REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 26, Pages 432–436 (March 25, 2022)
https://doi.org/10.1090/ert/607

# REPRESENTATIONS OF 2-TRANSITIVE LOCALLY COMPACT GROUPS

ROBERT A. BEKES

ABSTRACT. We show that noncompact representations of 2-transitive locally compact groups are irreducible.

## INTRODUCTION

Let $G$ be a locally compact group acting on a topological space $X$ such that the map $G \times X \to X$ is continuous. Then there exists a Radon measure $\mu$ on $X$ such that the action of $G$ on $X$ can be extended to a continuous unitary representation of $G$ on the Hilbert space $L_2(X, \mu)$. See Folland [5].

The action of $G$ is transitive if for $x$ and $y$ in $X$ there exists $g$ in $G$ such that $gx = y$. The action of $G$ is 2-transitive if for $x_1 \neq x_2$ and $y_1 \neq y_2$ in $X$ there exists $g$ in $G$ such that $gx_1 = y_1$ and $gx_2 = y_2$.

Assume that $G$ acts 2-transitively on $X$. If $G$ is finite, using Burnside's Lemma, the unitary representation of $G$ on $X$ splits into two subrepresentations, the identity representation and an irreducible representation orthogonal to the identity representation; see Serre [6, Section 2.3, problem 2.6]. For infinite discrete $G$ and $X$, Chernoff [2] showed that the unitary representation of $G$ on $X$ is irreducible. The purpose of this paper is to show that for noncompact $G$ and $X$ the unitary representation of $G$ on $X$ is irreducible.

## 1. NONCOMPACT $G$ AND $X$

In this section we prove Theorem 1:

**Theorem 1.** *Let $G$ be a noncompact nondiscrete locally compact and $\sigma$-compact topological transformation group acting faithfully and 2-transitively on a locally compact noncompact not totally disconnected space $X$. Then the unitary representation of $G$ on the Hilbert space $L_2(X, \mu)$ is irreducible.*

Throughout this section $G$ and $X$ satisfy the hypothesis of Theorem 1 and all group operations are written multiplicatively.

Let $H$ be the stabilizer of a point $o \in X$. Then $H$ acts transitively on $X \setminus \{o\}$. By Theorem C in Kramer [5], $X$ carries the structure of a finite dimensional vector space, with basepoint $o = 0$. The group $H$ is a matrix group, acting transitively on the set of nonzero vectors. The group $G$ is then the semi-direct product $G = H \ltimes X$, in its natural action on $X$.

---

Received by the editors September 1, 2020, and, in revised form, March 25, 2021, and December 9, 2021.

2020 *Mathematics Subject Classification.* Primary 22A25; Secondary 43A65.

Let $f$ be a continuous function $X$ with compact support and let $h \in H$. Define $\pi(h)f$ on $X$ by $\pi(h)f(x) = \sqrt{\rho(h)^{-1}}f(h^{-1}x)$ where $\rho(h)$ is the absolute value of the determinant of $h$ acting on $X$. Then $\pi(h)$ can be extended to a unitary operator on the Hilbert space $L_2(X, \mu)$ and the map $h \to \pi(h)$ to a continuous unitary representation of $H$ on $L_2(X, \mu)$. See Folland [4, Section 6.1, pg. 154]. Since $G = H \ltimes X$ we can extend $\rho$ to all of $G$ by $\rho(hx) = \rho(h)$ and so $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$ for $g_1, g_2 \in G$. The unitary representation $\pi$ can then be extended to all of $G$ by $\pi(g)f(x) = \sqrt{\rho(g)^{-1}}f(g^{-1}x)$.

**Lemma 1.** *Let $x_1 \neq x_2$ in $X$. Then there exist $x_3$, $x_4$, $\ldots$ and $\delta > 0$ such that the $x_i$ are all distinct and for $i \neq j$ there exists $g \in G$ with $gx_i = x_1$, $gx_j = x_2$, and $\rho(g) \leq \delta$.*

*Proof.* Let $\pi$ be the unitary representation of $G$ on $L_2(X, \mu)$ defined above. Since $G$ is 2-transitive, there exists a $g_0$ such that $g_0x_2 = x_1$ and $g_0x_1 = x_2$. Let $\delta = \rho(g_0)$. Suppose we have $x_1$, $x_2$, $x_3$, $\ldots$, $x_n$ such that for $i \neq j$ there is $g \in G$ with $gx_i = x_1$, $gx_j = x_2$, and $\rho(g) \leq \delta$. Choose $x_{n+1}$ as follows: For each $1 \leq i < n$ let $H_i$ be the stabilizer of $x_i$. Since $G$ is 2-transitive, $X = \{gx_n \mid g \in H_i\}$. Let $A = \{gx_n \mid \rho(g) \geq 1\}$. Then A has nonempty interior and so $\mu(A) > 0$. Therefore also $\mu(A \setminus \{x_1, \ldots x_n\}) > 0$. Choose $x_{n+1} \in A \setminus \{x_1, \ldots x_n\}$. Then for each $1 \leq i < n$ there exists $g_i$ such that $x_{n+1} = g_ix_n$, $g_ix_i = x_i$, and $\rho(g_i) \geq 1$. So ${g_i}^{-1}x_{n+1} = x_n$, ${g_i}^{-1}x_i = x_i$, and $\rho(g_i) \geq 1$. Therefore $\rho({g_i}^{-1}) = \rho(g_i)^{-1} \leq 1$. By the choice of $x_1$, $x_2$, $x_3$, $\ldots$, $x_n$, there exists $g'$ such that $g'x_n = x_2$, $g'x_i = x_1$, and $\rho(g') \leq \delta$. Then setting $g = g'{g_i}^{-1}$ we get $gx_{n+1} = x_2$, $gx_i = x_1$, and $\rho(g) = \rho(g'{g_i}^{-1}) = \rho(g')\rho({g_i}^{-1}) \leq \delta$. Therefore the set $x_1$, $x_2$, $x_3$, $\ldots$, $x_n$, $x_{n+1}$ has the desired property. ☐

*Proof of Theorem* 1. Let $\langle \quad , \quad \rangle$ be the inner product on $L_2(X, \mu)$. Let $U$ be a measurable set in $X$ with compact closure. Then $\mu(U) < \infty$. Let $x_1$ and $x_2$ be such that $x_1U \simeq x_1 + U$ and $x_2U \simeq x_2 + U$ are disjoint. Using Lemma 1 we get distinct $x_1$, $x_2$, $x_3$, $\ldots$ in $X$ and $\delta > 0$ such that for $i \neq j$ there exists $g_{ij} \in G$ with $g_{ij}x_i = x_1$, $g_{ij}x_j = x_2$ and $\rho(g_{ij}) \leq \delta$. Therefore $g_{ij}(x_iU) = (g_{ij}x_i)U = x_1U$, $g_{ij}(x_jU) = (g_{ij}x_j)U = x_2U$, and so $\{x_iU\}$ is a sequence of disjoint subsets.

For any subset $W$ of $X$ let $\xi_W$ denote the characteristic function of $W$. Let $f_n = \sum_{i=1}^{n} c_i\xi_{x_iU}$ with $c_i \geq 0$. Since $X$ acts transitively on itself, there exists $v_i \in X$ such that $v_ix_i = x_1$. Since $\rho \equiv 1$ on $X$, $\mu(x_iU) = \mu(x_1U)$. Therefore $\langle f_n, f_n\rangle = \sum_{i=1}^{n} {c_i}^2\mu(x_1U)$. Let $T$ be a positive intertwining operator for the action $\pi$ of $G$ on $X$. Then

$$\begin{aligned}\langle T\xi_{x_iU}, \xi_{x_iU}\rangle &= \langle \pi(v_i)T\xi_{x_iU}, \pi(v_i)\xi_{x_iU}\rangle \\ &= \langle T\xi_{v_ix_iU}, \xi_{v_ix_iU}\rangle \\ &= \langle T\xi_{x_1U}, \xi_{x_1U}\rangle\end{aligned}$$

and for $i \neq j$,

$$\begin{aligned}\langle T\xi_{x_iU}, \xi_{x_jU}\rangle &= \langle \pi(g_{ij})T\xi_{x_iU}, \pi(g_{ij})\xi_{x_jU}\rangle \\ &= \rho(g_{ij})^{-1}\langle T\xi_{g_{ij}x_iU}, \xi_{g_{ij}x_jU}\rangle \\ &= \rho(g_{ij})^{-1}\langle T\xi_{x_1U}, \xi_{x_2U}\rangle.\end{aligned}$$

Therefore

$$
\begin{aligned}
\langle Tf_n, f_n\rangle &= \sum_{i=1}^{n} {c_i}^2 \langle T\xi_{x_iU}, \xi_{x_iU}\rangle + \sum_{i\neq j} c_i c_j \langle T\xi_{x_iU}, \xi_{x_jU}\rangle \\
&= \sum_{i=1}^{n} {c_i}^2 \langle T\xi_{x_1U}, \xi_{x_1U}\rangle + \sum_{i\neq j} c_i c_j \rho(g_{ij})^{-1} \langle T\xi_{x_1U}, \xi_{x_2U}\rangle.
\end{aligned} \tag{1}
$$

Since $T$ is positive $\langle T\xi_{x_1U}, \xi_{x_2U}\rangle$ is real.

If $\langle T\xi_{x_1U}, \xi_{x_2U}\rangle \geq 0$, from (1) we get

$$
\begin{aligned}
\langle Tf_n, f_n\rangle &\geq \sum_{i\neq j} c_i c_j\, \rho(g_{ij})^{-1} \langle T\xi_{x_1U}, \xi_{x_2U}\rangle \\
&\geq \left[\sum_{i\neq j} c_i c_j \delta^{-1}\right] \langle T\xi_{x_1U}, \xi_{x_2U}\rangle \\
&= \left[\left[\sum_{i=1}^{n} c_i\right]^2 - \sum_{i=1}^{n} {c_i}^2\right] \delta^{-1} \langle T\xi_{x_1U}, \xi_{x_2U}\rangle.
\end{aligned} \tag{2}
$$

Now let $c_i = \frac{1}{i}$ and $f = \sum_{i=1}^{\infty} c_i \xi_{x_iU}$. Then since $\sum_{i=1}^{\infty} {c_i}^2 < \infty$ we have $f \in L_2(X, \mu)$ and $\lim_{n\to\infty}\langle Tf_n, f_n\rangle = \langle Tf, f\rangle < \infty$. Since $\sum_{i=1}^{\infty} c_i = \infty$, letting $n \to \infty$ in (2) we must have $\langle T\xi_{x_1U}, \xi_{x_2U}\rangle = 0$.

If $\langle T\xi_{x_1U}, \xi_{x_2U}\rangle \leq 0$, from (1) we get

$$
\begin{aligned}
\langle Tf_n, f_n\rangle &= \sum_{i=1}^{n} {c_i}^2 \langle T\xi_{x_1U}, \xi_{x_1U}\rangle + \sum_{i\neq j} c_i c_j \rho(g_{ij})^{-1} \langle T\xi_{x_1U}, \xi_{x_2U}\rangle \\
&\leq \sum_{i=1}^{n} {c_i}^2 \langle T\xi_{x_1U}, \xi_{x_1U}\rangle + \sum_{i\neq j} c_i c_j \delta^{-1} \langle T\xi_{x_1U}, \xi_{x_2U}\rangle \\
&= \sum_{i=1}^{n} {c_i}^2 \langle T\xi_{x_1U}, \xi_{x_1U}\rangle + \left[\left[\sum_{i=1}^{n} c_i\right]^2 - \sum_{i=1}^{n} {c_i}^2\right] \delta^{-1} \langle T\xi_{x_1U}, \xi_{x_2U}\rangle
\end{aligned} \tag{3}
$$

As above, let $c_i = \frac{1}{i}$. Then $\sum_{i=1}^{\infty} {c_i}^2 < \infty$, $\sum_{i=1}^{\infty} c_i = \infty$, and $\lim_{n\to\infty}\langle Tf_n, f_n\rangle = \langle Tf, f\rangle < \infty$. So with $\langle T\xi_{x_1U}, \xi_{x_2U}\rangle \leq 0$, letting $n \to \infty$ in (3) we must have $\langle T\xi_{x_1U}, \xi_{x_2U}\rangle = 0$. Therefore if $\mu(U) < \infty$, $\langle T\xi_{x_1U}, \xi_{x_2U}\rangle = 0$ when $x_1U$ and $x_2U$ are disjoint.

Since $X$ is homeomorphic with $\mathbb{R}^n$, there is a sequence $\{U_k\}_{k=1}^{\infty}$ of subsets of $X$ with $\mu(U_k) \to 0$ such that each $U_k$ is the disjoint union of $U_{k+1}$ and a translate of $U_{k+1}$ and finite linear combinations of characteristic functions of disjoint translates of $U_k$, $k \geq 1$, are dense in $L_2(X, \mu)$.

Now suppose $W = U \cup xU$ where $0 < \mu(U) < \infty$ and $U$ and $xU$ are disjoint. Then by the above argument, $\langle T\xi_U, \xi_{xU}\rangle = \langle T\xi_{xU}, \xi_U\rangle = 0$. There exists $v$ in $X$ such that $v(xU) = U$. Since $\rho \equiv 1$ on $X$, we also get

$$\langle T\xi_{xU}, \xi_{xU}\rangle = \langle \pi(v)T\xi_{xU}, \pi(v)\xi_{xU}\rangle = \langle T\xi_{v(xU)}, \xi_{v(xU)}\rangle = \langle T\xi_U, \xi_U\rangle.$$

Let

$$\lambda = \frac{\langle T\xi_W, \xi_W\rangle}{\mu(W)}.$$

Then

$$\lambda = \frac{\langle T\xi_U, \xi_U\rangle + \langle T\xi_{xU}, \xi_{xU}\rangle}{2\mu(U)} = \frac{\langle T\xi_U, \xi_U\rangle}{\mu(U)}.$$

So for any such decomposition, $\lambda$ is independent of $U$ and $W$ and so $\langle T\xi_W, \xi_W\rangle = \lambda\langle \xi_W, \xi_W\rangle$ and $\langle T\xi_U, \xi_U\rangle = \lambda\langle \xi_U, \xi_U\rangle$. Therefore $\langle T\xi_{U_k}, \xi_{U_k}\rangle = \lambda\langle \xi_{U_k}, \xi_{U_k}\rangle$ for all $k$ and so $T = \lambda I$. It then follows that the representation $\pi$ is irreducible. □

By Theorem 1, the noncompact representations of the 2-transitive groups classified by Tits in [7] are all irreducible. For a complete classification see Kramer [5, Theorem 5.14 and 6.17].

**Examples.** Let $G$ be the $ax + b$ group acting on $\mathbb{R}$ by $x \mapsto ax + b$ where $a \neq 0$. If $x_1 \neq x_2$ and $y_1 \neq y_2$ the system $\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ has a solution. Therefore the action of $G$ is 2-transitive on $\mathbb{R}$, see Conrad [3, example 4.3]. By Theorem 1, the unitary representation of $G$ on $L_2(\mathbb{R}, \mu)$ is irreducible. This result also follows from the representation theory of semi-direct products, see Folland [4, Section 6.7, pg. 189].

Some examples from Kramer [5, Theorem 5.14 and 6.17] of groups acting 2-transitively and hence irreducibly on $X \cong \mathbb{R}^n$, are $G = \mathrm{SO}(n) \cdot \mathbb{R}_{>0} \ltimes \mathbb{R}^n$ and $G = \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ with $n \geq 3$, and $G = \mathrm{Sp}(n) \cdot \mathbb{C}^* \ltimes \mathbb{R}^n$ and $G = \mathrm{Sp}_{2n}(\mathbb{R}) \ltimes \mathbb{R}^n$ with $n \geq 2$.

The argument in the proof of Theorem 1 simplifies for infinite discrete groups acting on an infinite discrete set $X$. To prove irreducibility, start by selecting distinct $x_1$, $x_2$, … in $X$ and replacing $x_iU$ by the singleton $\{x_i\}$ and $W$ with $\{x_i, x_1\}$. This case is proved in Chernoff [2].

For example let $G$ be the group of permutations on $\mathbb{Z}$ that move only a finite number of integers. Then $G$ acts 2-transitively and so the unitary representation of $G$ on $l_2(\mathbb{Z}, \mu)$ is irreducible.

## 2. Compact $X$

Suppose $G$ is a locally compact group acting 2-transitively on a compact topological space $X$. Let $\pi$ be the unitary representation of $G$ on the Hilbert space $L_2(X, \mu)$. Unlike the situation for finite groups, $\pi$ restricted to the orthogonal compliment of the constant functions in $L_2(X, \mu)$ may not be irreducible as the following example illustrates.

**Example.** Let $G = \mathrm{SL}_2(\mathbb{R})$ and $X = \mathbb{RP}^1$, the real projective line. Then it is shown in Conrad [3, Theorem 4.21] that the action of $G$ on $X$ is 2-transitive. It follows from Casselman [1, page 16] that $X \cong G/B$ where $B$ is the Borel subgroup of $G$ and the representation on $X$ is, via normalized induction, $\mathrm{Ind}_B^G \delta_B^{-1/2}$ where $\delta \begin{bmatrix} t & x \\ 0 & t^{-1} \end{bmatrix} = t^2$. By Casselman [1, Proposition 8.7] with $s = -1$, $m = 1$, and $n = 0$, the orthogonal complement of the projection onto the space of constant functions on $X$ splits into two infinite dimensional subrepresentations.

If $G$ is compact, let $H$ be the stabilizer of a point $o \in X$. Then $H$ is also compact. But $H$ acts transitively on the open set $X\backslash\{o\}$, so $X\backslash\{o\}$ is clopen. Therefore $\{o\}$ is open and so $X$ is discrete and hence finite.

## Acknowledgments

I would like to thank Kenneth A. Ross for his critique of this paper, Stephen DeBacker for the reference to Casselman [1], and the referee for helpful suggestions.

## References

[1] Bill Casselman, *Analysis on* SL(2) *representations of* $SL_2(\mathbb{R})$, `https://pdfs.semanticscholar.org/3556/1bd7456d3fa2892d087e497472109a993f1f.pdf`, 2013.

[2] Paul R. Chernoff, *Irreducible representations of infinite-dimensional transformation groups and Lie algebras. I*, J. Funct. Anal. **130** (1995), no. 2, 255–282, DOI 10.1006/jfan.1995.1069. MR1335380

[3] Keith Conrad, *Transitive group actions*, `https://www.math.uconn.edu/~kconrad/blurbs/grouptheory/transitive.pdf`, 2009.

[4] Gerald B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1397028

[5] Linus Kramer, *Two-transitive Lie groups*, J. Reine Angew. Math. **563** (2003), 83–113, DOI 10.1515/crll.2003.085. MR2009240

[6] Jean-Pierre Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, Vol. 42, Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott. MR0450380

[7] J. Tits, *Sur certaines classes d'espaces homogènes de groupes de Lie* (French), Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **29** (1955), no. 3, 268. MR76286

Department of Mathematics and Computer Science, Santa Clara University, Santa Clara, California 95053

*Email address*: `rbekes@scu.edu`