AN INTEGRAL SECOND FUNDAMENTAL THEOREM OF INVARIANT THEORY FOR PARTITION ALGEBRAS

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ABSTRACT. We prove that the kernel of the action of the group algebra of the Weyl group acting on tensor space (via restriction of the action from the general linear group) is a cell ideal with respect to the alternating Murphy basis. This provides an analogue of the second fundamental theory of invariant theory for the partition algebra over an arbitrary commutative ring and proves that the centraliser algebras of the partition algebra are cellular. We also prove similar results for the half partition algebras.

INTRODUCTION

The partition algebra $\mathcal{P}_r(n)$ over the complex field $\mathbb{C}$ arose in work of Paul Martin [Mar91, Mar94] and (independently) Vaughan Jones [Jon94] as a generalisation of the Temperley–Lieb algebra for n-state r-site Potts models in statistical mechanics. Suppose that $\mathbf{V}$ is an n-dimensional complex vector space. The algebra $\mathcal{P}_r(n)$ arises as the (generic) centraliser of the group of permutation matrices $W_n \leq \mathrm{GL}(\mathbf{V})$ acting on tensor space $\mathbf{V}^{\otimes r}$. By the general theory of finite dimensional algebras it follows that the $(\mathbb{C}W_n, \mathcal{P}_r(n))$-bimodule $\mathbf{V}^{\otimes r}$ satisfies Schur–Weyl duality (see [HR05]), in the sense that the image of each representation coincides with the centraliser algebra of the other action.

The partition algebra has found surprising applications to Deligne's tensor categories (see [Del07, CO11, CW12, CO14]) and the study of the Kronecker coefficients (see [BDVO15]). Heuristically speaking, this is because the partition algebra controls stability phenomena arising in the representation theory of symmetric groups.

More generally, let $\mathbf{V}$ be a free $\Bbbk$-module of rank n over an arbitrary (unital) commutative ring $\Bbbk$. The partition algebra makes sense as an algebra over $\Bbbk$, and $\mathbf{V}^{\otimes r}$ is a $(\Bbbk W_n, \mathcal{P}_r(n))$-bimodule. In a companion paper [BDM22], the authors have shown that Schur–Weyl duality holds over $\Bbbk$. Therefore, it is natural to expect that the partition algebra will continue to influence stability phenomena of symmetric groups over fields of positive characteristic.

The main result of this paper, Theorem 7.4, is that for any commutative ring $\Bbbk$, the annihilator of the $\Bbbk W_n$-action on $\mathbf{V}^{\otimes r}$ is a cell ideal with respect to the alternating Murphy basis of $\Bbbk W_n$. (The theory of cellular algebras was introduced in [GL96].) In light of Schur–Weyl duality, our main result implies (Corollary 7.6) that the centraliser algebra $\mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$ inherits a cellular structure from that of $\Bbbk W_n$ — even better, we obtain an explicit cellular basis. Thus our main result provides an analogue of the second fundamental theory of invariant theory for the

partition algebra over an arbitrary commutative ring $\Bbbk$. Similar results for other diagram algebras have been obtained in [Här99, LZ12, BEG20].

One can ask for conditions under which a centraliser algebra of a cellular algebra is again cellular; this question seems to be poorly understood in general. Our result establishes another positive occurrence of such a phenomenon. In a different direction, the cellularity of the centraliser algebra $\operatorname{End}_{W_n}(\mathbf{V}^{\otimes r})$ was only recently established [Don21]; see also [HR05, BH19a, BH19b] for explicit descriptions of the annihilator of the action of $\mathcal{P}_r(n)$.

Finally, we remark that Paul Martin [Mar00] introduced the half partition algebras $\mathcal{P}_{r+1/2}(n)$ in order to collate the individual (ordinary and half) partition algebras together in a tower of recollement structure. Our results treat both algebras $\operatorname{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$ and $\operatorname{End}_{\mathcal{P}_{r+1/2}(n)}(\mathbf{V}^{\otimes r})$ uniformly.

1. Combinatorics of symmetric groups

We write Sym_S for the *symmetric group* of permutations of a set S (the bijections of S under composition). We write Sym_d for $\operatorname{Sym}_{\{1,\dots,d\}}$. For any set S with $|S| = d$ we identify Sym_S with Sym_d via the obvious isomorphism. We let $*$ denote the anti-involution which sends w to w^{-1}, for any $w \in \operatorname{Sym}_d$, and we extend this $\Bbbk$-linearly to the group algebra.

A *weak composition* of a non-negative integer d is a way of writing d as the sum of a sequence of non-negative integers. There are infinitely many weak compositions of d, because we can always append 0 to any weak composition. Weak compositions are usually identified with infinite sequences with finite support (finitely many non-zero terms). The *length* of a weak composition $\lambda = (\lambda_1, \lambda_2, \dots)$ is the largest ℓ for which $\lambda_\ell \neq 0$. Thus we write λ as $\lambda = (\lambda_1, \dots, \lambda_\ell)$. There are finitely many weak compositions with a specified upper bound on length.

A *composition* of d is a way of writing d as the sum of a sequence of (strictly) positive integers. So a composition is a weak composition with positive parts, and its length is the number of parts. We stipulate that the integer 0 has one composition, of length 0, defined by the empty sequence. We write $\lambda \vDash d$ to mean that $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a composition of d.

Two sequences that differ in the order of their terms define different compositions of their sum, while they are considered to define the same *partition* of that number. Thus, partitions may be identified with ordered compositions $(\lambda_1, \dots, \lambda_\ell)$ satisfying $\lambda_1 \geq \cdots \lambda_{\ell-1} \geq \lambda_\ell$. We write $\lambda \vdash d$ to mean that λ is a partition of d. We write $\lambda \trianglerighteq \mu$ and say that λ dominates μ if

$$\sum_{1 \leq i \leq k} \lambda_i \geq \sum_{1 \leq i \leq k} \mu_i \quad \text{for all } k \geq 1.$$

Given a weak composition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of d, a Young diagram of shape λ is a planar arrangement of boxes into rows, with λ_i boxes in the ith row, for each $i = 1$, $\dots, \ell$. A λ-tableau t is a numbering of the boxes by the numbers $1, \dots, d$; i.e., a map from $\{1, \dots, d\}$ to the boxes. A tableau is *row standard* if the numbers in each row are increasing when read from left to right, and *standard* if row standard and the numbers in each column are increasing when read from top to bottom. Given $1 \leq k \leq n$, we let $\mathsf{t}\!\downarrow_{\{1,\dots,k\}}$ be the subtableau of t whose entries belong to the set $\{1, \dots, k\}$. We write $\mathsf{t} \trianglerighteq \mathsf{s}$ if $\mathsf{t}\!\downarrow_{\{1,\dots,k\}} \trianglerighteq \mathsf{s}\!\downarrow_{\{1,\dots,k\}}$ for all $1 \leq k \leq n$ and refer to this as the dominance order on standard λ-tableaux.

If H is a subgroup of a finite group G and V a left or right $\Bbbk H$-module, where $\Bbbk$ is a commutative ring, we respectively have the left or right induced module

$$\operatorname{ind}_H^G V = \Bbbk G \otimes_{\Bbbk H} V \text{ or } V \otimes_{\Bbbk H} \Bbbk G.$$

To each weak composition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of d there corresponds the following data:

(i) The *row-reading tableau* t^λ of shape λ, in which the numbers $1, \ldots, d$ are written from left to right in the rows.

(ii) The Young subgroup $\operatorname{Sym}_\lambda$ of Sym_d; this is the subgroup of Sym_d stabilising the rows of t^λ.

(iii) The permutation module M^λ, defined by

$$M^\lambda = \operatorname{ind}_{\operatorname{Sym}_\lambda}^{\operatorname{Sym}_d} \Bbbk.$$

It has a (tabloid) basis [Jam78] indexed by the set of row-standard tableaux of shape λ.

Permutation modules (both as left and right modules) play an important role in what follows. As a left $\Bbbk \operatorname{Sym}_d$-module, it is well known that M^λ is isomorphic to the left ideal $\Bbbk \operatorname{Sym}_d x_\lambda$, where x_λ is defined in the next paragraph.

Given a row-standard λ-tableau t, let $d(\mathsf{t})$ be the unique element of Sym_d such that $\mathsf{t} = d(\mathsf{t})\mathsf{t}^\lambda$. Let $\operatorname{sgn} w$ be the sign of a permutation w. Given any pair s, t of row-standard λ-tableaux, following Murphy, we set

$$(1) \qquad x_{\mathsf{st}}^\lambda = d(\mathsf{s})x_\lambda d(\mathsf{t})^{-1}; \quad y_{\mathsf{st}}^\lambda = d(\mathsf{s})y_\lambda d(\mathsf{t})^{-1},$$

where $x_\lambda = \sum_{w \in W_\lambda} w$ and $y_\lambda = \sum_{w \in W_\lambda} (\operatorname{sgn} w)\, w$. If λ is already specified in context, then we may omit the superscript λ from the notation, writing x_{st}, y_{st} instead of the more cumbersome x_{st}^λ, y_{st}^λ. Write $[\mathsf{t}]$ for the shape λ of a tableau t. Then $x_{\mathsf{st}} = x_{\mathsf{st}}^\lambda$ where $\lambda = [\mathsf{s}] = [\mathsf{t}]$, and similarly for the y_{st}.

Graham and Lehrer [GL96] introduced cellular algebras in order to axiomatise certain common features of certain classes of finite dimensional algebras. A cellular algebra is an algebra with a distinguished basis (the cellular basis) indexed by triples $(\lambda, \mathsf{s}, \mathsf{t})$ where λ varies over a poset. The basis yields canonical pairwise non-isomorphic cell modules $\Delta(\lambda)$, one for each λ in the indexing set. For each λ, s and t belong to an index set in bijection with a basis of $\Delta(\lambda)$.

Murphy [Mur92, Mur95] found two cellular bases of the Iwahori–Hecke algebra associated to $\Bbbk \operatorname{Sym}_d$. By specialising the deformation parameter to 1, we obtain cellular bases of $\Bbbk \operatorname{Sym}_d$ as follows.

Theorem 1.1 (Murphy). *Let $\Bbbk$ be a commutative ring. Each of the two disjoint unions*

$$\mathcal{X} = \bigsqcup_{\lambda \vdash d} \{x_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \text{ standard}, \ [\mathsf{s}] = [\mathsf{t}] = \lambda\},$$
$$\mathcal{Y} = \bigsqcup_{\lambda \vdash d} \{y_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \text{ standard}, \ [\mathsf{s}] = [\mathsf{t}] = \lambda\}$$

is a cellular $\Bbbk$-basis of the group algebra $\Bbbk \operatorname{Sym}_d$, with respect to the anti-involution $*$. The cells are ordered by the dominance order $\trianglerighteq$ (so $\leq$ in [GL96] must be replaced by $\trianglerighteq$ here) with the least dominant partition (1^d) at the top and the most dominant partition (d) at the bottom. The cell module $\Delta(\lambda)$ indexed by λ is isomorphic to the dual Specht module S_λ for the x-basis and the Specht module $S^{\lambda'}$ for the y-basis, where λ' is the transpose of λ.

Note that x_{st} and y_{st} are interchanged by the $\Bbbk$-linear algebra involution $\#$ of $\Bbbk \operatorname{Sym}_d$ defined on basis elements by $w \mapsto (\operatorname{sgn} w)w$, for $w \in \operatorname{Sym}_d$. This involution converts results about one basis into results about the other.

Remark 1.2. We need to distinguish notationally between two symmetric groups in this paper: $W_n \cong \operatorname{Sym}_n$ and $\mathfrak{S}_r \cong \operatorname{Sym}_r$. We write maps on the left in the former, on the right in the latter. Thus, we compose from right-to-left in W_n and from left-to-right in $\mathfrak{S}_r$. These groups act on $\mathbf{V}^{\otimes r}$ on the left and right by value and place-permutation, respectively. We will make this explicit in the next section. Any notation applicable to Sym_n will be extended to both W_n and $\mathfrak{S}_r$; in particular we have the Young subgroups W_λ and $\mathfrak{S}_\mu$ for any weak compositions λ, μ of n, r respectively.

2. The $(\Bbbk W_n, \mathcal{P}_r(n))$-bimodule $\mathbf{V}^{\otimes r}$

For the rest of the paper we fix a free $\Bbbk$-module $\mathbf{V}$ of rank n, with a given $\Bbbk$-basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, where $\Bbbk$ is a commutative ring. We identify $\mathbf{V}$ with $\Bbbk^n$ by taking coordinates in the basis. For any positive integer r, the set

$$(2) \qquad \{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r} \mid i_1, \ldots, i_r = 1, \ldots, n\}$$

is a basis of the rth tensor power $\mathbf{V}^{\otimes r}$. The general linear group $\operatorname{GL}(\mathbf{V})$ of $\Bbbk$-linear automorphisms of $\mathbf{V}$ acts naturally on the left on $\mathbf{V}$; this action extends diagonally to an action on $\mathbf{V}^{\otimes r}$. The symmetric group $\mathfrak{S}_r$ acts on the right on $\mathbf{V}^{\otimes r}$ by permuting the tensor positions; this action is known as the *place-permutation* action, defined by

$$(3) \qquad (\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r})^\sigma = \mathbf{v}_{i_{1\sigma^{-1}}} \otimes \cdots \otimes \mathbf{v}_{i_{r\sigma^{-1}}}, \text{ for } \sigma \in \mathfrak{S}_r$$

extended linearly. Note that we write maps in $\mathfrak{S}_r$ on the *right* of their arguments, so that we may compose permutations in $\mathfrak{S}_r$ from left-to-right. The actions of the groups $\operatorname{GL}(\mathbf{V})$, $\mathfrak{S}_r$ on $\mathbf{V}^{\otimes r}$ commute, thus their linear extensions to the group algebras make $\mathbf{V}^{\otimes r}$ into a $(\Bbbk \operatorname{GL}(\mathbf{V}), \Bbbk \mathfrak{S}_r)$-bimodule.

Basis elements $\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r}$ of $\mathbf{V}^{\otimes r}$ are indexed by multi-indices $(i_1, \ldots, i_r)$ in the set

$$I(n, r) = \{1, \ldots, n\}^r.$$

In the sequel, we will be careful to distinguish between values i_α in $\{1, \ldots, n\}$ and places α in $\{1, \ldots, r\}$. For example, the multi-index $(2, 7, 7, 6, 2)$ takes the value 7 in places $2, 3$, the value 2 in places $1, 5$, and the value 6 in place 4. In general, we will use Latin letters such as i, j to denote values and Greek letters such as α, β to denote places.

Let W_n be the Weyl group of $\operatorname{GL}(\mathbf{V})$, i.e., the group of elements of $\operatorname{GL}(\mathbf{V})$ permuting the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. We identify W_n with the group of permutation matrices, regarded as matrices with entries from $\Bbbk$. By restricting the action of $\operatorname{GL}(\mathbf{V})$ to W_n, we obtain a left action of W_n on $\mathbf{V}^{\otimes r}$. To be explicit, $w \in W_n$ acts by

$$(4) \qquad w(\mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_r}) = \mathbf{v}_{w(j_1)} \otimes \cdots \otimes \mathbf{v}_{w(j_r)}.$$

Note that we write maps on the *left* of their arguments when considering this action. Extended linearly, the action of W_n defines a linear representation $\Bbbk W_n \to \operatorname{End}_\Bbbk(\mathbf{V}^{\otimes r})$ of the group algebra $\Bbbk W_n$.

For a positive integer r, and any $\delta \in \Bbbk$, we let $\mathcal{P}_r(\delta)$ denote the $\Bbbk$-module with basis given by all set-partitions of $\{1, 2, \ldots, r, 1', 2', \ldots, r'\}$. By a *set-partition* we mean a pairwise disjoint covering of the set. An element of a set-partition is called a *block*. For example,

$$d = \{\{1, 2, 4, 2', 5'\}, \{3\}, \{5, 6, 7, 3', 4', 6', 7'\}, \{8, 8'\}, \{1'\}\}$$

is a set-partition of the set $\{1, \ldots, 8, 1', \ldots, 8'\}$ with five blocks.

We can depict each set-partition by a *partition diagram*, consisting of r northern nodes indexed by 1, 2, $\ldots$, r and r southern nodes indexed by $1'$, $2'$, $\ldots$, r', with edges between nodes, such that the nodes in the connected components give the blocks of the set-partition. In general there are many partition diagrams depicting a given set-partition, which we identify as equivalent.

We define the product $x \cdot y$ of two diagrams x and y by stacking x above y, where we identify the southern nodes of x with the northern nodes of y; these identified nodes then become the middle nodes. If there are m connected middle components, then the product xy is set equal to δ^m times the diagram obtained by deleting the middle components (including middle vertices). An example is given in Figure 1. Extending the product linearly defines a multiplication on $\mathcal{P}_r(\delta)$, making
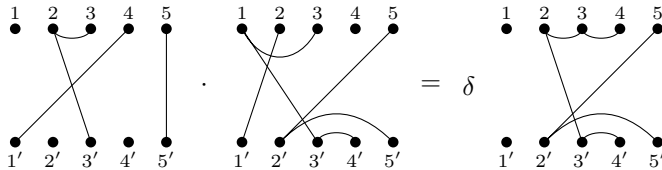

FIGURE 1. Multiplication of two diagrams in $\mathcal{P}_5(\delta)$

it an associative algebra. Note that $\Bbbk\mathfrak{S}_r \subset \mathcal{P}_r(\delta)$ is the subalgebra spanned by the *permutation diagrams*, the diagrams depicting set-partitions with r blocks, each of which contains exactly one element of $\{1, \ldots, r\}$ and one of $\{1', \ldots, r'\}$.

To obtain an action of the partition algebra on $\mathbf{V}^{\otimes r}$ it is necessary to specialise δ to $n = \operatorname{rank}_{\Bbbk} \mathbf{V}$. Following [HR05], we define a generalised Kronecker delta symbol $(d)_{i_{1'}, \ldots, i_{r'}}^{i_1, \ldots, i_r}$ corresponding to a diagram d and any $(i_1, \ldots, i_r)$, $(i_{1'}, \ldots, i_{r'})$ in $I(n, r)$, by

$$(d)_{i_{1'}, \ldots, i_{r'}}^{i_1, \ldots, i_r} = \begin{cases} 1 & \text{if } i_\alpha = i_\beta \text{ whenever } \alpha \neq \beta \text{ are in the same block of } d, \\ 0 & \text{otherwise,} \end{cases}$$

Then the diagram d acts on $\mathbf{V}^{\otimes r}$, on the right, by the rule

$$(5) \qquad (\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r})^d = \sum_{(i_{1'}, \ldots, i_{r'}) \in I(n, r)} (d)_{i_{1'}, \ldots, i_{r'}}^{i_1, \ldots, i_r} (\mathbf{v}_{i_{1'}} \otimes \cdots \otimes \mathbf{v}_{i_{r'}}).$$

Extended linearly, this action defines a linear representation $\mathcal{P}_r(n)^{\mathrm{op}} \to \operatorname{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$. If $d \in \mathfrak{S}_r$ is a permutation diagram, then d acts by the place-permutation action defined in (3). The actions of W_n and $\mathcal{P}_r(n)$ defined in (4) and (5) commute, thus we have a $(\Bbbk W_n, \mathcal{P}_r(n))$-bimodule structure on $\mathbf{V}^{\otimes r}$.

3. THE $(\Bbbk W_{n-1}, \mathcal{P}_{r+1/2}(n))$-BIMODULE $\mathbf{V}^{\otimes r}$

Let $\mathcal{P}_{r+1/2}(\delta)$ denote the submodule of $\mathcal{P}_{r+1}(\delta)$ with $\Bbbk$-basis given by all set-partitions such that $r+1$ and $(r+1)'$ belong to the same block. The submodule $\mathcal{P}_{r+1/2}(\delta)$ is closed under the multiplication and therefore is a subalgebra of $\mathcal{P}_{r+1}(\delta)$. The $\Bbbk$-submodule $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n \subset \mathbf{V}^{\otimes(r+1)}$ is stable under the action of $\mathcal{P}_{r+1/2}(n)$. Therefore, by identifying $\mathbf{V}^{\otimes r}$ with

$$\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n \subset \mathbf{V}^{\otimes(r+1)},$$

we regard $\mathbf{V}^{\otimes r}$ as a right $\mathcal{P}_{r+1/2}(n)$-module. We also regard it as a left $\Bbbk W_{n-1}$-module by restriction from $\Bbbk W_n$, where

$$W_{n-1} = \{w \in W_n \mid w(n) = n\}.$$

Thus, after identifying $\mathbf{V}^{\otimes r}$ with $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n$, we have a $(\Bbbk W_{n-1}, \mathcal{P}_{r+1/2}(n))$-bimodule structure on $\mathbf{V}^{\otimes r}$.

4. DECOMPOSITIONS OF $\mathbf{V}^{\otimes r}$

Henceforth we study $\mathbf{V}^{\otimes r}$ as left $\Bbbk W_n$-module and also as left $\Bbbk W_{n-1}$-module (subject to the identification of $\mathbf{V}^{\otimes r}$ with $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n$, discussed in Section 3). We write

$$\Phi_{n,r} : \Bbbk W_n \to \operatorname{End}_{\Bbbk}(\mathbf{V}^{\otimes r}), \quad \Phi_{n,r+1/2} : \Bbbk W_{n-1} \to \operatorname{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$$

for the $\Bbbk$-linear representations corresponding to the left actions in the two bimodule structures on $\mathbf{V}^{\otimes r}$. Our goal is to understand the annihilator of each action. In this section we obtain direct sum decompositions of $\mathbf{V}^{\otimes r}$, as both left $\Bbbk W_n$ and $\Bbbk W_{n-1}$-modules, but first we record the following elementary fact.

Lemma 4.1. *If $r \geq n$ then the representation $\Phi_{n,r} : \Bbbk W_n \to \operatorname{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$ is faithful. If $r+1 \geq n$ then the representation $\Phi_{n,r+1/2} : \Bbbk W_{n-1} \to \operatorname{End}_{\Bbbk}(\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n)$ is faithful.*

Proof. Suppose that $a = \sum_{w \in W_n} a_w w$ belongs to the kernel of $\Phi_{n,r}$. Then a acts as zero on all elements of $\mathbf{V}^{\otimes r}$. First suppose that $r = n$. Consider the simple tensor $v = \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n$. We have

$$w \cdot v = \mathbf{v}_{w(1)} \otimes \mathbf{v}_{w(2)} \otimes \cdots \otimes \mathbf{v}_{w(n)}$$

and each $w \cdot v$ is a simple tensor obtained from v by permuting its factors according to w. In particular, the set $\{w \cdot v \mid w \in W_n\}$ is linearly independent over $\Bbbk$. Thus the fact that $a \cdot v = 0$ implies that

$$\sum_{w \in W_n} a_w \left(\mathbf{v}_{w(1)} \otimes \mathbf{v}_{w(2)} \otimes \cdots \otimes \mathbf{v}_{w(n)} \right) = 0.$$

By linear independence, this forces $a_w = 0$ for all $w \in W_n$; that is, $a = 0$. This proves the first claim in case $r = n$.

For the general case, $r \geq n$, replace v by $v \otimes \mathbf{v}_n^{\otimes(r-n)}$ and repeat the argument. This proves the first claim. The proof of the second claim is similar. $\qquad\square$

Remark 4.2. Lemma 4.1 will be sharpened in Corollary 4.13. The preceding argument can be modified to prove the sharpened result; we leave the details to the interested reader.

Although not needed in this paper, for purposes of comparison we recall the standard multiplicity-free decomposition of $\mathbf{V}^{\otimes r}$ as a right $\Bbbk\mathfrak{S}_r$-module:

$$\mathbf{V}^{\otimes r} \cong \bigoplus_\lambda M^\lambda, \tag{6}$$

where the sum is over all weak compositions of r of length at most n. To see this, observe that each M^λ may be identified with the weight space $\mathbf{V}^{\otimes r}_\lambda$ consisting of all tensors of weight λ for the action of the diagonal torus $T \subset \mathrm{GL}(\mathbf{V})$ (elements of $\mathrm{GL}(\mathbf{V})$ acting diagonally on the basis vectors $\mathbf{v}_i$). The identification $\mathbf{V}^{\otimes r}_\lambda \cong M^\lambda$ is given on basis elements by

$$\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r} \mapsto \mathsf{t}(i_1, \ldots, i_r), \tag{7}$$

where $\mathsf{t}(i_1, \ldots, i_r)$ is the row-standard λ-tableau whose jth row contains all the tensor places in which $\mathbf{v}_j$ appears.

We now provide a language for decomposing $\mathbf{V}^{\otimes r}$, both as a left $\Bbbk W_n$ and $\Bbbk W_{n-1}$-module. The direct summands of this decomposition will be labelled by *hook partitions*, and this will be important in what follows. The combinatorics used here is completely different to that of the characteristic zero $(\Bbbk W_{n-1}, \mathcal{P}_{r+1/2}(n))$-bimodule decomposition, instead it uses the language from Section 1 (and ultimately goes back to ideas from [Jam78]).

Definition 4.3. The *value-type* of a multi-index $(i_1, \ldots, i_r)$ in $I(n, r)$ is the set-partition Λ of $\{1, \ldots, r\}$ defined by $\Lambda = \{\Lambda_1, \ldots, \Lambda_n\}$, where Λ_j is the set of places $\alpha = 1, \ldots r$ such that $i_\alpha = j$. By convention, we usually omit any empty subsets Λ_j from Λ. The non-empty subsets in Λ are called *parts*; their number is denoted by $\ell(\Lambda)$ and is called the *length* of Λ.

The non-empty parts of the value-type Λ associated to $(i_1, \ldots, i_r)$ are the subsets given by the non-empty rows of the associated tableau $\mathsf{t}(i_1, \ldots, i_r)$ defined in (7). For instance, the value-type of the multi-index

$$(i_1, \ldots, i_7) = (9, 8, 8, 1, 9, 8, 1)$$

is $\Lambda = \{\{1, 5\}, \{2, 3, 6\}, \{4, 7\}\}$, a set-partition of $\{1, \ldots, 7\}$ with three parts.

We remark that value-type may be visualised as a half-diagram, in which nodes are linked by edges if and only if they have the same value in the multi-index (although we do not use this visualisation in the paper).

Definition 4.4. Let Λ be a set-partition of $\{1, \ldots, r\}$ with not more than n parts. We define $\mathbf{V}(\Lambda)$ to be the $\Bbbk$-span of the tensors $\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r}$ such that the value-type of $(i_1, \ldots, i_r)$ is equal to Λ.

Similarly, for each set-partition Λ' of $\{1, \ldots, r+1\}$ with not more than n parts, we define $\mathbf{V}'(\Lambda')$ to be the $\Bbbk$-span of the tensors $\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r} \otimes \mathbf{v}_n$ such that the value-type of $(i_1, \ldots, i_r, n)$ is equal to Λ'.

Both $\mathbf{V}(\Lambda)$, $\mathbf{V}'(\Lambda')$ are free modules over the commutative ring $\Bbbk$. It is useful to understand how $\mathbf{V}'(\Lambda')$ is related to $\mathbf{V}(\Lambda')$. Of course we have $\mathbf{V}'(\Lambda') \subset \mathbf{V}(\Lambda')$ by definition. To be more specific, we have the following result.

Lemma 4.5. *For any set-partition Λ' of $\{1, \ldots, r+1\}$ with not more than n parts,* $\mathrm{rank}_\Bbbk \mathbf{V}'(\Lambda') = \frac{1}{n} \mathrm{rank}_\Bbbk \mathbf{V}(\Lambda')$.

Proof. Consider a simple tensor of the form $v = \mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r} \otimes \mathbf{v}_n$ in $\mathbf{V}'(\Lambda')$. Let Λ'_n be the part of Λ' recording the places in v at which $\mathbf{v}_n$ appears (so of course

$r + 1 \in \Lambda'_n$). For any $j = 1, \ldots, n - 1$ the tensor

$$s \cdot v = s \cdot (\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r} \otimes \mathbf{v}_n)$$

belongs to $\mathbf{V}(\Lambda')$, where $s \in W_n$ is the transposition $s = (j, n)$. For each $j = 1, \ldots, n - 1$, the map f_j from $\mathbf{V}'(\Lambda')$ into $\mathbf{V}(\Lambda')$ defined on simple tensors by $v \mapsto s(v)$, then extended linearly, is injective. Furthermore, the image of each f_j gives an isomorphic copy of $\mathbf{V}'(\Lambda')$ inside $\mathbf{V}(\Lambda')$. Finally, the embedding f_n of $\mathbf{V}'(\Lambda')$ in $\mathbf{V}(\Lambda')$ defined by $v \mapsto v$ gives another isomorphic copy of $\mathbf{V}'(\Lambda')$ in $\mathbf{V}(\Lambda')$. Since $\mathbf{V}(\Lambda')$ is equal to the direct sum of the images of the n maps $f_1, \ldots, f_n$, it follows that $\mathrm{rank}_{\Bbbk} \mathbf{V}(\Lambda') = n \cdot \mathrm{rank}_{\Bbbk} \mathbf{V}'(\Lambda')$. $\qquad\square$

Remark 4.6. It is straightforward to check that $\mathbf{V}(\Lambda') \cong \mathrm{ind}_{W_{n-1}}^{W_n} \mathbf{V}'(\Lambda')$, the module obtained by inducing $\mathbf{V}'(\Lambda')$ from W_{n-1} to W_n.

Note that, by definition, simple tensors in $\mathbf{V}'(\Lambda')$ always have the fixed vector $\mathbf{v}_n$ appearing in place $r + 1$. We also note that whenever Λ has more than n parts there are no simple tensors in $\mathbf{V}^{\otimes r}$ of value-type Λ, so $\mathbf{V}(\Lambda) = 0$. On the other hand, it is easy to see that $\mathbf{V}(\Lambda) \neq 0$ if $\ell(\Lambda) \leq n$. It follows from Lemma 4.5 that $\mathbf{V}'(\Lambda') \neq 0$ if and only if $\ell(\Lambda') \leq n$.

Here then are the promised decompositions.

Proposition 4.7. *The action of W_n on $\mathbf{V}^{\otimes r}$ preserves value-type of simple tensors, so $\mathbf{V}(\Lambda)$ is a left $\Bbbk W_n$-module. Similarly, $\mathbf{V}'(\Lambda')$ is a left $\Bbbk W_{n-1}$-module. Furthermore, we have direct sum decompositions*

$$\mathbf{V}^{\otimes r} = \bigoplus\nolimits_{\ell(\Lambda) \leq n} \mathbf{V}(\Lambda), \quad \mathbf{V}^{\otimes r} \otimes \mathbf{v}_n = \bigoplus\nolimits_{\ell(\Lambda') \leq n} \mathbf{V}'(\Lambda'),$$

where Λ, Λ' vary over all set-partitions (with not more than n parts) of $\{1, \ldots, r\}$, $\{1, \ldots, r + 1\}$, respectively. The decompositions are multiplicity-free, in the sense that each $\mathbf{V}(\Lambda)$, $\mathbf{V}'(\Lambda')$ appears exactly once in the direct sum.

Proof. All the claims are easily verified. The point is that the classification of simple tensors by value-type describes the orbits of simple tensors under the left action of W_n. The claims for W_{n-1} are just variations on this theme. $\qquad\square$

The problem with the decompositions in Proposition 4.7 is that the summands are not pairwise non-isomorphic (as $\Bbbk W_n$ or $\Bbbk W_{n-1}$-modules). By looking at examples one quickly observes that $\mathbf{V}(\Lambda)$ depends, up to isomorphism, only on the number of parts $\ell(\Lambda)$ and not on Λ itself. Similar remarks apply to the $\mathbf{V}'(\Lambda')$ in the second decomposition.

To overcome this difficulty, we introduce minimal prototypes for the isomorphism classes that can occur as summands in the above decompositions of tensor space.

Definition 4.8. For any $l = 1, \ldots, n$ let $H_n(l)$ be the $\Bbbk W_n$-submodule of $\mathbf{V}^{\otimes l}$ generated by $\mathbf{v}_{n-l+1} \otimes \mathbf{v}_{n-l+2} \otimes \cdots \otimes \mathbf{v}_n$.

Here are some elementary properties of the $H_n(l)$.

Lemma 4.9. *As left $\Bbbk W_n$-modules, we have isomorphisms:*
  (a) $H_n(l) \cong M^{(n-l,1^l)}$, *for any* $l = 1, \ldots, n - 1$.
  (b) $H_n(n) \cong M^{(1^n)} \cong \Bbbk W_n$.
  (c) $H_n(n - 1) \cong H_n(n)$.

Proof. As (c) follows from (a), (b) we only need to prove (a), (b). To prove (a), observe that $H_n(l)$ is a transitive permutation module, because its basis elements $\{w \cdot (\mathbf{v}_{n-l+1} \otimes \cdots \otimes \mathbf{v}_n) \mid w \in W_n\}$ are permuted transitively by the action of W_n. The stabiliser of $\mathbf{v}_{n-l+1} \otimes \cdots \otimes \mathbf{v}_n$ is the Young subgroup $W_{n-l} = \{w \in W_n \mid w(k) = k \text{ for all } k = n - l + 1, \ldots, n\}$. Since

$$W_{n-l} \cong W_{\{1,\ldots,n-l\}} \times W_{\{n-l+1\}} \times \cdots \times W_{\{n\}},$$

this is the Young subgroup indexed by the partition $(n - l, 1^l)$. Hence $H_n(l) \cong \mathrm{ind}_{W_{n-l}}^{W_n} \Bbbk \cong M^{(n-l,1^l)}$, and (a) is proved. For (b), observe that the stabiliser of the generator $\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n$ is the trivial subgroup, which is the Young subgroup indexed by (1^n). For another way to prove (c), observe that the stabiliser of $\mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n$ is also trivial, because any permutation in W_n fixing $n - 1$ points must fix all n points. $\square$

Proposition 4.10. *Let Λ, Λ' be set-partitions of $\{1, \ldots, r\}$, $\{1, \ldots, r+1\}$ respectively, with not more than n parts. Then:*

(a) $\mathbf{V}(\Lambda) \cong H_n(\ell(\Lambda))$, *as left $\Bbbk W_n$-modules.*
(b) $\mathbf{V}'(\Lambda') \cong H_{n-1}(\ell(\Lambda') - 1)$, *as left $\Bbbk W_{n-1}$-modules.*

Proof. (a) The isomorphism is given by sending each $\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_l}$ in $H_n(l)$ to the simple tensor in $\mathbf{V}(\Lambda)$ obtained by writing $\mathbf{v}_{i_k}$ into all tensor places in Λ_k, where $\Lambda = \{\Lambda_1, \ldots, \Lambda_l\}$.

(b) The proof of (b) is similar to the proof of (a), except that $\mathbf{v}_n$ is fixed wherever it appears. Write $\Lambda' = \{\Lambda_1, \ldots, \Lambda_l\}$ and assume that $r + 1$ belongs to Λ_l, so Λ_l records the places containing a $\mathbf{v}_n$. Then the isomorphism is defined by sending $\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_{l-1}}$ in $H_{n-1}(l - 1)$ to the simple tensor in $\mathbf{V}'(\Lambda')$ obtained by writing $\mathbf{v}_{i_k}$ into all tensor places in Λ_k, for $k = 1, \ldots, l - 1$, and writing $\mathbf{v}_n$ in the places in Λ_l. $\square$

Corollary 4.11. *For any commutative ring $\Bbbk$, we have isomorphisms*

$$\mathbf{V}^{\otimes r} \cong \bigoplus_{1 \leq l \leq \min(n,r)} H_n(l)^{\left\{{r \atop l}\right\}}, \quad \mathbf{V}^{\otimes r} \otimes \mathbf{v}_n \cong \bigoplus_{1 \leq l \leq \min(n-1,r)} H_{n-1}(l)^{\left\{{r+1 \atop l}\right\}}$$

as left $\Bbbk W_n$-modules, $\Bbbk W_{n-1}$-modules, respectively. The Stirling numbers $\left\{{r \atop l}\right\}$, $\left\{{r+1 \atop l}\right\}$ give the multiplicities of the direct summands in the decompositions.

Proof. By Proposition 4.10 the direct summands in Proposition 4.7 are isomorphic to $H_n(l)$, $H_{n-1}(l)$, where $1 \leq l \leq \min(n,r)$, $\min(n-1,r)$ respectively. The number of set-partitions Λ of $\{1, \ldots, r\}$ for which $\ell(\Lambda) = l$ is given by $\left\{{r \atop l}\right\}$, the Stirling number of the second kind [Sta12, §1.9]. It follows that the number of set-partitions Λ' of $\{1, \ldots, r+1\}$ for which $\ell(\Lambda') = l$ is given by $\left\{{r+1 \atop l}\right\}$. $\square$

Remark 4.12. If $n \leq r$, $n - 1 \leq r$, respectively, then the direct summands in the decompositions in Corollary 4.11 are still not pairwise non-isomorphic. In those cases, we have to take the isomorphism in Lemma 4.9(c) into account. This is worth living with in order to have a uniform formula for the multiplicities of the summands.

As another consequence of these results, we easily obtain the promised sharpening (see Remark 4.2) of Lemma 4.1.

Corollary 4.13. *For any commutative ring $\Bbbk$, the representations $\Phi_{n,r}$, $\Phi_{n,r+1/2}$ are faithful when $n - 1 \leq r$, $n - 2 \leq r$, respectively.*

Proof. This follows immediately from the decompositions in Corollary 4.11 and the isomorphisms in Lemma 4.9(b), (c) which imply that $H_n(n-1) \cong \Bbbk W_n$, $H_{n-1}(n-2) \cong \Bbbk W_{n-1}$ are faithful modules. □

Remark 4.14. Equation (16) in Section 7 implies that $\Phi_{n,r}$, $\Phi_{n,r+1/2}$ are not faithful when $n - 1 > r$, $n - 2 > r$ respectively, so the bounds in Corollary 4.13 are best possible.

5. THE ANNIHILATOR OF $\mathbf{V}^{\otimes r}$ IN CHARACTERISTIC ZERO

Recall that we always identify $\mathbf{V}^{\otimes r}$ with $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n$ when considering the representation $\Phi_{n,r+1/2} : \Bbbk W_{n-1} \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$. We also have the representation $\Phi_{n,r} : \Bbbk W_n \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$. From now on we treat the two representations uniformly, writing them as

$$(8) \qquad \Phi_{n,r+\varepsilon} : \Bbbk W_d \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r}),$$

where $\varepsilon \in \{0, 1/2\}$ and $d = d(n, \varepsilon) := n - 2\varepsilon$. Henceforth, the symbols ε, d will always have these fixed interpretations. We wish to study $\ker \Phi_{n,r+\varepsilon}$, so from now on we always assume that $d > r + 1$, because otherwise $\Phi_{n,r+\varepsilon}$ is faithful, by Corollary 4.13.

In this section we assume that $\Bbbk$ is a field of characteristic zero. This implies that $\Bbbk W_d$ is semisimple. By the Artin–Wedderburn theorem and the fact that all irreducible representations of W_d are absolutely irreducible, we have a (split) semisimple decomposition

$$(9) \qquad \Bbbk W_d \cong \bigoplus_{\mu \vdash d} \mathrm{End}_{\Bbbk}(S^{\mu}),$$

where S^{μ} is the (irreducible) Specht module indexed by $\mu \vdash d$. The assumption $d > r + 1$ implies that $\min(d, r) = r$, so we may combine the two cases of Corollary 4.11 as: $\mathbf{V}^{\otimes r} \cong \bigoplus_{1 \leq l \leq r} H_d(l)^{\left\{ {r+2\varepsilon \atop l} \right\}}$. In light of the isomorphism from Lemma 4.9(a), this gives the decomposition

$$(10) \qquad \mathbf{V}^{\otimes r} \cong \bigoplus_{\lambda \in \mathcal{H}(d,r)} (M^{\lambda})^{m_{\lambda}} \quad (m_{\lambda} > 0)$$

as left $\Bbbk W_d$-modules, where $\mathcal{H}(d, r) = \{(d - l, 1^l) \mid l = 1, \ldots, r\}$. Note that $\mathcal{H}(d, r)$ is a set of hook partitions and the M^{λ} such that $\lambda \in \mathcal{H}(d, r)$ are pairwise non-isomorphic. The multiplicities m_{λ} in (10) are given by Stirling numbers, but we only need that they are positive integers.

Definition 5.1. Write $\alpha(d, r) = (d - r, 1^r)$ for the minimum element (with respect to the dominance order, $\rhd$, defined in Section 1) of the set $\mathcal{H}(d, r)$.

If S is a simple module and M is a module satisfying the Jordan–Hölder theorem, write $[M : S]$ for the multiplicity of S in a composition series of M. Our next result provides a lower bound on $\ker \Phi_{n,r+\varepsilon}$ in characteristic zero.

Proposition 5.2. *Assume that $\Bbbk$ is a field of characteristic zero. Let $d = n - 2\varepsilon$ and assume that $d > r + 1$, where $\varepsilon \in \{0, 1/2\}$. The set of $\lambda \vdash d$ such that $[\mathbf{V}^{\otimes r} : S^{\lambda}] \neq 0$ is contained in $\{\lambda \vdash d \mid \lambda \unrhd \alpha(d, r)\}$. Hence, $\ker \Phi_{n,r+\varepsilon}$ contains an isomorphic copy of $\bigoplus_{\lambda \vdash d, \, \lambda \ntrianglerighteq \alpha(d,r)} \mathrm{End}_{\Bbbk}(S^{\lambda})$.*

Proof. The indexing set $\mathcal{H}(d,r)$ in the decomposition (10) forms a well-ordered chain under the dominance order. By Young's rule (see for instance [Jam78, 4.13 or 14.1]), for any $\mu \vdash d$, the set of $\lambda \vdash d$ such that $[M^\mu : S^\lambda] \neq 0$ is contained in $\{\lambda \vdash d \mid \lambda \trianglerighteq \mu\}$. The first claim follows, since

$$\{\lambda \trianglerighteq \mu \mid \mu \in \mathcal{H}(d,r)\} = \{\lambda \vdash d \mid \lambda \trianglerighteq \alpha(d,r)\}$$

because $\alpha(d,r)$ is the minimum element of $\mathcal{H}(d,r)$. Hence, as a $\Bbbk W_d$–module, the decomposition (10) takes the form

$$\mathbf{V}^{\otimes r} \cong \bigoplus_{\lambda \trianglerighteq \alpha(d,r)} (S^\lambda)^{n_\lambda},$$

where $n_\lambda \geq 0$ is the multiplicity of S^λ in the decomposition. The semisimple decomposition $\Bbbk W_d \cong \bigoplus_{\lambda \vdash d} \mathrm{End}_\Bbbk(S^\lambda)$ then implies the final statement in the proposition, since the only summands acting non-trivially on $\mathbf{V}^{\otimes r}$ are the $\mathrm{End}_\Bbbk(S^\lambda)$ such that $\lambda \trianglerighteq \alpha(d,r)$ and $n_\lambda > 0$. $\qquad\square$

The following fact from [DN11] can be applied to obtain the opposite inclusion and thus prove equality in Proposition 5.2.

Lemma 5.3 ([DN11, Lemma 6.4])**.** *Assume that $\Bbbk$ is a field of characteristic zero. For any partitions $\lambda \trianglerighteq \mu$ there is an embedding of M^λ in M^μ, as $\Bbbk W_d$-modules.*

If M^λ embeds in M^μ then clearly $\mathrm{ann}_{\Bbbk W_d} M^\lambda \supseteq \mathrm{ann}_{\Bbbk W_d} M^\mu$.

Proposition 5.4. *Assume that $\Bbbk$ is a field of characteristic zero. Let $d = n - 2\varepsilon$ and assume that $d > r + 1$, where $\varepsilon \in \{0, 1/2\}$. Then:*
   (a) $\ker \Phi_{n,r+\varepsilon}$ *is isomorphic to* $\bigoplus_{\lambda \vdash d,\, \lambda \ntrianglerighteq \alpha(d,r)} \mathrm{End}_\Bbbk(S^\lambda)$.
   (b) $\ker \Phi_{n,r+\varepsilon} = \mathrm{ann}_{\Bbbk W_d} M^{\alpha(d,r)} = \bigcap_{\lambda \vdash d,\, \lambda \trianglerighteq \alpha(d,r)} \mathrm{ann}_{\Bbbk W_d} M^\lambda$.

Proof. (a) Lemma 5.3 implies that we do not change the annihilator by adding extra permutation module terms in the $\Bbbk W_d$-module decomposition (10), enlarging it to $\bigoplus_{\mu \vdash d,\, \mu \trianglerighteq \alpha(d,r)} (M^\mu)^{m_\mu}$, where $m_\mu := 1$ for any $\mu \notin \mathcal{H}(d,r)$. But then any S^λ for $\lambda \vdash d$ such that $\lambda \trianglerighteq \alpha(d,r)$ must occur with multiplicity at least one, since $[M^\lambda : S^\lambda] = 1$. This implies that the annihilator cannot be larger than $\bigoplus_{\lambda \vdash d,\, \lambda \ntrianglerighteq \alpha(d,r)} \mathrm{End}_\Bbbk(S^\lambda)$. The equality in part (a) then follows from Proposition 5.2.
   (b) By Lemma 5.3, for each $\lambda \trianglerighteq \alpha(d,r)$ we have an embedding $M^\lambda \subseteq M^{\alpha(d,r)}$. It follows that there is an embedding $\mathrm{ann}_{\Bbbk W_d} M^\lambda \supseteq \mathrm{ann}_{\Bbbk W_d} M^{\alpha(d,r)}$. Hence the intersection

$$\mathrm{ann}_{\Bbbk W_d} \mathbf{V}^{\otimes r} = \bigcap_{\lambda \vdash d,\, \lambda \trianglerighteq \alpha(d,r)} \mathrm{ann}_{\Bbbk W_d} M^\lambda$$

collapses to the single term $\mathrm{ann}_{\Bbbk W_d} M^{\alpha(d,r)}$. This proves part (b). $\qquad\square$

6. Reformulation of the characteristic zero result

As in Section 5, we still assume that $\Bbbk$ is a field of characteristic zero. Our standing assumption that $d > r + 1$ remains in force. Finally, we remind the reader that $d = n - 2\varepsilon$, where $\varepsilon \in \{0, 1/2\}$.

It is straightforward to see by looking at examples that the y-basis $\mathcal{Y}$ (of Theorem 1.1) is the right one to use in order to describe the kernel of $\Phi_{n,r+\varepsilon}$ as a

cell ideal. For instance, if $n = 3$ and $r = 1$ then it is easy to check that the map $\Phi_{3,1} : \Bbbk W_3 \to \mathrm{End}(\mathbf{V})$ has kernel generated by the element

$$y_{\mathsf{t}^{(3)}\mathsf{t}^{(3)}} = \sum_{w \in W_3} \mathrm{sgn}(w)\, w.$$

Note that this example is compatible with Proposition 5.4(a), which says that the kernel is isomorphic to $\mathrm{End}_{\Bbbk}(S^{(1^3)}) = \mathrm{End}_{\Bbbk}(\Delta(3))$.

From now on we make heavy use of the fact that transposing reverses the dominance order: $\lambda \trianglerighteq \mu$ if and only if $\lambda' \trianglelefteq \mu'$. We now reformulate Proposition 5.4 by replacing λ by its transpose λ', in light of the identification $\Delta(\lambda) = S^{\lambda'}$ from Theorem 1.1. Then Proposition 5.4(a) takes the form

$$(11) \qquad \ker \Phi_{n,r+\varepsilon} \cong \bigoplus_{\lambda' \ntrianglerighteq \alpha(d,r)} \mathrm{End}_{\Bbbk}(S^{\lambda'}) = \bigoplus_{\lambda \ntrianglelefteq \alpha(d,r)'} \mathrm{End}_{\Bbbk}(\Delta(\lambda)),$$

where the rightmost equality follows from the equivalence

$$\lambda' \ntrianglerighteq \alpha(d,r) \iff \lambda \ntrianglelefteq \alpha(d,r)'.$$

Note that $\alpha(d,r)' = (r+1, 1^{d-r-1}) = \alpha(d, d-r-1)$. Combining the above with the equality

$$(12) \qquad \mathrm{ann}_{\Bbbk W_d}\, \mathbf{V}^{\otimes r} = \bigcap_{\lambda \vdash d,\, \lambda \trianglerighteq \alpha(d,r)} \mathrm{ann}_{\Bbbk W_d}\, M^{\lambda}$$

in Proposition 5.4(b), we obtain the following.

Proposition 6.1. *Assume that $\Bbbk$ is a field of characteristic zero. Let $d = n - 2\varepsilon$ and assume that $d > r + 1$, where $\varepsilon \in \{0, 1/2\}$. Then*

$$\ker \Phi_{n,r+\varepsilon} = \bigcap_{\lambda \vdash d,\, \lambda \trianglerighteq \alpha(d,r)} \mathrm{ann}_{\Bbbk W_d}\, M^{\lambda} = \Bbbk\{y_{\mathsf{st}} \mid [\mathsf{s}] = [\mathsf{t}] \ntrianglelefteq \alpha(d,r)'\}.$$

Proof. Most of the proof is in the remarks preceding the statement. To complete the proof, we only need to observe that in the semisimple case, the cell ideal spanned by all y_{st} for $[\mathsf{s}] = [\mathsf{t}] \ntrianglelefteq \alpha(d,r)'$ is isomorphic to $\bigoplus_{\lambda \ntrianglelefteq \alpha(d,r)'} \mathrm{End}_{\Bbbk}(\Delta(\lambda))$. $\square$

Note the similarity, and the difference, with Lemma 3 of [Här99]. Later we will show that the description of $\ker \Phi_{n,r+\varepsilon}$ in the rightmost equality of Proposition 6.1 is characteristic-free.

The following characterisation of the key inequalities in Proposition 6.1 will be useful in the next section. We write $\mathrm{rows}(\lambda)$, $\mathrm{cols}(\lambda)$ for the number of rows, columns in the Young diagram of λ, respectively.

Lemma 6.2. *Let $\lambda \vdash d$. Then:*
  (a) $\lambda \trianglerighteq \alpha(d,r) \iff \mathrm{cols}(\lambda) \geq d - r$.
  (b) $\lambda \trianglelefteq \alpha(d,r)' \iff \mathrm{rows}(\lambda) \geq d - r$.

Remark 6.3. Part (b) implies that $\lambda \ntrianglelefteq \alpha(d,r)' \iff \mathrm{rows}(\lambda) < d - r$.

Proof. (a) Since $\alpha(d,r) = (d-r, 1^r)$, it follows by the definition of the dominance order that $\lambda \trianglerighteq \alpha(d,r)$ if and only if

$$\lambda_1 \geq d - r, \quad \lambda_1 + \lambda_2 \geq d - r + 1, \quad \ldots, \quad \lambda_1 + \cdots + \lambda_{r+1} \geq d.$$

Since $\lambda_1 = \mathrm{cols}(\lambda)$, it follows that λ has at least $d - r$ columns if and only if $\lambda \trianglerighteq \alpha(d,r)$. This proves (a).

(b) This follows from (a), since for $\mu, \nu \vdash d$ we have $\mu \trianglerighteq \nu \iff \mu' \trianglelefteq \nu'$. First, replace λ with μ in (a) to get $\mu \trianglerighteq \alpha(d, r) \iff \text{cols}(\mu) \geq d - r$. This is equivalent to the statement

$$\mu' \trianglelefteq \alpha(d, r)' \iff \text{cols}(\mu) \geq d - r.$$

Using the fact that $\text{cols}(\mu) = \text{rows}(\mu')$, we obtain the desired result by setting $\lambda = \mu'$ in the above. $\qquad \square$

Suppose that Ω is any set of partitions of d. For convenience of notation, we set $A = \Bbbk W_d$. Following Graham and Lehrer [GL96], we define

$$A^y[\Omega] = \sum_{\lambda \in \Omega : [\mathsf{s}] = [\mathsf{t}] = \lambda} \Bbbk y_{\mathsf{st}}.$$

A set Ω of partitions of d is an *ideal* if $\lambda \in \Omega$ and $\mu \trianglerighteq \lambda$ (for $\mu \vdash d$) always implies $\mu \in \Omega$. It is clear that the set

(13) $$\Omega = \{\lambda \vdash d \mid \lambda \ntrianglelefteq \alpha(d, r)'\}$$

is an ideal, because if λ is a partition of strictly fewer than $d - r$ parts and if $\mu \trianglerighteq \lambda$ then the diagram of μ is obtained from the diagram of λ by moving boxes up, which cannot increase the number of rows. Alternatively, if $\lambda \ntrianglelefteq \alpha(d, r)'$ and if $\mu \trianglerighteq \lambda$ then the assumption $\mu \trianglelefteq \alpha(d, r)'$ implies that $\lambda \trianglelefteq \mu \trianglelefteq \alpha(d, r)'$, which forces $\lambda \trianglelefteq \alpha(d, r)'$, a contradiction.

The importance of this is that Graham and Lehrer proved that the set $A^y[\Omega]$ is an ideal in A whenever Ω is an ideal.

Proposition 6.4 ([GL96, Lemma (1.5)]). *For $\Omega = \{\lambda \vdash d \mid \lambda \ntrianglelefteq \alpha(d, r)'\}$ as above,*

$$A^y[\Omega] = A^y[\ntrianglelefteq \alpha(d, r)']$$

is a cell ideal in $A = \Bbbk W_d$, with basis $\{y_{\mathsf{st}} : \mathsf{s}, \mathsf{t} \text{ standard}, [\mathsf{s}] = [\mathsf{t}] \ntrianglelefteq \alpha(d, r)'\}$.

The cell ideal $A^y[\ntrianglelefteq \alpha(d, r)']$ is equal to $\ker \Phi_{n, r + \varepsilon}$ in Proposition 6.1.

7. The annihilator in general

Now we revert back to the general case, where $\Bbbk$ is once again an arbitrary commutative ring. Note that Proposition 6.4 holds in this generality. We continue to set $A = \Bbbk W_d$.

Let $v = \mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_l}$ in $H_d(l)$ be a simple tensor with distinct tensor factors (i.e., $i_\alpha \neq i_\beta$ for $1 \leq \alpha \neq \beta \leq l$). We start by computing $y_\lambda \cdot v$, where $\lambda \vdash d$ and $1 \leq l \leq d$. (See equation (1) for the definition of y_λ.) Let

$$W_d^v = \{w \in W_d \mid w \cdot v = v\}$$

be the stabiliser of v in W_d. Clearly we have $W_d^v = W_B$ where $B = \{1, \ldots, d\} \setminus \{i_1, \ldots, i_l\}$. Let W_λ be the Young subgroup determined by λ, and write it as $W_{C_1} \times \cdots \times W_{C_k}$, where λ has k parts. Here C_j is the subset of $\{1, \ldots, d\}$ defined by the numbers in the jth row of t^λ. Note that $\{C_1, \ldots, C_k\}$ is a set-partition of $\{1, \ldots, d\}$. We have

(14) $$W_\lambda \cap W_B = (W_{C_1} \times \cdots \times W_{C_k}) \cap W_B = W_{C_1 \cap B} \times \cdots \times W_{C_k \cap B}.$$

The stabiliser of v in W_λ is $S = W_\lambda \cap W_d^v = W_\lambda \cap W_B$ as above. Fix a left coset decomposition

$$w_1 S \sqcup w_2 S \sqcup \cdots \sqcup w_t S = W_\lambda.$$

of W_λ/S, where $\{w_1, \ldots, w_t\}$ is a set of coset representatives. Then by the definition of y_λ we have

$$y_\lambda \cdot v = \sum_{i=1}^{t} \sum_{s \in S} \mathrm{sgn}(w_i s)\, w_i s \cdot v = \sum_{i=1}^{t} \mathrm{sgn}(w_i) \Big(\sum_{s \in S} \mathrm{sgn}(s) \Big)\, w_i \cdot v.$$

This proves that if we set $F_S = \sum_{s \in S} \mathrm{sgn}(s)$ then we have

(15)
$$y_\lambda \cdot v = F_S \sum_{i=1}^{t} \mathrm{sgn}(w_i)\, w_i \cdot v.$$

We note that F_S is a scalar depending only on S. Since S is a product of symmetric groups by (14), it follows that F_S is either zero or one, with the latter case occurring if and only if S is the trivial subgroup. We have shown that:

(16)
$$S = W_d^v \cap W_\lambda \neq \{1\} \implies y_\lambda \cdot v = 0.$$

From the above analysis, we now obtain a lower bound for the annihilator of $\mathbf{V}^{\otimes r}$ as a $\Bbbk W_d$-module, where $d = n - 2\varepsilon$ as usual. Later we will show the bound is precise.

Proposition 7.1. *Let $\Bbbk$ be a commutative ring and assume that $d > r + 1$. Let $d = n - 2\varepsilon$, where $\varepsilon \in \{0, 1/2\}$. Then the kernel of $\Phi_{n, r+\varepsilon}$ contains the cell ideal $A^y[\ntrianglelefteq \alpha(d, r)']$, where $A = \Bbbk W_d$.*

Proof. Let $\lambda \vdash d$ have fewer than $d - r$ parts. In light of Corollary 4.11, we need to show that $y_{\mathsf{st}} \cdot v = 0$, for any $v \in H_d(l)$, where $1 \leq l \leq r - 2\varepsilon$. Since $y_{\mathsf{st}} = d(\mathsf{s}) y_\lambda d(\mathsf{t})^{-1}$ it clearly suffices to show that $y_\lambda \cdot v = 0$ for any such v. By (16), this will follow if we can show that $W_d^v \cap W_\lambda$ is not trivial. By (14), we have

$$W_d^v \cap W_\lambda = W_\lambda \cap W_B = W_{C_1 \cap B} \times \cdots \times W_{C_k \cap B},$$

where $W_\lambda = W_{C_1} \times \cdots \times W_{C_k}$ and $B = \{1, \ldots, d\} \setminus \{i_1, \ldots, i_l\}$. Since $\{1, \ldots, d\} = C_1 \sqcup \cdots \sqcup C_k$, we have

$$B = (C_1 \cap B) \sqcup \cdots \sqcup (C_k \cap B).$$

Furthermore, we have $|B| = d - l$. The condition $l \leq r$ (from above) forces

$$|B| = d - l \geq d - r > k.$$

Hence at least one $C_j \cap B$ has more than one element, and thus $W_d^v \cap W_\lambda$ is not trivial. $\square$

To obtain the opposite inclusion, we will adapt a result of Härterich.

Proposition 7.2. *Suppose that $\Bbbk$ is a commutative ring. For any $\mu \vdash d$,*

$$\bigcap_{\lambda \vdash d,\, \lambda \trianglerighteq \mu} \mathrm{ann}_A M^\lambda = A^y[\ntrianglelefteq \mu'], \text{ with basis } \{y_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \text{ standard}, [\mathsf{s}] = [\mathsf{t}] \ntrianglelefteq \mu'\}.$$

Proof. (Compare with the proof of [Här99, Lemma 3]. We are setting $q = 1$ and replacing right modules with left ones.) Let Ω be the set of all $\lambda \vdash d$ such that $\lambda \ntrianglelefteq \mu'$. The set Ω is that of equation (13), where we remark that this is an ideal in the poset of partitions of d. It follows from [Mur95, Lemma 4.12] that $A^y[\Omega]$ is contained in $\bigcap_{\lambda \vdash d,\, \lambda \trianglerighteq \mu} \mathrm{ann}_A M^\lambda$, so we need only prove the reverse containment.

Suppose that $h^{\#} = \sum_{(\mathsf{s},\mathsf{t})} \alpha_{\mathsf{st}} y_{\mathsf{st}}$ belongs to $\bigcap_{\lambda \vdash d,\, \lambda \trianglerighteq \mu} \mathrm{ann}_A M^\lambda$. We need to show that $\alpha_{\mathsf{st}} \neq 0$ implies $[\mathsf{s}] = [\mathsf{t}] \ntrianglelefteq \mu'$. Let $(\mathsf{s}_0, \mathsf{t}_0)$ be a *minimal* pair (with respect to

$\unrhd$) such that $\alpha_{\mathsf{s}_0\mathsf{t}_0} \neq 0$. Then $\alpha_{\mathsf{st}} = 0$ for all pairs $(\mathsf{s},\mathsf{t}) \lhd (\mathsf{s}_0,\mathsf{t}_0)$, and since the set Ω is an ideal in the poset, it suffices to show that $[\mathsf{s}_0] = [\mathsf{t}_0] \not\unlhd \mu'$.

Let $\lambda_0' = [\mathsf{s}] = [\mathsf{t}]$. The calculation in the proof of [Här99, Lemma 3] shows that $h^\# x_{\mathsf{s}_0'\lambda_0} \neq 0$. Since $h^\# \in \bigcap_{\lambda \vdash d, \, \lambda \unrhd \mu} \operatorname{ann}_A M^\lambda$, it follows that $\lambda_0 \not\unrhd \mu$. Equivalently, $\lambda_0' \not\unlhd \mu'$, so $[\mathsf{s}_0] = [\mathsf{t}_0] \not\unlhd \mu'$, as required. $\qquad\square$

To finish, we will specialise μ to $\alpha(d,r)$ in Proposition 7.2. We also need one more fact. We need to show that the intersection of annihilators in Proposition 7.2 in the case $\mu = \alpha(d,r)$ is in fact the same as the annihilator of $\mathbf{V}^{\otimes r}$. This is far from obvious, although we already proved this is so if $\Bbbk$ is a field of characteristic zero. If Lemma 5.3 were true over any commutative ground ring $\Bbbk$ then we would be able to apply the same argument that produced Proposition 5.4(a), but unfortunately there is an example in [DN11] showing that Lemma 5.3 can fail in positive characteristic.

It turns out, however, that there is a version of Lemma 5.3, valid for any commutative ring $\Bbbk$, in the special case where the second partition is taken to be $\alpha(d,r)$.

Proposition 7.3. *For any commutative ring $\Bbbk$, and any $\lambda \vdash d$ such that $\lambda \unrhd \alpha(d,r)$, we have an embedding $M^\lambda \subseteq M^{\alpha(d,r)}$, as $\Bbbk W_d$-modules.*

Proof. Write $A = \Bbbk W_d$ as before. For any $\lambda \vdash d$, the permutation module M^λ is isomorphic to the left ideal Ax_λ. For ease of typography set $\mu = \alpha(d,r)$, and assume that $\lambda \unrhd \mu$. Then by applying [Mur95, Lemma 4.1] we see that x_μ is a left factor of $x_\lambda = x_{\mathsf{tt}}$ for $\mathsf{t} = \mathsf{t}^\lambda$ the canonical λ-tableaux; i.e., there is some $z \in A$ such that $x_\lambda = zx_\mu$. This immediately implies that $Ax_\lambda = Azx_\mu \subseteq Ax_\mu$, as required. $\qquad\square$

Now we finally obtain our main result. Recall that $\alpha(d,r) = (d-r,1^r)$.

Theorem 7.4. *Let $d = n - 2\varepsilon$, where $\varepsilon \in \{0,1/2\}$. For any commutative ring $\Bbbk$, set $A = \Bbbk W_d$.*

(a) *The kernel of $\Phi_{n,r+\varepsilon}$ over $\Bbbk$ is equal to the cell ideal*

$$A^y[\not\unlhd \alpha(d,r)'] = \Bbbk\{y_{\mathsf{st}} \mid \mathsf{s},\mathsf{t} \text{ standard}, [\mathsf{s}] = [\mathsf{t}] \not\unlhd \alpha(d,r)'\},$$

that is, the $\Bbbk$-span of all $y_{\mathsf{st}} = y_{\mathsf{st}}^\lambda$ for which $\operatorname{rows}(\lambda) < d-r$. This spanning set is a cellular basis of the ideal, which is thus free over $\Bbbk$ of rank which is independent of $\Bbbk$.

(b) *The image in $A/(\ker \Phi_{n,r+\varepsilon})$ of the set*

$$\Bbbk\{y_{\mathsf{st}} \mid \mathsf{s},\mathsf{t} \text{ standard}, [\mathsf{s}] = [\mathsf{t}] \unlhd \alpha(d,r)'\},$$

that is, the $\Bbbk$-span of all $y_{\mathsf{st}} = y_{\mathsf{st}}^\lambda$ for which $\operatorname{rows}(\lambda) \geq d-r$, is a cellular basis of $\operatorname{im}\Phi_{n,r+\varepsilon}$, which is thus free over $\Bbbk$ of rank which is independent of $\Bbbk$.

(c) $\ker \Phi_{n,r+\varepsilon} = \operatorname{ann}_{\Bbbk W_d} \mathbf{V}^{\otimes r} = \operatorname{ann}_{\Bbbk W_d} M^{\alpha(d,r)}$.

Proof. (a) This is just a matter of putting the various pieces together. We use Proposition 7.3 to show that the annihilator

$$\operatorname{ann}_A \mathbf{V}^{\otimes r} = \bigcap_{\lambda \in \mathcal{H}(d,r)} \operatorname{ann}_A M^\lambda$$

remains unchanged when we include extra terms of the form $\operatorname{ann}_A M^\lambda$ for any $\lambda \unrhd \alpha(d,r)$. This implies that

$$\operatorname{ann}_A \mathbf{V}^{\otimes r} = \bigcap_{\lambda \vdash d, \, \lambda \unrhd \alpha(d,r)} \operatorname{ann}_A M^\lambda.$$

Now the first claim in (a) follows from Proposition 7.2. The equality of the displayed set and its alternative description as the $\Bbbk$-span of the y_{st}^λ for which $\mathrm{rows}(\lambda) < d - r$ is Lemma 6.2; see Remark 6.3.

    (b) is an immediate consequence of (a) and the structure of cellular algebras.

    (c) follows from Proposition 7.3. $\square$

Example 7.5. Let $d = n - 2\varepsilon$, where $\varepsilon \in \{0, 1/2\}$.

    (i) If $d - r \leq 1$ then $\ker \Phi_{n,r+\varepsilon} = 0$, by Corollary 4.13.

    (ii) If $d - r = 2$ then $\ker \Phi_{n,r+\varepsilon} = \Bbbk y_{\mathsf{tt}}^{(d)} = \Bbbk \sum_{w \in W_n} \mathrm{sgn}(w)\, w$, of rank 1, where $\mathsf{t} = \mathsf{t}^{(d)}$ is the unique standard tableau of shape (d).

    (iii) If $d - r = 3$ then $\ker \Phi_{n,r+\varepsilon}$ is the $\Bbbk$-span of all y_{st}^λ (where $[\mathsf{s}] = [\mathsf{t}] = \lambda$) for which λ has at most two rows.

By combining Theorem 7.4 with Schur–Weyl duality, we obtain the following consequence.

Corollary 7.6. *Let $\varepsilon \in \{0, 1/2\}$. For any commutative ring $\Bbbk$, the centraliser algebra $\mathrm{End}_{\mathcal{P}_{r+\varepsilon}(n)}(\mathbf{V}^{\otimes r})$ is cellular.*

Proof. By the main result of [BDM22], that Schur–Weyl duality holds over a commutative ring $\Bbbk$, the representation $\Phi_{n,r+\varepsilon}$ surjects onto the algebra $\mathrm{End}_{\mathcal{P}_{r+\varepsilon}(n)}(\mathbf{V}^{\otimes r})$. By Theorem 7.4, that image is cellular. $\square$

Remark 7.7. Stability phenomena for the symmetric group in characteristic zero are well-documented (see for example [BDVO15, Del07]). In characteristic $p > 0$ much less is known, but instances of such phenomena have already been discovered using partition algebra ideas [MW98]. We hope that the Schur–Weyl duality of [BDM22] and resulting Schur functors will provide the rigorous framework necessary to explore these ideas further.

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