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CHARACTERS OF IRREDUCIBLE UNITARY REPRESENTATIONS OF $\mathrm{U}(n, n+1)$ VIA DOUBLE LIFTING FROM $\mathrm{U}(1)$

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Abstract. In this paper, we obtained character formulas of irreducible unitary representations of $\mathrm{U}(n, n+1)$ by using Howe's correspondence and the Cauchy–Harish-Chandra integral. The representations of $\mathrm{U}(n, n+1)$ we are dealing with are obtained from a double lifting of a representation of $\mathrm{U}(1)$ via the dual pairs $(\mathrm{U}(1), \mathrm{U}(1, 1))$ and $(\mathrm{U}(1, 1), \mathrm{U}(n, n+1))$.

1. Introduction

For a finite dimensional representation (Π, V) of a group G, one can associate a function Θ_Π on G given by

$$\Theta_\Pi : \mathrm{G} \ni g \to \mathrm{tr}(\Pi(g)) \in \mathbb{C}.$$

The function Θ_Π is the character of the representation (Π, V), and it determines entirely the representation. Obviously, if we remove the assumption that V is finite dimensional, the map Θ_Π does not necessarily makes sense in general. In [9, Section 5], Harish-Chandra extended the concept of character for a particular class of representations of a real reductive Lie group. More precisely, he proved that for a quasi-simple representation $(\Pi, \mathscr{H})$ (see [10, Section 10]) of a real reductive Lie group G, the operator $\Pi(\psi), \psi \in \mathscr{C}^\infty_c(\mathrm{G})$, given by

$$\Pi(\psi) = \int_{\mathrm{G}} \psi(g)\Pi(g)dg,$$

is a trace class operator and the corresponding map

$$\Theta_\Pi : \mathscr{C}^\infty_c(\mathrm{G}) \ni \psi \to \mathrm{tr}(\Pi(\psi)) \in \mathbb{C}$$

is a distribution; Θ_Π is usually called the distribution character of Π. Moreover, Harish-Chandra proved (see [12, Theorem 2]) that there exists a locally integrable function Θ_Π on G, analytic on $\mathrm{G}^{\mathrm{reg}}$ (where $\mathrm{G}^{\mathrm{reg}}$ is the set of regular points of G, see [12, Section 3]), such that

$$\Theta_\Pi(\psi) = \int_{\mathrm{G}} \Theta_\Pi(g)\psi(g)dg, \qquad \left(\psi \in \mathscr{C}^\infty_c(\mathrm{G})\right).$$

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The locally integrable function Θ_Π is the character of Π. In few cases, the value of the character Θ_Π is well-known. For example (the following list is not exhaustive):

(1) G compact (H. Weyl),
(2) $(\Pi, \mathscr{H})$ a discrete series representation:
  (a) Harish-Chandra (see [11]) established a formula for Θ_Π on the compact Cartan subgroup T of G,
  (b) Hecht (see [13]) determined the value of Θ_Π on every Cartan subgroup of G for holomorphic discrete series representation,
(3) $(\Pi, \mathscr{H})$ an irreducible principal series representation (see [21, Proposition 10.18]).
(4) $(\Pi, \mathscr{H})$ irreducible unitary highest weight module (Enright [8, Corollary 2.3], see also [26]).

The goal of this paper is to explain how to use Howe's correspondence and the Cauchy–Harish-Chandra integral introduced by T. Przebinda to get explicit values of characters for some particular irreducible unitary non-highest weight modules of $\mathrm{U}(n, n+1)$ starting from a representation of $\mathrm{U}(1)$. Our method is as follows. Let $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(1), \mathrm{U}(p,q)), p, q \geq 1$, be a dual pair in $\mathrm{Sp}(2(p+q), \mathbb{R})$, $\widetilde{\mathrm{Sp}}(2(p+q), \mathbb{R})$ be the corresponding metaplectic group (see Equation (1)), $\omega^1_{p,q}$ be the metaplectic representation of $\widetilde{\mathrm{Sp}}(2(p+q), \mathbb{R})$ (see Theorem 2.2), $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}'}$ be the preimages of G and G′ in $\widetilde{\mathrm{Sp}}(2(p+q), \mathbb{R})$ and Π be a representation of $\mathrm{U}(1)$. We denote by $\theta^1_{p,q}$ the map coming from Howe's duality theorem (see Equation (3))

$$\theta^1_{p,q} : \mathscr{R}(\widetilde{\mathrm{U}}(1), \omega^1_{p,q}) \to \mathscr{R}(\widetilde{\mathrm{U}}(p,q), \omega^1_{p,q}),$$

where $\mathscr{R}(\widetilde{\mathrm{U}}(1), \omega^1_{p,q})$ and $\mathscr{R}(\widetilde{\mathrm{U}}(p,q), \omega^1_{p,q})$ are defined in Notation 3.4. By assumption on p and q, $\Pi' := \theta^1_{p,q}(\Pi)$ is a non-zero irreducible unitary highest weight module of $\widetilde{\mathrm{U}}(p,q)$. In Appendix A, we computed the value of the character $\Theta_{\Pi'}$ of Π' on every Cartan subgroup of $\widetilde{\mathrm{U}}(p,q)$. A similar result was obtained in [27] for $p = q = 1$.

We now consider the dual pair $(\mathrm{G}', \mathrm{G}_n) = (\mathrm{U}(p,q), \mathrm{U}(n, n+1))$ in $\mathrm{Sp}(2(p+q)(2n+1), \mathbb{R})$. As before, we denote by $\widetilde{\mathrm{Sp}}(2(p+q)(2n+1), \mathbb{R})$ the metaplectic group of $\mathrm{Sp}(2(p+q)(2n+1), \mathbb{R})$, $\omega^{p,q}_{n,n+1}$ the metaplectic representation of $\widetilde{\mathrm{Sp}}(2(p+q)(2n+1), \mathbb{R})$ and

$$\theta^{p,q}_{n,n+1} : \mathscr{R}(\widetilde{\mathrm{U}}(p,q), \omega^{p,q}_{n,n+1}) \to \mathscr{R}(\widetilde{\mathrm{U}}(n,n+1), \omega^{p,q}_{n,n+1})$$

the map obtained from Howe's correspondence (see Equation (3)). Using a result of Kudla (see [23] or [36]), it follows that $\Pi^n := \theta^{p,q}_{n,n+1}(\Pi') \neq 0$ for every $n \geq 2$, i.e. $\Pi^n \in \mathscr{R}(\widetilde{\mathrm{U}}(n,n+1), \omega^{p,q}_{n,n+1})$. Note that by using [25], it follows that $\Pi^n_1 = \Pi^n$, where Π^n_1 is usually called the "big theta" and is defined in Section 3.

As explained in [34] (see also Remark 4.5), by using that Π' is unitary, we get that if $p+q \leq n$, the distribution character Θ_{Π^n} of Π^n can be obtained by using the Cauchy–Harish-Chandra integral (see Section 4). More precisely, because $\Pi^n_1 = \Pi^n$, we get from Equation (4), Theorem 5.6 and Equation (5) that:

$$\Theta_{\Pi^n}(\psi) = \sum_{i=1}^{\min(p,q)+1} \int_{\widetilde{\mathrm{H}}'_i} \overline{\Theta_{\Pi'}(\tilde{h}'_i)} \left|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}'_i)^{-1})_{\mathfrak{g}'/\mathfrak{h}'_i}\right|^{\frac{1}{2}} \mathrm{Chc}_{\tilde{h}'_i}(\psi) d\tilde{h}'_i, \qquad \left(\psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}_n})\right),$$

where $\mathrm{H}'_1, \ldots, \mathrm{H}'_{\min(p,q)+1}$ is a maximal set of non-conjugate Cartan subgroups of G' and $\mathrm{Chc}_{\tilde{h}'_i}$ is a family of distributions on $\widetilde{\mathrm{G}_n}$ parametrized by regular elements on the different Cartan subgroups of $\widetilde{\mathrm{G}'}$ as recalled in Section 4.

In this paper, we compute explicitly the value of Θ_{Π^n} on every Cartan subgroup of $\widetilde{\mathrm{G}_n}$ for $p = q = 1$. We keep the notations of Appendix B and parametrize the $n+1$ Cartan subgroups of G_n by subsets $\mathrm{S}_0 = \{\emptyset\}, \mathrm{S}_1, \ldots, \mathrm{S}_n$ of strongly orthogonal imaginary roots of $\mathfrak{g}_n$ (see also [35, Section 2]) and let $\mathrm{H}_n(\mathrm{S}_0), \ldots, \mathrm{H}_n(\mathrm{S}_n)$ be the corresponding Cartan subgroups of G_n and $\mathrm{H}_{n,\mathrm{S}_0}, \ldots, \mathrm{H}_{n,\mathrm{S}_n}$ be the diagonal subgroups of $\mathrm{GL}(2n+1, \mathbb{C})$ given, for $0 \leq k \leq n$, by $\mathrm{H}_{n,\mathrm{S}_k} = c(\mathrm{S}_k)^{-1}\,\mathrm{H}_n(\mathrm{S}_k)c(\mathrm{S}_k)$, where $c(\mathrm{S}_k)$ is the Cayley transform defined in Equation (26).

In Theorem 6.5, we explain how to go from the distribution character Θ_{Π^n} to the locally integrable function Θ_{Π^n} by using results of Bernon and Przebinda from [3] and [2] (see also Section 5). In Theorem 6.12, we computed the value of Θ_{Π^n} of the different Cartan subgroups of G_n and get for every $k \in [|0, n|]$,

$$\Theta_{\Pi^n}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}) =$$
$$\mathrm{C}\begin{cases} \widetilde{\mathrm{A}} \sum_{\substack{j \in K(h) \cup \mathrm{A}_k \\ i \in J(h) \cup \mathrm{B}_k}} h_i^n h_j^{n+m} \Omega_{i,j}(h) + \delta_{k,0}\, \mathrm{B}\, e^{-(m+1)\,\mathrm{sgn}(X_{2n+1})X_{2n+1}} \Sigma(h) & \text{if } m \geq 1, \\ \widetilde{\mathrm{A}} \sum_{\substack{i,j \in J(h) \cup \mathrm{B}_t \\ i \neq j}} h_i^n h_j^n \Omega_{i,j}(h) + \delta_{k,0}\, \mathrm{B}\, e^{-(m+1)\,\mathrm{sgn}(X_{2n+1})X_{2n+1}} \Sigma(h) & \text{if } m = 0, \\ \widetilde{\mathrm{A}} \sum_{\substack{i \in K(h) \cup \mathrm{A}_k \\ j \in J(h) \cup \mathrm{B}_k}} h_i^{n+m} h_j^n \Omega_{i,j}(h) + \delta_{k,0}\, \mathrm{B}\, e^{(m-1)\,\mathrm{sgn}(X_{2n+1})X_{2n+1}} \Sigma(h) & \text{if } m \leq -1, \end{cases}$$

where C is a constant, m is the highest weight of Π (see Notation A.1), $\check{\mathrm{H}}_{n,\mathrm{S}_k}$ is a double cover of $\mathrm{H}_{n,\mathrm{S}_k}$ defined in Equation (5), $\check{p} : \check{\mathrm{H}}_{n,\mathrm{S}_k} \to \widetilde{\mathrm{H}}_{n,\mathrm{S}_k}$ is defined in Section 5, h is the element of $\mathrm{H}_{n,\mathrm{S}_k}$ given, as in Equation (19), by

$$\begin{aligned} h &= (h_1, \ldots, h_{2n+1}) \\ &= \mathrm{diag}(e^{iX_1 - X_{2n+1}}, \ldots, e^{iX_k - X_{2n+2-k}}, e^{iX_{k+1}}, \ldots, \\ &\qquad e^{iX_{2n+1-k}}, e^{iX_k + X_{2n+2-k}}, \ldots, e^{iX_1 + X_{2n+1}}), \end{aligned}$$

where $X_j \in \mathbb{R}$, $\Omega_{i,j}, 1 \leq i \neq j \leq 2n+1$, and Σ are the functions on $\mathrm{H}^{\mathrm{reg}}_{n,\mathrm{S}_k}$ given by

$$\Omega_{i,j}(h) = \frac{\prod_{\substack{d=1 \\ d \neq i,j}}^{2n+1} h_d}{\prod_{\substack{d=1 \\ d \neq i}}^{2n+1}(h_i - h_d) \prod_{\substack{d=1 \\ d \neq i,j}}^{2n+1}(h_j - h_d)},$$

$$\Sigma(h) = \frac{\mathrm{sgn}(X_{2n+1}) e^{imX_1} \left|e^{(2n-2)X_{2n+1}}\right| \left(1 - e^{-2X_{2n+1}}\right)}{\left|\prod_{d=2}^{2n}\left(1 - h_1 h_d^{-1}\right) \prod_{d=2}^{2n}\left(1 - h_d h_{2n+1}^{-1}\right)\right| \left|1 - e^{-2X_{2n+1}}\right|^2},$$

$K(h)$ and $J(h)$ are the subsets of $\{1,\dots,k\}$ defined by

$$J(h) = \{j \in \{1,\dots,k\}, \mathrm{sgn}(X_{2n+2-j}) = 1\},$$
$$K(h) = \{j \in \{1,\dots,k\}, \mathrm{sgn}(X_{2n+2-j}) = -1\},$$

A_k and B_k are the subsets of $\{1,\dots,2n+1\}$ given by

$$\mathrm{A}_k = \{k+1,\dots,n\}, \qquad \mathrm{B}_k = \{n+1,\dots,2n+1-k\},$$

sgn is the sign-function on $\mathbb{R}^*$ given by

$$\mathrm{sgn}(X) = \begin{cases} 1 & \text{if } X > 0 \\ -1 & \text{if } X < 0 \end{cases}, \qquad (X \in \mathbb{R}^*),$$

and $\widetilde{\mathrm{A}}$ and B are constants defined in Theorems 6.5 and 6.12. As explained in Lemma 6.14, the denominator

$$\prod_{d=2}^{2n}\left(1 - h_1h_d^{-1}\right)\prod_{d=2}^{2n}\left(1 - h_dh_{2n+1}^{-1}\right)$$

is real and its sign is constant on every Weyl chamber.

Note that the representation Π^n is irreducible and unitary (see Remark 5.11) but neither a highest weight module (see Proposition D.6) nor a discrete series representation (see Remark 5.11). In particular, its character Θ_{Π^n} cannot be obtained by using Enright's formula (see [8, Corollary 2.3]) or [26].

Our formula for the character is Weyl denominator free. The method we used in this paper gives a general procedure to give characters of non-highest weight representations by starting from a highest weight representation of a compact group. In particular, proving Conjecture 4.4 could make our method more general, by removing the assumption of stable range for the second lifting.

The paper is organised as follows. In Section 2, we recall a construction of the metaplectic representation given by Aubert and Przebinda in [1]. The goal is to define the embedding T of the metaplectic group into the set of tempered distributions on the symplectic space W (see Equation (2)) which is crucial in the construction of the Cauchy–Harish-Chandra integral. After recalling Harish-Chandra's character theory and Howe's correspondence in Section 3, we define in Section 4 the Cauchy–Harish-Chandra integral and explain a conjecture of Przebinda on the transfer of characters in the theta correspondence (see Conjecture 4.4). In Section 5, we summarize the results of [3] and [2] on how to compute the Cauchy–Harish-Chandra integral on the different Cartan subgroups, and we adapt the results to unitary groups, which are the ones we consider in this paper, and make the computations for Θ_{Π^n} in Section 6 (see Theorems 6.5 and 6.12). The document contains four appendices: in Appendix A, we make computations for Π' for the dual pair $(\mathrm{G},\mathrm{G}') = (\mathrm{U}(1),\mathrm{U}(p,q))$ on every Cartan subgroup of G', in Appendix B, we recall how to parametrize the Cartan subgroups of $\mathrm{U}(p,q)$ by using strongly orthogonal roots (see also [35, Section 2]), in Appendix C, we define a character ε appearing in the formulas for the Cauchy–Harish-Chandra integrals and prove a lemma for ε useful in the proof of Lemma 6.8 and in Appendix D, we recall the concept of wave front set of a representation and prove that the representations Π^n we are dealing with in Section 6 are not highest weight modules.

2. Metaplectic representation

Let χ be the character of $\mathbb{R}$ given by $\chi(t) = e^{2i\pi t}$ and let W be a finite dimensional vector over $\mathbb{R}$ endowed with a non-degenerate, skew-symmetric, bilinear form $\langle\cdot,\cdot\rangle$. We denote by Sp(W) the corresponding group of isometries, i.e.

$$\mathrm{Sp}(\mathrm{W}) = \{g \in \mathrm{GL}(\mathrm{W}), \langle g(w_1), g(w_2)\rangle = \langle w_1, w_2\rangle, (\forall w_1, w_2 \in \mathrm{W})\},$$

and by $\mathfrak{sp}(\mathrm{W})$ its Lie algebra given by:

$$\mathfrak{sp}(\mathrm{W}) = \{X \in \mathrm{End}(\mathrm{W}), \langle X(w_1), w_2\rangle + \langle w_1, X(w_2)\rangle = 0, (\forall w_1, w_2 \in \mathrm{W})\}.$$

We first start by recalling the construction of the metaplectic group $\widetilde{\mathrm{Sp}}(\mathrm{W})$: it is a connected two-fold cover of Sp(W). We use the formalism of [1]. Let J be a compatible positive complex structure on W, i.e. an element of the Lie algebra $\mathfrak{sp}(\mathrm{W})$ satisfying $\mathrm{J}^2 = -\mathrm{Id}_{\mathrm{W}}$ and such that the symmetric form $\langle \mathrm{J}\,\cdot,\cdot\rangle$ is positive definite. For every element $g \in \mathrm{Sp}(W)$, we denote by J_g the element of End(W) given by $\mathrm{J}_g = \mathrm{J}^{-1}(g-1)$. One can easily check that the restriction of J_g to $\mathrm{J}_g(\mathrm{W})$ is invertible and let $\widetilde{\mathrm{Sp}}(\mathrm{W})$ be the subset of $\mathrm{Sp}(\mathrm{W}) \times \mathbb{C}^\times$ defined by

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \left\{\tilde{g} = (g,\xi) \in \mathrm{Sp}(\mathrm{W}) \times \mathbb{C}^\times, \xi^2 = i^{\dim_{\mathbb{R}}(\mathrm{W})} \det(\mathrm{J}_g)^{-1}_{\mathrm{J}_g(\mathrm{W})}\right\}, \tag{1}$$

where $\det(\mathrm{J}_g)_{\mathrm{J}_g(\mathrm{W})}$ denotes the determinant of the endomorphism J_g restricted to $\mathrm{J}_g(\mathrm{W})$. On $\widetilde{\mathrm{Sp}}(\mathrm{W})$, we define a multiplication by:

$$(g_1,\xi_1)(g_2,\xi_2) = (g_1g_2, \xi_1\xi_2\,\mathrm{C}(g_1,g_2)), \qquad \left(g_1, g_2 \in \mathrm{Sp}(\mathrm{W}), \xi_1,\xi_2 \in \mathbb{C}^\times\right),$$

where $\mathrm{C} : \mathrm{Sp}(\mathrm{W}) \times \mathrm{Sp}(\mathrm{W}) \to \mathbb{C}^\times$ is a cocycle defined in [1, Proposition 4.13]. Let Θ be the map defined by:

$$\Theta : \widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \tilde{g} = (g,\xi) \to \xi \in \mathbb{C}^\times.$$

One can check easily that $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a connected two-fold cover of Sp(W), where the covering map $\pi : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{Sp}(\mathrm{W})$ is given by $\pi((g,\xi)) = g$.

For every $g \in \mathrm{End}(\mathrm{W})$, we denote by $c(g)$ the Cayley transform of g defined by:

$$c(g) : (g-1)\,\mathrm{W} \ni (g-1)w \to (g+1)w + \mathrm{Ker}(g-1) \in \mathrm{W}/\mathrm{Ker}(g-1).$$

We denote by S(W) the Schwartz space of W and by $\mathrm{S}^*(\mathrm{W})$ the corresponding space of tempered distributions. We define the map $t : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{S}^*(\mathrm{W})$ by $t(g) = \chi_{c(g)}\mu_{(g-1)\,\mathrm{W}}$, where $\chi_{c(g)}$ is the function on $(g-1)\,\mathrm{W}$ given by

$$\chi_{c(g)}(w) : (g-1)\,\mathrm{W} \to \chi\left(\frac{1}{4}\langle c(g)w, w\rangle\right), \qquad (w \in (g-1)\,\mathrm{W}),$$

and $\mu_{(g-1)\,\mathrm{W}}$ is the Lebesgue measure on $(g-1)\,\mathrm{W}$ such that the volume of the unit cube with respect to the bilinear form $\langle \mathrm{J}\,\cdot,\cdot\rangle$ is 1. More precisely,

$$t(g)\phi = \int_{(g-1)\,\mathrm{W}} \chi_{c(g)}(w)\phi(w)d\mu_{(g-1)\,\mathrm{W}}(w), \qquad (\phi \in \mathrm{S}(\mathrm{W})).$$

We define the map $\mathrm{T} : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{S}^*(\mathrm{W})$ given by

$$\mathrm{T}(\tilde{g}) = \Theta(\tilde{g})t(g), \qquad \left(\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}), g = \pi(\tilde{g})\right). \tag{2}$$

Remark 2.1. Let $\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}}(\mathrm{W})$. The question of the relation between the distributions $\mathrm{T}(\tilde{g}_1), \mathrm{T}(\tilde{g}_2)$ and $\mathrm{T}(\tilde{g}_1\tilde{g}_2)$ arises naturally. In order to explain this link, we need to recall the notion of twisted convolution.

For two functions $\phi_1, \phi_2 \in \mathrm{S}(\mathrm{W})$, we define $\phi_1 \natural \phi_2$ the function on W given by

$$\phi_1 \natural \phi_2(w) = \int_{\mathrm{W}} \phi_1(u)\phi_2(w-u)\chi\left(\frac{1}{2}\langle u, w\rangle\right) d\mu_{\mathrm{W}}(u), \qquad (w \in \mathrm{W}).$$

One can easily check that $\phi_1 \natural \phi_2 \in \mathrm{S}(\mathrm{W})$. We extend $\natural$ to some tempered distributions on W. In fact, for every $g \in \mathrm{Sp}(W)$, the twisted convolution

$$t(g)\natural\phi(w) = \int_{(g-1)\,\mathrm{W}} \chi_{c(g)}(u)\phi(w-u)\chi\left(\frac{1}{2}\langle u, w\rangle\right) d\mu_{\mathrm{W}}(u), \ \ (w \in \mathrm{W}, \phi \in \mathrm{S}(\mathrm{W})),$$

is still a Schwartz function and the map:

$$\mathrm{S}(\mathrm{W}) \ni \phi \to t(g)\natural\phi \in \mathrm{S}(\mathrm{W})$$

is well-defined and continuous (see [1, Proposition 4.11]). Similarly, $\mathrm{T}(\tilde{g})\natural\phi \in \mathrm{S}(\mathrm{W})$ for every $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and $\phi \in \mathrm{S}(\mathrm{W})$. In particular, it makes sense to consider $\mathrm{T}(\tilde{g}_1)\natural\,(\mathrm{T}(\tilde{g}_2)\natural\phi)$ for every $\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and $\phi \in \mathrm{S}(\mathrm{W})$ and one can prove that $\mathrm{T}(\tilde{g}_1)\natural(\mathrm{T}(\tilde{g}_2)\natural\phi) = \mathrm{T}(\tilde{g}_1\tilde{g}_2)\natural\phi$.

Let $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ be a complete polarization of the space W and we denote by dx, dy the Lebesgue measures on X and Y respectively such that $d\mu_{\mathrm{W}} = dxdy$. Using the Weyl transform $\mathscr{K}$, we have a natural isomorphism between the spaces S(W) and $\mathrm{S}(\mathrm{X} \times \mathrm{X})$ given by

$$\mathscr{K} : \mathrm{S}(\mathrm{W}) \ni \phi \to \mathscr{K}(\phi)(x_1, x_2) = \int_{\mathrm{Y}} \phi(x_1 - x_2 + y)\chi\left(\frac{1}{2}\langle y, x_1 + x_2\rangle\right) dy \in \mathrm{S}(\mathrm{X} \times \mathrm{X}),$$

which extends to an isomorphism on the corresponding spaces of distributions. Similarly, every tempered distribution on $\mathrm{X} \times \mathrm{X}$ can be identified to an element of $\mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}^*(\mathrm{X}))$ using the Schwartz Kernel Theorem (see [1, Equation 146]). The corresponding isomorphism will be denoted by Op and let $\omega : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}^*(\mathrm{X}))$ be the map given by

$$\omega = \mathrm{Op} \circ \mathscr{K} \circ \mathrm{T}.$$

As proved in [1, Section 4], we get that for every $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and $v \in \mathrm{S}(\mathrm{X})$, $\omega(\tilde{g})v \in \mathrm{S}(\mathrm{X})$ and that $\omega(\tilde{g}\tilde{h}) = \omega(\tilde{g}) \circ \omega(\tilde{h})$ for every $\tilde{g}, \tilde{h} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$. The operator $\omega(\tilde{g}) \in \mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}(\mathrm{X}))$ can be extended to $\mathrm{L}^2(\mathrm{X})$ by

$$\omega(\tilde{g})\phi = \lim_{\substack{\|\phi - v\|_2 \to 0 \\ v \in \mathrm{S}(\mathrm{X})}} \omega(\tilde{g})v, \qquad \left(\phi \in \mathrm{L}^2(\mathrm{X})\right).$$

Theorem 2.2. *For every $\phi \in \mathrm{L}^2(\mathrm{X})$, the map*

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \tilde{g} \mapsto \omega(\tilde{g})\phi \in \mathrm{L}^2(\mathrm{X})$$

is well-defined and continuous. Moreover, $\omega(\tilde{g}) \in \mathrm{U}(\mathrm{L}^2(\mathrm{X}))$, i.e. ω is a faithful unitary representation of $\widetilde{\mathrm{Sp}}(\mathrm{W})$, and for every $\psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$, we get:

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde{g})\psi(\tilde{g})d\tilde{g} = \mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \psi(\tilde{g})\omega(\tilde{g})d\tilde{g},$$

where $d\tilde{g}$ is a Haar measure on $\widetilde{\mathrm{Sp}}(\mathrm{W})$.

Remark 2.3.

(1) Let $\mathrm{Sp}^c(\mathrm{W})$ be the subset of $\mathrm{Sp}(\mathrm{W})$ given by

$$\mathrm{Sp}^c(\mathrm{W}) = \{g \in \mathrm{Sp}(\mathrm{W}), \det(g-1) \neq 0\}.$$

This is the domain of the Cayley transform. We will denote by $\widetilde{\mathrm{Sp}}^c(\mathrm{W})$ the preimage of $\mathrm{Sp}^c(\mathrm{W})$ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$.

For every $g \in \mathrm{Sp}^c(\mathrm{W})$, $c(g) \in \mathfrak{sp}(\mathrm{W})$. We denote by $\mathfrak{sp}^c(\mathrm{W})$ the subspace of $\mathfrak{sp}(\mathrm{W})$ defined by $c(\mathrm{Sp}^c(\mathrm{W}))$. Obviously, $c^2(g) = g$. It defines a bijective map $c : \mathfrak{sp}^c(\mathrm{W}) \to \mathrm{Sp}^c(\mathrm{W})$. Fix an element $\widetilde{-1}$ in $\pi^{-1}(\{-1\})$. In particular, there exists a unique map $\tilde{c} : \mathfrak{sp}^c(\mathrm{W}) \to \widetilde{\mathrm{Sp}}^c(\mathrm{W})$ such that $c = \pi \circ \tilde{c}$ and $\tilde{c}(0) = \widetilde{-1}$.

Moreover, for every $\psi \in \mathrm{Sp}(\mathrm{W})$ whose support is included in $\widetilde{\mathrm{Sp}}^c(\mathrm{W})$, we get:

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \psi(\tilde{g})d\tilde{g} = \int_{\mathfrak{sp}(\mathrm{W})} \psi(\tilde{c}(X))j_{\mathfrak{sp}(\mathrm{W})}(X)dX,$$

where $j_{\mathfrak{sp}(\mathrm{W})}(X) = |\det(1-X)|^r$, where $r = \frac{2\dim_{\mathbb{R}}(\mathfrak{sp}(\mathrm{W}))}{\dim_{\mathbb{R}}(\mathrm{W})}$ (see [31, Section 3]).

(2) For every $\psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$, we can consider the following distribution on W

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \psi(\tilde{g})\, \mathrm{T}(\tilde{g})d\tilde{g}.$$

This distribution is in fact given by a Schwartz function. Indeed, let's first assume that the support of ψ is included in $\widetilde{\mathrm{Sp}}^c(\mathrm{W})$. For every $\phi \in \mathrm{S}(\mathrm{W})$, we get:

$$\begin{aligned}
\left(\int_{\widetilde{\mathrm{Sp}}(W)} \psi(\tilde{g})\, \mathrm{T}(\tilde{g})d\tilde{g}\right)(\phi) &= \int_{\widetilde{\mathrm{Sp}}(W)} \psi(\tilde{g})\, \mathrm{T}(\tilde{g})\phi d\tilde{g} \\
&= \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \psi(\tilde{g}) \int_{\mathrm{W}} \Theta(\tilde{g})\chi_{c(g)}(w)\phi(w)dwd\tilde{g} \\
&= \int_{\mathfrak{sp}(\mathrm{W})}\int_{\mathrm{W}} \psi(\tilde{c}(X))\Theta(\tilde{c}(X))\chi_X(w)j_{\mathfrak{sp}(\mathrm{W})}(X)\phi(w)dwdX \\
&= \int_{\mathrm{W}} \left(\int_{\mathfrak{sp}(\mathrm{W})} \Phi_\psi(X)\chi(\tau_{\mathfrak{sp}(\mathrm{W})}(w)(X))dX\right)\phi(w)dw \\
&= \int_{\mathrm{W}} \mathscr{F}(\Phi_\psi)\circ\tau_{\mathfrak{sp}(\mathrm{W})}(w)\phi(w)dw,
\end{aligned}$$

where $\Phi_\psi(X) = \psi(\tilde{c}(X))\Theta(\tilde{c}(X))j_{\mathfrak{sp}(\mathrm{W})}(X), X \in \mathfrak{sp}(\mathrm{W})$, is smooth and compactly supported function on $\mathfrak{sp}(\mathrm{W})$ such that $\mathrm{supp}(\Phi_\psi) \subseteq \mathfrak{sp}^c(\mathrm{W})$, $\mathscr{F}(\Phi_\psi)$ is the Fourier transform of Φ_ψ and $\tau_{\mathfrak{sp}(\mathrm{W})} : \mathrm{W} \to \mathfrak{sp}(\mathrm{W})^*$ is the moment map defined by

$$\tau_{\mathfrak{sp}(\mathrm{W})}(w)(X) = \langle X(w), w\rangle, \qquad (w \in \mathrm{W}, X \in \mathfrak{sp}(\mathrm{W})).$$

In particular, $\mathscr{F}(\Phi_\psi)\circ\tau_{\mathfrak{sp}(\mathrm{W})}$ is a Schwarz function on W.

We can remove the assumption on the support of ψ by using the previous result. Indeed, the Zariski topology on $\mathrm{Sp}(W)$ is noetherian. In particular, as explained in [1, Equation 141], there exists $g_1, \dots, g_m \in \mathrm{Sp}(W)^c$ such that

$$\widetilde{\mathrm{Sp}}(W) = \bigcup_{i=1}^{m} \tilde{g}_i \widetilde{\mathrm{Sp}}^c(W).$$

Let $\psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}}(W))$. We can find functions $\psi_1, \dots, \psi_m \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}}^c(W))$ such that in an open neighbourhood of $\mathrm{supp}(\psi)$ we get

$$1 = \sum_{i=1}^{m} \psi_i(\tilde{g}_i{}^{-1}\tilde{g})$$

for every $\tilde{g} \in \widetilde{\mathrm{Sp}}(W)$. Then

$$\int_{\widetilde{\mathrm{Sp}}(W)} \psi(\tilde{g})\, \mathrm{T}(\tilde{g}) d\tilde{g} = \sum_{i=1}^{m} \int_{\widetilde{\mathrm{Sp}}(W)} \psi_i(\tilde{g}_i{}^{-1}\tilde{g})\psi(\tilde{g})\, \mathrm{T}(\tilde{g}) d\tilde{g}$$

$$= \sum_{i=1}^{m} \int_{\widetilde{\mathrm{Sp}}(W)} \psi_i(\tilde{g})\psi(\tilde{g}_i\tilde{g})\, \mathrm{T}(\tilde{g}_i\tilde{g}) d\tilde{g} = \sum_{i=1}^{m} \mathrm{T}(\tilde{g}_i)\natural \left(\int_{\widetilde{\mathrm{Sp}}(W)} \psi_i(\tilde{g})\psi(\tilde{g}_i\tilde{g})\, \mathrm{T}(\tilde{g}) d\tilde{g} \right).$$

The result follows from Remark 2.1.

3. Character theory and Howe's correspondence

Let G be a real connected reductive Lie group, $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ its Lie algebra and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. We denote by $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ the enveloping algebra of $\mathfrak{g}$ (see [22, Chapter 3.1]), $\mathrm{Z}(\mathscr{U}(\mathfrak{g}_{\mathbb{C}}))$ its center and $\mathrm{D}(\mathrm{G})$ the set of differential operators on G and by $\mathrm{D}_{\mathrm{G}}(\mathrm{G})$ the set of left-invariant differential operators on G. As explained in [14, Chapter 2], $\mathrm{D}_{\mathrm{G}}(\mathrm{G})$ is isomorphic to $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$. Let $\mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\mathrm{G})$ be the set of bi-invariant differential operators on G (which is isomorphic to $\mathrm{Z}(\mathscr{U}(\mathfrak{g}_{\mathbb{C}}))$, see [14]), $\mathscr{D}'(\mathrm{G})$ be the set of distributions on G and $\mathscr{D}'(\mathrm{G})^{\mathrm{G}}$ the set of G-invariant distributions.

Definition 3.1. We say that $T \in \mathscr{D}'(\mathrm{G})$ is an eigendistribution if there exists $\chi_T : \mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\mathrm{G}) \to \mathbb{C}$ a homomorphism of algebras such that $\mathrm{D}(T) = \chi_T(\mathrm{D})T$ for every $\mathrm{D} \in \mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\mathrm{G})$.

We will denote by Eig(G) the set of eigendistributions on G.

Theorem 3.2 (Harish-Chandra, [9]). *For every* G*-invariant eigendistribution* T *on* G*, there exists a locally integrable function* f_T *on* G*, analytic on* $\mathrm{G}^{\mathrm{reg}}$*, such that* $T = T_{f_T}$*, i.e. for every function* $\psi \in \mathscr{C}_c^\infty(\mathrm{G})$*,*

$$T(\psi) = \int_{\mathrm{G}} f_T(g)\psi(g) dg,$$

where dg *is a Haar measure on* G*.*

Let $(\Pi, \mathscr{H})$ be an irreducible quasi-simple representation (see [10, Section 10]). As explained in [9], the map

$$\Theta_\Pi : \mathscr{C}_c^\infty(\mathrm{G}) \ni \psi \to \mathrm{tr}(\Pi(\psi)) \in \mathbb{C}$$

is well-defined and is a distribution (in the sense of Laurent Schwartz). In particular, by assumption on Π, it follows from Theorem 3.2 that there exists $\Theta_\Pi \in \mathscr{L}^1_{\mathrm{loc}}(\mathrm{G})$ such that

$$\Theta_\Pi(\psi) = \int_{\mathrm{G}} \Theta_\Pi(g)\psi(g) dg$$

for every $\psi \in \mathscr{C}_c^\infty(\mathrm{G})$. The function Θ_Π is called the character of Π.

We now recall Howe's duality theorem and how it can be studied through characters. Let W be a finite dimensional vector space over $\mathbb{R}$ endowed with a non-degenerate, skew-symmetric, bilinear form $\langle \cdot, \cdot \rangle$. As in Section 2, we denote by Sp(W) the group of isometries of $(W, \langle \cdot, \cdot \rangle)$, by $(\widetilde{Sp}(W), \pi)$ the metaplectic cover of Sp(W) (see Equation (1)), by $(\omega, \mathscr{H})$ the corresponding Weil representation (see Theorem 2.2) and by $(\omega^\infty, \mathscr{H}^\infty)$ the corresponding smooth representation (see [37, Chapter 0]).

A dual pair in Sp(W) is a pair of subgroups (G, G') of Sp(W) which are mutually centralizer in $Sp(W)$. The dual pair is called irreducible if we cannot find any orthogonal decomposition $W = W_1 \oplus W_2$ where both spaces W_1 and W_2 are $G \cdot G'$-invariant, and called reductive if the actions of G and G' on W are both reductive. The set of irreducible reductive dual pairs in Sp(W) had been classified by Howe in [17].

Remark 3.3. In this paper, we will focus our attention on a dual pair consisting of two unitary groups. More precisely, let V and V' be two complex vector spaces endowed with a hermitian form $(\cdot, \cdot)$ and skew-hermitian form $(\cdot, \cdot)'$ respectively. We denote by U(V) and $U(V')$ the corresponding group of isometries and by W the complex vector space given by $W = V \otimes_{\mathbb{C}} V'$. The space W can naturally be seen as a real vector space, and to avoid any confusion, we will denote by $W_{\mathbb{R}}$ the corresponding real vector space. The skew-hermitian form $b = (\cdot, \cdot) \otimes (\cdot, \cdot)'$ on W defines a skew-symmetric form $\langle \cdot, \cdot \rangle$ on $W_{\mathbb{R}}$ by $\langle \cdot, \cdot \rangle = \mathrm{Im}(b)$. In particular, $(U(V), U(V'))$ is a dual pair in $Sp(W_{\mathbb{R}}, \langle \cdot, \cdot \rangle)$.

If we denote by (p, q) and (r, s) the signatures of $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ respectively, we get that $(U(p, q), U(r, s))$ form a dual pair in $Sp(2(p+q)(r+s), \mathbb{R})$.

Notation 3.4. For a subgroup H of Sp(W), we denote by $\widetilde{H} = \pi^{-1}(H)$ the preimage of H in $\widetilde{Sp}(W)$ and let $\mathscr{R}(\widetilde{H}, \omega)$ be the set of equivalence classes of irreducible admissible representations of $\tilde{G}$ which are infinitesimally equivalent to a quotient of $\mathscr{H}^\infty$ by a closed $\omega^\infty(\widetilde{H})$-invariant subspace.

Theorem 3.5 (R. Howe, [19])**.** *For every reductive dual pair* (G, G') *of* Sp(W)*, we get a bijection between* $\mathscr{R}(\widetilde{G}, \omega)$ *and* $\mathscr{R}(\widetilde{G'}, \omega)$*, whose graph is* $\mathscr{R}(\widetilde{G} \cdot \widetilde{G'}, \omega)$*.*

More precisely, if $\Pi \in \mathscr{R}(\widetilde{G}, \omega)$, we denote by $N(\Pi)$ the intersection of all the closed $\widetilde{G}$-invariant subspaces $\mathscr{N}$ such that $\Pi \approx \mathscr{H}^\infty / \mathscr{N}$. Then, the space $\mathscr{H}(\Pi) = \mathscr{H}^\infty / N(\Pi)$ is a $\widetilde{G} \cdot \widetilde{G'}$-module; more precisely, $\mathscr{H}(\Pi) = \Pi \otimes \Pi'_1$, where Π'_1 is a $\widetilde{G'}$-module, not irreducible in general, but Howe's duality theorem says that there exists a unique irreducible quotient Π' of Π'_1 with $\Pi' \in \mathscr{R}(\widetilde{G'}, \omega)$ and $\Pi \otimes \Pi' \in \mathscr{R}(\widetilde{G} \cdot \widetilde{G'}, \omega)$.

We will denote by

$$\theta : \mathscr{R}(\widetilde{G}, \omega) \to \mathscr{R}(\widetilde{G'}, \omega) \tag{3}$$

the corresponding bijection.

Remark 3.6. If G is compact, the situation turns out to be slightly easier. The action of $\widetilde{G}$ on $\mathscr{H}^\infty$ can be decomposed as

$$\mathscr{H}^\infty = \bigoplus_{(\Pi, \mathscr{H}_\Pi) \in \mathscr{R}(\widetilde{G}, \omega)} \mathscr{H}(\Pi),$$

where $\mathscr{H}(\Pi)$ is the closure of $\left\{T(\mathscr{H}_\Pi), T \in \mathrm{Hom}_{\widetilde{\mathrm{G}}}(\mathscr{H}_\Pi, \mathscr{H}^\infty) \neq \{0\}\right\}$ and $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ is the set of representations (Π, V_Π) of $\widetilde{\mathrm{G}}$ such that $\mathrm{Hom}_{\widetilde{\mathrm{G}}}(\mathrm{V}_\Pi, \omega^\infty) \neq \{0\}$. Obviously, $\widetilde{\mathrm{G}'}$ acts on $\mathscr{H}(\Pi)$ and we get that $\mathscr{H}(\Pi) = \Pi \otimes \Pi'$ where Π' is an irreducible unitary representation of $\widetilde{\mathrm{G}'}$.

4. Cauchy Harish-Chandra integral and transfer of invariant eigendistributions

We start this section by recalling the construction of the Cauchy–Harish-Chandra integral introduced in [33, Section 2].

Let $(\mathrm{G}, \mathrm{G}')$ be an irreducible reductive dual pair in $\mathrm{Sp}(\mathrm{W})$ and $\mathrm{T} : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{S}^*(\mathrm{W})$ the map defined in Equation (2). Let $\mathrm{H}_1, \dots, \mathrm{H}_n$ be a maximal set of non-conjugate Cartan subgroups of G and let $\mathrm{H}_i = \mathrm{T}_i\,\mathrm{A}_i$ be the decomposition of H_i as in [38, Section 2.3.6], where T_i is maximal compact in H_i. For every $1 \leq i \leq n$, we denote by A'_i the subgroup of $\mathrm{Sp}(\mathrm{W})$ given by $\mathrm{A}'_i = \mathrm{C}_{\mathrm{Sp}(\mathrm{W})}(\mathrm{A}_i)$ and let $\mathrm{A}''_i = \mathrm{C}_{\mathrm{Sp}(\mathrm{W})}(\mathrm{A}'_i)$. As recalled in [33, Section 1], there exists an open and dense subset $\mathrm{W}_{\mathrm{A}''_i} \subseteq \mathrm{W}$, which is A''_i-invariant and such that $\mathrm{A}''_i \setminus \mathrm{W}_{\mathrm{A}''_i}$ is a manifold, endowed with a measure $\underline{dw}$ such that for every $\phi \in \mathscr{C}^\infty_c(\mathrm{W})$ such that $\mathrm{supp}(\phi) \subseteq \mathrm{W}_{\mathrm{A}''_i}$,

$$\int_{\mathrm{W}_{\mathrm{A}''_i}} \phi(w)dw = \int_{\mathrm{A}''_i \setminus \mathrm{W}_{\mathrm{A}''_i}} \int_{\mathrm{A}''_i} \phi(aw)da\underline{dw}.$$

For every $\psi \in \mathscr{C}^\infty_c(\widetilde{\mathrm{A}'_i})$, we denote by $\mathrm{Chc}(\psi)$ the following integral:

$$\mathrm{Chc}(\psi) = \int_{\mathrm{A}''_i \setminus \mathrm{W}_{\mathrm{A}''_i}} \mathrm{T}(\psi)(w)\underline{dw}.$$

According to Remark 2.3, the previous integral is well-defined and as proved in [33, Lemma 2.9], the corresponding map $\mathrm{Chc} : \mathscr{C}^\infty_c(\widetilde{\mathrm{A}'_i}) \to \mathbb{C}$ defines a distribution on $\widetilde{\mathrm{A}'_i}$.

Remark 4.1. We say few words about the dual pair $(\mathrm{A}'_i, \mathrm{A}''_i)$ and the space $\mathrm{W}_{\mathrm{A}''_i}$. Let $\mathrm{V}_{0,i}$ be the subspace of V on which A_i acts trivially and $\mathrm{V}_{1,i} = \mathrm{V}_{0,i}^\perp$. The restriction of $(\cdot,\cdot)$ to $\mathrm{V}_{1,i}$ is non-degenerate and even dimensional. In particular, there exists a complete polarization of $\mathrm{V}_{1,i}$ of the form $\mathrm{V}_{1,i} = \mathrm{X}_i \oplus \mathrm{Y}_i$, where both spaces X_i and Y_i are H_i-invariant.

By looking at the action of A_i on $\mathrm{V}_{1,i}$, we get:

$$\mathrm{X}_i = \mathrm{X}^1_i \oplus \ldots \oplus \mathrm{X}^k_i, \qquad \mathrm{Y}_i = \mathrm{Y}^1_i \oplus \ldots \oplus \mathrm{Y}^k_i,$$

where all the spaces $\mathrm{X}^j_i, 1 \leq i \leq n, 1 \leq j \leq k$, are A_i-invariant and mutually non-equivalent. In particular,

$$\begin{aligned}\mathrm{W} = \mathrm{Hom}(\mathrm{V}, \mathrm{V}') &= \mathrm{Hom}(\mathrm{V}_{0,i}, \mathrm{V}') \oplus \mathrm{Hom}(\mathrm{V}_{1,i}, \mathrm{V}') \\ &= \mathrm{Hom}(\mathrm{V}_{0,i}, \mathrm{V}') \oplus \bigoplus_{j=1}^{k} \left(\mathrm{Hom}(\mathrm{X}^j_i, \mathrm{V}') \oplus \mathrm{Hom}(\mathrm{Y}^j_i, \mathrm{V}')\right).\end{aligned}$$

To simplify the notations, we denote by W^i_j the subspace of W given by $\mathrm{Hom}(\mathrm{X}^j_i,\mathrm{V}')\oplus\mathrm{Hom}(\mathrm{Y}^j_i,\mathrm{V}')$ and $\mathrm{W}_{0,i}=\mathrm{Hom}(\mathrm{V}_{0,i},\mathrm{V}')$. One can easily check that:

$$\mathrm{A}'_i=\mathrm{Sp}(\mathrm{W}_{0,i})\times\mathrm{GL}(\mathrm{Hom}(\mathrm{X}^1_i,\mathrm{V}')_{\mathbb{R}})\times\ldots\times\mathrm{GL}(\mathrm{Hom}(\mathrm{X}^k_i,\mathrm{V}')_{\mathbb{R}})$$

and

$$\mathrm{A}''_i=\mathrm{O}(1)\times\mathrm{GL}(1,\mathbb{R})\times\ldots\times\mathrm{GL}(1,\mathbb{R}).$$

Moreover,

$$\mathrm{W}_{\mathrm{A}''_i}=(\mathrm{W}_{0,i}\setminus\{0\})\times\widetilde{\mathrm{W}}_{1,i}\times\ldots\times\widetilde{\mathrm{W}}_{n,i},$$

where $\widetilde{\mathrm{W}}_{j,i}=\Big\{(x,y)\in\mathrm{Hom}(\mathrm{X}^j_i,\mathrm{V}')\oplus\mathrm{Hom}(\mathrm{Y}^j_i,\mathrm{V}'),x\neq 0,y\neq 0\Big\}$, $1\leq j\leq k$.

For every $\tilde{h}_i\in\widetilde{\mathrm{H}_i}$, we denote by $\tau_{\tilde{h}_i}$ the map:

$$\tau_{\tilde{h}_i}:\widetilde{\mathrm{G}'}\ni\tilde{g}'\to\tilde{h}_i\tilde{g}'\in\widetilde{\mathrm{A}'_i}.$$

As proved in [33], for every $\tilde{h}_i\in\widetilde{\mathrm{H}_i}^{\mathrm{reg}}$, the pull-back $\tau^*_{\tilde{h}_i}(\mathrm{Chc})$ of Chc via $\tau_{\tilde{h}_i}$ (see [16, Theorem 8.2.4]) is a well-defined distribution on $\widetilde{\mathrm{G}'}$.

For every $\tilde{h}_i\in\widetilde{\mathrm{H}_i}^{\mathrm{reg}}$, we denote by $\mathrm{Chc}_{\tilde{h}_i}:=\tau^*_{\tilde{h}_i}(\mathrm{Chc})$ the corresponding distribution on $\widetilde{\mathrm{G}'}$.

Notation 4.2. For every reductive group G, we denote by $\mathcal{I}(\mathrm{G})$ the space of orbital integrals on G as in [4, Section 3], endowed with a natural topology defined in [4, Section 3.3]. We denote by J_G the map $\mathrm{J}_\mathrm{G}:\mathscr{C}^\infty_c(\mathrm{G})\to\mathscr{C}^\infty(\mathrm{G}^{\mathrm{reg}})^{\mathrm{G}}$ given as follows: for every $\gamma\in\mathrm{G}^{\mathrm{reg}}$, there exists a unique, up to conjugation, Cartan subgroup $\mathrm{H}(\gamma)$ of G such that $\gamma\in\mathrm{H}(\gamma)$, and for every $\psi\in\mathscr{C}^\infty_c(\mathrm{G})$, we define $\mathrm{J}_\mathrm{G}(\psi)(\gamma)$ by:

$$\mathrm{J}_\mathrm{G}(\psi)(\gamma)=\left|\det(\mathrm{Id}-\mathrm{Ad}(\gamma^{-1}))_{\mathfrak{g}/\mathfrak{h}(\gamma)}\right|^{\frac{1}{2}}\int_{\mathrm{G}\,/\,\mathrm{H}(\gamma)}\psi(g\gamma g^{-1})\overline{dg}.$$

As proved in [4, Theorem 3.2.1], the map

$$\mathrm{J}_\mathrm{G}:\mathscr{C}^\infty_c(\mathrm{G})\to\mathcal{I}(\mathrm{G})$$

is well-defined and surjective. We denote by $\mathcal{I}(\mathrm{G})^*$ the set of continuous linear forms on $\mathcal{I}(\mathrm{G})$ and let $\mathrm{J}^t_\mathrm{G}:\mathcal{I}(\mathrm{G})^*\to\mathscr{D}'(\mathrm{G})$ be the transpose of J_G. In [4, Theorem 3.2.1], Bouaziz proved that the map

$$\mathrm{J}^t_\mathrm{G}:\mathcal{I}(\mathrm{G})^*\to\mathscr{D}'(\mathrm{G})^{\mathrm{G}}$$

is bijective.

We now apply these results to construct a map Chc^*, transferring the invariant distributions for a given dual pair (G,G'). Let (G,G') be an irreducible dual pair in $\mathrm{Sp}(W)$ such that $\mathrm{rk}(\mathrm{G})\leq\mathrm{rk}(\mathrm{G}')$. For every function $\psi\in\mathscr{C}^\infty_c(\widetilde{\mathrm{G}'})$, we denote by $\widetilde{\mathrm{Chc}}(\psi)$ the $\widetilde{\mathrm{G}}$-invariant function on $\widetilde{\mathrm{G}}^{\mathrm{reg}}$ given by:

$$\widetilde{\mathrm{Chc}}(\psi)(\tilde{h}_i)=\mathrm{Chc}_{\tilde{h}_i}(\psi),\qquad(\tilde{h}_i\in\widetilde{\mathrm{H}_i}^{\mathrm{reg}}).$$

As proved in [3], the corresponding map

$$\widetilde{\mathrm{Chc}}:\mathscr{C}^\infty_c(\widetilde{\mathrm{G}'})\to\mathcal{I}(\widetilde{\mathrm{G}})$$

is well-defined and continuous and factors through $\mathcal{I}(\widetilde{\mathrm{G}'})$, i.e.

$$\widetilde{\mathrm{Chc}} : \mathcal{I}(\widetilde{\mathrm{G}'}) \to \mathcal{I}(\widetilde{\mathrm{G}})$$

and the corresponding map is continuous. In particular, we get a map

$$\mathrm{Chc}^* : \mathscr{D}'(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}} \ni T \to \mathrm{J}^t_{\widetilde{\mathrm{G}'}} \circ \widetilde{\mathrm{Chc}}^t \circ (\mathrm{J}^t_{\widetilde{\mathrm{G}}})^{-1}(T) \in \mathscr{D}'(\widetilde{\mathrm{G}'})^{\widetilde{\mathrm{G}'}}.$$

Theorem 4.3. *The map* Chc^* *sends* $\mathrm{Eig}(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}}$ *into* $\mathrm{Eig}(\widetilde{\mathrm{G}'})^{\widetilde{\mathrm{G}'}}$. *Moreover, if* Θ *is a distribution on* $\widetilde{\mathrm{G}}$ *given by a locally integrable function* Θ *on* $\widetilde{\mathrm{G}}$, *we get for every* $\psi \in \mathscr{C}^\infty_c(\widetilde{\mathrm{G}'})$ *that:*

$$\mathrm{Chc}^*(\Theta)(\psi) = \sum_{i=1}^n \frac{1}{|\mathscr{W}(\mathrm{H}_i)|} \int_{\widetilde{\mathrm{H}_i}^{\mathrm{reg}}} \Theta(\tilde{h}_i) |\det(1 - \mathrm{Ad}(\tilde{h}_i^{-1}))_{\mathfrak{g}/\mathfrak{h}_i}| \, \mathrm{Chc}(\psi)(\tilde{h}_i) d\tilde{h}_i, \tag{4}$$

where $\mathrm{H}_1, \dots, \mathrm{H}_n$ *is a maximal set of non-conjugate Cartan subgroups of* G.

In [33], T. Przebinda conjectured that the correspondence of characters in the theta correspondence should be obtained via Chc^*. More precisely,

Conjecture 4.4. *Let* G_1 *and* G'_1 *be the Zariski identity components of* G *and* G′ *respectively. Let* $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ *satisfying* $\Theta_{\Pi|_{\widetilde{\mathrm{G}} \setminus \widetilde{\mathrm{G}_1}}} = 0$ *if* $\mathrm{G} = \mathrm{O}(\mathrm{V})$, *where* V *is an even dimensional vector space over* $\mathbb{R}$ *or* $\mathbb{C}$. *Then, up to a constant,* $\mathrm{Chc}^*(\overline{\Theta_\Pi}) = \Theta_{\Pi'_1}$ *on* $\widetilde{\mathrm{G}'_1}$.

Remark 4.5. The conjecture is known to be true in few cases:

(1) G compact,
(2) $(\mathrm{G}, \mathrm{G}')$ in the stable range and Π a unitary representation of $\widetilde{\mathrm{G}}$ (see [34]),
(3) $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(p,q), \mathrm{U}(r,s))$, with $p+q = r+s$ and Π a discrete series representation of $\widetilde{\mathrm{G}}$ (see [28]).

5. Explicit formulas of Chc for unitary groups

In this section, we quickly explain how to compute explicitly the Cauchy–Harish-Chandra integral on the different Cartan subgroups. Because our paper will only concern characters of some unitary groups, we will adapt the results of [3] and [2] in this context, but similar results can be obtained for other dual pairs.

Let $\mathrm{V} = \mathbb{C}^{p+q}$ and $\mathrm{V}' = \mathbb{C}^{r+s}$ be two complex vector spaces endowed with non-degenerate bilinear forms $(\cdot,\cdot)$ and $(\cdot,\cdot)'$ respectively, with $(\cdot,\cdot)$ hermitian and $(\cdot,\cdot)'$ skew-hermitian, and let (p,q) (resp. (r,s)) be the signature of $(\cdot,\cdot)$ (resp. $(\cdot,\cdot)'$). We assume that $p+q \leq r+s$. Let $\mathscr{B}_{\mathrm{V}} = \{f_1, \dots, f_n\}$, $n = p+q$ (resp. $\mathscr{B}_{\mathrm{V}'} = \{f'_1, \dots, f'_{n'}\}$, $n' = r+s$) be a basis of V (resp. V′) such that $\mathrm{Mat}\left((\cdot,\cdot), \mathscr{B}_{\mathrm{V}}\right) = \mathrm{Id}_{p,q}$ (resp. $\mathrm{Mat}\left((\cdot,\cdot)', \mathscr{B}_{\mathrm{V}'}\right) = i\,\mathrm{Id}_{r,s}$). Let G and G′ be the corresponding groups of isometries, i.e.

$$\mathrm{G} = \mathrm{G}(\mathrm{V}, (\cdot,\cdot)) \approx \left\{g \in \mathrm{GL}(n, \mathbb{C}), \overline{g}^t \,\mathrm{Id}_{p,q}\, g = \mathrm{Id}_{p,q}\right\},$$

$$\mathrm{G}' = \mathrm{G}(\mathrm{V}', (\cdot,\cdot)') \approx \left\{g \in \mathrm{GL}(n', \mathbb{C}), \overline{g}^t \,\mathrm{Id}_{r,s}\, g = \mathrm{Id}_{r,s}\right\},$$

where $\approx$ is a Lie group isomorphism.

Let $\mathrm{K} = \mathrm{U}(p) \times \mathrm{U}(q)$ and $\mathrm{K}' = \mathrm{U}(r) \times \mathrm{U}(s)$ be the maximal compact subgroups of G and G′ respectively and let H and H′ be the diagonal Cartan subgroups of

K and K$'$ respectively. By looking at the action of H on the space V, we get a decomposition of V of the form:

$$\mathrm{V} = \mathrm{V}_1 \oplus \ldots \oplus \mathrm{V}_n,$$

where the spaces V_a given by $\mathrm{V}_a = \mathbb{C}if_a$ are irreducible H-modules. We denote by J the element of $\mathfrak{h}$ such that $\mathrm{J} = i\,\mathrm{Id}_{\mathrm{V}}$ and let $\mathrm{J}_j = i\,\mathrm{E}_{j,j}$. Similarly, we write

$$\mathrm{V}' = \mathrm{V}'_1 \oplus \ldots \oplus \mathrm{V}'_{n'},$$

with $\mathrm{V}'_b = \mathbb{C}if'_b$, J$'$ the element of $\mathfrak{h}'$ given by $\mathrm{J}' = i\,\mathrm{Id}_{\mathrm{V}'}$ and $\mathrm{J}'_j = i\,\mathrm{E}_{j,j}$. Let $\mathrm{W} = \mathrm{Hom}_{\mathbb{C}}(\mathrm{V}', \mathrm{V})$ endowed with the symplectic form $\langle\cdot,\cdot\rangle$ given by:

$$\langle w_1, w_2\rangle = \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(w_2^* w_1), \qquad (w_1, w_2 \in \mathrm{W}),$$

where w_2^* is the element of $\mathrm{Hom}_{\mathbb{C}}(\mathrm{V}, \mathrm{V}')$ satisfying:

$$(w_2^*(v), v')' = (v, w_2(v')), \qquad (v \in \mathrm{V}, v' \in \mathrm{V}').$$

The space W can be seen as a complex vector space by

$$\mathrm{i}\,w = \mathrm{J} \circ w, \qquad (w \in \mathrm{W}).$$

We define a double cover $\widetilde{\mathrm{GL}}_{\mathbb{C}}(\mathrm{W})$ of the complex group $\mathrm{GL}_{\mathbb{C}}(\mathrm{W})$ by:

$$\widetilde{\mathrm{GL}}_{\mathbb{C}}(\mathrm{W}) = \left\{\tilde{g} = (g, \xi) \in \mathrm{GL}_{\mathbb{C}}(\mathrm{W}) \times \mathbb{C}^{\times}, \xi^2 = \det(g)\right\}.$$

Because $p + q \leq r + s$, we get a natural embedding of $\mathfrak{h}_{\mathbb{C}}$ into $\mathfrak{h}'_{\mathbb{C}}$ and we denote by $\mathrm{Z}' = \mathrm{G}'^{\mathfrak{h}}$ the centralizer of $\mathfrak{h}$ in G$'$.

Notation 5.1. We denote by Δ (resp. $\Delta(\mathfrak{k})$) the root system corresponding to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ (resp. $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$), by Ψ (resp. $\Psi(\mathfrak{k})$) a system of positive roots of Δ (resp. $\Delta(\mathfrak{k})$) and let $\Phi = -\Psi$ (resp. $\Phi(\mathfrak{k}) = -\Psi(\mathfrak{k})$) the set of negative roots.

Let e_i be the linear form on $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}^{p+q}$ given by $e_i(\lambda_1, \ldots, \lambda_{p+q}) = \lambda_i$. As explained in [22, Chapter 2], we know that:

$$\Delta = \{\pm(e_i - e_j), 1 \leq i < j \leq p+q\}, \qquad \Psi = \{e_i - e_j, 1 \leq i < j \leq p+q\},$$

and

$$\Delta(\mathfrak{k}) = \{\pm(e_i - e_j), 1 \leq i < j \leq p\} \cup \{\pm(e_i - e_j), p+1 \leq i < j \leq p+q\},$$
$$\Psi(\mathfrak{k}) = \Psi \cap \Delta(\mathfrak{k}).$$

We define $\Delta', \Delta'(\mathfrak{k}), \Psi', \Psi'(\mathfrak{k}), \Phi', \Phi'(\mathfrak{k})$ similarly and denote by $e'_i, 1 \leq i \leq r+s$, the linear form on $\mathfrak{h}'_{\mathbb{C}} = \mathbb{C}^{r+s}$ given by $e_i(\lambda_1, \ldots, \lambda_{r+s}) = \lambda_i$.

Let $\mathrm{H}'_{\mathbb{C}}$ be the complexification of H$'$ in $\mathrm{GL}_{\mathbb{C}}(\mathrm{W})$. In particular, $\mathrm{H}'_{\mathbb{C}}$ is isomorphic to

$$\mathfrak{h}'_{\mathbb{C}} / \left\{\sum_{j=1}^{n'} 2\pi x_j\,\mathrm{J}_j, x_j \in \mathbb{Z}\right\}.$$

We denote by $\check{\mathrm{H}}'_{\mathbb{C}}$ the connected two-fold cover of $\mathrm{H}'_{\mathbb{C}}$ isomorphic to

$$\mathfrak{h}'_{\mathbb{C}} / \left\{\sum_{j=1}^{n'} 2\pi x_j\,\mathrm{J}'_j, \sum_{j=1}^{n'} x_j \in 2\mathbb{Z}, x_j \in \mathbb{Z}\right\}. \tag{5}$$

We will denote by $p : \check{\mathrm{H}}'_{\mathbb{C}} \to \mathrm{H}'_{\mathbb{C}}$ the covering map. One can easily check that $\rho' = \frac{1}{2}\sum_{\alpha \in \Psi'} \alpha$ is analytic integral on $\check{\mathrm{H}}'_{\mathbb{C}}$. As explained in [33, Section 2], we construct a map $\check{p} : \check{\mathrm{H}}'_{\mathbb{C}} \to \widetilde{\mathrm{H}}'_{\mathbb{C}}$ which is bijective (but not an isomorphism of covering of $\mathrm{H}'_{\mathbb{C}}$ in general).

As explained in Appendix B, every Cartan subgroup of G' can be parametrized by a subset $\mathrm{S} \subseteq \Psi'^{\mathrm{st}}_n$ consisting of non-compact strongly orthogonal roots. We denote by $\mathrm{H}'(\mathrm{S})$ the corresponding Cartan subgroup and by H'_{S} the subgroup of $\mathrm{H}'_{\mathbb{C}}$ as in Appendix B. Let $\mathrm{S} \subseteq \Psi'^{\mathrm{st}}_n$ and $\check{\mathrm{H}}'_{\mathrm{S}}$ the preimage of the Cartan subgroup H'_{S} in $\check{\mathrm{H}}'_{\mathbb{C}}$ (see Appendix B). For every $\varphi \in \mathscr{C}^{\infty}_c(\widetilde{\mathrm{G}'})$, we denote by $\mathscr{H}_{\mathrm{S}}\varphi$ the function of $\check{\mathrm{H}}'_{\mathrm{S}}$ defined, for $\check{h}' \in \check{\mathrm{H}}'_{\mathrm{S}}$, by:

$$\begin{aligned}&\mathscr{H}_{\mathrm{S}}\varphi(\check{h}')\\&= \varepsilon_{\Psi'_{\mathrm{S},\mathbb{R}}}(\check{h}')\check{h}'^{\frac{1}{2}\sum_{\alpha\in\Psi'}\alpha} \prod_{\alpha\in\Psi'}(1-\check{h}'^{-\alpha}) \int_{\mathrm{G}'/\mathrm{H}'(\mathrm{S})} \varphi(g'c(\mathrm{S})\check{p}(\check{h}')c(\mathrm{S})^{-1}g'^{-1})dg'\,\mathrm{H}'(\mathrm{S}),\end{aligned}$$

where $\Psi'_{\mathrm{S},\mathbb{R}}$ is the subset of Ψ' consisting of real roots for H'_{S} and $\varepsilon_{\Psi'_{\mathrm{S},\mathbb{R}}}$ is the function defined on $\check{\mathrm{H}}'^{\mathrm{reg}}_{\mathrm{S}}$ by

$$\varepsilon_{\Psi'_{\mathrm{S},\mathbb{R}}}(\check{h}') = \mathrm{sign}\left(\prod_{\alpha\in\Psi'_{\mathrm{S},\mathbb{R}}}(1-\check{h}'^{-\alpha})\right).$$

We denote by $\Delta_{\Psi'}(\check{h}')$ the quantity

$$\Delta_{\Psi'}(\check{h}') = \check{h}'^{\frac{1}{2}\sum_{\alpha\in\Psi'}\alpha}\prod_{\alpha\in\Psi'}(1-\check{h}'^{-\alpha}), \qquad (\check{h}' \in \check{\mathrm{H}}'_{\mathrm{S}}),$$

and by $\Delta_{\Phi'}$ the function on $\check{\mathrm{H}}'^{\mathrm{reg}}_{\mathrm{S}}$ given by $\Delta_{\Phi'}(\check{h}') = \check{h}'^{\frac{1}{2}\sum_{\alpha\in\Phi'}\alpha}\prod_{\alpha\in\Phi'}(1-\check{h}'^{-\alpha})$.

Remark 5.2. For every $\check{h}' \in \check{\mathrm{H}}'^{\mathrm{reg}}_{\mathrm{S}}$, $\Delta_{\Phi'}(\check{h}')\Delta_{\Psi'}(\check{h}') = \prod_{\alpha\in\Psi'^+}(1-\check{h}'^{\alpha})(1-\check{h}'^{-\alpha}) = \prod_{\alpha\in\Psi'^+}(1-\check{h}'^{\alpha})\overline{(1-\check{h}'^{\alpha})}$. We denote by $|\Delta_{\mathrm{G}'}(\check{h}')|^2 = \Delta_{\Phi'}(\check{h}')\Delta_{\Psi'}(\check{h}')$.

Proposition 5.3 (Weyl's integration formula)**.** *For every $\varphi \in \mathscr{C}^{\infty}_c(\widetilde{\mathrm{G}'})$, we get:*

$$\int_{\widetilde{\mathrm{G}'}} \varphi(\tilde{g}')d\tilde{g}' = \sum_{\mathrm{S}\in\Psi'^{\mathrm{st}}_n} m_{\mathrm{S}} \int_{\check{\mathrm{H}}'_{\mathrm{S}}} \varepsilon_{\Psi'_{\mathrm{S},\mathbb{R}}}(\check{h}')\Delta_{\Phi'}(\check{h}')\mathscr{H}_{\mathrm{S}}\varphi(\check{h}')d\check{h}', \tag{6}$$

where m_{S} are complex numbers. Here, the subsets S *of Ψ'^{st}_n are defined up to equivalence (see Remark* B.2*).*

Proof. See [3, Section 2, Page 3830]. □

Remark 5.4. One can easily see that for every $\mathrm{S} \subseteq \Psi'^{\mathrm{st}}_n$ and $\varphi \in \mathscr{C}^{\infty}_c(\widetilde{\mathrm{G}'})$ such that $\mathrm{supp}(\varphi) \subseteq \widetilde{\mathrm{G}'} \cdot \widetilde{\mathrm{H}'}(\mathrm{S})^{\mathrm{reg}}$, Equation (6) can be written as follows:

$$\int_{\widetilde{\mathrm{G}'}} \varphi(\tilde{g}')d\tilde{g}' = m_{\mathrm{S}} \int_{\check{\mathrm{H}}'_{\mathrm{S}}} \varepsilon_{\Psi'_{\mathrm{S},\mathbb{R}}}(\check{h}')\Delta_{\Phi'}(\check{h}')\mathscr{H}_{\mathrm{S}}\varphi(\check{h}')d\check{h}'.$$

Let $\mathrm{W}^{\mathfrak{h}}$ be the set of elements of W commuting with $\mathfrak{h}$.

Remark 5.5. One can easily check that the space $\mathrm{W}^{\mathfrak{h}}$ is given by

$$\mathrm{W}^{\mathfrak{h}} = \sum_{i=1}^{n} \mathrm{Hom}_{\mathbb{C}}(\mathrm{V}'_i, \mathrm{V}_i).$$

For every $\mathrm{S} \subseteq \Psi_n'^{\mathrm{st}}$, we denote by $\underline{\mathrm{S}}$ the subset of $\{1, \dots, r+s\}$ given by

$$\underline{\mathrm{S}} = \left\{j, \exists \alpha \in \mathrm{S} \text{ such that } \alpha(\mathrm{J}_j') \neq 0\right\}.$$

Let $\sigma \in \mathscr{S}_{n'}$ and $\mathrm{S} \subseteq \Psi_n'^{\mathrm{st}}$, we denote by $\Gamma_{\sigma,\mathrm{S}}$ the subset of $\mathfrak{h}'$ defined as

$$\Gamma_{\sigma,\mathrm{S}} = \left\{Y \in \mathfrak{h}', \langle Y\cdot,\cdot\rangle_{\sigma \mathrm{W}^{\mathfrak{h}} \cap \sum_{j \notin \underline{\mathrm{S}}} \mathrm{Hom}(\mathrm{V}_j', \mathrm{V})} > 0\right\},$$

and let $\mathrm{E}_{\sigma,\mathrm{S}} = \exp(i\Gamma_{\sigma,\mathrm{S}})$ be the corresponding subset of $\mathrm{H}_{\mathbb{C}}'$, where $\exp : \mathfrak{h}_{\mathbb{C}}' \to \mathrm{H}_{\mathbb{C}}'$ is the exponential map.

Theorem 5.6. *For every* $\check{h} \in \check{\mathrm{H}} = \check{\mathrm{H}}_{\emptyset}$ *and* $\varphi \in \mathscr{C}(\widetilde{\mathrm{G}'})$, *we get:*

$$\det{}^{\frac{k}{2}}(\check{h})_{\mathrm{W}^{\mathfrak{h}}} \Delta_{\Psi}(\check{h}) \int_{\widetilde{\mathrm{G}'}} \Theta(\check{p}(\check{h})\tilde{g}')\varphi(\tilde{g}')d\tilde{g}'$$
$$= \sum_{\sigma \in \mathscr{W}(\mathrm{H}_{\mathbb{C}}')} \sum_{\mathrm{S} \subseteq \Psi_n'^{\mathrm{st}}} \mathrm{M}_{\mathrm{S}}(\sigma) \lim_{\substack{r \in \mathrm{E}_{\sigma,\mathrm{S}} \\ r \to 1}} \int_{\check{\mathrm{H}}_{\mathrm{S}}'} \frac{\det^{-\frac{k}{2}}(\sigma^{-1}(\check{h}'))_{\mathrm{W}^{\mathfrak{h}}} \Delta_{\Phi'(\mathrm{Z}')}(\sigma^{-1}(\check{h}'))}{\det(1 - p(\check{h})rp(\check{h}'))_{\sigma \mathrm{W}^{\mathfrak{h}}}}$$
$$\varepsilon_{\Phi_{\mathrm{S},\mathbb{R}}'}(\check{h}')\mathscr{H}_{\mathrm{S}}(\varphi)(\check{h}')d\check{h}',$$

where $\mathrm{M}_{\mathrm{S}}(\sigma) = \dfrac{(-1)^u \alpha \varepsilon(\sigma) m_{\mathrm{S}}}{|\mathscr{W}(\mathrm{Z}_{\mathbb{C}}', \mathrm{H}_{\mathbb{C}}')|}$, *u is a positive integer defined in* [3, Section 2], $\alpha \in \{1, -1\}$ *depends only on the choice of the positive roots* Ψ *and* Ψ' *(see* [3, Proposition 2.1]*) and* $k \in \{0, -1\}$ *is defined by*

$$k = \begin{cases} -1 & \textit{if } n' - n \in 2\mathbb{Z} \\ 0 & \textit{otherwise} \end{cases}.$$

Theorem 5.6 tells us how to compute $\mathrm{Chc}_{\tilde{h}}$ for an element $\tilde{h}$ in the compact Cartan $\widetilde{\mathrm{H}} = \widetilde{\mathrm{H}}(\emptyset)$. Using [2], it follows that the value of Chc on the other Cartan subgroups can be computed explicitly by knowing how to do it for the compact Cartan.

From now on, we assume that $p \leq q, r \leq s$ and $p \leq r$.

Notation 5.7. For every $t \in [|1, p|]$, we denote by S_t and S_t' the subsets of Ψ_n^{st} and $\Psi_n'^{\mathrm{st}}$ respectively given by

$$\mathrm{S}_t = \{e_1 - e_{p+1}, \dots, e_t - e_{p+t}\}, \qquad \mathrm{S}_t' = \left\{e_1' - e_{r+1}', \dots, e_t' - e_{r+t}'\right\},$$

where the linear forms e_k, e_h' have been introduced in Notation 5.1.

For every $t \in [|0, p|]$, we denote by $\mathrm{H}(\mathrm{S}_t)$ and $\mathrm{H}'(\mathrm{S}_t)$ the Cartan subgroups of G and G$'$ respectively and let $\mathrm{H}(\mathrm{S}_t) = \mathrm{T}(\mathrm{S}_t)\,\mathrm{A}(\mathrm{S}_t)$ (resp. $\mathrm{H}'(\mathrm{S}_t) = \mathrm{T}'(\mathrm{S}_t)\,\mathrm{A}'(\mathrm{S}_t)$) be the decompositions of $\mathrm{H}(\mathrm{S}_t)$ (resp. $\mathrm{H}'(\mathrm{S}_t)$) as in [38, Section 2.3.6].

As in Remark 4.1, we denote by $\mathrm{V}_{0,t}$ the subspace of V on which $\mathrm{A}(\mathrm{S}_t)$ acts trivially, by $\mathrm{V}_{1,t}$ the orthogonal complement of $\mathrm{V}_{0,t}$ in V and by $\mathrm{V}_{1,t} = \mathrm{X}_t \oplus \mathrm{Y}_t$ a complete polarization of $\mathrm{V}_{1,t}$. Because $p \leq r$, we have a natural embedding of $\mathrm{V}_{1,t}$ into V' such that $\mathrm{X}_t \oplus \mathrm{Y}_t$ is a complete polarization with respect to $(\cdot,\cdot)'$. We denote by U_t the orthogonal complement of $\mathrm{V}_{1,t}$ in V'; in particular, we get a natural embedding:

$$\mathrm{GL}(\mathrm{X}_t) \times \mathrm{G}(\mathrm{U}_t) \subseteq \mathrm{G}' = \mathrm{U}(r, s).$$

We denote by $\mathrm{T}_1(\mathrm{S}_t)$ the maximal subgroup of $\mathrm{T}(\mathrm{S}_t)$ which acts trivially on $\mathrm{V}_{0,t}$ and let $\mathrm{T}_2(\mathrm{S}_t)$ be the subgroup of $\mathrm{T}(\mathrm{S}_t)$ such that $\mathrm{T}(\mathrm{S}_t) = \mathrm{T}_1(\mathrm{S}_t) \times \mathrm{T}_2(\mathrm{S}_t)$ with $\mathrm{T}_2(\mathrm{S}_t) \subseteq \mathrm{G}(\mathrm{V}_{0,t})$. In particular,

$$\mathrm{H}(\mathrm{S}_t) = \mathrm{T}_1(\mathrm{S}_t) \times \mathrm{A}(\mathrm{S}_t) \times \mathrm{T}_2(\mathrm{S}_t). \tag{7}$$

Similarly, we get a decomposition of $\mathrm{H}'(\mathrm{S}'_t)$ of the form:

$$\mathrm{H}'(\mathrm{S}'_t) = \mathrm{T}'_1(\mathrm{S}'_t) \times \mathrm{A}'(\mathrm{S}'_t) \times \mathrm{T}'_2(\mathrm{S}'_t).$$

Remark 5.8. One can easily see that for every $0 \leq j < i \leq r$, we get:

$$\mathrm{H}'(\mathrm{S}'_i) = \mathrm{T}'_1(\mathrm{S}'_j) \times \mathrm{A}'(\mathrm{S}'_j) \times \mathrm{H}'(\widetilde{\mathrm{S}}'_{i-j}),$$

where $\widetilde{\mathrm{S}}'_{i-j} = \mathrm{S}'_i \setminus \mathrm{S}'_j$ and $\mathrm{H}'(\widetilde{\mathrm{S}}'_{i-j})$ is the Cartan subgroup of $\mathrm{U}(r-j, s-j)$ whose split part has dimension $i-j$. In particular,

$$\mathrm{H}'_{\mathrm{S}'_i} = \mathrm{T}'_{1,\mathrm{S}'_j} \times \mathrm{A}'_{\mathrm{S}'_j} \times \mathrm{H}'_{\widetilde{\mathrm{S}}'_{i-j}}. \tag{8}$$

Let $\eta(\mathrm{S}_t)$ and $\eta'(\mathrm{S}'_t)$ be the nilpotent Lie subalgebras of $\mathfrak{u}(p,q)$ and $\mathfrak{u}(r,s)$ respectively given by

$$\eta(\mathrm{S}'_t) = \mathrm{Hom}(\mathrm{X}_t, \mathrm{V}_{0,t}) \oplus \mathrm{Hom}(\mathrm{X}_t, \mathrm{Y}_t) \cap \mathfrak{u}(p,q),$$
$$\eta'(\mathrm{S}'_t) = \mathrm{Hom}(\mathrm{U}_t, \mathrm{X}_t) \oplus \mathrm{Hom}(\mathrm{X}_t, \mathrm{Y}_t) \cap \mathfrak{u}(r,s).$$

We will denote by $\mathrm{W}_{0,t}$ the subspace of W defined by $\mathrm{Hom}(\mathrm{U}_t, \mathrm{V}_{0,t})$ and by $\mathrm{P}(\mathrm{S}_t)$ and $\mathrm{P}'(\mathrm{S}'_t)$ the parabolic subgroups of G and G′ respectively whose Levi factors $\mathrm{L}(\mathrm{S}_t)$ and $\mathrm{L}'(\mathrm{S}'_t)$ are given by

$$\mathrm{L}(\mathrm{S}_t) = \mathrm{GL}(\mathrm{X}_t) \times \mathrm{G}(\mathrm{V}_{0,t}), \qquad \mathrm{L}'(\mathrm{S}'_t) = \mathrm{GL}(\mathrm{X}_t) \times \mathrm{G}(\mathrm{U}_t),$$

and by $\mathrm{N}(\mathrm{S}_t) := \exp(\eta(\mathrm{S}_t))$ and $\mathrm{N}'(\mathrm{S}'_t) := \exp(\eta'(\mathrm{S}'_t))$ the unipotent radicals of $\mathrm{P}(\mathrm{S}_t)$ and $\mathrm{P}'(\mathrm{S}'_t)$ respectively.

Remark 5.9. One can easily check that the forms on $\mathrm{V}_{0,t}$ and U_t have signature $(p-t, q-t)$ and $(r-t, s-t)$ respectively.

As proved in [2, Theorem 0.9], for every $\tilde{h} = \tilde{t}_1\tilde{a}\tilde{t}_2 \in \widetilde{\mathrm{H}}(\mathrm{S}_t)^{\mathrm{reg}}$ (using the decomposition of $\mathrm{H}(\mathrm{S}_t)$ given in Equation (7)) and $\varphi \in \mathscr{C}^\infty_c(\widetilde{\mathrm{G}'})$, we get:

$$|\det(\mathrm{Ad}(\tilde{h}) - \mathrm{Id})_{\eta(\mathrm{S}_t)}|\,\mathrm{Chc}_{\tilde{h}}(\varphi) = \mathrm{C}_t\, \mathrm{d}_{\mathrm{S}_t}(\tilde{h})\varepsilon(\tilde{t}_1\tilde{a}) \int_{\mathrm{GL}(\mathrm{X}_t)/\,\mathrm{T}_1(\mathrm{S}_t)\times\mathrm{A}(\mathrm{S}_t)} \int_{\widetilde{\mathrm{G}}(\mathrm{U}_t)} \varepsilon(\tilde{t}_1\tilde{a}\tilde{y})\,\mathrm{Chc}_{\mathrm{W}_{0,t}}(\tilde{t}_2\tilde{y})\,\mathrm{d}'_{\mathrm{S}_t}(g\tilde{t}_1\tilde{a}g^{-1}\tilde{y}) \varphi^{\widetilde{\mathrm{K}'}}_{\widetilde{\mathrm{N}'}(\mathrm{S}_t)}(g\tilde{t}_1\tilde{a}g^{-1}\tilde{y})d\tilde{y}\overline{dg}, \tag{9}$$

where C_t is the constant defined given by

$$\mathrm{C}_t = \left(\frac{2^{t(2(p+q+r+s)-4t+1)}}{(r+s)^t}\right)\left(\frac{r+s-2t}{r+s}\right)^{\frac{p+q-2t}{2}} \frac{1}{\mu(\mathrm{K}' \cap \mathrm{L}'(\mathrm{S}'_t))2^{t(r+s-2t)}}, \tag{10}$$

ε is the character defined in [2, Lemma 6.3], $\mathrm{d}_{\mathrm{S}_t} : \widetilde{\mathrm{L}}(\mathrm{S}_t) \to \mathbb{R}$ and $\mathrm{d}'_{\mathrm{S}'_t} : \widetilde{\mathrm{L}'}(\mathrm{S}'_t) \to \mathbb{R}$ are given by

$$\mathrm{d}_{\mathrm{S}_t}(\tilde{l}) = |\det(\mathrm{Ad}(\tilde{l})_{\eta(\mathrm{S}_t)})|^{\frac{1}{2}}, \qquad \mathrm{d}'_{\mathrm{S}'_t}(\tilde{l}') = |\det(\mathrm{Ad}(\tilde{l}')_{\eta'(\mathrm{S}_t)})|^{\frac{1}{2}},$$
$$\left(\tilde{l} \in \widetilde{\mathrm{L}}(\mathrm{S}_t), \tilde{l}' \in \widetilde{\mathrm{L}'}(\mathrm{S}'_t)\right),$$

and $\varphi^{\widetilde{\mathrm{K}'}}_{\widetilde{\mathrm{N}'(\mathrm{S}'_t)}}$ is the Harish-Chandra transform of φ, i.e. the function on $\widetilde{\mathrm{L}'(\mathrm{S}'_t)}$ defined by:

$$\varphi^{\widetilde{\mathrm{K}'}}_{\widetilde{\mathrm{N}'(\mathrm{S}'_t)}}(\tilde{l}') = \int_{\widetilde{\mathrm{N}'(\mathrm{S}'_t)}} \int_{\widetilde{\mathrm{K}'}} \varphi(\tilde{k}\tilde{l}'\tilde{n}\tilde{k}^{-1}) d\tilde{k} d\tilde{n}, \qquad \left(\tilde{l}' \in \widetilde{\mathrm{L}'(\mathrm{S}'_t)}\right).$$

Let's explain the method we will use in the next section to get character formulas of representations of $\mathrm{U}(n, n+1)$ by using the Cauchy–Harish-Chandra in the stable range.

Let $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(p), \mathrm{U}(r, s))$ be a dual pair in $\mathrm{Sp}(2p(r+s), \mathbb{R})$. To avoid any confusions, we will denote by $\omega^p_{r,s}$ the metaplectic representation of $\mathrm{Sp}(2p(r+s), \mathbb{R})$ and by $\theta^p_{r,s} : \mathscr{R}(\widetilde{\mathrm{U}}(p), \omega^p_{r,s}) \to \mathscr{R}(\widetilde{\mathrm{U}}(r, s), \omega^p_{r,s})$ the map defined in Equation (3).

Let $\Pi \in \mathscr{R}(\widetilde{\mathrm{U}}(p), \omega^p_{r,s})$. In this case, as explained in Remark 3.6, we get that $\Pi' = \Pi'_1$ and the corresponding representation Π' is an irreducible unitary representation of $\widetilde{\mathrm{U}}(r, s)$ (and $\Pi' \in \mathscr{R}(\widetilde{\mathrm{U}}(r, s)), \omega^p_{r,s})$. In particular, Theorem 5.6 tells us how to compute $\Theta_{\Pi'}$ on every Cartan subgroup of $\widetilde{\mathrm{U}}(r, s)$ (see Appendix A for $p = 1$).

For every $n \geq 0$, we denote by G_n the unitary group corresponding to a hermitian form of signature $(n, n+p)$, i.e. $\mathrm{G}_n = \mathrm{U}(n, p+n)$, by $\omega^{r,s}_{n,n+p}$ the metaplectic representation of $\widetilde{\mathrm{Sp}}(\mathrm{W}_n)$, where $\mathrm{W}_n = (\mathbb{C}^{r+s} \otimes_{\mathbb{C}} \mathbb{C}^{2n+p})_{\mathbb{R}}$ and by $\theta^{r,s}_{n,n+p} : \mathscr{R}(\widetilde{\mathrm{U}}(r, s), \omega^{r,s}_{n,n+p}) \to \mathscr{R}(\widetilde{\mathrm{U}}(n, n+p), \omega^{r,s}_{n,n+p})$ the map as in (3).

Remark 5.10. As explained in [29, Section 1.2], for the dual pair $(\mathrm{U}(a, b), \mathrm{U}(c, d))$ in $\mathrm{Sp}(2(a+b)(c+d), \mathbb{R})$, the double cover $\widetilde{\mathrm{U}}(a, b) \subseteq \widetilde{\mathrm{Sp}}(2(a+b)(c+d), \mathbb{R})$ of $\mathrm{U}(a, b)$ is isomorphic to the $\det^{c-d}$-cover, i.e.

$$\left\{(g, \xi) \in \mathrm{U}(a, b) \times \mathbb{C}^*, \xi^2 = \det(g)^{c-d}\right\}.$$

In particular, it follows that the double covers $\widetilde{\mathrm{U}}(p, q) \subseteq \widetilde{\mathrm{Sp}}(2p(r+s), \mathbb{R})$ and $\widetilde{\mathrm{U}}(p, q) \subseteq \widetilde{\mathrm{Sp}}(\mathrm{W}_n)$ are isomorphic.

Using Kudla's persistence principle (see [23] (p-adic case) or [36, Page 944], where the authors mentioned, without proof, that Kudla's persistence principle in the real case is still valid and follows easily from their results), we know that the representation $\Pi' \in \mathscr{R}(\widetilde{\mathrm{U}}(r, s), \omega^p_{r,s})$ satisfies $\theta^{n,p+n}_{r,s}(\Pi') \neq \{0\}$, i.e. $\Pi' \in \mathscr{R}(\widetilde{\mathrm{U}}(r, s), \omega^{r,s}_{n,n+p})$. We denote by Π^n_1 the corresponding representation of $\widetilde{\mathrm{G}_n}$ as in Section 3, by $\Pi^n \in \mathscr{R}(\widetilde{\mathrm{U}}(n, n+p), \omega^{r,s}_{n,n+p})$ its unique irreducible quotient, and by $\Theta_{\Pi^n_1}$ and Θ_{Π^n} the characters of Π^n_1 and Π^n respectively.

Using Theorems 3.2 and 4.3, we know that the $\widetilde{\mathrm{G}_n}$-invariant eigendistribution $\Theta'_{n,\Pi'} := \mathrm{Chc}^*(\Theta_{\Pi'})$ is given by a locally integrable function $\Theta'_{n,\Pi'}$ on $\widetilde{\mathrm{G}_n}$, analytic on $\widetilde{\mathrm{G}_n}^{\mathrm{reg}}$. Note that an explicit value of $\Theta'_{n,\Pi'}$ on every Cartan subgroup of $\widetilde{\mathrm{G}_n}$ can be obtained using Equation (4), Theorem 5.6 and Equation (5).

According to [25], if $n \geq r+s$, we get that $\Pi^n_1 = \Pi^n$ because Π' is unitary and by Remark 4.5, it follows that $\Theta'_{n,\Pi'} = \Theta_{\Pi^n}$.

In the next section, we are going to make Θ_{Π^n} explicit for $p = r = s = 1$.

Remark 5.11. Let $p = r = s = 1$ and $n \geq 2$. Since the pair $(\mathrm{U}(1, 1), \mathrm{U}(n, n+1))$ is in the stable range, it follows from [24] that the representations Π^n are unitary. Moreover, one can see that the representations Π^n are not in the discrete series of $\widetilde{\mathrm{G}_n}$ by using Paul's paper [30] on the first occurrence for unitary groups. Indeed, if we assume that Π^n is a discrete series representation, its first non-trivial lift will be Π', which is impossible by using [30, Proposition 3.4].

Finally, as explained in Proposition D.6, the representations Π^n are not highest weight modules. In particular, the character Θ_{Π^n} of Π^n cannot be obtained by using Enright's formula [8, Corollary 2.3].

6. Character formulas for some representations of $\mathrm{U}(n, n+1)$

We first start with the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(1), \mathrm{U}(1,1))$ in $\mathrm{Sp}(4, \mathbb{R})$. Because the set of irreducible genuine representations of $\widetilde{\mathrm{U}}(1)$ is isomorphic to $\mathbb{Z}$, the corresponding representation of $\mathscr{R}(\widetilde{\mathrm{U}}(1), \omega^1_{1,1})$ will be denoted by $\Pi_m, m \in \mathbb{Z}$ and let Π'_m be the corresponding representation of $\widetilde{\mathrm{G}'}$. Moreover, as explained in Section 5, for $n \geq 2$, the lift $\Pi^n = \theta^{1,1}_{n,n+1}(\Pi'_m)$ of Π'_m on $\widetilde{\mathrm{G}_n}$ is non-zero and its character Θ_{Π^n} is, up to a constant, equal to $\mathrm{Chc}^*(\Theta_{\Pi'_m})$. In this section, we are going to give an explicit formula for Θ_{Π^n} on every Cartan subgroup of $\widetilde{\mathrm{G}_n}$.

Remark 6.1. We denote by $\mathfrak{g}, \mathfrak{g}'$ and $\mathfrak{g}_n$ the Lie algebras of G, G' and G_n respectively. The Lie algebra $\mathfrak{g}'$ is given by

$$\begin{aligned}\mathfrak{g}' &= \left\{\begin{pmatrix} a & b \\ \overline{b} & d\end{pmatrix}, a, d \in i\mathbb{R}, b \in \mathbb{C}\right\} \\ &= \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix},\end{aligned}$$

and the two Cartan subgroups of G', up to conjugation, are of the form

$$\mathrm{H}' = \mathrm{H}'(\mathrm{S}'_0) = \{\mathrm{diag}(h_1, h_2), h_1, h_2 \in \mathrm{U}(1)\}, \tag{11}$$

$$\mathrm{H}'(\mathrm{S}'_1) = \exp\left(\mathbb{R}\begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right) = \left\{\begin{pmatrix} e^{i\theta}\,\mathrm{ch}(X) & \mathrm{sh}(X) \\ \mathrm{sh}(X) & e^{i\theta}\,\mathrm{ch}(X)\end{pmatrix}, \theta, X \in \mathbb{R}\right\},$$

where $\mathrm{S}'_0 = \{\emptyset\}$ and $\mathrm{S}'_1 = \{e_1 - e_2\}$ (see Appendix B). We denote by $(\mathrm{V}', (\cdot,\cdot))$ the skew-hermitian form corresponding to G' and by $(\mathrm{V}_n, (\cdot,\cdot)_n)$ the hermitian form corresponding to G_n. Let $\mathscr{B}_{\mathrm{V}'} = \{f'_1, f'_2\}$ be a basis of V' such that $\mathrm{Mat}_{\mathscr{B}'}(\cdot,\cdot)' = i\,\mathrm{Id}_{1,1}$. We have the following complete polarization of V'

$$\mathrm{V}' = \mathrm{X}'_1 \oplus \mathrm{Y}'_1, \qquad \mathrm{X}'_1 = \mathbb{C}(f'_1 + f'_2), \qquad \mathrm{Y}'_1 = \mathbb{C}(f'_1 - f'_2),$$

where both X'_1 and Y'_1 are H'_2-invariant. Let $\mathscr{B}_{\mathrm{V}_n} = \{f^n_1, \ldots, f^n_{2n+1}\}$ be a basis of V_n such that $\mathrm{Mat}_{\mathscr{B}_{\mathrm{V}_n}}(\cdot,\cdot)_n = \mathrm{Id}_{n,n+1}$.

We consider the embedding of V' onto V_n sending f'_1 onto f^n_1 and f'_2 onto f^n_{2n+1}. Obviously, $\mathrm{X}'_1 \oplus \mathrm{Y}'_1$ is a complete polarization of $\mathrm{V}' \subseteq \mathrm{V}_n$ with respect to $(\cdot,\cdot)_n$. We consider the subspace U_1 of V_n given by

$$\mathrm{V}_n = \mathrm{V}' \oplus \mathrm{U}_1, \qquad \mathrm{U}_1 = \mathrm{V}'^{\perp},$$

where $\mathrm{V}'^{\perp}$ is the orthogonal complement of V' in V_n with respect to $(\cdot,\cdot)_n$.

Let $\mathrm{G}(\mathrm{U}_1)$ be the group of isometries corresponding to the hermitian space $(\mathrm{U}_1, (\cdot,\cdot)_{n|_{\mathrm{U}_1}})$. Note that $\mathrm{G}(\mathrm{U}_1) \approx \mathrm{U}(n-1, n)$.

As explained in Appendix B, for every $0 \leq k \leq n$ and $\mathrm{S}_k = \{e_1 - e_{2n+1}, \ldots, e_k - e_{2n+2-k}\}$, we denote by $\mathrm{H}_n(\mathrm{S}_k)$ the Cartan subgroup of G_n whose split part is of dimension k and by $\mathrm{H}_{n,\mathrm{S}_k}$ the subgroup of $\mathrm{H}_n(\emptyset)_{\mathbb{C}} = \{h = \mathrm{diag}(h_1, \ldots, h_{2n+1}), h_i \in \mathbb{C}\}$ given by $\mathrm{H}_{n,\mathrm{S}_k} = c(\mathrm{S}_k)\,\mathrm{H}_n(\mathrm{S}_k)c(\mathrm{S}_k)^{-1}$, where $c(\mathrm{S}_k)$ is the Cayley transform corresponding to S_k (see Appendix B). We denote by $\mathrm{P}_n(\mathrm{S}_1)$ the parabolic subgroup of G_n whose Levi factor $\mathrm{L}_n(\mathrm{S}_1)$ is given by $\mathrm{L}_n(\mathrm{S}_1) = \mathrm{GL}(\mathrm{X}_1) \times \mathrm{G}(\mathrm{U}_1)$.

Lemma 6.2. *We get* $\mathrm{GL}(\mathrm{X}'_1) = \mathrm{H}'(\mathrm{S}'_1)$.

Proof. The Lie algebra of $\mathrm{GL}(\mathrm{X}'_1)$ is the set of matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathfrak{g}'$ such that:

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \qquad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \beta \\ -\beta \end{pmatrix}, \qquad (\alpha, \beta \in \mathbb{C}). \tag{12}$$

We first assume that $A \in \mathrm{End}(\mathrm{V}')$ satisfies the conditions of Equation (12). Then, we get:

$$\begin{cases} a+b & = \alpha, \\ c+d & = \alpha, \end{cases} \qquad \begin{cases} a-b & = \beta, \\ c-d & = -\beta. \end{cases}$$

In particular, $a+b = c+d$ and $a-b = -c+d$. Then, $a = d$ and $b = c$. In particular, if $A \in \mathfrak{g}'$, we get that $a \in i\mathbb{R}$ and $b \in \mathbb{R}$. In particular,

$$\mathrm{GL}(\mathrm{X}'_1) = \exp\left(\mathbb{R}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \exp(\mathfrak{h}'(\mathrm{S}'_1)) = \mathrm{H}'(\mathrm{S}'_1). \qquad \square$$

In this section, we are going to determine the value of the character Θ_{Π^n} on the $n+1$ different Cartan subgroups of G_n.

Notation 6.3. We denote by Δ_n the set of roots corresponding to $(\mathfrak{g}_n, \mathfrak{h}_n)$, where $\mathrm{H}_n = \mathrm{H}_n(\emptyset)$ is the compact Cartan of G_n, by Ψ_n a set of positive roots of Δ_n, by $\Phi_n = -\Psi_n$, by $\Psi^{\mathrm{st}}_n(n) = \{e_b - e_{2n+2-b}, 1 \leq b \leq n\}$ the corresponding set of strongly orthogonal roots of Ψ_n and by Z_n the subgroup of G_n defined by $\mathrm{Z}_n = \mathrm{G}^{\mathfrak{h}'}_n$, where $\mathfrak{h}' = \mathrm{Lie}(\mathrm{H}')$ is the Lie algebra of H' seen as a subspace of $\mathfrak{h}_n$.

We denote by $\eta'(\mathrm{S}'_1)$ the subspace of $\mathfrak{g}'$ defined by $\mathrm{Hom}(\mathrm{X}'_1, \mathrm{Y}'_1) \cap \mathfrak{g}'$ and by $\eta_n(\mathrm{S}_1)$ the subspace of $\mathfrak{g}_n = \mathrm{Lie}(\mathrm{G}_n)$ given by $\mathrm{Hom}(\mathrm{X}'_1, \mathrm{U}_1) \oplus \mathrm{Hom}(\mathrm{X}'_1, \mathrm{Y}'_1) \cap \mathfrak{g}_n$.

Remark 6.4. As explained in Remark 5.8, we get for every $k \geq 1$ that

$$\mathrm{H}_n(\mathrm{S}_k) = \mathrm{T}_1(\mathrm{S}_1) \times \mathrm{A}(\mathrm{S}_1) \times \mathrm{H}_{n-1}(\widetilde{\mathrm{S}}_{k-1}),$$

where $\widetilde{\mathrm{S}}_{k-1} = \mathrm{S}_k \setminus \{e_1 - e_{2n+1}\}$ and $\mathrm{H}_{n-1}(\widetilde{\mathrm{S}}_{k-1})$ is a Cartan subgroup of $\mathrm{G}(\mathrm{U}_1)$ whose split part is of dimension $k-1$. As in Equation (8), we get:

$$\mathrm{H}_{n,\mathrm{S}_k} = \mathrm{T}_{1,\mathrm{S}_1} \times \mathrm{A}_{\mathrm{S}_1} \times \mathrm{H}_{n-1,\widetilde{\mathrm{S}}_{k-1}}, \tag{13}$$

and we denote by $\check{\mathrm{T}}_{1,\mathrm{S}_1}$, $\check{\mathrm{A}}_{\mathrm{S}_1}$ and $\check{\mathrm{H}}_{n-1,\widetilde{\mathrm{S}}_{k-1}}$ the preimages of $\mathrm{T}_{1,\mathrm{S}_1}, \mathrm{A}_{\mathrm{S}_1}$ and $\mathrm{H}_{n-1,\widetilde{\mathrm{S}}_{k-1}}$ respectively via the map $\check{p} : \check{\mathrm{H}}_{n,\mathrm{S}_k} \to \mathrm{H}_{n,\mathrm{S}_k}$.

Using Equation (13), every element $h \in \mathrm{H}_{n,\mathrm{S}_k}$ can be written as $h = tah_1$ (where, by convention, $t = a = \mathrm{Id}$ and $h = h_1$ if $k = 0$). In particular, if $\check{h} \in p^{-1}(\{h\})$, $h = tah_1 \in \mathrm{H}_{n,\mathrm{S}_k}$, there exists $\check{t} \in p^{-1}(\{t\}), \check{a} \in p^{-1}(\{a\})$ and $\check{h}_1 \in p^{-1}(\{h_1\})$ such that $\check{h} = \check{t}\check{a}\check{h}_1$. Note that the decomposition of $\check{h}$ as $\check{h} = \check{t}\check{a}\check{h}_1$ is not unique.

Theorem 6.5. *For every $k \in [|0, n|]$ and $\check{h} = \check{t}\check{a}\check{h}_1 \in \check{\mathrm{H}}_{n,\mathrm{S}_k}$ as in Remark 6.4, we get, up to a constant, that:*

$$\begin{aligned}
&\Theta_{\Pi^n}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}) \\
&= \mathrm{A}\sum_{\sigma \in \mathscr{W}(\mathrm{H}_n^{\mathbb{C}})} \varepsilon(\sigma)\frac{\Delta_{\Phi(\mathrm{Z}_n)}(\sigma^{-1}(\check{h}))}{\Delta_{\Phi_n}(\check{h})} \lim_{\substack{r \in \mathrm{E}_{\sigma,\mathrm{S}_k} \\ r \to 1}} \int_{\check{\mathrm{H}}'} \frac{\overline{\Theta_{\Pi'_m}(\check{p}(\check{h}'))\Delta_{\Psi'}(\check{h}')}}{\det(1 - p(\check{h}')rp(\check{h}))_{\sigma \mathrm{W}^{\mathfrak{h}}}} d\check{h}' + \delta_{k,0}\,\mathrm{B} \\
&\frac{\overline{\Theta_{\Pi'_m}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})c(\mathrm{S}_1)^{-1})}|\Delta_{\Psi'}(\check{t}\check{a})|^2 \mathrm{d}_{\mathrm{S}'_1}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})c(\mathrm{S}_1)^{-1})\,\mathrm{d}'_{\mathrm{S}_1}(c(\mathrm{S}_k)\check{p}(\check{t}\check{a}\check{h}_1)c(\mathrm{S}_k)^{-1})}{\varepsilon(\widetilde{(-1)}c(\widetilde{\mathrm{S}}_{k-1})\check{p}(\check{h}_1)c(\widetilde{\mathrm{S}}_{k-1})^{-1})^{-1}|\det(\mathrm{Ad}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})c(\mathrm{S}_1)^{-1})^{-1} - 1)_{|_{\eta'(\mathrm{S}_1)}}|} \\
&\qquad\qquad \frac{\mathrm{D}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})|\Delta_{\mathrm{G}(\mathrm{U}_1)}(\check{h})|^2}{\mathrm{D}_1(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})|\Delta_{\mathrm{G}_n}(\check{h})|^2},
\end{aligned}$$

where $\delta_{k,0} = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{otherwise} \end{cases}$, D *and* D_1 *are functions on* $\widetilde{\mathrm{H}}_n(\mathrm{S}_k)$, $1 \leq k \leq n$, *given, for* $\tilde{h} \in \widetilde{\mathrm{H}}_n(\mathrm{S}_t)$, *by*

$$\mathrm{D}_1(\tilde{h}) = |\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h})^{-1})_{\mathfrak{l}_n(\mathrm{S}_1)/\mathfrak{h}_n(\mathrm{S}_1)}|^{\frac{1}{2}},\ \mathrm{D}(\tilde{h}) = |\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h})^{-1})_{\mathfrak{g}_n/\mathfrak{h}_n(\mathrm{S}_1)}|^{\frac{1}{2}},$$

and where A *and* B *are constants given by*

$$\mathrm{A} = \frac{(-1)^u}{2(2n-1)!}, \qquad \mathrm{B} = \frac{2^{4n+3}(2n-1)m_{\widetilde{\mathrm{S}}_{k-1}}}{2(2n+1)^2 m_{\mathrm{S}_k}}.$$

Remark 6.6.

(1) In Theorem 6.5, the value of the character Θ_{Π^n} is given up to a constant. As conjectured in [33] and proved in [34], this constant is equal to $\chi_{\Pi'_m}(\widetilde{-1})\Theta(\widetilde{-1})$, where $\chi_{\Pi'_m}$ is the central character of Π'_m and $\widetilde{-1}$ is in the preimage of -1 in $\widetilde{\mathrm{Sp}}(2(2n+1), \mathbb{R})$ (see [33, Page 301]).

(2) As mentioned in Remark 6.4, the decomposition of $\check{h}$ as $\check{h} = \check{t}\check{a}\check{h}_1$ is not unique. But one can see that it doesn't affect the formula given in Theorem 6.5. Assume for example that $\check{h} = \check{t}\check{a}\check{h}_1 = \check{t}_1\check{a}\check{h}_2$, where $\check{p}^{-1}(\{t\}) = \{\check{t}, \check{t}_1\}$, $\check{p}^{-1}(\{h\}) = \{\check{h}_1, \check{h}_2\}$. One can see that

$$\Theta_{\Pi'_m}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})c(\mathrm{S}_1)^{-1}) = -\Theta_{\Pi'_m}(c(\mathrm{S}_1)\check{p}(\check{t}_1\check{a})c(\mathrm{S}_1)^{-1}),$$

$$\varepsilon(\widetilde{(-1)}c(\widetilde{\mathrm{S}}_{k-1})\check{p}(\check{h}_1)c(\widetilde{\mathrm{S}}_{k-1})^{-1}) = -\varepsilon(\widetilde{(-1)}c(\widetilde{\mathrm{S}}_{k-1})\check{p}(\check{h}_2)c(\widetilde{\mathrm{S}}_{k-1})^{-1}),$$

and the other factors are not affected.

Before proving Theorem 6.5, we recall a lemma concerning orbital integrals.

Lemma 6.7. *For every $k \geq 1$, $\tilde{h} \in \widetilde{\mathrm{H}}_n(\mathrm{S}_t)^{\mathrm{reg}}$ and $\psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}_n})$, we get:*

$$\int_{\mathrm{G}_n\,/\,\mathrm{H}_n(\mathrm{S}_k)} \psi(g\tilde{h}g^{-1})\overline{dg} = \mathrm{C}_{n,1}\,\frac{\mathrm{D}_1(\tilde{h})}{\mathrm{D}(\tilde{h})}\int_{\mathrm{L}_1(\mathrm{S}_1)/\,\mathrm{H}_n(\mathrm{S}_k)} \psi^{\widetilde{\mathrm{K}_n}}_{\widetilde{\mathrm{N}}_n(\mathrm{S}_1)}(m\tilde{h}m^{-1})\overline{dm},$$

where K_n *is the maximal compact subgroup of* G_n, $\mathrm{N}_n(\mathrm{S}_1) = \exp(\eta_n(\mathrm{S}_1))$ *and* $\mathrm{C}_{n,1}$ *is the constant given by*

$$\mathrm{C}_{n,1} = \frac{1}{\mu(\mathrm{K}_n \cap \mathrm{L}_n(\mathrm{S}_1))\sqrt{2}^{\dim_{\mathbb{R}}(\eta_n(\mathrm{S}_1))}} = \frac{1}{\mu(\mathrm{K}_n \cap \mathrm{L}_n(\mathrm{S}_1))\sqrt{2}^{4n-1}}.$$

Proof. We see easily that for $k \geq 1$, $\mathrm{H}_n(\mathrm{S}_k)$ is a Cartan subgroup of $\mathrm{L}_n(\mathrm{S}_1)$, and then the result follows from [2, Corollary A.4]. ☐

Proof of Theorem 6.5. Fix $k \in [|0, n|]$ and $\psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}_n})$ such that $\mathrm{supp}(\psi) \subseteq \widetilde{\mathrm{G}_n} \cdot \widetilde{\mathrm{H}_n}(\mathrm{S}_k)$. Using Remark 5.4, it follows that:

(14)
$$\begin{aligned}
\Theta_{\Pi^n}(\psi) &= \int_{\widetilde{\mathrm{G}_n}} \Theta_{\Pi^n}(\tilde{g})\psi(\tilde{g})d\tilde{g} \\
&= m_{\mathrm{S}_k} \int_{\check{\mathrm{H}}_{n,\mathrm{S}_k}} \Theta_{\Pi^n}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})\varepsilon_{\Psi_{n,\mathrm{S}_k,\mathbb{R}}}(\check{h})\Delta_{\Phi(n)}(\check{h})\mathscr{H}_{\mathrm{S}_k}\psi(\check{h})d\check{h} \\
&= m_{\mathrm{S}_k} \int_{\check{\mathrm{H}}_{n,\mathrm{S}_k}} \Theta_{\Pi^n}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})|\Delta_{\mathrm{G}_n}(\check{h})|^2 \\
&\qquad \int_{\mathrm{G}_n\,/\,\mathrm{H}_n(\mathrm{S}_k)} \psi(gc(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}g^{-1})\overline{dg}d\check{h}.
\end{aligned}$$

According to Remark 4.5, the global character Θ_{Π^n} of Π^n is given by

$$\begin{aligned}
\Theta_{\Pi^n}(\psi) &= \frac{1}{2}\int_{\widetilde{\mathrm{H}'}} \Theta_{\Pi'_m}(\tilde{h}'_0)|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}'_0)^{-1})_{\mathfrak{g}'/\mathfrak{h}'}|\,\mathrm{Chc}_{\tilde{h}'_0}(\psi)d\tilde{h}'_0 \\
&\quad + \frac{1}{2}\int_{\widetilde{\mathrm{H}'(\mathrm{S}'_1)}} \Theta_{\Pi'_m}(\tilde{h}'_1)|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}'_1)^{-1})_{\mathfrak{g}'/\mathfrak{h}'(\mathrm{S}'_1)}|\,\mathrm{Chc}_{\tilde{h}'_1}(\psi)d\tilde{h}'_1,
\end{aligned}$$

where $\mathrm{H}', \mathrm{H}'(\mathrm{S}_1)$ are the two Cartan subgroups of G' (up to conjugation) defined in Equation (11). Using that $\mathrm{supp}(\psi) \subseteq \widetilde{\mathrm{G}_n} \cdot \widetilde{\mathrm{H}_n}(\mathrm{S}_k)$, we get from Theorem 5.6 and [3, Equation 8] that:

(15)
$$\begin{aligned}
&\int_{\widetilde{\mathrm{H}'}} \overline{\Theta_{\Pi'_m}(\tilde{h}'_0)}|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}'_0)^{-1})_{\mathfrak{g}'/\mathfrak{h}'}|\,\mathrm{Chc}_{\tilde{h}'_0}(\psi)d\tilde{h}'_0 \\
&= \int_{\widetilde{\mathrm{H}'}} \int_{\widetilde{\mathrm{G}_n}} \overline{\Theta_{\Pi'_m}(\tilde{h}'_0)}|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}'_0)^{-1})_{\mathfrak{g}'/\mathfrak{h}'}|^2\Theta(\widetilde{(-1)}\tilde{g}\tilde{h}'_0)\psi(\tilde{g})d\tilde{g}d\tilde{h}'_0 \\
&= \int_{\check{\mathrm{H}}'} \overline{\Theta_{\Pi'_m}(\check{p}(\check{h}'_0))\Delta_{\Psi'}(\check{h}'_0)} \left(\Delta_{\Psi'}(\check{h}'_0)\int_{\widetilde{\mathrm{G}_n}} \Theta(\check{p}(\check{h}'_0)\tilde{g})\psi(\tilde{g})d\tilde{g}\right) d\check{h}'_0 \\
&= \sum_{\sigma \in \mathscr{W}(\mathrm{H}_n^{\mathbb{C}})} \mathrm{M}_{\mathrm{S}_k}(\sigma) \lim_{\substack{r \in \mathrm{E}_{\sigma,\mathrm{S}_k} \\ r \to 1}} \int_{\check{\mathrm{H}}'} \overline{\Theta_{\Pi'_m}(\check{p}(\check{h}'_0))\Delta_{\Psi'}(\check{h}'_0)} \int_{\check{\mathrm{H}}^{\mathrm{reg}}_{n,\mathrm{S}_k}} \frac{\Delta_{\Psi_n}(\check{h})\Delta_{\Phi(\mathrm{Z}_n)}(\sigma^{-1}(\check{h}))}{\det(1 - p(\check{h}'_0)rp(\check{h}))_{\sigma\,\mathrm{W}^{\mathfrak{h}'}}} \\
&\int_{\mathrm{G}_n\,/\,\mathrm{H}_n(\mathrm{S}_k)} \psi(gc(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}g^{-1})\overline{dg}d\check{h}'_0.
\end{aligned}$$

Similarly, by using Equation (5), we get:

$$(16)\quad \int_{\widetilde{\mathrm{H}'}(\mathrm{S}'_1)} \overline{\Theta_{\Pi'_m}(\tilde{h}'_1)} |\det(\mathrm{Id}-\mathrm{Ad}(\tilde{h}'_1)^{-1})_{\mathfrak{g}'/\mathfrak{h}'(\mathrm{S}'_1)}|^2 \,\mathrm{Chc}_{\tilde{h}'_1}(\psi) d\tilde{h}'_1 =$$

$$\begin{cases} 0 & \text{if } \mathrm{S}=\emptyset \\ \mathrm{C}_1 \int_{\widetilde{\mathrm{H}'}(\mathrm{S}'_1)} \overline{\Theta_{\Pi'_m}(\tilde{h}'_1)} |\det(\mathrm{Id}-\mathrm{Ad}(\tilde{h}'_1)^{-1})_{\mathfrak{g}'/\mathfrak{h}'(\mathrm{S}'_1)} \\ \quad |\frac{|\det(\mathrm{Ad}(\tilde{h}'_1))_{|_{\eta'(\mathrm{S}'_1)}}|^{\frac{1}{2}}}{|\det(\mathrm{Ad}(\tilde{h}'_1)-1)_{|_{\eta'(\mathrm{S}'_1)}}|} \int_{\widetilde{\mathrm{G}}(\mathrm{U}_1)} |\det(\tilde{h}'_1\tilde{u})_{|_{\eta_n(\mathrm{S}_1)}}|^{\frac{1}{2}} \varepsilon(\widetilde{(-1)}\tilde{u}) & \text{otherwise} \\ \quad \psi^{\widetilde{\mathrm{K}}_n}_{\widetilde{\mathrm{N}}_n(\mathrm{S}_1)}(\tilde{h}'_1\tilde{u}) d\tilde{u} d\tilde{h}'_1 \end{cases}$$

where the constant C_1 defined in Equation (10) is given by

$$\mathrm{C}_1 = \frac{2^{4n+3}(2n-1)}{(2n+1)^2}\mathrm{C}_{n,1},$$

and $\mathrm{C}_{n,1}$ is defined in Lemma 6.7. Note that in this case the formula (5) is slightly simplified because $\mathrm{W}_{0,k}=\{0\}$, i.e. $\mathrm{Chc}_{\mathrm{W}_{0,k}}=1$.

In particular, the theorem follows for $k=0$, i.e. $\mathrm{S}_0=\{\emptyset\}$. From now on, we assume that $k\geq 1$, i.e. without loss of generality that $e_1-e_{2n+1}\in\mathrm{S}_k$. In this case, using Remark 5.8, we get:

$$\mathrm{H}_n(\mathrm{S}_k) = \mathrm{T}_1(\mathrm{S}_1)\times\mathrm{A}(\mathrm{S}_1)\times\mathrm{H}_{n-1}(\widetilde{\mathrm{S}}_{k-1}).$$

The Cartan subgroup $\mathrm{H}_n(\mathrm{S}_k)$ is included in the Levi $\mathrm{L}_n(\mathrm{S}_1)=\mathrm{GL}(\mathrm{X}'_1)\times\mathrm{G}(\mathrm{U}_1)$ of $\mathrm{P}_n(\mathrm{S}_1)$. In particular, using Lemma 6.7, Equation (15) can be written as:

$$\sum_{\sigma\in\mathscr{W}(\mathrm{H}_n^{\mathbb{C}})} \mathrm{M}_{\mathrm{S}_k}(\sigma)\,\mathrm{C}_{n,1} \lim_{\substack{r\in\mathrm{E}_{\sigma,\mathrm{S}_k}\\ r\to 1}} \int_{\check{\mathrm{H}}'} \overline{\Theta_{\Pi'_m}(\check{p}(\check{h}'_0))\Delta_{\Psi'}(\check{h}'_0)}$$
$$\int_{\check{\mathrm{H}}^{\mathrm{reg}}_{n,\mathrm{S}_k}} \frac{\Delta_{\Psi_n}(\check{h})\Delta_{\Phi(\mathrm{Z}_n)}(\sigma^{-1}(\check{h}))}{\det(1-p(\check{h}'_0)rp(\check{h}))_{\sigma\,\mathrm{W}^{\mathfrak{h}}}} \frac{\mathrm{D}_1(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})}{\mathrm{D}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})}$$
$$\int_{\mathrm{L}_n(\mathrm{S}_1)/\,\mathrm{H}_n(\mathrm{S}_k)} \psi^{\widetilde{\mathrm{K}}_n}_{\widetilde{\mathrm{N}}_n(\mathrm{S}_1)}(gc(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}g^{-1})\overline{dg}d\check{h}d\check{h}'_0.$$

Similarly, Equation (16) is equal to

$$(17)\quad \mathrm{C}_1\, m_{\widetilde{\mathrm{S}}_{k-1}} \int_{\check{\mathrm{T}}'_{1,\mathrm{S}_1}} \int_{\check{\mathrm{A}}'_{\mathrm{S}_1}} \overline{\Theta_{\Pi'_m}(c(\mathrm{S}_1)\check{p}(\check{t}'\check{a}')c(\mathrm{S}_1)^{-1})}$$
$$|\det(\mathrm{Id}-\mathrm{Ad}(c(\mathrm{S}_1)\check{p}(\check{t}'\check{a}')c(\mathrm{S}_1)^{-1})^{-1})_{\mathfrak{g}'/\mathfrak{h}'(\mathrm{S}_1)}| \frac{|\det(\mathrm{Ad}(c(\mathrm{S}_1)\check{p}(\check{t}'\check{a}')c(\mathrm{S}_1)^{-1}))_{|_{\eta'(\mathrm{S}_1)}}|^{\frac{1}{2}}}{|\det(\mathrm{Ad}(c(\mathrm{S}_1)\check{p}(\check{t}'\check{a}')c(\mathrm{S}_1)^{-1})-\mathrm{Id})_{|_{\eta'(\mathrm{S}_1)}}|}$$
$$\int_{\check{\mathrm{H}}_{n-1,\widetilde{\mathrm{S}}_{k-1}}} \int_{\mathrm{G}(\mathrm{U}_1)/\,\mathrm{H}_{n-1}(\widetilde{\mathrm{S}}_{k-1})} |\Delta_{\mathrm{G}(\mathrm{U}_1)}(\check{h})|^2$$
$$|\det(c(\mathrm{S}_1)\check{p}(\check{t}'\check{a}')c(\mathrm{S}_1)^{-1}gc(\widetilde{\mathrm{S}}_{k-1})\check{p}(\check{h})c(\widetilde{\mathrm{S}}_{k-1})^{-1}g^{-1})_{|_{\eta_n(\mathrm{S}_1)}}|^{\frac{1}{2}}\varepsilon(\widetilde{(-1)}c(\widetilde{\mathrm{S}}_{k-1})\check{p}(\check{h})c(\widetilde{\mathrm{S}}_{k-1})^{-1})$$
$$\psi^{\widetilde{\mathrm{K}_n}}_{\widetilde{\mathrm{N}}_n(\mathrm{S}_1)}(c(\mathrm{S}_1)\check{p}(\check{t}'\check{a}')c(\mathrm{S}_1)^{-1}gc(\widetilde{\mathrm{S}}_{t-1})\check{p}(\check{h})c(\widetilde{\mathrm{S}}_{t-1})^{-1}g^{-1})\overline{dg}d\check{h}d\check{a}'d\check{t}',$$

where $\widetilde{\mathrm{S}}_{k-1} = \mathrm{S}_k \setminus \{e_1 - e_{2n+1}\}$. From (14), we get:
(18)

$$\Theta_{\Pi^n}(\psi) = m_{\mathrm{S}_k}\,\mathrm{C}_{n,1} \int_{\check{\mathrm{H}}'_{n,\mathrm{S}_k}} \Theta_{\Pi^n}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}) |\Delta_{\mathrm{G}_n}(\check{h})|^2 \frac{\mathrm{D}_1(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})}{\mathrm{D}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})}$$

$$\int_{\mathrm{L}_n(\mathrm{S}_1)/\,\mathrm{H}_n(\mathrm{S}_k)} \psi(gc(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}g^{-1})\overline{dg}d\check{h}$$

and using that:

$$\begin{aligned}\mathrm{L}_n(\mathrm{S}_1)/\,\mathrm{H}_n(\mathrm{S}_k) &= \mathrm{GL}(\mathrm{X}_1)/(\mathrm{T}_1(\mathrm{S}_1)\times \mathrm{A}(\mathrm{S}_1)) \times \mathrm{G}(\mathrm{U}_1)/\,\mathrm{H}_{n-1}(\widetilde{\mathrm{S}}_{k-1}) \\ &= \mathrm{G}(\mathrm{U}_1)/\,\mathrm{H}_{n-1}(\widetilde{\mathrm{S}}_{k-1}),\end{aligned}$$

it follows from Equations (15), (17) and (18) that for every $\check{h} = \check{t}\check{a}\check{h}_1 \in \check{\mathrm{H}}_{n,\mathrm{S}_k} = \check{\mathrm{T}}_{1,\mathrm{S}_1} \times \check{\mathrm{A}}_{\mathrm{S}_1} \times \check{\mathrm{H}}_{\widetilde{\mathrm{S}}_{k-1}}$,

$$\begin{aligned}& m_{\mathrm{S}_k}\,\mathrm{C}_{n,1}\,\Theta_{\Pi^n}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})|\Delta_{\mathrm{G}_n}(\check{h})|^2 \frac{\mathrm{D}_1(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})}{\mathrm{D}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})} \\ &= \frac{\mathrm{C}_{n,1}(-1)^u m_{\mathrm{S}_k}}{2(2n-1)!} \frac{\Delta_{\Psi_n}(\check{h})\,\mathrm{D}_1(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})}{\mathrm{D}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})} \sum_{\sigma\in\mathscr{W}(\mathrm{H}_n^{\mathbb{C}})} \varepsilon(\sigma)\Delta_{\Phi(\mathrm{Z}_n)}(\sigma^{-1}(\check{h})) \\ &\qquad \lim_{\substack{r\in\mathrm{E}_{\sigma,\mathrm{S}_k}\\ r\to 1}} \int_{\check{\mathrm{H}}'} \frac{\overline{\Theta_{\Pi'_m}(\check{p}(\check{h}'_0))\Delta_{\Psi'}(\check{h}'_0)}}{\det(1-p(\check{h}'_0)rp(\check{h}))_{\sigma\,\mathrm{W}^\mathfrak{h}}} d\check{h}'_0 + \frac{m_{\widetilde{\mathrm{S}}_{k-1}}\,\mathrm{C}_1}{2} \\ & \frac{\overline{\Theta_{\Pi'_m}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})c(\mathrm{S}_1)^{-1})|\Delta_{\Psi'}(\check{t}\check{a})|^2}\,\mathrm{d}_{\mathrm{S}'_1}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})c(\mathrm{S}_1)^{-1})\,\mathrm{d}'_{\mathrm{S}_1}(c(\mathrm{S}_k)\check{p}(\check{t}\check{a}\check{h}_1)c(\mathrm{S}_k)^{-1})}{|\det(\mathrm{Ad}(c(\mathrm{S}_1)\check{p}(\check{t}'\check{a}')c(\mathrm{S}_1)^{-1})-\mathrm{Id})_{|_{\eta'(\mathrm{S}_1)}}|} \\ &\qquad \varepsilon(\widetilde{(-1)}c(\widetilde{\mathrm{S}}_{k-1})\check{p}(\check{h}_1)c(\widetilde{\mathrm{S}}_{k-1})))|\Delta_{\mathrm{G}(\mathrm{U}_1)}(\check{h})|^2,\end{aligned}$$

and the result follows. □

Lemma 6.8. *For every $\tilde{g} \in \widetilde{\mathrm{G}_n}$, $\varepsilon(\tilde{g}) = \pm 1$.*

Proof. The space $\mathrm{X}'_1 \otimes_{\mathbb{C}} \mathrm{V}_n \oplus \mathrm{Y}'_1 \otimes_{\mathbb{C}} \mathrm{V}_n$ is a complete polarization of $\mathrm{V}' \otimes_{\mathbb{C}} \mathrm{V}_n$. In particular, $\widetilde{\mathrm{X}'} = (\mathrm{X}'_1 \otimes \mathrm{V}_n)_{\mathbb{R}}$ is a maximal isotropic subspace of $\mathrm{W} = (\mathrm{V}' \otimes_{\mathbb{C}} \mathrm{V}_n)_{\mathbb{R}}$.

According to Equation (28), the character ε is defined on $\mathrm{GL}(\widetilde{\mathrm{X}'})^c$ by the following formula:

$$\varepsilon(\tilde{g}) = \frac{\Theta(\tilde{g})}{|\Theta(\tilde{g})|}, \qquad \left(g \in \mathrm{GL}(\widetilde{\mathrm{X}'})^c\right).$$

In particular, using Equation (29), for every $\tilde{g} \in \widetilde{\mathrm{G}}_n^c$, we get:

$$\varepsilon(\tilde{g}) = \frac{\det_{\widetilde{\mathrm{X}'}}(\tilde{g})^{-\frac{1}{2}}}{|\det_{\widetilde{\mathrm{X}'}}(\tilde{g})^{-\frac{1}{2}}|} = \pm\frac{|\det_{\mathrm{X}'}(g)|^{-1}}{|\det_{\mathrm{X}'}(g)|^{-1}}.$$

Using the fact that $|\det_{\mathrm{X}'}(g)| = 1$, it follows that $\varepsilon(\tilde{g}) = \pm 1$. □

As explained in Appendix B (see Equation (27)), for every $0 \leq k \leq n$ and $\mathrm{S}_t = \{e_1 - e_{2n+2-1}, \dots, e_k - e_{2n+2-k}\}$,

$$\text{(19)} \quad \mathrm{H}_{n,\mathrm{S}_k} = \{\mathrm{diag}(e^{iX_1 - X_{2n+1}}, \dots, e^{iX_k - X_{2n+2-k}}, e^{iX_{k+1}}, \dots, e^{iX_{2n+1-k}}, e^{iX_k + X_{2n+2-k}}, \dots, e^{iX_1 + X_{2n+1}}), X_j \in \mathbb{R}\}.$$

In particular, using Remark 6.4, we get that h can be written as $h = tah_1$, where

$$\text{(20)} \qquad t = \mathrm{diag}(e^{iX_1}, \underbrace{1, \dots, 1}_{2n-1}, e^{iX_1}), \qquad a = \mathrm{diag}(e^{-X_{2n+1}}, \underbrace{1, \dots, 1}_{2n-1}, e^{X_{2n+1}})$$

and

$$\text{(21)} \quad h_1 = \mathrm{diag}(1, e^{iX_2 - X_{2n}}, \dots, e^{iX_k - X_{2n+2-k}}, e^{iX_{k+1}}, \dots, e^{iX_{2n+1-k}}, e^{iX_k + X_{2n+2-k}}, \dots, e^{iX_2 + X_{2n}}, 1).$$

We get $\Phi(\mathrm{Z}_n) = \{e_i - e_j, 2 \leq i < j \leq 2n\}$. In particular, for every $\sigma \in \mathscr{S}_{2n+1}$ and $\check{h} \in \check{\mathrm{H}}_n(\mathrm{S}_k)$, we get:

$$\text{(22)} \quad \begin{aligned} \frac{\Delta_{\Phi(\mathrm{Z}_n)}(\sigma^{-1}(\check{h}))}{\Delta_{\Phi(n)}(\check{h})} &= \frac{\prod_{2 \leq i < j \leq 2n} \left(h_{\sigma(j)}^{\frac{1}{2}} h_{\sigma(i)}^{-\frac{1}{2}} - h_{\sigma(j)}^{-\frac{1}{2}} h_{\sigma(i)}^{\frac{1}{2}}\right)}{\prod_{1 \leq i < j \leq 2n+1} \left(h_i^{\frac{1}{2}} h_j^{-\frac{1}{2}} - h_i^{-\frac{1}{2}} h_j^{\frac{1}{2}}\right)} \\ &= \varepsilon(\sigma) \frac{h_{\sigma(1)}^n h_{\sigma(2n+1)}^n \prod_{\substack{i=1 \\ i \neq \sigma(1), \sigma(2n+1)}}^{2n+1} h_i}{\prod_{\substack{j=1 \\ j \neq \sigma(1)}}^{2n+1} \left(h_{\sigma(1)} - h_j\right) \prod_{\substack{j=1 \\ j \neq \sigma(1), \sigma(2n+1)}}^{2n+1} \left(h_{\sigma(2n+1)} - h_j\right)}. \end{aligned}$$

Remark 6.9. Let $\check{\exp} : \mathfrak{h}_{n,\mathrm{S}_k} \to \check{\mathrm{H}}_{n,\mathrm{S}_k}$ be the exponential map (lift of the usual exponential $\exp : \mathfrak{h}_{n,\mathrm{S}_k} \to \mathrm{H}_{n,\mathrm{S}_k}$). Let $e_i, 1 \leq i \leq 2n+1$ be the linear forms on $\mathbb{C}^{2n+1}$ defined in Notation 5.1. Using (20) and (21), for every element $\check{h} = \check{\exp}(X)$ with X given by

$$X = (X_1 - X_{2n+1}, iX_2 - X_{2n}, \dots, iX_k - X_{2n+2-k}, iX_{k+1}, \dots, iX_{2n+1-k}, iX_k + X_{2n+2-k}),$$

we get $\check{h}^{\frac{e_i - e_j}{2}} = e^{\frac{e_i - e_j}{2}(X)} = e^{\frac{e_i(X)}{2}} e^{-\frac{e_j(X)}{2}} := h_i^{\frac{1}{2}} h_j^{-\frac{1}{2}}$. This is how the square root in Equation (22) was defined.

Up to a sign, the quotient $\dfrac{\Delta_{\Phi(\mathrm{Z}_n)}(\sigma^{-1}(\check{h}))}{\Delta_{\Phi(n)}(\check{h})}$ is "uniquely" determined by $\sigma(1)$ and $\sigma(2n+1)$. For $\sigma \in \mathscr{S}_{2n+1}$ and $i, j \in \{1, \dots, 2n+1\}$ such that $\sigma(1) = i, \sigma(2n+1) = j$, we denote by $\Delta(i, j, \check{h})$ the following quantity:

$$\Delta(i, j, \check{h}) = \varepsilon(\sigma) \frac{\Delta_{\Phi(\mathrm{Z}_n)}(\sigma^{-1}(\check{h}))}{\Delta_{\Phi(n)}(\check{h})} = \frac{h_i^n h_j^n \prod_{\substack{k=1 \\ k \neq i,j}}^{2n+1} h_k}{\prod_{\substack{k=1 \\ k \neq i}}^{2n+1} (h_i - h_k) \prod_{\substack{l=1 \\ l \neq i,j}}^{2n+1} (h_j - h_l)}.$$

Lemma 6.10. *Let $b \in \mathbb{Z}$ and $a \in \mathbb{C}^* \setminus \mathrm{S}^1$. Then,*

$$\frac{1}{2i\pi} \int_{\mathrm{S}^1} \frac{z^b}{z-a} dz = \begin{cases} a^b & \text{if } k \geq 0 \text{ and } |a| < 1, \\ -a^b & \text{if } k < 0 \text{ and } |a| > 1, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by C_2 the constant $\frac{(-1)^u}{(p+q-2)!}$.

Notation 6.11. For an element $h \in \mathrm{H}^{\mathrm{reg}}_{n,\mathrm{S}_k}$ as in Equation (19), we denote by $\mathrm{J}(h)$ and $\mathrm{K}(h)$ the subsets of $\{1, \dots, k\}$ given by

$$J(h) = \{j \in \{1, \dots, k\}, \mathrm{sgn}(X_{2n+2-j}) = 1\},$$
$$K(h) = \{j \in \{1, \dots, k\}, \mathrm{sgn}(X_{2n+2-j}) = -1\},$$

where $\mathrm{sgn}(X)$ is defined for every $X \in \mathbb{R}^*$ by

$$\mathrm{sgn}(X) = \begin{cases} 1 & \text{if } X > 0 \\ -1 & \text{if } X < 0 \end{cases}.$$

To simplify the notations, we will denote by $(h_1, \dots, h_{2n+1})$ the components of h.

Finally, we denote by A_k and B_k the subsets of $\{1, \dots, 2n+1\}$ given by

$$\mathrm{A}_k = \{k+1, \dots, n\}, \qquad \mathrm{B}_k = \{n+1, \dots, 2n+1-k\}.$$

Theorem 6.12. *For every $0 \leq k \leq n$, the value of Θ_{Π^n} on $\widetilde{\mathrm{H}}_n(\mathrm{S}_k)^{\mathrm{reg}}$ is given by*

$$\Theta_{\Pi^n}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}) = \tag{23}$$

$$\mathrm{C}\begin{cases} \widetilde{\mathrm{A}} \sum_{\substack{j \in K(h) \cup \mathrm{A}_k \\ i \in J(h) \cup \mathrm{B}_k}} h_i^n h_j^{n+m} \Omega_{i,j}(h) + \delta_{k,0}\, \mathrm{B}\, e^{-(m+1)\,\mathrm{sgn}(X_{2n+1})X_{2n+1}} \Sigma(h) & \textit{if } m \geq 1, \\ \widetilde{\mathrm{A}} \sum_{\substack{i,j \in J(h) \cup \mathrm{B}_k \\ i \neq j}} h_i^n h_j^n \Omega_{i,j}(h) + \delta_{k,0}\, \mathrm{B}\, e^{-(m+1)\,\mathrm{sgn}(X_{2n+1})X_{2n+1}} \Sigma(h) & \textit{if } m = 0, \\ \widetilde{\mathrm{A}} \sum_{\substack{i \in K(h) \cup \mathrm{A}_k \\ j \in J(h) \cup \mathrm{B}_k}} h_i^{n+m} h_j^n \Omega_{i,j}(h) + \delta_{k,0}\, \mathrm{B}\, e^{(m-1)\,\mathrm{sgn}(X_{2n+1})X_{2n+1}} \Sigma(h) & \textit{if } m \leq -1, \end{cases}$$

where $\Omega_{i,j}, 1 \leq i \neq j \leq 2n+1$, and Σ are the functions on $\mathrm{H}^{\mathrm{reg}}_{n,\mathrm{S}_k}$ given by

$$\Omega_{i,j}(h) = \frac{\prod_{\substack{d=1 \\ d \neq i,j}}^{2n+1} h_d}{\prod_{\substack{d=1 \\ d \neq i}}^{2n+1} (h_i - h_d) \prod_{\substack{d=1 \\ d \neq i,j}}^{2n+1} (h_j - h_d)},$$

$$\Sigma(h) = \frac{\mathrm{sgn}(X_{2n+1}) e^{imX_1} \left| e^{(2n-2)X_{2n+1}} \right| \left(1 - e^{-2X_{2n+1}}\right)}{\left| \prod_{d=2}^{2n} \left(1 - h_1 h_d^{-1}\right) \prod_{d=2}^{2n} \left(1 - h_d h_{2n+1}^{-1}\right) \right| \left|1 - e^{-2X_{2n+1}}\right|^2},$$

and where $\check{h} \in \check{\mathrm{H}}_{n,\mathrm{S}_k}$ is as in Equation (19), $\widetilde{\mathrm{A}} = \dfrac{2\,\mathrm{A}}{(2n-1)!}$ *and* C *is the constant given in Remark* 6.6.

Proof. Let $h = tah_1 \in \mathrm{H}_{n,\mathrm{S}_k}$. We denote by $(h_1, \dots, h_{2n+1})$ the components of h. In particular,

$$h_c = \begin{cases} e^{iX_c - X_{2n+2-c}} & \text{if } 1 \leq c \leq k \\ e^{iX_c} & \text{if } k+1 \leq c \leq 2n+1-k \\ e^{iX_{2n+2-c} + X_c} & \text{if } 2n+2-k \leq c \leq 2n+1 \end{cases}.$$

We denote by $\Delta_n(\mathfrak{l}) := \Delta_n(\mathfrak{l}_n(\mathrm{S}_1))$ the set of roots of $\mathfrak{l}_n(\mathrm{S}_1)$ and let $\Psi_n(\mathfrak{l}) := \Delta_n(\mathfrak{l}) \cap \Psi_n$. One can easily check that

$$\Psi_n(\mathfrak{l}) = \{e_i - e_j, 2 \leq i < j \leq 2n\}.$$

Similarly, let $\Psi_n(\eta_n(\mathrm{S}_1)) = \{e_1 - e_d, 2 \leq d \leq 2n+1\} \cup \{e_d - e_{2n+1}, 2 \leq d \leq 2n\}$ the roots of $\eta_n(\mathrm{S}_1)$. Then,

$$\frac{\mathrm{D}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})}{\mathrm{D}_1(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})} = \frac{\left|\det(\mathrm{Id} - \mathrm{Ad}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})^{-1})_{\mathfrak{g}_n/\mathfrak{h}_n(\mathrm{S}_k)}\right|^{\frac{1}{2}}}{\left|\det(\mathrm{Id} - \mathrm{Ad}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})^{-1})_{\mathfrak{l}_n(\mathrm{S}_1)/\mathfrak{h}_n(\mathrm{S}_k)}\right|^{\frac{1}{2}}}$$

$$= \mathrm{d}_{\mathrm{S}_1}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})^{-2} \frac{\prod_{\alpha \in \Psi_n(\mathfrak{l})} \left|1 - \check{h}^\alpha\right|}{\prod_{\alpha \in \Psi_n} \left|1 - \check{h}^\alpha\right|}$$

$$= \mathrm{d}_{\mathrm{S}_1}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})^{-2} \prod_{\alpha \in \Psi_n(\eta_n(\mathrm{S}_1))} \left|1 - \check{h}^\alpha\right|$$

$$= \mathrm{d}_{\mathrm{S}_1}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})^{-2} \left|1 - h_1 h_{2n+1}^{-1}\right| \prod_{d=2}^{2n} \left|1 - h_1 h_d^{-1}\right| \prod_{d=2}^{2n} \left|1 - h_d h_{2n+1}^{-1}\right|.$$

Moreover,

$$\frac{\left|\Delta_{\mathrm{G}}(\check{h})\right|^2}{\left|\Delta_{\mathrm{G}(\mathrm{U}_1)}(\check{h})\right|^2} = \frac{\prod_{\alpha \in \Psi_n} \left|1 - \check{h}^\alpha\right|^2}{\prod_{\alpha \in \Psi(\mathfrak{g}(\mathrm{U}_1))} \left|1 - \check{h}^\alpha\right|^2}$$

$$= \left|1 - h_1 h_{2n+1}^{-1}\right|^2 \prod_{d=2}^{2n} \left|1 - h_1 h_d^{-1}\right|^2 \prod_{d=2}^{2n} \left|1 - h_d h_{2n+1}^{-1}\right|^2.$$

In particular,

$$\frac{\mathrm{D}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}) \left|\Delta_{\mathrm{G}(\mathrm{U}_1)}(\check{h})\right|^2}{\mathrm{D}_1(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}) \left|\Delta_{\mathrm{G}}(\check{h})\right|^2}$$

$$= \mathrm{d}_{\mathrm{S}_1}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})^{-2} \left|1 - e^{-2X_{2n+1}}\right|^{-1} \left|\prod_{d=2}^{2n} \left(1 - h_1 h_d^{-1}\right) \prod_{d=2}^{2n} \left(1 - h_d h_{2n+1}^{-1}\right)\right|^{-1}.$$

Similarly,

$$\left|\Delta_{\Psi'}(\check{t}\check{a})\right|^2 = \left|1 - h_1 h_{2n+1}^{-1}\right|^2 = \left|1 - e^{-2X_{2n+1}}\right|^2$$

and it follows from Remark A.5 that

$$\Theta_{\Pi'_m}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})c(\mathrm{S}_1)^{-1}) = \pm 2 \begin{cases} \mathrm{sgn}(X_{2n+1}) \dfrac{e^{-imX_1} e^{m\,\mathrm{sgn}(X_{2n+1})X_{2n+1}}}{e^{X_{2n+1}} - e^{-X_{2n+1}}} & \text{if } m \leq -1 \\ \mathrm{sgn}(X_{2n+1}) \dfrac{e^{-imX_1} e^{-m\,\mathrm{sgn}(X_{2n+1})X_{2n+1}}}{e^{X_{2n+1}} - e^{-X_{2n+1}}} & \text{if } m \geq 0 \end{cases},$$

i.e.

$$\left|\Delta_{\Psi'}(\check{t}\check{a})\right|^2 \overline{\Theta_{\Pi'_m}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})c(\mathrm{S}_1)^{-1})}$$

$$= \pm 2 \begin{cases} \mathrm{sgn}(X_{2n+1}) e^{imX_1} e^{(m-1)\,\mathrm{sgn}(X_{2n+1})X_{2n+1}} (1 - e^{-2X_{2n+1}}) & \text{if } m \leq -1 \\ \mathrm{sgn}(X_{2n+1}) e^{imX_1} e^{-(m+1)\,\mathrm{sgn}(X_{2n+1})X_{2n+1}} (1 - e^{-2X_{2n+1}}) & \text{if } m \geq 0 \end{cases}.$$

Finally, using that

$$\left|\det(\mathrm{Ad}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})\ c(\mathrm{S}_1)^{-1})^{-1} - 1)_{|_{\eta'(\mathrm{S}_1)}}\right|$$
$$= |1 - e^{-2X_{2n+1}}|, \quad \mathrm{d}_{\mathrm{S}'_1}(c(\mathrm{S}_1)\check{p}(\check{t}\check{a})c(\mathrm{S}_1)^{-1})$$
$$= \left|e^{-X_{2n+1}}\right|,$$

and

$$\mathrm{d}'_{\mathrm{S}_1}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1}) = \left|\prod_{j=2}^{2n+1}\left(e^{iX_1 - X_{2n+1}}h_j^{-1}\right)\prod_{j=2}^{2n}\left(h_j e^{-iX_1 - X_{2n+1}}\right)\right|^{\frac{1}{2}}$$
$$= \left|e^{-(2n-1)X_{2n+1}}\right|,$$

we get the second member of Equation (23).

We now look at the first member of Equation (23). One can easily check that for every $\sigma \in \mathscr{S}_{2n+1}$, $w = w_{1,1}\,\mathrm{E}_{1,1} + w_{2n+1,2}\,\mathrm{E}_{2n+1,2} \in \mathrm{W}^{\mathfrak{h}'}$ and $y \in \mathfrak{h}_n$ such that $y = (y_1, \ldots, y_{2n+1}) = (iX_1, \ldots, iX_{2n+1})$, we get:

$$\langle y\sigma(w), \sigma(w)\rangle = \begin{cases} X_{\sigma(1)}|w_{1,1}|^2 + X_{\sigma(2n+1)}|w_{2n+1,2}|^2 \\ \qquad \text{if } \sigma(1), \sigma(2n+1) \in \{1, \ldots, n\}, \\ -X_{\sigma(1)}|w_{1,1}|^2 - X_{\sigma(2n+1)}|w_{2n+1,2}|^2 \\ \qquad \text{if } \sigma(1), \sigma(2n+1) \in \{n+1, \ldots, 2n+1\}, \\ X_{\sigma(1)}|w_{1,1}|^2 - X_{\sigma(2n+1)}|w_{2n+1,2}|^2 \\ \qquad \text{if } \sigma(1) \in \{1, \ldots, n\} \text{ and } \sigma(2n+1) \in \{n+1, \ldots, 2n+1\}, \\ -X_{\sigma(1)}|w_{1,1}|^2 + X_{\sigma(2n+1)}|w_{2n+1,2}|^2 \\ \qquad \text{if } \sigma(1) \in \{n+1, \ldots, 2n+1\} \text{ and } \sigma(2n+1) \in \{1, \ldots, n\} \end{cases}$$

(see the proof of Proposition A.3 for an easier computation) and then

$$\Gamma_{\sigma,\mathrm{S}_k} = \begin{cases} \mathfrak{h}_n \\ \qquad \text{if } \{\sigma(1), \sigma(2n+1)\} \subseteq \underline{\mathrm{S}_k} \\ \{y = (y_1, \ldots, y_{2n+1}) \in \mathfrak{h}_n, X_{\sigma(2n+1)} > 0\} \\ \qquad \text{if } \sigma(1) \in \underline{\mathrm{S}_k} \text{ and } \sigma(2n+1) \in \mathrm{A}_k \\ \{y = (y_1, \ldots, y_{2n+1}) \in \mathfrak{h}_n, X_{\sigma(2n+1)} < 0\} \\ \qquad \text{if } \sigma(1) \in \underline{\mathrm{S}_k} \text{ and } \sigma(2n+1) \in \mathrm{B}_k \\ \{y = (y_1, \ldots, y_{2n+1}) \in \mathfrak{h}_n, X_{\sigma(1)} > 0\} \\ \qquad \text{if } \sigma(2n+1) \in \underline{\mathrm{S}_k} \text{ and } \sigma(1) \in \mathrm{A}_k \\ \{y = (y_1, \ldots, y_{2n+1}) \in \mathfrak{h}_n, X_{\sigma(1)} < 0\} \\ \qquad \text{if } \sigma(2n+1) \in \underline{\mathrm{S}_k} \text{ and } \sigma(1) \in \mathrm{B}_k \\ \{y = (y_1, \ldots, y_{2n+1}) \in \mathfrak{h}_n, X_{\sigma(1)} > 0, X_{\sigma(2n+1)} > 0\} \\ \qquad \text{if } \sigma(1), \sigma(2n+1) \in \mathrm{A}_k \\ \{y = (y_1, \ldots, y_{2n+1}) \in \mathfrak{h}_n, X_{\sigma(1)} > 0, X_{\sigma(2n+1)} < 0\} \\ \qquad \text{if } \sigma(1) \in \mathrm{A}_k, \sigma(2n+1) \in \mathrm{B}_k \\ \{y = (y_1, \ldots, y_{2n+1}) \in \mathfrak{h}_n, X_{\sigma(1)} < 0, X_{\sigma(2n+1)} > 0\} \\ \qquad \text{if } \sigma(2n+1) \in \mathrm{A}_k, \sigma(1) \in \mathrm{B}_k \\ \{y = (y_1, \ldots, y_{2n+1}) \in \mathfrak{h}_n, X_{\sigma(1)} < 0, X_{\sigma(2n+1)} < 0\} \\ \qquad \text{if } \sigma(1), \sigma(2n+1) \in \mathrm{B}_k\,. \end{cases}$$

Let $\check{h} \in \check{\mathrm{H}}_n(\mathrm{S}_k)$ and $h = p(\check{h}) = (h_1, \dots, h_{2n+1})$ as in Notation 6.11. Assume that $m \geq 1$. Using Corollary A.4, we get that

$$\Theta_{\Pi'_k}(\check{p}(\check{h}'_0))\Delta_{\Psi'}(\check{h}'_0) = h_1'^{\frac{1}{2}} h_2'^{\frac{1}{2}} \frac{h_2'^{-m}}{h'_2 - h'_1} h_1'^{-\frac{1}{2}} h_2'^{-\frac{1}{2}} (h'_1 - h'_2) = -h_2'^{-m}, \qquad (h'_0 = \mathrm{diag}(h'_1, h'_2)).$$

By keeping the normalizations of [27, Section 5] and identifying H' with $\mathrm{S}^1 \times \mathrm{S}^1$, we get:

$$\begin{aligned}\sum_{\sigma \in \mathscr{W}(\mathrm{H}_n^{\mathbb{C}})} \varepsilon(\sigma) \frac{\Delta_{\Phi(\mathrm{Z}_n)}(\sigma(\check{h}))}{\Delta_{\Phi(n)}(\check{h})} \lim_{\substack{r \in \mathrm{E}_{\sigma, \mathrm{S}_k} \\ r \to 1}} \int_{\check{\mathrm{H}}'} \frac{\overline{\Theta_{\Pi'_m}(\check{p}(\check{h}'_0))\Delta(\check{p}(\check{h}'_0))}}{\det(1 - p(\check{h}'_0) r p(\check{h}))_{\sigma \mathrm{W}^{\mathfrak{h}}}} d\check{h}'_0 \\ = 2 \sum_{\sigma \in \mathscr{W}(\mathrm{H}_n^{\mathbb{C}})} \varepsilon(\sigma) \frac{\Delta_{\Phi(\mathrm{Z}_n)}(\sigma(\check{h}))}{\Delta_{\Phi(n)}(\check{h})} \\ \lim_{\substack{r \in \mathrm{E}_{\sigma, \mathrm{S}_k} \\ r \to 1}} \int_{\mathrm{H}'} \frac{-\, h_2'^{m}}{(1 - h'_1 (rh)^{-1}_{\sigma(1)})(1 - h'_2 (rh)^{-1}_{\sigma(2n+1)})} dh'_1 dh'_2,\end{aligned}$$

where $d\check{h}'_0$ and $dh'_1 dh'_2$ are the normalized Haar measures on $\check{\mathrm{H}}'$ and H' respectively. Using that

$$dh'_1 dh'_2 = \frac{dz_1 dz_2}{(2i\pi)^2 z_1 z_2},$$

we get:

$$\begin{aligned}&\sum_{\sigma \in \mathscr{W}(\mathrm{H}_n^{\mathbb{C}})} \varepsilon(\sigma) \frac{\Delta_{\Phi(\mathrm{Z}_n)}(\sigma(\check{h}))}{\Delta_{\Phi(n)}(\check{h})} \lim_{\substack{r \in \mathrm{E}_{\sigma, \mathrm{S}_k} \\ r \to 1}} \int_{\check{\mathrm{H}}'} \frac{\overline{\Theta_{\Pi'_m}(\check{p}(\check{h}'_0))\Delta(\check{p}(\check{h}'_0))}}{\det(1 - p(\check{h}'_0) r p(\check{h}))_{\sigma \mathrm{W}^{\mathfrak{h}}}} d\check{h}'_0 \\ &= \frac{-2}{(2i\pi)^2} \sum_{\sigma \in \mathscr{S}_{2n+1}} \varepsilon(\sigma) \frac{h_{\sigma(1)} h_{\sigma(2n+1)} \Delta_{\Phi(\mathrm{Z}_n)}(\sigma(\check{h}))}{\Delta_{\Phi(n)}(\check{h})} \\ &\quad \lim_{\substack{r \in \mathrm{E}_{\sigma, \mathrm{S}_k} \\ r \to 1}} \int_{\mathrm{S}^1} \frac{z_1^{-1} dz_1}{z_1 - r h_{\sigma(1)}} \int_{\mathrm{S}^1} \frac{z_2^{m-1} dz_2}{z_2 - r h_{\sigma(2n+1)}} \\ &= \frac{-2}{(2i\pi)^2 (2n-1)!} \sum_{1 \leq i \neq j \leq 2n+1} h_i h_j \Delta(i, j, \check{h}) \lim_{\substack{r \in \mathrm{E}_{\sigma, \mathrm{S}_k} \\ r \to 1}} \int_{\mathrm{S}^1} \frac{z_1^{-1} dz_1}{z_1 - r h_i} \int_{\mathrm{S}^1} \frac{z_2^{m-1} dz_2}{z_2 - r h_j} \\ &= \frac{2}{(2n-1)!} \sum_{i \in J(h) \cup \mathrm{B}_k} \sum_{j \in K(h) \cup \mathrm{A}_k} h_i h_j \Delta(i, j, \check{h}) h_i^{-1} h_j^{m-1} \\ &= \frac{2}{(2n-1)!} \sum_{i \in J(h) \cup \mathrm{B}_k} \sum_{j \in K(h) \cup \mathrm{A}_k} \frac{h_i^n h_j^{n+m} \prod_{\substack{d=1 \\ d \neq i,j}}^{2n+1} h_d}{\prod_{\substack{d=1 \\ d \neq i}}^{2n+1} (h_i - h_d) \prod_{\substack{d=1 \\ d \neq i,j}}^{2n+1} (h_j - h_d)}.\end{aligned}$$

Similarly, if $m = 0$, we get:

$$\sum_{\sigma\in\mathscr{W}(\mathrm{H}_n^{\mathbb{C}})}\varepsilon(\sigma)\frac{\Delta_{\Phi(\mathrm{Z}_n)}(\sigma(\check{h}))}{\Delta_{\Phi(n)}(\check{h})}\lim_{\substack{r\in\mathrm{E}_{\sigma,\mathrm{S}_k}\\ r\to 1}}\int_{\check{\mathrm{H}}'}\frac{\overline{\Theta_{\Pi'_m}(\check{p}(\check{h}'_0))\Delta_{\Psi'}(\check{h}'_0)}}{\det(1-p(\check{h}'_0)p(\check{h}))_{\sigma\,\mathrm{W}^{\mathfrak{h}}}}d\check{h}'_1$$
$$= \frac{2}{(2i\pi)^2(2n-1)!}\sum_{1\leq i\neq j\leq 2n+1}h_ih_j\Delta(i,j,\check{h})\lim_{\substack{r\in\mathrm{E}_{\sigma,\mathrm{S}_k}\\ r\to 1}}\int_{\mathrm{S}^1}\frac{z_1^{-1}dz_1}{z_1-rh_i}\int_{\mathrm{S}^1}\frac{z_2^{-1}dz_2}{z_2-rh_j}$$
$$= \frac{2}{(2n-1)!}\sum_{\substack{i,j\in J(h)\cup\mathrm{B}_k\\ i\neq j}}h_ih_j\Delta(i,j,\check{h})h_i^{-1}h_j^{-1}$$
$$= \frac{1}{(2n-1)!}\sum_{\substack{i,j\in J(h)\cup\mathrm{B}_k\\ i\neq j}}\frac{h_i^nh_j^n\prod_{\substack{d=1\\ d\neq i,j}}^{2n+1}h_d}{\prod_{\substack{d=1\\ d\neq i}}^{2n+1}(h_i-h_d)\prod_{\substack{d=1\\ d\neq i,j}}^{2n+1}(h_j-h_d)}.$$

Finally, if $m \leq -1$, it follows from Corollary A.4 that:

$$\Theta_{\Pi'_m}(\check{p}(\check{h}'_0))\Delta(\check{p}(\check{h}'_0)) = h_1'^{\frac{1}{2}}h_2'^{\frac{1}{2}}\frac{h_1'^{-m}}{h'_2-h'_1}h_1'^{-\frac{1}{2}}h_2'^{-\frac{1}{2}}(h'_1-h'_2) = h_1'^{-m},$$
$$(h'_0 = (h'_1, h'_2))\,.$$

Then,

$$\sum_{\sigma\in\mathscr{W}(\mathrm{H}_n^{\mathbb{C}})}\varepsilon(\sigma)\frac{\Delta_{\Phi(\mathrm{Z}_n)}(\sigma(\check{h}))}{\Delta_{\Phi(n)}(\check{h})}\lim_{\substack{r\in\mathrm{E}_{\sigma,\mathrm{S}_k}\\ r\to 1}}\int_{\check{\mathrm{H}}'}\frac{\overline{\Theta_{\Pi'_m}(\check{p}(\check{h}'_0))\Delta(\check{h}'_0)}}{\det(1-p(\check{h}'_0)p(\check{h}))_{\sigma\,\mathrm{W}^{\mathfrak{h}}}}d\check{h}'_0$$
$$= \frac{2}{(2i\pi)^2}\sum_{\sigma\in\mathscr{S}_{2n+1}}\varepsilon(\sigma)\frac{h_{\sigma(1)}h_{\sigma(2n+1)}\Delta_{\Phi(\mathrm{Z}(n))}(\sigma(\check{h}))}{\Delta_{\Phi(n)}(\check{h})}$$
$$\lim_{\substack{r\in\mathrm{E}_{\sigma,\mathrm{S}_k}\\ r\to 1}}\int_{\mathrm{S}^1}\frac{z_1^{m-1}dz_1}{z_1-rh_{\sigma(1)}}\int_{\mathrm{S}^1}\frac{z_2^{-1}dz_2}{z_2-rh_{\sigma(2n+1)}}$$
$$= \frac{2}{(2n-1)!}\sum_{i\in K(h)\cup\mathrm{A}_k}\sum_{j\in J(h)\cup\mathrm{B}_k}h_ih_j\Delta(i,j,\check{h})h_i^{k-1}h_j^{-1}$$
$$= \frac{2}{(2n-1)!}\sum_{i\in K(h)\cup\mathrm{A}_k}\sum_{j\in J(h)\cup\mathrm{B}_k}\frac{h_i^{n+m}h_j^n\prod_{\substack{d=1\\ d\neq i,j}}^{2n+1}h_d}{\prod_{\substack{d=1\\ d\neq i}}^{2n+1}(h_i-h_d)\prod_{\substack{d=1\\ d\neq i,j}}^{2n+1}(h_j-h_d)},$$

and the theorem follows. □

Remark 6.13. From Theorem 6.12, it follows that the value of Θ_{Π_n} on the compact Cartan H_n is given, up to a constant, by

$$\Theta_{\Pi^n}(c(\mathrm{S}_k)\check{p}(\check{h})c(\mathrm{S}_k)^{-1})$$
$$= \begin{cases} \sum_{\substack{j\in K(h)\cup \mathrm{A}_k \\ i\in J(h)\cup \mathrm{B}_k}} h_i^n h_j^{n+m}\Omega_{i,j}(h) & \text{if } m\geq 1, \\ \sum_{\substack{i,j\in J(h)\cup \mathrm{B}_k \\ i\neq j}} h_i^n h_j^n \Omega_{i,j}(h) & \text{if } m = 0, \\ \sum_{\substack{i\in K(h)\cup \mathrm{A}_k \\ j\in J(h)\cup \mathrm{B}_k}} h_i^{n+m} h_j^n \Omega_{i,j}(h) & \text{if } m\leq -1, \end{cases}$$
$$= \begin{cases} \sum_{j=1}^{n}\sum_{i=n+1}^{2n+1} \dfrac{h^{\lambda_{i,j}}}{\prod_{\substack{d=1\\ d\neq i}}^{2n+1}(h_i-h_d)\prod_{\substack{d=1\\ d\neq i,j}}^{2n+1}(h_j-h_d)} & \text{if } m\geq 1, \\ \sum_{\substack{i,j=n+1\\ i\neq j}}^{2n+1} \dfrac{h^{\lambda_{i,j}}}{\prod_{\substack{d=1\\ d\neq i}}^{2n+1}(h_i-h_d)\prod_{\substack{d=1\\ d\neq i,j}}^{2n+1}(h_j-h_d)} & \text{if } m = 0, \\ \sum_{i=1}^{n}\sum_{j=n+1}^{2n+1} \dfrac{h^{\tau_{i,j}\lambda_{i,j}}}{\prod_{\substack{d=1\\ d\neq i}}^{2n+1}(h_i-h_d)\prod_{\substack{d=1\\ d\neq i,j}}^{2n+1}(h_j-h_d)} & \text{if } m\leq -1, \end{cases}$$

where $\lambda_{i,j}$ is the linear form given by $\lambda_{i,j} = ne_i + (n+m)e_j + \sum_{\substack{d=1\\ d\neq i,j}}^{2n+1} e_d$ and $\tau_{i,j}$ is the permutation (i,j).

We finish this section with a Lemma concerning the formulas we got in Theorem 6.12.

Lemma 6.14. *For every* $h \in \mathrm{H}_{n,\mathrm{S}_k}$*,* $\prod_{d=2}^{2n}\left(1-h_1h_d^{-1}\right)\prod_{d=2}^{2n}\left(1-h_dh_{2n+1}^{-1}\right) \in \mathbb{R}$*, and its sign is constant on every Weyl chamber.*

Proof. This result was obtained in [27, Lemma 6.9] for $k = 1$. We prove this lemma for $k = 2$ (the proof of the general statement is similar). Assume that $k = 2$. We get:

$$\prod_{d=2}^{2n}\left(1-h_1h_d^{-1}\right)\prod_{d=2}^{2n}\left(1-h_dh_{2n+1}^{-1}\right)$$
$$= (1-h_1h_2^{-1})(1-h_1h_{2n}^{-1})(1-h_2h_{2n+1}^{-1})(1-h_{2n}h_{2n+1}^{-1})$$
$$\left(\prod_{d=3}^{2n-1}\left(1-h_1h_d^{-1}\right)\left(1-h_dh_{2n+1}^{-1}\right)\right).$$

Firstly,

$$\left(\prod_{j=3}^{2n-1}\left(1-h_1h_j^{-1}\right)\left(1-h_jh_{2n+1}^{-1}\right)\right)$$
$$= \prod_{j=3}^{2n-1}\left(1-e^{iX_1-X_{2n+1}}e^{-iX_j}\right)\left(1-e^{iX_j}e^{-iX_1-X_{2n+1}}\right)$$
$$= \prod_{j=3}^{2n-1}\left|1-e^{iX_1-X_{2n+1}}e^{-iX_j}\right|^2.$$

Moreover,

$$\begin{aligned}(1 - h_1 h_{2n}^{-1})&(1 - h_{2n} h_{2n+1}^{-1})\\ &= (1 - e^{iX_1 - X_{2n+1}} e^{-iX_2 + X_{2n}})(1 - e^{iX_2 + X_{2n}} e^{-iX_1 - X_{2n+1}})\\ &= 1 - 2\cos(X_1 - X_2) e^{X_{2n} - X_{2n+1}} + e^{2(X_{2n} - X_{2n+1})}\end{aligned}$$

and

$$\begin{aligned}(1 - h_2 h_{2n+1}^{-1})&(1 - h_1 h_{2n}^{-1})\\ &= (1 - e^{iX_2 - X_{2n}} e^{-iX_1 + X_{2n+1}})(1 - e^{iX_1 + X_{2n+1}} e^{-iX_2 - X_{2n}})\\ &= 1 - 2\cos(X_2 - X_1) e^{X_{2n+1} - X_{2n+1}} + e^{2(X_{2n+1} - X_{2n})},\end{aligned}$$

so the lemma follows. □

Appendix A. The dual pair $(\mathrm{G} = \mathrm{U}(1), \mathrm{G}' = \mathrm{U}(p, q))$

In [27, Proposition 6.4], the author gave explicit formulas for the value of the character $\Theta_{\Pi'}$ on the compact Cartan $\widetilde{\mathrm{H}'} = \widetilde{\mathrm{H}'}(\emptyset)$ of $\widetilde{\mathrm{G}}'$. Moreover, for $p = q = 1$, he computed the character $\Theta_{\Pi'}$ on $\widetilde{\mathrm{H}}'(\mathrm{S}_1)$, where $\mathrm{H}'(\mathrm{S}_1)$ is the non-compact Cartan subgroup of $\widetilde{\mathrm{U}}(1,1)$ as in Equation (11). In this section, we recover the results proved in [27] using the results of Section 5 and get formulas for $\Theta_{\Pi'}$ on every Cartan subgroup of $\widetilde{\mathrm{G}'}$.

By keeping the notations of Section 5, we get $\mathrm{V} = \mathbb{C}$ with the hermitian form $(\cdot, \cdot)$ given by

$$(u, v) = u\overline{v}, \qquad (u, v \in \mathrm{V}),$$

$\mathrm{V}' = \mathrm{M}_{n',1}(\mathbb{C})$, where $n' = p + q$, with the skew-hermitian form $(\cdot, \cdot)'$ given by:

$$(u, v) = \overline{v}^t i \,\mathrm{Id}_{p,q}\, u,$$

and $\mathrm{W} = \mathrm{V} \otimes_{\mathbb{C}} \mathrm{V}'$ the symplectic space defined by

$$\langle w, w' \rangle = \mathrm{Re}(\mathrm{tr}(w'^* w)) = \mathrm{Im}(\overline{w'}^t \,\mathrm{Id}_{p,q}\, w).$$

Similarly,

$$\mathrm{H} = \mathrm{G} = \mathrm{U}(1) = \{h \in \mathbb{C}, |h| = 1\}, \qquad \mathrm{J}_1 = i, \qquad \mathfrak{h} = \mathbb{R}\,\mathrm{J}_1,$$

and the group $\mathrm{GL}_{\mathbb{C}}(\mathrm{W})$ is given by:

$$\mathrm{GL}_{\mathbb{C}}(\mathrm{W}) = \left\{g \in \mathrm{GL}(\mathrm{W}),\ \mathrm{J}'_1\, g = g\, \mathrm{J}'_1\right\} = \mathrm{G}'_{\mathbb{C}},$$

$$\widetilde{\mathrm{GL}}_{\mathbb{C}}(\mathrm{W}) = \left\{\tilde{g} = (g, \xi), g \in \mathrm{GL}_{\mathbb{C}}(\mathrm{W}), \det(g) = \xi^2\right\}.$$

Using that $\mathrm{V}' = \mathrm{V}'_1 \oplus \ldots \oplus \mathrm{V}'_{n'}$, with $\mathrm{V}'_j = \mathbb{C}\,\mathrm{E}_{j,1}$, and the embedding

$$\mathfrak{h}_{\mathbb{C}} \ni \lambda \to (\lambda, 0, \ldots, 0) \in \mathfrak{h}'_{\mathbb{C}},$$

we get that

$$\mathrm{W}^{\mathfrak{h}} = \{w = (w_{1,1}, 0, 0, \ldots, 0),\ w_{1,1} \in \mathbb{C}\},$$

$$\mathrm{Z}' = \mathrm{G}'^{\mathfrak{h}} = \left\{g' \in \mathrm{G}', g' = \begin{pmatrix}\lambda & 0\\ 0 & X\end{pmatrix}, \lambda \in \mathrm{U}(1), X \in \mathrm{GL}(n'-1, \mathbb{C})\right\}.$$

In particular, $\Phi'(\mathrm{Z}') = \{\pm(e_i - e_j), 2 \leq i < j \leq n'\}$, where e_j is the form defined in Notation 5.1. For every $\check{h}' \in \check{\mathrm{H}}'_{\mathbb{C}}$, with $h' = (h'_1, \dots, h'_{n'})$, we get:

$$\Delta_{\Phi'}(\check{h}') = \prod_{\alpha>0}(\check{h}'^{\frac{\alpha}{2}} - \check{h}'^{-\frac{\alpha}{2}}) = \prod_{1\leq i<j\leq n'}(h_i'^{\frac{1}{2}} h_j'^{-\frac{1}{2}} - h_i'^{-\frac{1}{2}} h_j'^{\frac{1}{2}})$$
$$= \prod_{i=1}^{n'} h_i'^{-\frac{n'-1}{2}} \prod_{1\leq i<j\leq n'}(h'_i - h'_j),$$

and for every $\sigma \in \mathscr{S}_{n'}$,

$$\Delta_{\Phi'(\mathrm{Z}')}(\sigma(\check{h}')) = \prod_{i=2}^{n'} h'^{-\frac{n'-2}{2}}_{\sigma(i)} \prod_{2\leq i<j\leq n'}(h'_{\sigma(i)} - h'_{\sigma(j)})$$
$$= \varepsilon(\sigma) \prod_{\substack{i=1 \\ i\neq\sigma(1)}}^{n'} h_i'^{-\frac{n'-2}{2}} \prod_{\substack{1\leq i<j\leq n' \\ i,j\neq\sigma(1)}}(h'_i - h'_j).$$

In particular,

$$\frac{\Delta_{\Phi'(\mathrm{Z}')}(\sigma(\check{h}'))}{\Delta_{\Phi'}(\check{h}')} = \varepsilon(\sigma)\frac{h'^{\frac{n'-1}{2}}_{\sigma(1)} \prod_{\substack{i=1 \\ i\neq\sigma(1)}}^{n'} h_i'^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq\sigma(1)}}^{n'}(h'_{\sigma(1)} - h'_i)} = \varepsilon(\sigma)\frac{h'^{\frac{n'-2}{2}}_{\sigma(1)} \prod_{i=1}^{n'} h_i'^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq\sigma(1)}}^{n'}(h'_{\sigma(1)} - h'_i)}, \qquad (\check{h}' \in \check{\mathrm{H}}'_{\mathbb{C}}).$$

Notation A.1. As in Section 6, because the set of genuine representations of $\widetilde{\mathrm{U}}(1)$ is isomorphic to $\mathbb{Z}$, we will denote by $\Pi_m, m \in \mathbb{Z}$, the representations of $\mathscr{R}(\widetilde{\mathrm{U}}(1), \omega)$. Using [20], we get that $\Pi_m(\tilde{h}) = \pm h^{m+\frac{q-p}{2}}$. We will denote by Π'_m the corresponding representation of $\widetilde{\mathrm{U}}(p, q)$ and by $\Theta_{\Pi'_m}$ its character.

Proposition A.2. *For every* $\mathrm{S} \subseteq \Psi_n'^{\mathrm{st}}$ *(see Appendix* B*), the value of the character* $\Theta_{\Pi'_m}$ *on* $\widetilde{\mathrm{H}'(\mathrm{S})}^{\mathrm{reg}}$ *is given by the following formula:*

$$\Delta_{\Phi'}(\check{h}')\Theta_{\Pi'_m}(c(\mathrm{S})\check{p}(\check{h}')c(\mathrm{S})^{-1}) =$$
$$\sum_{\sigma\in\mathscr{W}(\mathrm{H}'_{\mathbb{C}})} \frac{(-1)^u \varepsilon(\sigma)}{|\mathscr{W}(\mathrm{Z}'_{\mathbb{C}}, \mathrm{H}'_{\mathbb{C}})|} \Delta_{\Phi'(\mathrm{Z}')}(\sigma^{-1}(\check{h}')) \det{}^{-\frac{k}{2}}(\sigma^{-1}(\check{h}'))_{\mathrm{W}^{\mathfrak{h}}}$$
$$\lim_{\substack{r\in\mathrm{E}_{\sigma,\mathrm{S}} \\ r\to1}} \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))} \det^{-\frac{k}{2}}(\check{h})}{\det(1 - p(\check{h})rp(\check{h}'))_{\sigma\,\mathrm{W}^{\mathfrak{h}}}} d\check{h}$$

for every $\check{h}' \in \check{\mathrm{H}}_{\mathrm{S}}'^{\mathrm{reg}}$.

Proof. Let $\psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})$ such that $\mathrm{supp}(\psi) \subseteq \widetilde{\mathrm{G}'} \cdot \widetilde{\mathrm{H}}'(\mathrm{S})$, we get:

$$\Theta_{\Pi'_m}(\psi) = \mathrm{tr}(\mathscr{P}_{\Pi_m} \circ \omega(\psi)) = \int_{\widetilde{\mathrm{G}'}} \left(\int_{\widetilde{\mathrm{U}}(1)} \overline{\Theta_{\Pi_m}(\tilde{g})}\Theta(\tilde{g}\tilde{g}')d\tilde{g}\right)\psi(\tilde{g}')d\tilde{g}'$$
$$= \int_{\widetilde{\mathrm{G}'}} \left(\int_{\check{\mathrm{U}}(1)} \overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}\Theta(\check{p}(\check{h})\tilde{g}')d\check{h}\right)\psi(\tilde{g}')d\tilde{g}',$$

where $\mathscr{P}_{\Pi_m} : \mathscr{H} \to \mathscr{H}(\Pi_m)$ is the projection onto the Π_m-isotypic component given by $\mathscr{P}_{\Pi_m} = \omega(\overline{\Theta_{\Pi_m}})$ (see [38, Section 1.4.6]), i.e. as a generalized function on $\widetilde{\mathrm{G}'}$,

$$\Theta_{\Pi'_m}(\tilde{g}') = \int_{\check{\mathrm{U}}(1)} \overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}\Theta(\check{p}(\check{h})\tilde{g}')d\tilde{g}, \qquad (\tilde{g}' \in \widetilde{\mathrm{G}'}).$$

Using Remark 5.4, we get:

(24)

$$\begin{aligned}\Theta_{\Pi'_m}(\psi) &= \int_{\widetilde{\mathrm{G}'}} \Theta_{\Pi'_m}(\tilde{g}')\psi(\tilde{g}')d\tilde{g}' = m_{\mathrm{S}} \int_{\check{\mathrm{H}}'_{\mathrm{S}}} \varepsilon_{\Psi'_{\mathrm{S},\mathbb{R}}}(\check{h}')\Delta_{\Phi'}(\check{h}')\mathscr{H}_{\mathrm{S}}(\Theta_{\Pi'_m}\psi)(\check{h}')d\check{h}' \\ &= m_{\mathrm{S}} \int_{\check{\mathrm{H}}'_{\mathrm{S}}} \varepsilon_{\Psi'_{\mathrm{S},\mathbb{R}}}(\check{h}')\Delta_{\Phi'}(\check{h}')\Theta_{\Pi'_m}(c(\mathrm{S})\check{p}(\check{h}')c(\mathrm{S})^{-1})\mathscr{H}_{\mathrm{S}}\psi(\check{h}')d\check{h}'.\end{aligned}$$

Using Theorem 5.6, Equation (24) can be written as:

(25)

$$\begin{aligned}\Theta_{\Pi'_m}(\psi) &= \int_{\check{\mathrm{U}}(1)} \overline{\Theta_{\Pi_m}(\check{p}(\check{h}))} \int_{\widetilde{\mathrm{G}'}} \Theta(\check{p}(\check{h})\tilde{g}')\psi(\tilde{g}')d\tilde{g}'d\check{h} \\ &= \int_{\check{\mathrm{U}}(1)} \overline{\Theta_{\Pi_m}(\check{p}(\check{h}))} \det{}^{-\frac{k}{2}}(\check{h}) \left(\det{}^{\frac{k}{2}}(\check{h}) \int_{\widetilde{\mathrm{G}'}} \Theta(\check{p}(\check{h})\tilde{g}')\psi(\tilde{g}')d\tilde{g}'\right) d\check{h} \\ &= \int_{\check{\mathrm{U}}(1)} \overline{\Theta_{\Pi_m}(\check{p}(\check{h}))} \det{}^{-\frac{k}{2}}(\check{h}) \sum_{\sigma\in\mathscr{W}(\mathrm{H}'_{\mathbb{C}})} \mathrm{M}_{\mathrm{S}}(\sigma) \\ &\quad \lim_{\substack{r\in\mathrm{E}_{\sigma,\mathrm{S}} \\ r\to 1}} \int_{\check{\mathrm{H}}'_{\mathrm{S}}} \frac{\det^{-\frac{k}{2}}(\sigma^{-1}(\check{h}'))_{\mathrm{W}^{\mathfrak{h}}}\Delta_{\Phi'(\mathrm{Z}')}(\sigma^{-1}(\check{h}'))}{\det(1-p(\check{h})rp(\check{h}'))_{\sigma\,\mathrm{W}^{\mathfrak{h}}}} \varepsilon_{\Psi'_{\mathrm{S},\mathbb{R}}}(\check{h}')\mathscr{H}_{\mathrm{S}}(\psi)(\check{h}')d\check{h}'d\check{h} \\ &= \int_{\check{\mathrm{H}}'_{\mathrm{S}}} \varepsilon_{\Psi'_{\mathrm{S},\mathbb{R}}}(\check{h}') \sum_{\sigma\in\mathscr{W}(\mathrm{H}'_{\mathbb{C}})} \mathrm{M}_{\mathrm{S}}(\sigma)\Delta_{\Phi'(\mathrm{Z}')}(\sigma^{-1}(\check{h}'))\det{}^{-\frac{k}{2}}(\sigma^{-1}(\check{h}'))_{\mathrm{W}^{\mathfrak{h}}} \\ &\quad \lim_{\substack{r\in\mathrm{E}_{\sigma,\mathrm{S}} \\ r\to 1}} \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}\det^{-\frac{k}{2}}(\check{h})}{\det(1-p(\check{h})rp(\check{h}'))_{\sigma\,\mathrm{W}^{\mathfrak{h}}}} d\check{h}\mathscr{H}_{\mathrm{S}}(\psi)(\check{h}')d\check{h}'.\end{aligned}$$

The result follows by comparing Equations (24) and (25). □

Without loss of generality, we assume that $p \leq q$ and keep the notations of Appendix B (see Equation (27)). In particular, for every $h' \in \mathrm{H}'_{\mathrm{S}_t}$, $0 \leq t \leq p$, h' is of the form

$$\begin{aligned}h' &= (h'_1, \dots, h'_{n'}) \\ &= \mathrm{diag}(e^{iX_1 - X_{p+1}}, \dots, e^{iX_t - X_{p+t}}, e^{iX_{t+1}}, \dots, e^{iX_p}, \\ &\qquad e^{iX_1 + X_{p+1}}, \dots, e^{iX_t + X_{p+t}}, e^{iX_{p+t+1}}, \dots, e^{iX_{p+q}}),\end{aligned}$$

where $X_j \in \mathbb{R}$ and $\mathrm{S}_t = \{e_1 - e_{p+1}, \dots, e_t - e_{p+t}\}$.

Proposition A.3. *The value of the character* $\Theta_{\Pi'_m}$ *is given, for every* $\check{h}' \in \check{\mathrm{H}}'_{\mathrm{S}_t}$, $0 \leq t \leq p$, *by the formula:*

$$\Theta_{\Pi'_m}(c(\mathrm{S}_t)\check{p}(\check{h}')c(\mathrm{S}_t)^{-1}) = \pm 2\,\mathrm{C}^2$$
$$\begin{cases} -\sum_{\substack{j=1 \\ j\in J(h')}}^{t} \dfrac{h_j'^{-m+p-1}\left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ i\neq j}}^{n'}(h'_j - h'_i)} - \sum_{\substack{j=1 \\ j\in K(h')}}^{t} \dfrac{h_{p+j}'^{-m+p-1}\left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ i\neq p+j}}^{n'}(h'_{p+j} - h'_i)} \\ \quad - \sum_{i=t+1}^{p} \dfrac{h_i'^{-m+p-1}\left(\prod_{j=1}^{n'} h_j'^{\frac{1}{2}}\right)}{\prod_{\substack{j=1\\ j\neq i}}^{n'}(h'_i - h'_j)} & \text{if } m \leq \alpha^p_q \\ \sum_{\substack{j=1 \\ j\in K(h')}}^{t} \dfrac{h_j'^{-m+p-1}\left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ i\neq j}}^{n'}(h'_j - h'_i)} + \sum_{\substack{j=1 \\ j\in K(h')}}^{t} \dfrac{h_{p+j}'^{-m+p-1}\left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ i\neq p+j}}^{n'}(h'_{p+j} - h'_i)} \\ \quad + \sum_{j=p+t+1}^{p+q} \dfrac{h_j'^{-m+p-1}\left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ j\neq i}}^{n'}(h'_j - h'_i)} & \text{otherwise} \end{cases}$$

where $\alpha^p_q = -1 - \frac{q-p}{2}$, $\mathrm{C} = \frac{1}{(p+q-1)!}$, $h' = (h'_1, \dots, h'_{n'})$ *and* $K(h')$, $J(h')$ *are given by:*

$$J(h') = \{j \in \{1, \dots, t\}, \mathrm{sgn}(X_{p+j}) = 1\}, K(h') = \{j \in \{1, \dots, t\}, \mathrm{sgn}(X_{p+j}) = -1\}.$$

To make the equation shorter, we will denote by C the constant $\mathrm{C} = \frac{(-1)^u}{(p+q-1)!}$.

Proof. We start by determining the space $\mathrm{E}_{\sigma,\emptyset}$ for $\sigma \in \mathscr{S}_{n'}$. For every $w = w_{1,1}\,\mathrm{E}_{1,1}$ and $y' = (y'_1, \dots, y'_{n'}) \in \mathfrak{h}'$ with $y'_j = iX'_j, X'_j \in \mathbb{R}$, we get:

$$\begin{aligned} \langle y'\sigma(w), \sigma(w)\rangle &= \langle y'(w_{1,1}\,\mathrm{E}_{\sigma(1),1}), w_{1,1}\,\mathrm{E}_{\sigma(1),1}\rangle = \langle (w_{1,1}y'_{\sigma(1)}\,\mathrm{E}_{\sigma(1),1}), w_{1,1}\,\mathrm{E}_{\sigma(1),1}\rangle \\ &= \mathrm{Im}(\overline{w_{1,1}}\,\mathrm{E}_{1,\sigma(1)}\,\mathrm{Id}_{p,q}\,w_{1,1}y'_{\sigma(1)}\,\mathrm{E}_{\sigma(1),1}) \\ &= \begin{cases} |w_{1,1}|^2\,\mathrm{Im}(y'_{\sigma(1)}\,\mathrm{E}_{1,\sigma(1)}\,\mathrm{E}_{\sigma(1),1}) & \text{if } 1 \leq \sigma(1) \leq p, \\ -|w_{1,1}|^2\,\mathrm{Im}(y'_{\sigma(1)}\,\mathrm{E}_{1,\sigma(1)}\,\mathrm{E}_{\sigma(1),1}) & \text{if } p+1 \leq \sigma(1) \leq p+q, \end{cases} \\ &= \begin{cases} X'_{\sigma(1)}|w_{1,1}|^2 & \text{if } 1 \leq \sigma(1) \leq p, \\ -X'_{\sigma(1)}|w_{1,1}|^2 & \text{if } p+1 \leq \sigma(1) \leq p+q. \end{cases} \end{aligned}$$

In particular,

$$\Gamma_{\sigma,\emptyset} = \begin{cases} \left\{y' \in \mathfrak{h}', X'_{\sigma(1)} > 0\right\} & \text{if } 1 \leq \sigma(1) \leq p, \\ \left\{y' \in \mathfrak{h}', X'_{\sigma(1)} < 0\right\} & \text{if } p+1 \leq \sigma(1) \leq n' \end{cases}$$

and then

$$\mathrm{E}_{\sigma,\emptyset} = \exp(i\Gamma_{\sigma,\emptyset}) = \begin{cases} \Big\{h' \in \mathrm{H}'_{\mathbb{C}}, h' = (e^{-X'_1}, \ldots, e^{-X'_{n'}}), X'_{\sigma(1)} > 0\Big\} \\ \hfill \text{if } 1 \leq \sigma(1) \leq p, \\ \Big\{h' \in \mathrm{H}'_{\mathbb{C}}, h' = (e^{-X'_1}, \ldots, e^{-X'_{n'}}), X'_{\sigma(1)} < 0\Big\} \\ \hfill \text{if } p+1 < \sigma(1) \leq n'. \end{cases}$$

More generally, for every $\sigma \in \mathscr{S}_n$, we get:

$$\Gamma_{\sigma,\mathrm{S}_t} = \begin{cases} \mathfrak{h}' & \text{if } \sigma(1) \in \underline{\mathrm{S}_t}, \\ \Big\{y \in \mathfrak{h}', X'_{\sigma(1)} > 0\Big\} & \text{if } \sigma(1) \in \{t+1, \ldots, p\}, \\ \Big\{y \in \mathfrak{h}', X'_{\sigma(1)} < 0\Big\} & \text{if } \sigma(1) \in \{p+t+1, \ldots, n\}. \end{cases}$$

In particular,

$$\mathrm{E}_{\sigma,\mathrm{S}} = \exp \begin{cases} \Big\{h' \in \mathrm{H}'_{\mathbb{C}}, h = \mathrm{diag}(e^{-X'_1}, \ldots, e^{-X'_n}), X'_i \in \mathbb{R}\Big\} \\ \hfill \text{if } \sigma(1) \in \underline{\mathrm{S}_t}, \\ \Big\{h' \in \mathrm{H}'_{\mathbb{C}}, h' = \mathrm{diag}(e^{-X'_1}, \ldots, e^{-X'_n}), X'_i \in \mathbb{R}, X'_{\sigma(1)} > 0\Big\} \\ \hfill \text{if } \sigma(1) \in \{t+1, \ldots, p\}, \\ \Big\{h' \in \mathrm{H}'_{\mathbb{C}}, h = \mathrm{diag}(e^{-X'_1}, \ldots, e^{-X'_n}), X'_i \in \mathbb{R}, X'_{\sigma(1)} < 0\Big\} \\ \hfill \text{if } \sigma(1) \in \{p+t+1, \ldots, n\}. \end{cases}$$

Because the space $\mathrm{E}_{\sigma,\mathrm{S}_t}$ only depends on $\sigma(1)$, we will denote this space by $\mathrm{E}_{i,\mathrm{S}_t}$ for a $\sigma \in \mathscr{S}_{n'}$ such that $\sigma(1) = i$.

We first assume that n' is even, i.e. $k = 0$. Then, according to Proposition A.2 and that $\det(1 - p(\check{h})rp(\check{h}'))_{\sigma \mathrm{W}^\flat} = 1 - h(rh')^{-1}_{\sigma(1)}$, we get (up to a constant):

$$\begin{aligned}
&\Theta_{\Pi'}(c(\mathrm{S}_t)p(\check{h}')c(\mathrm{S}_t)^{-1}) \\
&= \mathrm{C} \sum_{\sigma \in \mathscr{S}_{n'}} \varepsilon(\sigma) \frac{\Delta_{\Phi'(\mathrm{Z}')}(\sigma^{-1}(\check{h}'))}{\Delta_{\Phi'}(\check{h}')} \lim_{\substack{r \to 1 \\ r \in \mathrm{E}_{\sigma,\mathrm{S}}}} \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}}{1 - h(rh')^{-1}_{\sigma(1)}} d\check{h} \\
&= \mathrm{C}^2 \sum_{j=1}^{t} \frac{h_j'^{\frac{n'-2}{2}} \left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ i \neq j}}^{n'} (h_j' - h_i')} \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}}{1 - he^{-iX_1 + X_{p+1}}} d\check{h} \\
&+ \mathrm{C}^2 \sum_{j=1}^{t} \frac{h_{p+j}'^{\frac{n'-2}{2}} \left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ i \neq p+j}}^{n'} (h_{p+j}' - h_i')} \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}}{1 - he^{-iX_j - X_{p+j}}} d\check{h} \\
&+ \mathrm{C}^2 \sum_{j=t+1}^{p} \frac{h_j'^{\frac{n'-2}{2}} \left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ j \neq i}}^{n'} (h_j' - h_i')} \lim_{\substack{r \to 1 \\ 0<r<1}} \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}}{1 - h(rh_j')^{-1}} d\check{h} \\
&+ \mathrm{C}^2 \sum_{j=p+t+1}^{n'} \frac{h_j'^{\frac{n'-2}{2}} \left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ j \neq i}}^{n'} (h_j' - h_i')} \lim_{\substack{r \to 1 \\ r>1}} \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}}{1 - h(rh_i')^{-1}} d\check{h} \\
&= -\mathrm{C}^2 \sum_{j=1}^{t} \frac{h_j'^{\frac{n'-2}{2}} \left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ i \neq j}}^{n'} (h_j' - h_i')} \frac{1}{e^{-iX_j + X_{p+j}}} \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}}{h - e^{iX_j - X_{p+j}}} d\check{h} \\
&- \mathrm{C}^2 \sum_{j=1}^{t} \frac{h_{p+j}'^{\frac{n'-2}{2}} \left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ i \neq p+j}}^{n'} (h_{p+j}' - h_i')} \frac{1}{e^{-iX_j - X_{p+j}}} \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}}{h - e^{iX_j + X_{p+j}}} d\check{h} \\
&- \mathrm{C}^2 \sum_{j=t+1}^{p} \frac{h_j'^{\frac{n'-2}{2}} \left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ j \neq i}}^{n'} (h_j' - h_i')} \lim_{\substack{r \to 1 \\ 0<r<1}} h_i' \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}}{h - rh_j'} d\check{h} \\
&- \mathrm{C}^2 \sum_{j=p+t+1}^{n'} \frac{h_j'^{\frac{n'-2}{2}} \left(\prod_{i=1}^{n'} h_i'^{\frac{1}{2}}\right)}{\prod_{\substack{i=1\\ j \neq i}}^{n'} (h_j' - h_i')} \lim_{\substack{r \to 1 \\ r>1}} h_i' \int_{\check{\mathrm{U}}(1)} \frac{\overline{\Theta_{\Pi_m}(\check{p}(\check{h}))}}{h - rh_j'} d\check{h} \\
&= -\frac{2\,\mathrm{C}^2}{2i\pi} \sum_{j=1}^{t} \frac{h_j'^{\frac{n'-2}{2}} \prod_{i=1}^{n'} h_i'^{\frac{1}{2}}}{\prod_{\substack{i=1\\ i \neq j}}^{n'} (h_j' - h_i')} \frac{1}{e^{-iX_j + X_{p+j}}} \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z - e^{iX_j - X_{p+j}}} dz \\
&- \frac{2\,\mathrm{C}^2}{2i\pi} \sum_{j=1}^{t} \frac{h_{p+j}'^{\frac{n'-2}{2}} \prod_{i=1}^{n'} h_i'^{\frac{1}{2}}}{\prod_{\substack{i=1\\ i \neq p+j}}^{n'} (h_{p+j}' - h_i')} \frac{1}{e^{-iX_j - X_{p+j}}} \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z - e^{iX_j + X_{p+j}}} dz \\
&- \frac{2\,\mathrm{C}^2}{2i\pi} \sum_{j=t+1}^{p} \frac{h_j'^{\frac{n'-2}{2}} \prod_{i=1}^{n'} h_i'^{\frac{1}{2}}}{\prod_{\substack{i=1\\ j \neq i}}^{n'} (h_j' - h_j')} \lim_{\substack{r \to 1 \\ 0<r<1}} h_i' \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z - rh_j'} dz \\
&- \frac{2\,\mathrm{C}^2}{2i\pi} \sum_{j=p+t+1}^{n'} \frac{h_j'^{\frac{n'-2}{2}} \prod_{i=1}^{n'} h_i'^{\frac{1}{2}}}{\prod_{\substack{i=1\\ j \neq i}}^{n'} (h_j' - h_i')} \lim_{\substack{r \to 1 \\ r>1}} h_i' \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z - rh_j'} dz
\end{aligned}$$

If $-m-1-\frac{q-p}{2} \geq 0$, i.e. $m \leq -1-\frac{q-p}{2}$. Then, according to Lemma 6.10, we get:

$$\begin{aligned}
&\Theta_{\Pi'_m}(c(\mathrm{S}_t)p(\check{h}')c(\mathrm{S}_t)^{-1}) \\
&= -\frac{2\,\mathrm{C}^2}{2i\pi} \sum_{\substack{j=1 \\ j\in J(h')}}^{t} \frac{{h'_j}^{\frac{n'-2}{2}} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq j}}^{n'} (h'_j - h'_i)} \frac{1}{e^{-iX_j+X_{p+j}}} \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z-e^{iX_j-X_{p+j}}} dz \\
&\quad - \frac{2\,\mathrm{C}^2}{2i\pi} \sum_{\substack{j=1 \\ j\in K(h')}}^{t} \frac{{h'_{p+j}}^{\frac{n'-2}{2}} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq p+j}}^{n'} (h'_{p+j} - h'_i)} \frac{1}{e^{-iX_j-X_{p+j}}} \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z-e^{iX_j+X_{p+j}}} dz \\
&\quad - \frac{2\,\mathrm{C}^2}{2i\pi} \sum_{j=t+1}^{p} \frac{{h'_j}^{\frac{n'-2}{2}} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ j\neq i}}^{n'} (h'_j - h'_i)} \lim_{\substack{r\to 1 \\ 0<r<1}} h'_i \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z-rh'_i} dz \\
&= -2\,\mathrm{C}^2 \sum_{\substack{j=1 \\ j\in J(h')}}^{t} \frac{{h'_j}^{-m+p-1} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq j}}^{n'} (h'_j - h'_i)} - 2\,\mathrm{C}^2 \sum_{\substack{j=1 \\ j\in K(h')}}^{t} \frac{{h'_{p+j}}^{-m+p-1} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq p+j}}^{n'} (h'_{p+j} - h'_i)} \\
&\quad - 2\,\mathrm{C}^2 \sum_{j=t+1}^{p} \frac{{h'_j}^{-m+p-1} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ j\neq i}}^{n'} (h'_j - h'_i)}.
\end{aligned}$$

Similarly, if $m > -1-\frac{q-p}{2}$, we get:

$$\begin{aligned}
&\Theta_{\Pi'}(c(\mathrm{S}_t)p(\check{h}')c(\mathrm{S}_t)^{-1}) \\
&= -\frac{2\,\mathrm{C}^2}{2i\pi} \sum_{\substack{j=1 \\ j\in K(h')}}^{t} \frac{{h'_j}^{\frac{n'-2}{2}} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq j}}^{n'} (h'_j - h'_i)} \frac{1}{e^{-iX_j+X_{p+j}}} \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z-e^{iX_j-X_{p+j}}} dz \\
&\quad - \frac{2\,\mathrm{C}^2}{2i\pi} \sum_{\substack{j=1 \\ j\in J(h')}}^{t} \frac{{h'_{p+j}}^{\frac{n'-2}{2}} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq p+j}}^{n'} (h'_{p+j} - h'_i)} \frac{1}{e^{-iX_j-X_{p+j}}} \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z-e^{iX_j+X_{p+j}}} dz \\
&\quad - \frac{2\,\mathrm{C}^2}{2i\pi} \sum_{j=p+t+1}^{n'} \frac{{h'_j}^{\frac{n'-2}{2}} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ j\neq i}}^{n'} (h'_j - h'_i)} \lim_{\substack{r\to 1 \\ r>1}} h'_i \int_{\mathrm{U}(1)} \frac{z^{-m-1-\frac{q-p}{2}}}{z-rh'_j} dz \\
&= 2\,\mathrm{C}^2 \sum_{\substack{j=1 \\ j\in J(h')}}^{t} \frac{{h'_j}^{-m+p-1} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq j}}^{n'} (h'_j - h'_i)} + 2\,\mathrm{C}^2 \sum_{\substack{j=1 \\ j\in K(h')}}^{t} \frac{{h'_{p+j}}^{-m+p-1} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ i\neq p+j}}^{n'} (h'_{p+j} - h'_i)} \\
&\quad + 2\,\mathrm{C}^2 \sum_{j=p+t+1}^{n'} \frac{{h'_j}^{-m+p-1} \prod_{i=1}^{n'} {h'_i}^{\frac{1}{2}}}{\prod_{\substack{i=1 \\ j\neq i}}^{n'} (h'_j - h'_i)}.
\end{aligned}$$

The computations are similar if n' is odd. $\square$

Corollary A.4. *The value of* $\Theta_{\Pi'_m}$ *on* $\widetilde{\mathrm{H}}' = \check{p}(\check{\mathrm{H}}'_\emptyset)$ *is given, up to a constant, by:*

$$\Theta_{\Pi'_m}(\check{p}(\check{h}')) = \pm 2\,\mathrm{C}^2 \begin{cases} \prod_{i=1}^{n'} h_i'^{\frac{1}{2}} \sum_{i=1}^{p} \dfrac{h_i'^{-m+p-1}}{\prod_{j\neq i}(h'_i - h'_j)} & \text{if } m \leq -1 - \frac{q-p}{2}, \\ -\prod_{i=1}^{n'} h_i'^{\frac{1}{2}} \sum_{i=p+1}^{n'} \dfrac{h_i'^{-m+p-1}}{\prod_{j\neq i}(h'_i - h'_j)} & \text{otherwise} \end{cases},$$

where $\mathrm{C} \in \mathbb{R}$.

This result was obtained in [27, Section 6].

Remark A.5. Assume that $p = 1$, $q = 1$. Then,

$$\Theta_{\Pi'_m}(\tilde{h}') = 2 \pm \begin{cases} \dfrac{(e^{i\theta - X})^{-m}}{e^X - e^{-X}} & \text{if } m \leq -1 \text{ and } X > 0 \\ -\dfrac{(e^{i\theta + X})^{-m}}{e^X - e^{-X}} & \text{if } m \leq -1 \text{ and } X < 0 \\ \dfrac{(e^{i\theta + X})^{-m}}{e^X - e^{-X}} & \text{if } m \geq 0 \text{ and } X > 0 \\ -\dfrac{(e^{i\theta - X})^{-m}}{e^X - e^{-X}} & \text{if } m \geq 0 \text{ and } X < 0 \end{cases},$$

where $h' = \begin{pmatrix} e^{i\theta}\operatorname{ch}(X) & \operatorname{sh}(X) \\ \operatorname{sh}(X) & e^{i\theta}\operatorname{ch}(X) \end{pmatrix}$. We recover the results of [27, Section 7].

Appendix B. Cartan subgroups for unitary groups

It is well-known that the number of non-conjugated Cartan subgroups of $\mathrm{G} = \mathrm{U}(p, q)$, up to equivalence, is $\min(p, q) + 1$ (see [15]). We recall in this appendix how Cartan subgroups can be parametrised using strongly orthogonal roots (see [35, Section 2]).

Let $\mathrm{K} = \mathrm{U}(p) \times \mathrm{U}(q)$ be the maximal compact subgroup of G and H be the (diagonal) compact Cartan subgroup of K. We denote by $\mathfrak{h}$, $\mathfrak{k}$ and $\mathfrak{g}$ the Lie algebras of H, K and G respectively and $\mathfrak{h}_\mathbb{C}$, $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}$ their complexifications.

We denote by $\Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the set of roots, by $\Delta_c := \Delta_c(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the set of compact roots and by $\Delta_n = \Delta \setminus \Delta_c$ the set of non-compact roots. Similarly, we denote by Ψ a set of positive roots of Δ and let Ψ_c and Ψ_n the subsets of Ψ given by $\Psi_c = \Delta_c \cap \Psi$ and $\Psi_n = \Delta_n \cap \Psi$. In particular,

$$\mathfrak{g}_\mathbb{C} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C},\alpha},$$

where $\mathfrak{g}_{\mathbb{C},\alpha} = \{X \in \mathfrak{g}_\mathbb{C}, [H, X] = \alpha(H)X, H \in \mathfrak{h}_\mathbb{C}\}$.

Notation B.1.

(1) For every $\alpha \in \Delta$, we fix $X_\alpha \in \mathfrak{g}_{\mathbb{C},\alpha}$, $Y_\alpha \in \mathfrak{g}_{\mathbb{C},-\alpha}$ and $H_\alpha \in i\mathfrak{h}$ such that:

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha, \quad [X_\alpha, Y_\alpha] = H_\alpha, \quad \overline{H_\alpha} = -H_\alpha = H_{-\alpha},$$

and such that $\overline{X_\alpha} = -Y_\alpha$ if $\alpha \in \Delta_c$ and $\overline{X_\alpha} = Y_\alpha$ if $\alpha \in \Delta_n$.

(2) We say that $\alpha, \beta \in \Delta$ are strongly orthogonal if $\alpha \neq \pm\beta$ and $\alpha \pm \beta \notin \Delta$. We denote by Ψ_n^{st} a maximal family of strongly orthogonal roots of Ψ_n (i.e. a subset of Ψ_n such that every pair $\alpha, \beta \in \Psi_n^{\mathrm{st}}$ is strongly orthogonal).

For every $\alpha \in \Psi_n^{\mathrm{st}}$, we denote by $c(\alpha)$ the element of $\mathrm{GL}(p+q,\mathbb{C})$ given by:

$$c(\alpha) = \exp\left(\frac{\pi}{4}(Y_\alpha - X_\alpha)\right).$$

For every subset S of Ψ_n^{st}, we denote by $c(\mathrm{S})$ the element of $\mathrm{GL}(p+q,\mathbb{C})$ defined by

$$c(\mathrm{S}) = \prod_{\alpha\in\mathrm{S}} c(\alpha), \tag{26}$$

and let

$$\mathfrak{h}(\mathrm{S}) = \mathfrak{g} \cap \mathrm{Ad}(c(\mathrm{S}))(\mathfrak{h}_{\mathbb{C}}).$$

We denote by H(S) the analytic subgroup of G whose Lie algebra is $\mathfrak{h}(\mathrm{S})$. Then, H(S) is a Cartan subgroup of G and one can prove that all the Cartan subgroups are of this form (up to conjugation).

For every $\mathrm{S} \subseteq \Psi_n^{\mathrm{st}}$, we will denote by $\mathrm{H_S}$ the subgroup of $\mathrm{H}_{\mathbb{C}}$ given by:

$$\mathrm{H_S} = c(\mathrm{S})^{-1}\,\mathrm{H}(\mathrm{S})c(\mathrm{S}),$$

where $\mathrm{H}_{\mathbb{C}} = \{\mathrm{diag}(\lambda_1,\dots,\lambda_{p+q}), \lambda_i \in \mathbb{C}\}$.

Without loss of generality, we assume that $p \le q$. The set of roots Δ is given by $\Delta = \{\pm(e_i - e_j), 1 \le i < j \le p+q\}$, where e_i is the linear form on $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}^{p+q}$ given by

$$e_i(\lambda_1,\dots,\lambda_{p+q}) = \lambda_i.$$

In this case,

$$\Delta_c = \{\pm(e_i-e_j), 1 \le i < j \le p\} \cup \{\pm(e_i - e_j), p+1 \le i < j \le p+q\},$$

$$\Delta_n = \{\pm(e_i - e_j), 1 \le i \le p, p+1 \le j \le p+q\},$$

and the set Ψ_n^{st} can be chosen as $\{e_t - e_{p+t}, 1 \le t \le p\}$. In particular, $\mathrm{H}(\emptyset) = \mathrm{H}$ and if $\mathrm{S}_t = \{e_1 - e_{p+1},\dots,e_t - e_{p+t}\}, 1 \le t \le p$, we get:

$$\mathrm{H}(\mathrm{S}_t) = \exp\Big(\bigoplus_{j=t+1}^{p} i\mathbb{R}\,\mathrm{E}_{j,j} \oplus \bigoplus_{j=p+t+1}^{p+q} i\mathbb{R}\,\mathrm{E}_{j,j} \oplus \bigoplus_{j=1}^{t} i\mathbb{R}(\mathrm{E}_{j,j} + \mathrm{E}_{p+j,p+j}) \oplus \bigoplus_{j=1}^{t} \mathbb{R}(\mathrm{E}_{j,p+j} + \mathrm{E}_{p+j,j})\Big),$$

and the corresponding group $\mathrm{H}_{\mathrm{S}_t}$ is given by

$$\big\{\mathrm{diag}(e^{iX_1 - X_{p+1}},\dots,e^{iX_t - X_{p+t}}, e^{iX_{t+1}},\dots,e^{iX_p}, e^{iX_1+X_{p+1}},\dots,e^{iX_t+X_{p+t}}, e^{iX_{p+t+1}},\dots,e^{iX_{p+q}}), X_j \in \mathbb{R}\big\}. \tag{27}$$

Remark B.2. As explained in [35, Proposition 2.16], two Cartan subalgebras $\mathfrak{h}(\mathrm{S}_1)$ and $\mathfrak{h}(\mathrm{S}_2)$, with $\mathrm{S}_1, \mathrm{S}_2 \subseteq \Psi_n^{\mathrm{st}}$, are conjugate if and only if there exists an element of $\sigma \in \mathscr{W}$ sending $\mathrm{S}_1 \cup (-\mathrm{S}_1)$ onto $\mathrm{S}_2 \cup (-\mathrm{S}_2)$.

Appendix C. The character ε

Let $(\mathrm{W}, \langle\cdot,\cdot\rangle)$ be a real symplectic space, $\mathrm{Sp}(\mathrm{W})$ the corresponding group of isometries and $\widetilde{\mathrm{Sp}}(\mathrm{W})$ its metaplectic cover as in Equation (1).

Let $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ be a complete polarization of W. We denote by Z the subgroup of Sp(W) preserving both X and Y. In particular, we get that

$$\mathrm{Z} = \left\{ \begin{pmatrix} g & 0 \\ 0 & (g^{-1})^t \end{pmatrix}, g \in \mathrm{GL}(\mathrm{X}) \right\} \approx \mathrm{GL}(\mathrm{X}).$$

We define a double cover $\widetilde{\mathrm{GL}}(\mathrm{X})$ of $\mathrm{GL}(\mathrm{X})$ by

$$\widetilde{\mathrm{GL}}(\mathrm{X}) = \left\{(g,\eta) \in \mathrm{GL}(\mathrm{X}) \times \mathbb{C}^\times, \eta^2 = \det(g)\right\}, \qquad \widetilde{\mathrm{GL}}(\mathrm{X}) \ni (g,\eta) \to g \in \mathrm{GL}(\mathrm{X}).$$

As recalled in [2, Section 6], the restriction map

$$\mathrm{Z} \ni g \to g_{|\mathrm{X}} \in \mathrm{GL}(\mathrm{X})$$

lifts to a group isomorphism

$$\widetilde{\mathrm{Z}} \ni \tilde{g} \to (g_{|\mathrm{X}}, \eta) \in \widetilde{\mathrm{GL}}(\mathrm{X}),$$

where $\eta = \eta(\tilde{g})$ is defined on $\mathrm{Z}^c = \left\{\tilde{g} \in \widetilde{\mathrm{Z}}, \det(g-1)_{\mathrm{W}} \neq 0\right\}$ by the following formula:

$$\eta(\tilde{g}) = \frac{\Theta(\tilde{g})}{|\Theta(\tilde{g})|} |\det(g_{|\mathrm{X}})|^{\frac{1}{2}}.$$

We denote by ε the function on $\widetilde{\mathrm{Z}}$ given by:

$$\varepsilon : \widetilde{\mathrm{Z}} \ni \tilde{g} \to \varepsilon(\tilde{g}) = \frac{\eta(\tilde{g})}{|\eta(\tilde{g})|} \in \mathbb{C}.$$

One can easily prove that ε is a character of $\widetilde{\mathrm{Z}}$ with values in the set $\{\pm 1, \pm i\}$ such that

$$\varepsilon(\tilde{g}) = \frac{\Theta(\tilde{g})}{|\Theta(\tilde{g})|}, \qquad \left(\tilde{g} \in \widetilde{\mathrm{Z}^c}\right). \tag{28}$$

Lemma C.1. *For every $\tilde{g} \in \widetilde{\mathrm{Z}^c}$, we get:*

$$\Theta(\tilde{g})^2 = \det(g_{|X})^{-1} \det\left(\frac{1}{2}(c(g_{|X}) + 1)\right)^2.$$

Proof. Then, for every $\tilde{g} \in \widetilde{\mathrm{Z}^c}$, we get:

$$\begin{aligned}\Theta(\tilde{g})^2 &= \det(i(g-1))^{-1} = (-1)^{\frac{\dim_{\mathbb{R}}(W)}{2}} \det(g_{|X} - 1)^{-1} \det(g_{|Y} - 1)^{-1} \\ &= \det(g_{|X} - 1)^{-1} \det(1 - g_{|Y})^{-1} = \det(g_{|X} - 1)^{-1} \det(1 - g^{-1}_{|X})^{-1} \\ &= \det(g_{|X} - 1)^{-1} \det(1 - g^{-1}_{|X})^{-1} = \det(g_{|X}) \det(g_{|X} - 1)^{-2}.\end{aligned}$$

We have:

$$\begin{aligned}\frac{1}{2}\left(c(g_{|X}) + 1\right) &= \frac{1}{2}\left((g_{|X} + 1)(g_{|X} - 1)^{-1} + 1\right) \\ &= \frac{1}{2}\left((g_{|X} + 1)(g_{|X} - 1)^{-1} + (g_{|X} - 1)(g_{|X} - 1)^{-1}\right) = g_{|X}(g_{|X} - 1)^{-1}.\end{aligned}$$

Then,

$$\Theta(\tilde{g})^2 = \det(g_{|X}) \det(g_{|X} - 1)^{-2} = \left(\det(g_{|X}) \det(g_{|X} - 1)^{-1}\right)^2 \det(g_{|X})^{-1}$$
$$= \det(g_{|X})^{-1} \det\left(\frac{1}{2}(c(g_{|X}) + 1)\right)^2 .$$

□

We define by $\det_{\mathrm{X}}^{-\frac{1}{2}}(\tilde{g})$ the following quantity:

$$\det{}_{\mathrm{X}}^{-\frac{1}{2}}(\tilde{g}) = \Theta(\tilde{g}) \left|\det\left(\frac{1}{2}(c(g_{|X}) + 1)\right)\right|^{-1} .$$

In particular, for every $\tilde{g} \in \widetilde{\mathrm{Z}^c}$, we get:

$$\varepsilon(\tilde{g}) = \frac{\Theta(\tilde{g})}{|\Theta(\tilde{g})|} = \frac{\det_{\mathrm{X}}^{-\frac{1}{2}}(\tilde{g})}{|\det_{\mathrm{X}}^{-\frac{1}{2}}(\tilde{g})|}, \qquad \left(\tilde{g} \in \widetilde{\mathrm{Z}^c}\right). \tag{29}$$

Appendix D. The representations Π^n are not highest weight modules

D.1. **Wave front set of a representation.** The main reference here is [18] (for more results concerning distributions, one can also check [16, Chapter 8]). Let G be a real reductive Lie group and $(\rho, \mathscr{H})$ be a unitary representation of G. We denote by $\mathrm{J}(\mathscr{H})$ the set of trace class operators on $\mathscr{H}$.

We denote by $\mathrm{tr}_\rho : \mathrm{J}(\mathscr{H}) \to \mathbb{C}_b(\mathrm{G})$ the map given by

$$\mathrm{tr}_\rho(T)(g) := \mathrm{tr}(\rho(g)(T)), \qquad (g \in \mathrm{G}, T \in \mathrm{J}(\mathscr{H})),$$

where $\mathbb{C}_b(\mathrm{G})$ is the set of bounded functions on G. For every $T \in \mathrm{J}(\mathscr{H})$, one can easily see that $\mathrm{tr}_\rho(T)$ defines a distribution on G, where

$$\mathrm{tr}_\rho(T)(\psi) = \int_{\mathrm{G}} \psi(g) \, \mathrm{tr}_\rho(T)(g) dg, \qquad (\psi \in \mathscr{C}_c^\infty(\mathrm{G})).$$

In particular, we can consider the wave front set $\mathrm{WF}(\mathrm{tr}_\rho(T))$ of the distribution $\mathrm{tr}_\rho(T)$. The wave front set $\mathrm{WF}(\mathrm{tr}_\rho(T))$ is a closed set of the cotangent bundle $\mathrm{T}^* \mathrm{G}$ of G that we will identify with $\mathrm{G} \times \mathfrak{g}^*$.

Definition D.1. The wave front set $\mathrm{WF}(\rho)$ of the representation ρ is defined as the closure of the union of $\mathrm{WF}(\mathrm{tr}_\rho(T))$ as T varies over $\mathrm{J}(\mathscr{H})$.

One can easily verify that $\mathrm{WF}(\rho)$ is invariant under left and right invariant translations of G on $\mathrm{T}^*(\mathrm{G})$. As explained in [18, Section 1], a bi-invariant set in $\mathrm{T}^* \mathrm{G} = \mathrm{G} \times \mathfrak{g}^*$ is identified with $\mathrm{G} \times \mathrm{X}$, where $\mathrm{X} \subseteq \mathfrak{g}^*$ is an $\mathrm{Ad}^*(\mathrm{G})$-invariant set. In particular, we associate to $\mathrm{WF}(\rho)$ an $\mathrm{Ad}^*(\mathrm{G})$-invariant subset $\mathrm{WF}^\circ(\rho)$ of $\mathfrak{g}^*$, which determines $\mathrm{WF}(\rho)$.

We denote by Θ_ρ the character of the representation ρ and by $\mathrm{WF}(\Theta_\rho)$ its wave front set. As before, one can easily see that $\mathrm{WF}(\Theta_\rho)$ is invariant under left and right translations of G on $\mathrm{T}^* \mathrm{G}$, and we associate to $\mathrm{WF}(\Theta_\rho)$ a closed $\mathrm{Ad}^*(\mathrm{G})$-invariant set $\mathrm{WF}^\circ(\Theta_\rho)$ of $\mathfrak{g}^*$. As proved in [18, Theorem 1.8],

$$\mathrm{WF}^\circ(\Theta_\rho) = \mathrm{WF}^\circ(\rho).$$

Remark D.2. The group G is reductive. We can find a non-degenerate invariant form on $\mathfrak{g}$ and then identify the G-orbits on $\mathfrak{g}$ with the G-orbits on $\mathfrak{g}^*$.

We denote by $\mathscr{N}$ the nilpotent set in $\mathfrak{g}$. As proved in [18, Proposition 2.4], if ρ is an irreducible unitary representation of G, then

$$\mathrm{WF}^\circ(\rho) \subseteq \mathscr{N}.$$

D.2. **Wave front set of a highest weight module.** Let $(\mathrm{G}, \mathrm{G}')$ be an irreducible reductive dual pair in $\mathrm{Sp}(\mathrm{W})$ and $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ and $\mathfrak{g}' = \mathrm{Lie}(\mathrm{G}')$ the Lie algebras of G and G′ respectively. We know that $\mathrm{W} = \mathrm{Hom}_{\mathbb{D}}(\mathrm{V}', \mathrm{V})$ (where $\mathbb{D}$ is a division algebra over $\mathbb{R}$) and $\mathfrak{g} \subseteq \mathrm{End}_{\mathbb{D}}(\mathrm{V}), \mathfrak{g}' \subseteq \mathrm{End}_{\mathbb{D}}(\mathrm{V}')$ (see [17]). We denote by $(\cdot,\cdot)$ and $(\cdot,\cdot)'$ the $\mathbb{D}$-hermitian and $\mathbb{D}$-skew-hermitian forms on V and V′ respectively.

For every $w \in \mathrm{Hom}_{\mathbb{D}}(\mathrm{V}', \mathrm{V})$, we denote by w^* the unique element of $\mathrm{Hom}_{\mathbb{D}}(\mathrm{V}, \mathrm{V}')$ satisfying

$$(w(v'), v) = (v', w^*(v))', \qquad (v \in \mathrm{V}, v' \in \mathrm{V}').$$

We denote by $\tau_{\mathfrak{g}} : \mathrm{W} \to \mathfrak{g}$ and $\tau_{\mathfrak{g}'} : \mathrm{W} \to \mathfrak{g}'$ the unnormalized (moment) maps given by

$$\tau_{\mathfrak{g}}(w) = ww^*, \qquad \tau_{\mathfrak{g}'}(w) = w^*w, \qquad (w \in \mathrm{W}).$$

Lemma D.3. *Let* $w \in \mathrm{W}$ *such that* $\tau_{\mathfrak{g}}(w) = 0$. *Then* $\tau_{\mathfrak{g}'}(w)^2 = 0$.

Proof. Assume that $\tau_{\mathfrak{g}}(w) = ww^* = 0$. Then

$$\tau_{\mathfrak{g}'}(w) = (w^*w)(w^*w) = w^*(ww^*)w = w^*\tau_{\mathfrak{g}}(w)w = 0.$$

□

Lemma D.4. *Assume that* G *is compact. Then* $\mathrm{WF}^\circ(\Pi')^2 = \left\{X^2, X \in \mathrm{WF}^\circ(\Pi')\right\} = \{0\}$.

Proof. As explained in [32, Theorem 6.11], we get $\mathrm{WF}^\circ(\Pi') = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}\{0\})$. The result follows from Lemma D.3. □

Lemma D.5. *Let* $k \in \mathbb{Z}^+$ *and* Π' *be a highest weight representation of* $\widetilde{\mathrm{U}}(p,q)$, *where* $\widetilde{\mathrm{U}}(p,q)$ *is the* $\det^{\frac{k}{2}}$*-cover of* $\mathrm{U}(p,q)$, *i.e.*

$$\widetilde{\mathrm{U}}(p,q) = \left\{(g,\xi) \in \mathrm{U}(p,q) \times \mathbb{C}^*, \xi^2 = \det{}^k(g)\right\}.$$

Then $\mathrm{WF}(\Pi')^2 = \{0\}$.

Proof. By using [7], there exists $n \in \mathbb{Z}^+$ such that $n - k \in 2\mathbb{Z}$ and a representation Π of $\widetilde{\mathrm{U}}(n)$ (where $\widetilde{\mathrm{U}}(n)$ is the $\det^{\frac{p-q}{2}}$-cover of $\mathrm{U}(n)$) such that $\Pi' = \theta^n_{p,q}(\Pi)$. The lemma follows from Lemma D.4. □

We now use the notations of Section 6. Let $(\mathrm{G}', \mathrm{G}_n) = (\mathrm{U}(1,1), \mathrm{U}(n, n+1))$ be the dual pair of $\mathrm{Sp}(\mathrm{W}_n)$, with $\mathrm{W}_n = (\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^{2n+1})_{\mathbb{R}}$, and $\Pi^n = \theta^{1,1}_{n,n+1}(\Pi')$. We denote by $\mathfrak{g}'$ and $\mathfrak{g}_n$ the Lie algebras of G′ and G_n respectively. Similarly, let $\tau_{\mathfrak{g}'} : \mathrm{W}_n \to \mathfrak{g}'$ and $\tau_{\mathfrak{g}_n} : \mathrm{W}_n \to \mathfrak{g}_n$ the unnormalized moment maps.

Proposition D.6. *The representations* Π^n *constructed in Section* 6 *are not highest weight modules of* $\widetilde{\mathrm{U}}(n, n+1)$.

Proof. By using Lemma D.5, the theorem follows if we prove that $\mathrm{WF}^\circ(\Pi^n) \neq \{0\}$. As proved in [34, Theorem 2], we have $\mathrm{WF}^\circ(\Pi^n) = \tau_{\mathfrak{g}_n}(\tau_{\mathfrak{g}'}^{-1}(\mathrm{WF}^\circ(\Pi')))$. Let $\mathscr{O}$ be the G'-nilpotent orbit such that $\overline{\mathscr{O}} = \mathrm{WF}^\circ(\Pi')$. As explained in [32], $\tau_{\mathfrak{g}_n}(\tau_{\mathfrak{g}'}^{-1}(\overline{\mathscr{O}})) = \overline{\mathscr{O}_n}$, where $\mathscr{O}_n$ is a unique nilpotent G_n-orbit and $\mathrm{WF}^\circ(\Pi^n) = \overline{\mathscr{O}_n}$.

The moment maps $\tau_{\mathfrak{g}'}$ and $\tau_{\mathfrak{g}_n}$ can be extended canonically to the complexifications and let $\mathscr{O}_{\mathbb{C}}$ (resp. $\mathscr{O}_{n,\mathbb{C}}$) be the corresponding $\mathrm{GL}(2,\mathbb{C})$-orbit (resp. $\mathrm{GL}(2n+1,\mathbb{C})$). The correspondence of orbits in the stable range has been studied in [6]. As explained in [5], nilpotent orbits can be parametrized by partitions. Clearly, the partition corresponding to $\mathscr{O}_{\mathbb{C}}$ is (2) and it follows from [6, Theorem 4.2] that the partition corresponding to $\mathscr{O}_{n,\mathbb{C}}$ is $(3, \underbrace{1, \ldots, 1}_{2n-2})$.

In particular, one can easily see that for any $X \in \mathscr{O}_{n,\mathbb{C}}$, $X^2 \neq 0$ and it follows that $\mathscr{O}_n$ contain elements X_1 such that $X_1^2 \neq 0$, so $\mathrm{WF}^\circ(\Pi^n)^2 \neq \{0\}$. □

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