$ because a priori we have no information about the center of the category $\mathcal{C}_{Q,\beta}$. To remedy this situation, we rely on Nakajima’s construction [35] of $U_q(\mathfrak{Lg})$ -modules using convolution algebras of equivariant K -groups of quiver varieties. For us, an advantage of this kind of convolution construction is that the representation ring of the group of equivariance automatically gives a large central subalgebra of the convolution algebra. For a Dynkin quiver Q and an element $\beta \in \mathbb{Q}^+$, we have the corresponding graded quiver varieties $\mathfrak{M}_\beta^\bullet, \mathfrak{M}_{0,\beta}^\bullet$ with actions of a linear algebraic group $\mathbb{G}(\beta)$ determined by β . We remark that our $\mathfrak{M}_{0,\beta}^\bullet$ is the same graded quiver variety as Hernandez-Leclerc studied in [21], where it was proved that $\mathfrak{M}_{0,\beta}^\bullet$ is isomorphic to the space of representations of the quiver Q of dimension vector β . From the canonical projective $\mathbb{G}(\beta)$ -equivariant morphism $\mathfrak{M}_\beta^\bullet \rightarrow \mathfrak{M}_{0,\beta}^\bullet$, we form the corresponding Steinberg type variety $Z_\beta^\bullet = \mathfrak{M}_\beta^\bullet \times_{\mathfrak{M}_{0,\beta}^\bullet} \mathfrak{M}_\beta^\bullet$. Thanks to the construction by Nakajima [35], we obtain an algebra homomorphism from $U_q(\mathfrak{Lg})$ to a suitable central completion $\widehat{\mathcal{K}}^{\mathbb{G}(\beta)}(Z_\beta^\bullet)$ of the $\mathbb{G}(\beta)$ -equivariant K -group with the convolution product. Then we define our “completion” $\widehat{\mathcal{C}}_{Q,\beta}$ of Hernandez-Leclerc’s category $\mathcal{C}_{Q,\beta}$ as the category of finitely generated $\widehat{\mathcal{K}}^{\mathbb{G}(\beta)}(Z_\beta^\bullet)$ -modules. We prove that this category $\widehat{\mathcal{C}}_{Q,\beta}$ actually plays a role of a completion of $\mathcal{C}_{Q,\beta}$ and that it has a structure of affine highest weight category.

Theorem A (= Theorem 4.9). *The following hold.*

- (1) *The convolution algebra $\widehat{\mathcal{K}}^{\mathbb{G}(\beta)}(Z_\beta^\bullet)$ is an affine quasi-hereditary algebra and hence the category $\widehat{\mathcal{C}}_{Q,\beta}$ has a structure of affine highest weight category.*
- (2) *Via Nakajima’s homomorphism, standard modules in $\widehat{\mathcal{C}}_{Q,\beta}$ are identified with the deformed local Weyl modules (see Definition 2.12) and the full subcategory of all the finite-dimensional modules in $\widehat{\mathcal{C}}_{Q,\beta}$ is identified with the category $\mathcal{C}_{Q,\beta}$.*

Kang-Kashiwara-Kim’s functor $\mathcal{F}_{Q,\beta}$ is extended to an exact functor $\widehat{\mathcal{M}}_{Q,\beta} \rightarrow \widehat{\mathcal{C}}_{Q,\beta}$ between the two affine highest weight categories. There is a simple sufficient condition (= Theorem 1.7) obtained in [17] for an exact functor between two affine highest weight categories to give an equivalence of categories. We prove that the functor $\mathcal{F}_{Q,\beta}$ satisfies this sufficient condition. Restricting the equivalence $\mathcal{F}_{Q,\beta}$ to the subcategory $\mathcal{M}_{Q,\beta}$ of finite-dimensional modules and summing up over $\beta \in \mathbb{Q}^+$, we obtain the following result.

Theorem B (= Corollary 5.16). *Under a certain assumption about simpleness of poles of normalized R -matrices for type E , Kang-Kashiwara-Kim’s generalized*

quantum affine Schur-Weyl duality functor $\mathcal{F}_Q: \mathcal{M}_Q \rightarrow \mathcal{C}_Q$ gives an equivalence of monoidal categories.

0.3. Remark. Theorem A(1) can be obtained as a consequence of the theory of geometric extension algebras developed by Kato [27] and by McNamara [32], although in this paper we give an alternative proof which does not use geometric extension algebras. In fact, our convolution algebra $\widehat{\mathcal{K}}^{\mathbb{G}(\beta)}(Z_\beta^\bullet)$ is isomorphic to the completion of the geometric extension algebra associated with $\mathfrak{M}_\beta^\bullet \rightarrow \mathfrak{M}_{0,\beta}^\bullet$, which satisfies some conditions presented in [27, 32]. Moreover, we can see that this geometric extension algebra is Morita equivalent to the quiver Hecke algebra $H_Q(\beta)$ thanks to the result of Varagnolo-Vasserot [42]. From this point of view, we already know an abstract equivalence of categories $\widehat{\mathcal{M}}_{Q,\beta} \simeq \widehat{\mathcal{C}}_{Q,\beta}$. See Remark 4.19 for more details. However, one should note that it is still not obvious that the equivalence is realized concretely by Kang-Kashiwara-Kim’s functor $\mathcal{F}_{Q,\beta}$. In addition, to prove the other statement (2) in Theorem A, it is necessary to compare the quantum loop algebra $U_q(L\mathfrak{g})$ and the convolution algebra $\widehat{\mathcal{K}}^{\mathbb{G}(\beta)}(Z_\beta^\bullet)$ via Nakajima’s homomorphism, which is neither injective nor surjective in general.

Organization. This paper is organized as follows. In Section 1, we recall the definition and some properties of affine highest weight categories and affine quasi-hereditary algebras. Section 2 is concerned with the representation theory of quantum loop algebras $U_q(L\mathfrak{g})$ of type ADE. In Section 3, we recall and prove some geometric properties of quiver varieties, which are needed in the sequel. Section 4 is the main part of this paper. After recalling Nakajima’s construction in Subsection 4.2, we prove Theorem A in Subsection 4.3. We study the Kang-Kashiwara-Kim functor in Section 5. Theorem B is proved in Subsection 5.3.

Convention. For an algebra A , the category of finitely generated left A -modules is denoted by $A\text{-mod}_{\text{fg}}$. If \mathbb{k} is a field and A is a \mathbb{k} -algebra, the category of finite-dimensional left A -modules is denoted by $A\text{-mod}_{\text{fd}}$. For a two-sided ideal $\mathfrak{a} \subset A$ and a left A -module M , the quotient $M/\mathfrak{a}M$ is denoted by M/\mathfrak{a} for simplicity. Working over a field \mathbb{k} , the symbol \otimes stands for $\otimes_{\mathbb{k}}$ if there is no other clarification. For $i = 1, 2$, let R_i be a complete local commutative \mathbb{k} -algebra with maximal ideal $\mathfrak{r}_i \subset R_i$ satisfying $R_i/\mathfrak{r}_i \cong \mathbb{k}$. For any R_i -module M_i ($i = 1, 2$), the symbol $M_1 \hat{\otimes} M_2$ denotes the completion of the $(R_1 \otimes R_2)$ -module $M_1 \otimes M_2$ with respect to the maximal ideal $\mathfrak{r}_1 \otimes R_2 + R_1 \otimes \mathfrak{r}_2$. Note that $M_1 \hat{\otimes} M_2$ is a module over the complete local algebra $R_1 \hat{\otimes} R_2$.

1. AFFINE HIGHEST WEIGHT CATEGORIES

In this section, we recall the definitions and some properties of (topologically complete) affine highest weight categories and affine quasi-hereditary algebras following Kleshchev [30].

Let A be a left Noetherian algebra over an algebraically closed field \mathbb{k} and $J \subset A$ the Jacobson radical of A . Throughout this section, we assume that $\dim(A/J) < \infty$ and A is complete with respect to the J -adic topology, i.e., $\varprojlim A/J^n \cong A$. Let $\mathcal{C} := A\text{-mod}_{\text{fg}}$ be the \mathbb{k} -linear abelian category of all finitely generated left A -modules. Our assumption guarantees that any simple module of \mathcal{C} is finite-dimensional and the number of isomorphism classes of simple modules in \mathcal{C} is finite. We parametrize the set $\text{lrr } \mathcal{C}$ of simple isomorphism classes in \mathcal{C} by a finite set Π as

$\text{lrr } \mathcal{C} = \{L(\pi) \in \mathcal{C} \mid \pi \in \Pi\}$. For each $\pi \in \Pi$, we fix a projective cover $P(\pi)$ of the simple module $L(\pi)$.

Definition 1.1. A two-sided ideal $I \subset A$ is said to be *affine heredity* if the following three conditions are satisfied:

- (1) We have $\text{Hom}_{\mathcal{C}}(I, A/I) = 0$;
- (2) As a left A -module, we have $I \cong P(\pi)^{\oplus m}$ for some $\pi \in \Pi$ and $m \in \mathbb{Z}_{>0}$;
- (3) The endomorphism \mathbb{k} -algebra $\text{End}_A(P(\pi))$ is isomorphic to a ring of formal power series $\mathbb{k}[[z_1, \dots, z_n]]$ for some $n \in \mathbb{Z}_{\geq 0}$, and $P(\pi)$ is free of finite rank over $\text{End}_A(P(\pi))$.

Definition 1.2. We say that the algebra A is *affine quasi-hereditary* if there is a chain of ideals:

$$(1.1) \quad 0 = I_l \subsetneq I_{l-1} \subsetneq \dots \subsetneq I_1 \subsetneq I_0 = A$$

such that, for each $i \in \{1, 2, \dots, l\}$, the ideal I_{i-1}/I_i is an affine heredity ideal of the algebra A/I_i . We refer to such a chain (1.1) as an *affine heredity chain*.

Let \leq be a partial order of Π .

Definition 1.3. The category $\mathcal{C} = A\text{-mod}_{\text{fg}}$ is called an *affine highest weight category* for the poset (Π, \leq) if, for each $\pi \in \Pi$, there exists an indecomposable module $\Delta(\pi)$ which is a non-zero quotient of $P(\pi)$ (i.e., $P(\pi) \twoheadrightarrow \Delta(\pi) \twoheadrightarrow L(\pi)$) satisfying the following three conditions:

- (1) The endomorphism \mathbb{k} -algebra $B_\pi := \text{End}_{\mathcal{C}}(\Delta(\pi))$ is isomorphic to a ring of formal power series $\mathbb{k}[[z_1, \dots, z_{n_\pi}]]$ for some $n_\pi \in \mathbb{Z}_{\geq 0}$, and $\Delta(\pi)$ is free of finite rank over B_π ;
- (2) Define $\bar{\Delta}(\pi) := \Delta(\pi)/\text{rad } B_\pi$, where $\text{rad } B_\pi$ denotes the maximal ideal of B_π . Then each composition factor of the kernel of the natural quotient map $\bar{\Delta}(\pi) \twoheadrightarrow L(\pi)$ is isomorphic to $L(\sigma)$ for some $\sigma < \pi$;
- (3) The kernel of natural quotient map $P(\pi) \twoheadrightarrow \Delta(\pi)$ is filtered by various $\Delta(\sigma)$'s with $\sigma > \pi$.

We refer to the module $\Delta(\pi)$ (resp. $\bar{\Delta}(\pi)$) as the *standard module* (resp. *proper standard module*) associated with $\pi \in \Pi$.

Theorem 1.4 is an analog of a famous result by Cline-Parshall-Scott [12].

Theorem 1.4 ([30, Theorem 6.7]). *The following are mutually equivalent.*

- (1) *The algebra A is an affine quasi-hereditary algebra;*
- (2) *There is a partial order \leq of Π such that the category $\mathcal{C} = A\text{-mod}_{\text{fg}}$ is an affine highest weight category for (Π, \leq) .*

Remark 1.5. Let A be an affine quasi-hereditary algebra with $\text{lrr } \mathcal{C} = \{L(\pi) \in \pi \in \Pi\}$. Then the partial order \leq of Π in Theorem 1.4 can be chosen as follows. The standard modules of the affine highest weight category \mathcal{C} are obtained as an indecomposable direct summand of subquotients I_{i-1}/I_i of an affine heredity chain (1.1). Thus an affine heredity chain (1.1) gives a total ordering $\{\pi_1, \pi_2, \dots, \pi_l\}$ of the parameter set Π by $I_{i-1}/I_i \cong \Delta(\pi_i)^{\oplus m_i}$. Using this notation, we define a partial order \leq on the set Π so that the following condition is satisfied:

- (*) For $\sigma, \tau \in \Pi$, we have $\sigma < \tau$ if and only if for any affine heredity chain we have $\sigma = \pi_i, \tau = \pi_j$ for some i, j with $1 \leq i < j \leq l$.

Then we can prove that the category \mathcal{C} is an affine highest weight category for this partial order \leq on Π .

Theorem 1.6 is the Ext-version of BGG type reciprocity.

Theorem 1.6 ([30, Lemmas 7.2 and 7.4]). *Let \mathcal{C} be an affine highest weight category for a poset (Π, \leq) . Then, for each $\pi \in \Pi$, there exists an indecomposable module $\bar{\nabla}(\pi) \in \mathcal{C}$, uniquely up to isomorphism, characterized by the following Ext-orthogonality:*

$$\text{Ext}_{\mathcal{C}}^i(\Delta(\sigma), \bar{\nabla}(\pi)) = \begin{cases} \mathbb{k} & i = 0, \sigma = \pi; \\ 0 & \text{else.} \end{cases}$$

We refer to the module $\bar{\nabla}(\pi)$ as the *proper costandard module* associated with $\pi \in \Pi$. The following criterion is proved by using a theory of tilting modules in affine highest weight categories.

Theorem 1.7 ([17, Theorem 3.9]). *For $i = 1, 2$, let $\mathcal{C}_i = A_i\text{-mod}_{\text{fg}}$ be an affine highest weight category for a poset (Π_i, \leq_i) . Assume that we have an exact functor $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and the following conditions are satisfied:*

- (1) *The algebra A_i is a finitely generated module over its center ($i = 1, 2$);*
- (2) *There exists a bijection $f: \Pi_1 \xrightarrow{\sim} \Pi_2$ preserving the partial orders and we have the following isomorphisms for each $\pi \in \Pi_1$:*

$$F(\Delta(\pi)) \cong \Delta(f(\pi)), \quad F(\bar{\nabla}(\pi)) \cong \bar{\nabla}(f(\pi)).$$

Then the functor F gives an equivalence of categories $F: \mathcal{C}_1 \simeq \mathcal{C}_2$.

2. REPRESENTATIONS OF QUANTUM LOOP ALGEBRAS

In this section, we recall and prove some basic facts on the representation theory of quantum loop algebras $U_q(\text{Lg})$ of type ADE.

2.1. Quivers and root systems. From now on, we fix a Dynkin quiver $Q = (I, \Omega)$ (of type ADE) with its set of vertices $I = \{1, 2, \dots, n\}$ and its set of arrows Ω . We write $i \sim j$ if $i, j \in I$ are adjacent in Q . We also fix a function $\xi: I \rightarrow \mathbb{Z}; i \mapsto \xi_i$ such that $\xi_j = \xi_i - 1$ if $i \rightarrow j \in \Omega$. Given a Dynkin quiver Q , such a function ξ is determined uniquely up to adding a constant and called a *height function* on Q .

Let \mathfrak{g} be the complex simple Lie algebra whose Dynkin diagram is the underlying graph of the quiver Q . The Cartan matrix $A = (a_{ij})_{i,j \in I}$ of \mathfrak{g} is given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \sim j; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{P}^\vee = \bigoplus_{i \in I} \mathbb{Z}h_i$ be the coroot lattice of \mathfrak{g} . The fundamental weights $\{\varpi_i \mid i \in I\}$ form a basis of the weight lattice $\mathbb{P} := \text{Hom}_{\mathbb{Z}}(\mathbb{P}^\vee, \mathbb{Z})$ which is dual to $\{h_i \mid i \in I\}$. Let $\mathbb{P}^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i$ be the set of dominant weights. The simple roots $\{\alpha_i \mid i \in I\}$ are defined by $\alpha_i := \sum_{j \in I} a_{ij} \varpi_j$. We define the root lattice by $\mathbb{Q} := \bigoplus_{i \in I} \mathbb{Z} \alpha_i \subset \mathbb{P}$ and put $\mathbb{Q}^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. We define a partial order \leq called the dominance order on \mathbb{P} by the condition that for $\lambda, \mu \in \mathbb{P}$, we have $\lambda \leq \mu$ if and only if $\mu - \lambda \in \mathbb{Q}^+$.

The Weyl group W of \mathfrak{g} is a group of linear transformations of \mathbb{P} generated by the set $\{s_i \mid i \in I\}$ of simple reflections, which are defined by $s_i(\lambda) := \lambda - \lambda(h_i) \alpha_i$ for $\lambda \in \mathbb{P}$. For an element $w \in W$, its length $l(w)$ is the smallest number $l \in \mathbb{Z}_{\geq 0}$

such that there is an expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$. We say that an expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ is reduced if $l = l(w)$. Let w_0 denote the unique longest element of W . Let $R := W\{\alpha_i \mid i \in I\}$ be the set of roots, which decomposes as $R = R^+ \sqcup (-R^+)$, where $R^+ := R \cap Q^+$ denotes the set of positive roots.

For each $i \in I$, we denote by $s_i Q$ the quiver obtained from Q by changing the orientations of all arrows incident to i . A vertex $i \in I$ is called a source (resp. sink) if there is no arrow of the form $j \rightarrow i$ (resp. $i \rightarrow j$) in Q . A reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ is said to be adapted to Q if the vertex i_k is a source of the quiver $s_{i_{k-1}} \cdots s_{i_1} Q$ for every $k \in \{1, 2, \dots, l\}$. For any Dynkin quiver Q , we can choose a total ordering $I = \{i_1, i_2, \dots, i_n\}$ of the vertex set I such that we have $a < b$ whenever $i_a \rightarrow i_b \in Q$. Then the reduced expression $s_{i_1} s_{i_2} \cdots s_{i_n}$ is adapted to Q . We define the corresponding Coxeter element $\tau \in W$ to be the product $\tau := s_{i_1} s_{i_2} \cdots s_{i_n}$. The element τ does not depend on the choice of such a total ordering $I = \{i_1, i_2, \dots, i_n\}$ (i.e., depends only on the orientation of the quiver Q).

Let $\widehat{I} := \{(i, p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}$, where $\xi: I \rightarrow \mathbb{Z}$ is a fixed height function on Q . Following Hernandez-Leclerc [21, Section 2.2], we define the bijection³ $\phi: R^+ \times \mathbb{Z} \rightarrow \widehat{I}$ by the following rule:

- (a) For each $i \in I$, we put $\gamma_i := \sum_j \alpha_j$ where j runs over all the vertices $j \in I$ such that there is an oriented path in Q from j to i . Then we define $\phi(\gamma_i, 0) := (i, \xi_i)$;
- (b) Inductively, if $\phi(\alpha, k) = (i, p)$ for $(\alpha, k) \in R^+ \times \mathbb{Z}$, then we define

$$\begin{aligned} \phi(\tau^{\pm 1}(\alpha), k) &:= (i, p \mp 2) && \text{if } \tau^{\pm 1}(\alpha) \in R^+, \\ \phi(-\tau^{\pm 1}(\alpha), k \mp 1) &:= (i, p \mp 2) && \text{if } \tau^{\pm 1}(\alpha) \in -R^+. \end{aligned}$$

Let \widehat{Q} be the infinite quiver whose set of vertices is \widehat{I} and whose set of arrows consists of all the arrows $(i, p) \rightarrow (j, p + 1)$ for $(i, p) \in \widehat{I}$ and $j \sim i$. Note that the quiver \widehat{Q} does not depend on the orientation of Q . This quiver \widehat{Q} is called the *repetition quiver*. Recall that by Gabriel’s theorem, taking dimension vector gives a bijection between the set of isomorphism classes of indecomposable modules of the path algebra $\mathbb{C}Q$ and the set R^+ of positive roots. For a positive root $\alpha \in R^+$, let $M(\alpha)$ denote the indecomposable $\mathbb{C}Q$ -module whose dimension vector is α . It is known that the full subquiver Γ_Q of \widehat{Q} whose vertex set is the subset $\widehat{I}_Q := \phi(R^+ \times \{0\})$ is isomorphic to the *Auslander-Reiten quiver* of $\mathbb{C}Q$. The indecomposable module $M(\alpha)$ corresponds to the vertex $\phi(\alpha) := \phi(\alpha, 0)$. Moreover, the action of the Coxeter element τ on R corresponds to the Auslander-Reiten translation.

2.2. Quantum loop algebras. Let q be an indeterminate. From now on, let \mathbb{k} denote the algebraic closure of $\mathbb{Q}(q)$ in the ambient field $\bigcup_{m \in \mathbb{Z}_{\geq 1}} \overline{\mathbb{Q}}(\!(q^{1/m})\!)$.

Definition 2.1. The quantum loop algebra $U_q \equiv U_q(L\mathfrak{g})$ associated with \mathfrak{g} is a \mathbb{k} -algebra given by the generators:

$$\{e_{i,r}, f_{i,r} \mid i \in I, r \in \mathbb{Z}\} \cup \{q^h \mid h \in P^\vee\} \cup \{h_{i,m} \mid i \in I, m \in \mathbb{Z} \setminus \{0\}\}$$

³Our bijection ϕ is the inverse of the bijection φ in [21].

and the following relations:

$$\begin{aligned}
 q^0 &= 1, & q^h q^{h'} &= q^{h+h'}, & [q^h, h_{i,m}] &= [h_{i,m}, h_{j,l}] = 0, \\
 q^h e_{i,r} q^{-h} &= q^{\alpha_i(h)} e_{i,r}, & q^h f_{i,r} q^{-h} &= q^{-\alpha_i(h)} f_{i,r}, \\
 (z - q^{\pm a_{ij}} w) \psi_i^\varepsilon(z) x_j^\pm(w) &= (q^{\pm a_{ij}} z - w) x_j^\pm(w) \psi_i^\varepsilon(z), \\
 [x_i^+(z), x_j^-(w)] &= \frac{\delta_{ij}}{q - q^{-1}} \left(\delta \left(\frac{w}{z} \right) \psi_i^+(w) - \delta \left(\frac{z}{w} \right) \psi_i^-(z) \right), \\
 (z - q^{\pm a_{ij}} w) x_i^\pm(z) x_j^\pm(w) &= (q^{\pm a_{ij}} z - w) x_j^\pm(w) x_i^\pm(z), \\
 \{x_i^\pm(z_1) x_j^\pm(z_2) x_j^\pm(w) - (q + q^{-1}) x_i^\pm(z_1) x_j^\pm(w) x_i^\pm(z_2) \\
 &\quad + x_j^\pm(w) x_i^\pm(z_1) x_i^\pm(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0 \quad \text{if } i \sim j,
 \end{aligned}$$

where $\varepsilon \in \{+, -\}$ and $\delta(z), \psi_i^\pm(z), x_i^\pm(z)$ are the formal series defined as follows:

$$\begin{aligned}
 \delta(z) &:= \sum_{r=-\infty}^{\infty} z^r, & \psi_i^\pm(z) &:= q^{\pm h_i} \exp \left(\pm (q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} z^{\mp m} \right), \\
 x_i^+(z) &:= \sum_{r=-\infty}^{\infty} e_{i,r} z^{-r}, & x_i^-(z) &:= \sum_{r=-\infty}^{\infty} f_{i,r} z^{-r}.
 \end{aligned}$$

In the last relation, the second term $\{z_1 \leftrightarrow z_2\}$ means the exchange of z_1 with z_2 in the first term.

Remark 2.2. By [2], we have a \mathbb{k} -algebra isomorphism $U_q(L\mathfrak{g}) \cong U'_q(\widehat{\mathfrak{g}})/\langle q^c - 1 \rangle$, where the RHS is a quotient of the quantum affine algebra $U'_q(\widehat{\mathfrak{g}})$ (without the degree operator), which is presented by the Chevalley type generators $\{e_i, f_i \mid i \in I \cup \{0\}\} \cup \{q^h \mid h \in \mathbb{P}^\vee \oplus \mathbb{Z}c\}$, by the ideal generated by a central element $q^c - 1$. Although this isomorphism depends on a function $o: I \rightarrow \{\pm 1\}$ such that $o(i) = -o(j)$ if $i \sim j$, the choice does not affect the results of this paper. Via this isomorphism, the quantum loop algebra $U_q(L\mathfrak{g})$ inherits a structure of Hopf algebra from the quantum affine algebra $U'_q(\widehat{\mathfrak{g}})$. In terms of the Chevalley type generators, the coproduct Δ is given by:

$$\Delta(e_i) = e_i \otimes q^{-h_i} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + q^{h_i} \otimes f_i, \quad \Delta(q^h) = q^h \otimes q^h$$

for $i \in I \cup \{0\}, h \in \mathbb{P}^\vee$. By [13, Proposition 7.1] (see also [19, Remark 3.4]), for each $i \in I$ and $r \in \mathbb{Z}_{>0}$, we have

$$(2.1) \quad \Delta(h_{i,\pm r}) - (h_{i,\pm r} \otimes 1 + 1 \otimes h_{i,\pm r}) \in \bigoplus_{\gamma \in \mathbb{Q}^+ \setminus \{0\}} (U_q)_{-\gamma} \otimes (U_q)_\gamma,$$

where $(U_q)_\gamma := \{x \in U_q \mid q^h x q^{-h} = q^{\gamma(h)} x \ (\forall h \in \mathbb{P}^\vee)\}$. The antipode S is given by

$$S(e_i) = -e_i q^{h_i}, \quad S(f_i) = -q^{-h_i} f_i, \quad S(q^h) = q^{-h},$$

which we use to define the dual modules.

A U_q -module M is said to be of type **1** if it has a decomposition $M = \bigoplus_{\lambda \in \mathbb{P}} M_\lambda$, where $M_\lambda := \{v \in M \mid q^h v = q^{\lambda(h)} v \ (\forall h \in \mathbb{P}^\vee)\}$. A non-zero subspace M_λ is called a weight space of M . Let $\mathcal{C}_\mathfrak{g}$ denote the category of finite-dimensional U_q -modules of type **1**. This is an abelian \mathbb{k} -linear monoidal category.

We also use the modified quantum loop algebra $\widetilde{U}_q(L\mathfrak{g})$ defined by

$$\widetilde{U}_q \equiv \widetilde{U}_q(L\mathfrak{g}) := \bigoplus_{\lambda \in \mathcal{P}} U_q a_\lambda, \quad U_q a_\lambda := U_q \left/ \sum_{h \in \mathcal{P}^\vee} U_q (q^h - q^{\lambda(h)}), \right.$$

where a_λ stands for the image of 1 in the quotient. This is a non-unital \mathbb{k} -algebra, whose multiplication is given by

$$a_\lambda a_\mu = \delta_{\lambda\mu} a_\lambda, \quad a_\lambda x = x a_{\lambda-\gamma},$$

where $x \in (U_q)_\gamma, \gamma \in \mathcal{Q}$. By definition, considering a \widetilde{U}_q -module is the same as considering a U_q -module of type $\mathbf{1}$.

We use the following notation. Let $\mathcal{P} := \bigoplus_{(i,a) \in I \times \mathbb{k}^\times} \mathbb{Z} \varpi_{i,a}$ be the set of ℓ -weights, which is a free abelian group with a basis $\{\varpi_{i,a} \mid i \in I, a \in \mathbb{k}^\times\}$. We call a basis element $\varpi_{i,a}$ a *fundamental ℓ -weight*. An element in the submonoid $\mathcal{P}^+ := \sum \mathbb{Z}_{\geq 0} \varpi_{i,a}$ is said to be ℓ -dominant. We define a \mathbb{Z} -linear map $\text{cl}: \mathcal{P} \rightarrow \mathcal{P}$ by $\varpi_{i,a} \mapsto \varpi_i$. Following [16], we define the ℓ -root $\alpha_{i,a}$ for each $(i,a) \in I \times \mathbb{k}^\times$ by

$$\alpha_{i,a} := \varpi_{i,aq} + \varpi_{i,aq^{-1}} - \sum_{j \sim i} \varpi_{j,a}.$$

We define the ℓ -root lattice by $\mathcal{Q} := \bigoplus_{(i,a) \in I \times \mathbb{k}^\times} \mathbb{Z} \alpha_{i,a} \subset \mathcal{P}$ and set $\mathcal{Q}^+ := \sum \mathbb{Z}_{\geq 0} \alpha_{i,a}$. Note that $\text{cl}: \mathcal{P} \rightarrow \mathcal{P}$ induces a map $\text{cl}: \mathcal{Q} \rightarrow \mathcal{Q}$ since $\text{cl}(\alpha_{i,a}) = \alpha_i$. We define a partial order \leq on \mathcal{P} called the *Nakajima partial ordering* by the condition that for $\hat{\lambda}, \hat{\mu} \in \mathcal{P}$, we have $\hat{\lambda} \leq \hat{\mu}$ if and only if $\hat{\mu} - \hat{\lambda} \in \mathcal{Q}^+$.

Let $U_q(L\mathfrak{h})$ denote the commutative \mathbb{k} -subalgebra of $U_q(L\mathfrak{g})$ generated by $\{q^h \mid h \in \mathcal{P}^\vee\} \cup \{h_{i,r} \mid i \in I, r \in \mathbb{Z} \setminus \{0\}\}$. A module $M \in \mathcal{C}_{\mathfrak{g}}$ decomposes into a direct sum of generalized eigenspaces for $U_q(L\mathfrak{h})$ as $M = \bigoplus M_\Psi$, where $\Psi = (\Psi_i^+(z), \Psi_i^-(z))_{i \in I} \in \mathbb{k}[[z^{-1}]]^I \times \mathbb{k}[[z]]^I$ and M_Ψ is the subspace on which all the coefficients of the series $\psi_i^\pm(z) - \Psi_i^\pm(z) \text{id}_M$ act nilpotently. By [16], it is known that if $M_\Psi \neq 0$, there is a unique ℓ -weight $\hat{\lambda} = \sum l_{i,a} \varpi_{i,a} \in \mathcal{P}$ such that we have

$$(2.2) \quad \Psi_i^\pm(z) = q^{\text{cl}(\hat{\lambda})(h_i)} \left(\prod_{a \in \mathbb{k}^\times} \left(\frac{1 - aq^{-2}z^{-1}}{1 - az^{-1}} \right)^{l_{i,a}} \right)^\pm,$$

where $(-)^{\pm}$ denotes the formal expansion at $z = \infty$ and 0 respectively. In this case, we write $M_{\hat{\lambda}} = M_\Psi$ and call it the ℓ -weight space of ℓ -weight $\hat{\lambda}$.

We say a module $M \in \mathcal{C}_{\mathfrak{g}}$ is an ℓ -highest weight module of ℓ -highest weight $\hat{\lambda} \in \mathcal{P}$ if there exists a generating vector $v_0 \in M$ satisfying

$$x_i^+(z) \cdot v_0 = 0, \quad \psi_i^\pm(z) \cdot v_0 = q^{\text{cl}(\hat{\lambda})(h_i)} \left(\prod_{a \in \mathbb{k}^\times} \left(\frac{1 - aq^{-2}z^{-1}}{1 - az^{-1}} \right)^{l_{i,a}} \right)^\pm v_0$$

in $M[[z, z^{-1}]]$ for any $i \in I$. Compare the latter equation with (2.2). In this case, it is known by [7, 8] that the ℓ -highest weight $\hat{\lambda}$ automatically becomes ℓ -dominant, i.e., $\hat{\lambda} \in \mathcal{P}^+$ and we have $M_{\hat{\lambda}} = \mathbb{k} \cdot v_0$. Moreover, for any $\hat{\lambda} \in \mathcal{P}^+$, there is a simple ℓ -highest weight module $L(\hat{\lambda}) \in \mathcal{C}_{\mathfrak{g}}$ uniquely up to isomorphism and any simple module in $\mathcal{C}_{\mathfrak{g}}$ is of this form. By [15, 35], it is known that for an element $\hat{\mu} \in \mathcal{P}$, we have $L(\hat{\lambda})_{\hat{\mu}} \neq 0$ only if $\hat{\mu} \leq \hat{\lambda}$. When $\hat{\lambda} = \varpi_{i,a}$ for some $(i,a) \in I \times \mathbb{k}^\times$, the simple module $L(\varpi_{i,a})$ is called an ℓ -fundamental module.

Recall that for two simple modules $M_1, M_2 \in \mathcal{C}_{\mathfrak{g}}$, we say that M_1 and M_2 are *linked* if there is no splitting $\mathcal{C}_{\mathfrak{g}} \cong \mathcal{C}_1 \oplus \mathcal{C}_2$ such that $M_1 \in \mathcal{C}_1$ and $M_2 \in \mathcal{C}_2$.

Theorem 2.3 (Chari-Moura [6]). *For any ℓ -dominant ℓ -weight $\hat{\lambda}, \hat{\mu} \in \mathcal{P}^+$, the simple modules $L(\hat{\lambda})$ and $L(\hat{\mu})$ are linked if and only if $\hat{\lambda} - \hat{\mu} \in \mathcal{Q}$.*

For each $M \in \mathcal{C}_{\mathfrak{g}}$, we define its left dual module M^* (resp. right dual module *M) as the dual space $\text{Hom}_{\mathbb{k}}(M, \mathbb{k})$ equipped with the left U_q -action obtained by twisting the natural right action with the antipode S (resp. S^{-1}). For any $M_1, M_2 \in \mathcal{C}_{\mathfrak{g}}$, we have

$$(M_1 \otimes M_2)^* \cong M_2^* \otimes M_1^*, \quad {}^*(M_1 \otimes M_2) \cong {}^*M_2 \otimes {}^*M_1.$$

We define the following \mathbb{Z} -linear maps on \mathcal{P} :

$$\begin{aligned} (-)^*: \mathcal{P} &\rightarrow \mathcal{P}, & \varpi_{i,a} &\mapsto \varpi_{i^*,a}^{*} := \varpi_{i^*,aq^{-h}}, \\ {}^*(-): \mathcal{P} &\rightarrow \mathcal{P}, & \varpi_{i,a} &\mapsto {}^*\varpi_{i,a} := \varpi_{i^*,aq^h}, \end{aligned}$$

where $i \mapsto i^*$ denotes the involution on I defined by $\alpha_{i^*} := -w_0\alpha_i$ and h is the Coxeter number (the order of a Coxeter element of W). Then we have

$$L(\varpi_{i,a})^* \cong L(\varpi_{i^*,a}^*), \quad {}^*L(\varpi_{i,a}) \cong L({}^*\varpi_{i,a}).$$

See [15, Corollary 6.10].

2.3. Weyl modules. In this subsection, we recall the global and local Weyl modules of U_q introduced by Chari-Pressley [10]. Also we define the deformed local Weyl modules, which will play a role of standard modules of an affine highest weight category later.

Definition 2.4. A U_q -module M of type **1** is said to be ℓ -integrable if the following property is satisfied: For each $v \in M$, there exists an integer $n_0 \geq 1$ such that we have $e_{i,r_1}e_{i,r_2} \cdots e_{i,r_N}v = f_{i,r_1}f_{i,r_2} \cdots f_{i,r_N}v = 0$ for any $N \geq n_0$ and any $i \in I, r_1, \dots, r_N \in \mathbb{Z}$.

Remark 2.5. In this paper, we do not impose that $\dim M_{\lambda} < \infty$ for ℓ -integrability. Note that any finite-dimensional modules of type **1**, i.e., any objects of the category $\mathcal{C}_{\mathfrak{g}}$, are automatically ℓ -integrable.

First we define the global Weyl modules.

Definition 2.6. Let $\lambda \in \mathcal{P}^+$ be a dominant weight. We define the *global Weyl module* associated with λ to be the left U_q -module $\mathbb{W}(\lambda)$ generated by a cyclic vector w_{λ} satisfying the following defining relations:

$$e_{i,r}w_{\lambda} = 0, \quad q^h w_{\lambda} = q^{\lambda(h)} w_{\lambda}, \quad (f_{i,r})^{\lambda(h_i)+1} w_{\lambda} = 0,$$

where $i \in I, r \in \mathbb{Z}$ and $h \in \mathbb{P}^{\vee}$.

In what follows, \mathfrak{S}_d denotes the symmetric group of degree $d \in \mathbb{Z}_{\geq 1}$. For a dominant weight $\lambda = \sum_{i \in I} l_i \varpi_i \in \mathcal{P}^+$, we define the following \mathbb{k} -algebra of partially symmetric Laurent polynomials:

$$(2.3) \quad \mathcal{R}(\lambda) := \bigotimes_{i \in I} (\mathbb{k}[z_i^{\pm 1}]^{\otimes l_i})^{\mathfrak{S}_{l_i}} = \bigotimes_{i \in I} \mathbb{k}[z_{i,1}^{\pm 1}, \dots, z_{i,l_i}^{\pm 1}]^{\mathfrak{S}_{l_i}}.$$

Theorem 2.7 (Chari-Pressley [10], Nakajima). *Write $\lambda = \sum_{i \in I} l_i \varpi_i \in \mathcal{P}^+$.*

- (1) *The global Weyl module $\mathbb{W}(\lambda)$ is ℓ -integrable and has the following universal property: If M is an ℓ -integrable U_q -module with a vector $v \in M_\lambda$ of weight λ satisfying $x_i^+(z) \cdot v = 0$ for any $i \in I$, then there is a unique U_q -homomorphism $\mathbb{W}(\lambda) \rightarrow M$ such that $w_\lambda \mapsto v$;*
- (2) *There is a unique isomorphism $\text{End}_{U_q}(\mathbb{W}(\lambda)) \cong \mathcal{R}(\lambda)$ such that we have*

$$\psi_i^\pm(z)w_\lambda = q^{l_i} \prod_{k=1}^{l_i} \left(\frac{1 - q^{-2}z_{i,k}z^{-1}}{1 - z_{i,k}z^{-1}} \right)^\pm w_\lambda$$

for each $i \in I$;

- (3) $\mathbb{W}(\lambda)$ is free over $\mathcal{R}(\lambda)$ of finite rank.

Proof. See [10, Section 4]. The assertion (3) is proved by the geometric realization due to Nakajima. For details, see Theorem 4.2 and Theorem 4.3(2). □

Remark 2.8. The global Weyl module $\mathbb{W}(\lambda)$ is known to be isomorphic to the level 0 extremal weight module $V^{\max}(\lambda)$ of extremal weight λ , in the sense of Kashiwara [24]. See [10, Proposition 4.5] and [37, Remark 2.15].

Next we discuss the local Weyl modules. We identify a point of the quotient space $(\mathbb{k}^\times)^N / \mathfrak{S}_N$ with a $(\mathbb{Z}_{\geq 0})$ -linear combination of the formal symbols $\{[a] \mid a \in \mathbb{k}^\times\}$ whose coefficients sum up to N . Note that we have $\text{Specm } \mathcal{R}(\lambda) \cong \prod_{i \in I} ((\mathbb{k}^\times)^{l_i} / \mathfrak{S}_{l_i})$. Let $\hat{\lambda} = \sum_{(i,a) \in I \times \mathbb{k}^\times} l_{i,a} \varpi_{i,a} \in \mathcal{P}^+$ be an ℓ -dominant ℓ -weight and put $\lambda := \text{cl}(\hat{\lambda}) \in \mathcal{P}^+$. Let $\mathfrak{r}_{\lambda, \hat{\lambda}}$ denote the maximal ideal of $\mathcal{R}(\lambda)$ corresponding to the point

$$\left(\sum_{a \in \mathbb{k}^\times} l_{i,a} [a] \right)_{i \in I} \in \prod_{i \in I} ((\mathbb{k}^\times)^{l_i} / \mathfrak{S}_{l_i}).$$

Definition 2.9. We define the *local Weyl module* $W(\hat{\lambda})$ associated with $\hat{\lambda} \in \mathcal{P}^+$ by $W(\hat{\lambda}) := \mathbb{W}(\lambda) / \mathfrak{r}_{\lambda, \hat{\lambda}}$. The image of the cyclic vector $w_\lambda \in \mathbb{W}(\lambda)$ is denoted by $w_{\hat{\lambda}} \in W(\hat{\lambda})$.

Theorem 2.10 (Chari-Pressley [10]). *Let $\hat{\lambda} \in \mathcal{P}^+$ be an ℓ -dominant ℓ -weight.*

- (1) *The local Weyl module $W(\hat{\lambda})$ is a finite-dimensional ℓ -highest weight module of ℓ -highest weight $\hat{\lambda}$ with $W(\hat{\lambda})_{\hat{\lambda}} = \mathbb{k} \cdot w_{\hat{\lambda}}$. Moreover it has the following universal property: If $M \in \mathcal{C}_{\mathfrak{g}}$ is an ℓ -highest weight module of ℓ -highest weight $\hat{\lambda}$ with $M_{\hat{\lambda}} = \mathbb{k} \cdot v_0$, then there is a unique surjective U_q -homomorphism $W(\hat{\lambda}) \rightarrow M$ such that $w_{\hat{\lambda}} \mapsto v_0$;*
- (2) $W(\hat{\lambda})$ has a simple head isomorphic to $L(\hat{\lambda})$.

Proof. The assertions follow immediately from Theorem 2.7. □

Remark 2.11. The local Weyl module $W(\hat{\lambda})$ is known to be isomorphic to the standard module in the sense of Nakajima [35, Section 13] and Varagnolo-Vasserot [43]. See Theorem 2.15 and [43, Corollary 7.16].

Finally, we introduce the deformed local Weyl modules. Let $\hat{\lambda} = \sum l_{i,a} \varpi_{i,a} \in \mathcal{P}^+$ be an ℓ -dominant ℓ -weight and set $\lambda := \text{cl}(\hat{\lambda})$. We define

$$\widehat{\mathcal{R}}(\lambda, \hat{\lambda}) := \varprojlim_N \mathcal{R}(\lambda) / \mathfrak{r}_{\lambda, \hat{\lambda}}^N.$$

Definition 2.12. We define the *deformed local Weyl module* $\widehat{W}(\hat{\lambda})$ associated with $\hat{\lambda} \in \mathcal{P}^+$ by

$$\widehat{W}(\hat{\lambda}) := \mathbb{W}(\lambda) \otimes_{\mathcal{R}(\lambda)} \widehat{\mathcal{R}}(\lambda, \hat{\lambda}) \cong \varprojlim_N \mathbb{W}(\lambda) / \mathfrak{r}_{\lambda, \hat{\lambda}}^N.$$

We set $\widehat{w}_{\hat{\lambda}} := w_{\lambda} \otimes 1 \in \widehat{W}(\hat{\lambda})$.

We also use the following algebra:

$$(2.4) \quad \mathcal{R}(\hat{\lambda}) := \bigotimes_{i \in I} \bigotimes_{a \in \mathbb{k}^\times} (\mathbb{k}[z_i^{\pm 1}]^{\otimes l_{i,a}})^{\mathfrak{S}_{l_{i,a}}}.$$

Note that the algebra $\mathcal{R}(\lambda)$ is a subalgebra of $\mathcal{R}(\hat{\lambda})$. Let $\mathfrak{r}_{\hat{\lambda}}$ denote a maximal ideal of $\mathcal{R}(\hat{\lambda})$ corresponding to the point

$$(l_{i,a}[a])_{(i,a) \in I \times \mathbb{k}^\times} \in \prod_{(i,a) \in I \times \mathbb{k}^\times} ((\mathbb{k}^\times)^{l_{i,a}} / \mathfrak{S}_{l_{i,a}}) = \text{Specm } \mathcal{R}(\hat{\lambda}).$$

Then we have $\mathfrak{r}_{\lambda, \hat{\lambda}} = \mathcal{R}(\lambda) \cap \mathfrak{r}_{\hat{\lambda}}$ and there is a natural isomorphism

$$(2.5) \quad \widehat{\mathcal{R}}(\hat{\lambda}) := \varprojlim_N \mathcal{R}(\hat{\lambda}) / \mathfrak{r}_{\hat{\lambda}}^N \cong \widehat{\mathcal{R}}(\lambda, \hat{\lambda}).$$

Hereafter we identify $\widehat{\mathcal{R}}(\lambda, \hat{\lambda})$ with $\widehat{\mathcal{R}}(\hat{\lambda})$ via the isomorphism (2.5). For simplicity, we will use the same symbol $\mathfrak{r}_{\hat{\lambda}}$ to denote the maximal ideal of the local ring $\widehat{\mathcal{R}}(\hat{\lambda})$.

Proposition 2.13. *The deformed local Weyl module $\widehat{W}(\hat{\lambda})$ satisfies the following properties:*

- (1) *For each $M \in \mathcal{C}_{\mathfrak{g}}$, taking the image of $\widehat{w}_{\hat{\lambda}}$ gives a natural isomorphism:*

$$\text{Hom}_{U_q}(\widehat{W}(\hat{\lambda}), M) \cong \{v \in M_{\hat{\lambda}} \mid e_{i,r}v = 0 \text{ for any } i \in I, r \in \mathbb{Z}\};$$

- (2) *$\text{End}_{U_q}(\widehat{W}(\hat{\lambda})) = \widehat{\mathcal{R}}(\hat{\lambda})$ and $\widehat{W}(\hat{\lambda})$ is free over $\widehat{\mathcal{R}}(\hat{\lambda})$ of finite rank;*

- (3) *$\widehat{W}(\hat{\lambda}) / \mathfrak{r}_{\hat{\lambda}} \cong W(\hat{\lambda})$.*

Proof. The assertions follow immediately from Theorem 2.7. □

Remark 2.14. Let z be an indeterminate. For any U_q -module M , we can equip the $\mathbb{k}[z^{\pm 1}]$ -module $M[z^{\pm 1}] := M \otimes \mathbb{k}[z^{\pm 1}]$ with a structure of U_q -module by setting

$$(2.6) \quad \begin{aligned} e_{i,r} \cdot (v \otimes \varphi) &= e_{i,r}v \otimes z^r \varphi, & f_{i,r} \cdot (v \otimes \varphi) &= f_{i,r}v \otimes z^r \varphi, \\ q^h \cdot (v \otimes \varphi) &= q^h v \otimes \varphi, & h_{i,l} \cdot (v \otimes \varphi) &= h_{i,l}v \otimes z^l \varphi, \end{aligned}$$

for any $v \in M, \varphi \in \mathbb{k}[z^{\pm 1}]$ and the defining generators $e_{i,r}, f_{i,r}, q^h, h_{i,l}$ of U_q . The resulting U_q -module $M[z^{\pm 1}]$ is called the *affinization of M* (cf. [25, Section 4.2]).

For each $i \in I$, we have an isomorphism $\mathbb{W}(\varpi_i) \cong L(\varpi_{i,1})[z^{\pm 1}]$ of U_q -modules, under which the generating vector w_{ϖ_i} corresponds to the vector $v_0 \otimes 1$ with $v_0 \in L(\varpi_{i,1})$ being a fixed ℓ -highest weight vector. This isomorphism yields the isomorphism $\text{End}_{U_q}(\mathbb{W}(\varpi_i)) \cong \mathbb{k}[z^{\pm 1}]$ in Theorem 2.7(2). In particular, the deformed local Weyl module $\widehat{W}(\varpi_{i,a})$ is isomorphic to the $\mathbb{k}[[z-a]]$ -module $L(\varpi_{i,1}) \otimes \mathbb{k}[[z-a]]$ equipped with the action of U_q by the same formulas as (2.6).

2.4. R -matrices and factorization of deformed local Weyl modules. In this subsection, we recall some facts about R -matrices between ℓ -fundamental modules following [1, 23, 25] and describe a factorization of deformed local Weyl modules.

For any pair $(i_1, i_2) \in I^2$, let us write $\text{End}_{U_q}(\mathbb{W}(\varpi_{i_j})) = \mathbb{k}[z_j^{\pm 1}]$ for $j = 1, 2$ as in Theorem 2.7(2). Then there is a unique homomorphism of $(U_q, \mathbb{k}[z_1^{\pm 1}, z_2^{\pm 1}])$ -bimodules, called the *normalized R -matrix*

$$R_{i_1, i_2}^{\text{norm}} : \mathbb{W}(\varpi_{i_1}) \otimes \mathbb{W}(\varpi_{i_2}) \rightarrow \mathbb{k}(z_2/z_1) \otimes_{\mathbb{k}[(z_2/z_1)^{\pm 1}]} (\mathbb{W}(\varpi_{i_2}) \otimes \mathbb{W}(\varpi_{i_1})),$$

such that $R_{i_1, i_2}^{\text{norm}}(w_{\varpi_{i_1}} \otimes w_{\varpi_{i_2}}) = w_{\varpi_{i_2}} \otimes w_{\varpi_{i_1}}$. The *denominator* of $R_{i_1, i_2}^{\text{norm}}$ is defined to be the monic polynomial $d_{i_1, i_2}(u) \in \mathbb{k}[u]$ with the smallest degree among polynomials satisfying

$$\text{Im } R_{i_1, i_2}^{\text{norm}} \subset d_{i_1, i_2}(z_2/z_1)^{-1} \otimes (\mathbb{W}(\varpi_{i_2}) \otimes \mathbb{W}(\varpi_{i_1})).$$

By [25, Proposition 9.3], we have

$$(2.7) \quad d_{i_1, i_2}(a) = 0 \implies a \in \bigcup_{m \in \mathbb{Z}_{>0}} q^{1/m} \overline{\mathbb{Q}}[[q^{1/m}]].$$

Theorem 2.15 ([6, 25, 43]). *Let $\hat{\lambda} = \sum_{j=1}^l \varpi_{i_j, a_j} \in \mathcal{P}^+$ be an ℓ -dominant ℓ -weight. Then the following three conditions are mutually equivalent:*

- (1) *The tensor product module $L(\varpi_{i_1, a_1}) \otimes L(\varpi_{i_2, a_2}) \otimes \cdots \otimes L(\varpi_{i_l, a_l})$ is generated by the tensor product of ℓ -highest weight vectors;*
- (2) *$W(\hat{\lambda}) \cong L(\varpi_{i_1, a_1}) \otimes L(\varpi_{i_2, a_2}) \otimes \cdots \otimes L(\varpi_{i_l, a_l})$;*
- (3) *$d_{i_j, i_k}(a_k/a_j) \neq 0$ for any $1 \leq j < k \leq l$.*

We can always make these conditions satisfied by reordering $\{(i_j, a_j)\}_j$ suitably.

Proof. The equivalence of (1) and (2) was proved by Chari-Moura [6, Theorem 6.4] using the results from geometry due to Nakajima [35]. The equivalence of (1) and (3) was proved by Kashiwara [25, Proposition 9.4]. The last assertion follows from (2.7). See also [43, Corollary 7.15]. □

Definition 2.16. Let $\hat{\lambda} \in \mathcal{P}^+$ be an ℓ -dominant ℓ -weight. We define the *dual local Weyl module* associated with $\hat{\lambda}$ by $W^\vee(\hat{\lambda}) := W(*\hat{\lambda})^*$.

Proposition 2.17. *Let $\hat{\lambda} = \sum_{j=1}^l \varpi_{i_j, a_j} \in \mathcal{P}^+$ be an ℓ -dominant ℓ -weight. Assume $W(\hat{\lambda}) \cong L(\varpi_{i_1, a_1}) \otimes L(\varpi_{i_2, a_2}) \otimes \cdots \otimes L(\varpi_{i_l, a_l})$. Then we have*

$$W^\vee(\hat{\lambda}) \cong L(\varpi_{i_l, a_l}) \otimes L(\varpi_{i_{l-1}, a_{l-1}}) \otimes \cdots \otimes L(\varpi_{i_1, a_1}).$$

Proof. Use the equivalence of (2) and (3) in Theorem 2.15 and the fact $d_{i_1, i_2}(u) = d_{i_1^*, i_2^*}(u)$ (see [1, Appendix A]). □

For any two ℓ -dominant ℓ -weights $\hat{\lambda} = \sum l_{i,a} \varpi_{i,a}$, $\hat{\lambda}' = \sum l'_{i,a} \varpi_{i,a} \in \mathcal{P}^+$, we have the following injective homomorphism:

$$\begin{aligned} \mathcal{R}(\hat{\lambda} + \hat{\lambda}') &= \bigotimes_{i \in I} \bigotimes_{a \in \mathbb{k}^\times} \left(\mathbb{k}[z_i^{\pm 1}]^{\otimes l_{i,a} + l'_{i,a}} \right)^{\mathfrak{S}^{(l_{i,a} + l'_{i,a})}} \\ &\hookrightarrow \bigotimes_{i \in I} \bigotimes_{a \in \mathbb{k}^\times} \left(\mathbb{k}[z_i^{\pm 1}]^{\otimes l_{i,a}} \otimes \mathbb{k}[z_i^{\pm 1}]^{\otimes l'_{i,a}} \right)^{\mathfrak{S}^{l_{i,a}} \times \mathfrak{S}^{l'_{i,a}}} = \mathcal{R}(\hat{\lambda}) \otimes \mathcal{R}(\hat{\lambda}'). \end{aligned}$$

We have similar injective homomorphisms

$$\mathcal{R}(\hat{\lambda}_1 + \cdots + \hat{\lambda}_l) \hookrightarrow \mathcal{R}(\hat{\lambda}_1) \otimes \cdots \otimes \mathcal{R}(\hat{\lambda}_l)$$

for any $\hat{\lambda}_1, \dots, \hat{\lambda}_l \in \mathcal{P}^+$. They induce the following injective homomorphisms for the completions:

$$\widehat{\mathcal{R}}(\hat{\lambda}_1 + \dots + \hat{\lambda}_l) \hookrightarrow \widehat{\mathcal{R}}(\hat{\lambda}_1) \hat{\otimes} \dots \hat{\otimes} \mathcal{R}(\hat{\lambda}_l).$$

We refer to these kinds of injective homomorphisms as the *standard inclusions*.

Let $\hat{\lambda} := \sum_{j=1}^l m_j \varpi_{i_j, a_j} \in \mathcal{P}^+$ be an l -dominant l -weight with $(i_j, a_j) \neq (i_k, a_k)$ for $j \neq k$. We see that the standard inclusion is an isomorphism $\widehat{\mathcal{R}}(\hat{\lambda}) \cong \widehat{\mathcal{R}}(m_1 \varpi_{i_1, a_1}) \hat{\otimes} \dots \hat{\otimes} \widehat{\mathcal{R}}(m_l \varpi_{i_l, a_l})$ in this case. Thus the completed tensor product $\widehat{W}(m_1 \varpi_{i_1, a_1}) \hat{\otimes} \dots \hat{\otimes} \widehat{W}(m_l \varpi_{i_l, a_l})$ is regarded as a $(U_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodule.

Proposition 2.18. *With the above notation, we further assume that $d_{i_j, i_k}(a_k/a_j) \neq 0$ for any $1 \leq j < k \leq l$. Then we have an isomorphism of $(U_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodules:*

$$\widehat{W}(\hat{\lambda}) \cong \widehat{W}(m_1 \varpi_{i_1, a_1}) \hat{\otimes} \dots \hat{\otimes} \widehat{W}(m_l \varpi_{i_l, a_l}).$$

Proof. By the universal property of the global Weyl module $\mathbb{W}(\lambda)$ (Theorem 2.7(1)), there is the U_q -homomorphism

$$(2.8) \quad \mathbb{W}(\lambda) \rightarrow \mathbb{W}(m_1 \varpi_{i_1}) \otimes \dots \otimes \mathbb{W}(m_l \varpi_{i_l}); w_\lambda \mapsto w_{m_1 \varpi_{i_1}} \otimes \dots \otimes w_{m_l \varpi_{i_l}}.$$

By (2.1), we have

$$\psi_i^\pm(z)(w_{m_1 \varpi_{i_1}} \otimes \dots \otimes w_{m_l \varpi_{i_l}}) = (\psi_i^\pm(z)w_{m_1 \varpi_{i_1}}) \otimes \dots \otimes (\psi_i^\pm(z)w_{m_l \varpi_{i_l}})$$

for any $i \in I$. Therefore, by Theorem 2.7(2), the homomorphism (2.8) is actually a homomorphism of $(U_q, \mathcal{R}(\hat{\lambda}))$ -bimodules, where the RHS of (2.8) is regarded as an $\mathcal{R}(\hat{\lambda})$ -module via the standard inclusion. Completing with respect to the maximal ideal $\mathfrak{r}_{\lambda, \hat{\lambda}} \subset \mathcal{R}(\lambda)$, we get

$$(2.9) \quad \widehat{W}(\hat{\lambda}) \rightarrow \widehat{W}(m_1 \varpi_{i_1, a_1}) \hat{\otimes} \dots \hat{\otimes} \widehat{W}(m_l \varpi_{i_l, a_l}).$$

Note that both sides of (2.9) are free over $\widehat{\mathcal{R}}(\hat{\lambda})$. By Theorem 2.15, taking the quotients by $\mathfrak{r}_{\hat{\lambda}}$ induces an isomorphism

$$W(\hat{\lambda}) \xrightarrow{\cong} W(m_1 \varpi_{i_1, a_1}) \otimes \dots \otimes W(m_l \varpi_{i_l, a_l}) \cong L(\varpi_{i_1, a_1})^{\otimes m_1} \otimes \dots \otimes L(\varpi_{i_l, a_l})^{\otimes m_l}.$$

Therefore we see that (2.9) is an isomorphism by Nakayama's lemma. \square

Note that if $d_{i_1, i_2}(a_2/a_1) \neq 0$, the R -matrix $R_{i_1, i_2}^{\text{norm}}$ induces a homomorphism of $(U_q, \mathbb{k}[[z_1 - a_1, z_2 - a_2]])$ -bimodules

$$R_{i_1, i_2}^{\text{norm}} : \widehat{W}(\varpi_{i_1, a_1}) \hat{\otimes} \widehat{W}(\varpi_{i_2, a_2}) \rightarrow \widehat{W}(\varpi_{i_2, a_2}) \hat{\otimes} \widehat{W}(\varpi_{i_1, a_1}).$$

Let NH_m denote the *nil-Hecke algebra* of degree $m \in \mathbb{Z}_{>0}$. For its basic properties, see [29, Examples 2.2.3] and references therein. It is a \mathbb{k} -algebra presented by the generators $\{x_1, \dots, x_m\} \cup \{\tau_1, \dots, \tau_{m-1}\}$ satisfying the following relations:

$$x_k x_l = x_l x_k, \quad \tau_k^2 = 0, \quad \tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_{k+1}, \quad \tau_k \tau_l = \tau_l \tau_k \quad \text{if } |k - l| > 1, \\ \tau_k x_{k+1} - x_k \tau_k = x_{k+1} \tau_k - \tau_k x_k = 1, \quad \tau_k x_l = x_l \tau_k \quad \text{if } l \neq k, k + 1.$$

The third and fourth relations above are called the braid relations. Let $\sigma_i \in \mathfrak{S}_m$ denote the transposition of i and $i + 1$ ($1 \leq i < m$). For each $\sigma \in \mathfrak{S}_m$, we define $\tau_\sigma := \tau_{i_1} \dots \tau_{i_l} \in NH_m$, where $\sigma = \sigma_{i_1} \dots \sigma_{i_l}$ is a reduced expression of σ . Thanks to the braid relations, the element τ_σ does not depend on the choice of a reduced expression. It is known that the algebra NH_m is free as a left (or a right) $\mathbb{k}[x_1, \dots, x_m]$ -module with a free basis $\{\tau_\sigma \mid \sigma \in \mathfrak{S}_m\}$ and the center of NH_m coincides with the subalgebra $\mathfrak{S}_m := \mathbb{k}[x_1, \dots, x_m]^{\mathfrak{S}_m}$ of symmetric polynomials.

Moreover, it is known that NH_m is isomorphic to the matrix algebra of rank $m!$ over its center \mathbb{S}_m . A primitive idempotent e_m is given by $e_m := \tau_{\sigma_0} x_2 x_3^2 \cdots x_m^{m-1}$, where $\sigma_0 \in \mathfrak{S}_m$ is the longest element. The nil-Hecke algebra NH_m is a graded algebra by setting $\deg x_k = 2, \deg \tau_k = -2$. Since the grading is bounded from below, the completion \widehat{NH}_m with respect to the grading naturally becomes an algebra. The completed nil-Hecke algebra \widehat{NH}_m is isomorphic to the matrix algebra over its center $\widehat{\mathbb{S}}_m := \mathbb{k}[[x_1, \dots, x_m]]^{\mathfrak{S}_m}$ of rank $m!$.

Let $\hat{\lambda} := m\varpi_{i,a} \in \mathcal{P}^+$ for a pair $(i, a) \in I \times \mathbb{k}^\times$ and $m \in \mathbb{Z}_{>0}$. Consider the completed tensor product $\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m}$, which is regarded as a $(U_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodule, via the standard inclusion $\widehat{\mathcal{R}}(\hat{\lambda}) \hookrightarrow \widehat{\mathcal{R}}(\varpi_{i,a})^{\hat{\otimes} m} = \mathbb{k}[[z_1 - a, \dots, z_m - a]]$. Since we know that $d_{i,i}(1) \neq 0$ (recall (2.7)), the $(U_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodule automorphism

$$r_k := 1^{\hat{\otimes} k-1} \hat{\otimes} \sigma_k R_{i,i}^{\text{norm}} \hat{\otimes} 1^{\hat{\otimes} m-k-1} \in \text{End}(\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m})$$

is well-defined for $1 \leq k < m$, where σ_k denotes the permutation of z_k and z_{k+1} . Then the formulas

$$x_k \mapsto z_k - a, \quad \tau_k \mapsto (z_k - z_{k+1})^{-1}(r_k - 1)$$

define a right action of \widehat{NH}_m , which makes $\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m}$ into a (U_q, \widehat{NH}_m) -bimodule.

Remark 2.19. The nil-Hecke algebra HN_m coincides with the quiver Hecke algebra of type A_1 . The above construction of the \widehat{NH}_m -action is a special case of the construction of Kang-Kashiwara-Kim [23], which we will review later in Section 5.2.

Proposition 2.20. *Let $\hat{\lambda} = m\varpi_{i,a} \in \mathcal{P}^+$. As $(U_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodules, we have*

$$\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} \cong \widehat{W}(\hat{\lambda})^{\oplus m!}.$$

Proof. Since $\widehat{NH}_m \cong (\widehat{NH}_m e_m)^{\oplus m!}$ as a left \widehat{NH}_m -module, we have

$$\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} = \widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} \otimes_{\widehat{NH}_m} \widehat{NH}_m \cong \left(\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} e_m \right)^{\oplus m!}.$$

A summand $\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} e_m$ is a $(U_q, \widehat{\mathbb{S}}_m)$ -bimodule. Note that we have $\widehat{\mathbb{S}}_m = \widehat{\mathcal{R}}(\hat{\lambda})$ inside $\text{End}(\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} e_m)$. Since $\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m}$ is free over $\widehat{\mathcal{R}}(\varpi_{i,a})^{\hat{\otimes} m}$ of rank $(\dim L(\varpi_{i,a}))^m$, the summand $\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} e_m$ is free over $\widehat{\mathcal{R}}(\hat{\lambda})$ of rank $(\dim L(\varpi_{i,a}))^m$. Using the universal property, we have the $(U_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -homomorphism

$$\widehat{W}(\hat{\lambda}) \rightarrow \widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} e_m; \quad \hat{w}_{\hat{\lambda}} \mapsto (\hat{w}_{\varpi_{i,a}})^{\hat{\otimes} m} e_m.$$

Taking the quotients by $\tau_{\hat{\lambda}}$, we get a non-zero U_q -homomorphism $W(\hat{\lambda}) \rightarrow \widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} e_m / \tau_{\hat{\lambda}}$. This should be an isomorphism because $W(\hat{\lambda}) \cong L(\varpi_{i,a})^{\hat{\otimes} m} \cong W^\vee(\hat{\lambda})$ is simple (recall Theorem 2.15 and Proposition 2.17) and $\dim W(\hat{\lambda}) = \dim(\widehat{W}(\varpi_{i,a})^{\hat{\otimes} m} e_m / \tau_{\hat{\lambda}})$. Thus we complete the proof by Nakayama's lemma. \square

Corollary 2.21. *Let $\hat{\lambda} := \sum_{j=1}^l m_j \varpi_{i_j, a_j} \in \mathcal{P}^+$ be an ℓ -dominant ℓ -weight with $(i_j, a_j) \neq (i_k, a_k)$ for $j \neq k$. Assume that $d_{i_j, i_k}(a_k/a_j) \neq 0$ for any $1 \leq j < k \leq l$. Then there is an isomorphism of $(U_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodules*

$$\widehat{W}(\varpi_{i_1, a_1})^{\hat{\otimes} m_1} \hat{\otimes} \cdots \hat{\otimes} \widehat{W}(\varpi_{i_l, a_l})^{\hat{\otimes} m_l} \cong \widehat{W}(\hat{\lambda})^{\oplus m_1! \cdots m_l!},$$

where $\widehat{\mathcal{R}}(\hat{\lambda})$ acts on the LHS via the standard inclusion.

Proof. It follows from Propositions 2.18 and 2.20. □

2.5. Affine cellular structure. In this subsection, we recall the affine cellular algebra structure (in the sense of Koenig-Xi [31]) of the modified quantum loop algebra \tilde{U}_q , following [3, 39]. Note that the notion of affine cellular algebra is closely related to the notion of affine quasi-hereditary algebra as explained in [30, Section 9]. What we discuss in this subsection will be a key ingredient in the proof of our main theorem in Section 4.3.

Let $\lambda = \sum_{i \in I} l_i \varpi_i \in P^+$. By Theorem 2.7(2), the global Weyl module $\mathbb{W}(\lambda)$ is regarded as a $(\tilde{U}_q, \mathcal{R}(\lambda))$ -bimodule. We obtain an $(\mathcal{R}(\lambda), \tilde{U}_q)$ -bimodule $\mathbb{W}(\lambda)^\sharp$ from $\mathbb{W}(\lambda)$ by twisting the actions of \tilde{U}_q and $\mathcal{R}(\lambda)$ with the anti-involution \sharp on $\tilde{U}_q \otimes \mathcal{R}(\lambda)$ given by

$$\sharp(e_i) = f_i, \quad \sharp(f_i) = e_i, \quad \sharp(q^h) = q^h, \quad \sharp(a_\lambda) = a_\lambda, \quad \sharp(z_{j,k}) = z_{j,k}^{-1},$$

where e_i, f_i ($i \in I \cup \{0\}$) are the Chevalley generators (see Remark 2.2), $h \in P^\vee$, $\lambda \in P$ and $z_{j,k}$ ($j \in I, 1 \leq k \leq l_j$) are as in (2.3).

Fix a dominant weight $\lambda \in P^+$. Let $U_{\leq \lambda}$ be the following quotient of the modified quantum loop algebra \tilde{U}_q :

$$(2.10) \quad U_{\leq \lambda} := \tilde{U}_q \Big/ \bigcap_{\mu \leq \lambda} \text{Ann}_{\tilde{U}_q} \mathbb{W}(\mu),$$

where $\text{Ann}_{\tilde{U}_q} M$ denotes the annihilator of a \tilde{U}_q -module M . We fix a total ordering $\{\lambda_1, \lambda_2, \dots, \lambda_l\}$ of the finite set $P_{\leq \lambda}^+ := \{\mu \in P^+ \mid \mu \leq \lambda\}$ such that we have $\lambda_i = \lambda$ and $i < j$ whenever $\lambda_i < \lambda_j$. For each $i \in \{1, 2, \dots, l-1\}$, we define a two-sided ideal I_i of $U_{\leq \lambda}$ by

$$(2.11) \quad I_i := \bigcap_{j \leq i} \text{Ann}_{U_{\leq \lambda}} \mathbb{W}(\lambda_j).$$

We also define $I_0 := U_{\leq \lambda}$ and $I_l := 0$. By definition, we have $I_i \subset I_{i-1}$ for each $i \in \{1, \dots, l\}$.

Theorem 2.22 (Beck-Nakajima [3, 39]). *For each $i \in \{1, \dots, l\}$, there is an isomorphism of $(U_{\leq \lambda}, U_{\leq \lambda})$ -bimodules*

$$I_{i-1}/I_i \cong \mathbb{W}(\lambda_i) \otimes_{\mathcal{R}(\lambda_i)} \mathbb{W}(\lambda_i)^\sharp.$$

Under this isomorphism, the image of the element $a_{\lambda_i} \in I_{i-1}$ corresponds to the generating vector $w_{\lambda_i} \otimes w_{\lambda_i} \in \mathbb{W}(\lambda_i) \otimes_{\mathcal{R}(\lambda_i)} \mathbb{W}(\lambda_i)^\sharp$.

Proof. See [39, Section A(ii), A(iii)]. □

2.6. Hernandez-Leclerc category \mathcal{C}_Q . In this subsection, we define the Hernandez-Leclerc category \mathcal{C}_Q following [21].

Henceforth, we only consider ℓ -weights $\varpi_{i,a}$ with $a = q^p$ for some $p \in \mathbb{Z}$. To simplify the notation, we write $\varpi_{i,p}$ and $\alpha_{i,p}$ for $(i, p) \in I \times \mathbb{Z}$ instead of ϖ_{i,q^p} and α_{i,q^p} respectively. Recall that we defined the subsets $\widehat{I}_Q \subset \widehat{I} \subset I \times \mathbb{Z}$ in Subsection

2.1. Now, we consider the following sublattices:

$$\begin{aligned} \mathcal{P} &\supset \mathcal{P}_{\mathbb{Z}} := \bigoplus_{(i,p) \in \widehat{I}} \mathbb{Z}\varpi_{i,p} \supset \mathcal{P}_Q := \bigoplus_{(i,p) \in \widehat{I}_Q} \mathbb{Z}\varpi_{i,p}, \\ \mathcal{Q} &\supset \mathcal{Q}_{\mathbb{Z}} := \bigoplus_{(i,p) \in \widehat{J}} \mathbb{Z}\alpha_{i,p} \supset \mathcal{Q}_Q := \bigoplus_{(i,p) \in \widehat{J}_Q} \mathbb{Z}\alpha_{i,p}, \end{aligned}$$

where $\widehat{J} := (I \times \mathbb{Z}) \setminus \widehat{I}$ and $\widehat{J}_Q := \{(i, p) \in I \times \mathbb{Z} \mid (i, p - 1), (i, p + 1) \in \widehat{I}_Q\}$. By definition, we have $\mathcal{Q}_{\mathbb{Z}} = \mathcal{P}_{\mathbb{Z}} \cap \mathcal{Q}$. We set $\mathcal{P}_{\mathbb{Z}}^+ := \mathcal{P}_{\mathbb{Z}} \cap \mathcal{P}^+$, $\mathcal{P}_Q^+ := \mathcal{P}_Q \cap \mathcal{P}^+$, $\mathcal{Q}_{\mathbb{Z}}^+ := \mathcal{Q}_{\mathbb{Z}} \cap \mathcal{Q}^+$, $\mathcal{Q}_Q^+ := \mathcal{Q}_Q \cap \mathcal{Q}^+$.

Lemma 2.23. *The following hold.*

- (1) $\mathcal{Q}_Q = \mathcal{P}_Q \cap \mathcal{Q}$.
- (2) Assume that $\hat{\lambda} - \hat{\nu} \in \mathcal{P}_{\mathbb{Z}}^+$ for $\hat{\lambda} \in \mathcal{P}_Q^+$, $\hat{\nu} \in \mathcal{Q}_{\mathbb{Z}}^+$. Then we have $\hat{\nu} \in \mathcal{Q}_Q^+$.

Proof. As mentioned in the last paragraph of Subsection 2.1, the full subquiver Γ_Q with its vertex set \widehat{I}_Q inside \widehat{Q} is isomorphic to the Auslander-Reiten quiver of the path algebra $\mathbb{C}Q$. In particular, the following properties are satisfied:

- (i) If both (i, p_1) and (i, p_2) belong to \widehat{I}_Q with $p_1 < p_2$, then (i, p) also belongs to \widehat{I}_Q for any p with $p_1 < p < p_2$ and $p - \xi_i \in 2\mathbb{Z}$.
- (ii) If both $(i, p - 1)$ and $(i, p + 1)$ belong to \widehat{I}_Q , then (j, p) also belongs to \widehat{I}_Q for any j with $i \sim j$.

From these properties, we obtain the assertions. □

The following combinatorial lemma will be used in Subsection 5.3.

Lemma 2.24. *Suppose that Q is not of type A_1 .⁴*

- (1) Let $(i, p), (j, r) \in \widehat{I}$ be two vertices of the repetition quiver \widehat{Q} . If $d_{i,j}(q^{r-p}) = 0$, there is an oriented path in \widehat{Q} from (i, p) to (j, r) .
- (2) Let $(i, p), (j, r) \in \widehat{I}_Q$ be two vertices of the Auslander-Reiten quiver Γ_Q . If $d_{i,j}(q^{r-p}) = 0$, there is an oriented path in Γ_Q from (i, p) to (j, r) .

Proof. We prove the assertion (1) first. By (2.7), we have $r > p$. By Theorem 2.15, we see that the module $L(\varpi_{i,p}) \otimes L(\varpi_{j,r})$ is not simple. Therefore there is a non-zero element $\hat{\nu} \in \mathcal{Q}_{\mathbb{Z}}^+$ such that

$$(2.12) \quad \varpi_{i,p} + \varpi_{j,r} - \hat{\nu} \in \mathcal{P}_{\mathbb{Z}}^+,$$

which imposes $r - p \geq 2$. We write $\hat{\nu} = \sum_{(k,s) \in X} n_{k,s} \alpha_{k,s}$ with $X := \{(k, s) \in \widehat{J} \mid n_{k,s} > 0\}$. Then from (2.12), we can easily see the following:

- (a) $p < s < r$ holds whenever $(k, s) \in X$;
- (b) $(i, p + 1) \in X$;
- (c) $k = j$ holds when $(k, r - 1) \in X$.

Set $\hat{\nu}_0 := \hat{\nu} - \alpha_{i,p+1}$. Then we have $\hat{\nu}_0 \in \mathcal{Q}_{\mathbb{Z}}^+$ by the property (b). Rewrite (2.12) as

$$(2.13) \quad \sum_{i_0 \sim i} \varpi_{i_0,p+1} - \varpi_{i,p+2} + \varpi_{j,r} - \hat{\nu}_0 \in \mathcal{P}_{\mathbb{Z}}^+.$$

⁴The type A_1 is exceptional here because its repetition quiver is disconnected.

If $p + 2 = r$, we have $(i, p + 2) = (j, r)$ by (c) and find a path $(i, p) \rightarrow (i_0, p + 1) \rightarrow (j, r)$ in \widehat{Q} for an i_0 with $i_0 \sim i$. If $p + 2 < r$, (2.13) implies that there is some $(k_1, p + 2) \in X$ with $k_1 \sim i$. We set $\hat{\nu}_1 := \hat{\nu}_0 - \alpha_{k_1, p+2} \in \mathcal{Q}_{\mathbb{Z}}^+$ and rewrite (2.13) as

$$\sum_{i_0 \sim i, i_0 \neq k_1} \varpi_{i_0, p+1} + \sum_{i_1 \sim k_1, i_1 \neq i} \varpi_{i_1, p+2} - \varpi_{k_1, p+3} + \varpi_{j, r} - \hat{\nu}_1 \in \mathcal{P}_{\mathbb{Z}}^+.$$

If $p + 3 = r$, we have $(k_1, p + 3) = (j, r)$ and find a path $(i, p) \rightarrow (k_1, p + 1) \rightarrow (i_1, p + 2) \rightarrow (j, r)$ for an $i_1 \in I$ with $i_1 \sim k_1$. If $p + 3 < r$, we find another $(k_2, p + 3) \in X$ with $k_2 \sim k_1$ and $k_2 \neq i$, and repeat a similar argument. After repeating a similar argument finitely many times, we get the assertion (1). A proof of the assertion (2) can be completely similar, thanks to Lemma 2.23(2). \square

Definition 2.25 (Hernandez-Leclerc [20, 21]). We define the category \mathcal{C}_Q (resp. $\mathcal{C}_{\mathbb{Z}}$) to be the full subcategory of the category $\mathcal{C}_{\mathfrak{g}}$ consisting of modules whose composition factors are isomorphic to $L(\hat{\lambda})$ for some $\hat{\lambda} \in \mathcal{P}_Q^+$ (resp. $\hat{\lambda} \in \mathcal{P}_{\mathbb{Z}}^+$).

The categories \mathcal{C}_Q and $\mathcal{C}_{\mathbb{Z}}$ are proved to be monoidal subcategories of $\mathcal{C}_{\mathfrak{g}}$ by a similar reason as in the proof of Lemma 2.23. See [21, Lemma 5.8] and [20, Proposition 5.8] respectively.

Using the bijection $\phi: \mathbb{R}^+ \times \{0\} \rightarrow \widehat{I}_Q$, we can write as $\mathcal{P}_Q = \bigoplus_{\alpha \in \mathbb{R}^+} \mathbb{Z} \varpi_{\phi(\alpha)}$, where $\phi(\alpha) := \phi(\alpha, 0)$. We define a \mathbb{Z} -linear map $\mathbf{deg}: \mathcal{P}_Q \rightarrow \mathbb{Q}$ by $\mathbf{deg} \varpi_{\phi(\alpha)} := \alpha$ for $\alpha \in \mathbb{R}^+$. For each $\beta \in \mathbb{Q}^+$, we define the finite subset

$$\mathcal{P}_{Q, \beta}^+ := \{\hat{\lambda} \in \mathcal{P}_Q^+ \mid \mathbf{deg}(\hat{\lambda}) = \beta\}$$

of ℓ -dominant ℓ -weights of degree β . Let $\mathcal{C}_{Q, \beta}$ be the full subcategory of \mathcal{C}_Q consisting of modules whose composition factors are isomorphic to $L(\hat{\lambda})$ for some $\hat{\lambda} \in \mathcal{P}_{Q, \beta}^+$.

Lemma 2.26. *The block decomposition of the category $\mathcal{C}_{\mathfrak{g}}$ in Theorem 2.3 induces a direct decomposition*

$$\mathcal{C}_Q \cong \bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{C}_{Q, \beta}.$$

Moreover, we have $\mathcal{C}_{Q, \beta_1} \otimes \mathcal{C}_{Q, \beta_2} \subset \mathcal{C}_{Q, \beta_1 + \beta_2}$ for $\beta_1, \beta_2 \in \mathbb{Q}^+$.

Proof. Let $(i, p) \in \widehat{I}_Q$. Then the indecomposable module $M(\phi^{-1}(i, p + 1))$ is non-projective and its Auslander-Reiten translation is $M(\phi^{-1}(i, p - 1))$, where we regard $\phi^{-1}: \widehat{I}_Q \rightarrow \mathbb{R}^+$. By the Auslander-Reiten theory, there is an almost split sequence

$$0 \rightarrow M(\phi^{-1}(i, p - 1)) \rightarrow \bigoplus_{j \sim i} M(\phi^{-1}(j, p)) \rightarrow M(\phi^{-1}(i, p + 1)) \rightarrow 0.$$

Because the dimension vector function $\underline{\dim}(-)$ is additive, we have

$$\mathbf{deg} \alpha_{i, p} = \phi^{-1}(i, p - 1) + \phi^{-1}(i, p + 1) - \sum_{j \sim i} \phi^{-1}(j, p) = 0.$$

Therefore we have $\mathbf{deg} \hat{\nu} = 0$ for any $\hat{\nu} \in \mathcal{Q}_Q$. Combining this observation with Theorem 2.3 and Lemma 2.23(1), we obtain the former assertion.

The latter assertion follows from the fact

$$\dim(M_1 \otimes M_2)_{\hat{\lambda}} = \sum_{\hat{\lambda}_1 + \hat{\lambda}_2 = \hat{\lambda}} \dim(M_1)_{\hat{\lambda}_1} \cdot \dim(M_2)_{\hat{\lambda}_2},$$

which holds for any $M_1, M_2 \in \mathcal{C}_{\mathfrak{g}}$. This is due to [16, Theorem 3]. \square

Remark 2.27. The decomposition $\mathcal{C}_Q = \bigoplus_{\beta \in Q^+} \mathcal{C}_{Q,\beta}$ in Lemma 2.26 turns out to be a block decomposition, i.e., $L(\hat{\lambda}_1)$ and $L(\hat{\lambda}_2)$ are linked in $\mathcal{C}_{Q,\beta}$ for any $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathcal{P}_{Q,\beta}^+$. Indeed, the composition multiplicity of the simple module $L(\hat{\lambda})$ in the local Weyl module $W(\hat{\lambda}_\beta)$ is non-zero for each $\hat{\lambda} \in \mathcal{P}_{Q,\beta}^+$ (see Section 3.5 for the notation). This follows from the geometric fact $\mathfrak{M}_0^*(\hat{\lambda}_\beta - \hat{\lambda}, \hat{\lambda}_\beta) \neq \emptyset$ (see Lemma 3.6).

3. QUIVER VARIETIES

In this section, we review the definitions and some properties of the (graded) quiver varieties associated with a Dynkin quiver Q . Basic references are [33–35]. We keep the notation in Section 2.

3.1. Quiver varieties of Dynkin types. Fix an element $\nu = \sum_{i \in I} n_i \alpha_i \in Q^+$ and a dominant weight $\lambda = \sum_{i \in I} l_i \varpi_i \in P^+$. Consider I -graded \mathbb{C} -vector spaces $V^\nu = \bigoplus_{i \in I} V_i^\nu, W^\lambda = \bigoplus_{i \in I} W_i^\lambda$ such that $\dim V_i^\nu = n_i, \dim W_i^\lambda = l_i$ for each $i \in I$. We form the following space of linear maps:

$$\mathbf{N}(V^\nu, W^\lambda) := \left(\bigoplus_{i \rightarrow j \in \Omega} \text{Hom}(V_i^\nu, V_j^\nu) \right) \oplus \left(\bigoplus_{i \in I} \text{Hom}(W_i^\lambda, V_i^\nu) \right),$$

which is considered as the space of framed representations of the quiver Q of dimension vector (ν, λ) . On the space $\mathbf{N}(V^\nu, W^\lambda)$, the group $G(\nu) := \prod_{i \in I} GL(V_i^\nu)$ acts by conjugation. Let

$$\mathbf{M}(V^\nu, W^\lambda) := T^*\mathbf{N}(V^\nu, W^\lambda) = \mathbf{N}(V^\nu, W^\lambda) \oplus \mathbf{N}(V^\nu, W^\lambda)^*$$

be the cotangent space of $\mathbf{N}(V^\nu, W^\lambda)$, which is naturally a symplectic vector space. We identify the space $\mathbf{M}(V^\nu, W^\lambda)$ with the direct sum

$$\left(\bigoplus_{(i,j): i \sim j} \text{Hom}(V_j^\nu, V_i^\nu) \right) \oplus \left(\bigoplus_{i \in I} \text{Hom}(W_i^\lambda, V_i^\nu) \right) \oplus \left(\bigoplus_{i \in I} \text{Hom}(V_i^\nu, W_i^\lambda) \right).$$

According to this direct sum expression, we write an element of $\mathbf{M}(V^\nu, W^\lambda)$ as a triple (B, a, b) of linear maps $B = \bigoplus B_{ij}, a = \bigoplus a_i$ and $b = \bigoplus b_i$. Let $\mu = \bigoplus_{i \in I} \mu_i: \mathbf{M}(V^\nu, W^\lambda) \rightarrow \bigoplus_{i \in I} \mathfrak{gl}(V_i^\nu)$ be the moment map with respect to the $G(\nu)$ -action. Explicitly, it is given by the formula

$$\mu_i(B, a, b) = a_i b_i + \sum_{j \sim i} \varepsilon(i, j) B_{ij} B_{ji},$$

where $\varepsilon(i, j) := 1$ (resp. -1) if we have $j \rightarrow i$ (resp. $i \rightarrow j$) in Ω . A point $(B, a, b) \in \mu^{-1}(0)$ is said to be stable if there exists no non-zero I -graded subspace $V' \subset V^\nu$ such that $B(V') \subset V'$ and $V' \subset \text{Ker } b$. Let $\mu^{-1}(0)^{\text{st}}$ be the set of stable points, on which $G(\nu)$ acts freely. Then we consider a set-theoretic quotient $\mathfrak{M}(\nu, \lambda) := \mu^{-1}(0)^{\text{st}}/G(\nu)$. It is known that this quotient has a structure of a non-singular quasi-projective variety which is isomorphic to a quotient in the geometric invariant theory. On the other hand, we also consider the affine algebro-geometric quotient $\mathfrak{M}_0(\nu, \lambda) := \mu^{-1}(0)//G(\nu) = \text{Spec } \mathbb{C}[\mu^{-1}(0)]^{G(\nu)}$, together with the canonical projective morphism $\pi: \mathfrak{M}(\nu, \lambda) \rightarrow \mathfrak{M}_0(\nu, \lambda)$. We refer to these varieties $\mathfrak{M}(\nu, \lambda), \mathfrak{M}_0(\nu, \lambda)$ as *quiver varieties*.

On the linear space $\mathbf{M}(V^\nu, W^\lambda)$, the group $G(\lambda) := \prod_{i \in I} GL(W_i^\lambda)$ acts by conjugation and \mathbb{C}^\times acts as the scalar multiplication. The combined action of the

group $\mathbb{G}(\lambda) := G(\lambda) \times \mathbb{C}^\times$ on $\mathbf{M}(V^\nu, W^\lambda)$ commutes with the action of the group $G(\nu)$. Thus we have the induced $\mathbb{G}(\lambda)$ -actions on the quotients $\mathfrak{M}(\nu, \lambda), \mathfrak{M}_0(\nu, \lambda)$, which make the canonical morphism π into a $\mathbb{G}(\lambda)$ -equivariant morphism.

For $\nu, \nu' \in \mathbb{Q}^+$ with $\nu \leq \nu'$, we fix a direct sum decomposition $V^{\nu'} = V^\nu \oplus V^{\nu'-\nu}$. Extending by 0 on $V^{\nu'-\nu}$, we have an injective linear map $\mathbf{M}(V^\nu, W^\lambda) \hookrightarrow \mathbf{M}(V^{\nu'}, W^\lambda)$. This induces a natural closed embedding $\mathfrak{M}_0(\nu, \lambda) \hookrightarrow \mathfrak{M}_0(\nu', \lambda)$, which does not depend on the choice of decomposition $V^{\nu'} = V^\nu \oplus V^{\nu'-\nu}$. Via this natural embedding, we regard $\mathfrak{M}_0(\nu, \lambda)$ as a closed subvariety of $\mathfrak{M}_0(\nu', \lambda)$. We consider the union of them and obtain the following combined morphism:

$$\pi: \mathfrak{M}(\lambda) := \bigsqcup_{\nu} \mathfrak{M}(\nu, \lambda) \rightarrow \mathfrak{M}_0(\lambda) := \bigcup_{\nu} \mathfrak{M}_0(\nu, \lambda).$$

For each $x \in \mathfrak{M}_0(\lambda)$, let $\mathfrak{M}(\lambda)_x := \pi^{-1}(x)$ denote the fiber of x . The fiber $\mathfrak{L}(\lambda) := \pi^{-1}(0)$ of the origin $0 \in \mathfrak{M}_0(\lambda)$ is called the *central fiber*. We also set $\mathfrak{M}(\nu, \lambda)_x := \mathfrak{M}(\lambda)_x \cap \mathfrak{M}(\nu, \lambda)$ and $\mathfrak{L}(\nu, \lambda) := \mathfrak{L}(\lambda) \cap \mathfrak{M}(\nu, \lambda)$.

Recall that the geometric points of $\mathfrak{M}_0(\nu, \lambda)$ correspond to the closed $G(\nu)$ -orbits in $\mu^{-1}(0)$. Let $\mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$ be the subset of $\mathfrak{M}_0(\nu, \lambda)$ consisting of the closed $G(\nu)$ -orbits containing the elements $\mathbf{x} = (B, a, b) \in \mu^{-1}(0)$ with trivial stabilizers (i.e., $\text{Stab}_{G(\nu)} \mathbf{x} = \{1\}$). This is a (possibly empty) non-singular open subset of $\mathfrak{M}_0(\nu, \lambda)$, on which the morphism π becomes an isomorphism of varieties $\pi^{-1}(\mathfrak{M}_0^{\text{reg}}(\nu, \lambda)) \xrightarrow{\cong} \mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$. It is known that $\mathfrak{M}_0^{\text{reg}}(\nu, \lambda) \neq \emptyset$ if and only if $\lambda - \nu$ is a dominant weight appearing in the finite-dimensional irreducible \mathfrak{g} -module of highest weight λ . They form a stratification:

$$(3.1) \quad \mathfrak{M}_0(\lambda) = \bigsqcup_{\nu \in \mathbb{Q}^+; \lambda - \nu \in \mathbb{P}^+} \mathfrak{M}_0^{\text{reg}}(\nu, \lambda).$$

The closure inclusion $\mathfrak{M}_0^{\text{reg}}(\nu, \lambda) \subset \overline{\mathfrak{M}_0^{\text{reg}}(\nu', \lambda)}$ implies $\nu \leq \nu'$.

3.2. Graded quiver varieties. Fix an element $\hat{\nu} = \sum_{(i,p) \in \hat{\mathcal{J}}} n_{i,p} \alpha_{i,p} \in \mathcal{Q}_{\mathbb{Z}}^+$ and an ℓ -dominant ℓ -weight $\hat{\lambda} = \sum_{(i,p) \in \hat{\mathcal{I}}} l_{i,p} \varpi_{i,p} \in \mathcal{P}_{\mathbb{Z}}^+$. Consider a $\hat{\mathcal{J}}$ -graded \mathbb{C} -vector space $V^{\hat{\nu}} = \bigoplus_{(i,p) \in \hat{\mathcal{J}}} V_i^{\hat{\nu}}(p)$ with $\dim V_i^{\hat{\nu}}(p) = n_{i,p}$ for $(i,p) \in \hat{\mathcal{J}}$, and an $\hat{\mathcal{I}}$ -graded \mathbb{C} -vector space $W^{\hat{\lambda}} = \bigoplus_{(i,p) \in \hat{\mathcal{I}}} W_i^{\hat{\lambda}}(p)$ with $\dim W_i^{\hat{\lambda}}(p) = l_{i,p}$ for $(i,p) \in \hat{\mathcal{I}}$. We consider the following space of linear maps:

$$\mathbf{M}^\bullet(V^{\hat{\nu}}, W^{\hat{\lambda}}) := \left(\bigoplus_{(i,p) \in \hat{\mathcal{J}}, j \in \hat{\mathcal{I}}; i \sim j} \text{Hom}(V_i^{\hat{\nu}}(p), V_j^{\hat{\nu}}(p-1)) \right) \oplus \left(\bigoplus_{(i,p) \in \hat{\mathcal{I}}} \text{Hom}(W_i^{\hat{\lambda}}(p), V_i^{\hat{\nu}}(p-1)) \right) \oplus \left(\bigoplus_{(i,p) \in \hat{\mathcal{J}}} \text{Hom}(V_i^{\hat{\nu}}(p), W_i^{\hat{\lambda}}(p-1)) \right).$$

According to this direct sum expression, we write an element of $\mathbf{M}^\bullet(V^{\hat{\nu}}, W^{\hat{\lambda}})$ as a triple (B, a, b) of linear maps $B = \bigoplus B_{ji}(p)$, $a = \bigoplus a_i(p)$ and $b = \bigoplus b_i(p)$. Let $\mu^\bullet = \bigoplus_{(i,p) \in \widehat{\mathcal{J}}} \mu_{i,p}^\bullet: \mathbf{M}^\bullet(V^{\hat{\nu}}, W^{\hat{\lambda}}) \rightarrow \bigoplus_{(i,p) \in \widehat{\mathcal{J}}} \text{Hom}(V_i^{\hat{\nu}}(p), V_i^{\hat{\nu}}(p-2))$ be the map defined by the formula

$$\mu_{i,p}^\bullet(B, a, b) = a_i(p-1)b_i(p) + \sum_{j \sim i} \varepsilon(i, j) B_{ij}(p-1) B_{ji}(p),$$

where $\varepsilon(i, j)$ is the same as in Subsection 3.1. The map μ^\bullet is equivariant with respect to the conjugate action of the group $G(\hat{\nu}) := \prod_{(i,p) \in \widehat{\mathcal{J}}} GL(V_i^{\hat{\nu}}(p))$. A point $(B, a, b) \in \mu^{\bullet-1}(0)$ is said to be stable if there exists no non-zero $\widehat{\mathcal{J}}$ -graded subspace $V' \subset V^{\hat{\nu}}$ such that $B(V') \subset V'$ and $V' \subset \text{Ker } b$. Let $\mu^{\bullet-1}(0)^{\text{st}}$ be the set of stable points. Similarly as in Subsection 3.1, we consider two kinds of quotients $\mathfrak{M}^\bullet(\hat{\nu}, \hat{\lambda}) := \mu^{\bullet-1}(0)^{\text{st}}/G(\hat{\nu})$ and $\mathfrak{M}_0^\bullet(\hat{\nu}, \hat{\lambda}) := \mu^{\bullet-1}(0)/G(\hat{\nu})$, together with the canonical projective morphism $\pi^\bullet: \mathfrak{M}^\bullet(\hat{\nu}, \hat{\lambda}) \rightarrow \mathfrak{M}_0^\bullet(\hat{\nu}, \hat{\lambda})$. We refer to these varieties $\mathfrak{M}^\bullet(\hat{\nu}, \hat{\lambda}), \mathfrak{M}_0^\bullet(\hat{\nu}, \hat{\lambda})$ as *graded quiver varieties*.

On the space $\mathbf{M}^\bullet(V^{\hat{\nu}}, W^{\hat{\lambda}})$, we have the conjugate action of the group $G(\hat{\lambda}) := \prod_{(i,p) \in \widehat{\mathcal{I}}} GL(W_i^{\hat{\lambda}}(p))$ and the scalar action of \mathbb{C}^\times . The combined action of the group $\mathbb{G}(\hat{\lambda}) := G(\hat{\lambda}) \times \mathbb{C}^\times$ on $\mathbf{M}(V^{\hat{\nu}}, W^{\hat{\lambda}})$ induces actions on the quotients $\mathfrak{M}(\hat{\nu}, \hat{\lambda}), \mathfrak{M}_0(\hat{\nu}, \hat{\lambda})$ which make the canonical morphism π^\bullet into a $\mathbb{G}(\hat{\lambda})$ -equivariant morphism. As in Subsection 3.1, we can form the unions:

$$\pi^\bullet: \mathfrak{M}^\bullet(\hat{\lambda}) := \bigsqcup_{\hat{\nu}} \mathfrak{M}^\bullet(\hat{\nu}, \hat{\lambda}) \rightarrow \mathfrak{M}_0^\bullet(\hat{\lambda}) := \bigsqcup_{\hat{\nu}} \mathfrak{M}_0^\bullet(\hat{\nu}, \hat{\lambda}).$$

Let $\mathfrak{M}^\bullet(\hat{\lambda})_x := \pi^{\bullet-1}(x)$ denote the fiber of a point $x \in \mathfrak{M}_0(\hat{\lambda})$. We set $\mathfrak{L}^\bullet(\hat{\lambda}) := \pi^{\bullet-1}(0)$.

3.3. Identification with fixed point subvarieties. Let $\hat{\lambda} = \sum l_{i,p} \varpi_{i,p} \in \mathcal{P}_{\mathbb{Z}}^+$ be an ℓ -dominant ℓ -weight. In this subsection, we recall that the graded quiver varieties $\mathfrak{M}^\bullet(\hat{\lambda}), \mathfrak{M}_0^\bullet(\hat{\lambda})$ can be realized as fixed point subvarieties of the usual quiver varieties $\mathfrak{M}(\lambda), \mathfrak{M}_0(\lambda)$ with $\lambda := \text{cl}(\hat{\lambda})$ with respect to a certain torus action.

We have $\lambda = \sum_{i \in I} l_i \varpi_i$ with $l_i = \sum_{p \in 2\mathbb{Z} + \xi_i} l_{i,p}$ by the definition of cl . For each $i \in I$, we choose a direct sum decomposition $W_i^\lambda = \bigoplus_{p \in 2\mathbb{Z} + \xi_i} W_i^{\hat{\lambda}}(p)$ such that $\dim W_i^{\hat{\lambda}}(p) = l_{i,p}$. Note that this choice specifies a group embedding $G(\hat{\lambda}) \hookrightarrow G(\lambda)$. Define a 1-parameter subgroup $f_i: \mathbb{C}^\times \rightarrow GL(W_i^\lambda)$ by $f_i(t)|_{W_i^{\hat{\lambda}}(p)} := t^p \cdot \text{id}_{W_i^{\hat{\lambda}}(p)}$ for $t \in \mathbb{C}^\times$ and a 1-dimensional subtorus $\mathbb{T} := (\prod_{i \in I} f_i \times \text{id})(\mathbb{C}^\times)$ of $\mathbb{G}(\lambda)$. Then we consider the subvarieties $\mathfrak{M}(\lambda)^\mathbb{T}, \mathfrak{M}_0(\lambda)^\mathbb{T}$ consisting of \mathbb{T} -fixed points and the induced canonical morphism $\pi^\mathbb{T}: \mathfrak{M}(\lambda)^\mathbb{T} \rightarrow \mathfrak{M}_0(\lambda)^\mathbb{T}$. Since the centralizer of \mathbb{T} in $\mathbb{G}(\lambda)$ is identical to the subgroup $\mathbb{G}(\hat{\lambda}) = G(\hat{\lambda}) \times \mathbb{C}^\times \subset \mathbb{G}(\lambda)$, we have the induced action of $\mathbb{G}(\hat{\lambda})$ on the \mathbb{T} -fixed point subvarieties $\mathfrak{M}(\lambda)^\mathbb{T}, \mathfrak{M}_0(\lambda)^\mathbb{T}$. The morphism $\pi^\mathbb{T}$ is $\mathbb{G}(\hat{\lambda})$ -equivariant.

On the other hand, for each $\hat{\nu} = \sum n_{i,p} \alpha_{i,p} \in \mathcal{Q}_{\mathbb{Z}}^+$, we fix a direct sum decomposition $V_i^\nu = \bigoplus_{p \in 2\mathbb{Z} + \xi_i + 1} V_i^{\hat{\nu}}(p)$ of I -graded vector space V^ν with $\nu := \text{cl}(\hat{\nu})$ such that $\dim V_i^{\hat{\nu}}(p) = n_{i,p}$, just as we have done for W^λ in the last paragraph. These direct sum decompositions induce an embedding $\iota_{\hat{\nu}, \hat{\lambda}}: \mathbf{M}^\bullet(V^{\hat{\nu}}, W^{\hat{\lambda}}) \hookrightarrow \mathbf{M}(V^\nu, W^\lambda)$.

After taking the quotients, this embedding $\iota_{\hat{\nu}, \hat{\lambda}}$ yields the morphisms $\mathfrak{M}^\bullet(\hat{\nu}, \hat{\lambda}) \rightarrow \mathfrak{M}(\nu, \lambda)^\mathbb{T}$ and $\mathfrak{M}_0^\bullet(\hat{\nu}, \hat{\lambda}) \rightarrow \mathfrak{M}_0(\nu, \lambda)^\mathbb{T}$.

Lemma 3.1. *The above morphisms induce $\mathbb{G}(\hat{\lambda})$ -equivariant isomorphisms $\mathfrak{M}^\bullet(\hat{\lambda}) \xrightarrow{\cong} \mathfrak{M}(\lambda)^\mathbb{T}, \mathfrak{M}_0^\bullet(\hat{\lambda}) \xrightarrow{\cong} \mathfrak{M}_0(\lambda)^\mathbb{T}$ which make the following diagram commute:*

$$\begin{array}{ccc} \mathfrak{M}^\bullet(\hat{\lambda}) & \xrightarrow{\cong} & \mathfrak{M}(\lambda)^\mathbb{T} \\ \pi^\bullet \downarrow & & \downarrow \pi^\mathbb{T} \\ \mathfrak{M}_0^\bullet(\hat{\lambda}) & \xrightarrow{\cong} & \mathfrak{M}_0(\lambda)^\mathbb{T}. \end{array}$$

In particular, we have a $\mathbb{G}(\hat{\lambda})$ -equivariant isomorphism $\mathfrak{L}^\bullet(\hat{\lambda}) \cong \mathfrak{L}(\lambda)^\mathbb{T}$.

Proof. See [35, Section 4]. □

Hereafter, we identify the graded quiver varieties $\mathfrak{M}_0^\bullet(\hat{\lambda}), \mathfrak{M}^\bullet(\hat{\lambda})$ with the \mathbb{T} -fixed point subvarieties $\mathfrak{M}_0(\lambda)^\mathbb{T}, \mathfrak{M}(\lambda)^\mathbb{T}$ via the isomorphisms in Lemma 3.1. Then we have

$$(3.2) \quad \mathfrak{M}(\nu, \lambda)^\mathbb{T} = \bigsqcup_{\hat{\nu} \in \mathcal{Q}_\mathbb{Z}^+; \text{cl}(\hat{\nu})=\nu} \mathfrak{M}^\bullet(\hat{\nu}, \hat{\lambda}), \quad \mathfrak{M}_0(\nu, \lambda)^\mathbb{T} = \bigsqcup_{\hat{\nu} \in \mathcal{Q}_\mathbb{Z}^+; \text{cl}(\hat{\nu})=\nu} \mathfrak{M}_0^\bullet(\hat{\nu}, \hat{\lambda}).$$

We define $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda}) := \mathfrak{M}_0^\bullet(\hat{\nu}, \hat{\lambda}) \cap \mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$. It is known that $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda}) \neq \emptyset$ if and only if $\hat{\lambda} - \hat{\nu}$ is an ℓ -dominant ℓ -weight appearing in the local Weyl module $W(\hat{\lambda})$. By (3.1) and (3.2), we get a stratification:

$$(3.3) \quad \mathfrak{M}_0^\bullet(\hat{\lambda}) = \bigsqcup_{\hat{\nu} \in \mathcal{Q}_\mathbb{Z}^+; \hat{\lambda} - \hat{\nu} \in \mathcal{P}_\mathbb{Z}^+} \mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda}).$$

The closure inclusion $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}_1, \hat{\lambda}) \subset \overline{\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}_2, \hat{\lambda})}$ implies $\hat{\nu}_1 \leq \hat{\nu}_2$.

3.4. Structure of non-central fibers. In this subsection, we recall the structure of (non-central) fibers of the canonical morphisms π and π^\bullet . Our exposition is based on [33, Section 6], [35, Section 3] and [38, Section 2.7] with some more details on the group actions.

Let $(\nu, \lambda) \in \mathbb{Q}^+ \times \mathbb{P}^+$ be a pair. For any triple $\mathbf{x} = (B, a, b) \in \mu^{-1}(0) \subset \mathbf{M}(V^\nu, W^\lambda)$, we consider the following two kinds of complexes of vector spaces:

$$(3.4) \quad C_i(\nu, \lambda)_\mathbf{x}: \quad V_i^\nu \xrightarrow{\sigma_i} W_i^\lambda \oplus \bigoplus_{j \sim i} V_j^\nu \xrightarrow{\tau_i} V_i^\nu \quad \text{for each } i \in I,$$

where we define $\sigma_i := b_i \oplus \bigoplus_j B_{ji}$ and $\tau_i := a_i + \sum_j \varepsilon(i, j) B_{ij}$;

$$(3.5) \quad \mathcal{C}(\nu, \lambda)_\mathbf{x}: \quad \bigoplus_{i \in I} \text{End}(V_i^\nu) \xrightarrow{\iota} \mathbf{M}(V^\nu, W^\lambda) \xrightarrow{d\mu} \bigoplus_{i \in I} \text{End}(V_i^\nu),$$

where ι is given by $\iota(\xi) = (B\xi - \xi B) \oplus (-\xi a) \oplus (b\xi)$ and $d\mu$ is the differential of the moment map $\mu = \bigoplus_i \mu_i$ at the point $\mathbf{x} = (B, a, b)$. Note that the middle cohomology $H^0(\mathcal{C}(\nu, \lambda)_\mathbf{x})$ of the complex (3.5) is identical to the quotient space $(T_\mathbf{x}G(\nu)_\mathbf{x})^\perp / T_\mathbf{x}G(\nu)_\mathbf{x}$, where $(T_\mathbf{x}G(\nu)_\mathbf{x})^\perp$ is the symplectic perpendicular of the tangent space $T_\mathbf{x}G(\nu)_\mathbf{x}$ of the $G(\nu)$ -orbit of \mathbf{x} . In particular, if \mathbf{x} is stable, the space $H^0(\mathcal{C}(\nu, \lambda)_\mathbf{x})$ is isomorphic to the tangent space $T_x\mathfrak{M}(\nu, \lambda)$ of the point $x \in \mathfrak{M}(\nu, \lambda)$ corresponding to \mathbf{x} .

Let $(\nu, \lambda) \in \mathbf{Q}^+ \times \mathbf{P}^+$ be a pair such that $\mathfrak{M}_0^{\text{reg}}(\nu, \lambda) \neq \emptyset$. Recall that we have $\lambda - \nu \in \mathbf{P}^+$ in this case. We fix a point $x_\nu \in \mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$ and its lift $\mathbf{x}_\nu \in \mu^{-1}(0) \subset \mathbf{M}(V^\nu, W^\lambda)$ whose $G(\nu)$ -orbit is closed. Then in the complex $C_i(\nu, \lambda)_{\mathbf{x}_\nu}$, the map σ_i is injective [34, Proposition 3.24] and the map τ_i is surjective [34, Lemma 4.7]. In particular, the dimension of the middle cohomology $H^0(C_i(\nu, \lambda)_{\mathbf{x}_\nu}) = \text{Ker } \tau_i / \text{Im } \sigma_i$ is equal to $(\lambda - \nu)(h_i)$. Therefore we can identify $W_i^{\lambda - \nu} = H^0(C_i(\nu, \lambda)_{\mathbf{x}_\nu})$.

We pick an arbitrary element ν' such that $\nu \leq \nu'$. In order to construct the natural embedding $\mathfrak{M}_0(\nu, \lambda) \hookrightarrow \mathfrak{M}_0(\nu', \lambda)$, we fix a direct sum decomposition $V^{\nu'} = V^\nu \oplus V^{\nu' - \nu}$. Extending by 0 on $V^{\nu' - \nu}$, we have an injective linear map $\mathbf{M}(V^\nu, W^\lambda) \hookrightarrow \mathbf{M}(V^{\nu'}, W^\lambda)$, by which our fixed element $\mathbf{x}_\nu = (B, a, b)$ is regarded as an element of $\mu^{-1}(0) \subset \mathbf{M}(V^{\nu'}, W^\lambda)$. Then we can calculate as

$$(3.6) \quad H^0(\mathcal{C}(\nu', \lambda)_{\mathbf{x}_\nu}) \cong \mathbf{M}(V^{\nu' - \nu}, W^{\lambda - \nu}) \oplus H^0(\mathcal{C}(\nu, \lambda)_{\mathbf{x}_\nu}),$$

where we have $W_i^{\lambda - \nu} = H^0(C_i(\nu, \lambda)_{\mathbf{x}_\nu})$. We also see that the space $H^0(\mathcal{C}(\nu, \lambda)_{\mathbf{x}_\nu})$ is isomorphic to the tangent space $T := T_{x_\nu} \mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$.

The stabilizer $\text{Stab}_{G(\nu')} \mathbf{x}_\nu$ is naturally isomorphic to $G(\nu' - \nu)$. Under this isomorphism, the action of $\text{Stab}_{G(\nu')} \mathbf{x}_\nu$ on the LHS of (3.6) coincides with the action of $G(\nu' - \nu)$ on the RHS of (3.6), which is the direct sum of the natural action on $\mathbf{M}(V^{\nu' - \nu}, W^{\lambda - \nu})$ and the trivial action on $T \cong H^0(\mathcal{C}(\nu, \lambda)_{\mathbf{x}_\nu})$.

An appropriate Hamiltonian reduction with respect to the action of the group $\text{Stab}_{G(\nu')} \mathbf{x}_\nu \cong G(\nu' - \nu)$ on the RHS of (3.6) yields the following canonical map:

$$\pi \times \text{id}: \mathfrak{M}(\nu' - \nu, \lambda - \nu) \times T \rightarrow \mathfrak{M}_0(\nu' - \nu, \lambda - \nu) \times T.$$

According to the discussion in [35, Section 3], this gives a local description of $\pi: \mathfrak{M}(\nu', \lambda) \rightarrow \mathfrak{M}_0(\nu', \lambda)$ around the point $x_\nu \in \mathfrak{M}_0^{\text{reg}}(\nu, \lambda) \subset \mathfrak{M}_0(\nu', \lambda)$. In precise, we have Theorem 3.2.

Theorem 3.2 (Nakajima [35, Theorem 3.3.2]). *Let $x_\nu \in \mathfrak{M}_0^{\text{reg}}(\nu, \lambda) \subset \mathfrak{M}_0(\nu', \lambda)$. Then there exist neighborhoods U, U_S, U_T of $x_\nu \in \mathfrak{M}_0(\nu', \lambda)$, $0 \in \mathfrak{M}_0(\nu' - \nu, \lambda - \nu)$, $0 \in T := T_{x_\nu} \mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$ respectively and biholomorphic maps $U \xrightarrow{\cong} U_S \times U_T$; $x_\nu \mapsto (0, 0)$ and $\pi^{-1}(U) \xrightarrow{\cong} \pi^{-1}(U_S) \times U_T$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{M}(\nu', \lambda) & \supset & \pi^{-1}(U) \xrightarrow{\cong} \pi^{-1}(U_S) \times U_T & \subset & \mathfrak{M}(\nu' - \nu, \lambda - \nu) \times T \\ & & \pi \downarrow & & \downarrow \pi \times \text{id} \\ \mathfrak{M}_0(\nu', \lambda) & \supset & U \xrightarrow{\cong} U_S \times U_T & \subset & \mathfrak{M}_0(\nu' - \nu, \lambda - \nu) \times T \end{array}$$

Now let us consider the action of the group $\text{Stab}_{\mathbb{G}(\lambda)} x_\nu$ on the fiber $\mathfrak{M}(\lambda)_{x_\nu}$. By the definition of $\mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$, we have $\text{Stab}_{G(\nu)} \mathbf{x}_\nu = \{1\}$. Therefore the second projection $G(\nu) \times \mathbb{G}(\lambda) \rightarrow \mathbb{G}(\lambda)$ restricts to an isomorphism $r: \text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu \xrightarrow{\cong} \text{Stab}_{\mathbb{G}(\lambda)} x_\nu$. Via the fixed decomposition $V^{\nu'} = V^\nu \oplus V^{\nu' - \nu}$, we regard the group $\text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu$ as a subgroup of $\text{Stab}_{G(\nu') \times \mathbb{G}(\lambda)} \mathbf{x}_\nu$. In fact, we have

$$(3.7) \quad \text{Stab}_{G(\nu') \times \mathbb{G}(\lambda)} \mathbf{x}_\nu \cong \text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu \times G(\nu' - \nu).$$

Thus the group $\text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu$ acts on the vector space $\mathbf{M}(\nu', \lambda)$ and also on the middle cohomology (3.6) of the complex $\mathcal{C}(\nu', \lambda)_{\mathbf{x}_\nu}$. Note that this induced action preserves each summand of the RHS of (3.6). In particular, we obtain an action of the group $\text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu$ on the vector space $\mathbf{M}(V^{\nu' - \nu}, W^{\lambda - \nu})$. By

the construction, we can easily see that this action factors through the natural action of $\mathbb{G}(\lambda - \nu)$ on $\mathbf{M}(V^{\nu'-\nu}, W^{\lambda-\nu})$. The corresponding group homomorphism $\text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu \rightarrow \mathbb{G}(\lambda - \nu) = G(\lambda - \nu) \times \mathbb{C}^\times$ is the direct product of two homomorphisms $\varphi: \text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu \rightarrow G(\lambda - \nu)$ and $\rho: \text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu \rightarrow \mathbb{C}^\times$. The homomorphism φ is given as the induced action of the group $\text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu$ on the middle cohomology of the complex $C_i(\nu, \lambda)_{\mathbf{x}_\nu}$ under the identification $W_i^{\lambda-\nu} = H^0(C_i(\nu, \lambda)_{\mathbf{x}_\nu})$. The homomorphism ρ is obtained by the projection

$$\rho: \text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu \hookrightarrow G(\nu) \times \mathbb{G}(\lambda) = G(\nu) \times G(\lambda) \times \mathbb{C}^\times \xrightarrow{\text{pr}_3} \mathbb{C}^\times.$$

The action of the group $\text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu$ on the space $\mathbf{M}(V^{\nu'-\nu}, W^{\lambda-\nu})$ commutes with the action of group $G(\nu' - \nu)$. After taking the Hamiltonian reductions with respect to the action of the group $G(\nu' - \nu)$, we obtain an action of the group $\text{Stab}_{\mathbb{G}(\lambda)} x_\nu$ on the central fiber $\mathfrak{L}(\nu' - \nu, \lambda - \nu)$. The above argument says that this action factors through the group homomorphism

$$(3.8) \quad \text{Stab}_{\mathbb{G}(\lambda)} x_\nu \xrightarrow{\cong} \text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \mathbf{x}_\nu \xrightarrow{\varphi \times \rho} \mathbb{G}(\lambda - \nu).$$

This homomorphism (3.8) does not depend on ν' .

By Theorem 3.2, there is an isomorphism $\mathfrak{M}(\nu', \lambda)_{x_\nu} \xrightarrow{\cong} \mathfrak{L}(\nu' - \nu, \lambda - \nu)$. As stated in [35, Remark 3.3.3], this isomorphism can be equivariant with respect to the actions of the group $\text{Stab}_{\mathbb{G}(\lambda)} x_\nu$. Summing up over ν' , we obtain the following.

Lemma 3.3. *Let $(\nu, \lambda) \in \mathbb{Q}^+ \times \mathbb{P}^+$ be a pair such that $\mathfrak{M}_0^{\text{reg}}(\nu, \lambda) \neq \emptyset$ and $\pi: \mathfrak{M}(\lambda) \rightarrow \mathfrak{M}_0(\lambda)$ be the canonical morphism. Then for each point $x_\nu \in \mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$, there exists a $(\text{Stab}_{\mathbb{G}(\lambda)} x_\nu)$ -equivariant isomorphism*

$$\mathfrak{M}(\lambda)_{x_\nu} \cong \mathfrak{L}(\lambda - \nu),$$

where the group $\text{Stab}_{\mathbb{G}(\lambda)} x_\nu$ acts on the RHS $\mathfrak{L}(\lambda - \nu)$ via the group homomorphism $(\varphi \times \rho) \circ r^{-1}$ in (3.8).

Next we consider the graded versions. Let $(\hat{\nu}, \hat{\lambda}) \in \mathbb{Q}_{\mathbb{Z}}^+ \times \mathbb{P}_{\mathbb{Z}}^+$ be a pair. For any triple $\mathbf{x} = (B, a, b) \in \mu^{\bullet-1}(0) \subset \mathbf{M}^\bullet(V^{\hat{\nu}}, W^{\hat{\lambda}})$ and $(i, p) \in \hat{I}$, we can consider a complex of vector spaces

$$C_{i,p}(\hat{\nu}, \hat{\lambda})_{\mathbf{x}}: V_i^{\hat{\nu}}(p+1) \xrightarrow{\sigma_{i,p}} W_i^{\hat{\lambda}}(p) \oplus \bigoplus_{j \sim i} V_j^{\hat{\nu}}(p) \xrightarrow{\tau_{i,p}} V_i^{\hat{\nu}}(p-1),$$

where we define $\sigma_{i,p} := b_i(p+1) \oplus \bigoplus_j B_{ji}(p+1)$ and $\tau_{i,p} := a_i(p) + \sum_j \varepsilon(i, j) B_{ij}(p)$.

Now we assume that $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda}) \neq \emptyset$. In particular, we have $\hat{\lambda} - \hat{\nu} \in \mathbb{P}_{\mathbb{Z}}^+$. We fix a point $x_{\hat{\nu}} \in \mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda})$ and its lift $\mathbf{x}_{\hat{\nu}} \in \mu^{\bullet-1}(0) \subset \mathbf{M}^\bullet(V^{\hat{\nu}}, W^{\hat{\lambda}})$ whose $G(\hat{\nu})$ -orbit is closed. By the same reason as in the non-graded case, in the complex $C_{i,p}(\hat{\nu}, \hat{\lambda})_{\mathbf{x}_{\hat{\nu}}}$, the map $\sigma_{i,p}$ is injective and the map $\tau_{i,p}$ is surjective. Therefore the dimension vector of the \hat{I} -graded vector space $\bigoplus_{(i,p) \in \hat{I}} H^0(C_{i,p}(\hat{\nu}, \hat{\lambda})_{\mathbf{x}_{\hat{\nu}}})$ is equal to $\hat{\lambda} - \hat{\nu}$. This allows us to identify $W_i^{\hat{\lambda}-\hat{\nu}}(p)$ with $H^0(C_{i,p}(\hat{\nu}, \hat{\lambda})_{\mathbf{x}_{\hat{\nu}}})$ for each $(i, p) \in \hat{I}$. Similarly as in (3.8), we consider the following group homomorphism

$$(3.9) \quad \text{Stab}_{\mathbb{G}(\hat{\lambda})} x_{\hat{\nu}} \xrightarrow{\cong} \text{Stab}_{G(\hat{\nu}) \times \mathbb{G}(\hat{\lambda})} \mathbf{x}_{\hat{\nu}} \xrightarrow{\hat{\varphi} \times \hat{\rho}} \mathbb{G}(\hat{\lambda} - \hat{\nu}),$$

where \hat{r} is the isomorphism obtained as the restriction of the projection $G(\hat{\nu}) \times \mathbb{G}(\hat{\lambda}) \rightarrow \mathbb{G}(\hat{\lambda})$, $\hat{\varphi}$ is given as the induced action of the group $\text{Stab}_{G(\hat{\nu}) \times \mathbb{G}(\hat{\lambda})} \mathbf{x}_{\hat{\nu}}$ on

$W_i^{\hat{\lambda}-\hat{\nu}}(p) = H^0(C_{i,p}(\hat{\nu}, \hat{\lambda})_{\mathbf{x}_{\hat{\nu}}})$ and $\hat{\rho}$ is the restriction of the projection $G(\hat{\nu}) \times G(\hat{\lambda}) \times \mathbb{C}^\times \rightarrow \mathbb{C}^\times$.

Lemma 3.4. *Let $(\hat{\nu}, \hat{\lambda}) \in \mathcal{Q}_\mathbb{Z}^+ \times \mathcal{P}_\mathbb{Z}^+$ be a pair such that $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda}) \neq \emptyset$. Then for each point $x_{\hat{\nu}} \in \mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda})$, there exists a $(\text{Stab}_{\mathbb{G}(\hat{\lambda})} x_{\hat{\nu}})$ -equivariant isomorphism*

$$\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}} \cong \mathfrak{L}^\bullet(\hat{\lambda} - \hat{\nu}),$$

where the group $\text{Stab}_{\mathbb{G}(\hat{\lambda})} x_{\hat{\nu}}$ acts on the RHS $\mathfrak{L}^\bullet(\hat{\lambda} - \hat{\nu})$ via the group homomorphism $(\hat{\varphi} \times \hat{\rho}) \circ \hat{r}^{-1}$ in (3.9).

For a proof, we use the description in the non-graded case (Lemma 3.3) and the identification with the \mathbb{T} -fixed point subvarieties (Lemma 3.1).

Proof. We put $\nu := \text{cl}(\hat{\nu}), \lambda := \text{cl}(\hat{\lambda})$. We make identifications of vector spaces: $W_i^\lambda = \bigoplus_{p \in 2\mathbb{Z} + \xi_i} W_i^\lambda(p), V_i^\nu = \bigoplus_{p \in 2\mathbb{Z} + \xi_i + 1} V_i^\nu(p)$, which specifies an embedding $\iota \equiv \iota_{\hat{\nu}, \hat{\lambda}}: \mathbf{M}^\bullet(V^\nu, W^\lambda) \hookrightarrow \mathbf{M}(V^\nu, W^\lambda)$. Using these direct sum decompositions, we define a group homomorphism $f_i: \mathbb{C}^\times \rightarrow GL(W_i^\lambda)$ (resp. $g_i: \mathbb{C}^\times \rightarrow GL(V_i^\nu)$) for each $i \in I$ by $f_i(t)|_{W_i^\lambda(p)} := t^p \cdot \text{id}_{W_i^\lambda(p)}$ (resp. $g_i(t)|_{V_i^\nu(p)} := t^p \cdot \text{id}_{V_i^\nu(p)}$). Recall we have $\mathfrak{M}^\bullet(\hat{\lambda}) \cong \mathfrak{M}(\lambda)^\mathbb{T}$ and $\mathfrak{M}_0^\bullet(\hat{\lambda}) \cong \mathfrak{M}_0(\lambda)^\mathbb{T}$ by Lemma 3.1, where $\mathbb{T} := (\prod_{i \in I} f_i \times \text{id})(\mathbb{C}^\times)$. Under this identification, we also regard $x_{\hat{\nu}}$ is a point of $\mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$. We can easily see that the image $\iota(\mathbf{x}_{\hat{\nu}}) \in \mu^{-1}(0) \subset \mathbf{M}(\nu, \lambda)$ has a closed $G(\nu)$ -orbit corresponding to the point $x_{\hat{\nu}} \in \mathfrak{M}_0^{\text{reg}}(\nu, \lambda)$ and in particular $\text{Stab}_{G(\nu)} \iota(\mathbf{x}_{\hat{\nu}}) = \{1\}$. Let $\tilde{\mathbb{T}} := (\prod_{i \in I} g_i \times \prod_{i \in I} f_i \times \text{id})(\mathbb{C}^\times)$. This is a 1-dimensional subtorus of $\text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \iota(\mathbf{x}_{\hat{\nu}})$. Under the isomorphism $r: \text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \iota(\mathbf{x}_{\hat{\nu}}) \xrightarrow{\cong} \text{Stab}_{\mathbb{G}(\lambda)} x_{\hat{\nu}}$, the torus $\tilde{\mathbb{T}}$ is isomorphic to \mathbb{T} . In fact, we have $r \circ (\prod_{i \in I} g_i \times \prod_{i \in I} f_i \times \text{id}) = (\prod_{i \in I} f_i \times \text{id})$.

On the other hand, we have a decomposition $C_i(\nu, \lambda)_{\iota(\mathbf{x}_{\hat{\nu}})} \cong \bigoplus_{p \in 2\mathbb{Z} + \xi_i} C_{i,p}(\hat{\nu}, \hat{\lambda})_{\mathbf{x}_{\hat{\nu}}}$ of complexes and hence $H^0(C_i(\nu, \lambda)_{\iota(\mathbf{x}_{\hat{\nu}})}) \cong \bigoplus_{p \in 2\mathbb{Z} + \xi_i} H^0(C_{i,p}(\hat{\nu}, \hat{\lambda})_{\mathbf{x}_{\hat{\nu}}})$, which is identified with $W_i^{\lambda-\nu} = \bigoplus_{p \in 2\mathbb{Z} + \xi_i} W_i^{\lambda-\nu}(p)$. Let $h_i: \mathbb{C}^\times \rightarrow GL(W_i^{\lambda-\nu})$ be a group homomorphism defined by $h_i(t)|_{W_i^{\lambda-\nu}(p)} := t^p \cdot \text{id}_{W_i^{\lambda-\nu}(p)}$ and set $\mathbb{T}' := (\prod_{i \in I} h_i \times \text{id})(\mathbb{C}^\times) \subset \mathbb{G}(\lambda - \nu)$. Then we have

$$(\varphi \times \rho) \circ r^{-1} \circ \left(\prod_{i \in I} f_i \times \text{id} \right) = (\varphi \times \rho) \circ \left(\prod_{i \in I} g_i \times \prod_{i \in I} f_i \times \text{id} \right) = \prod_{i \in I} h_i \times \text{id}.$$

Therefore, under the isomorphism in Lemma 3.3, the action of the torus \mathbb{T} on $\mathfrak{M}(\lambda)_{x_{\hat{\nu}}}$ coincides with the action of the torus \mathbb{T}' on $\mathfrak{L}(\lambda - \nu)$. Therefore, by Lemma 3.1, we have

$$(3.10) \quad \mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}} = \mathfrak{M}(\lambda)_{x_{\hat{\nu}}}^\mathbb{T} \cong \mathfrak{L}(\lambda - \nu)^{\mathbb{T}'} \cong \mathfrak{L}^\bullet(\hat{\lambda} - \hat{\nu}).$$

It remains to show that this isomorphism (3.10) is $(\text{Stab}_{\mathbb{G}(\hat{\lambda})} x_{\hat{\nu}})$ -equivariant. Note that the centralizer of the torus \mathbb{T} (resp. $\tilde{\mathbb{T}}, \mathbb{T}'$) in $\text{Stab}_{\mathbb{G}(\lambda)} x_{\hat{\nu}}$ (resp. $\text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \iota(\mathbf{x}_{\hat{\nu}}), \mathbb{G}(\lambda - \nu)$) is the subgroup $\text{Stab}_{\mathbb{G}(\hat{\lambda})} x_{\hat{\nu}}$ (resp. $\text{Stab}_{G(\hat{\nu}) \times \mathbb{G}(\hat{\lambda})} \mathbf{x}_{\hat{\nu}}$,

$\mathbb{G}(\hat{\lambda} - \hat{\nu})$). We have the following commutative diagram:

$$\begin{array}{ccccc}
 \text{Stab}_{\mathbb{G}(\lambda)} x_{\hat{\nu}} & \xleftarrow[\cong]{r} & \text{Stab}_{G(\nu) \times \mathbb{G}(\lambda)} \iota(\mathbf{x}_{\hat{\nu}}) & \xrightarrow{\varphi \times \rho} & \mathbb{G}(\lambda - \nu) \\
 \uparrow & & \uparrow & & \uparrow \\
 \text{Stab}_{\mathbb{G}(\hat{\lambda})} x_{\hat{\nu}} & \xleftarrow[\cong]{\hat{r}} & \text{Stab}_{G(\hat{\nu}) \times \mathbb{G}(\hat{\lambda})} \mathbf{x}_{\hat{\nu}} & \xrightarrow{\hat{\varphi} \times \hat{\rho}} & \mathbb{G}(\hat{\lambda} - \hat{\nu}).
 \end{array}$$

Because the isomorphism in Lemma 3.3 is $(\text{Stab}_{\mathbb{G}(\lambda)} x_{\hat{\nu}})$ -equivariant via the homomorphism $(\varphi \times \rho) \circ r^{-1}$, the induced isomorphism (3.10) on the torus fixed parts is $(\text{Stab}_{\mathbb{G}(\hat{\lambda})} x_{\hat{\nu}})$ -equivariant via the homomorphism $(\hat{\varphi} \times \hat{\rho}) \circ \hat{r}^{-1}$ from the above commutative diagram. \square

3.5. Graded quiver varieties for the category $\mathcal{C}_{Q,\beta}$. Fix an element $\beta = \sum_{i \in I} d_i \alpha_i \in \mathbb{Q}^+$. Let

$$E_\beta := \bigoplus_{i \rightarrow j \in \Omega} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

be the space of representations of the quiver Q of dimension vector β . The group $G_\beta := \prod_{i \in I} GL_{d_i}(\mathbb{C})$ acts on E_β by conjugation. By Gabriel’s theorem, the set of G_β -orbits is in bijection with the set

$$\text{KP}(\beta) := \left\{ \mathbf{m} = (m_\alpha)_{\alpha \in \mathbb{R}^+} \in (\mathbb{Z}_{\geq 0})^{\mathbb{R}^+} \mid \sum_{\alpha \in \mathbb{R}^+} m_\alpha \alpha = \beta \right\}$$

of Kostant partitions of β . For each Kostant partition $\mathbf{m} \in \text{KP}(\beta)$, let $\mathbb{O}_{\mathbf{m}}$ denote the corresponding G_β -orbit.

Set $\hat{\lambda}_\beta := \sum_{i \in I} d_i \varpi_{\phi(\alpha_i)} \in \mathcal{P}_Q^+$, where ϕ is the bijection $\mathbb{R}^+ \rightarrow \hat{I}_Q$ defined in Subsection 2.1. We consider the corresponding graded quiver variety $\mathfrak{M}_0^\bullet(\hat{\lambda}_\beta)$. We identify $G(\hat{\lambda}_\beta)$ with G_β in the natural way.

We define a map $\mathfrak{p}: \mathbb{R}^+ \rightarrow \mathbb{Z}$ by $\mathfrak{p}(\alpha) := \text{pr}_2 \circ \phi(\alpha)$, where $\text{pr}_2: I \times \mathbb{Z} \rightarrow \mathbb{Z}$ is the second projection. Using this notation, we define a homomorphism $\rho_i: \mathbb{C}^\times \rightarrow GL(\mathbb{C}^{d_i})$ for each $i \in I$ by $\rho_i(t) := t^{-\mathfrak{p}(\alpha_i)} \cdot \text{id}_{\mathbb{C}^{d_i}}$. Then we have a group homomorphism $(\text{id} \times \prod_{i \in I} \rho_i): \mathbb{G}(\hat{\lambda}_\beta) = G(\hat{\lambda}_\beta) \times \mathbb{C}^\times \rightarrow G_\beta$, via which E_β is equipped with a $\mathbb{G}(\hat{\lambda}_\beta)$ -action.

In [21], Hernandez-Leclerc constructed a $\mathbb{G}(\hat{\lambda}_\beta)$ -equivariant isomorphism of varieties $\mathfrak{M}_0^\bullet(\hat{\lambda}_\beta) \cong E_\beta$. Their original motivation was to give a geometric interpretation to their isomorphism between the quantum Grothendieck ring of \mathcal{C}_Q and the quantized coordinate ring of the unipotent group associated with the positive part of \mathfrak{g} . Note that the isomorphism $\mathfrak{M}_0^\bullet(\hat{\lambda}_\beta) \cong E_\beta$ is very special for the ℓ -weight $\hat{\lambda}_\beta$ because a graded quiver variety $\mathfrak{M}_0^\bullet(\hat{\lambda})$ is not an affine space in general. In particular, the restriction to the subcategory $\mathcal{C}_Q \subset \mathcal{C}_{\mathfrak{g}}$ is crucial here.

Let us recall Hernandez-Leclerc’s construction. By Lemma 2.23(2), it is enough to discuss the graded quiver varieties $\mathfrak{M}_0^\bullet(\hat{\nu}, \hat{\lambda}_\beta)$ with $\hat{\nu} \in \mathcal{Q}_Q^+$. We define a \mathbb{C} -algebra $\tilde{\Lambda}_Q$ given by the following quiver $\tilde{\Gamma}_Q$ with relations. The quiver $\tilde{\Gamma}_Q$ consists of two types of vertices $\{v_j(p) \mid (j, p) \in \hat{J}_Q\} \cup \{w_j(p) \mid (j, p) = \phi(\alpha_i) \text{ for some } i \in I\}$

and three types of arrows:

$$\begin{aligned} \mathbf{a}_i(p) &: w_i(p) \rightarrow v_i(p-1), & \mathbf{b}_i(p) &: v_i(p) \rightarrow w_i(p-1), \\ \mathbf{B}_{ji}(p) &: v_i(p) \rightarrow v_j(p-1) \text{ for } i \sim j. \end{aligned}$$

The relations are

$$\mathbf{a}_i(p-1)\mathbf{b}_i(p) + \sum_{j \sim i} \varepsilon(i, j)\mathbf{B}_{ij}(p-1)\mathbf{B}_{ji}(p) = 0 \text{ for each } i \in I.$$

For each $i \in I$, let $\epsilon_i \in \tilde{\Lambda}_Q$ denote the idempotent corresponding to the vertex $w_j(p)$ with $(j, p) = \phi(\alpha_i)$. Then [21, Lemma 9.6] proved that the algebra $\bigoplus_{i,j \in I} \epsilon_i \tilde{\Lambda}_Q \epsilon_j$ is identical to the path algebra $\mathbb{C}Q$. By definition, each element $\mathbf{x} = (B, a, b) \in \mu^{\bullet-1}(0) \subset \mathbf{M}^\bullet(V^{\hat{\nu}}, W^{\hat{\lambda}_\beta})$ gives a representation of $\tilde{\Lambda}_Q$. Then restricted to $\bigoplus_{i,j \in I} \epsilon_i \tilde{\Lambda}_Q \epsilon_j$, it gives a representation of $\mathbb{C}Q$ of dimension vector β . This defines a morphism $\mathfrak{M}_0^\bullet(\hat{\nu}, \hat{\lambda}_\beta) \rightarrow E_\beta$.

Theorem 3.5 (Hernandez-Leclerc [21, Theorem 9.11]). *The morphism constructed above gives a $\mathbb{G}(\hat{\lambda}_\beta)$ -equivariant isomorphism of varieties*

$$\Psi_\beta: \mathfrak{M}_0^\bullet(\hat{\lambda}_\beta) \xrightarrow{\cong} E_\beta.$$

In the proof of [21, Theorem 9.11], it was also proved that the stratification (3.3) of $\mathfrak{M}_0^\bullet(\hat{\lambda}_\beta)$ coincides with the $\mathbb{G}(\hat{\lambda}_\beta)$ -orbit stratification of E_β . Here we shall give a precise correspondence between the strata. Define a bijection $f: \text{KP}(\beta) \rightarrow \mathcal{P}_{Q,\beta}^+$ by

$$f(\mathbf{m}) := \sum_{\alpha \in R^+} m_\alpha \varpi_{\phi(\alpha)}$$

for $\mathbf{m} = (m_\alpha)_{\alpha \in R^+} \in \text{KP}(\beta)$.

Lemma 3.6. *For each $\mathbf{m} \in \text{KP}(\beta)$, we have*

$$\Psi_\beta(\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\lambda}_\beta - f(\mathbf{m}), \hat{\lambda}_\beta)) = \mathbb{O}_{\mathbf{m}}.$$

Proof. For each $\mathbf{m} \in \text{KP}(\beta)$, there is a unique $\hat{\nu} \in \mathcal{Q}_Q^+$ (recall Lemma 2.23) such that $\Psi_\beta(\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda}_\beta)) \supset \mathbb{O}_{\mathbf{m}}$ since each stratum $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda}_\beta)$ is stable under the action of $\mathbb{G}(\hat{\lambda}_\beta)$. It suffices to prove that $\hat{\nu} = \hat{\lambda}_\beta - f(\mathbf{m})$.

First we consider the case when $\beta = \alpha \in R^+$ and \mathbf{m} is the Kostant partition $\mathbf{m}_\alpha := (\delta_{\alpha,\alpha'})_{\alpha' \in R^+}$ consisting of the single root α . In this case, the orbit $\mathbb{O}_{\mathbf{m}_\alpha}$ is the unique open dense orbit of E_α as $\dim \mathbb{O}_{\mathbf{m}_\alpha} = \dim E_\alpha$ (see [14, Proposition 4.4.9 (2)] for example). Recall that $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda}_\beta) \subset \mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}', \hat{\lambda}_\beta)$ implies $\hat{\lambda}_\beta - \hat{\nu} \geq \hat{\lambda}_\beta - \hat{\nu}'$. Since the ℓ -weight $\varpi_{\phi(\alpha)}$ is minimum in $\mathcal{P}_{Q,\alpha}^+$, the corresponding stratum $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}_\alpha, \hat{\lambda}_\alpha)$ is maximum, where we put $\hat{\nu}_\alpha := \hat{\lambda}_\alpha - \varpi_{\phi(\alpha)}$. Therefore we have $\Psi_\beta(\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}_\alpha, \hat{\lambda}_\alpha)) \supset \mathbb{O}_{\mathbf{m}_\alpha}$ as desired.

Next we consider general $\mathbf{m} = (m_\alpha)_{\alpha \in R^+} \in \text{KP}(\beta)$. For each $\alpha \in R^+$, we fix an element $\mathbf{y}_\alpha \in \mu^{\bullet-1}(0) \subset \mathbf{M}^\bullet(V^{\hat{\nu}_\alpha}, W^{\hat{\lambda}_\alpha})$ such that \mathbf{y}_α has a closed $G(\hat{\nu}_\alpha)$ -orbit and $\text{Stab}_{G(\hat{\nu}_\alpha)} \mathbf{y}_\alpha = \{1\}$ holds. By the previous paragraph, the element \mathbf{y}_α , which is regarded as a representation of the algebra $\tilde{\Lambda}_Q$, restricts to give an indecomposable

representation of $\mathbb{C}Q$ isomorphic to $M(\alpha)$. We put

$$\begin{aligned} \mathbf{x}_m &:= \bigoplus_{\alpha \in \mathbb{R}^+} \mathbf{y}_\alpha^{\oplus m_\alpha} \in \mu^{\bullet-1}(0) \\ &\subset \mathbf{M}^\bullet \left(\bigoplus_{\alpha \in \mathbb{R}^+} (V^{\hat{\nu}_\alpha})^{\oplus m_\alpha}, \bigoplus_{\alpha \in \mathbb{R}^+} (W^{\hat{\lambda}_\alpha})^{\oplus m_\alpha} \right) = \mathbf{M}^\bullet(V^{\hat{\nu}_m}, W^{\hat{\lambda}_\beta}), \end{aligned}$$

where $\hat{\nu}_m := \sum_{\alpha \in \mathbb{R}^+} m_\alpha \hat{\nu}_\alpha$. Then \mathbf{x}_m defines a closed $G(\hat{\nu}_m)$ -orbit and has a trivial stabilizer. Hence, the corresponding point x_m belongs to $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}_m, \hat{\lambda}_\beta)$. On the other hand, the element \mathbf{x}_m , which is regarded as a representation of Λ_Q , restricts to give a representation of $\mathbb{C}Q$ isomorphic to $\bigoplus_{\alpha \in \mathbb{R}^+} M(\alpha)^{\oplus m_\alpha}$. This means that $\Psi_\beta(x_m) \in \mathcal{O}_m$. Therefore we have $\Psi_\beta(\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}_m, \hat{\lambda}_\beta)) \supset \mathcal{O}_m$ as desired. \square

We define a partial order \leq on the set $\text{KP}(\beta)$ of Kostant partitions of β by the condition that for $\mathbf{m}, \mathbf{m}' \in \text{KP}(\beta)$, we have $\mathbf{m} \leq \mathbf{m}'$ if and only if $\overline{\mathcal{O}}_{\mathbf{m}} \supset \mathcal{O}_{\mathbf{m}'}$. From Lemma 3.6, we conclude that the bijection $f: \text{KP}(\beta) \rightarrow \mathcal{P}_{Q,\beta}^+$ preserves the partial orders.

Using the isomorphism Ψ_β and the representation theory of Q , we can obtain more precise information on the stabilizer group of a point of $\mathfrak{M}_0^{\bullet}(\hat{\lambda}_\beta)$ as follows.

Lemma 3.7. *Let $\hat{\nu} \in \mathcal{Q}_Q^+$ such that $\hat{\lambda}_\beta - \hat{\nu} \in \mathcal{P}_{Q,\beta}^+$. Fix an arbitrary point $x_{\hat{\nu}} \in \mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda}_\beta)$. Then the maximal reductive quotient (= the quotient by the unipotent radical) of the group $\text{Stab}_{\mathbb{G}(\hat{\lambda}_\beta)} x_{\hat{\nu}}$ is isomorphic to $\mathbb{G}(\hat{\lambda}_\beta - \hat{\nu})$. Moreover the group homomorphism $(\hat{\varphi} \times \hat{\rho}) \circ \hat{r}^{-1}: \text{Stab}_{\mathbb{G}(\hat{\lambda}_\beta)} x_{\hat{\nu}} \rightarrow \mathbb{G}(\hat{\lambda}_\beta - \hat{\nu})$ defined in (3.9) is identical to the canonical quotient map.*

Proof. Define $\mathbf{m} = (m_\alpha)_{\alpha \in \mathbb{R}^+} \in \text{KP}(\beta)$ by $f(\mathbf{m}) = \hat{\lambda}_\beta - \hat{\nu}$. By Lemma 3.6, the point $\Psi_\beta(x_{\hat{\nu}})$ corresponds to a $\mathbb{C}Q$ -module $M(\mathbf{m}) \cong \bigoplus_{\alpha \in \mathbb{R}^+} M(\alpha)^{\oplus m_\alpha}$. Then we have $\text{Stab}_{\mathbb{G}(\hat{\lambda}_\beta)} x_{\hat{\nu}} = \text{Stab}_{G_\beta} \Psi_\beta(x_{\hat{\nu}}) = \text{End}_{\mathbb{C}Q}(M(\mathbf{m}))^\times$. We consider a subgroup

$$G_1 := \prod_{\alpha \in \mathbb{R}^+} \text{End}_{\mathbb{C}Q}(M(\alpha)^{\oplus m_\alpha})^\times \subset \text{End}_{\mathbb{C}Q}(M(\mathbf{m}))^\times.$$

Note that we have $\text{End}_{\mathbb{C}Q}(M(\alpha)) = \mathbb{C}$ for any root $\alpha \in \mathbb{R}^+$. We can easily see that this subgroup G_1 is a Levi subgroup of $\text{End}_{\mathbb{C}Q}(M(\mathbf{m}))^\times$ and therefore is isomorphic to the maximal reductive quotient of $\text{Stab}_{\mathbb{G}(\hat{\lambda}_\beta)} x_{\hat{\nu}}$ by the canonical quotient map. This shows that the maximal reductive quotient of $\text{Stab}_{\mathbb{G}(\hat{\lambda}_\beta)} x_{\hat{\nu}} = \text{Stab}_{G(\hat{\lambda}_\beta)} x_{\hat{\nu}} \times \mathbb{T}$ is isomorphic to $G(\hat{\lambda}_\beta - \hat{\nu}) \times \mathbb{C}^\times = \mathbb{G}(\hat{\lambda}_\beta - \hat{\nu})$.

Let us prove the latter assertion. Corresponding to the decomposition $M(\mathbf{m}) \cong \bigoplus_{\alpha \in \mathbb{R}^+} M(\alpha)^{\oplus m_\alpha}$, we choose an element $\mathbf{x}_{\hat{\nu}} = \bigoplus_{\alpha \in \mathbb{R}^+} \mathbf{y}_\alpha^{\oplus m_\alpha}$ as a lift of the point $x_{\hat{\nu}}$, where \mathbf{y}_α 's are the same as in the proof of Lemma 3.6. Then we have $\text{Stab}_{G(\hat{\nu}) \times G(\hat{\lambda}_\beta)} \mathbf{x}_{\hat{\nu}} = \text{End}_{\tilde{\Lambda}_Q}(\mathbf{x}_{\hat{\nu}})^\times$. We consider a subgroup

$$\tilde{G}_1 := \prod_{\alpha \in \mathbb{R}^+} GL_{m_\alpha}(\mathbb{C} \cdot \text{id}_{\mathbf{y}_\alpha}) \subset \prod_{\alpha \in \mathbb{R}^+} \text{End}_{\tilde{\Lambda}_Q}(\mathbf{y}_\alpha^{\oplus m_\alpha})^\times \subset \text{End}_{\tilde{\Lambda}_Q}(\mathbf{x}_{\hat{\nu}})^\times.$$

Note that the homomorphism \hat{r} gives an isomorphism $\tilde{G}_1 \xrightarrow{\cong} G_1$. On the other hand, we can easily see that the homomorphism $\hat{\varphi}: \text{Stab}_{G(\hat{\nu}) \times G(\hat{\lambda}_\beta)} \hat{\nu} \rightarrow G(\hat{\lambda}_\beta - \hat{\nu})$

induces the isomorphism

$$\tilde{G}_1 \xrightarrow{\cong} \prod_{\alpha \in \mathbb{R}^+} GL(H^0(C_{\phi(\alpha)}(\hat{\nu}, \hat{\lambda}_\beta)_{\mathbf{x}_{\hat{\nu}}})) \cong G(\hat{\lambda}_\beta - \hat{\nu}).$$

As a result, we have the following commutative diagram:

$$(3.11) \quad \begin{array}{ccccc} \text{Stab}_{\mathbb{G}(\hat{\lambda}_\beta)} x_{\hat{\nu}} & \xleftarrow[\cong]{\hat{r}} & \text{Stab}_{G(\hat{\nu}) \times \mathbb{G}(\hat{\lambda}_\beta)} \mathbf{x}_{\hat{\nu}} & \xrightarrow{\hat{\varphi} \times \hat{\rho}} & \mathbb{G}(\hat{\lambda}_\beta - \hat{\nu}) \\ \uparrow & & \uparrow & & \parallel \\ G_1 \times \mathbb{T} & \xleftarrow[\cong]{} & \tilde{G}_1 \times \tilde{\mathbb{T}} & \xrightarrow[\cong]{} & \mathbb{G}(\hat{\lambda}_\beta - \hat{\nu}) \end{array}$$

where the torus $\tilde{\mathbb{T}}$ is defined as in the proof of Lemma 3.4. This diagram completes a proof. □

For a linear algebraic group G , we denote its representation ring by $R(G)$. For $G = \mathbb{C}^\times$, we always identify $R(\mathbb{C}^\times) = \mathbb{A} := \mathbb{Z}[q^{\pm 1}]$ so that q corresponds to the natural 1-dimensional representation of \mathbb{C}^\times . With this notation, we have the standard identifications

$$R(\mathbb{G}(\hat{\lambda}_\beta)) = \bigotimes_{i \in I} (\mathbb{A}[z_i^{\pm 1}]^{\otimes d_i})^{\mathfrak{S}_{d_i}}, \quad R(\mathbb{G}(\hat{\mu})) = \bigotimes_{\alpha \in \mathbb{R}^+} (\mathbb{A}[z_\alpha^{\pm 1}]^{\otimes m_\alpha})^{\mathfrak{S}_{m_\alpha}},$$

where $\hat{\mu} = \sum_{\alpha \in \mathbb{R}^+} m_\alpha \varpi_{\phi(\alpha)} \in \mathcal{P}_{Q,\beta}^+$. Here \otimes 's are taken over \mathbb{A} .

For a positive root $\alpha = \sum_{i \in I} c_i \alpha_i \in \mathbb{R}^+$, we define the following algebra homomorphism:

$$\theta_\alpha: \bigotimes_{i \in I} \mathbb{A}[z_i^{\pm 1}]^{\otimes c_i} \rightarrow \mathbb{A}[z_\alpha^{\pm 1}], \quad z_i \mapsto q^{p(\alpha_i) - p(\alpha)} z_\alpha.$$

Now we return to the setting of Lemma 3.7. By the commutative diagram (3.11) in the proof of Lemma 3.7, we have the following group embedding:

$$\mathbb{G}(\hat{\lambda}_\beta - \hat{\nu}) \cong G_1 \times \mathbb{T} \hookrightarrow \text{Stab}_{\mathbb{G}(\hat{\lambda}_\beta)} x_{\hat{\nu}} \hookrightarrow \mathbb{G}(\hat{\lambda}_\beta),$$

which induces the following homomorphism:

$$(3.12) \quad \theta_{\hat{\lambda}_\beta - \hat{\nu}}: R(\mathbb{G}(\hat{\lambda}_\beta)) \rightarrow R(\text{Stab}_{\mathbb{G}(\hat{\lambda}_\beta)} x_{\hat{\nu}}) \xrightarrow{\cong} R(\mathbb{G}(\hat{\lambda}_\beta - \hat{\nu})).$$

From the proof of Lemma 3.7, we have the following.

Corollary 3.8. *Write $\hat{\lambda}_\beta - \hat{\nu} = \sum_{\alpha \in \mathbb{R}^+} m_\alpha \varpi_{\phi(\alpha)} \in \mathcal{P}_{Q,\beta}^+$. Then the above homomorphism $\theta_{\hat{\lambda}_\beta - \hat{\nu}}: R(\mathbb{G}(\hat{\lambda}_\beta)) \rightarrow R(\mathbb{G}(\hat{\lambda}_\beta - \hat{\nu}))$ coincides with the restriction of the homomorphism*

$$\bigotimes_{\alpha} \theta_\alpha^{\otimes m_\alpha}: \bigotimes_{i \in I} \mathbb{A}[z_i^{\pm 1}]^{\otimes d_i} \rightarrow \bigotimes_{\alpha \in \mathbb{R}^+} \mathbb{A}[z_\alpha^{\pm 1}]^{\otimes m_\alpha}.$$

4. CENTRAL COMPLETION OF CONVOLUTION ALGEBRA

In this section, we study the structure of the Hernandez-Leclerc category $\mathcal{C}_{Q,\beta}$ using the geometry of graded quiver varieties.

4.1. Notation for equivariant K -theory. For a quasi-projective variety X over \mathbb{C} equipped with an action of a linear algebraic group G , we denote by $K^G(X)$ the Grothendieck group of the abelian category of G -equivariant coherent sheaves on X . For a G -equivariant coherent sheaf \mathcal{F} on X , we denote by $[\mathcal{F}]$ the corresponding element in $K^G(X)$. The structure sheaf of X is denoted by \mathcal{O}_X . For a G -equivariant vector bundle \mathcal{E} on X , the map $K^G(X) \ni [\mathcal{F}] \mapsto [\mathcal{E}] \cdot [\mathcal{F}] := [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}] \in K^G(X)$ is well-defined. For a G -equivariant vector bundle \mathcal{E} on X , we also define $\bigwedge_u[\mathcal{E}] := \sum_{i=0}^{\text{rank } \mathcal{E}} u^i [\bigwedge^i \mathcal{E}] \in [\mathcal{O}_X] + uK^G(X)[u]$. If $0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0$ is an exact sequence of G -equivariant vector bundles on X , we have $\bigwedge_u[\mathcal{E}] = \bigwedge_u[\mathcal{E}_1] \cdot \bigwedge_u[\mathcal{E}_2]$. Therefore we set $\bigwedge_u([\mathcal{E}_1] + [\mathcal{E}_2]) := \bigwedge_u[\mathcal{E}_1] \cdot \bigwedge_u[\mathcal{E}_2]$ and $\bigwedge_u(-[\mathcal{E}]) := (\bigwedge_u[\mathcal{E}])^{-1} \in [\mathcal{O}_X] + uK^G(X)[u]$.

When $X = \text{pt}$, we have $K^G(\text{pt}) = R(G)$. For a general X , the group $K^G(X)$ is a module over $R(G)$. When G is written in the form $G = G_0 \times \mathbb{C}^\times$ where G_0 is another linear algebraic group, we regard $K^G(X)$ as a $\mathbb{Z}[q^{\pm 1}]$ -module through the standard identification $\mathbb{Z}[q^{\pm 1}] \cong R(\mathbb{C}^\times)$ and the algebra inclusion $R(\mathbb{C}^\times) \hookrightarrow R(G)$ induced from the natural projection $G = G_0 \times \mathbb{C}^\times \rightarrow \mathbb{C}^\times$. Then we write $\mathcal{K}^G(X) := K^G(X) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{k}$ and $\mathcal{R}(G) := R(G) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{k}$, where $\mathbb{k} = \overline{\mathbb{Q}(q)}$ as before. For each $m \in \mathbb{Z}$, let L_m denote the 1-dimensional \mathbb{C}^\times -module of weight $m \in \mathbb{Z}$. Namely $[L_m] = q^m$ in $R(\mathbb{C}^\times)$. Although it is an abuse of notation, we write $q^m \mathcal{F} := L_m \otimes_{\mathcal{O}_X} \mathcal{F}$ for any G -equivariant coherent sheaf \mathcal{F} on X so that we have $q^m[\mathcal{F}] = [q^m \mathcal{F}]$ in $\mathcal{K}^G(X)$.

We also use the equivariant topological K -homologies denoted by $K_{i,\text{top}}^G(X)$ ($i = 0, 1$). There is a canonical comparison map $K^G(X) \rightarrow K_{0,\text{top}}^G(X)$ (see [11, Section 5.5.5]).

Let Y be a G -invariant closed subvariety of X and $U = X \setminus Y$ be the complement of Y . Then the inclusions $Y \xrightarrow{i} X \xleftarrow{j} U$ induce the following:

- (1) an exact sequence:

$$(4.1) \quad K^G(Y) \xrightarrow{i_*} K^G(X) \xrightarrow{j^*} K^G(U) \longrightarrow 0,$$

- (2) an exact hexagon:

$$(4.2) \quad \begin{array}{ccccc} K_{0,\text{top}}^G(Y) & \xrightarrow{i_*} & K_{0,\text{top}}^G(X) & \xrightarrow{j^*} & K_{0,\text{top}}^G(U) \\ \uparrow & & & & \downarrow \\ K_{1,\text{top}}^G(U) & \xleftarrow{j^*} & K_{1,\text{top}}^G(X) & \xleftarrow{i_*} & K_{1,\text{top}}^G(Y). \end{array}$$

4.2. The Nakajima homomorphism and its completion. Fix a dominant weight $\lambda \in P^+$ and consider the corresponding quiver variety $\pi: \mathfrak{M}(\lambda) \rightarrow \mathfrak{M}_0(\lambda)$. We define the *Steinberg type variety* $Z(\lambda)$ as

$$Z(\lambda) := \mathfrak{M}(\lambda) \times_{\mathfrak{M}_0(\lambda)} \mathfrak{M}(\lambda) = \bigsqcup_{\nu_1, \nu_2 \in Q^+} \mathfrak{M}(\nu_1, \lambda) \times_{\mathfrak{M}_0(\lambda)} \mathfrak{M}(\nu_2, \lambda),$$

together with the canonical map $\pi: Z(\lambda) \rightarrow \mathfrak{M}_0(\lambda)$. By the convolution product, the equivariant K -group

$$\mathcal{K}^{\mathbb{G}(\lambda)}(Z(\lambda)) = \bigoplus_{\nu_1, \nu_2 \in Q^+} \mathcal{K}^{\mathbb{G}(\lambda)}(\mathfrak{M}(\nu_1, \lambda) \times_{\mathfrak{M}_0(\lambda)} \mathfrak{M}(\nu_2, \lambda))$$

becomes an algebra over the commutative \mathbb{k} -algebra $\mathcal{R}(\lambda) := \mathcal{R}(\mathbb{G}(\lambda))$. Note that this notation is consistent with (2.3).

We consider the following tautological vector bundles on $\mathfrak{M}(\nu, \lambda)$. The vector bundle \mathcal{V}_i^ν is defined by $\mathcal{V}_i^\nu := \mu^{-1}(0)^{\text{st}} \times_{G(\nu)} V_i^\nu$ for each $i \in I$. We regard \mathcal{V}_i^ν as a $\mathbb{G}(\lambda)$ -equivariant vector bundle with the trivial action. On the other hand, we consider the trivial vector bundle $\mathcal{W}_i^\lambda := \mathfrak{M}(\nu, \lambda) \times W_i^\lambda$ with fiber W_i^λ for each $i \in I$. We regard \mathcal{W}_i^λ as a $\mathbb{G}(\lambda)$ -equivariant vector bundle with the natural $G(\lambda)$ -action and the trivial \mathbb{C}^\times -action. Recall the complex of vector spaces $\mathcal{C}_i(\nu, \lambda)_{\mathbf{x}}$ for each $\mathbf{x} \in \mu^{-1}(0) \subset \mathbf{M}(V^\nu, W^\lambda)$ defined in (3.4). This complex yields the complex $\mathcal{C}_i(\nu, \lambda)$ of $\mathbb{G}(\lambda)$ -equivariant vector bundles on $\mathfrak{M}(\nu, \lambda)$:

$$\mathcal{C}_i(\nu, \lambda): q^{-2}\mathcal{V}_i^\nu \xrightarrow{\sigma_i} q^{-1} \left(\mathcal{W}_i^\lambda \oplus \bigoplus_{j \sim i} \mathcal{V}_j^\nu \right) \xrightarrow{\tau_i} \mathcal{V}_i^\nu.$$

Note that the class of the complex $\mathcal{C}_i(\nu, \lambda)$ in $\mathcal{K}^{\mathbb{G}(\lambda)}(\mathfrak{M}(\nu, \lambda))$ is calculated as

$$[\mathcal{C}_i(\nu, \lambda)] = q^{-1} \left([\mathcal{W}_i^\lambda] - (q + q^{-1})[\mathcal{V}_i^\nu] + \sum_{j \sim i} [\mathcal{V}_j^\nu] \right).$$

Then we have the following fundamental result due to Nakajima [35].

Theorem 4.1 (Nakajima [35, Theorem 9.4.1]). *There is a \mathbb{k} -algebra homomorphism*

$$\Phi_\lambda: \tilde{U}_q(L\mathfrak{g}) \rightarrow \mathcal{K}^{\mathbb{G}(\lambda)}(Z(\lambda)),$$

such that

$$\Phi_\lambda(a_\mu) = \begin{cases} \Delta_*[\mathcal{O}_{\mathfrak{M}(\nu, \lambda)}] & \text{if } \nu := \lambda - \mu \in \mathbf{Q}^+; \\ 0 & \text{otherwise,} \end{cases}$$

and it sends the series $\psi_i^\pm(z)a_\mu$ with $\nu := \lambda - \mu \in \mathbf{Q}^+, i \in I$ to the series

$$q^{\mu(h_i)} \Delta_* \left(\frac{\Lambda_{-1/qz}[\mathcal{C}_i(\nu, \lambda)]}{\Lambda_{-q/z}[\mathcal{C}_i(\nu, \lambda)]} \right)^\pm,$$

where $\Delta: \mathfrak{M}(\nu, \lambda) \rightarrow \mathfrak{M}(\nu, \lambda) \times_{\mathfrak{m}_0(\lambda)} \mathfrak{M}(\nu, \lambda)$ is the diagonal embedding and $(-)^{\pm}$ denotes the formal expansion at $z = \infty$ and 0 respectively.

We refer to the homomorphism Φ_λ as the *Nakajima homomorphism*.

By construction, the equivariant K -group $\mathcal{K}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda))$ of the central fiber $\mathfrak{L}(\lambda)$ becomes a module over the convolution algebra $\mathcal{K}^{\mathbb{G}(\lambda)}(Z(\lambda))$. Via the Nakajima homomorphism Φ_λ , we regard $\mathcal{K}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda))$ as a $(U_q, \mathcal{R}(\lambda))$ -bimodule.

Theorem 4.2 (Nakajima). *As a $(U_q, \mathcal{R}(\lambda))$ -bimodule, the module $\mathcal{K}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda))$ is isomorphic to the global Weyl module $\mathbb{W}(\lambda)$. The element $[\mathcal{O}_{\mathfrak{L}(0, \lambda)}] \in \mathcal{K}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda))$ corresponds to the cyclic vector $w_\lambda \in \mathbb{W}(\lambda)$. (Recall $\mathfrak{L}(0, \lambda) = \mathfrak{M}(0, \lambda) = \text{pt}$ from Subsection 3.1.)*

Proof. See [37, Theorem 2]. □

For future references, we collect some important properties of the equivariant K -groups of central fibers.

Theorem 4.3 (Nakajima). *Let G' be a closed subgroup of $\mathbb{G}(\lambda)$. The following hold.*

- (1) We have $K_{1,\text{top}}^{G'}(\mathfrak{L}(\lambda)) = 0$;
- (2) $K_{0,\text{top}}^{G'}(\mathfrak{L}(\lambda))$ is a free $R(G')$ -module and the comparison map $K^{G'}(\mathfrak{L}(\lambda)) \rightarrow K_{0,\text{top}}^{G'}(\mathfrak{L}(\lambda))$ is an isomorphism;
- (3) The natural map $K^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda)) \otimes_{R(\mathbb{G}(\lambda))} R(G') \rightarrow K^{G'}(\mathfrak{L}(\lambda))$ is an isomorphism;
- (4) The Künneth homomorphisms

$$K^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda)) \otimes_{R(\mathbb{G}(\lambda))} K^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda)) \rightarrow K^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda) \times \mathfrak{L}(\lambda)),$$

$$K_{0,\text{top}}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda)) \otimes_{R(\mathbb{G}(\lambda))} K_{i,\text{top}}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda)) \rightarrow K_{i,\text{top}}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda) \times \mathfrak{L}(\lambda))$$

are isomorphisms, where $i = 0, 1$.

Proof. The properties (1), (2), (3) are the same as the property $(T_{\mathbb{G}(\lambda)})$ in [35, Section 7]. The assertion for $K^{\mathbb{G}(\lambda)}$ in (4) follows from [36, Theorem 3.4]. The assertion for $K_{i,\text{top}}^{\mathbb{G}(\lambda)}$ in (4) follows from the properties (1), (2) and the property (n3) in [28, Section 1.2]. \square

Next we fix an ℓ -dominant ℓ -weight $\hat{\lambda} \in \mathcal{P}_{\mathbb{Z}}^+$ and put $\lambda := \text{cl}(\hat{\lambda})$. We consider the corresponding graded quiver varieties $\pi^\bullet: \mathfrak{M}^\bullet(\hat{\lambda}) \rightarrow \mathfrak{M}_0^\bullet(\hat{\lambda})$ and the corresponding Steinberg type variety

$$Z^\bullet(\hat{\lambda}) := \mathfrak{M}^\bullet(\hat{\lambda}) \times_{\mathfrak{M}_0^\bullet(\hat{\lambda})} \mathfrak{M}^\bullet(\hat{\lambda}).$$

The $\mathbb{G}(\hat{\lambda})$ -equivariant K -group $\mathcal{K}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ is an algebra over $\mathcal{R}(\mathbb{G}(\hat{\lambda}))$ by the convolution product. We set $\mathcal{R}(\hat{\lambda}) := \mathcal{R}(\mathbb{G}(\hat{\lambda}))$, which is consistent with (2.4).

We choose the 1-dimensional subtorus $\mathbb{T} \subset \mathbb{G}(\hat{\lambda}) \subset \mathbb{G}(\lambda)$ as in Lemma 3.1. Then we have the identification

$$(4.3) \quad Z^\bullet(\hat{\lambda}) \cong Z(\lambda)^\mathbb{T}.$$

Let $\mathfrak{r}_{\hat{\lambda}}$ denote the maximal ideal of the commutative ring $\mathcal{R}(\hat{\lambda})$ corresponding to the subtorus $\mathbb{T} \subset \mathbb{G}(\hat{\lambda})$ and form the completion $\widehat{\mathcal{R}}(\hat{\lambda}) := \varprojlim \mathcal{R}(\hat{\lambda})/\mathfrak{r}_{\hat{\lambda}}^N$. Note that the notation is consistent with (2.5). For any $\mathbb{G}(\hat{\lambda})$ -variety X , we write the corresponding completions of K -groups by

$$\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(X) := \mathcal{K}^{\mathbb{G}(\hat{\lambda})}(X) \otimes_{\mathcal{R}(\hat{\lambda})} \widehat{\mathcal{R}}(\hat{\lambda}), \quad \widehat{\mathcal{K}}_{i,\text{top}}^{\mathbb{G}(\hat{\lambda})}(X) := \mathcal{K}_{i,\text{top}}^{\mathbb{G}(\hat{\lambda})}(X) \otimes_{\mathcal{R}(\hat{\lambda})} \widehat{\mathcal{R}}(\hat{\lambda}).$$

When X is the Steinberg type variety $Z^\bullet(\hat{\lambda})$, the completed K -group $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ is an algebra over $\widehat{\mathcal{R}}(\hat{\lambda})$ with respect to the convolution product. Thanks to the localization theorem, this algebra can be obtained from the convolution algebra $\mathcal{K}^{\mathbb{G}(\hat{\lambda})}(Z(\lambda))$ by the central completion $- \otimes_{\mathcal{R}(\hat{\lambda})} \widehat{\mathcal{R}}(\hat{\lambda})$.

Definition 4.4. We define the *completed Nakajima homomorphism* $\widehat{\Phi}_{\hat{\lambda}}: \widetilde{U}_q \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ as the following composition:

$$\widetilde{U}_q \xrightarrow{\Phi_\lambda} \mathcal{K}^{\mathbb{G}(\lambda)}(Z(\lambda)) \rightarrow \mathcal{K}^{\mathbb{G}(\hat{\lambda})}(Z(\lambda)) \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z(\lambda)) \xrightarrow{\cong} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda})),$$

where the first homomorphism is the Nakajima homomorphism Φ_λ in Theorem 4.1, the second is the restriction to the subgroup $\mathbb{G}(\hat{\lambda}) \subset \mathbb{G}(\lambda)$, the third is canonical and the last is due to the localization theorem and (4.3).

Let $\hat{\nu} \in \mathcal{Q}_{\mathbb{Z}}^+$ be an element such that $\mathfrak{M}_0^{\text{reg}}(\hat{\nu}, \hat{\lambda}) \neq \emptyset$. We pick an arbitrary point $x_{\hat{\nu}} \in \mathfrak{M}_0^{\text{reg}}(\hat{\nu}, \hat{\lambda})$ and consider the (non-equivariant) K -group $\mathcal{K}(\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}}) := K(\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}}) \otimes_{\mathbb{Z}} \mathbb{k}$ of the fiber $\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}}$. This is a module over the convolution algebra $\mathcal{K}(\mathcal{Z}^\bullet(\hat{\lambda})) := K(\mathcal{Z}^\bullet(\hat{\lambda})) \otimes_{\mathbb{Z}} \mathbb{k}$. We regard $\mathcal{K}(\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}})$ as a \tilde{U}_q -module via the following composition:

$$\tilde{U}_q \xrightarrow{\hat{\Phi}_{\hat{\lambda}}} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}^\bullet(\hat{\lambda})) \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}^\bullet(\hat{\lambda}))/\mathfrak{r}_{\hat{\lambda}} \xrightarrow{\cong} \mathcal{K}^{\mathbb{T}}(\mathcal{Z}^\bullet(\hat{\lambda})) \cong \mathcal{K}(\mathcal{Z}^\bullet(\hat{\lambda})),$$

where the third arrow is an isomorphism by Theorem 4.3(3).

Proposition 4.5. *The \tilde{U}_q -module $\mathcal{K}(\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}})$ is isomorphic to the local Weyl module $W(\hat{\lambda} - \hat{\nu})$.*

Proof. When $\hat{\nu} = 0$, we have

$$\begin{aligned} \mathcal{K}(\mathcal{Z}^\bullet(\hat{\lambda})) &\cong K^{\mathbb{T}}(\mathcal{Z}(\lambda)) \otimes_{R(\mathbb{T})} \mathbb{k} && \text{by the localization theorem} \\ &\cong \mathcal{K}^{\mathbb{G}(\lambda)}(\mathcal{Z}(\lambda))/\mathfrak{r}_{\lambda, \hat{\lambda}} && \text{by Theorem 4.3(3)} \\ &\cong \mathbb{W}(\lambda)/\mathfrak{r}_{\lambda, \hat{\lambda}} && \text{by Theorem 4.2} \\ &= W(\hat{\lambda}). \end{aligned}$$

For a general $\hat{\nu}$, we know that the \tilde{U}_q -module $\mathcal{K}(\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}})$ is a quotient of $W(\hat{\lambda} - \hat{\nu})$ by [35, Proposition 13.3.1] and by the universality of the local Weyl module. Because there is an isomorphism $\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}} \cong \mathcal{Z}^\bullet(\hat{\lambda} - \hat{\nu})$ by Lemma 3.4, we have $\dim \mathcal{K}(\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}}) = \dim \mathcal{K}(\mathcal{Z}^\bullet(\hat{\lambda} - \hat{\nu})) = \dim W(\hat{\lambda} - \hat{\nu})$ and hence the isomorphism $W(\hat{\lambda} - \hat{\nu}) \xrightarrow{\cong} \mathcal{K}(\mathfrak{M}^\bullet(\hat{\lambda})_{x_{\hat{\nu}}})$. □

4.3. Completed Hernandez-Leclerc category $\widehat{\mathcal{C}}_{Q, \beta}$. Throughout this subsection, we fix an element $\beta = \sum_{i \in I} d_i \alpha_i \in \mathbb{Q}^+$ and set

$$\hat{\lambda} \equiv \hat{\lambda}_\beta := \sum_{i \in I} d_i \varpi_{\phi(\alpha_i)} \in \mathcal{P}_Q^+, \quad \lambda := \text{cl}(\hat{\lambda}) \in \mathbb{P}^+,$$

as in Section 3.5. For simplicity, we mainly use the notation $\hat{\lambda}$ rather than $\hat{\lambda}_\beta$, suppressing β .

Fix an element $\hat{\nu} \in \mathcal{Q}_Q^+$ such that $\hat{\mu} := \hat{\lambda} - \hat{\nu} \in \mathcal{P}_{Q, \beta}^+$ and put $\nu := \text{cl}(\hat{\nu}) \in \mathbb{Q}^+$, $\mu := \text{cl}(\hat{\mu}) \in \mathbb{P}^+$. Recall that we have an algebra homomorphism $\theta_{\hat{\mu}}: R(\mathbb{G}(\hat{\lambda})) \rightarrow R(\mathbb{G}(\hat{\mu}))$ defined in Subsection 3.5 (3.12). After the localization $(-)\otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{k}$, we get a homomorphism

$$\theta_{\hat{\mu}}: \mathcal{R}(\hat{\lambda}) \rightarrow \mathcal{R}(\hat{\mu}),$$

for which we use the same symbol $\theta_{\hat{\mu}}$. Through this homomorphism $\theta_{\hat{\mu}}$, the algebra $\mathcal{R}(\hat{\mu})$ is regarded as an $\mathcal{R}(\hat{\lambda})$ -algebra.

Lemma 4.6. *The ideal $\langle \theta_{\hat{\mu}}(\mathfrak{r}_{\hat{\lambda}}) \rangle \subset \mathcal{R}(\hat{\mu})$ generated by the image of $\mathfrak{r}_{\hat{\lambda}}$ is a primary ideal whose associated prime is the maximal ideal $\mathfrak{r}_{\hat{\mu}}$. In particular, we have*

$$\mathcal{R}(\hat{\mu}) \otimes_{\mathcal{R}(\hat{\lambda})} \widehat{\mathcal{R}}(\hat{\lambda}) \cong \widehat{\mathcal{R}}(\hat{\mu}).$$

Proof. This is a direct consequence of Corollary 3.8. □

Proposition 4.7. *With the above notation, let \mathfrak{M}_μ^\bullet denote the inverse image of the stratum $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda})$ along the canonical morphism $\pi^\bullet : \mathfrak{M}^\bullet(\hat{\lambda}) \rightarrow \mathfrak{M}_0^\bullet(\hat{\lambda})$. Then we have the following isomorphism of $(\tilde{U}_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodules:*

$$\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet) \cong \widehat{W}(\hat{\mu}),$$

where the action of $\widehat{\mathcal{R}}(\hat{\lambda})$ on the RHS is given via the homomorphism $\theta_{\hat{\mu}}$.

Proof. Fix a point $x \in \mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda})$. Since $\mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}, \hat{\lambda})$ consists of a single $\mathbb{G}(\hat{\lambda})$ -orbit by Lemma 3.6 and the morphism $\pi^\bullet : \mathfrak{M}^\bullet(\hat{\lambda}) \rightarrow \mathfrak{M}_0^\bullet(\hat{\lambda})$ is $\mathbb{G}(\hat{\lambda})$ -equivariant, we have an isomorphism $\mathfrak{M}_\mu^\bullet \cong \mathbb{G}(\hat{\lambda}) \times_{(\text{Stab}_{\mathbb{G}(\hat{\lambda})} x)} \mathfrak{M}^\bullet(\hat{\lambda})_x$. Then we have

$$\begin{aligned} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet) &\cong \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})} \left(\mathbb{G}(\hat{\lambda}) \times_{(\text{Stab}_{\mathbb{G}(\hat{\lambda})} x)} \mathfrak{M}^\bullet(\hat{\lambda})_x \right) \\ &\cong \mathcal{K}^{(\text{Stab}_{\mathbb{G}(\hat{\lambda})} x)}(\mathfrak{M}^\bullet(\hat{\lambda})_x) \otimes_{\widehat{\mathcal{R}}(\hat{\lambda})} \widehat{\mathcal{R}}(\hat{\lambda}) \\ &\cong \mathcal{K}^{\mathbb{G}(\hat{\mu})}(\mathfrak{L}^\bullet(\hat{\mu})) \otimes_{\widehat{\mathcal{R}}(\hat{\mu})} \widehat{\mathcal{R}}(\hat{\mu}) \\ &\cong \mathcal{K}^{\mathbb{G}(\mu)}(\mathfrak{L}(\mu)) \otimes_{\mathcal{R}(\mu)} \widehat{\mathcal{R}}(\hat{\mu}), \end{aligned}$$

where the second isomorphism is by the induction (see [11, 5.2.16]), the third is due to Lemma 3.4, Lemma 3.7 and Lemma 4.6, the last is due to the localization and Theorem 4.3(3). Through this isomorphism $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet) \cong \mathcal{K}^{\mathbb{G}(\mu)}(\mathfrak{L}(\mu)) \otimes_{\mathcal{R}(\mu)} \widehat{\mathcal{R}}(\hat{\mu})$, we see that the action of $\widehat{\mathcal{R}}(\hat{\lambda})$ on $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet)$ extends to an action of $\widehat{\mathcal{R}}(\hat{\mu})$, which commutes with the action of U_q . By Proposition 4.5, the module $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet)/\tau_{\hat{\mu}} \cong \mathcal{K}(\mathfrak{M}(\lambda)_x)$ is isomorphic to the local Weyl module $W(\hat{\mu})$. Therefore, by Nakayama's lemma, we see that the vector in $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet)$ which corresponds to $[\mathcal{O}_{\mathfrak{L}(0,\mu)}]$ in $\mathcal{K}^{\mathbb{G}(\mu)}(\mathfrak{L}(\mu)) \otimes_{\mathcal{R}(\mu)} \widehat{\mathcal{R}}(\hat{\mu})$ generates $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet)$ as $(U_q, \widehat{\mathcal{R}}(\hat{\mu}))$ -bimodule. Moreover, from the construction of isomorphism $\mathfrak{M}(\lambda)_x \cong \mathfrak{L}(\mu)$ in Lemma 3.3, we see that the restriction of the class $[\mathcal{C}_i(\lambda, \nu)]$ in $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet)$ corresponds to the class $[\mathcal{W}_i^{\lambda-\nu}]|_{\mathfrak{L}(\mu)}$ in $\mathcal{K}^{\mathbb{G}(\mu)}(\mathfrak{L}(\mu)) \otimes_{\mathcal{R}(\mu)} \widehat{\mathcal{R}}(\hat{\mu})$. Therefore, by the universal property of the global Weyl module, we find a surjection $\widehat{W}(\hat{\mu}) \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet)$ of $(U_q, \widehat{\mathcal{R}}(\hat{\mu}))$ -bimodules. Since both are free over $\widehat{\mathcal{R}}(\hat{\mu})$ of the same rank $\dim W(\hat{\mu})$, this surjection should be an isomorphism $\widehat{W}(\hat{\mu}) \cong \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathfrak{M}_\mu^\bullet)$. \square

Definition 4.8. We define the *completed Hernandez-Leclerc category* $\widehat{\mathcal{C}}_{Q,\beta}$ to be the category of finitely generated $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ -modules, i.e.,

$$\widehat{\mathcal{C}}_{Q,\beta} := \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))\text{-mod}_{\text{fg}}.$$

Let $(\widehat{\mathcal{C}}_{Q,\beta})_f$ denote the full subcategory of finite-dimensional modules in $\widehat{\mathcal{C}}_{Q,\beta}$, i.e.,

$$(\widehat{\mathcal{C}}_{Q,\beta})_f := \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))\text{-mod}_{\text{fd}}.$$

We regard the deformed local Weyl module $\widehat{W}(\hat{\mu})$ corresponding to $\hat{\mu} \in \mathcal{P}_{Q,\beta}^+$ as a $(\tilde{U}_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodule on which the algebra $\widehat{\mathcal{R}}(\hat{\lambda})$ acts via the homomorphism $\theta_{\hat{\mu}}$. By Proposition 4.7, we can regard $\widehat{W}(\hat{\mu}) \in \widehat{\mathcal{C}}_{Q,\beta}$ for any $\hat{\mu} \in \mathcal{P}_{Q,\beta}^+$. Now we state the main theorem of this section.

Theorem 4.9. *With the above notation, the following hold:*

- (1) Via the completed Nakajima homomorphism $\widehat{\Phi}_\lambda$, we can identify the category $(\widehat{\mathcal{C}}_{Q,\beta})_f$ with the Hernandez-Leclerc category $\mathcal{C}_{Q,\beta}$. More precisely, the pullback functor $(\widehat{\mathcal{C}}_{Q,\beta})_f \rightarrow \mathcal{C}_{\mathfrak{g}}$; $M \mapsto (\widehat{\Phi}_\lambda)^*M$ gives an equivalence of categories $(\widehat{\mathcal{C}}_{Q,\beta})_f \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$;
- (2) The completed Hernandez-Leclerc category $\widehat{\mathcal{C}}_{Q,\beta}$ is an affine highest weight category for the poset $(\mathcal{P}_{Q,\beta}^+, \leq)$. The standard module (resp. proper standard module, proper costandard module) associated with $\hat{\mu} \in \mathcal{P}_{Q,\beta}^+$ is given by the deformed local Weyl module $\widehat{W}(\hat{\mu})$ (resp. local Weyl module $W(\hat{\mu})$, dual local Weyl module $W^\vee(\hat{\mu})$).

Remark 4.10. Note that there is a small conflict of terminologies. As we noted in Remark 2.11, local Weyl modules $W(\hat{\mu})$ are the same as the standard modules in the sense of Nakajima and Varagnolo-Vasserot. But, in our context of affine highest weight category, they are not standard modules but proper standard modules.

A proof of Theorem 4.9 is given after Corollary 4.17. We need some lemmas.

Lemma 4.11. For $\hat{\nu} \in \mathcal{Q}_Q^+$ with $\hat{\mu} := \hat{\lambda} - \hat{\nu} \in \mathcal{P}_{Q,\beta}^+$, let $Z_{\hat{\mu}}^\bullet$ denote the inverse image of the stratum $\mathfrak{M}_0^{\text{reg}}(\hat{\nu}, \hat{\lambda})$ along the canonical morphism $\pi^\bullet: Z^\bullet(\hat{\lambda}) \rightarrow \mathfrak{M}_0^\bullet(\hat{\lambda})$.

- (1) As a $(\widetilde{U}_q, \widetilde{U}_q)$ -bimodule, we have $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\hat{\mu}}^\bullet) \cong \widehat{W}(\hat{\mu}) \otimes_{\widehat{\mathcal{R}}(\hat{\mu})} \widehat{W}(\hat{\mu})^\sharp$;
- (2) We have $\widehat{\mathcal{K}}_{1,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\hat{\mu}}^\bullet) = 0$;
- (3) The comparison map gives an isomorphism $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\hat{\mu}}^\bullet) \cong \widehat{\mathcal{K}}_{0,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\hat{\mu}}^\bullet)$.

Proof. Fix a point $x \in \mathfrak{M}_0^{\text{reg}}(\hat{\nu}, \hat{\lambda})$. Then we have an isomorphism

$$Z_{\hat{\mu}}^\bullet \cong \mathbb{G}(\hat{\lambda}) \times_{(\text{Stab}_{\mathbb{G}(\hat{\lambda})} x)} \left(\mathfrak{M}^\bullet(\hat{\lambda})_x \times \mathfrak{M}^\bullet(\hat{\lambda})_x \right).$$

A similar computation as in the proof of Proposition 4.7 yields:

$$\begin{aligned} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\hat{\mu}}^\bullet) &\cong \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})} \left(\mathbb{G}(\hat{\lambda}) \times_{(\text{Stab}_{\mathbb{G}(\hat{\lambda})} x)} \left(\mathfrak{M}^\bullet(\hat{\lambda})_x \times \mathfrak{M}^\bullet(\hat{\lambda})_x \right) \right) \\ &\cong \mathcal{K}^{\mathbb{G}(\mu)}(\mathfrak{L}(\mu) \times \mathfrak{L}(\mu)) \otimes_{\mathcal{R}(\mu)} \widehat{\mathcal{R}}(\hat{\mu}) \\ &\cong \mathcal{K}^{\mathbb{G}(\mu)}(\mathfrak{L}(\mu)) \otimes_{\mathcal{R}(\mu)} \widehat{\mathcal{R}}(\hat{\mu}) \otimes_{\mathcal{R}(\mu)} \mathcal{K}^{\mathbb{G}(\mu)}(\mathfrak{L}(\mu)), \end{aligned}$$

where the last isomorphism is due to Theorem 4.3(4). Then Proposition 4.7 proves the assertion (1). Because the same computation is valid for equivariant K -homologies, we have

$$\widehat{\mathcal{K}}_{i,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\hat{\mu}}^\bullet) \cong \mathcal{K}_{0,\text{top}}^{\mathbb{G}(\mu)}(\mathfrak{L}(\mu)) \otimes_{\mathcal{R}(\mu)} \widehat{\mathcal{R}}(\hat{\mu}) \otimes_{\mathcal{R}(\mu)} \mathcal{K}_{i,\text{top}}^{\mathbb{G}(\mu)}(\mathfrak{L}(\mu)),$$

for $i = 0, 1$. Then Theorem 4.3(1), (2) prove the assertions (2), (3). □

We fix a total ordering $\{\lambda_1, \lambda_2, \dots, \lambda_l\}$ of the set $\mathcal{P}_{\leq \lambda}^+ := \{\mu \in \mathcal{P}^+ \mid \mu \leq \lambda\}$ such that $\lambda = \lambda_l$ and $i < j$ whenever $\lambda_i < \lambda_j$. Let $\nu_i := \lambda - \lambda_i \in \mathcal{Q}^+$ and $\mathfrak{N}_i := \mathfrak{M}_0^{\text{reg}}(\nu_i, \lambda)$ for $i \in \{1, \dots, l\}$. Then the stratification (3.1) is written as:

$$\mathfrak{M}_0(\lambda) = \mathfrak{N}_1 \sqcup \mathfrak{N}_2 \sqcup \dots \sqcup \mathfrak{N}_l$$

with $\mathfrak{N}_l = \{0\}$. For each $i \in \{1, \dots, l\}$, we set $\mathfrak{N}_{\leq i} := \bigsqcup_{j \leq i} \mathfrak{N}_j \subset \mathfrak{M}_0(\lambda)$. Note that \mathfrak{N}_i is a closed subvariety of $\mathfrak{N}_{\leq i}$ and its complement is $\mathfrak{N}_{\leq i-1}$.

We set $\mathfrak{N}_i^\bullet := \mathfrak{N}_i \cap \mathfrak{M}_0^\bullet(\hat{\lambda})$ and $\mathfrak{N}_{\leq i}^\bullet := \mathfrak{N}_{\leq i} \cap \mathfrak{M}_0^\bullet(\hat{\lambda})$ for each $i \in \{1, \dots, l\}$. We fix a total ordering $\{\hat{\lambda}_{i,1}, \hat{\lambda}_{i,2}, \dots, \hat{\lambda}_{i,k_i}\}$ of the set $\text{cl}^{-1}(\lambda_i) \cap \mathcal{P}_{Q,\beta}^+$, where we

define $k_i = 0$ if $\text{cl}^{-1}(\lambda_i) \cap \mathcal{P}_{Q,\beta}^+ = \emptyset$. We simplify the notation by setting $\mathcal{R}_{i,s} := \mathcal{R}(\hat{\lambda}_{i,s})$, $\widehat{\mathcal{R}}_{i,s} := \widehat{\mathcal{R}}(\hat{\lambda}_{i,s})$, $\theta_{i,s} := \theta_{\hat{\lambda}_{i,s}}$ for each $i \in I$ and $s \in \{1, \dots, k_i\}$. Set $\hat{\nu}_{i,s} := \hat{\lambda} - \hat{\lambda}_{i,s} \in \mathcal{Q}_Q^+$ and $\mathfrak{N}_{i,s}^\bullet := \mathfrak{M}_0^{\bullet \text{reg}}(\hat{\nu}_{i,s}, \hat{\lambda})$. Note that we have $\text{cl}(\hat{\nu}_{i,s}) = \nu_i$ and that $\mathfrak{N}_{i,s}^\bullet$ gives a connected component of \mathfrak{N}_i^\bullet for any $s \in \{1, \dots, k_i\}$. Namely, we get the decomposition

$$(4.4) \quad \mathfrak{N}_i^\bullet = \bigsqcup_{s=1}^{k_i} \mathfrak{N}_{i,s}^\bullet$$

of \mathfrak{N}_i^\bullet into connected components. We define a subvariety Z_i^\bullet (resp. $Z_{\leq i}^\bullet, Z_{i,s}^\bullet$) of $Z^\bullet(\hat{\lambda})$ to be the inverse image of the subvariety \mathfrak{N}_i^\bullet (resp. $\mathfrak{N}_{\leq i}^\bullet, \mathfrak{N}_{i,s}^\bullet$) along the canonical morphism $\pi^\bullet: Z^\bullet(\hat{\lambda}) \rightarrow \mathfrak{M}_0^\bullet(\hat{\lambda})$. From the decomposition (4.4), we have

$$(4.5) \quad Z_i^\bullet = \bigsqcup_{s=1}^{k_i} Z_{i,s}^\bullet.$$

By construction, Z_i^\bullet is a closed subvariety of $Z_{\leq i}^\bullet$ and its complement is $Z_{\leq i-1}^\bullet$ for each $i \in \{2, \dots, l\}$. From (4.1), we have an exact sequence:

$$(4.6) \quad \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet) \xrightarrow{\iota_*} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet) \xrightarrow{j^*} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i-1}^\bullet) \longrightarrow 0,$$

where $\iota: Z_i^\bullet \hookrightarrow Z_{\leq i}^\bullet$ and $j: Z_{\leq i-1}^\bullet \hookrightarrow Z_{\leq i}^\bullet$ are the inclusions.

Lemma 4.12. *The map ι_* in the sequence (4.6) is injective. Therefore we have the following short exact sequence:*

$$0 \longrightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet) \xrightarrow{\iota_*} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet) \xrightarrow{j^*} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i-1}^\bullet) \longrightarrow 0,$$

for each $i \in \{2, \dots, l\}$.

Proof. We shall prove that $\widehat{\mathcal{K}}_{1,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet) = 0$ and the comparison map $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet) \rightarrow \widehat{\mathcal{K}}_{0,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet)$ is an isomorphism for each $i \in \{1, \dots, l\}$. If we prove them, the exact hexagon (4.2) yields the assertion. We proceed by induction on i .

When $i = 1$, from (4.5), we have

$$\widehat{\mathcal{K}}_{j,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq 1}^\bullet) = \widehat{\mathcal{K}}_{j,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_1^\bullet) = \bigoplus_{s=1}^{k_1} \widehat{\mathcal{K}}_{j,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{1,s}^\bullet),$$

where $j = 0, 1$. From Lemma 4.11, we know that $\widehat{\mathcal{K}}_{1,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{1,s}^\bullet) = 0$ and $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{1,s}^\bullet) \cong \widehat{\mathcal{K}}_{0,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{1,s}^\bullet)$ for each $s \in \{1, \dots, k_1\}$. Therefore we are done in this case.

Let $i > 1$ and assume that we know that $\widehat{\mathcal{K}}_{1,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i-1}^\bullet) = 0$ and $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i-1}^\bullet) \cong \widehat{\mathcal{K}}_{0,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i-1}^\bullet)$. By the same reason for the case $i = 1$ above, we have $\widehat{\mathcal{K}}_{1,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet) = 0$ and $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet) \cong \widehat{\mathcal{K}}_{0,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet)$. Then we see that $\widehat{\mathcal{K}}_{1,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet) = 0$ from the exact

hexagon (4.2). Moreover we have the following commutative diagram:

$$\begin{array}{ccccccc}
 \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet) & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet) & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i-1}^\bullet) & \longrightarrow & 0 \\
 \downarrow \cong & & \downarrow & & \downarrow \cong & & \\
 0 & \longrightarrow & \widehat{\mathcal{K}}_{0,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet) & \longrightarrow & \widehat{\mathcal{K}}_{0,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet) & \longrightarrow & \widehat{\mathcal{K}}_{0,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i-1}^\bullet) \longrightarrow 0,
 \end{array}$$

where the upper row is the exact sequence (4.6), the lower row is the exact sequence coming from the exact hexagon (4.2). All vertical arrows are the comparison maps. Applying the five lemma, we see that the middle comparison map $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet) \rightarrow \widehat{\mathcal{K}}_{0,\text{top}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet)$ is also an isomorphism. \square

Corollary 4.13. *The completed convolution algebra $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ is finitely generated as an $\widehat{\mathcal{R}}(\hat{\lambda})$ -module.*

Proof. The assertion follows from Lemmas 4.11 and 4.12. \square

Recall we have defined a quotient algebra $U_{\leq \lambda}$ of the modified quantum loop algebra \widetilde{U}_q in Subsection 2.5 by (2.10).

Lemma 4.14. *The completed Nakajima homomorphism $\widehat{\Phi}_\lambda: \widetilde{U}_q \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ factors through the quotient $\widetilde{U}_q \rightarrow U_{\leq \lambda}$.*

Proof. Since we have $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda})) \cong \varprojlim \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))/\mathfrak{t}_\lambda^N$, it is enough to prove that the composition

$$(4.7) \quad \widetilde{U}_q \xrightarrow{\widehat{\Phi}_\lambda} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda})) \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))/\mathfrak{t}_\lambda^N$$

factors through the quotient $U_{\leq \lambda}$ for every $N \in \mathbb{Z}_{>0}$. We can discuss the composition factors of $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))/\mathfrak{t}_\lambda^N$ as a left \widetilde{U}_q -module because it is finite-dimensional by Corollary 4.13. By Lemma 4.11(1) and the exact sequence (4.6), we see that every composition factor is a subquotient of the global Weyl modules $\mathbb{W}(\mu)$ for some $\mu \leq \lambda$. Therefore the ideal $\bigcap_{\mu \leq \lambda} \text{Ann}_{\widetilde{U}_q} \mathbb{W}(\mu)$ is included in the kernel of the map (4.7). \square

By Lemma 4.14, we have the induced homomorphism $\widehat{\Phi}_\lambda: U_{\leq \lambda} \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$, which we denote by the same symbol $\widehat{\Phi}_\lambda$.

Proposition 4.15. *For each $N \in \mathbb{Z}_{>0}$, the homomorphism $\widehat{\Phi}_\lambda^N$ given by the composition*

$$\widehat{\Phi}_\lambda^N: U_{\leq \lambda} \xrightarrow{\widehat{\Phi}_\lambda} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda})) \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))/\mathfrak{t}_\lambda^N$$

is surjective. In particular, the forgetful functor from $\widehat{\mathcal{C}}_{Q,\beta}$ to the category of $(U_{\leq \lambda}, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodules is fully faithful.

Proof. Fix $N \in \mathbb{Z}_{>0}$. Using the homomorphism $\widehat{\Phi}_\lambda$, we compare the affine cellular structure of the algebra $U_{\leq \lambda}$ with the filtration of the algebra $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ coming from the geometric stratification of $\mathfrak{M}_0^\bullet(\hat{\lambda})$ as in Lemma 4.12. First, for each i ,

we observe that there is the following isomorphism of $(U_{\leq \lambda}, U_{\leq \lambda})$ -bimodules by Lemma 4.11(1):

$$\begin{aligned}
 \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet)/\mathfrak{r}_\lambda^N &\cong \bigoplus_{s=1}^{k_i} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{i,s}^\bullet)/\mathfrak{r}_\lambda^N \\
 &\cong \bigoplus_{s=1}^{k_i} \widehat{W}(\hat{\lambda}_{i,s}) \otimes_{\widehat{\mathcal{R}}_{i,s}} \left(\frac{\widehat{\mathcal{R}}_{i,s}}{\langle \theta_{i,s}(\mathfrak{r}_\lambda^N) \rangle} \right) \otimes_{\widehat{\mathcal{R}}_{i,s}} \widehat{W}(\hat{\lambda}_{i,s})^\# \\
 &\cong \bigoplus_{s=1}^{k_i} \mathbb{W}(\lambda_i) \otimes_{\mathcal{R}(\lambda_i)} \left(\frac{\mathcal{R}(\lambda_i)}{\mathcal{R}(\lambda_i) \cap \langle \theta_{i,s}(\mathfrak{r}_\lambda^N) \rangle} \right) \otimes_{\mathcal{R}(\lambda_i)} \mathbb{W}(\lambda_i)^\# \\
 (4.8) \quad &\cong \mathbb{W}(\lambda_i) \otimes_{\mathcal{R}(\lambda_i)} \left(\frac{\mathcal{R}(\lambda_i)}{\prod_{s=1}^{k_i} \mathcal{R}(\lambda_i) \cap \langle \theta_{i,s}(\mathfrak{r}_\lambda^N) \rangle} \right) \otimes_{\mathcal{R}(\lambda_i)} \mathbb{W}(\lambda_i)^\#,
 \end{aligned}$$

where we apply the Chinese remainder theorem for the last isomorphism. This is possible because the maximal ideals associated with the primary ideals $\mathcal{R}(\lambda_i) \cap \langle \theta_{i,s}(\mathfrak{r}_\lambda^N) \rangle$ are distinct by Lemma 4.6. By (4.8), we see that the K -group $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet)$ is cyclic as $(U_{\leq \lambda}, U_{\leq \lambda})$ -bimodule. By construction, we can easily see that the class in $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet)$ obtained as the restriction of the class $\Delta_*[\mathcal{O}_{\mathfrak{M}(\nu_i, \lambda)}]$ corresponds to the cyclic vector $w_{\lambda_i} \otimes 1 \otimes w_{\lambda_i}$ of the RHS of (4.8).

Recall the ideals I_i of $U_{\leq \lambda}$ defined by (2.11). By downward induction on $i \in \{1, \dots, l\}$, we shall construct algebra homomorphisms $f_i^N: U_{\leq \lambda}/I_i \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet)/\mathfrak{r}_\lambda^N$ and $(U_{\leq \lambda}, U_{\leq \lambda})$ -bimodule homomorphisms $g_i^N: I_{i-1}/I_i \rightarrow \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet)/\mathfrak{r}_\lambda^N$, which make the following diagrams commute:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{W}(\lambda_i) \otimes_{\mathcal{R}(\lambda_i)} \mathbb{W}(\lambda_i)^\# & \xrightarrow{a_i} & U_{\leq \lambda}/I_i & \xrightarrow{b_i} & U_{\leq \lambda}/I_{i-1} \longrightarrow 0 \\
 & & \downarrow g_i^N & & \downarrow f_i^N & & \downarrow f_{i-1}^N \\
 & & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet)/\mathfrak{r}_\lambda^N & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet)/\mathfrak{r}_\lambda^N & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i-1}^\bullet)/\mathfrak{r}_\lambda^N \longrightarrow 0,
 \end{array}$$

where the upper row is the exact sequence coming from the ideal chain and the lower row is the exact sequence coming from (4.6).

We start from $i = l$. Define f_l^N to be the homomorphism $\widehat{\Phi}_\lambda^N$. Recall that the Nakajima homomorphism Φ_λ sends the element a_λ to the class $\Delta_*[\mathcal{O}_{\mathfrak{M}(0, \lambda)}]$ (see Theorem 4.1). Therefore, by our observation in the previous paragraph, if we define the homomorphism g_l^N to give the quotient map from $\mathbb{W}(\lambda) \otimes_{\mathcal{R}(\lambda)} \mathbb{W}(\lambda)^\#$ to the RHS of (4.8) via the isomorphism (4.8), the left square in (4.9) commutes. Then we have the induced homomorphism f_{l-1}^N between the cokernels.

The induction step is similar. Assume that we have defined f_i^N . By construction, f_i^N sends the image of the element a_{λ_i} to the restriction of the class $\Delta_*[\mathcal{O}_{\mathfrak{M}(\nu_i, \lambda)}]$. Then if we define g_i^N to be the quotient map via the isomorphism (4.8), the left square in (4.9) commutes. We get f_{i-1}^N as the induced homomorphism between the cokernels.

Note that we have $f_1^N = g_1^N$ and all the homomorphisms g_i^N are surjective by construction. Therefore we can apply the five lemma to the diagram (4.9) inductively, starting from the case $i = 2$, to prove that every f_i^N is also surjective. Eventually we see that the homomorphism $f_l^N = \widehat{\Phi}_\lambda^N$ is surjective. \square

Using the notation in the above proof of Proposition 4.15, we define $K_{\leq i}^N := \text{Ker } f_i^N, K_i^N := \text{Ker } g_i^N$ for each $i \in \{1, \dots, l\}$ and $N \in \mathbb{Z}_{>0}$.

Proposition 4.16. *For each $i \in \{1, \dots, l\}$ and for any $N_1, N_2 \in \mathbb{Z}_{>0}$, there exists a positive integer $N > N_1 + N_2$ satisfying*

- (1) $K_i^N \subset K_i^{N_1} \cdot K_i^{N_2}$;
- (2) $K_{\leq i}^N \subset K_{\leq i}^{N_1} \cdot K_{\leq i}^{N_2}$.

Proof. We first prove the assertion (1). Assume that $k_i = 0$. Thus we have $g_i^N = 0$ and hence $K_i^N = I_{i-1}/I_i$ for any $N \in \mathbb{Z}_{>0}$. In this case, the assertion (1) is equivalent to the assertion $(I_{i-1}/I_i)^2 = I_{i-1}/I_i$, which follows from Theorem 2.22. Next we consider the case $k_i \neq 0$. By (4.8), we have

$$K_i^N = \mathbb{W}(\lambda_i) \otimes_{\mathcal{R}(\lambda_i)} \mathbb{W}(\lambda_i)^\sharp \left(\prod_{s=1}^{k_i} (\mathcal{R}(\lambda_i) \cap \langle \theta_{i,s}(\mathfrak{r}_{\hat{\lambda}})^N \rangle) \right) \otimes_{\mathcal{R}(\lambda_i)} \mathbb{W}(\lambda_i)^\sharp,$$

for each $N \in \mathbb{Z}_{>0}$. By Lemma 4.6, the ideal $\mathcal{R}(\lambda_i) \cap \langle \theta_{i,a}(\mathfrak{r}_{\hat{\lambda}})^N \rangle$ is a primary ideal whose associated prime is the maximal ideal $\mathfrak{r}_{\lambda_i, \hat{\lambda}_{i,a}}$. Thus for a sufficiently large $N > 0$, we have $\mathcal{R}(\lambda_i) \cap \langle \theta_{i,s}(\mathfrak{r}_{\hat{\lambda}})^N \rangle \subset (\mathcal{R}(\lambda_i) \cap \langle \theta_{i,s}(\mathfrak{r}_{\hat{\lambda}})^{N_1} \rangle)(\mathcal{R}(\lambda_i) \cap \langle \theta_{i,s}(\mathfrak{r}_{\hat{\lambda}})^{N_2} \rangle)$. Then we obtain the assertion $K_i^N \subset K_i^{N_1} \cdot K_i^{N_2}$.

We prove the assertion (2) by induction on i . The case $i = 1$ follows from (1) since $f_1^N = g_1^N$. We assume that $i > 1$ and the assertion (2) is true for $i - 1$. For given $N_1, N_2 \in \mathbb{Z}_{>0}$, we can find an integer $N' > N_1 + N_2$ such that $K_i^{N'} \subset K_i^{N_1} \cdot K_i^{N_2}$ by (1). By the Artin-Rees lemma, we find an integer $N'' > N'$ such that we have

$$\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet) \cap (\mathfrak{r}_{\hat{\lambda}}^{N''} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet)) \subset \mathfrak{r}_{\hat{\lambda}}^{N'} \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_i^\bullet),$$

which implies

$$(4.10) \quad \text{Ker}(f_i^{N''} \circ a_i) \subset K_i^{N'} \subset K_i^{N_1} \cdot K_i^{N_2},$$

where a_i is the inclusion $I_{i-1}/I_i \hookrightarrow U_{\leq \lambda}/I_i$ as in the diagram (4.9). Applying the snake lemma to the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{W}(\lambda_i) \otimes_{\mathcal{R}(\lambda_i)} \mathbb{W}(\lambda_i)^\sharp & \xrightarrow{a_i} & U_{\leq \lambda}/I_i & \xrightarrow{b_i} & U_{\leq \lambda}/I_{i-1} \longrightarrow 0 \\ & & \downarrow f_i^{N''} \circ a_i & & \downarrow f_i^{N''} & & \downarrow f_{i-1}^{N''} \\ 0 & \longrightarrow & \text{Im}(f_i^{N''} \circ a_i) & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i}^\bullet)/\mathfrak{r}_{\hat{\lambda}}^{N''} & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z_{\leq i-1}^\bullet)/\mathfrak{r}_{\hat{\lambda}}^{N''} \longrightarrow 0, \end{array}$$

we get an exact sequence

$$(4.11) \quad 0 \longrightarrow \text{Ker}(f_i^{N''} \circ a_i) \longrightarrow K_{\leq i}^{N''} \longrightarrow K_{\leq i-1}^{N''} \longrightarrow 0.$$

Let N'_1, N'_2 be any two integers larger than N'' . By induction hypothesis, there is an integer $N > N'_1 + N'_2$ such that $K_{\leq i-1}^N \subset K_{\leq i-1}^{N'_1} \cdot K_{\leq i-1}^{N'_2}$. We shall prove that the assertion (2) holds for this N . Let $x \in K_{\leq i}^N$ be an arbitrary element. Note that we have $b_i(x) \in K_{\leq i-1}^N \subset K_{\leq i-1}^{N'_1} \cdot K_{\leq i-1}^{N'_2}$. Since the quotient map $b_i: U_{\leq \lambda}/I_i \rightarrow U_{\leq \lambda}/I_{i-1}$ induces the surjection $K_{\leq i}^{N'_1} \cdot K_{\leq i}^{N'_2} \rightarrow K_{\leq i-1}^{N'_1} \cdot K_{\leq i-1}^{N'_2}$, we can choose an element $y \in K_{\leq i}^{N'_1} \cdot K_{\leq i}^{N'_2}$ so that $b_i(x - y) = 0$. By the exact sequence (4.11), there is an element $y' \in \text{Ker}(f_i^{N''} \circ a_i)$ such that $x = y + a_i(y')$. By (4.10), we see that $a_i(y') \in K_{\leq i}^{N_1} \cdot K_{\leq i}^{N_2}$. Therefore we have $x \in K_{\leq i}^{N_1} \cdot K_{\leq i}^{N_2}$ as desired. \square

As the special case $i = l$ of Proposition 4.16(2), we obtain the following.

Corollary 4.17. *For any $N_1, N_2 \in \mathbb{Z}_{>0}$, there is a positive integer N such that $\text{Ker } \widehat{\Phi}_\lambda^N \subset (\text{Ker } \widehat{\Phi}_\lambda^{N_1}) \cdot (\text{Ker } \widehat{\Phi}_\lambda^{N_2})$.*

Proof of Theorem 4.9(1). Note that we have $(\widehat{\mathcal{C}}_{Q,\beta})_f = \bigcup_N \widehat{\mathcal{C}}_{Q,\beta}^N$, where $\widehat{\mathcal{C}}_{Q,\beta}^N := (\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))/\mathfrak{r}_\lambda^N)$ -mod_{fg} is the full subcategory of $\widehat{\mathcal{C}}_{Q,\beta}$ consisting of modules M with $\mathfrak{r}_\lambda^N M = 0$. Thus, by Proposition 4.15, we see that the pullback functor $(\widehat{\mathcal{C}}_{Q,\beta})_f \rightarrow \mathcal{C}_\mathfrak{g}; M \mapsto (\widehat{\Phi}_\lambda)^* M$ is fully faithful. To prove that it is essentially surjective onto $\mathcal{C}_{Q,\beta}$, it is enough to show that for each module $M \in \mathcal{C}_{Q,\beta}$ there is a positive integer $N \in \mathbb{Z}_{>0}$ such that $(\text{Ker } \widehat{\Phi}_\lambda^N)M = 0$. We proceed by induction on the length of M . When $M = L(\hat{\mu})$ is a simple module of $\mathcal{C}_{Q,\beta}$ of ℓ -highest weight $\hat{\mu} \in \mathcal{P}_{Q,\beta}^+$, we have $(\text{Ker } \widehat{\Phi}_\lambda^1)L(\hat{\mu}) = 0$ because we know $(\text{Ker } \widehat{\Phi}_\lambda^1)W(\hat{\mu}) = 0$. For induction step, we write the given module M as an extension of two non-zero modules $M_1, M_2 \in \mathcal{C}_{Q,\beta}$. By induction hypothesis, there are integers $N_1, N_2 \in \mathbb{Z}_{>0}$ such that $(\text{Ker } \widehat{\Phi}_\lambda^{N_1})M_1 = (\text{Ker } \widehat{\Phi}_\lambda^{N_2})M_2 = 0$. We can find an integer $N \in \mathbb{Z}_{>0}$ such that $\text{Ker } \widehat{\Phi}_\lambda^N \subset (\text{Ker } \widehat{\Phi}_\lambda^{N_1}) \cdot (\text{Ker } \widehat{\Phi}_\lambda^{N_2})$ by Corollary 4.17. Then we have $(\text{Ker } \widehat{\Phi}_\lambda^N)M = 0$. \square

Proof of Theorem 4.9(2). We construct an affine quasi-heredity chain of the algebra $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$. Let $\{\hat{\lambda}_1, \dots, \hat{\lambda}_m\}$ be a total ordering of the set $\mathcal{P}_{Q,\beta}^+$ satisfying the condition that we have $j < k$ whenever $\hat{\lambda}_j < \hat{\lambda}_k$. Note that $\hat{\lambda}_m = \hat{\lambda}$. Put $\hat{\nu}_i := \hat{\lambda} - \hat{\lambda}_i$ and set $\mathcal{N}_i := \mathfrak{M}_0^{\bullet, \text{reg}}(\hat{\nu}_i, \hat{\lambda})$, $\mathcal{N}_{\geq i} := \bigsqcup_{j \geq i} \mathcal{N}_j$, $\mathcal{N}_{\leq i} := \bigsqcup_{j \leq i} \mathcal{N}_j$. We denote the inverse image of \mathcal{N}_i (resp. $\mathcal{N}_{\geq i}, \mathcal{N}_{\leq i}$) along the canonical morphism $\pi: Z^\bullet(\hat{\lambda}) \rightarrow \mathfrak{M}_0^{\bullet}(\hat{\lambda})$ by \mathcal{Z}_i (resp. $\mathcal{Z}_{\geq i}, \mathcal{Z}_{\leq i}$). By construction, $\mathcal{Z}_{\geq i+1}$ (resp. \mathcal{Z}_i) is a closed subvariety of $\mathcal{Z}_{\geq i}$ (resp. $\mathcal{Z}_{\leq i}$) and its complement is \mathcal{Z}_i (resp. $\mathcal{Z}_{\leq i-1}$). We also note that $\mathcal{Z}_{\geq i}$ is a closed subvariety of $Z^\bullet(\hat{\lambda})$ whose complement is $\mathcal{Z}_{\leq i-1}$. Then we consider the following commutative diagram arising from (4.1):

$$(4.12) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}_{\geq i+1}) & = & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}_{\geq i+1}) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}_{\geq i}) & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda})) & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}_{\leq i-1}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}_i) & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}_{\leq i}) & \longrightarrow & \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}_{\leq i-1}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Arguing as in Lemma 4.12, we see that the left column and the lower row in (4.12) are exact. By downward induction on i and diagram chases, we see that the middle row (and hence the middle column) in the diagram (4.12) is also exact.

Therefore we can regard $\mathcal{I}_i := \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}_{\geq i+1})$ for each $i \in \{0, \dots, m\}$ as a two-sided ideal of the algebra $\mathcal{A} := \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}^\bullet(\hat{\lambda}))$. What we have to do is to prove that the chain of ideals

$$(4.13) \quad 0 = \mathcal{I}_m \subsetneq \mathcal{I}_{m-1} \subsetneq \cdots \subsetneq \mathcal{I}_1 \subsetneq \mathcal{I}_0 = \mathcal{A}$$

gives an affine quasi-heredity chain. Observe that

$$\mathcal{I}_{i-1}/\mathcal{I}_i \cong \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(\mathcal{Z}_i) \cong \widehat{W}(\hat{\lambda}_i) \otimes_{\widehat{\mathcal{R}}(\hat{\lambda}_i)} \widehat{W}(\hat{\lambda}_i)^\# \cong \widehat{W}(\hat{\lambda}_i)^{\oplus s_i}$$

as a left \mathcal{A} -module, where $s_i := \dim W(\hat{\lambda}_i)$. By Theorem 4.9(1), the category of finite-dimensional modules over $\mathcal{A}/\mathcal{I}_i$ is identified with the full subcategory of $\mathcal{C}_{Q,\beta}$ consisting of the modules M whose ℓ -weights belong to the set $\bigcup_{j \leq i} \{\hat{\mu} \in \mathcal{P}_{\mathbb{Z}} \mid \hat{\mu} \leq \hat{\lambda}_j\}$. For such a module M , we have $\text{Hom}_{\widehat{\mathcal{C}}_{Q,\beta}}(\widehat{W}(\hat{\lambda}_i), M) \cong M_{\hat{\lambda}_i}$ by Proposition 2.13(1). In particular, the functor $\text{Hom}_{\widehat{\mathcal{C}}_{Q,\beta}}(\widehat{W}(\hat{\lambda}_i), -)$ is exact on the subcategory $\mathcal{A}/\mathcal{I}_i\text{-mod}_{\text{fd}}$. Since any module $M \in \mathcal{A}/\mathcal{I}_i\text{-mod}_{\text{fg}}$ can be written as a projective limit of finite-dimensional modules (i.e., $M \cong \varprojlim M/\mathfrak{t}_\lambda^N$), we see that the deformed local Weyl module $\widehat{W}(\hat{\lambda}_i)$ is a projective module in the category $\mathcal{A}/\mathcal{I}_i\text{-mod}_{\text{fg}}$ with its simple head $L(\hat{\lambda}_i)$. Moreover, we have $\text{Hom}_{\widehat{\mathcal{C}}_{Q,\beta}}(\mathcal{I}_{i-1}/\mathcal{I}_i, \mathcal{A}/\mathcal{I}_{i-1}) = 0$. From these observations and Proposition 2.13, we conclude that the chain (4.13) is an affine quasi-heredity chain.

By Theorem 1.4 and Remark 1.5, we see that the category $\widehat{\mathcal{C}}_{Q,\beta}$ is an affine highest weight category for the poset (\mathcal{P}^+, \leq) , whose standard module (resp. proper standard module) associated with $\hat{\mu} \in \mathcal{P}_{Q,\beta}^+$ is the deformed local Weyl module $\widehat{W}(\hat{\mu})$ (resp. local Weyl module $W(\hat{\mu})$). To prove the assertion for proper costandard modules, we have to show the Ext-orthogonality as in Theorem 1.6. This will be done in Proposition 4.18. \square

In the remainder of this section, we show the Ext-orthogonality between deformed local Weyl modules and dual local Weyl modules in the category $\widehat{\mathcal{C}}_{Q,\beta}$. Note that the dual local Weyl module $W^\vee(\hat{\mu})$ associated with $\hat{\mu} \in \mathcal{P}_{Q,\beta}^+$ actually belongs to $\widehat{\mathcal{C}}_{Q,\beta}$ by Proposition 2.17 and Theorem 4.9(1).

We need to prepare some duality functors. We temporarily use the ambient category \mathcal{B} of all $(\widetilde{U}_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodules. Note that each $M \in \mathcal{B}$ has the weight space decomposition $M = \bigoplus_{\mu \in \mathcal{P}} M_\mu$, where $M_\mu = a_\mu M$ and each weight space M_μ is preserved by the action of $\widehat{\mathcal{R}}(\hat{\lambda})$. We define the full dual and the topological dual of $M \in \mathcal{B}$ by $M^* := \text{Hom}_{\mathbb{k}}(M, \mathbb{k})$ and $D(M) := \bigcup_N \text{Hom}_{\mathbb{k}}(M/\mathfrak{t}_\lambda^N, \mathbb{k})$ respectively. We equip a left \widetilde{U}_q -module structure on M^* (resp. on $D(M)$) by twisting the natural right \widetilde{U}_q -module structure with the antipode S (resp. S^{-1}). Thus we obtain an exact functor $(-)^*: \mathcal{B} \rightarrow \mathcal{B}^{op}$ and a right exact functor $D: \mathcal{B} \rightarrow \mathcal{B}^{op}$. Moreover, if $M \in \mathcal{B}$ is finitely generated over $\widehat{\mathcal{R}}(\hat{\lambda})$, we have $(D(M))^* \cong M$. If $M \in \mathcal{C}_{\mathfrak{g}}$ (with the trivial $\widehat{\mathcal{R}}(\hat{\lambda})$ -action), the dual M^* (resp. $D(M)$) coincides with the left dual M^* (resp. the right dual *M) of M .

Proposition 4.18. *For any $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathcal{P}_{Q,\beta}^+$ and $i \in \mathbb{Z}_{\geq 0}$, we have*

$$\text{Ext}_{\widehat{\mathcal{C}}_{Q,\beta}}^i(\widehat{W}(\hat{\lambda}_1), W^\vee(\hat{\lambda}_2)) = \begin{cases} \mathbb{k} & i = 0, \hat{\lambda}_1 = \hat{\lambda}_2; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The case $i = 0$ follows from Proposition 2.13(1) and the fact that the module $W^\vee(\hat{\lambda}_2)$ has a simple socle $L(\hat{\lambda}_2)$ and $\dim W^\vee(\hat{\lambda}_2)_{\hat{\lambda}_2} = 1$.

For $i = 1$, we consider an extension in $\widehat{\mathcal{C}}_{Q,\beta}$:

$$(4.14) \quad 0 \rightarrow W^\vee(\hat{\lambda}_2) \rightarrow E \rightarrow \widehat{W}(\hat{\lambda}_1) \rightarrow 0.$$

Put $\lambda_j := \text{cl}(\hat{\lambda}_j)$ for $j = 1, 2$. If $\lambda_1 \not\prec \lambda_2$, then the ℓ -weight $\hat{\lambda}_1$ is maximal in E/\mathfrak{t}_λ^N for any $N \in \mathbb{Z}_{>0}$. By Proposition 2.13(1), we see that the sequence (4.14) splits. If $\lambda_1 < \lambda_2$, we apply the topological duality functor D to the sequence (4.14) to get the following exact sequence:

$$(4.15) \quad 0 \rightarrow D(\widehat{W}(\hat{\lambda}_1)) \rightarrow D(E) \rightarrow W(*\hat{\lambda}_2).$$

Since $\lambda_1 < \lambda_2$, we have $\widehat{W}(\hat{\lambda}_1)_{\lambda_2} = 0 = D(\widehat{W}(\hat{\lambda}_1))_{-w_0\lambda_2}$ and hence

$$\dim D(E)_{-w_0\lambda_2} = \dim E_{w_0\lambda_2} = \dim E_{\lambda_2} = \dim W^\vee(\hat{\lambda}_2)_{\lambda_2} = 1 = \dim W(*\hat{\lambda}_2)_{-w_0\lambda_2},$$

where the second equality is due to Weyl group symmetry coming from integrability. In particular, the image of the weight space $D(E)_{-w_0\lambda_2}$ coincides with $W(*\hat{\lambda}_2)_{-w_0\lambda_2}$, which generates $W(*\hat{\lambda}_2)$. Therefore the most right arrow in the sequence (4.15) is surjective. Moreover, by the universal property of the local Weyl module $W(*\hat{\lambda}_2)$, we see that the sequence (4.15) is split. By applying the full duality functor $(-)^*$, we find that the sequence (4.14) is also split. Therefore we have $\text{Ext}_{\widehat{\mathcal{C}}_{Q,\beta}}^1(\widehat{W}(\hat{\lambda}_1), W^\vee(\hat{\lambda}_2)) = 0$.

The cases $i > 1$ follow from the case $i = 1$ by a standard argument in (affine) highest weight categories. □

Remark 4.19. There is an alternative proof of Theorem 4.9(2) by the general theory of geometric extension algebras due to Kato [27] and McNamara [32]. In fact, we have the following $\widehat{\mathcal{R}}(\hat{\lambda})$ -algebra isomorphisms:

$$(4.16) \quad \widehat{\mathcal{K}}^{G(\hat{\lambda})}(Z^\bullet(\hat{\lambda})) \cong H_\bullet^{G(\hat{\lambda})}(Z^\bullet(\hat{\lambda}), \mathbb{k})^\wedge \cong \text{Ext}_{G_\beta}^\bullet(\mathcal{L}, \mathcal{L})^\wedge,$$

where the middle term $H_\bullet^{G(\hat{\lambda})}(Z^\bullet(\hat{\lambda}), \mathbb{k})$ denotes the convolution algebra of the $G(\hat{\lambda})$ -equivariant Borel-Moore homology of $Z^\bullet(\hat{\lambda})$, the right term $\text{Ext}_{G_\beta}^\bullet(\mathcal{L}, \mathcal{L})$ denotes the Yoneda algebra in the G_β -equivariant derived category $D_{G_\beta}(E_\beta; \mathbb{k})$ of constructible complexes on E_β and $\mathcal{L} := \pi_*\underline{\mathbb{k}}$ is the derived push-forward of the constant sheaf along the proper morphism $\pi: \mathfrak{M}^\bullet(\hat{\lambda}) \rightarrow \mathfrak{M}_0^\bullet(\hat{\lambda}) = E_\beta$. The symbol $(-)^\wedge$ stands for the completion with respect to the degree. Here, we make a suitable identification $\widehat{\mathcal{R}}(\hat{\lambda}) \cong H_\bullet^{G(\hat{\lambda})}(\text{pt}, \mathbb{k})^\wedge$. The first isomorphism in (4.16) is given by the equivariant Chern character map and the second one is as in [11, Section 8.6]. By [35, Theorem 14.3.2], we have

$$\mathcal{L} = \bigoplus_{\mathbf{m} \in \text{KP}(\beta)} L_{\mathbf{m}} \otimes \text{IC}(\mathbb{O}_{\mathbf{m}}),$$

where $L_{\mathbf{m}}$ is a non-zero finite-dimensional graded vector space and $\text{IC}(\mathbb{O}_{\mathbf{m}})$ is the intersection cohomology complex associated with the constant sheaf on the orbit $\mathbb{O}_{\mathbf{m}}$ for each $\mathbf{m} \in \text{KP}(\beta)$. Therefore, the Yoneda algebra $\text{Ext}_{G_\beta}^\bullet(\mathcal{L}, \mathcal{L})$ is Morita equivalent to the quiver Hecke algebra $H_Q(\beta)$ (cf. Definition 5.1) by [42] and the completed algebra $\widehat{\mathcal{K}}^{G(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ is affine quasi-hereditary (cf. Theorem 5.6).

5. APPLICATION TO KANG-KASHIWARA-KIM'S FUNCTOR

In this section, we apply the results obtained so far to prove that Kang-Kashiwara-Kim's generalized quantum affine Schur-Weyl duality functor gives an equivalence of categories assuming the simpleness of some poles of normalized R -matrices between ℓ -fundamental modules.

5.1. Quiver Hecke algebra. In this subsection, we recall the definition of the quiver Hecke algebra and the affine highest weight structure of its module category.

We keep the notation in the previous sections. In particular, we fix a Dynkin quiver $Q = (I, \Omega)$. Let $\beta = \sum_{i \in I} d_i \alpha_i \in \mathbb{Q}^+$ with $\text{ht } \beta := \sum_{i \in I} d_i = d$. Set

$$I^\beta := \{\mathbf{i} = (i_1, \dots, i_d) \in I^d \mid \alpha_{i_1} + \dots + \alpha_{i_d} = \beta\}.$$

Note that the symmetric group \mathfrak{S}_d of degree d acts on the set I^β by $\sigma \cdot \mathbf{i} := (i_{\sigma^{-1}(1)}, \dots, i_{\sigma^{-1}(d)})$ for $\mathbf{i} = (i_1, \dots, i_d) \in I^\beta$ and $\sigma \in \mathfrak{S}_d$. For each $k \in \{1, \dots, d-1\}$, we denote the transposition of k and $k+1$ by $\sigma_k \in \mathfrak{S}_d$.

Definition 5.1 (Khovanov-Lauda [29], Rouquier [41]). The quiver Hecke algebra $H_Q(\beta)$ is defined to be a \mathbb{k} -algebra with the generators:

$$\{e(\mathbf{i}) \mid \mathbf{i} \in I^\beta\} \cup \{x_1, \dots, x_d\} \cup \{\tau_1, \dots, \tau_{d-1}\},$$

satisfying the following relations:

$$e(\mathbf{i})e(\mathbf{i}') = \delta_{\mathbf{i}, \mathbf{i}'}e(\mathbf{i}), \quad \sum_{\mathbf{i} \in I^\beta} e(\mathbf{i}) = 1, \quad x_k x_l = x_l x_k, \quad x_k e(\mathbf{i}) = e(\mathbf{i})x_k,$$

$$\tau_k e(\mathbf{i}) = e(\sigma_k \cdot \mathbf{i})\tau_k, \quad \tau_k \tau_l = \tau_l \tau_k \quad \text{if } |k-l| > 1,$$

$$\tau_k^2 e(\mathbf{i}) = \begin{cases} (x_k - x_{k+1})e(\mathbf{i}) & \text{if } i_k \leftarrow i_{k+1} \in \Omega, \\ (x_{k+1} - x_k)e(\mathbf{i}) & \text{if } i_k \rightarrow i_{k+1} \in \Omega, \\ e(\mathbf{i}) & \text{if } a_{i_k, i_{k+1}} = 0, \\ 0 & \text{if } i_k = i_{k+1}, \end{cases}$$

$$(\tau_k x_l - x_{s_k(l)} \tau_k) e(\mathbf{i}) = \begin{cases} -e(\mathbf{i}) & \text{if } l = k, i_k = i_{k+1}, \\ e(\mathbf{i}) & \text{if } l = k+1, i_k = i_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\mathbf{i}) = \begin{cases} e(\mathbf{i}) & \text{if } i_k = i_{k+2}, (i_k \leftarrow i_{k+1}) \in \Omega, \\ -e(\mathbf{i}) & \text{if } i_k = i_{k+2}, (i_k \rightarrow i_{k+1}) \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{P}_\beta := \bigoplus_{\mathbf{i} \in I^\beta} \mathbb{k}[x_1, \dots, x_d]e(\mathbf{i})$ with a commutative $\mathbb{k}[x_1, \dots, x_d]$ -algebra structure $e(\mathbf{i})e(\mathbf{i}') = \delta_{\mathbf{i}, \mathbf{i}'}e(\mathbf{i})$. The symmetric group \mathfrak{S}_d acts on \mathbb{P}_β by

$$\sigma(f(x_1, \dots, x_d)e(\mathbf{i})) := f(x_{\sigma(1)}, \dots, x_{\sigma(d)})e(\sigma \cdot \mathbf{i})$$

for $f(x_1, \dots, x_d) \in \mathbb{k}[x_1, \dots, x_d]$ and $\sigma \in \mathfrak{S}_d$. The \mathfrak{S}_d -invariant part $(\mathbb{P}_\beta)^{\mathfrak{S}_d}$ is identified with the algebra

$$\mathbb{S}_\beta := \bigotimes_{i \in I} (\mathbb{k}[x_i]^{\otimes d_i})^{\mathfrak{S}_{d_i}} = \bigotimes_{i \in I} \mathbb{k}[x_{i,1}, \dots, x_{i,d_i}]^{\mathfrak{S}_{d_i}},$$

via the isomorphism $\mathbb{S}_\beta \xrightarrow{\cong} (\mathbb{P}_\beta)^{\mathfrak{S}_d}$ given by the formula

$$(5.1) \quad f_1(x_{1,1}, \dots, x_{1,d_1}) \otimes \cdots \otimes f_n(x_{n,1}, \dots, x_{n,d_n}) \\ \mapsto \frac{1}{d_1! \cdots d_n!} \sum_{\sigma \in \mathfrak{S}_d} \sigma(f_1(x_{1,1}, \dots, x_{1,d_1}) \cdots f_n(x_{n,1}, \dots, x_{n,d_n}) e(1^{d_1}, \dots, n^{d_n})),$$

where we put $x_{i,j} := x_{d_1 + \cdots + d_{i-1} + j}$ on the RHS for $i \in \{1, \dots, n\}, j \in \{1, \dots, d_i\}$.

For each $\sigma \in \mathfrak{S}_d$, we fix a reduced expression $\sigma = \sigma_{k_1} \cdots \sigma_{k_p}$. Then we define $\tau_\sigma := \tau_{k_1} \cdots \tau_{k_p} \in H_Q(\beta)$. Note that this element τ_σ depends on the choice of a reduced expression of σ in general because τ_k 's do not satisfy the braid relations.

Theorem 5.2 (Khovanov-Lauda [29]). *The following hold.*

- (1) *The quiver Hecke algebra $H_Q(\beta)$ is a left (or right) free module over the commutative subalgebra \mathbb{P}_β with a free basis $\{\tau_\sigma \mid \sigma \in \mathfrak{S}_d\}$;*
- (2) *The center of $H_Q(\beta)$ coincides with the subalgebra $(\mathbb{P}_\beta)^{\mathfrak{S}_d}$ of \mathfrak{S}_d -invariant polynomials, which is identified with \mathbb{S}_β . In particular, $H_Q(\beta)$ is a free module over its center \mathbb{S}_β of rank $(d!)^2$.*

Proof. See [29, Proposition 2.7] for (1) and [29, Theorem 2.9] for (2). □

The quiver Hecke algebra $H_Q(\beta)$ is equipped with a \mathbb{Z} -grading given by $\deg e(\mathbf{i}) = 0, \deg x_k = 2, \deg \tau_k e(\mathbf{i}) = -a_{i_k, i_{k+1}}$. Let $\widehat{H}_Q(\beta)$ (resp. $\widehat{\mathbb{P}}_\beta$) be the completion of the algebra $H_Q(\beta)$ (resp. \mathbb{P}_β) with respect to the grading. More explicitly, by Theorem 5.2(1), we have

$$\widehat{H}_Q(\beta) = \bigoplus_{\sigma \in \mathfrak{S}_d} \widehat{\mathbb{P}}_\beta \tau_\sigma = \bigoplus_{\sigma \in \mathfrak{S}_d} \tau_\sigma \widehat{\mathbb{P}}_\beta,$$

where $\widehat{\mathbb{P}}_\beta := \bigoplus_{\mathbf{i}} \mathbb{k}[x_1, \dots, x_d] e(\mathbf{i})$. The center of $\widehat{H}_Q(\beta)$ is the \mathfrak{S}_d -invariant part $(\widehat{\mathbb{P}}_\beta)^{\mathfrak{S}_d}$ of $\widehat{\mathbb{P}}_\beta$, which is identified with the completion $\widehat{\mathbb{S}}_\beta$ of \mathbb{S}_β at the maximal ideal 0 via the isomorphism given by the formula (5.1).

Definition 5.3. We define the category $\widehat{\mathcal{M}}_{Q,\beta}$ to be the category of finitely generated $\widehat{H}_Q(\beta)$ -modules, i.e.,

$$\widehat{\mathcal{M}}_{Q,\beta} := \widehat{H}_Q(\beta)\text{-mod}_{\text{fg}}.$$

Let $\mathcal{M}_{Q,\beta}$ denote the full subcategory of finite-dimensional modules in $\widehat{\mathcal{M}}_{Q,\beta}$, i.e.,

$$\mathcal{M}_{Q,\beta} := \widehat{H}_Q(\beta)\text{-mod}_{\text{fd}}.$$

Remark 5.4. By Theorem 5.2(2), we see that the completion $\widehat{H}_Q(\beta)$ is naturally isomorphic to the central completion $H_Q(\beta) \otimes_{\mathbb{S}_\beta} \widehat{\mathbb{S}}_\beta$ along the trivial central character. Therefore the category $\mathcal{M}_{Q,\beta}$ is the same as the category $H_Q(\beta)\text{-mod}_{\text{fd}}^0$ of finite-dimensional modules on which the elements x_k 's act nilpotently. Also note that we have the forgetful functor $H_Q(\beta)\text{-gmod}_{\text{fd}} \rightarrow \mathcal{M}_{Q,\beta}$, where $H_Q(\beta)\text{-gmod}_{\text{fd}}$ denotes the category of finite-dimensional graded $H_Q(\beta)$ -modules with homogeneous morphisms.

For $\beta, \beta' \in \mathbb{Q}^+$ with $\text{ht } \beta = d, \text{ht } \beta' = d'$, we have an embedding

$$H_Q(\beta) \otimes H_Q(\beta') \hookrightarrow H_Q(\beta + \beta')$$

given by $e(\mathbf{i}) \otimes e(\mathbf{i}') \mapsto e(\mathbf{i} \circ \mathbf{i}')$, $x_k \otimes 1 \mapsto x_k$, $\tau_k \otimes 1 \mapsto \tau_k$, $1 \otimes x_k \mapsto x_{d+k}$, $1 \otimes \tau_k \mapsto \tau_{d+k}$, where $\mathbf{i} \circ \mathbf{i}' := (i_1, \dots, i_d, i'_1, \dots, i'_{d'}) \in I^{\beta_1 + \beta_2}$. This embedding naturally induces its completed version

$$\widehat{H}_Q(\beta) \otimes \widehat{H}_Q(\beta') \hookrightarrow \widehat{H}_Q(\beta + \beta'),$$

which equips the direct sum category $\widehat{\mathcal{M}}_Q := \bigoplus_{\beta \in Q^+} \widehat{\mathcal{M}}_{Q,\beta}$ with a structure of monoidal category whose tensor product $(-) \circ (-): \widehat{\mathcal{M}}_{Q,\beta} \times \widehat{\mathcal{M}}_{Q,\beta'} \rightarrow \widehat{\mathcal{M}}_{Q,\beta + \beta'}$ is given by

$$M \circ M' := \widehat{H}_Q(\beta + \beta') \otimes_{\widehat{H}_Q(\beta) \otimes \widehat{H}_Q(\beta')} (M \otimes M').$$

The subcategory $\mathcal{M}_Q := \bigoplus_{\beta \in Q^+} \mathcal{M}_{Q,\beta}$ of finite-dimensional modules is closed under this monoidal structure.

Write $\beta = \sum_{i \in I} d_i \alpha_i$, $\beta' = \sum_{i \in I} d'_i \alpha_i$. Then we have the following injective homomorphism

$$\mathbb{S}_{\beta + \beta'} = \bigotimes_{i \in I} (\mathbb{k}[x_i]^{\otimes d_i + d'_i})^{\mathfrak{S}_{d_i + d'_i}} \hookrightarrow \bigotimes_{i \in I} (\mathbb{k}[x_i]^{\otimes d_i} \otimes \mathbb{k}[x_i]^{\otimes d'_i})^{\mathfrak{S}_{d_i} \times \mathfrak{S}_{d'_i}} \cong \mathbb{S}_\beta \otimes \mathbb{S}_{\beta'},$$

which induces an injective homomorphism $\widehat{\mathbb{S}}_{\beta + \beta'} \hookrightarrow \widehat{\mathbb{S}}_\beta \widehat{\otimes} \widehat{\mathbb{S}}_{\beta'}$ for the completions. We refer to this kind of injective homomorphism as the standard inclusion. Note that for any $M \in \widehat{\mathcal{M}}_{Q,\beta}$ and $M' \in \widehat{\mathcal{M}}_{Q,\beta'}$, actions of the centers $\widehat{\mathbb{S}}_\beta$ and $\widehat{\mathbb{S}}_{\beta'}$ induce the natural homomorphism $\widehat{\mathbb{S}}_\beta \widehat{\otimes} \widehat{\mathbb{S}}_{\beta'} \rightarrow \text{End}_{\widehat{H}_Q(\beta + \beta')}(M \circ M')$. Lemma 5.5 is easily deduced from the definition.

Lemma 5.5. *For any $M \in \widehat{\mathcal{M}}_{Q,\beta}$ and $M' \in \widehat{\mathcal{M}}_{Q,\beta'}$, the action of the center $\widehat{\mathbb{S}}_{\beta + \beta'}$ on the module $M \circ M'$ is given via the standard inclusion $\widehat{\mathbb{S}}_{\beta + \beta'} \hookrightarrow \widehat{\mathbb{S}}_\beta \widehat{\otimes} \widehat{\mathbb{S}}_{\beta'}$.*

We define the affinization M_{af} of an $H_Q(\beta)$ -module M following [23, Section 1.3]. Let y be an indeterminate. We set $M_{\text{af}} := \mathbb{k}[[y]] \otimes M$ and define an action of $H_Q(\beta)$ by $e(\mathbf{i})(a \otimes m) := a \otimes (e(\mathbf{i})m)$, $x_k(a \otimes m) := (ya) \otimes m + a \otimes (x_k m)$, $\tau_k(a \otimes m) := a \otimes (\tau_k m)$, where $a \in \mathbb{k}[[y]]$, $m \in M$. Note that if $M \in \mathcal{M}_{Q,\beta}$, the completion $\widehat{H}_Q(\beta)$ naturally acts on the affinization M_{af} and we have $M_{\text{af}} \in \widehat{\mathcal{M}}_{Q,\beta}$.

Recall from Subsection 3.5 that the set $\text{KP}(\beta)$ of Kostant partitions of β is equipped with a partial order \leq , which is the opposite of the orbit closure ordering in the G_β -space E_β . Then we have the following important result.

Theorem 5.6 (Kato [26], Brundan-Kleshchev-McNamara [5]). *The category $\widehat{\mathcal{M}}_{Q,\beta}$ has a structure of affine highest weight category for $(\text{KP}(\beta), \leq)$.*

Proof. Actually, [26] and [5] proved that the finitely generated graded module category of $H_Q(\beta)$ satisfies the conditions of affine highest weight category in the setting of graded algebras, which Kleshchev originally concerned in his paper [30]. We can translate the results to the completed version $\widehat{H}_Q(\beta)$ by applying [17, Theorem 4.6], which is available now thanks to Theorem 5.2(2).

The orderings of the set $\text{KP}(\beta)$ used in [26] and [5] are stronger than ours. However, our ordering is enough to control the affine highest weight structure of $\widehat{\mathcal{M}}_{Q,\beta}$. See [26, Remark 4.1] and [27], or [32, Theorem 6.12]. \square

In particular, Theorem 5.6 tells us that simple modules of the category $\widehat{\mathcal{M}}_{Q,\beta}$ are labelled by Kostant partitions of β . Let $S(\mathbf{m})$ denote the simple module corresponding to $\mathbf{m} \in \text{KP}(\beta)$. To simplify the notation, we identify a positive root

α with the Kostant partition $\mathbf{m}_\alpha := (\delta_{\alpha, \alpha'})_{\alpha' \in R^+} \in \text{KP}(\alpha)$ consisting of the single root α . For instance, we write $S(\alpha)$ instead of $S(\mathbf{m}_\alpha)$. By convention, our $S(\alpha)$ is the same as “the dual root module of weight α ” in [22, Section 4.3] (see also [22, Remark 4.3.3(b)] for a comparison with [5]).

According to [5, 26], we can construct the standard module $\Delta(\mathbf{m})$, the proper standard module $\bar{\Delta}(\mathbf{m})$ and the proper costandard module $\bar{\nabla}(\mathbf{m})$ in the affine highest weight category $\widehat{\mathcal{M}}_{Q, \beta}$ for $(\text{KP}(\beta), \leq)$ from the dual root modules $\{S(\alpha) \mid \alpha \in R^+\}$ as in the next paragraph. Although the construction a priori depends on the choice of a reduced expression $w_0 = s_{i_1} \cdots s_{i_r}$ of the longest element $w_0 \in W$ adapted to the quiver Q , the resulting modules $\Delta(\mathbf{m}), \bar{\Delta}(\mathbf{m}), \bar{\nabla}(\mathbf{m})$ do not depend on the choice up to isomorphism. See [26, Remark 4.1].

Fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_r}$ adapted to Q . It yields a total ordering $R^+ = \{\gamma_1, \dots, \gamma_r\}$ among the positive roots, where $\gamma_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Using this total ordering, we write $\text{KP}(\beta) = \{\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r \mid \sum_{i=1}^r m_i \gamma_i = \beta\}$. Then for each $\mathbf{m} = (m_1, \dots, m_r) \in \text{KP}(\beta)$, we have

$$\begin{aligned} \bar{\Delta}(\mathbf{m}) &= S(\gamma_1)^{\circ m_1} \circ S(\gamma_2)^{\circ m_2} \circ \cdots \circ S(\gamma_r)^{\circ m_r}, \\ \bar{\nabla}(\mathbf{m}) &= S(\gamma_r)^{\circ m_r} \circ S(\gamma_{r-1})^{\circ m_{r-1}} \circ \cdots \circ S(\gamma_1)^{\circ m_1}. \end{aligned}$$

The standard module $\Delta(\mathbf{m})$ is obtained as an indecomposable direct summand of the product $S(\gamma_1)_{\text{af}}^{\circ m_1} \circ S(\gamma_2)_{\text{af}}^{\circ m_2} \circ \cdots \circ S(\gamma_r)_{\text{af}}^{\circ m_r}$. Here, all the summands are mutually isomorphic. Namely, we have

$$(5.2) \quad \Delta(\mathbf{m})^{\oplus m_1! \cdots m_r!} \cong S(\gamma_1)_{\text{af}}^{\circ m_1} \circ S(\gamma_2)_{\text{af}}^{\circ m_2} \circ \cdots \circ S(\gamma_r)_{\text{af}}^{\circ m_r}.$$

In particular, we have $\bar{\Delta}(\alpha) = \bar{\nabla}(\alpha) = S(\alpha)$ and $\Delta(\alpha) = S(\alpha)_{\text{af}}$ for each $\alpha \in R^+$.

5.2. Kang-Kashiwara-Kim’s functor. Recall the bijection $\phi: R^+ \rightarrow \widehat{I}_Q$ from Subsection 2.1. For each $i \in I$, we set $\mathbb{V}_i := \mathbb{W}(\text{cl}(\varpi_{\phi(\alpha_i)}))$.

Fix $\beta = \sum_{i \in I} d_i \alpha_i \in \mathbb{Q}^+$. For each $\mathbf{i} = (i_1, \dots, i_d) \in I^\beta$, we write

$$\mathcal{O}_{\mathbf{i}} := \bigotimes_{k=1}^d \text{End}_{\bar{U}_q}(\mathbb{V}_{i_k}) \cong \mathbb{k}[z_1^{\pm 1}, \dots, z_d^{\pm 1}],$$

where $\text{End}_{\bar{U}_q}(\mathbb{V}_{i_k}) = \mathbb{k}[z_k^{\pm 1}]$ as in Theorem 2.7(2). Using $\mathbf{p} = \text{pr}_2 \circ \phi$, we set

$$\widehat{\mathbb{V}}_{\mathbf{i}} := \mathbb{k}[[z_1 - q^{\mathbf{p}(\alpha_{i_1})}, \dots, z_d - q^{\mathbf{p}(\alpha_{i_d})}]] \otimes_{\mathcal{O}_{\mathbf{i}}} (\mathbb{V}_{i_1} \otimes \cdots \otimes \mathbb{V}_{i_d}).$$

Take the direct sum over $\mathbf{i} \in I^\beta$, we define

$$\widehat{\mathbb{V}}^{\otimes \beta} := \bigoplus_{\mathbf{i} \in I^\beta} \widehat{\mathbb{V}}_{\mathbf{i}}.$$

Let $e(\mathbf{i}): \widehat{\mathbb{V}}^{\otimes \beta} \rightarrow \widehat{\mathbb{V}}_{\mathbf{i}}$ denote the projection to the \mathbf{i} -th component. By construction, the algebra

$$\widehat{\mathcal{O}}_{\beta} := \bigoplus_{\mathbf{i} \in I^\beta} \mathbb{k}[[z_1 - q^{\mathbf{p}(\alpha_{i_1})}, \dots, z_d - q^{\mathbf{p}(\alpha_{i_d})}]] e(\mathbf{i})$$

naturally acts on $\widehat{\mathbb{V}}^{\otimes \beta}$.

Theorem 5.7 (Kang-Kashiwara-Kim [22, 23]). *Assume the following condition:*

- (\star) For any $(i, p), (j, r) \in \{\phi(\alpha_i) \mid i \in I\} \subset \widehat{I}_Q$, the zero order of the denominator $d_{i,j}(u)$ at $u = q^{r-p}$ is at most one.

Then there exists a \mathbb{k} -algebra homomorphism $\widehat{H}_Q(\beta) \rightarrow \text{End}_{\widetilde{U}_q}(\widehat{V}^{\otimes \beta})^{\text{op}}$ such that

$$e(\mathbf{i}) \mapsto e(\mathbf{i}), \quad x_k e(\mathbf{i}) \mapsto (q^{-p(\alpha_{i_k})z_k} - 1)e(\mathbf{i}),$$

and the images of the elements τ_k 's are defined in a suitable way using the normalized R -matrices. In particular, $\widehat{V}^{\otimes \beta}$ has a structure of $(\widetilde{U}_q, \widehat{H}_Q(\beta))$ -bimodule.

Remark 5.8. For a quiver Q of type AD, the condition (\star) is always satisfied thanks to [22, Lemma 3.2.4]. Henceforth, we keep assuming the condition (\star) in Theorem 5.7 until the end of this paper if our quiver Q is of type E.⁵

In the remainder of this subsection, we shall extend the $(\widetilde{U}_q, \widehat{H}_Q(\beta))$ -bimodule $\widehat{V}^{\otimes \beta}$ to a $(\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda})), \widehat{H}_Q(\beta))$ -bimodule. As before, we set

$$\hat{\lambda} \equiv \hat{\lambda}_\beta := \sum_{i \in I} d_i \varpi_{\phi(\alpha_i)} \in \mathcal{P}_{Q,\beta}^+, \quad \text{and} \quad \lambda := \text{cl}(\hat{\lambda}) \in \mathbf{P}^+.$$

Recall from Section 3 that we have the quiver varieties $\pi: \mathfrak{M}(\lambda) \rightarrow \mathfrak{M}_0(\lambda)$ with $\mathbb{G}(\lambda)$ -action. We realize the graded quiver varieties $\pi^\bullet: \mathfrak{M}^\bullet(\hat{\lambda}) \rightarrow \mathfrak{M}_0^\bullet(\hat{\lambda})$ with $\mathbb{G}(\hat{\lambda})$ -action as the \mathbb{T} -fixed point subvarieties, where $\mathbb{T} \subset \mathbb{G}(\lambda)$ is a certain 1-dimensional subtorus and $\mathbb{G}(\hat{\lambda})$ is regarded as the centralizer of \mathbb{T} in $\mathbb{G}(\lambda)$.

We fix a maximal torus $H(\hat{\lambda})$ of $G(\hat{\lambda})$:

$$H(\hat{\lambda}) \cong (\mathbb{C}^\times)^d = (\mathbb{C}^\times)^{d_1} \times \cdots \times (\mathbb{C}^\times)^{d_n} \subset GL_{d_1}(\mathbb{C}) \times \cdots \times GL_{d_n}(\mathbb{C}) \cong G(\hat{\lambda}),$$

and set $\mathbb{H}(\hat{\lambda}) := H(\hat{\lambda}) \times \mathbb{C}^\times \subset G(\hat{\lambda}) \times \mathbb{C}^\times = \mathbb{G}(\hat{\lambda})$. We have the standard identification $\mathcal{R}(\mathbb{H}(\hat{\lambda})) = \mathbb{k}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$. Then, the natural homomorphism $\mathcal{R}(\hat{\lambda}) = \mathcal{R}(\mathbb{G}(\hat{\lambda})) \hookrightarrow \mathcal{R}(\mathbb{H}(\hat{\lambda}))$ is given by the following composition

$$\bigotimes_{i \in I} \mathbb{k}[z_{i,1}^{\pm 1}, \dots, z_{i,d_i}^{\pm 1}]^{\mathfrak{S}_{d_i}} \hookrightarrow \bigotimes_{i \in I} \mathbb{k}[z_{i,1}^{\pm 1}, \dots, z_{i,d_i}^{\pm 1}] \cong \mathbb{k}[z_1^{\pm 1}, \dots, z_d^{\pm 1}],$$

where the second isomorphism sends $z_{i,j}$ to $z_{d_1+\dots+d_{i-1}+j}$ for each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, d_i\}$.

For each $\mathbf{i} = (i_1, \dots, i_d) \in I^\beta$, we choose an element $\sigma \in \mathfrak{S}_d$ such that $\mathbf{i} = \sigma \cdot (1^{d_1}, 2^{d_2}, \dots, n^{d_n})$. Then we define a group homomorphism $f_\sigma: \mathbb{C}^\times \rightarrow H(\hat{\lambda})$ by $f_\sigma(t) := (t^{\sigma(1)}, \dots, t^{\sigma(d)})$. Following Nakajima [36], we consider the following subvariety of $\mathfrak{M}(\lambda)$:

$$\widetilde{\mathfrak{Z}}(\lambda; \sigma) := \left\{ x \in \mathfrak{M}(\lambda) \mid \lim_{t \rightarrow 0} f_\sigma(t) \pi(x) = 0 \in \mathfrak{M}_0(\lambda) \right\}.$$

By construction, the group $\mathbb{H}(\hat{\lambda})$ acts on $\widetilde{\mathfrak{Z}}(\lambda; \sigma)$ and the $\mathbb{H}(\hat{\lambda})$ -equivariant K -group $\mathcal{K}^{\mathbb{H}(\hat{\lambda})}(\widetilde{\mathfrak{Z}}(\lambda; \sigma))$ becomes a left module over the convolution algebra $\mathcal{K}^{\mathbb{H}(\hat{\lambda})}(Z(\lambda))$. Via the homomorphism

$$\widetilde{U}_q \xrightarrow{\Phi_\lambda} \mathcal{K}^{\mathbb{G}(\lambda)}(Z(\lambda)) \xrightarrow{\text{forget}} \mathcal{K}^{\mathbb{H}(\hat{\lambda})}(Z(\lambda)),$$

we regard $\mathcal{K}^{\mathbb{H}(\hat{\lambda})}(\widetilde{\mathfrak{Z}}(\lambda; \sigma))$ as a left \widetilde{U}_q -module.

Theorem 5.9 (Nakajima [36]). *With the above notation, the following hold.*

- (1) *As a \widetilde{U}_q -module, we have $\mathcal{K}^{\mathbb{H}(\hat{\lambda})}(\widetilde{\mathfrak{Z}}(\lambda; \sigma)) \cong \mathbb{V}_{i_1} \otimes \cdots \otimes \mathbb{V}_{i_d}$;*

⁵After the initial submission of this paper, it was also proved that the condition (\star) is always satisfied for any quiver Q of type E by Oh-Scrimshaw [40] and by the author [18] independently.

(2) The isomorphism in (1) induces the following commutative diagram

$$\begin{array}{ccc} \text{End}(\mathcal{K}^{\mathbb{H}(\hat{\lambda})}(\tilde{\mathfrak{Z}}(\lambda; \sigma))) & \xrightarrow{\cong} & \text{End}(\mathbb{V}_{i_1} \otimes \cdots \otimes \mathbb{V}_{i_d}) \\ \uparrow & & \uparrow \\ \mathcal{R}(\mathbb{H}(\hat{\lambda})) & \xrightarrow{\sigma} & \mathcal{O}_{\mathbf{i}}, \end{array}$$

where the vertical arrows denote the actions on the modules and the bottom arrow $\sigma: \mathcal{R}(\mathbb{H}(\hat{\lambda})) = \mathbb{k}[z_1^{\pm 1}, \dots, z_d^{\pm 1}] \rightarrow \mathbb{k}[z_1^{\pm 1}, \dots, z_d^{\pm 1}] = \mathcal{O}_{\mathbf{i}}$ is given by $\sigma(f(z_1, \dots, z_d)) = f(z_{\sigma(1)}, \dots, z_{\sigma(d)})$.

Proof. See [36, Theorem 6.12]. □

Let $\widehat{\mathcal{R}}(\mathbb{H}(\hat{\lambda}))$ denote the completion of $\mathcal{R}(\mathbb{H}(\hat{\lambda}))$ with respect to the maximal ideal corresponding to $\mathbb{T} \subset \mathbb{H}(\hat{\lambda})$. We apply the corresponding completion $(-) \otimes_{\mathcal{R}(\mathbb{H}(\hat{\lambda}))} \widehat{\mathcal{R}}(\mathbb{H}(\hat{\lambda}))$ to the isomorphism in Theorem 5.9(1) to get the following isomorphism:

$$(5.3) \quad \mathcal{K}^{\mathbb{H}(\hat{\lambda})}(\tilde{\mathfrak{Z}}(\lambda; \sigma)) \otimes_{\mathcal{R}(\mathbb{H}(\hat{\lambda}))} \widehat{\mathcal{R}}(\mathbb{H}(\hat{\lambda})) \cong \widehat{\mathbb{V}}_{\mathbf{i}}.$$

Restricting to $\widehat{\mathcal{R}}(\hat{\lambda}) \subset \widehat{\mathcal{R}}(\mathbb{H}(\hat{\lambda}))$, we regard the isomorphism (5.3) as an isomorphism of $(\tilde{U}_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodules, which does not depend on the choice of σ such that $\mathbf{i} = \sigma \cdot (1^{d_1}, \dots, n^{d_n})$. By the construction, this $(\tilde{U}_q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodule structure on the LHS of (5.3) comes from the convolution action of $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$, and hence $\widehat{\mathbb{V}}_{\mathbf{i}} \in \widehat{\mathcal{C}}_{Q, \beta}$ via the isomorphism (5.3). As a result, we can regard the $(\tilde{U}_q, \widehat{H}_Q(\beta))$ -bimodule $\widehat{V}^{\otimes \beta}$ as a $(\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda})), \widehat{H}_Q(\beta))$ -bimodule.

By Theorem 5.9(2), the action of $\widehat{\mathcal{R}}(\hat{\lambda})$ on $\widehat{V}^{\otimes \beta}$ factors through the homomorphism

$$\widehat{\mathcal{R}}(\hat{\lambda}) = \bigotimes_{i \in I} \mathbb{k}[[z_{i,1} - q^{p(\alpha_i)}, \dots, z_{i,d_i} - q^{p(\alpha_i)}]]^{\mathfrak{S}_{d_i}} \rightarrow \widehat{\mathcal{O}}_{\beta},$$

which is given by the following formula:

$$(5.4) \quad f_1(z_{1,1}, \dots, z_{1,d_1}) \otimes \cdots \otimes f_n(z_{n,1}, \dots, z_{n,d_n}) \mapsto \frac{1}{d_1! \cdots d_n!} \sum_{\sigma \in \mathfrak{S}_d} \sigma(f_1(z_{1,1}, \dots, z_{1,d_1}) \cdots f_n(z_{n,1}, \dots, z_{n,d_n}) e(1^{d_1}, \dots, n^{d_n})),$$

where we put $z_{i,j} := z_{d_1 + \dots + d_{i-1} + j}$ for $i \in \{1, \dots, n\}$, $j \in \{1, \dots, d_i\}$ on the RHS.

Lemma 5.10. *We have the following commutative diagram:*

$$\begin{array}{ccc} \text{End}(\widehat{V}^{\otimes \beta}) & \xlongequal{\quad} & \text{End}(\widehat{V}^{\otimes \beta}) \\ \uparrow & & \uparrow \\ \widehat{\mathfrak{S}}_{\beta} & \xrightarrow{\cong} & \widehat{\mathcal{R}}(\hat{\lambda}), \end{array}$$

where the vertical arrows denote the actions of $\widehat{\mathfrak{S}}_{\beta} \subset \widehat{H}_Q(\beta)$ and $\widehat{\mathcal{R}}(\hat{\lambda}) \subset \widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ respectively. The bottom isomorphism is given by $x_{i,j} \mapsto q^{-p(\alpha_i)} z_{i,j} - 1$ for each $i \in I, j \in \{1, \dots, d_i\}$.

Proof. The assertion follows from the construction in Theorem 5.7 and the formulas (5.1) and (5.4). □

Definition 5.11. Under the assumption (\star) , we define the (completed version of) Kang-Kashiwara-Kim functor by

$$\mathcal{F}_{Q,\beta}: \widehat{\mathcal{M}}_{Q,\beta} \rightarrow \widehat{\mathcal{C}}_{Q,\beta}, \quad M \mapsto \widehat{V}^{\otimes \beta} \otimes_{\widehat{H}_Q(\beta)} M.$$

Theorem 5.12 (Kang-Kashiwara-Kim [22,23]). *The Kang-Kashiwara-Kim functor $\mathcal{F}_{Q,\beta}$ satisfies the following properties.*

- (1) *The bimodule $\widehat{V}^{\otimes \beta}$ is flat over $\widehat{H}_Q(\beta)$ and hence the functor $\mathcal{F}_{Q,\beta}$ is exact;*
- (2) *For any $M \in \widehat{\mathcal{M}}_{Q,\beta}$ and $M' \in \widehat{\mathcal{M}}_{Q,\beta'}$, there is a natural isomorphism*

$$\mathcal{F}_{Q,\beta+\beta'}(M \circ M') \cong \mathcal{F}_{Q,\beta}(M) \widehat{\otimes} \mathcal{F}_{Q,\beta'}(M')$$

of $(\widetilde{U}_q, \widehat{\mathcal{R}}(\widehat{\lambda}_{\beta+\beta'}))$ -bimodules, where the action of $\widehat{\mathcal{R}}(\widehat{\lambda}_{\beta+\beta'})$ on the RHS is given via the standard inclusion $\widehat{\mathcal{R}}(\widehat{\lambda}_{\beta+\beta'}) = \widehat{\mathcal{R}}(\widehat{\lambda}_\beta + \widehat{\lambda}_{\beta'}) \hookrightarrow \widehat{\mathcal{R}}(\widehat{\lambda}_\beta) \widehat{\otimes} \widehat{\mathcal{R}}(\widehat{\lambda}_{\beta'})$;

- (3) *For any $M \in \widehat{\mathcal{M}}_{Q,\beta}$, we have the following commutative diagram:*

$$\begin{array}{ccc} \text{End}_{\widehat{\mathcal{M}}_{Q,\beta}}(M) & \xrightarrow{\mathcal{F}_{Q,\beta}} & \text{End}_{\widehat{\mathcal{C}}_{Q,\beta}}(\mathcal{F}_{Q,\beta}(M)) \\ \uparrow & & \uparrow \\ \widehat{\mathcal{S}}_\beta & \xrightarrow{\cong} & \widehat{\mathcal{R}}(\widehat{\lambda}), \end{array}$$

where the vertical arrows denote the actions of the centers. The bottom arrow is the isomorphism in Lemma 5.10;

- (4) *For any $\alpha \in \mathbb{R}^+$, we have $\mathcal{F}_{Q,\alpha}(S(\alpha)) \cong L(\varpi_{\phi(\alpha)})$.*

Proof. (1) is [23, Theorem 3.3.3]. (2) follows from [23, Theorem 3.2.1] together with Lemma 5.5 and Lemma 5.10. (3) is a direct consequence of Lemma 5.10. (4) is [22, Theorem 4.3.4]. □

5.3. Comparison of affine highest weight structures. In this subsection, we prove the following.

Theorem 5.13. *Let $\beta \in \mathbb{Q}^+$ be an element. Under the assumption (\star) , the Kang-Kashiwara-Kim functor gives an equivalence of affine highest weight categories*

$$\mathcal{F}_{Q,\beta}: \widehat{\mathcal{M}}_{Q,\beta} \xrightarrow{\cong} \widehat{\mathcal{C}}_{Q,\beta}.$$

We keep the assumption (\star) until the end of this paper.

Lemma 5.14. *For any $\alpha \in \mathbb{R}^+$, we have $\mathcal{F}_{Q,\alpha}(\Delta(\alpha)) \cong \widehat{W}(\varpi_{\phi(\alpha)})$.*

Proof. Let $\mathfrak{m}_\alpha \subset \widehat{\mathcal{S}}_\alpha$ be the maximal ideal of the center $\widehat{\mathcal{S}}_\alpha \subset \widehat{H}_Q(\alpha)$. By the definition of the affinization $\Delta(\alpha) = S(\alpha)_{\text{af}}$, we have $\Delta(\alpha)/\mathfrak{m}_\alpha \cong S(\alpha)$. From Theorem 5.12(3) and (4), we see

$$\mathcal{F}_{Q,\alpha}(\Delta(\alpha))/\mathfrak{r}_{\widehat{\lambda}_\alpha} \cong \mathcal{F}_{Q,\alpha}(\Delta(\alpha)/\mathfrak{m}_\alpha) \cong \mathcal{F}_{Q,\alpha}(S(\alpha)) \cong L(\varpi_{\phi(\alpha)}).$$

In particular, the module $\mathcal{F}_{Q,\alpha}(\Delta(\alpha))$ has a simple head isomorphic to $L(\varpi_{\phi(\alpha)})$. Let us consider the Serre subcategory $\widehat{\mathcal{C}}(\alpha)$ of $\widehat{\mathcal{C}}_{Q,\alpha}$ generated by the standard module $\widehat{W}(\varpi_{\phi(\alpha)})$. Then $\mathcal{F}_{Q,\alpha}(\Delta(\alpha)) \in \widehat{\mathcal{C}}(\alpha)$. By a standard theory of affine highest weight category (see [30, Proposition 5.16]), this category $\widehat{\mathcal{C}}(\alpha)$ is equivalent to the category of finitely generated $\widehat{\mathcal{R}}(\varpi_{\phi(\alpha)})$ -modules under the functor

$\text{Hom}_{\widehat{\mathcal{C}}_{Q,\alpha}}(\widehat{W}(\varpi_{\phi(\alpha)}), -)$. Since $\widehat{W}(\varpi_{\phi(\alpha)})$ corresponds to the rank 1 free $\widehat{\mathcal{R}}(\varpi_{\phi(\alpha)})$ -module under this equivalence, it is a projective cover of $L(\varpi_{\phi(\alpha)})$ in $\widehat{\mathcal{C}}(\alpha)$. Therefore we have a surjective homomorphism $\widehat{W}(\varpi_{\phi(\alpha)}) \rightarrow \mathcal{F}_{Q,\alpha}(\Delta(\alpha))$. Note that any non-trivial quotient of $\widehat{W}(\varpi_{\phi(\alpha)})$ in $\widehat{\mathcal{C}}(\alpha)$ is finite-dimensional because $\widehat{\mathcal{R}}(\varpi_{\phi(\alpha)}) \cong \mathbb{k}\llbracket x \rrbracket$. On the other hand, the module $\mathcal{F}_{Q,\alpha}(\Delta(\alpha))$ is infinite-dimensional thanks to Theorem 5.12(1) and (4). Thus the surjective homomorphism $\widehat{W}(\varpi_{\phi(\alpha)}) \rightarrow \mathcal{F}_{Q,\alpha}(\Delta(\alpha))$ should be an isomorphism. \square

Recall that we have defined the order-preserving bijection $f: \text{KP}(\beta) \rightarrow \mathcal{P}_{Q,\beta}^+$ in Subsection 3.5 by $f(\mathbf{m}) := \sum_{\alpha \in \mathbb{R}^+} m_\alpha \varpi_{\phi(\alpha)}$ for $\mathbf{m} = (m_\alpha)_{\alpha \in \mathbb{R}^+} \in \text{KP}(\beta)$.

Proposition 5.15. *For each $\mathbf{m} \in \text{KP}(\beta)$, we have*

$$\mathcal{F}_{Q,\beta}(\Delta(\mathbf{m})) \cong \widehat{W}(f(\mathbf{m})), \quad \mathcal{F}_{Q,\beta}(\bar{\nabla}(\mathbf{m})) \cong W^\vee(f(\mathbf{m})).$$

Proof. Fix a reduced expression $w_0 = s_{j_1} \cdots s_{j_r}$ of the longest element w_0 adapted to Q . We set $\gamma_k := s_{j_1} \cdots s_{j_{k-1}}(\alpha_{j_k})$ and write $(i_k, p_k) := \phi(\gamma_k) \in \widehat{I}_Q$ for each $k \in \{1, \dots, r\}$. Then we can prove that $d_{i_j, i_k}(q^{p_k - p_j}) \neq 0$ holds for any $1 \leq j < k \leq r$. Indeed, if we suppose $d_{i_j, i_k}(q^{p_k - p_j}) = 0$ for some $1 \leq j < k \leq r$, there is an oriented path from (i_j, p_j) to (i_k, p_k) in the Auslander-Reiten quiver Γ_Q by Lemma 2.24(2). However, then [4, Theorem 2.17] implies that $j > k$, which is a contradiction.

Therefore, we have

$$\begin{aligned} \widehat{W}(f(\mathbf{m}))^{\oplus m_1! \cdots m_r!} &\cong \widehat{W}(\varpi_{i_1, p_1})^{\otimes m_1} \hat{\otimes} \cdots \hat{\otimes} \widehat{W}(\varpi_{i_r, p_r})^{\otimes m_r} \\ &\cong \mathcal{F}_{Q,\beta}(\Delta(\gamma_1)^{\circ m_1} \circ \cdots \circ \Delta(\gamma_r)^{\circ m_r}) \\ &\cong \mathcal{F}_{Q,\beta}(\Delta(\mathbf{m}))^{\oplus m_1! \cdots m_r!}, \end{aligned}$$

where the first isomorphism is Corollary 2.21, the second one is Theorem 5.12(2) and Lemma 5.14, the last one is due to (5.2). This isomorphism is an isomorphism of $(U_Q, \widehat{\mathcal{R}}(\hat{\lambda}))$ -bimodules. In fact, it is easy to check from Theorem 5.12(3) that the action of $\widehat{\mathcal{R}}(\hat{\lambda})$ on the RHS is also given via the homomorphism $\theta_{f(\mathbf{m})}$, as well as on the LHS. Thus we obtain $\mathcal{F}_{Q,\beta}(\Delta(\mathbf{m})) \cong \widehat{W}(f(\mathbf{m}))$. A proof of $\mathcal{F}_{Q,\beta}(\bar{\nabla}(\mathbf{m})) \cong W^\vee(f(\mathbf{m}))$ is similar and easier. \square

Proof of Theorem 5.13. Both of the affine quasi-hereditary algebras $\widehat{\mathcal{K}}^{\mathbb{G}(\hat{\lambda})}(Z^\bullet(\hat{\lambda}))$ and $\widehat{H}_Q(\beta)$ are finite over their centers by Corollary 4.13 and Theorem 5.2 respectively. Thanks to Proposition 5.15, we can apply Theorem 1.7 to the functor $\mathcal{F}_{Q,\beta}: \widehat{\mathcal{M}}_{Q,\beta} \rightarrow \widehat{\mathcal{C}}_{Q,\beta}$, which proves the assertion. \square

Finally, we consider the direct sum of the functors $\mathcal{F}_{Q,\beta}$ over $\beta \in \mathbb{Q}^+$ and restrict it to the full subcategory of finite-dimensional modules to obtain

$$\mathcal{F}_Q := \bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{F}_{Q,\beta}: \mathcal{M}_Q = \bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{M}_{Q,\beta} \rightarrow \mathcal{C}_Q = \bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{C}_{Q,\beta},$$

where we recall Lemma 2.26. By Theorem 5.12(2), this functor \mathcal{F}_Q is monoidal. As a corollary of Theorem 5.13, we have the following.

Corollary 5.16. *Under the assumption (\star) , the Kang-Kashiwara-Kim functor gives an equivalence of monoidal categories $\mathcal{F}_Q: \mathcal{M}_Q \simeq \mathcal{C}_Q$.*

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, OIWAKE-KITASHIRAKAWA, SAKYO, KYOTO 606-8502, JAPAN; AND INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, UNIVERSITÉ DE PARIS, F-75013 PARIS, FRANCE

Email address: rfujita@kurims.kyoto-u.ac.jp