HOMOLOGICAL INVARIANTS OF THE ARROW REMOVAL OPERATION

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ABSTRACT. In this paper we show that Gorsteinness, singularity categories and the finite generation condition Fg for the Hochschild cohomology are invariants under the arrow removal operation for a finite dimensional algebra.

CONTENTS

1. INTRODUCTION AND THE MAIN RESULT

In [7] the arrow removal operation on quotients of path algebras was investigated with respect to the finitistic dimension conjecture. The idea was to remove those arrows that don't contribute to the finitistic dimension. This new technique gave us a successful reduction method for actual computing the finitistic dimension in many examples. Our aim in this paper is to investigate further this operation, and in particular study the class of *arrow removal algebras* with respect to various homological invariants. The guiding problem can be formulated as follows:

Problem. How does the arrow removal behave with respect to Gorsteinness, singularity categories and the finite generation condition Fg for the Hochschild cohomology.

See Section 4 for the definition of Gorsteinness [1] and the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ of a finite dimensional algebra Λ [3], and Section 5 for the definition of Fg [5, 10].

In our main result we prove that the above three homological invariants remain the same under the arrow removal operation for an admissible path algebra over a field. Before we state our main result we briefly recall the arrow removal operation.

Let Λ be an admissible quotient kQ/I of a path algebra kQ over a field k. Consider an arrow a in Q such that a does not occur in a minimal generating set of I. Then the quotient algebra $\Gamma = \Lambda/\langle \overline{a} \rangle$ is called an *arrow removal algebra* of Λ.

This new algebra can be explicitly described as a trivial extension. More precisely, it has been proved in [7, Theorem A] that the arrow $a\colon v_e \to v_f$ in Q does not occur in a set of minimal generators of I in kQ if and only if $\Lambda \cong \Gamma \ltimes P$, where $P = \Gamma e \otimes_k f\Gamma$ with $\mathsf{Hom}_\Gamma(e\Gamma, f\Gamma) = (0)$. Here e is a trivial path in kQ and v_e denotes the corrsponding vertex in Q. We can also consider arrow removal for any finite number of arrows, but for simplicity we only review our results in the one arrow case in the introduction.

The module category of Λ can be described using the context of cleft extensions of abelian categories in the sense of Beligiannis [2]. This means that there are functors $\mathsf{i}\colon \mathsf{mod}\text{-}\Gamma \to \mathsf{mod}\text{-}\Lambda$ and $\mathsf{e}\colon \mathsf{mod}\text{-}\Lambda \to \mathsf{mod}\text{-}\Gamma$, induced by the natural surjection map $\Gamma \to \Lambda$ and by the inclusion map $\Gamma \to \Lambda$ respectively, the functor e is faithful exact and the composition $\mathsf{e} \circ \mathsf{i}$ is equivalent to the identity functor on $\mathsf{mod}\text{-}\Gamma$. The arrow removal operation gives even more homological properties on this cleft extension.

The notion of eventually homological isomorphism was introduced in [8]. Recall that a functor $F\colon \mathscr{B} \to \mathscr{C}$ between abelian categories is called an *eventually homological isomorphism*, if there is an integer t such that for every $j > t$ there is an isomorphism $\mathsf{Ext}^j_{\mathscr{B}}(X, Y) \cong \mathsf{Ext}^j_{\mathscr{C}}(FX, FY)$ for all objects $X, Y' \in \mathscr{B}$. Given the smallest such t, we call the functor a *t-eventually homological isomorphism*. The latter notion was used in comparing the algebras Λ and $e\Lambda e$ for an idempotent e in Λ, with respect to Gorensteinness, singularity categories and the finite generation condition Fg for the Hochschild cohomology ([8, Main Theorem]). We mention that this comparison theorem was achieved via recollements of abelian categories.

We summarize below our main results in the simplified setting of a one arrow removal.

Main Theorem. *Let $\Lambda = kQ/I$ be an admissible quotient of a path algebra kQ over a field k and let $\Gamma = \Lambda/\langle \overline{a} \rangle$ an arrow removal of Λ for an arrow a in Q. Then the following hold.*

  (i) *The functor $\mathsf{e}\colon \mathsf{mod}\text{-}\Lambda \to \mathsf{mod}\text{-}\Gamma$ is a 1-eventually homological isomorphism.*
 (ii) *Λ is Gorenstein if and only if Γ is Gorenstein.*
(iii) *The functor $\mathsf{e}\colon \mathsf{D}_{\mathsf{sg}}(\Lambda) \to \mathsf{D}_{\mathsf{sg}}(\Gamma)$ is a singular equivalence.*
(iv) *Λ satisfies Fg if and only if Γ satisfies Fg.*

We remark that the arrow removal operation has been also considered in [4]. They mainly worked on the converse process, i.e. add arrows to a path algebra, and they described the Hochschild (co)homology using different techniques.

We end the introduction with a short description of the contents of the paper section by section. In Section 2 we review relevant results on cleft extensions. As part of a cleft extension between abelian categories $\mathcal{A}, \mathcal{B}$ there is a functor $\mathsf{e}\colon \mathcal{A} \to \mathcal{B}$ which is faithful and exact. One would like this to be an eventually homological isomorphism. In Section 3, we show this is the case under certain conditions. Section 4 shows that Gorensteinness and singularity categories are invariant under arrow removal. In Section 5 we investigate the Fg condition, and prove that it is invariant under arrow removal. Finally, we would like to mention that QPA2 [9] was used to investigate examples, in particular with respect to Lemma 5.2.

2. Cleft extensions and arrow removals

We start this section by recalling and reviewing some results about cleft extensions of abelian categories from [2, 7] that we need in the sequel.

2.1. **Cleft extensions.** We first recall the definition of cleft extensions of abelian categories.

Definition 2.1 ([2, Definition 2.1]). A **cleft extension** of an abelian category $\mathscr{B}$ is an abelian category $\mathscr{A}$ together with functors:

$$\mathscr{B} \xrightarrow{\quad i \quad} \mathscr{A} \xrightarrow{\quad e \quad} \mathscr{B}$$

with l arcing over $\mathscr{A}$.
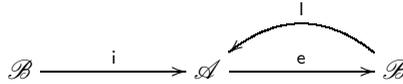
henceforth denoted by $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$, such that the following conditions hold:

  (a) The functor e is faithful exact.
  (b) The pair (l, e) is an adjoint pair of functors, where we denote the adjunction by
  $$\theta_{B,A}\colon \operatorname{\mathsf{Hom}}_{\mathscr{A}}(\mathsf{l}(B), A) \simeq \operatorname{\mathsf{Hom}}_{\mathscr{B}}(B, \mathsf{e}(A)).$$
  (c) There is a natural isomorphism $\varphi\colon \mathsf{ei} \longrightarrow \operatorname{\mathsf{Id}}_{\mathscr{B}}$ of functors.

Denote the unit $\theta_{B,\mathsf{l}(B)}(1_{\mathsf{l}(B)})$ and the counit $\theta^{-1}_{\mathsf{e}(A),A}(1_{\mathsf{e}(A)})$ of the adjoint pair (l, e) by $\nu\colon 1_{\mathscr{B}} \longrightarrow \mathsf{el}$ and $\mu\colon \mathsf{le} \longrightarrow 1_{\mathscr{A}}$, respectively. The unit and the counit satisfy the relations

$$(2.1) \qquad\qquad\qquad 1_{\mathsf{l}(B)} = \mu_{\mathsf{l}(B)}\mathsf{l}(\nu_B)$$

and

$$(2.2) \qquad\qquad\qquad 1_{\mathsf{e}(A)} = \mathsf{e}(\mu_A)\nu_{\mathsf{e}(A)}$$

for all B in $\mathscr{B}$ and A in $\mathscr{A}$. From (2.2) the morphism $\mathsf{e}(\mu_A)$ is an (split) epimorphism. Since e is faithful exact, it follows that μ_A is an epimorphism for all A in $\mathscr{A}$. This implies that for all A in $\mathscr{A}$ the following sequence is exact

$$(2.3) \qquad 0 \longrightarrow \operatorname{\mathsf{Ker}}\mu_A \longrightarrow \mathsf{le}(A) \xrightarrow{\ \mu_A\ } A \longrightarrow 0.$$

The next result collects some basic properties of a cleft extension which basically follows from Definition 2.1 and are discussed in [2]. For a detailed proof the reader is referred to [7, Lemma 2.2].

Lemma 2.2. *Let $\mathscr{A}$ be a cleft extension of $\mathscr{B}$. Then the following hold.*

  (i) *The functor $\mathsf{e}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is essentially surjective.*
  (ii) *The functor $\mathsf{i}\colon \mathscr{B} \longrightarrow \mathscr{A}$ is fully faithful and exact.*
  (iii) *The functor $\mathsf{l}\colon \mathscr{B} \longrightarrow \mathscr{A}$ is faithful and preserves projective objects.*
  (iv) *There is a functor $\mathsf{q}\colon \mathscr{A} \longrightarrow \mathscr{B}$ such that (q, i) is an adjoint pair.*
  (v) *There is a natural isomorphism $\mathsf{ql} \simeq \operatorname{\mathsf{Id}}_{\mathscr{B}}$ of functors.*

A cleft extension $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$ is equipped with three additional functors that are crucial in our investigations. We saw in (2.3) that there is a short exact sequence

$$0 \longrightarrow \operatorname{\mathsf{Ker}}\mu_A \longrightarrow \mathsf{le}(A) \xrightarrow{\ \mu_A\ } A \longrightarrow 0$$

for all A in $\mathscr{A}$. The assignment $A \mapsto \operatorname{Ker} \mu_A$ defines an endofunctor $G\colon \mathscr{A} \longrightarrow \mathscr{A}$ and therefore also an exact sequence of endofunctors on $\mathscr{A}$

$$(2.4) \qquad 0 \longrightarrow G \longrightarrow \mathsf{le} \xrightarrow{\mu_{(-)}} \mathsf{Id}_{\mathscr{A}} \longrightarrow 0.$$

Precompose the above exact sequence of functors with the functor $\mathsf{i}\colon \mathscr{B} \to \mathscr{A}$, and we obtain an exact sequence of functors

$$0 \longrightarrow G\mathsf{i} \longrightarrow \mathsf{lei} \xrightarrow{\mu_{\mathsf{i}(-)}} \mathsf{i} \longrightarrow 0.$$

Denote the functor $G\mathsf{i}\colon \mathscr{B} \to \mathscr{A}$ by H and view $\varphi\colon \mathsf{ei} \to \mathsf{Id}_{\mathscr{B}}$ as an identification. Then we have the exact sequence of funtors

$$(2.5) \qquad 0 \longrightarrow H \longrightarrow \mathsf{l} \xrightarrow{\mu_{\mathsf{i}(-)}} \mathsf{i} \longrightarrow 0.$$

Postcompose the above exact sequence with the functor $\mathsf{e}\colon \mathscr{A} \to \mathscr{B}$ and obtain the exact sequence

$$0 \to \mathsf{e}H \to \mathsf{el} \xrightarrow{\mathsf{e}(\mu_{\mathsf{i}(-)})} \mathsf{ei} \to 0.$$

Again, viewing $\varphi\colon \mathsf{ei} \to \mathsf{Id}_{\mathscr{B}}$ as an identification and denote the endofunctor $\mathsf{e}H$ on $\mathscr{B}$ by F. Then we obtain an exact sequence of endofunctors on $\mathscr{B}$

$$(2.6) \qquad 0 \longrightarrow F \longrightarrow \mathsf{el} \xrightarrow{\mathsf{e}(\mu_{\mathsf{i}(-)})} \mathsf{Id}_{\mathscr{B}} \longrightarrow 0.$$

The functors G, F and H introduced above are important functors associated to a cleft extension. We collect them in Definition 2.3.

Definition 2.3. Let $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$ be a cleft extension of abelian categories. Then we have endofunctors $G\colon \mathscr{A} \to \mathscr{A}$ and $F\colon \mathscr{B} \to \mathscr{B}$ given by the exact sequence (2.4) and (2.6)

$$(2.7) \qquad 0 \longrightarrow G \longrightarrow \mathsf{le} \xrightarrow{\mu_{(-)}} \mathsf{Id}_{\mathscr{A}} \longrightarrow 0$$

and

$$(2.8) \qquad 0 \longrightarrow F \longrightarrow \mathsf{el} \xrightarrow{\mathsf{e}(\mu_{\mathsf{i}(-)})} \mathsf{Id}_{\mathscr{B}} \longrightarrow 0.$$

In addition, there is a functor $H\colon \mathscr{B} \to \mathscr{A}$ given by the exact sequence (2.5)

$$(2.9) \qquad 0 \longrightarrow H \longrightarrow \mathsf{l} \xrightarrow{\mu_{\mathsf{i}(-)}} \mathsf{i} \longrightarrow 0.$$

Lemma 2.4 is an immediate consequence of (2.2).

Lemma 2.4. *Let $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$ be a cleft extension of abelian categories. Then the exact sequence (2.8) in Definition 2.3 splits.*

Another fact on cleft extensions we use later is the following result (see [7, Lemma 2.4]).

Lemma 2.5. *Let $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$ be a cleft extension of abelian categories. The following statements hold.*

   (i) *For any $n \geq 1$, there is a natural isomorphism $\mathsf{e}G^n \simeq F^n\mathsf{e}$.*
   (ii) *Let $n \geq 1$. Then $F^n = 0$ if and only if $G^n = 0$.*

In Sections 3 and 5 the following assumption on a cleft extension $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$ of abelian categories shall be of importance.

$$(2.10) \qquad \text{The functor } \mathsf{l} \text{ is exact and the functor } \mathsf{e} \text{ preserves projectives.}$$

2.2. **Cleft extensions arising from arrow removals.** In this subsection we recall from [7] that the arrow removal operation on an admissible path algebra induces a cleft extension with certain homological properties.

First we recall the definition of an arrow removal.

Definition 2.6. Let $\Lambda = kQ/I$ be an admissible quotient of a path algebra kQ over a field k. Suppose that there is a set of arrows $a_i\colon v_{e_i} \to v_{f_i}$ in Q for $i = 1, 2, \ldots, t$ which do not occur in a set of minimal generators of I in kQ and $\mathsf{Hom}_\Lambda(e_i\Lambda, f_j\Lambda) = 0$ for all i and j in $\{1, 2, \ldots, t\}$. Let $\Gamma = \Lambda/\Lambda\{\overline{a}_i\}_{i=1}^t\Lambda$. The natural projection $\pi\colon \Lambda \to \Gamma$ or just the pair Λ and $\Gamma = \Lambda/\Lambda\{\overline{a}_i\}_{i=1}^t\Lambda$ is an *arrow removal*.

The following result from [7] shows that the arrow removal operation induces a cleft extension between the corresponding module categories, i.e. mod-Λ is a cleft extension of mod-Γ, with certain homological properties.

Theorem 2.7 (For (i) [7, Corollary 4.3, Proposition 4.6], and for (ii) Proposition 4.6)**.** *Let $\Lambda = kQ/I$ be an admissible quotient of a path algebra kQ, where $\{a_i\}_{i=1}^t$ is a set of arrows in Q. Furthermore, let $\Gamma = \Lambda/\Lambda\{\overline{a}_i\}_{i=1}^t\Lambda$.*

(i) *The pair Λ and Γ is an arrow removal if and only if Λ is isomorphic to the trivial extension $\Gamma \ltimes P$, where $P = \oplus_{i=1}^t \Gamma e_i \otimes_k f_i\Gamma$ with $\mathsf{Hom}_\Gamma(e_i\Gamma, f_j\Gamma) = 0$ for all $i, j = 1, 2, \ldots, t$.*

(ii) *Let Λ and Γ be an arrow removal. Then the tuple $(\mathsf{mod}\text{-}\Gamma, \mathsf{mod}\text{-}\Lambda, \mathsf{e}, \mathsf{l}, \mathsf{i})$:*

(2.11)

$$
\begin{array}{ccccc}
F & \mathsf{q}=-\otimes_\Lambda\Lambda\Gamma_\Gamma & G & \mathsf{l}=-\otimes_\Gamma\Gamma\Lambda_\Lambda & F \\
& \mathsf{i}=\mathsf{Hom}_\Gamma({}_\Lambda\Gamma_\Gamma,-) & & \mathsf{e}=\mathsf{Hom}_\Lambda({}_\Gamma\Lambda_\Lambda,-) & \\
\mathsf{mod}\text{-}\Gamma & & \mathsf{mod}\text{-}\Lambda & & \mathsf{mod}\text{-}\Gamma \\
& \mathsf{p}=\mathsf{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda,-) & & \mathsf{r}=\mathsf{Hom}_\Gamma({}_\Lambda\Lambda_\Gamma,-) &
\end{array}
$$

is a cleft extension satisfying the following conditions, where F and G are as in Definition 2.3 *and sequences* (2.8) *and* (2.7):

(a) e *is faithful exact,*

(b) (l, e) *is an adjoint pair of functors,*

(c) $\mathsf{ei} \simeq 1_{\mathsf{mod}\text{-}\Gamma}$,

(d) l *and* r *are exact functors,*

(e) e *preserves projectives,*

(f) $\mathsf{Im}\, F \subseteq \mathsf{proj}(\Gamma)$ *and* $\mathsf{Im}\, G \subseteq \mathsf{Proj}(\Lambda)$,

(g) $F^2 = 0$.

From the proof of [7, Proposition 4.6 (iv)] we have the following description of the functors F and F^{op} (when we consider left modules).

Lemma 2.8. *Let $\Lambda = kQ/I$ be an admissible quotient of a path algebra kQ over a field k. For a set of arrows $a_i\colon v_{e_i} \to v_{f_i}$ in Q for $i = 1, 2, \ldots, t$ suppose that $\Lambda \to \Gamma = \Lambda/\Lambda\{\overline{a}\}_{i=1}^t\Lambda$ is an arrow removal. Then*

(a) *The endofunctor $F\colon \mathsf{mod}\text{-}\Gamma \to \mathsf{mod}\text{-}\Gamma$ is given as*

$$F = - \otimes_\Gamma \Gamma\{\overline{a}_i\}_{i=1}^t\Gamma\colon \mathsf{mod}\text{-}\Gamma \to \mathsf{mod}\text{-}\Gamma.$$

(b) *The endofunctor $F^{\mathsf{op}}\colon \mathsf{mod}\text{-}\Gamma^{\mathsf{op}} \to \mathsf{mod}\text{-}\Gamma^{\mathsf{op}}$ is given as*

$$F^{\mathsf{op}} = \Gamma\{\overline{a}_i\}_{i=1}^t\Gamma \otimes_\Gamma -\colon \mathsf{mod}\text{-}\Gamma^{\mathsf{op}} \to \mathsf{mod}\text{-}\Gamma^{\mathsf{op}}.$$

We remark that the above homological properties were used to show that the finiteness of the finitistic dimension of Λ can be reduced to the finiteness of the finitistic dimension of the arrow removal algebra Γ, see [7, Theorem A]. These homological properties are also used intensively in the sequel of the paper to show the invariance of Gorensteinness, singularity categories and the finite generation condition Fg for the Hochschild cohomology under the arrow removal operation. It is interesting that this operation gives rise to such a powerful cleft extension.

3. CLEFT EXTENSIONS AND EVENTUALLY HOMOLOGICAL ISOMORPHISMS

Let $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$ be a cleft extension of abelian categories. Then the functor $\mathsf{e} \colon \mathscr{A} \to \mathscr{B}$ is an exact functor, so it always induces homomorphism of the following Yoneda rings

$$\mathsf{e} \colon \mathsf{Ext}^*_{\mathscr{A}}(A, A) \to \mathsf{Ext}^*_{\mathscr{B}}(\mathsf{e}(A), \mathsf{e}(A))$$

for all A in $\mathscr{A}$. Recall from [8, Section 3] that e is called an *eventually homological isomorphism* if

$$\mathsf{Ext}^i_{\mathscr{A}}(A, A) \simeq \mathsf{Ext}^i_{\mathscr{B}}(\mathsf{e}(A), \mathsf{e}(A))$$

for every $i > t$ where t is some positive integer. For the minimal such t, the above isomorphism is called a *t-eventually homological isomorphism*. Note that in the definition we do not require that the isomorphism is induced by the functor e.

In this section we describe one situation where the functor e of a cleft extension is an eventually homological isomorphism. We start with the following result.

Lemma 3.1. *Let $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$ be a cleft extension of abelian categories such that condition (2.10) is satisfied. Then, for all $i \geq 1$ and for all $A, C \in \mathscr{A}$, the following diagram commutes*

$$\begin{array}{ccc}
\mathsf{Ext}^i_{\mathscr{A}}(C, A) & \xrightarrow{\;\mathsf{e}\;} & \mathsf{Ext}^i_{\mathscr{B}}(\mathsf{e}(C), \mathsf{e}(A)) \\
\| & & \simeq \Big\uparrow \theta_{\mathsf{e}(C), A} \\
\mathsf{Ext}^i_{\mathscr{A}}(C, A) & \xrightarrow{\;\mu^*_C\;} & \mathsf{Ext}^i_{\mathscr{A}}(\mathsf{le}(C), A)
\end{array}$$

where the vertical maps are isomorphisms.

Proof. Let $m \geq 1$, and let $f \in \mathsf{Ext}^m_{\mathscr{A}}(C, A)$ be represented by a morphism $f \colon \Omega^m_{\mathscr{A}}(C) \to A$. Then consider the following exact commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^m_{\mathscr{A}}(C) & \longrightarrow & P_{m-1} & \longrightarrow & P_{m-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow 0 \\
& & \downarrow f & & \downarrow & & \| & & & & \| & & \| \\
0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & P_{m-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow 0
\end{array}$$

Apply the exact functor e to this diagram and obtain the following diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathsf{e}(\Omega^m_{\mathscr{A}}(C)) & \longrightarrow & \mathsf{e}(P_{m-1}) & \longrightarrow & \mathsf{e}(P_{m-2}) & \longrightarrow & \cdots & \longrightarrow & \mathsf{e}(P_0) & \longrightarrow & \mathsf{e}(C) & \longrightarrow 0 \\
& & \downarrow \mathsf{e}(f) & & \downarrow & & \| & & & & \| & & \| \\
0 & \longrightarrow & \mathsf{e}(A) & \longrightarrow & \mathsf{e}(M) & \longrightarrow & \mathsf{e}(P_{m-2}) & \longrightarrow & \cdots & \longrightarrow & \mathsf{e}(P_0) & \longrightarrow & \mathsf{e}(C) & \longrightarrow 0
\end{array}$$

The lower row represents the image of f under the functor e.

By the adjunction $\theta\colon \mathsf{Hom}_{\mathscr{B}}(\mathsf{e}(\Omega^m_{\mathscr{A}}(C)), \mathsf{e}(A)) \simeq \mathsf{Hom}_{\mathscr{A}}(\mathsf{le}(\Omega^m_{\mathscr{A}}(C)), A)$ the morphism $\mathsf{e}(f)$ corresponds to $\theta(\mathsf{e}(f))\colon \mathsf{le}(\Omega^m_{\mathscr{A}}(C)) \to A$. This last morphism is equal to the composition of the morphisms

$$\mathsf{le}(\Omega^m_{\mathscr{A}}(C)) \xrightarrow{\mathsf{le}(f)} \mathsf{le}(A) \xrightarrow{\mu_A} A.$$

In addition we have the following two commutative diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{le}(A) & \longrightarrow & \mathsf{le}(M) & \longrightarrow & \mathsf{le}(\Omega^{m-1}_{\mathscr{A}}(C)) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle s} & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & \mathsf{le}(\Omega^{m-1}_{\mathscr{A}}(C)) & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \mu_{\Omega^{m-1}_{\mathscr{A}}(C)}} & & \\
0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & \Omega^{m-1}_{\mathscr{A}}(C) & \longrightarrow & 0
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{le}(\Omega^1_{\mathscr{A}}(C)) & \longrightarrow & \mathsf{le}(P_0) & \longrightarrow & \mathsf{le}(C) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mu_{\Omega^1_{\mathscr{A}}(C)}} & & \downarrow{\scriptstyle s'} & & \| & & \\
0 & \longrightarrow & \Omega^1_{\mathscr{A}}(C) & \longrightarrow & E' & \longrightarrow & \mathsf{le}(C) & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle \mu_C} & & \\
0 & \longrightarrow & \Omega^1_{\mathscr{A}}(C) & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

with $ts = \mu_M$ and $t's' = \mu_{P_0}$. Using these commutative diagrams we can construct the following commutative diagram

$$
\begin{array}{ccccccccccccccc}
0 & \to & \mathsf{le}(\Omega^m_{\mathscr{A}}(C)) & \to & \mathsf{le}(P_{m-1}) & \to & \mathsf{le}(P_{m-2}) & \to & \cdots & \to & \mathsf{le}(P_1) & \to & \mathsf{le}(P_0) & \to & \mathsf{le}(C) & \to & 0 \\
& & \downarrow{\scriptstyle \mathsf{le}(f)} & & \downarrow & & \| & & & & \| & & \| & & \| & & \\
0 & \to & \mathsf{le}(A) & \to & \mathsf{le}(M) & \to & \mathsf{le}(P_{m-2}) & \to & \cdots & \to & \mathsf{le}(P_1) & \to & \mathsf{le}(P_0) & \to & \mathsf{le}(C) & \to & 0 \\
& & \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle s} & & \| & & & & \| & & \| & & \| & & \\
0 & \to & A & \to & E & \to & \mathsf{le}(P_{m-2}) & \to & \cdots & \to & \mathsf{le}(P_1) & \to & \mathsf{le}(P_0) & \to & \mathsf{le}(C) & \to & 0 \\
& & \| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \mu_{P_{m-2}}} & & & & \downarrow{\scriptstyle \mu_{P_1}} & & \downarrow{\scriptstyle s'} & & \| & & \\
0 & \to & A & \to & M & \to & P_{m-2} & \to & \cdots & \to & P_1 & \to & E' & \to & \mathsf{le}(C) & \to & 0 \\
& & \| & & \| & & \| & & & & \| & & \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle \mu_C} & & \\
0 & \to & A & \to & M & \to & P_{m-2} & \to & \cdots & \to & P_1 & \to & P_0 & \to & C & \to & 0
\end{array}
$$

The third row in the above diagram corresponds to the $\theta^{-1}_{\mathsf{e}(C),A}(\mathsf{e}(f))$, and the fourth row in the above diagram corresponds to the image of f under the map μ_C^*. It follows that the diagram in the statement is commutative.

Finally, using the adjunction (l, e) and since both functors are exact and preserve projectives, it follows immediately that the map $\theta_{\mathsf{e}(C),A}$ is an isomorphism. $\qquad\square$

If the map induced by μ_C^* in Lemma 3.1 is an isomorphism for all C and all $i \gg 0$, it will follow that the functor e is an eventually homological isomorphism. Using a homological condition on the functor G (see Definition 2.3 and (2.7)) the next result describes a situation when μ_C^* induces such an isomorphism.

Theorem 3.2. *Let* $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$ *be a cleft extension of abelian categories satisfying condition* (2.10). *Assume that*

$$\sup\{\mathsf{pd}_{\mathscr{A}} G(A) \mid A \in \mathscr{A}\} \leq n_{\mathscr{A}}$$

for some integer $n_{\mathscr{A}}$. *Then the functor* $\mathsf{e} \colon \mathscr{A} \to \mathscr{B}$ *is an* $n_{\mathscr{A}} + 1$-*eventually homological isomorphism.*

Proof. Using the commutative diagram in Lemma 3.1 and the long exact sequence induced from the exact sequence

$$0 \to G(C) \to \mathsf{le}(C) \to C \to 0$$

applying the functor $\mathsf{Ext}_{\mathscr{A}}^*(-, A)$, in fact the functor e induces an isomorphism between $\mathsf{Ext}_{\mathscr{A}}^i(C, A)$ and $\mathsf{Ext}_{\mathscr{B}}^i(\mathsf{e}(C), \mathsf{e}(A))$ for $i > n_{\mathscr{A}} + 1$. The claim follows from this. $\qquad\square$

Applying Theorem 3.2 to the cleft extension of the arrow removal, see Theorem 2.7, we get the following consequence. This result constitutes part (i) of the Main Theorem presented in the Introduction.

Corollary 3.3. *Let* $\Lambda = kQ/I$ *be an admissible quotient of a path algebra* kQ *over a field* k, *and assume that* $\Gamma = \Lambda/\langle\{\overline{a}_i\}_{i=1}^t\rangle$ *is an arrow removal of* Λ *for the arrows* $\{a_i\}_{i=1}^t$ *in* Q. *Then the functor* $\mathsf{e} \colon \mathsf{mod}\text{-}\Lambda \to \mathsf{mod}\text{-}\Gamma$ *is a* 1-*eventually homological isomorphism.*

4. Gorenstein algebras and singular equivalences

In this section we show that Gorensteinness and singularity categories are invariant under the arrow removal operation. Recall from [1] that a finite dimensional algebra Δ is called *Gorenstein* if $\mathsf{id}_{\Delta^{\mathrm{op}}} {}_{\Delta}\Delta < \infty$ and $\mathsf{id}_\Delta \Delta_\Delta < \infty$. Furthermore, recall from [3] that the *singularity category* $\mathsf{D}_{\mathsf{sg}}(\mathscr{A})$ of an abelian category $\mathscr{A}$ with enough projectives is given by the Verdier quotient $\mathsf{D}^b(\mathscr{A})/\mathsf{D}^{\mathrm{perf}}(\mathscr{A})$. Here $\mathsf{D}^{\mathrm{perf}}(\mathscr{A})$ denotes the full triangulated subcategory of $\mathsf{D}^b(\mathscr{A})$ consisting of the perfect objects, i.e. complexes quasi-isomorphic to bounded complexes with components in $\mathsf{Proj}\,\mathscr{A}$.

4.1. **Gorenstein algebras.** Let $\Lambda = kQ/I$ be an admissible quotient of a path algebra kQ, and suppose $\{a_i\}_{i=1}^t$ is a set of arrows in Q such that $\Gamma = \Lambda/\langle\{\overline{a}_i\}_{i=1}^t\rangle$ is an arrow removal. The key fact for the invariance of Gorensteiness is that the functor $\mathsf{e} \colon \mathsf{mod}\text{-}\Lambda \to \mathsf{mod}\text{-}\Gamma$ is an eventually homological isomorphism, as shown in Corollary 3.3. The reason is the following result which we formulate, for simplicity, for module categories over finite dimensional algebras.

Theorem 4.1 ([8, Theorem 4.3 (v)]). *Let* $T \colon \mathsf{mod}\text{-}\Delta \to \mathsf{mod}\text{-}\Theta$ *be a functor which is essentially surjective and an eventually homological isomorphism. Then* Δ *is Gorenstein if and only if* Θ *is Gorenstein.*

We can now show that Gorensteinness is indeed invariant under the arrow removal operation. In particular, the following result is an immediate consequence of Corollary 3.3 and Theorem 4.1. This result constitutes part (ii) of the Main Theorem presented in the Introduction.

Corollary 4.2. *Let $\Lambda = kQ/I$ be an admissible quotient of a path algebra kQ, and suppose that $\Gamma = \Lambda/\langle\{\overline{a}_i\}_{i=1}^t\rangle$ is an arrow removal of Λ for the arrows $\{a_i\}_{i=1}^t$ in Q. Then Λ is Gorenstein if and only if Γ is Gorenstein.*

4.2. **Singularity categories.** Our aim in this subsection is to show that the singularity categories of the algebras under an arrow removal are triangle equivalent.

For this we have Lemma 4.3 in the abstract setting of cleft extensions of abelian categories.

Lemma 4.3. *Let $(\mathscr{B}, \mathscr{A}, \mathsf{e}, \mathsf{l}, \mathsf{i})$ be a cleft extension of abelian categories with enough projectives. Consider the following conditions.*

  (i) $\sup\{\mathsf{pd}_{\mathscr{B}}\,\mathsf{e}(P) \mid P \in \mathsf{Proj}(\mathscr{A})\} = p_{\mathscr{A}}$ *for some integer $p_{\mathscr{A}}$.*
  (ii) $\sup\{\mathsf{pd}_{\mathscr{A}}\,\mathsf{i}(F) \mid F \in \mathsf{Proj}(\mathscr{B})\} = p_{\mathscr{B}}$ *for some integer $p_{\mathscr{B}}$.*
  (iii) $\sup\{\mathsf{pd}_{\mathscr{A}}\,H(B) \mid B \in \mathscr{B}\} = n_H$ *for some integer n_H.*
  (iv) $\sup\{\mathsf{pd}_{\mathscr{A}}\,G(A) \mid A \in \mathscr{A}\} = n_G$ *for some integer n_G.*

  (a) *If* (ii) *holds, then* $\mathsf{i}\colon \mathscr{B} \to \mathscr{A}$ *induces a functor* $\mathsf{i}\colon \mathsf{D}_{\mathsf{sg}}(\mathscr{B}) \to \mathsf{D}_{\mathsf{sg}}(\mathscr{A})$.
  (b) *If* (i) *holds, then* $\mathsf{e}\colon \mathscr{A} \to \mathscr{B}$ *induces a functor* $\mathsf{e}\colon \mathsf{D}_{\mathsf{sg}}(\mathscr{A}) \to \mathsf{D}_{\mathsf{sg}}(\mathscr{B})$.
  (c) *If* (i) *and* (ii) *hold, then* $\mathsf{ei}\colon \mathsf{D}_{\mathsf{sg}}(\mathscr{B}) \to \mathsf{D}_{\mathsf{sg}}(\mathscr{B})$ *is isomorphic to the identity functor.*
  (d) *If* l *is an exact functor, then* $\mathsf{l}\colon \mathscr{B} \to \mathscr{A}$ *induces a functor* $\mathsf{l}\colon \mathsf{D}_{\mathsf{sg}}(\mathscr{B}) \to \mathsf{D}_{\mathsf{sg}}(\mathscr{A})$.
  (e) *If* (i) *and* (iv) *hold and* l *is an exact functor, then* $\mathsf{le}\colon \mathsf{D}_{\mathsf{sg}}(\mathscr{A}) \to \mathsf{D}_{\mathsf{sg}}(\mathscr{A})$ *is isomorphic to the identity functor.*
  (f) *If* (i)–(iv) *hold and the functor* l *is exact, then* $\mathsf{e}\colon \mathsf{D}_{\mathsf{sg}}(\mathscr{A}) \to \mathsf{D}_{\mathsf{sg}}(\mathscr{B})$ *is a singular equivalence.*

Proof. (a) Since the functor $\mathsf{i}\colon \mathscr{B} \to \mathscr{A}$ is exact, we have an induced functor $\mathsf{i}\colon \mathsf{D}(\mathscr{B}) \to \mathsf{D}(\mathscr{A})$. By property (ii) the functor i induce a functor $\mathsf{i}\colon \mathsf{D}^{\mathrm{perf}}(\mathscr{B}) \to \mathsf{D}^{\mathrm{perf}}(\mathscr{A})$. The claim follows from this.

(b) This follows as the claim in (a).

(c) This follows from (a) and (b) and the fact that $\mathsf{ei} \simeq \mathrm{Id}_{\mathscr{B}}$.

(d) Since the functor $\mathsf{l}\colon \mathscr{B} \to \mathscr{A}$ preserves projective objects, the claim is immediate.

(e) By (b) and (d) the functors e and l induce functors on the singularity categories. Having the exact sequence

$$0 \to G(A) \to \mathsf{le}(A) \to A \to 0$$

from (2.7) in Definition 2.3 and property (iv) ensure that the composition of l and e is isomorphic to the identity.

(f) By (c) the composition of e and i is the identity functor on $\mathsf{D}_{\mathsf{sg}}(\mathscr{B})$. From (e) the composition of l and e is isomorphic to the identity functor on $\mathsf{D}_{\mathsf{sg}}(\mathscr{A})$. Using the exact sequence of functors

$$0 \to H \to \mathsf{l} \to \mathsf{i} \to 0$$

from (2.9) in Definition 2.3 and property (iii), we infer that le and ie are isomorphic as endofunctors of $\mathsf{D}_{\mathsf{sg}}(\mathscr{A})$. The claim follows from this. $\square$

As a consequence of Lemma 4.3 and Theorem 2.7 we have the following. This result constitutes part (iii) of the Main Theorem presented in the Introduction. Below the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ of Λ is the Verdier quotient $\mathsf{D}^b(\mathsf{mod}\text{-}\Lambda)/\mathsf{D}^{\mathrm{perf}}(\Lambda)$.

Corollary 4.4. *Let $\Lambda = kQ/I$ be an admissible quotient of a path algebra kQ over a field k and suppose that $\Gamma = \Lambda/\langle\{\overline{a}_i\}_{i=1}^t\rangle$ is an arrow removal of Λ for the arrows $\{a_i\}_{i=1}^t$ in Q. Then the functor $\mathsf{e}\colon \mathsf{mod}\text{-}\Lambda \to \mathsf{mod}\text{-}\Gamma$ induces a singular equivalence between Λ and Γ:*

$$\mathsf{e}\colon \mathsf{D}_{\mathsf{sg}}(\Lambda) \xrightarrow{\ \simeq\ } \mathsf{D}_{\mathsf{sg}}(\Gamma).$$

Example 4.5 shows that algebras can be of finite, tame or wild representation type and still be singular equivalent to each other.

Example 4.5. Let Q_n be the quiver given by
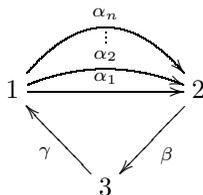
for $n \geq 1$. For a field k consider the relations $\rho = \{\alpha_1\beta, \beta\gamma, \gamma\alpha_1\}$ in kQ_n, and define the algebra $\Lambda_n = kQ_n/\langle\rho\rangle$. Then the algebras Λ_1 and Λ_n are related by arrow removal for all $n \geq 2$, so that they are all singular equivalent by the above corollary, where Λ_1 is of finite type, Λ_2 is of tame type and Λ_n is wild type for $n \geq 3$.

5. Cleft extensions and the Fg condition

This section is devoted to study the behaviour of the Fg condition for Hochschild cohomology under the arrow removal operation. As mentioned in the Main result of the Introduction, we prove that the Fg condition is invariant under an arrow removal. Recall from [5,10] that an algebra Λ over a commutative ring k such that Λ is flat as a module over k satisfies the Fg condition if the following is true:

   (i) The Hochschild cohomology ring $\mathsf{HH}^*(\Lambda)$ of Λ is noetherian.
   (ii) The $\mathsf{HH}^*(\Lambda)$-module $\mathsf{Ext}^*_\Lambda(\Lambda/\mathsf{rad}\Lambda, \Lambda/\mathsf{rad}\Lambda)$ is finitely generated.

Towards this we start with the following result where we show that starting with an arrow removal and passing to the corresponding enveloping algebras we still get a cleft extension.

Proposition 5.1. *Let $\Lambda = kQ/I$ be an admissible quotient of a path algebra kQ, and suppose $\{a_i\}_{i=1}^t$ is a set of arrows in Q such that $\Gamma = \Lambda/\langle\{\overline{a}_i\}_{i=1}^t\rangle$ is an arrow removal. Let $\nu\colon \Gamma \to \Lambda$ and $\pi\colon \Lambda \to \Gamma$ be the algebra homomorphism defining the cleft extension. Then the following assertions hold.*

   (i) *The algebra homomorphisms*

$$\nu \otimes \nu\colon \Gamma^{\mathsf{op}} \otimes_k \Gamma \to \Lambda^{\mathsf{op}} \otimes_k \Lambda$$

   and

$$\pi \otimes \pi\colon \Lambda^{\mathsf{op}} \otimes_k \Lambda \to \Gamma^{\mathsf{op}} \otimes_k \Gamma$$

   define $\Gamma^{\mathrm{env}} = \Gamma^{\mathsf{op}} \otimes_k \Gamma$ and $\Lambda^{\mathrm{env}} = \Lambda^{\mathsf{op}} \otimes_k \Lambda$ as a cleft extension.
   (ii) *$_{\Gamma^{\mathrm{env}}}\Lambda^{\mathrm{env}}$ and $\Lambda^{\mathrm{env}}_{\Gamma^{\mathrm{env}}}$ are projective modules.*

(iii) *The restriction functor* $\mathsf{e}^{\mathrm{env}}$ *along the algebra homomorphism* $\nu \otimes \nu$ *preserves projective modules (and is exact), and the functor*

$$\mathsf{l}^{\mathrm{env}} = - \otimes_{\Gamma^{\mathrm{env}}} \Lambda^{\mathrm{env}} \colon \ \mathsf{mod}\text{-}\Gamma^{\mathrm{env}} \to \mathsf{mod}\text{-}\Lambda^{\mathrm{env}}$$

is exact. In particular the condition (2.10) *is satisfied for the cleft extension* $\Lambda^{\mathrm{env}} \to \Gamma^{\mathrm{env}}$.

Proof. (i) It is straightforward to see that $(\pi \otimes \pi)(\nu \otimes \nu) = \mathsf{id}_{\Gamma^{\mathrm{env}}}$.

(ii) Since Λ and Γ is an arrow removal, $_\Gamma\Lambda$ and Λ_Γ are projective modules. Since $_{\Gamma^{\mathrm{env}}}\Lambda^{\mathrm{env}} \simeq {}_\Gamma\Lambda^{\mathsf{op}} \otimes_k \Lambda_\Gamma$, it follows that $_{\Gamma^{\mathrm{env}}}\Lambda^{\mathrm{env}}$ is a projective module over Γ^{env}. Similarly we infer that $\Lambda^{\mathrm{env}}_{\Gamma^{\mathrm{env}}}$ is a projective module over Γ^{env}.

(iii) Both of the claims follow from (ii). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The functors F and G are crucial for a cleft extension. Next we see how the F- and the G-functors are connected for a cleft extension of algebras and the corresponding cleft extension for the enveloping algebras.

Lemma 5.2. *Let Λ and Γ be a cleft extension given by the algebra homomorphisms $\Gamma \xrightarrow{\nu} \Lambda \xrightarrow{\pi} \Gamma$. Then for a Γ-bimodule B the following hold.*

(a) *The endofunctor F of $\mathsf{mod}\text{-}\Gamma$ applied to B defines a Γ-bimodule and the exact sequence*

$$0 \to F(B) \to B \otimes_\Gamma \Lambda \xrightarrow{\mathrm{mult}(1 \otimes \pi)} B \to 0$$

obtained from (2.8) *in Definition 2.3 splits as a sequence of Γ-bimodules.*

(b) *We have the isomorphism*

$$F(\Lambda \otimes_\Gamma B) \simeq \Lambda \otimes_\Gamma F(B).$$

(c) *Let F^{op} be the endofunctor of $\mathsf{mod}\text{-}\Gamma^{\mathsf{op}}$ considering Γ^{op} and Λ^{op} as a cleft extension of algebras. We have*

$$F^{\mathrm{env}}(B) \simeq (\Lambda \otimes_\Gamma F(B_\Gamma)) \oplus F^{\mathsf{op}}(_\Gamma B).$$

For a Λ-bimodule B the following hold.

(d) *The endofunctor G of $\mathsf{mod}\text{-}\Lambda$ applied to B defines a Λ-bimodule.*

(e) *When Λ_Γ is projective, we have*

$$G(\Lambda \otimes_\Gamma (_\Gamma B_\Lambda)) = \Lambda \otimes_\Gamma G(B).$$

(f) *Let G^{op} be the endofunctor of $\mathsf{mod}\text{-}\Lambda^{\mathsf{op}}$ considering Γ^{op} and Λ^{op} as a cleft extension of algebras. For a Λ-bimodule B we have an exact sequence*

$$0 \to \Lambda \otimes_\Gamma G(B_\Lambda) \to G^{\mathrm{env}}(B) \to G^{\mathsf{op}}(_\Lambda B) \to 0,$$

when Λ_Γ is projective.

(g) *In this final statement let Λ and Γ be an arrow removal given by a set of arrows $\{\overline{a}_i\}_{i=1}^t$ in the quiver of the algebra Λ as defined in subsection 2.2. Then the following hold.*

    (i) $(F^{\mathrm{env}})^2(\mathsf{e}^{\mathrm{env}}(\Lambda)) = 0$,

    (ii) $(G^{\mathrm{env}})^2(\Lambda) = 0$,

    (iii) $G^{\mathrm{env}}(\Lambda)$ *is a projective Λ-bimodule.*

Proof. (a) Let B be a Γ-bimodule and consider the exact sequence

$$0 \to F(B_\Gamma) \to B \otimes_\Gamma \Lambda \xrightarrow{\mathrm{mult}(1 \otimes \pi)} B_\Gamma \to 0,$$

where the map mult is a homomorphism of Γ-bimodules. This implies that $F(B_\Gamma)$ is a Γ-bimodule whenever B is a Γ-bimodule. The above exact sequence splits as right Γ-modules by Lemma 2.4, but the splitting $(1 \otimes \nu)\mathrm{mult}^{-1}$ is also a homomorphism of Γ-bimodules. Hence the final claim follows.

(b) Let B be a Γ-bimodule and consider the exact sequence

$$0 \to F(B_\Gamma) \to B \otimes_\Gamma \Lambda_\Gamma \xrightarrow{\mathrm{mult}(1 \otimes \pi)} B_\Gamma \to 0,$$

which splits as an exact sequence of Γ-bimodules. Tensoring this split exact sequence with $\Lambda \otimes_\Gamma -$ we get the following exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda \otimes_\Gamma F(B_\Gamma) & \longrightarrow & \Lambda \otimes_\Gamma B \otimes_\Gamma \Lambda_\Gamma & \xrightarrow{1 \otimes \mathrm{mult}(1 \otimes \pi)} & \Lambda \otimes_\Gamma B_\Gamma & \longrightarrow & 0 \\
& & \downarrow{\simeq} & & \| & & \| & & \\
0 & \longrightarrow & F(\Lambda \otimes_\Gamma B_\Gamma) & \longrightarrow & \Lambda \otimes_\Gamma B \otimes_\Gamma \Lambda_\Gamma & \xrightarrow{\mathrm{mult}((1_{\Lambda \otimes_\Gamma B}) \otimes \pi)} & \Lambda \otimes_\Gamma B_\Gamma & \longrightarrow & 0
\end{array}
$$

The claim follows from this.

(c) Recall that F^{env} is given by the exact sequence

$$0 \to F^{\mathrm{env}} \to \mathsf{e}^{\mathrm{env}} \mathsf{l}^{\mathrm{env}} \to \mathrm{Id}_{\mathsf{mod}\text{-}\Gamma^{\mathrm{env}}} \to 0.$$

Let B be a Γ-bimodule. Then $\mathsf{l}^{\mathrm{env}}(B) = \Lambda \otimes_\Gamma B \otimes_\Gamma \Lambda$, so that

$$\mathsf{e}^{\mathrm{env}} \mathsf{l}^{\mathrm{env}}(B) = {}_\Gamma\Lambda \otimes_\Gamma B \otimes_\Gamma \Lambda_\Gamma.$$

We construct the following commutative diagram

where the second row is split exact by Lemma 2.4. This implies the first isomorphism below

$$\Lambda \otimes_\Gamma B \otimes_\Gamma \Lambda \simeq F^{\mathrm{env}}(B) \oplus B$$
$$\simeq \Lambda \otimes_\Gamma F(B) \oplus F^{\mathrm{op}}(B) \oplus B.$$

Since the first row in the above diagram is a split exact sequence by (a) and the first column is a pullback of the first row, the second isomorphism follows. Cancelling the direct summand B on each side implies that $F^{\mathrm{env}}(B) \simeq \Lambda \otimes_\Gamma F(B) \oplus F^{\mathrm{op}}(B)$.

(d) Let B be a Λ-bimodule and consider the exact sequence

$$(5.1) \qquad 0 \to G(B_\Lambda) \to B \otimes_\Gamma \Lambda \xrightarrow{\mathrm{mult}} B_\Lambda \to 0,$$

where the map mult is a homomorphism of Λ-bimodules. This implies that $G(B_\Lambda)$ is a Λ-bimodule whenever B is a Λ-bimodule.

(e) Let B be a Λ-bimodule. Since Λ_Γ is projective, tensoring the exact sequence (5.1) with $\Lambda \otimes_\Gamma -$ leaves it exact and we obtain the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda \otimes_\Gamma G(B_\Lambda) & \longrightarrow & \Lambda \otimes_\Gamma B \otimes_\Gamma \Lambda & \xrightarrow{1_\Lambda \otimes \mathrm{mult}_B} & \Lambda \otimes_\Gamma B & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & G(\Lambda \otimes_\Gamma B) & \longrightarrow & \Lambda \otimes_\Gamma B \otimes_\Gamma \Lambda & \xrightarrow{\mathrm{mult}_{\Lambda \otimes_\Gamma B}} & \Lambda \otimes_\Gamma B & \longrightarrow & 0
\end{array}
$$

The claim follows from this.

(f) This follows in a similar way as for F^{env}, and it is left to the reader.

(g) Let Λ and Γ be an arrow removal, and let B be a Γ-bimodule. Then

$$
\begin{aligned}
(F^{\mathrm{env}})^2(B) &= F^{\mathrm{env}}(F^{\mathrm{env}}(B)), \\
&\simeq F^{\mathrm{env}}(\Lambda \otimes_\Gamma F(B) \oplus F^{\mathrm{op}}(B)), \quad \text{using (c)} \\
&\simeq F^{\mathrm{env}}(\Lambda \otimes_\Gamma F(B)) \oplus F^{\mathrm{env}}(F^{\mathrm{op}}(B)), \text{ using additivity} \\
&= \Lambda \otimes_\Gamma F((\Lambda \otimes_\Gamma F(B_\Gamma))_\Gamma) \oplus F^{\mathrm{op}}(_\Gamma(\Lambda \otimes_\Gamma F(B_\Gamma))) \\
&\qquad \oplus \Lambda \otimes_\Gamma F(F^{\mathrm{op}}(_\Gamma B)_\Gamma) \oplus F^{\mathrm{op}}(F^{\mathrm{op}}(_\Gamma B)), \quad \text{using (c)} \\
&= \Lambda \otimes_\Gamma F^2(\Lambda \otimes_\Gamma B_\Gamma) \oplus F^{\mathrm{op}} F(\Lambda \otimes_\Gamma B) \\
&\qquad \oplus \Lambda \otimes_\Gamma F(F^{\mathrm{op}}(_\Gamma B)_\Gamma) \oplus F^{\mathrm{op}}(F^{\mathrm{op}}(B)), \quad \text{using (b).}
\end{aligned}
$$

Since $F^2 = 0$ and $(F^{\mathrm{op}})^2 = 0$ for an arrow removal by Theorem 2.7 (ii) (g), we have

$$(F^{\mathrm{env}})^2(B) = F^{\mathrm{op}} F(\Lambda \otimes_\Gamma B) \oplus \Lambda \otimes_\Gamma F F^{\mathrm{op}}(B).$$

When we let $\langle \{\overline{a}_i\}_{i=1}^t \rangle$ denote the Γ-sub-bimodule of Λ generated by $\{\overline{a}_i\}_{i=1}^t$, we have by Lemma 2.8 that

$$F^{\mathrm{op}} F(\Lambda \otimes_\Gamma B) = \langle \{\overline{a}_i\}_{i=1}^t \rangle \otimes_\Gamma \Lambda \otimes_\Gamma B \otimes_\Gamma \langle \{\overline{a}_i\}_{i=1}^t \rangle$$

and

$$F F^{\mathrm{op}}(B) = \langle \{\overline{a}_i\}_{i=1}^t \rangle \otimes_\Gamma B \otimes_\Gamma \langle \{\overline{a}_i\}_{i=1}^t \rangle.$$

When we specialize to $B = {}_\Gamma \Lambda_\Gamma = \mathrm{e}^{\mathrm{env}}(\Lambda)$ and use that $\Lambda \simeq \Gamma \oplus \langle \{\overline{a}_i\}_{i=1}^t \rangle$, then

$$
\begin{aligned}
F F^{\mathrm{op}}(\mathrm{e}^{\mathrm{env}}(\Lambda)) &= \langle \{\overline{a}_i\}_{i=1}^t \rangle \otimes_\Gamma (\Gamma \oplus \langle \{\overline{a}_i\}_{i=1}^t \rangle) \otimes_\Gamma \langle \{\overline{a}_i\}_{i=1}^t \rangle \\
&\simeq \langle \{\overline{a}_i\}_{i=1}^t \rangle \otimes_\Gamma \langle \{\overline{a}_i\}_{i=1}^t \rangle \\
&\qquad \oplus \langle \{\overline{a}_i\}_{i=1}^t \rangle \otimes_\Gamma \langle \{\overline{a}_i\}_{i=1}^t \rangle \otimes_\Gamma \langle \{\overline{a}_i\}_{i=1}^t \rangle \\
&= 0,
\end{aligned}
$$

since $f_j \Gamma e_i = 0$ for all $i, j = 1, 2, \ldots, t$. For similar reasons we obtain that $F^{\mathrm{op}} F(\Lambda \otimes_\Gamma \Lambda) = 0$ and consequently

$$(F^{\mathrm{env}})^2(\mathrm{e}^{\mathrm{env}}(\Lambda)) = 0.$$

Since $\mathsf{e}^{\mathrm{env}}(G^{\mathrm{env}}(B))^2 \simeq (F^{\mathrm{env}})^2(\mathsf{e}^{\mathrm{env}}(B))$ by Lemma 2.5 (i) and $\mathsf{e}^{\mathrm{env}}$ is faithful, we infer that $(G^{\mathrm{env}})^2(\Lambda) = 0$. Using similar arguments as above $F^{\mathrm{env}}(\mathsf{e}^{\mathrm{env}}(\Lambda)) \simeq \langle\{\overline{a}_i\}_{i=1}^t\rangle^{\oplus 2}$ as a Γ-bimodule. For an arrow removal $\langle\{\overline{a}_i\}_{i=1}^t\rangle$ is a projective Γ-bimodule. Then by Lemma 2.5 (i) $\mathsf{e}^{\mathrm{env}}(G^{\mathrm{env}}(\Lambda)) \simeq F^{\mathrm{env}}(\mathsf{e}^{\mathrm{env}}(\Lambda))$ and it is projective. Since the functor $\mathsf{l}^{\mathrm{env}}$ preserves projective modules, the bimodule $\mathsf{l}^{\mathrm{env}}\mathsf{e}^{\mathrm{env}}G^{\mathrm{env}}(\Lambda)$ is projective. We have the exact sequence

$$0 \to (G^{\mathrm{env}})^2(\Lambda) \to \mathsf{l}^{\mathrm{env}}\mathsf{e}^{\mathrm{env}}G^{\mathrm{env}}(\Lambda) \to G^{\mathrm{env}}(\Lambda) \to 0,$$

which implies that $G^{\mathrm{env}}(\Lambda) \simeq \mathsf{l}^{\mathrm{env}}\mathsf{e}^{\mathrm{env}}G^{\mathrm{env}}(\Lambda)$ is a projective Λ-bimodule. $\qquad\square$

The following result establishes a close relationship between the Hochschild cohomology rings for the algebras in an arrow removal. The interested reader is suggested to compare the isomorphism below with [4, Theorem 4.6].

Proposition 5.3. *If $\pi\colon \Lambda \to \Gamma$ is an arrow removal, then*

$$\mathsf{Ext}^*_{\Lambda^{\mathrm{env}}}(\Lambda, \Lambda) \simeq \mathsf{Ext}^*_{\Gamma^{\mathrm{env}}}(\Gamma, \Gamma \oplus \mathsf{Ker}\,\pi)$$

is an isomorphism for $* > 1$.

Proof. As above we have the exact sequence

$$0 \to G^{\mathrm{env}}(\Lambda) \to \mathsf{l}^{\mathrm{env}}\mathsf{e}^{\mathrm{env}}(\Lambda) \to \Lambda \to 0.$$

By Lemma 5.2 (g) the bimodule $G^{\mathrm{env}}(\Lambda)$ is projective. The condition (2.10) is satisfied for the cleft extension $\Lambda^{\mathrm{env}} \to \Gamma^{\mathrm{env}}$ (see Proposition 5.1 (iii)), so that we can use Lemma 3.1 to obtain

$$\mathsf{Ext}^*_{\Lambda^{\mathrm{env}}}(\Lambda, \Lambda) \simeq \mathsf{Ext}^*_{\Gamma^{\mathrm{env}}}(\mathsf{e}^{\mathrm{env}}(\Lambda), \mathsf{e}^{\mathrm{env}}(\Lambda))$$

for $* > 1$. The restriction $\mathsf{e}^{\mathrm{env}}(\Lambda) \simeq \Gamma \oplus \mathsf{Ker}\,\pi$, where $\mathsf{Ker}\,\pi = \langle\{\overline{a}_i\}_{i=1}^t\rangle$ is a projective Γ-bimodule. This implies that

$$\mathsf{Ext}^*_{\Lambda^{\mathrm{env}}}(\Lambda, \Lambda) \simeq \mathsf{Ext}^*_{\Gamma^{\mathrm{env}}}(\Gamma, \Gamma \oplus \mathsf{Ker}\,\pi),$$

for $* > 1$ and it completes the proof. $\qquad\square$

For the Fg-property to be preserved for an arrow removal, not only the Hochschild cohomology rings need to be related, but also their action on the Ext-groups must respect each other, in order to apply general results from [8, Proposition 6.4]. The following two results prepare for this.

Lemma 5.4. *Let $\pi\colon \Lambda \to \Gamma$ be an arrow removal. Let M be a right Λ-module and B a Λ-bimodule. Then the map*

$$\mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(B) \xrightarrow{\varphi} \mathsf{e}(M \otimes_\Lambda B)$$

given by $m \otimes b \mapsto m \otimes b$ is well-defined, functorial in both variables, and an onto map of right Γ-modules.

Proof. The module $\mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(B) = M_\Gamma \otimes_{\Gamma} {}_\Gamma B_\Gamma$ and the module $\mathsf{e}(M \otimes_\Lambda B) = M \otimes_\Lambda B_\Gamma$. Therefore the map φ is the natural projection. $\qquad\square$

Proposition 5.5. *Let* $\pi\colon \Lambda \to \Gamma$ *be an arrow removal. The following diagram is commutative*

$$
\begin{array}{ccc}
\mathsf{Ext}^*_{\Lambda^{\mathrm{env}}}(\Lambda, \Lambda) & \xrightarrow{\ M\otimes_\Lambda -\ } & \mathsf{Ext}^*_\Lambda(M, M) \\
& & \downarrow{\scriptstyle \mathsf{e}} \\
& & \mathsf{Ext}^*_\Gamma(\mathsf{e}(M), \mathsf{e}(M)) \\
\downarrow{\scriptstyle \mathsf{e}^{\mathrm{env}}} & & \downarrow{\scriptstyle \mathsf{Ext}^*_\Gamma(\varphi,-)} \\
& & \mathsf{Ext}^*_\Gamma(\mathsf{e}(M)\otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda), \mathsf{e}(M)) \\
& & \uparrow{\scriptstyle \mathsf{Ext}^*_\Gamma(-,\varphi)} \\
\mathsf{Ext}^*_{\Gamma^{\mathrm{env}}}(\mathsf{e}^{\mathrm{env}}(\Lambda), \mathsf{e}^{\mathrm{env}}(\Lambda)) & \xrightarrow{\ \mathsf{e}(M)\otimes_\Gamma -\ } & \mathsf{Ext}^*_\Gamma(\mathsf{e}(M)\otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda), \mathsf{e}(M)\otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda))
\end{array}
$$

Proof. Let $\eta\colon \Omega^n_{\Lambda^{\mathrm{env}}}(\Lambda) \to \Lambda$ represent an element in $\mathsf{Ext}^n_{\Lambda^{\mathrm{env}}}(\Lambda, \Lambda)$. As an extension η correspond to the lower row in the following commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \Omega^n_{\Lambda^{\mathrm{env}}}(\Lambda) & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & \Lambda & \longrightarrow 0 \\
& & \downarrow{\scriptstyle \eta} & & \downarrow & & \| & & \| & & \| \\
0 & \longrightarrow & \Lambda & \longrightarrow & E & \longrightarrow & P_{n-2} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & \Lambda & \longrightarrow 0
\end{array}
$$

where the first row is the start of a projective resolution of Λ over Λ^{env}. Tensoring this diagram with M over Λ we obtain the extension $M \otimes_\Lambda \eta$ as the lower row in the following exact commutative diagram

$$
\begin{array}{ccccccccccc}
0 \rightarrowtail M \otimes_\Lambda \Omega^n_{\Lambda^{\mathrm{env}}}(\Lambda) & \rightarrowtail & M \otimes_\Lambda P_{n-1} & \rightarrowtail & M \otimes_\Lambda P_{n-2} & \rightarrowtail \cdots \rightarrowtail & M \otimes_\Lambda P_0 & \rightarrowtail & M \otimes_\Lambda \Lambda \rightarrowtail 0 \\
\downarrow{\scriptstyle M\otimes\eta} & & \downarrow & & \| & & \| & & \| \\
0 \longrightarrow M \otimes_\Lambda \Lambda & \longrightarrow & M \otimes_\Lambda E & \longrightarrow & M \otimes_\Lambda P_{n-2} & \rightarrowtail \cdots \rightarrowtail & M \otimes_\Lambda P_0 & \rightarrowtail & M \otimes_\Lambda \Lambda \rightarrowtail 0
\end{array}
$$

Restricting all the homomorphisms and all the modules to Γ in the above diagram we obtain the extension $\mathsf{e}(M \otimes_\Lambda \eta)$. We use similar arguments as in the proof of Lemma 3.1 to construct it. We first look at the case $n = 1$ to illustrate this. In the following commutative diagram, the second row is the image in $\mathsf{Ext}^1_\Gamma(\mathsf{e}(M), \mathsf{e}(M))$ and the third row is the image in $\mathsf{Ext}^1_\Gamma(\mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda), \mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda))$.

$$
\begin{array}{ccccccc}
0 \rightarrowtail \mathsf{e}(M \otimes_\Lambda \Omega^1_{\Lambda^{\mathrm{env}}}(\Lambda)) & \longrightarrow & \mathsf{e}(M \otimes_\Lambda P_0) & \longrightarrow & \mathsf{e}(M \otimes_\Lambda \Lambda) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathsf{e}(M\otimes\eta)} & & \downarrow & & \| & & \\
0 \longrightarrow \mathsf{e}(M \otimes_\Lambda \Lambda) & \longrightarrow & \mathsf{e}(M \otimes_\Lambda E) & \longrightarrow & \mathsf{e}(M \otimes_\Lambda \Lambda) & \longrightarrow & 0 \\
\uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \varphi} & & \\
0 \rightarrowtail \mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda) & \rightarrowtail & \mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(E) & \rightarrowtail & \mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda) & \rightarrowtail & 0
\end{array}
$$

Then the pullback of the second row along φ is equivalent to the pushout of the third row along φ, which shows the claim for $n = 1$. For $n > 1$ we have the following.

$$\begin{array}{ccccccccccccc}
0 & \rightarrowtail & M \otimes_\Lambda \Omega^n_{\Lambda^{\mathrm{env}}}(\Lambda)_\Gamma & \rightarrowtail & M \otimes_\Lambda P_{n-1} & \rightarrow & M \otimes_\Lambda P_{n-2} & \rightarrow & \cdots & \rightarrow & M \otimes_\Lambda P_1 & \twoheadrightarrow & M \otimes_\Lambda P_0 & \twoheadrightarrow & M \otimes_\Lambda \Lambda_\Gamma & \twoheadrightarrow & 0
\end{array}$$

(commutative diagram with rows as described)

As said above, the second row is $\mathsf{e}(M \otimes_\Lambda \eta)$, the third row is

$$\mathsf{Ext}^n_\Gamma(\varphi, -)(\mathsf{e}(M \otimes_\Lambda \eta)),$$

the fifth row is $\mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\eta)$, and the fourth row is

$$\mathsf{Ext}^n_\Gamma(-, \varphi)(\mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\eta)).$$

The diagram shows that the extension on the third row and the extension on the fourth row are equivalent. In other words, the diagram in the proposition is commutative. $\square$

Next we prove the main result of this section which shows that the Fg condition is invariant under the arrow removal operation. This is part (iv) of the Main Theorem presented in the Introduction.

Theorem 5.6. *Let* $\Lambda = kQ/I$ *be an admissible quotient of a path algebra* kQ, *and suppose that* $\Lambda \to \Gamma$ *is an arrow removal. Then* Λ *satisfies* Fg *if and only if* Γ *satisfies* Fg.

Proof. We use [8, Proposition 6.4] with $N = N' = \Gamma/\mathrm{rad}\,\Gamma$ and $M = M' = \mathsf{i}(\Gamma/\mathrm{rad}\,\Gamma)$, where N is the direct sum of all simple Γ-modules and M is the direct sum of all simple Λ-modules.

We have that

$$\begin{aligned}
\mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda) &\simeq \mathsf{e}(M) \otimes_\Gamma (\Gamma \oplus \mathsf{Ker}\,\pi) \\
&\simeq \mathsf{e}(M) \oplus (\mathsf{e}(M) \otimes_\Gamma \mathsf{Ker}\,\pi) \\
&\simeq \mathsf{e}(M) \oplus F(\mathsf{e}(M)).
\end{aligned}$$

Since $F(\mathsf{e}(M))$ is projective by Theorem 2.7 (f), the homomorphism $\mathsf{Ext}^*_\Gamma(\varphi, -)$ is an isomorphism for $* > 0$ in the commutative diagram in Proposition 5.5. Since Λ is Gorenstein if and only if Γ is Gorenstein by Theorem 4.2, we have that both Λ and Γ are Gorenstein whenever we assume one of them is Gorenstein. Hence if we assume that one of Λ and Γ has Fg, then Γ is Gorenstein by [5, Prop. 2.2]. So we can suppose Γ is Gorenstein. Then $\mathsf{Ker}\,\pi$ has finite injective dimension as a right Γ-module, say n. This implies that the homomorphism $\mathsf{Ext}^*_\Gamma(-, \varphi)$ is an

isomorphism for $* > n$ in the commutative of Proposition 5.5. Furthermore, Γ^{env} and Λ^{env} are both also Gorenstein. Suppose that Γ^{env} has Gorenstein dimension d.

Let $p_\Gamma = \pi \colon \mathsf{e}^{\mathrm{env}}(\Lambda) \to \Gamma$ be the natural projection. Then construct the following commutative diagram for $* > \max\{n, d\}$. The upper square is the the commutative square of Proposition 5.5.

$$
\begin{array}{ccc}
\mathsf{Ext}^*_{\Lambda^{\mathrm{env}}}(\Lambda, \Lambda) & \xrightarrow{\ M \otimes_\Lambda - \ } & \mathsf{Ext}^*_\Lambda(M, M) \\[2mm]
\Big\downarrow{\scriptstyle \mathsf{e}^{\mathrm{env}}} & & \Big\downarrow{\scriptstyle \simeq} \\[2mm]
\mathsf{Ext}^*_{\Gamma^{\mathrm{env}}}(\mathsf{e}^{\mathrm{env}}(\Lambda), \mathsf{e}^{\mathrm{env}}(\Lambda)) & \xrightarrow{\ \mathsf{e}(M) \otimes_\Gamma - \ } & \mathsf{Ext}^*_\Gamma(\mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda), \mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda)) \\[2mm]
\Big\downarrow{\scriptstyle \mathsf{Ext}^*_{\Gamma^{\mathrm{env}}}(-, p_\Gamma)} & & \Big\uparrow{\scriptstyle \mathsf{Ext}^*_\Gamma(-, 1 \otimes p_\Gamma)} \\[2mm]
\mathsf{Ext}^*_{\Gamma^{\mathrm{env}}}(\mathsf{e}^{\mathrm{env}}(\Lambda), \Gamma) & \xrightarrow{\ \mathsf{e}(M) \otimes_\Gamma - \ } & \mathsf{Ext}^*_\Gamma(\mathsf{e}(M) \otimes_\Gamma \mathsf{e}^{\mathrm{env}}(\Lambda), \mathsf{e}(M) \otimes_\Gamma \Gamma) \\[2mm]
\Big\uparrow{\scriptstyle \mathsf{Ext}^*_\Gamma(p_\Gamma, -)} & & \Big\uparrow{\scriptstyle \mathsf{Ext}^*_\Gamma(1 \otimes p_\Gamma, -)} \\[2mm]
\mathsf{Ext}^*_{\Gamma^{\mathrm{env}}}(\Gamma, \Gamma) & \xrightarrow{\ \mathsf{e}(M) \otimes_\Gamma - \ } & \mathsf{Ext}^*_\Gamma(\mathsf{e}(M) \otimes_\Gamma \Gamma, \mathsf{e}(M) \otimes_\Gamma \Gamma) \\[2mm]
\Big\| & & \Big\downarrow{\scriptstyle \simeq} \\[2mm]
\mathsf{Ext}^*_{\Gamma^{\mathrm{env}}}(\Gamma, \Gamma) & \xrightarrow{\ \mathsf{e}(M) \otimes_\Gamma - \ } & \mathsf{Ext}^*_\Gamma(\mathsf{e}(M), \mathsf{e}(M))
\end{array}
$$

All the vertical maps in this diagram are isomorphisms and the diagram is commutative. Then using [8, Proposition 6.4] with $N = N' = \Gamma/\mathsf{rad}\Gamma$ and $M = M' = \mathsf{i}(N)$, where M is the direct sum of all simple Λ-modules and noting that $\mathsf{e}(M) = \mathsf{ei}(N) \simeq N$, we obtain that Λ has Fg if and only if Γ has Fg. $\qquad\square$

Example 5.7. Let $\Lambda_n = kQ_n/\langle \rho \rangle$ be the algebra of Example 4.5. After removing the arrows $\alpha_2, \ldots, \alpha_n$, we obtain a radical square zero Nakayama algebra which satisfies Fg by [6, Proposition 1.4]. By Theorem 5.6, we infer that Λ_n satisfies Fg.

We end the paper with an example showing that a general arrow removal (factoring out an arrow) and preserving Fg is not possible.

Example 5.8. Consider the following example presented by Fei Xu [11, 3.1 The category $\mathcal{E}_0$]. Let Q be the quiver given by
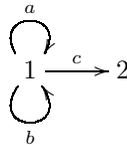
and the ideal $I = \langle a^2, ab - ba, b^2, ac \rangle$ in kQ for a field k. Denote by Λ the factor algebra kQ/I. By a result in a forthcoming paper or by direct computations, $\Lambda \xrightarrow{\pi} \Gamma = \Lambda/\langle c \rangle$ is a cleft extension. Then by [11] Λ does not satisfy Fg, while Γ do satisfy Fg (since Γ is a symmetric radical cube zero algebra satisfying Fg by [6]).

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