PARABOLIC INDUCTION AND
THE HARISH-CHANDRA $\mathcal{D}$-MODULE

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To the memory of Tom Nevins

ABSTRACT. Let G be a reductive group and L a Levi subgroup. Parabolic induction and restriction are a pair of adjoint functors between Ad-equivariant derived categories of either constructible sheaves or (not necessarily holonomic) $\mathcal{D}$-modules on G and L, respectively. Bezrukavnikov and Yom Din proved, generalizing a classic result of Lusztig, that these functors are exact. In this paper, we consider a special case where $L = T$ is a maximal torus. We give explicit formulas for parabolic induction and restriction in terms of the Harish-Chandra $\mathcal{D}$-module on $G \times T$. We show that this module is flat over $\mathcal{D}(T)$, which easily implies that parabolic induction and restriction are exact functors between the corresponding abelian categories of $\mathcal{D}$-modules.

CONTENTS

1. MAIN RESULTS

1.1. Throughout the paper, we fix a connected and simply-connected complex semisimple group G. Let T be the abstract Cartan torus, W the Weyl group, R_+ the set of positive roots, and ρ the half-sum of positive roots. Let

$$(1.1) \qquad \delta := \prod_{\alpha \in R_+} (e^{-\alpha} - 1) \in \mathbb{C}[T],$$

and $T_r := \{t \in T \mid \delta(t) \neq 0\}$, a W-stable Zariski open subset of T.

Let $\mathcal{D}_X$ denote the sheaf of differential operators on a smooth complex algebraic variety X and $\mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$. Let $\mathbb{C}[G]^G \subset \mathbb{C}[G]$ and $\mathcal{D}(G)^G \subset \mathcal{D}(G)$ be the subalgebras of $\operatorname{Ad} G$-invariant regular functions and differential operators on G, respectively.

Throughout the paper, we let W act on the algebra $\mathcal{D}(T)$ via the 'dot-action' $w : u \mapsto w{\cdot}u$, rather than the natural action $u \mapsto w^*u$. By definition, the differential operator $w{\cdot}u$ acts on $f \in \mathbb{C}[T]$ by the formula $w{\cdot}u : f \mapsto e^{-\rho} \cdot (w^*u)(e^\rho \cdot f)$. Let $\mathcal{D}(T)^W \subset \mathcal{D}(T)$ be the algebra of invariants of the dot-action. Note that for zero order differential operators, one has $w{\cdot}u = w^*u$. Thus, the natural algebra

imbedding $\mathbb{C}[T] \hookrightarrow \mathcal{D}(T)$ respects the W-actions so, we have $\mathbb{C}[T]^W = \mathbb{C}[T] \cap \mathcal{D}(T)^W$.

Harish-Chandra extended the Chevalley isomorphism $\mathbb{C}[G]^G \xrightarrow{\sim} \mathbb{C}[T]^W$, to be written as $f \mapsto f_T$, to a 'radial parts' homomorphism $r : \mathcal{D}(G)^G \to \mathcal{D}(T)^W$ such that Equation (1.2) holds, cf. [HC, Section 6],

$$(1.2) \qquad r(u)(\delta \cdot f_T) = \delta \cdot (u(f))_T, \qquad \forall u \in \mathcal{D}(G)^G, \ f \in \mathbb{C}[G]^G.$$

It is clear from the equation that $r(f) = f_T$ for all $f \in \mathbb{C}[G]^G \subset \mathcal{D}(G)^G$.

Remark 1.3. Equation (1.2) may be used as a definition of $r(u)$, viewed as a differential operator on T_r. It is not at all clear, however, that the differential operator thus defined has no poles at the divisor $\delta = 0$. $\Diamond$

Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T, respectively. Let $\mathcal{U}\mathfrak{g}$ and $\mathcal{U}\mathfrak{t}$ denote the corresponding enveloping algebras. The algebra $\mathcal{U}\mathfrak{t} \cong \operatorname{Sym}\mathfrak{t}$ is naturally identified with the algebra of translation invariant differential operators on T. Similarly, the center $Z\mathfrak{g}$, of $\mathcal{U}\mathfrak{g}$, may (and will) be identified with the algebra bi-invariant differential operators on G. With these identifications, the restriction of the radial parts map to the subalgebra $Z\mathfrak{g} \subset \mathcal{D}(G)^G$ reduces to the Harish-Chandra isomorphism $Z\mathfrak{g} \xrightarrow{\sim} (\operatorname{Sym}\mathfrak{t})^W$.

The differential of the action of G on itself by conjugation sends an element $a \in \mathfrak{g}$ to a vector field $\operatorname{ad} a$, on G. This gives a map $\operatorname{ad} : \mathfrak{g} \to \mathcal{D}(G)$. Let $\mathcal{D}(G) \operatorname{ad}\mathfrak{g}$ be a left ideal of the algebra $\mathcal{D}(G)$ generated by the image of the map ad. Thus, $\mathbf{N} := \mathcal{D}(G)/\mathcal{D}(G) \operatorname{ad}\mathfrak{g}$ is a left $\mathcal{D}(G)$-module.[1] The algebra $(\operatorname{End}_{\mathcal{D}(G)} \mathbf{N})^{op}$ can be naturally identified with $(\mathcal{D}(G)/\mathcal{D}(G) \operatorname{ad}\mathfrak{g})^G$, a quantum Hamiltonian reduction of the algebra $\mathcal{D}(G)$ with respect to the $\operatorname{Ad} G$-action. Further, it is immediate from (1.2) that the map r factors through a map $\left(\mathcal{D}(G)/\mathcal{D}(G) \operatorname{ad}\mathfrak{g}\right)^G \to \mathcal{D}(T)^W$, furthermore, the resulting map

$$(1.4) \qquad \mathsf{rad} : (\operatorname{End}_{\mathcal{D}(G)} \mathbf{N})^{op} = \left(\mathcal{D}(G)/\mathcal{D}(G) \operatorname{ad}\mathfrak{g}\right)^G \to \mathcal{D}(T)^W$$

is an algebra homomorphism. An important result due to Levasseur and Stafford [LS1]-[LS2], cf. also Wallach [Wa], says that the map (1.4) is, in fact, an isomorphism.

One can view $\mathbf{N}$ as a bimodule with respect to the left action of the algebra $\mathcal{D}(G)$ and the natural right action of the algebra $(\operatorname{End}_{\mathcal{D}(G)} \mathbf{N})^{op}$. Therefore, transporting the right action via the map rad gives $\mathbf{N}$ the structure of a $(\mathcal{D}(G), \mathcal{D}(T)^W)$-bimodule.

Remark 1.5. The surjectivity of the map rad follows from the fact, [LS1, Lemma 9], that the algebra $\mathcal{D}(T)^W$ is generated by the subalgebras $\mathbb{C}[T]^W$ and $(\operatorname{Sym}\mathfrak{t})^W$. Proving injectivity is much more difficult.

1.2. Given a morphism $f : X \to Y$ of smooth varieties, we follow the notation of [HTT, ch.1], and write $f^*(-) = \mathcal{O}_X \overset{L}{\otimes}_{f^\bullet \mathcal{O}_Y} f^\bullet(-)$, resp. $\int_f (-) = Rf_\bullet(\mathcal{D}_{Y \leftarrow X} \overset{L}{\otimes}_{\mathcal{D}_X} -)$, for derived pull-back, resp. push-forward, functors on $\mathcal{D}$-modules. Let $\int_f^i \mathcal{E} = \mathcal{H}^i(\int_f \mathcal{E})$ denote the ith cohomology $\mathcal{D}_Y$-module.

Let $Coh(\mathcal{D}_X)$ be the abelian category of coherent $\mathcal{D}_X$-modules and $D^b_{coh}(\mathcal{D}_X)$ a full subcategory of the bounded derived category of $\mathcal{D}_X$-modules whose objects

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[1]This $\mathcal{D}(G)$-module has been considered earlier by M. Kashiwara [Ka].

$\mathcal{E}$ have coherent cohomology $\mathcal{H}^\bullet(\mathcal{E})$. Thus, $Coh(\mathcal{D}_X)$ is the heart of the natural t-structure on $D^b_{coh}(\mathcal{D}_X)$. We also consider the abelian category $Coh(\mathcal{D}(X))$ of finitely generated $\mathcal{D}(X)$-modules, and the corresponding bounded derived category $D^b_{coh}(\mathcal{D}(X))$.

Let $\mathcal{B}$ be the flag variety, the variety of all Borel subgroups $B \subset G$. We have a diagram

$$G \xleftarrow{\quad p \quad} \widetilde{G} := \{(B, g) \in \mathcal{B} \times G \mid g \in B\} \xrightarrow{\quad q \quad} T.$$

Here, p is the second projection (the Grothendieck-Springer morphism) and the map q is defined by the assignment $(B, g) \mapsto g \bmod [B, B]$.

The functor of *parabolic induction* is defined by the formula $\int_p q^* : D^b_{coh}(\mathcal{D}_T) \to D^b_{coh}(\mathcal{D}_G)$.

The main result of the paper reads

Theorem 1.6.

(i) *There is an isomorphism of functors that makes the following diagram commute*

$$
\begin{array}{ccc}
D^b_{coh}(\mathcal{D}_T) & \xrightarrow{\int_p q^*(-)} & D^b_{coh}(\mathcal{D}_G), \\
\Big\Vert {\scriptstyle R\Gamma(T,-)} & & \Big\Vert {\scriptstyle R\Gamma(G,-)} \\
D^b_{coh}(\mathcal{D}(T)) & \xrightarrow[\mathbf{N} \overset{L}{\otimes}_{\mathcal{D}(T)^W}(-)]{} & D^b_{coh}(\mathcal{D}(G))
\end{array}
$$

(ii) *The $(\mathcal{D}(G), \mathcal{D}(T)^W)$-bimodule $\mathbf{N}$ is simple. Furthermore, $\mathbf{N}$ is flat as a right $\mathcal{D}(T)^W$-module; hence, the functor $\int_p q^*$ is t-exact (with respect to the natural t-structure).*

Remarks 1.7.

(i) Bezrukavnikov and Yom Din [BY] have shown that, in the general case of an arbitrary parabolic, the functor of parabolic induction (as well as parabolic restriction, cf. Section 4) is an exact functor between equivariant derived categories. The methods of [BY] are different from ours; in particular, the arguments in *loc cit* depend on the 'second adjointness' theorem of Drinfeld and Gaitsgory [DG] and on results of Raskin [Ra] on holonomic defects of $\mathcal{D}$-modules.

(ii) The flatness statement in Theorem 1.6(ii) is somewhat surprising since a natural commutative counterpart of this statement is false, cf. Remark 2.7(ii). A weaker flatness result has been established earlier by Kashiwara [Ka]. Specifically, Kashiwara proved that $\mathbf{N}$ is flat as a right $Z\mathfrak{g}$-module, where $Z\mathfrak{g}$ is identified with the commutative subalgebra $(\mathrm{Sym}\,\mathfrak{t})^W \subset \mathcal{D}(T)^W$. Levasseur and Stafford [LS2] proved that $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\,\mathrm{ad}\,\mathfrak{g}$ is flat as a right $(\mathrm{Sym}\,\mathfrak{g})^G$-module, an analogue of the flatness result from [Ka] in the Lie algebra setting.

(iii) Some constructions closely related to Theorem 1.6 have been considered earlier by D. Ben-Zvi and S. Gunningham [BZG].

(iv) Harish-Chandra's original construction of the radial parts map r can be translated into modern language as follows. Fix a Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b}$ and let B act on G by conjugation and act on T trivially. We have a diagram $T = B/[B, B] \xleftarrow{\alpha} B \xrightarrow{\beta} G$, of B-equivariant maps. The space $L = \Gamma(G, \int_\beta \alpha^* \mathcal{D}_T)$ has the natural structure of a $(\mathcal{D}(G), \mathcal{D}(T))$-bimodule. Furthermore,

$\int_\beta \alpha^* \mathcal{D}_T$ is strongly equivariant as a left $\mathcal{D}_G$-module and the resulting B-action on L commutes with the right $\mathcal{D}(T)$-action. Let $\gamma = 2\rho$ be the sum of positive roots viewed as a character of B, and $L_\gamma \subset L$ the γ-weight space of the B-action.

The bimodule L comes equipped with a canonical section $s_L = (dg)^{-1} db$, where we use the notation dk for a left invariant volume form on an algebraic group K. The section s_L has weight γ, hence the right action of $\mathcal{D}(T)$ on L gives a map $\mathcal{D}(T) \to L_\gamma$, $D \mapsto s_L D$. Using an associated graded of L with respect to a natural good filtration, it is not difficult to show that this map is actually a bijection. It follows that the image of the map $D \mapsto s_L D$ is $\mathcal{D}(G)^G$-stable. We deduce that for every $u \in \mathcal{D}(G)^G$ there is a unique differential operator $r(u) \in \mathcal{D}(T)$ such that, inside L, one has $u s_L = s_L \, r(u)$. It is immediate that the assignment $u \mapsto r(u)$ gives an algebra homomorphism $r : \mathcal{D}(G)^G \to \mathcal{D}(T)$. From the G-invariance of u one deduces that $r(u)$ is a W-invariant operator and formula (1.2) holds.

2. The Harish-Chandra $\mathcal{D}$-module

A key role in our approach to parabolic induction is played by the Harish-Chandra module, a left $\mathcal{D}_{G \times T}$-module defined as follows:

$$(2.1) \quad \boldsymbol{\mathcal{M}} := \mathcal{D}_{G \times T} / \big(\mathcal{D}_{G \times T} \, (\operatorname{ad} \mathfrak{g} \otimes 1) + \mathcal{D}_{G \times T} \, \{ u \otimes 1 - 1 \otimes \mathsf{rad}(u), \ u \in \mathcal{D}(G)^G \} \big).$$

The following result, essentially due to Hotta and Kashiwara [HK1], provides a geometric interpretation of the Harish-Chandra module in terms of the diagram $G \xleftarrow{p} \widetilde{G} \xrightarrow{q} T$.

Theorem 2.2. *There is an isomorphism $\int_{p \times q}^0 \mathcal{O}_{\widetilde{G}} \cong \boldsymbol{\mathcal{M}}$; furthermore, $\int_{p \times q}^i \mathcal{O}_{\widetilde{G}} = 0$ for all $i \neq 0$.*

Remark 2.3. In [HK1], the authors considered a Lie algebra version of Theorem 2.2, cf. [HK2] and [Ka] for closely related results concerning $\int_p \mathcal{O}_{\widetilde{G}}$. The strategy of the proof of Theorem 2.2 outlined below is similar to the strategy used in [HK1]. There are, however, two differences. First, the arguments in [HK1] involve Fourier transform, which is not available in the group setting. The second difference is that the definition of the Lie algebra counterpart of the Harish-Chandra module given in [HK1] doesn't quite match formula (2.1). Specifically, the elements of the form $u \otimes 1 - 1 \otimes \mathsf{rad}(u)$ that appeared in the definition in *loc cit* are the ones where u is taken to be either an element of the subalgebra $\mathbb{C}[\mathfrak{g}]^G \subset \mathcal{D}(\mathfrak{g})^G$ or of the subalgebra $(\operatorname{Sym} \mathfrak{g})^G \subset \mathcal{D}(\mathfrak{g})^G$, of G-invariant differential operators with *constant* coefficients. The difference between the (Lie algebra analogue of) (2.1) and the formula in [HK1] doesn't affect the resulting $\mathcal{D}$-module, thanks to [LS1, Lemma 9], cf. Remark 1.5.

The rest of this subsection is devoted to the proof of Theorem 2.2.

Let $G_{rs} \subset G$ be the regular semisimple locus of G. Let $\mathfrak{c} = \operatorname{Spec} \mathbb{C}[G]^G$. We consider a fiber product $G \times_{\mathfrak{c}} T$ and its open subset $Y := G_{rs} \times_{\mathfrak{c}} T_r$, which is a smooth connected closed subvariety of $G_{rs} \times T_r$. Let $N_Y \subset \mathcal{T}^*(G_{rs} \times T_r)$ be (the total space of) the conormal bundle on this subvariety, $\overline{N_Y}$ the closure of N_Y in $\mathcal{T}^*(G \times T)$, and $[\overline{N_Y}]$ the fundamental class of $\overline{N_Y}$.

Let SS$(-)$, resp. CC$(-)$, denote the characteristic variety, resp. characteristic cycle, of a $\mathcal{D}$-module.

Proposition 2.4. *One has* CC$(\boldsymbol{\mathcal{M}}) = [\overline{N_Y}]$. *Further, for all i, we have* SS$\big(\int_{p \times q}^i \mathcal{O}_{\widetilde{G}} \big) \subseteq \overline{N_Y}$.

The proof of the proposition is based on Lemma 2.5 that provides a description of the variety $\overline{N_Y}$ in terms of the commuting scheme $\mathcal{Z}$, a closed *not* necessarily reduced subscheme of $G \times \mathfrak{g}$ defined by the equations

$$\mathcal{Z} = \{(g, x) \in G \times \mathfrak{g} \mid \operatorname{Ad} g(x) = x\}.$$

A choice of imbedding $T \hookrightarrow G$ gives an imbedding $T \times \mathfrak{t} \hookrightarrow \mathcal{Z}$. We let $\mathcal{X}$ be a closed subscheme of $\mathcal{Z} \times T \times \mathfrak{t}$ defined by the equations

$$\mathcal{X} = \{(g, x, t, h) \in \mathcal{Z} \times T \times \mathfrak{t} \mid f(g, x) = f|_{T \times \mathfrak{t}}(t, h), \quad \forall f \in \mathbb{C}[\mathcal{Z}]^G\}.$$

The scheme $\mathcal{X}$ is independent of the choice of an imbedding $T \hookrightarrow G$.

To simplify notation, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$, resp. $\mathfrak{t}$ with $\mathfrak{t}^*$, using the Killing form. This gives natural identifications $G \times \mathfrak{g} = G \times \mathfrak{g}^* = \mathcal{T}^*G$, resp. $T \times \mathfrak{t} = T \times \mathfrak{t}^* = \mathcal{T}^*T$, where $\mathcal{T}^*X$ denotes the total space of the cotangent bundle on a smooth variety X. Thus, we may (and will) view $\mathcal{X}$ as a closed subscheme of $\mathcal{T}^*(G \times T)$.

The proof of Lemma 2.5 is similar to the proof of [HK1, Lemma 4.2.1], cf. also [Gi, Lemma 1.5].

Lemma 2.5. *The variety N_Y is a Zariski open and dense subscheme of $\mathcal{X}$; explicitly, we have*

$$N_Y = \mathcal{X} \cap \mathcal{T}^*(G_{rs} \times T_r).$$

Remark 2.6. It is known that the commuting variety $\mathcal{Z}$ is an irreducible and generically reduced scheme of dimension $\dim G + \dim T$, [Ri]. The scheme $\mathcal{X}$ is not reduced even in the case $G = SL_2$. It follows from Lemma 2.5 that the scheme $\mathcal{X}$ is generically reduced and irreducible; furthermore, the reduced scheme $\mathcal{X}_{\mathrm{red}}$, called the *isospectral commuting variety*, equals $\overline{N_Y}$.

Proof of Proposition 2.4. We write formula (2.1) in the form $\boldsymbol{\mathcal{M}} = \mathcal{D}_{G \times T}/\mathcal{J}$, where $\mathcal{J}$ is a left ideal of $\mathcal{D}_{G \times T}$. The standard ascending filtration on $\mathcal{D}_{G \times T}$ by order of the differential operator induces a quotient filtration on $\boldsymbol{\mathcal{M}}$ such that $\operatorname{gr} \boldsymbol{\mathcal{M}} = \mathcal{O}_{\mathcal{T}^*(G \times T)}/\operatorname{gr} \mathcal{J}$. It is immediate from (2.1) that the ideal $\operatorname{gr} \mathcal{J}$ contains the ideal of definition of the scheme $\mathcal{X}$, cf. [Gi, Lemma 2.4.3] for a Lie algebra counterpart. Hence, there is a surjective morphism $\mathcal{O}_{\mathcal{X}} \twoheadrightarrow \operatorname{gr} \boldsymbol{\mathcal{M}}$. By Lemma 2.5, we know that N_Y is open dense in $\mathcal{X}$ and, moreover, the scheme $\mathcal{X}$ is reduced at every closed point of N_Y. This implies the equation $\operatorname{CC}(\boldsymbol{\mathcal{M}}) = [\overline{N_Y}]$, since $\overline{N_Y}$ is an irreducible Lagrangian subvariety.

To prove the second statement of Proposition 2.4 we use a well known upper bound on the characteristic variety of a proper direct image, [HTT, Section 2.5]. By a straighforward calculation, one finds that this upper bound forces $\operatorname{SS}\left(\int_{p \times q}^i \mathcal{O}_{\widetilde{G}}\right)$ be set-theoretically contained in $\mathcal{X}_{\mathrm{red}}$, that is, in $\overline{N_Y}$. $\square$

Let $\mathcal{K}_X$ denote the canonical sheaf of a smooth variety X. We choose and fix translation invariant global sections dt and dg of $\mathcal{K}_T$ and $\mathcal{K}_G$, respectively.

One shows by computing Jacobians that there is a nowhere vanishing G-invariant global section ω of $\mathcal{K}_{\widetilde{G}}$ such that $p^*(dg) = q^*\delta \cdot \omega$, cf. [HK1, (4.1.4)] for a Lie algebra analogue.

Sketch of Proof of Theorem 2.2. Let $\widetilde{G}_{rs} := p^{-1}(G_{rs}) = q^{-1}(T_r)$. The map $p \times q$ restricts to an isomorphism $\widetilde{G}_{rs} \xrightarrow{\sim} Y$. It follows that $\left(\int_{p \times q}^i \mathcal{O}_{\widetilde{G}}\right)|_{G_{rs} \times T_r} \cong \int_\Delta^i \mathcal{O}_Y$, where Δ is the closed imbedding $Y \hookrightarrow G_{rs} \times T_r$.

If $i \neq 0$ we deduce that $\left(\int_{p \times q}^{i} \mathcal{O}_{\widetilde{G}}\right)|_{G_{rs} \times T_r} = \int_{\Delta}^{i} \mathcal{O}_Y = 0$. By Proposition 2.4, this forces $SS\left(\int_{p \times q}^{i} \mathcal{O}_{\widetilde{G}}\right)$ be contained in $\overline{N_Y} \smallsetminus N_Y$, which implies that $\int_{p \times q}^{i} \mathcal{O}_{\widetilde{G}} = 0$ for all $i \neq 0$.

Similarly, in the case $i = 0$, we find that the characteristic cycle of $\left(\int_{p \times q}^{0} \mathcal{O}_{\widetilde{G}}\right)|_{G_{rs} \times T_r}$ is equal to $\mathrm{CC}(\int_{\Delta}^{0} \mathcal{O}_Y) = [N_Y]$. Using this equation and the inclusion $SS(\int_{p \times q}^{0} \mathcal{O}_{\widetilde{G}}) \subset \overline{N_Y}$, cf. Proposition 2.4, we deduce that $\mathrm{CC}(\int_{\Delta}^{0} \mathcal{O}_Y) = [\overline{N_Y}]$. It follows that $\int_{\Delta}^{0} \mathcal{O}_Y$ is a simple $\mathcal{D}$-module; in paricular, this $\mathcal{D}$-module has no nonzero sections supported on the complement of the open set $G_{rs} \times T_r$.

Observe next that the $\mathcal{D}_{G \times T}$-module $\int_{p \times q}^{0} \mathcal{O}_{\widetilde{G}}$ comes equipped with a canonical global section $s = (dg\, dt)^{-1} \otimes (p \times q)_* \omega$. For any $u \in \mathcal{D}(G)^G$, using equation (1.2) one shows that the section $(u \otimes 1 - 1 \otimes \mathsf{rad}(u))s$ vanishes on the open set $G_{rs} \times T_r \subset G \times T$. Hence, this section is identically zero, by the previous paragraph. Further, the section s is G-invariant, hence it is annihilated by the elements $\mathrm{ad}\, a \otimes 1 \in \mathcal{D}(G \times T)$, for all $a \in \mathfrak{g}$. We conclude that the map $\mathcal{D}_{G \times T} \to \int_{p \times q}^{0} \mathcal{O}_{\widetilde{G}}$, $u \mapsto u(s)$, descends to a well-defined $\mathcal{D}$-module morphism $F : \boldsymbol{\mathcal{M}} \to \int_{p \times q}^{0} \mathcal{O}_{\widetilde{G}}$. It is straight-forward to check that the composition $\boldsymbol{\mathcal{M}}|_{G_{rs} \times T_r} \xrightarrow{F|_{G_{rs} \times T_r}} (\int_{p \times q}^{0} \mathcal{O}_{\widetilde{G}})|_{G_{rs} \times T_r} \xrightarrow{\sim} \int_{\Delta}^{0} \mathcal{O}_Y$ is an isomorphism of $\mathcal{D}$-modules on $G_{rs} \times T_r$. Hence, the $\mathcal{D}$-module $\mathrm{Ker}(F)$, resp. $\mathrm{Im}(F)$, is supported on the complement of $G_{rs} \times T_r$. Since the characteristic variety of $\boldsymbol{\mathcal{M}}$, resp. $\int_{p \times q}^{0} \mathcal{O}_{\widetilde{G}}$, is contained in $\overline{N_Y}$, this forces $SS(\mathrm{Ker}(F))$, resp. $SS(\mathrm{Im}(F))$, be contained in $\overline{N_Y} \smallsetminus N_Y$. We conclude that $\mathrm{Ker}(F) = 0$ and $\mathrm{Im}(F) = 0$. Thus, the map F yields an isomorphism $\boldsymbol{\mathcal{M}} \xrightarrow{\sim} \int_{p \times q}^{0} \mathcal{O}_{\widetilde{G}}$. $\qquad\square$

Remarks 2.7.

(i) The map $p \times q : \widetilde{G} \to G \times_{\mathfrak{c}} T$ is well known to be a small morphism. This yields an alternative proof of the fact that $\int_{p \times q} \mathcal{O}_{\widetilde{G}}$ is a simple holonomic $\mathcal{D}_{G \times T}$-module.

(ii) Sending a commuting pair $(g, x) \in \mathcal{Z}$ to the pair (g_{ss}, x_{ss}) of the corresponding semisimple components of the Jordan decomposition gives a map $\pi : \mathcal{Z} \to (T \times \mathfrak{t})/W$. This map is not flat (and neither is the map $\mathcal{Z}_{\mathrm{red}} \to (T \times \mathfrak{t})/W$). Indeed, the general fiber of π has dimension $\dim G - \dim T$, while the fiber over the W-orbit of the point $(1, 0) \in T \times \mathfrak{t}$ has dimension $\dim G$.

It is easy to show that $\mathrm{SS}(\mathbf{N}) = \mathcal{Z}$. Furthermore, the algebra map $\pi^* : \mathbb{C}[T \times \mathfrak{t}]^W \hookrightarrow \mathbb{C}[\mathcal{Z}]$, induced by π, may be viewed as a commutative analogue of the composite map $\mathcal{D}(T)^W \cong (\mathcal{D}(G)/\mathcal{D}(G)\, \mathrm{ad}\, \mathfrak{g})^G \hookrightarrow \mathcal{D}(G)/\mathcal{D}(G)\, \mathrm{ad}\, \mathfrak{g} = \mathbf{N}$. Thus, one may view the $\mathbb{C}[T \times \mathfrak{t}]^W$-module $\mathbb{C}[\mathcal{Z}]$ as a commutative analogue of the $\mathcal{D}(T)^W$-module $\mathbf{N}$. We know that $\mathbf{N}$ is flat over $\mathcal{D}(T)^W$, by Theorem 1.6. However, $\mathbb{C}[\mathcal{Z}]$ is not flat over $\mathbb{C}[T \times \mathfrak{t}]^W$ since the map π is not flat, cf. Remark 1.7(ii).
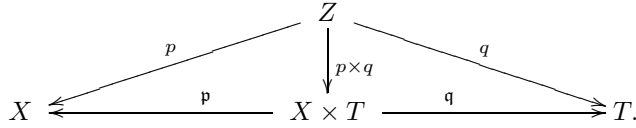
3. Proof of Theorem 1.6

3.1. Below, we make no distinction between left and right $\mathcal{D}_T$-modules; specifically, we identify a left $\mathcal{D}_T$-module $\mathcal{F}$ with the right $\mathcal{D}_T$-module $\mathcal{K}_T \otimes_{\mathcal{O}_T} \mathcal{F}$ using the map $\mathcal{F} \mapsto \mathcal{K}_T \otimes_{\mathcal{O}_T} \mathcal{F}$, $f \mapsto dt \otimes f$. The left and right actions are related by the formula $(dt \otimes f)u = dt \otimes u^t(f)$, where $u \mapsto u^t$ is an anti-involution $\mathcal{D}_T \to \mathcal{D}_T$ which sends $\frac{\partial}{\partial t}$ to $-\frac{\partial}{\partial t}$ and restricts to the identity on $\mathcal{O}_T \subset \mathcal{D}_T$.

More generally, let X be a smooth variety and write $\mathfrak{p} : X \times T \to X$ for the first, resp. $\mathfrak{q} : X \times T \to T$ for the second, projection. Then, we can (and will) identify

left $\mathcal{D}_{X \times T}$-modules with $(\mathfrak{p}^*\mathcal{D}_X, \mathfrak{q}^*\mathcal{D}_T)$-bimodules using a construction similar to the one above.

Now, fix a smooth variety Z and a pair of morphisms p, q, as depicted in the diagram

$$
\begin{array}{ccc}
 & Z & \\
\swarrow^{p} \quad \downarrow^{p \times q} \quad \searrow^{q} & & \\
X \xleftarrow{\quad \mathfrak{p} \quad} X \times T \xrightarrow{\quad \mathfrak{q} \quad} T.
\end{array}
$$

Let $\mathfrak{M} := \int_{p \times q} \mathcal{O}_Z$. The proof of Theorem 1.6 is based on the following general result.

Lemma 3.1.

   (1) *There are isomorphisms of functors*

$$
(3.2) \qquad \int_p q^*(\text{-}) \cong \int_{\mathfrak{p}} (\mathfrak{M} \overset{L}{\otimes}_{\mathfrak{q}^\bullet \mathcal{O}_T} \mathfrak{q}^\bullet(\text{-})) \cong R\mathfrak{p}_\bullet (\mathfrak{M} \overset{L}{\otimes}_{\mathfrak{q}^*\mathcal{D}_T} \mathfrak{q}^*(\text{-})).
$$

   (2) *If the morphism p is proper and the morphism q is smooth, then $\int_{p \times q}^i \mathcal{O}_Z$ is a flat $\mathfrak{q}^\bullet \mathcal{O}_T$-module, for all i.*

Proof. We will abuse notation and write $\otimes$ for $\overset{L}{\otimes}$, resp. $\mathcal{D}_T$ for $\mathfrak{q}^*\mathcal{D}_T$. Let $\mathbb{C}$ denote the trivial 1-dimensional representation of $\mathcal{U}\mathfrak{t}$. Recall that tensoring over $\mathcal{O}$ gives a tensor product operation on $\mathcal{D}$-modules.

For any left $\mathcal{D}_{X \times T}$-module $\mathcal{G}$, we have a chain of natural isomorphisms:

$$
\big(\mathcal{K}_T \otimes_{\mathcal{O}_T} (\mathfrak{M} \otimes_{\mathcal{O}_{X \times T}} \mathcal{G})\big) \otimes_{\mathcal{D}_T} \mathcal{O}_T \overset{\cong}{\longleftarrow} \big(\mathbb{C}dt \otimes (\mathfrak{M} \otimes_{\mathcal{O}_{X \times T}} \mathcal{G})\big) \otimes_{\mathcal{U}\mathfrak{t}} \mathbb{C}
$$
$$
\overset{\cong}{\longrightarrow} (\mathbb{C}dt \otimes \mathfrak{M}) \otimes_{\mathcal{U}\mathfrak{t}} \mathcal{G} \overset{\cong}{\longrightarrow} (\mathcal{K}_T \otimes \mathfrak{M}) \otimes_{\mathcal{D}_T} \mathcal{G}.
$$

Here, the first and the third isomorphism are induced by the natural imbeddings $\mathbb{C}dt \hookrightarrow \mathcal{K}_T$ and $\mathbb{C} \hookrightarrow \mathcal{O}_T$; the second isomorphism sends $dt \otimes (u \otimes u') \otimes 1$ to $(dt \otimes u) \otimes u'$.

We identify the left $\mathcal{D}_{X \times T}$-module $\mathfrak{M}$ with $\mathcal{K}_T \otimes \mathfrak{M}$, a $(p^*\mathcal{D}_X, q^*\mathcal{D}_T)$-bimodule. With this identification, the composite isomorphism above reads as follows $(\mathfrak{M} \otimes_{\mathcal{O}_{X \times T}} \mathcal{G}) \otimes_{\mathcal{D}_T} \mathcal{O}_T \cong \mathfrak{M} \otimes_{\mathcal{D}_T} \mathcal{G}$. Thus, by the definition of the functor $\int_{\mathfrak{p}}$, we have

$$
(3.3) \qquad \int_{\mathfrak{p}} (\mathfrak{M} \otimes_{\mathcal{O}_{X \times T}} \mathcal{G}) = \int_{\mathfrak{p}} \big((\mathfrak{M} \otimes_{\mathcal{O}_{X \times T}} \mathcal{G}) \otimes_{\mathcal{D}_T} \mathcal{O}_T\big) = R\mathfrak{p}_\bullet (\mathfrak{M} \otimes_{\mathcal{D}_T} \mathcal{G}).
$$

Next, let $\mathcal{F}$ be a $\mathcal{D}_T$-module. Writing $\varpi := p \times q$, and using the projection formula, cf. [HTT, Corollary 1.7.5], we compute

$$
(3.4)
$$
$$
\int_p q^*\mathcal{F} = \int_{\mathfrak{p}} \int_{\varpi} \varpi^* \mathfrak{q}^* \mathcal{F} = \int_{\mathfrak{p}} \Big(\big(\int_{\varpi} \varpi^* \mathcal{O}_{X \times T} \big) \otimes_{\mathcal{O}_{X \times T}} \mathfrak{q}^* \mathcal{F} \Big)
$$
$$
= \int_{\mathfrak{p}} \Big(\big(\int_{\varpi} \mathcal{O}_Z \big) \otimes_{\mathcal{O}_{X \times T}} \mathfrak{q}^* \mathcal{F} \Big) = \int_{\mathfrak{p}} (\mathfrak{M} \otimes_{\mathfrak{q}^\bullet \mathcal{O}_T} \mathfrak{q}^\bullet \mathcal{F}).
$$

Combining isomorphisms (3.3) and (3.4) yields (3.2).

We now prove part (2) of the lemma. Let $\mathcal{T}_V^* U$ denote (the total space of) the conormal bundle of a smooth subvariety V of a smooth variety U. We write

$\xi_u \in \mathcal{T}_u^* U$ for a covector at a point $u \in U$. In particular, $\mathcal{T}_U^* U$ is the zero section of $\mathcal{T}^* U$ and 0_u is the zero covector at u.

First, we claim that the assumptions in (2) imply an inclusion

$$(3.5) \qquad \mathrm{SS}(\mathfrak{M}) \cap (\mathcal{T}_X^* X \times \mathcal{T}^* T) \ \subseteq\ \mathcal{T}_X^* X \times \mathcal{T}_T^* T = \mathcal{T}_{X \times T}^* (X \times T).$$

To see this, we use the natural diagram

$$\mathcal{T}^* Z \ \xleftarrow{\ d^* p \times d^* q\ }\ Z \times_{X \times T} (\mathcal{T}^* X \times \mathcal{T}^* T) \ \xrightarrow{\ pr_X \times pr_T\ }\ \mathcal{T}^* X \times \mathcal{T}^* T.$$

The map $p \times q$ being proper, we deduce $\mathrm{SS}(\mathfrak{M}) \subseteq (pr_X \times pr_T)(d^* p \times d^* q)^{-1}(\mathrm{SS}(\mathcal{O}_Z))$. We compute

$$\mathrm{SS}(\mathfrak{M}) \cap (\mathcal{T}_X^* X \times \mathcal{T}^* T)$$
$$\subseteq \Big((pr_X \times pr_T)(d^* p \times d^* q)^{-1}(\mathcal{T}_Z^* Z) \Big) \bigcap (\mathcal{T}_X^* X \times \mathcal{T}^* T)$$
$$= \{ (\xi_{p(z)}, \xi_{q(z)}) \in \mathcal{T}^* X \times \mathcal{T}^* T \mid \xi_{p(z)} = 0, \ d^* q(\xi_{q(z)}) = 0, \ z \in Z \}$$
$$= \{ (0_{p(z)}, 0_{q(z)}) \in \mathcal{T}^* X \times \mathcal{T}^* T, \ z \in Z \} \ \subseteq\ \mathcal{T}_X^* X \times \mathcal{T}_T^* T,$$

where the last equality holds since the assumption that q be a smooth morphism implies that the map $d^* q$ is injective. This proves (3.5).

To complete the proof of (2) we must check that

$$(3.6) \qquad\qquad L^k(\mathrm{Id}_X \times i)^* \mathfrak{M} = 0, \qquad \forall k \neq 0,$$

for any imbedding $i : V \hookrightarrow T$, of a closed subscheme. By noetherian induction (aka 'devissage'), this is equivalent to proving that (3.6) holds for any imbedding $i : V \hookrightarrow T$, where V is a *smooth* locally-closed subvariety of T. For any such V, the inclusion in (3.5) implies that the $\mathcal{D}_{X \times T}$-module $\mathfrak{M}$ is noncharacteristic with respect to the subvariety $X \times V \subseteq X \times T$. It follows, [HTT, Section 2.4], that the $\mathcal{D}$-modules $L^k(\mathrm{Id}_X \times i)^* \mathfrak{M}$ vanish for all $k \neq 0$, as desired. $\qquad\square$

3.2. Let $\mathbf{M}$ be the $(\mathcal{D}(G), \mathcal{D}(T))$-bimodule that corresponds to the left $\mathcal{D}(G \times T)$-module $\Gamma(G \times T, \mathcal{M})$ via the identification explained at the beginning of Section 3.1. Taking global sections in (2.1) and writing $\langle - \rangle$ for $\mathbb{C}$-linear span, we find

$$\mathbf{M} = \frac{\mathcal{D}(G) \otimes \mathcal{D}(T)}{(\mathcal{D}(G)\,\mathrm{ad}\,\mathfrak{g}) \otimes \mathcal{D}(T) + (\mathcal{D}(G) \otimes \mathcal{D}(T))\{u \otimes 1 - 1 \otimes \mathsf{rad}(u), \ u \in \mathcal{D}(G)^G\}}$$
$$= (\mathbf{N} \otimes \mathcal{D}(T)) \big/ \big\langle nu \otimes s - n \otimes \mathsf{rad}(u)s, \ n \in \mathbf{N}, \ s \in \mathcal{D}(T), \ u \in \mathcal{D}(G)^G \big\rangle$$
$$= \mathbf{N} \otimes_{\mathcal{D}(T)^W} \mathcal{D}(T).$$

Thus, with the above identification, from Theorem 2.2 we obtain an isomorphism

$$(3.7) \qquad\qquad R\Gamma\big(G \times T, \int_{p \times q} \mathcal{O}_{\widetilde{G}}\big) \cong \mathbf{M} = \mathbf{N} \otimes_{\mathcal{D}(T)^W} \mathcal{D}(T).$$

It is known that the algebra $\mathcal{D}(T)$ is flat as a $\mathcal{D}(T)^W$-module, cf. Proposition 4.1(ii) of Section 4.1. Therefore, one has isomorphisms of functors

$$(3.8) \qquad \mathbf{N} \overset{L}{\otimes}_{\mathcal{D}(T)^W} (\text{-}) \cong \mathbf{N} \overset{L}{\otimes}_{\mathcal{D}(T)^W} \mathcal{D}(T) \overset{L}{\otimes}_{\mathcal{D}(T)} (\text{-})$$
$$\cong \big(\mathbf{N} \otimes_{\mathcal{D}(T)^W} \mathcal{D}(T) \big) \overset{L}{\otimes}_{\mathcal{D}(T)} (\text{-}) \cong \mathbf{M} \overset{L}{\otimes}_{\mathcal{D}(T)} (\text{-}).$$

The functors $R\Gamma(G, \text{-})$ and $R\Gamma(T, \text{-})$ being equivalences of categories, we see that Theorem 1.6 is equivalent to the following result.

Theorem 3.9.

(i) *The following functors $D_{coh}^b(\mathcal{D}_T) \to D_{coh}^b(\mathcal{D}_G)$ are isomorphic:*

$$(3.10) \qquad R\Gamma(G, \int_p q^*(\text{-})) \cong \mathbf{M} \overset{L}{\otimes}_{\mathcal{D}(T)} R\Gamma(T, \text{-}).$$

(ii) *The bimodule $\mathbf{M}$ is flat as a right $\mathcal{D}(T)$-module.*

Proof of Theorem 3.9. We apply Lemma 3.1 in the case where $X = G$, $Z = \widetilde{G}$, and the diagram above the statement of the lemma is $G \overset{p}{\leftarrow} \widetilde{G} \overset{q}{\to} T$. In this case, by Theorem 2.2 we have $\mathfrak{M} = \int_{p \times q}^0 \mathcal{O}_{\widetilde{G}} = \mathcal{M}$. Thus, part (2) of Lemma 3.1 implies that $\mathcal{M}$ is a flat $q^\bullet \mathcal{O}_T$-module. Hence, the functor $\mathcal{M} \overset{L}{\otimes}_{q^\bullet \mathcal{O}_T} q^\bullet(\text{-})$ is exact.

Observe next that any object in the essential image of the functor $\mathcal{M} \overset{L}{\otimes}_{q^\bullet \mathcal{O}_T} q^\bullet(\text{-})$ is supported on $G \times_{\mathfrak{c}} T$, since the support of $\mathcal{M}$ equals $G \times_{\mathfrak{c}} T$. The restriction of the first projection $\mathfrak{p} : G \times T \to G$ to the subvariety $G \times_{\mathfrak{c}} T$ is a finite morphism. Therefore, applying Lemma 3.11 below we deduce that the composite functor $\int_{\mathfrak{p}}(\mathcal{M} \overset{L}{\otimes}_{q^\bullet \mathcal{O}_T} q^\bullet(\text{-}))$ is exact. Thus, it follows from the first isomorphism in (3.2) that the functor $\int_p q^*$ is exact, which is part of the statement of Theorem 1.6.

Finally, for any $\mathcal{F} \in D_{coh}^b(\mathcal{D}_T)$, using the composite isomorphism in (3.2), we obtain

$$R\Gamma(G, \int_p q^*\mathcal{F}) \cong R\Gamma(G, R\mathfrak{p}_\bullet(\mathcal{M} \overset{L}{\otimes}_{\mathfrak{q}^* \mathcal{D}_T} \mathfrak{q}^*\mathcal{F}))$$

$$\cong R\Gamma(G \times T, \mathcal{M} \overset{L}{\otimes}_{\mathfrak{q}^* \mathcal{D}_T} \mathfrak{q}^*\mathcal{F}))$$

$$\cong R\Gamma(G \times T, \mathcal{M}) \overset{L}{\otimes}_{\mathcal{D}(T)} R\Gamma(T, \mathcal{F}) = \mathbf{M} \overset{L}{\otimes}_{\mathcal{D}(T)} R\Gamma(T, \mathcal{F}).$$

This proves part (i) of Theorem 3.9. Part (ii) follows from (i) since we have shown that the functor $\mathbf{M} \overset{L}{\otimes}_{\mathcal{D}(T)} R\Gamma(T, \text{-}) \cong R\Gamma(G, \int_p q^*(\text{-}))$ is exact. $\qquad\square$

Lemma 3.11 is, of course, well known in the case of holonomic $\mathcal{D}$-modules where it is proved using that finite morphisms are proper, hence, the corresponding push-forward functor $\int$ commutes with Verdier duality. This argument is not sufficient in the general case since the abelian category of not necessarily holonomic $\mathcal{D}$-modules is not stable under Verdier duality.

Lemma 3.11. *Let X be a smooth variety and $Y \subseteq X \times T$ a closed, not necessarily smooth, subvariety such that the first projection $\mathfrak{p} : Y \to X$ is a finite morphism. Define a full subcategory $D_{Y, Coh}^b(\mathcal{D}_{X \times T})$ of $D_{coh}^b(\mathcal{D}_{X \times T})$ to be the category formed by the objects $\mathcal{G}$ such that $\mathrm{Supp}(\mathcal{H}^i\mathcal{G}) \subseteq Y$ for all i. Then, the functor $\int_{\mathfrak{p}} : D_{Y, Coh}^b(\mathcal{D}_{X \times T}) \to D_{coh}^b(\mathcal{D}_X)$ is exact.*

Proof. Using induction on $\dim T$ we reduce the statement to the case of a 1-dimensional torus $T = \mathbb{C}^*$. Thus, we may assume that $\mathbb{C}[T] = \mathbb{C}[t, t^{-1}]$, so $\partial := t\frac{d}{dt}$ is a translation invariant vector field on T. Then, the object $\int_{\mathfrak{p}} \mathcal{G}$ is represented by a two-term complex $[\mathfrak{p}_\bullet \mathcal{G} \overset{\partial}{\to} \mathfrak{p}_\bullet \mathcal{G}]$ concentrated in degrees -1 and 0, and we have $\mathcal{H}^{-1}(\int_{\mathfrak{p}} \mathcal{G}) = \mathrm{Ker}\,\partial$. Thus, we must show that $\mathrm{Ker}\,\partial = 0$.

We may assume that X is affine and work with global sections. Let $u \in \mathrm{Ker}\,\partial$ be a nonzero global section. Let I be the annihilator of u in $\mathbb{C}[X \times T]$, resp. I_X

the annihilator of u in $\mathbb{C}[X]$. Clearly, one has $\mathbb{C}[X \times T]I_X \subset I$. Observe that the inclusion here is strict since the composite $\mathrm{Supp}(u) \hookrightarrow \mathrm{Supp}(\mathcal{G}) \to X$ is a finite morphism. Hence, there exists a function $f \in I \smallsetminus (\mathbb{C}[X \times T]I_X)$ such that $fu = 0$. Since ∂ kills u, it follows that the operator $(\mathrm{ad}\,\partial)^N(f)$ kills u, for all $N \geq 0$. We can write f as a finite sum $f = \sum_{j \in \mathbb{Z}} t^j \cdot f_j$, for some $f_j \in \mathbb{C}[X]$. Since $(\mathrm{ad}\,\partial)^N(t^j f_j) = j^N \cdot t^j f_j$, we deduce that $\sum_j j^N \cdot t^j f_j u = 0$, for all N. This implies that each of the functions f_j kills u, that is, $f_j \in I_X$. We conclude that $\sum_j t^j f_j \in \mathbb{C}[X \times T]I_X$, contradicting the choice of f. $\qquad\square$

Remark 3.12. One can check that neither the proof of Theorem 2.2 nor the proof of Theorem 3.9 relies on the fact that the algebra map rad in (1.4) is actually an isomorphism. This fact is only used to insure an implication

$$\mathbf{M} \text{ is flat over } \mathcal{D}(T) \implies \mathbf{N} \text{ is flat over } \mathcal{D}(T)^W.$$

Thus, the proof of the exactness of parabolic induction does not rely on the difficult results of Levasseur and Stafford [LS1]-[LS2].

4. Parabolic restriction

4.1. In this subsection we collect some results concerning W-equivariant $\mathcal{D}(T)$-modules.

Let K be an algebraic group. Given a smooth variety X with a K-action, we denote by $(\mathcal{D}(X), K)$-mod, resp. $(\mathcal{D}_X, K)$-mod, the abelian category of *strongly* K-*equivariant* $\mathcal{D}(X)$-modules, resp. $\mathcal{D}_X$-modules.

In the rest of this subsection we will use simplified notation $\mathcal{D} := \mathcal{D}(T)$. The Weyl group W acts on $\mathcal{D}$ by algebra automorphisms. It is clear that the category $(\mathcal{D}, W)$-mod is equivalent to the category $(W \ltimes \mathcal{D})$-mod of modules over $W \ltimes \mathcal{D}$, a smash-product algebra.

One has the following standard result.

Proposition 4.1.

(i) *The algebras $W \ltimes \mathcal{D}$ and $\mathcal{D}^W$ are Morita equivalent; specifically, one has the following mutually inverse equivalences:*

$$(4.2) \qquad (W \ltimes \mathcal{D})\text{-mod} \;\underset{\mathcal{D} \otimes_{\mathcal{D}^W} (\text{-})}{\overset{(\text{-})^W}{\rightleftarrows}}\; \mathcal{D}^W\text{-mod}.$$

(ii) *The algebra $\mathcal{D}$ is finitely generated and projective as a left, resp. right, $\mathcal{D}^W$-module.*

(iii) *There is an isomorphism $(W \ltimes \mathcal{D}) \cong \mathcal{D}(T) \otimes_{\mathcal{D}(T)^W} \mathcal{D}(T)$, of $W \times W$-equivariant $\mathcal{D}(T)$-bimodules.*

(iv) *For any $L \in D^b(\mathcal{D}^W\text{-mod})$, let $\mathcal{D}$ act on $R\,\mathrm{Hom}_{\mathcal{D}^W}(\mathcal{D}, L)$ via right multiplication on the domain $\mathcal{D}$. Then, in $D^b(\mathcal{D}\text{-mod})$, there is a canonical isomorphism*

$$R\,\mathrm{Hom}_{\mathcal{D}^W}(\mathcal{D}, L) \cong \mathcal{D} \overset{L}{\otimes}_{\mathcal{D}^W} L.$$

Proof. Part (i) is well known. In more detail, let $e = \frac{1}{|W|} \sum_{w \in W} w$ be the averaging idempotent of the group algebra $\mathbb{C}W$, and put $H := W \ltimes \mathcal{D}$. It is immediate to check that inside H, we have $\mathcal{D}^W = eHe$. Further, using that the algebra $\mathcal{D}$ is simple, one proves an equality $HeH = H$. This implies the Morita equivalence. Further, since H is projective as a left H-module, it follows from the Morita equivalence

that eH is projective as a left eHe-module, proving (ii). Part (iii) says that the
map $He \otimes_{eHe} eH \to H$ induced by multiplication is an isomorphism. The latter
statement follows from the Morita equivalence by multiplying both sides by e on
the left.

To prove (iv), we use the pairing $eH \otimes_H He \to eHe$, $eu \otimes ve \mapsto euve$. This pairing
is known to be perfect, that is, it induces an isomorphism $\mathcal{D} \to \operatorname{Hom}_{\mathcal{D}^W}(\mathcal{D}, \mathcal{D}^W)$,
of right $\mathcal{D}^W$-modules, cf. [EG, Theorem 1.5(iii)] for a more general result. Since
$\mathcal{D}$ is a projective $\mathcal{D}^W$-module, tensoring both sides of the isomorphism with L, we
obtain a chain isomorphisms

$$\mathcal{D} \otimes_{\mathcal{D}^W} L \xrightarrow{\sim} R\operatorname{Hom}_{\mathcal{D}^W}(\mathcal{D}, \mathcal{D}^W) \overset{L}{\otimes}_{\mathcal{D}^W} L \xrightarrow{\sim} R\operatorname{Hom}_{\mathcal{D}^W}(\mathcal{D}, L),$$

and (iv) follows. $\qquad\qquad\square$

4.2. The functor of *parabolic restriction* is defined as the functor $\int_q p^* : D^b(\mathcal{D}_G) \to$
$D^b(\mathcal{D}_T)$. The functor $\int_p q^*(-)$ of parabolic induction commutes with Verdier du-
ality, since the morphism $p : \widetilde{G} \to G$ is proper and the morphism $q : \widetilde{G} \to T$ is
smooth. It follows that the functor of parabolic restriction is a right adjoint of the
functor of parabolic induction.

Proposition 4.3. *There is an isomorphism of functors that makes the following
diagram commute*

$$
\begin{array}{ccc}
D^b(\mathcal{D}_G\text{-mod}) & \xrightarrow{\quad\int_q p^*(\text{-})\quad} & D^b(\mathcal{D}_T\text{-mod}), \\
\Big\| {\scriptstyle R\Gamma(G,\text{-})} & & \Big\| {\scriptstyle R\Gamma(T,\text{-})} \\
D^b(\mathcal{D}(G)\text{-mod}) & \xrightarrow[\ \mathcal{D}(T)\overset{L}{\otimes}_{\mathcal{D}(T)^W} R\operatorname{Hom}_{\mathcal{D}(G)}(\mathbf{N},\text{-})\ ]{} & D^b(\mathcal{D}(T)\text{-mod})
\end{array}
$$

Proof. Using a standard adjunction between $R\operatorname{Hom}$ and $\overset{L}{\otimes}$, we obtain the following
isomorphisms of functors

$$R\operatorname{Hom}_{\mathcal{D}(G)}(\mathbf{M},\text{-}) = R\operatorname{Hom}_{\mathcal{D}(G)}(\mathbf{N} \otimes_{\mathcal{D}(T)^W} \mathcal{D}(T),\text{-})$$

$$\cong R\operatorname{Hom}_{\mathcal{D}(T)^W}\big(\mathcal{D}(T), R\operatorname{Hom}_{\mathcal{D}(G)}(\mathbf{N},\text{-})\big)$$

$$\cong \mathcal{D}(T) \overset{L}{\otimes}_{\mathcal{D}(T)^W} R\operatorname{Hom}_{\mathcal{D}(G)}(\mathbf{N},\text{-}),$$

where the last isomorphism follows from Proposition 4.1(iv). Hence, for $E \in$
$D^b(\mathcal{D}(G)\text{-mod})$ and $F \in D^b(\mathcal{D}(T)\text{-mod})$, we find

$$R\operatorname{Hom}_{\mathcal{D}(G)}(\mathbf{M} \overset{L}{\otimes}_{\mathcal{D}(T)} F, E) = R\operatorname{Hom}_{\mathcal{D}(T)}\big(F, R\operatorname{Hom}_{\mathcal{D}(G)}\big(\mathbf{M}, E\big)\big)$$

$$= R\operatorname{Hom}_{\mathcal{D}(T)}\big(F, \mathcal{D}(T) \overset{L}{\otimes}_{\mathcal{D}(T)^W} R\operatorname{Hom}_{\mathcal{D}(G)}(\mathbf{N}, E)\big).$$

We conclude that the functor $\mathcal{D}(T) \overset{L}{\otimes}_{\mathcal{D}(T)^W} R\operatorname{Hom}_{\mathcal{D}(G)}(\mathbf{N},\text{-})$ is a right adjoint
of the functor $\mathbf{M} \overset{L}{\otimes}_{\mathcal{D}(T)} (\text{-})$. The result now follows from Theorem 3.9(i). $\qquad\square$

It is well known that parabolic restriction can be upgraded to a functor DRes :
$D(\mathcal{D}_G, G) \to D(\mathcal{D}_T, W)$ between equivariant derived categories. We restrict our
attention to abelian categories. Let Ind : $\mathcal{D}(T)\text{-mod} \to (\mathcal{D}_G, G)\text{-mod}$, resp. $\operatorname{Res}_W :$
$(\mathcal{D}_G, G)\text{-mod} \to (\mathcal{D}_T, W)\text{-mod}$, be the functor that corresponds to the functor
$\mathcal{H}^0\big(\int_p q^*(\text{-})\big)$, resp. $\mathcal{H}^0\big(\int_q p^*(\text{-})\big)$, via the equivalences $\Gamma(G,\text{-})$ and $\Gamma(T,\text{-})$. Further,

given a $\mathcal{D}(T)$-module F and $w \in W$, let F^w be the $\mathcal{D}(T)$-module obtained by twisting the $\mathcal{D}(T)$-action by the automorphism w. The $\mathcal{D}(T)$-module $\oplus_{w \in W} F^w$ has the natural W-equivariant structure such that the action of $y \in W$ is given by the identity maps $F^w \to F^{yw}$.

Our main result concerning parabolic restriction reads

Theorem 4.4.

(i) *The functor* Res_W *is isomorphic to the functor* $E \mapsto \mathcal{D}(T) \otimes_{\mathcal{D}(T)^w} E^G$.

(ii) *The composite functor* $\mathrm{Res}_W \circ \mathrm{Ind} : \mathcal{D}(T)$-mod $\to (\mathcal{D}_T, W)$-mod *is isomorphic to the functor* $F \mapsto \oplus_{w \in W} F^w$.

(iii) *The functor* Res_W *has a left adjoint functor*

$$(4.5) \qquad \mathrm{Ind}_W : \ (\mathcal{D}(T), W)\text{-mod} \to (\mathcal{D}(G), G), \quad F \mapsto \mathbf{N} \otimes_{\mathcal{D}(T)^w} F^W.$$

The functors Ind_W *and* Res_W *are exact; moreover, the functor* Res_W *induces an equivalence*

$$(4.6) \qquad\qquad (\mathcal{D}(G), G)\text{-mod}/\mathrm{Ker}(\mathrm{Res}_W) \ \overset{\sim}{\to} \ (\mathcal{D}_T, W)\text{-mod}.$$

Remarks 4.7.

(i) One can check that the W-action on $\mathcal{D}(T) \otimes_{\mathcal{D}(T)^w} \Gamma(G, \mathcal{E})$ induced by the natural W-action on the first tensor factor corresponds, via isomorphism of functors Res and $\mathcal{D}(T) \otimes_{\mathcal{D}(T)^w} (\text{-})^G$, to the W-action on $\Gamma(T, \mathrm{Res}_W \mathcal{E})$ that comes from the W-equivariant structure on $\mathrm{Res}_W \mathcal{E}$ induced from the one on $\mathrm{DRes}\,\mathcal{E} \in D(\mathcal{D}_T, W)$. Conversely, one can use the W-action on $\mathcal{D}(T) \otimes_{\mathcal{D}(T)^w} \Gamma(G, \mathcal{E})$ to give $\mathcal{H}^0(\int_q p^* \mathcal{E})$ a W-equivariant structure.

(ii) Part (ii) of the theorem was obtained by T.H. Chen [Ch, Proposition 3.2] by a different, less elementary method following an earlier result, [Gu], in the Lie algebra setting.

(iii) It follows from the theorem that an object $\mathcal{E}$ of $(\mathcal{D}_G, G)$-mod is killed by the functor Res_W iff $\mathcal{E}$ has no nonzero G-invariant global sections.

Proof of Theorem 4.4. For any $\mathcal{D}(G)$-module E, one has

$$(4.8) \qquad \mathrm{Hom}_{\mathcal{D}(G)}(\mathbf{N}, E) = \mathrm{Hom}_{\mathcal{D}(G)}(\mathcal{D}(G)/\mathcal{D}(G)\,\mathrm{ad}\,\mathfrak{g}, E) = E^{\mathrm{ad}\,\mathfrak{g}} = E^G.$$

The isomorphism of functors stated in (i) follows from this and Proposition 4.3.

Observe that for any $\mathcal{D}(T)$-module F we have $(\mathbf{M} \otimes_{\mathcal{D}(T)} F)^G = \mathbf{M}^G \otimes_{\mathcal{D}(T)} F = F$. Hence, from part (i), using Proposition 4.1(iii), we find

$$\mathrm{Res}_W(\mathrm{Ind}\,F) = \mathcal{D}(T) \otimes_{\mathcal{D}(T)^w} (\mathbf{M} \otimes_{\mathcal{D}(T)} F)^G = \mathcal{D}(T) \otimes_{\mathcal{D}(T)^w} F$$
$$= \big(\mathcal{D}(T) \otimes_{\mathcal{D}(T)^w} \mathcal{D}(T)\big) \otimes_{\mathcal{D}(T)} F = \big(W \ltimes \mathcal{D}(T)\big) \otimes_{\mathcal{D}(T)} F.$$

The last expression is isomorphic to $\oplus_{w \in W} F^w$, proving (ii).

The functor Ind is exact since $\mathbf{N}$ is a flat $\mathcal{D}(T)^W$-module. The functor

$$(\mathcal{D}(G), G)\text{-mod} \to \mathcal{D}(T)^W\text{-mod}, E \mapsto E^G,$$

is exact since the G-action on G-equivariant $\mathcal{D}(G)$-modules is semisimple. Since $\mathcal{D}(T)$ is a flat $\mathcal{D}(T)^W$-module, we deduce that the functor $E \mapsto \mathcal{D}(T) \otimes_{\mathcal{D}(T)^w} E^G$, hence the functor Res_W, is exact.

To prove that Ind_W is a left adjoint of Res_W, observe that by Morita equivalence, for any $F, F' \in (\mathcal{D}(T), W)$-mod, the natural map $\mathrm{Hom}_{(\mathcal{D}(T), W)\text{-mod}}(F, F') \to$

$\mathrm{Hom}_{\mathcal{D}(T)^W}(F^W, (F')^W)$ is an isomorphism. Hence, for any $E \in (\mathcal{D}(G), G)$-mod, using (4.8) we obtain

$$
\begin{aligned}
\mathrm{Hom}_{(\mathcal{D}(T),W)\text{-mod}}(F, \mathrm{Res}_W E) &= \mathrm{Hom}_{(\mathcal{D}(T),W)\text{-mod}}(F, \mathcal{D}(T) \otimes_{\mathcal{D}(T)^W} E^G) \\
&= \mathrm{Hom}_{\mathcal{D}(T)^W}(F^W, \mathrm{Hom}_{\mathcal{D}(G)}(\mathbf{N}, E)) \\
&= \mathrm{Hom}_{\mathcal{D}(G)}(\mathbf{N} \otimes_{\mathcal{D}(T)^W} F^W, E) \\
&= \mathrm{Hom}_{\mathcal{D}(G)}(\mathrm{Ind}_W F, E).
\end{aligned}
$$

It remains to show that (4.6) is an equivalence. It is sufficient to check that for any W-equivariant $\mathcal{D}(T)$-module F the adjunction morphism $F \to \mathrm{Res}_W \circ \mathrm{Ind}_W(F)$ is an isomorphism. To this end, we compute

$$(\mathbf{N} \otimes_{\mathcal{D}(T)^W} F^W)^G = (\mathbf{N}^G) \otimes_{\mathcal{D}(T)^W} F^W = \mathcal{D}(T)^W \otimes_{\mathcal{D}(T)^W} F^W = F^W.$$

Thus, using Proposition 4.1(i) we obtain

$$\mathrm{Res}_W(\mathrm{Ind}_W F) = \mathcal{D}(T) \otimes_{\mathcal{D}(T)^W} \left((\mathbf{N} \otimes_{\mathcal{D}(T)^W} F^W)^G \right) = \mathcal{D}(T) \otimes_{\mathcal{D}(T)^W} F^W = F. \quad \square$$

Let $\mathbf{M}^t$ be a $(\mathcal{D}(T), \mathcal{D}(G))$-bimodule obtained from the $(\mathcal{D}(G), \mathcal{D}(T))$-bimodule $\mathbf{M}$ by replacing the left $\mathcal{D}(G)$-action by a right $\mathcal{D}(G)$-action, resp. the right $\mathcal{D}(T)$-action by a left $\mathcal{D}(T)$-action, using the invariant volume form dg, resp. dt.

Corollary 4.9.

(i) *The functor* Res_W *is isomorphic to the functor* $E \mapsto \mathbf{M}^t \otimes_{\mathcal{D}(G)} E$.

(ii) *The functor* Res_W *takes finitely generated* $\mathcal{D}(G)$*-modules to finitely generated* $\mathcal{D}(T)$*-modules.*

Proof. For any $\mathcal{D}(G)$-module E, we have

$$\mathfrak{g}E \backslash E \cong (\mathrm{ad}\,\mathfrak{g}\,\mathcal{D}(G) \backslash \mathcal{D}(G)) \otimes_{\mathcal{D}(G)} E.$$

If E is, in addition, a semisimple as a G-representation. Then, the composite $E^{\mathrm{ad}\,\mathfrak{g}} \hookrightarrow E \twoheadrightarrow \mathfrak{g}E \backslash E$ is an isomorphism. Thus, using Theorem 4.4(i) we find

$$
\begin{aligned}
\mathrm{Res}_W(E) &\cong \mathcal{D}(T) \otimes_{\mathcal{D}(T)^W} E^G \\
&\cong \mathcal{D}(T) \otimes_{\mathcal{D}(T)^W} (\mathrm{ad}\,\mathfrak{g}\,\mathcal{D}(G) \backslash \mathcal{D}(G)) \otimes_{\mathcal{D}(G)} E \\
&\cong \mathbf{M}^t \otimes_{\mathcal{D}(G)} E.
\end{aligned}
$$

This proves (i). Part (ii) is a consequence of a well known result due to Hilbert; specifically, the result implies that E^G is finitely generated over $\mathcal{D}(G)^G$ provided E is finitely generated over $\mathcal{D}(G)$. $\qquad\square$

Remark 4.10. S. Gunningham [Gu] proved that Lie algebra analogues $D(\mathcal{D}_{\mathfrak{g}}, G) \leftrightarrows D(\mathcal{D}_{\mathfrak{t}}, W)$ of the functors DInd and DRes preserve coherence.

References

[BZG] D. Ben-Zvi and S. Gunningham, *Symmetries of categorical representations and the quantum Ngo action*, `arXiv:1712.01963`, 2017.

[BY] Roman Bezrukavnikov and Alexander Yom Din, *On parabolic restriction of perverse sheaves*, Publ. Res. Inst. Math. Sci. **57** (2021), no. 3, 1089–1107, DOI 10.4171/prims/57-3-12. MR4322008

[Ch] T.-H. Chen, *On the conjectures of Braverman-Kazhdan*, `arXiv:1909.05467`. J. Amer. Math. Soc., `https://doi.org/10.1090/jams/992`.

[DG] V. Drinfeld and D. Gaitsgory, *On a theorem of Braden*, Transform. Groups **19** (2014), no. 2, 313–358, DOI 10.1007/s00031-014-9267-8. MR3200429

[EG] Pavel Etingof and Victor Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), no. 2, 243–348, DOI 10.1007/s002220100171. MR1881922

[Gi] Victor Ginzburg, *Isospectral commuting variety, the Harish-Chandra $\mathcal{D}$-module, and principal nilpotent pairs*, Duke Math. J. **161** (2012), no. 11, 2023–2111, DOI 10.1215/00127094-1699392. MR2957698

[Gu] Sam Gunningham, *Generalized Springer theory for D-modules on a reductive Lie algebra*, Selecta Math. (N.S.) **24** (2018), no. 5, 4223–4277, DOI 10.1007/s00029-018-0443-x. MR3874694

[HC] Harish-Chandra, *Invariant eigendistributions on a semisimple Lie group*, Trans. Amer. Math. Soc. **119** (1965), 457–508, DOI 10.2307/1994080. MR180631

[HK1] R. Hotta and M. Kashiwara, *The invariant holonomic system on a semisimple Lie algebra*, Invent. Math. **75** (1984), no. 2, 327–358, DOI 10.1007/BF01388568. MR732550

[HK2] R. Hotta and M. Kashiwara, *Quotients of the Harish-Chandra system by primitive ideals*, Geometry today (Rome, 1984), Progr. Math., vol. 60, Birkhäuser Boston, Boston, MA, 1985, pp. 185–205. MR895154

[HTT] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008. Translated from the 1995 Japanese edition by Takeuchi, DOI 10.1007/978-0-8176-4523-6. MR2357361

[Ka] Masaki Kashiwara, *The invariant holonomic system on a semisimple Lie group*, Algebraic analysis, Vol. I, Academic Press, Boston, MA, 1988, pp. 277–286. MR992461

[LS1] T. Levasseur and J. T. Stafford, *Invariant differential operators and an homomorphism of Harish-Chandra*, J. Amer. Math. Soc. **8** (1995), no. 2, 365–372, DOI 10.2307/2152821. MR1284849

[LS2] T. Levasseur and J. T. Stafford, *The kernel of an homomorphism of Harish-Chandra*, Ann. Sci. École Norm. Sup. (4) **29** (1996), no. 3, 385–397. MR1386924

[Ra] Sam Raskin, *A generalization of the b-function lemma*, Compos. Math. **157** (2021), no. 10, 2199–2214, DOI 10.1112/S0010437X21007491. MR4311556

[Ri] R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Compositio Math. **38** (1979), no. 3, 311–327. MR535074

[Wa] Nolan R. Wallach, *Invariant differential operators on a reductive Lie algebra and Weyl group representations*, J. Amer. Math. Soc. **6** (1993), no. 4, 779–816, DOI 10.2307/2152740. MR1212243

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