REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 26, Pages 498–514 (April 25, 2022)
https://doi.org/10.1090/ert/609

BRANCHING OF METAPLECTIC REPRESENTATION OF $Sp(2,\mathbb{R})$ UNDER ITS PRINCIPAL $SL(2,\mathbb{R})$-SUBGROUP

GENKAI ZHANG

Abstract. We study the branching problem of the metaplectic representation of $Sp(2,\mathbb{R})$ under its principle subgroup $SL(2,\mathbb{R})$. We find the complete decomposition.

Introduction

The problem of finding the branching rule of a unitary representation (Λ, G) of a semisimple Lie group G under a subgroup H is of fundamental interests in representation theory. In the present paper we shall study this problem for the metaplectic representation Λ of the symplectic group $G = Sp(2,\mathbb{R})$ under its principal subgroup $SL(2,\mathbb{R})$.

The metaplectic representation for $G = Sp(2,\mathbb{R})$ can be realized on the Fock space $\mathcal{F}(\mathbb{C}^2)$ and the Lie algebra $\mathfrak{g}$ acts as differential operators with quadratic polynomial coefficients. We use the classical approach by finding the spectra of the Casimir operator C. To find the continuous spectrum we compute the action of the Casimir element on some orthogonal basis of weight vectors of $SO(2) \subset SL(2,\mathbb{R})$ of a fixed weight. We prove that Casimir element is unitary equivalent to the coordinate multiplication operator on certain hypergeometric orthogonal polynomials, more precisely the continuous dual Hahn polynomials, and it determines the continuous spectrum of C.

To find the discrete components we solve the equation of highest or lowest weight vectors for $\mathfrak{sl}(2,\mathbb{C})$-actions in the Fock space. We prove that lowest weight representations of $SL(2,\mathbb{R})$ will not appear in (Λ, G) and we find all highest weight representations.

For the even part of the metaplectic representation this decomposition can also be treated using the Berezin transform [20]; see Remark 3.8. The connection between hypergeometric orthogonal polynomials and branching problems has been observed in the earlier works [19, 21], it was proved that density for orthogonality is precisely the product of Harish-Chandra c-function and the symbol of the Berezin transform.

We remark that if we transfer our results in the Schrödinger $L^2(\mathbb{R}^2)$-realization, we have essentially found the spectrum of some partial differential operators of degree four involving squares of harmonic oscillator of the form $\alpha(A\partial, \partial) + \beta(Bx, x)$ with non-positive definite symmetric matrices A and B, and this might be of independent interests.

Received by the editors September 3, 2021, and, in revised form, December 12, 2021, and February 7, 2022.

2020 *Mathematics Subject Classification.* Primary 22E45, 43A80, 43A90.

The research for this work was supported by the Swedish Science Council (VR).

Finally we mention very briefly our motivation, related known results and some perspective questions. Firstly the exact appearance of $SL(2,\mathbb{R}) \subset G = Sp(2,\mathbb{R})$ and the corresponding branching problem can be used to study the induced representation of the group G_2 from its Heisenberg parabolic subgroup [7]; in [8] we shall study the non-compact realization of induced represenations for Hermitian Lie groups using the related branching of metaplectic representation, the compact picture being studied in [22]. Next this kind of branching problem can be formulated for any real split simple Lie group G. Indeed up to conjugation there is a unique principal subalgebra $\mathfrak{sl}(2,\mathbb{R})$ (and it might have more than one principal $\mathfrak{sl}(2,\mathbb{R})$-subgroups); see [14]. For general $G = Sp(n,\mathbb{R})$ we can use covariant differentiations to produce some discrete components, but the results are not complete as for $G = Sp(2,\mathbb{R})$ and will not be present here; see e.g. [3, 12, 16] for related study and references. It is also tempting but challenging to study the branching problems for the minimal representations of G under $SL(2,\mathbb{R})$, even for groups of lower rank such as $SL(3,\mathbb{R})$ and G_2. The principal $SL(2,\mathbb{R})$-subgroups in split real Lie groups correspond to certain nilpotent orbits [2]. On the other hand general nilpotent orbits of real Lie groups are closely related $SL(2,\mathbb{R})$-subgroups. One might ask if there is a general theory about the branching under $SL(2,\mathbb{R})$ of representations of G related to the corresponding nilpotent orbits. Next the branching of metaplectic representations under dual pairs, namely the dual correspondence, has been under intensive study and has found many applications; for the dual pair $SL(2,\mathbb{R}) \times O(p,q)$ in $Sp(p+q,\mathbb{R})$ there is some detailed study of Plancherel formula by Howe [10] using the results of Repka [17] on tensor products of representations of $SL(2,\mathbb{R})$; however our result here seems can not be deduced from [17]. From a classical analytic point of view explicit decompositions under $SL(2,\mathbb{R})$ provide methods to discover new orthogonal polynomials and spectral decomposition of higher order self-adjoint operators. Finally the appearance of certain specific $SL(2,\mathbb{R})$ in $Sp(n,\mathbb{R})$ and hyperbolic discs in locally symmetric spaces or Teichmüller space is of importance in higher Teichmüller theory [1].

The paper is organized as follows. In Section 1 we recall the realization of the metaplectic representation of $Sp(n,\mathbb{R})$ on the Fock space. We find the explicit description of the principal $SL(2,\mathbb{R})$-subgroup in $Sp(n,\mathbb{R})$ and its action on the metaplectic representation in Section 2. The complete decomposition of the metaplectic representation of $Sp(2,\mathbb{R})$ is done in Section 3.

1. Preliminaries

We recall some known facts on metaplectic representations of $G = Sp(n,\mathbb{R})$.

1.1. The group $G = Sp(n,\mathbb{R})$.

The symplectic group $G = Sp(n,\mathbb{R})$ has usually two different realizations, as the group of biholomorphic transformations of the Siegel upper half space or the bounded symmetric domain. We shall use the latter one, which is sometimes denoted by $G = Sp(n,\mathbb{R})_c$. See [4–6, 15].

Let $Sp(n,\mathbb{C})$ be the group of complex $2n \times 2n$-matrices preserving the standard complex symplectic form $\Omega(z,w) = z_1w_{n+1} + \cdots + z_nw_{2n} - z_{n+1}w_1 - \cdots - z_{2n}w_n$. Let $U(n,n)$ be the indefinite unitary group on $\mathbb{C}^{2n}$ preserving the Hermitian form $\langle z,w\rangle_{n,n} = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n - \cdots - z_{2n}\overline{w}_{2n}$ of signature (n,n). Let $G = Sp(n,\mathbb{R}) :=$

$U(n,n)\cap Sp(n,\mathbb{C})$. Elements in G will be represented as 2×2-block matrices

$$g=\begin{bmatrix} a & b\\ \bar{b} & \bar{a}\end{bmatrix}, \tag{1.1}$$

where a,b are complex $n\times n$-matrices. Let $K\simeq U(n)$ be the subgroup of block diagonal matrices $g=\mathrm{diag}(a,\bar{a})$, with $a\in U(n)$, $U(n)$ being in its standard realization. Then K is the maximal subgroup of G, and to save notation we identify $\mathrm{diag}(a,\bar{a})$ with a.

The Cartan decomposition is $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ with $\mathfrak{k}=\mathfrak{u}(n)$, $\mathfrak{p}=M_n^s(\mathbb{C})=\{B\in M_n(\mathbb{C}); B=B^t\}$; more precisely

$$\mathfrak{k}=\left\{diag(A,-A^t), A\in\mathfrak{u}(n)\right\}, \tag{1.2}$$

$$\mathfrak{p}=\left\{\xi_B=\begin{bmatrix} 0 & B\\ \bar{B} & 0\end{bmatrix}; B=B^t\in M_n(\mathbb{C})\right\}, \tag{1.3}$$

in the above matrix realization.

1.2. **Metaplectic representation.** Let $\mathcal{F}=\mathcal{F}(\mathbb{C}^n)$ be the Fock space of entire functions $f\in\mathcal{O}(\mathbb{C}^n)$ on $\mathbb{C}^n$ such that

$$\|f\|^2=\int_{\mathbb{C}^n}|f(z)|^2e^{-\pi|z|^2}dz<\infty,$$

where dz is the Lebesgue measure. The monomials $\{z^\alpha=z_1^{\alpha_1}\cdots z_n^{\alpha_n}\}$ form an orthogonal basis with norm square

$$\|z^\alpha\|^2=\frac{1}{\pi^{\alpha_1+\cdots+\alpha_n}}(\alpha_1!)\cdots(\alpha_n!), \tag{1.4}$$

and $\mathcal{F}$ has reproducing kernel $e^{\pi(z,\bar{w})}$, namely

$$f(z)=\langle f,e^{\pi(\cdot,\bar{z})}\rangle=\int f(w)e^{\pi(z,\bar{w})}e^{-\pi|w|^2}dw,\quad f\in\mathcal{F},\ z\in\mathbb{C}^n, \tag{1.5}$$

where $(z,w)=\sum z_jw_j$ is the $\mathbb{C}$-bilinear form.

The metaplectic representation $(\Lambda,\mathcal{F},Mp(n,\mathbb{R}))$ is a unitary representation of the double cover of $G=Sp(n,\mathbb{R})$ on the Fock space $\mathcal{F}$, and is explicitly given by [6, Theorem 4.37],

$$\begin{aligned}&\Lambda(g)f(z)\\ &\quad=\det{}^{-\frac{1}{2}}(a)\int\exp\left\{\frac{\pi}{2}\left[(\bar{b}a^{-1}z,z)+2(a^{-1}z,\bar{w})-(a^{-1}b\bar{w},\bar{w})\right]\right\}f(w)e^{-\pi|w|^2}dw,\end{aligned}$$

for g as in (1.1). In particular the group $U(n)$ acts as

$$\Lambda(a)f(z)=\det{}^{-\frac{1}{2}}(a)f(a^{-1}z),\qquad a\in U(n).$$

The space $\mathcal{F}=\mathcal{F}_+\oplus\mathcal{F}_-$ is sum of the subspaces of even and odd functions, with $\Lambda=\Lambda^+\oplus\Lambda^-$ a sum of two irreducible representations.

Lemma 1.1. *The Lie algebra action of $sp(n,\mathbb{C})$ on $\mathcal{F}$ is given as follows*

$$\Lambda(X)f=-\frac{1}{2\pi}(B\partial,\partial)f,\quad \Lambda(Y)f=\frac{\pi}{2}(\overline{B}z,z)f,$$

$$\Lambda(Z)f(z)=-\frac{\mathrm{tr}\,D}{2}f(z)-(\partial_{Dz}f)(z),$$

for

$$X = X_B = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}, \quad Y = Y_{\overline{B}} = \begin{bmatrix} 0 & 0 \\ \overline{B} & 0 \end{bmatrix}, \quad B = B^t,$$

$$Z = Z_D = \begin{bmatrix} D & 0 \\ 0 & -D^t \end{bmatrix}, \quad D \in \mathfrak{gl}(n).$$

Proof. Let ξ_B be as in (1.3) and perform the differentiation $\frac{d}{dt}$ of the action $\exp(t\xi_B) \in G$ at $t = 0$,

$$\Lambda(\xi_B)f(z) = \int \frac{\pi}{2}\left[(\overline{B}z, z) - (B\bar{w}, \bar{w})\right] e^{\pi(z,\bar{w})} f(w) e^{-\pi|w|^2} dw.$$

The integration of the first term is

$$\frac{\pi}{2}(\overline{B}z, z) f(z)$$

by the reproducing kernel formula (1.5). Differentiating the formula (1.5) by $(B\partial, \partial)$ we find

$$(B\partial, \partial) f(z) = \pi^2 \int (B\bar{w}, \bar{w}) e^{\pi(z,\bar{w})} f(w) e^{-\pi|w|^2} dw.$$

Thus the second integration above is

$$-\frac{\pi}{2}\frac{1}{\pi^2}(B\partial, \partial) f(z) = -\frac{1}{2\pi}(B\partial, \partial) f(z).$$

Taking the complex linear and conjugate linear parts in B we get the first two formulas. The last formula is a straightforward computation. $\square$

1.3. **Unitary representations of $SU(1,1)$ and their realizations.** Let $G_0 := SU(1,1)$ be the group of 2×2-matrices preserving the Hermitian form $\langle z, w\rangle_{1,1} = z_1\bar{w}_1 - z_2\bar{w}_2$ with determinant 1. We fix the maximal subgroup

$$L = U(1) = \left\{ u_\theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \right\} \subset G_0. \tag{1.6}$$

We fix also the Lie algebra elements

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e = e^+ + e^- \tag{1.7}$$

forming the standard basis of $sl(2,\mathbb{C})$. Let

$$C = e^+e^- + e^-e^+ + \frac{1}{2}h^2 = 2e^+e^- - h + \frac{1}{2}h^2$$

be the Casimir element.

Recall [9, Theorem 1.3] that any unitary irreducible representation σ of $SU(1,1)$ is one of the following

(1) Spherical principal or complementary series $\sigma_{i\lambda,+}$ with all weights of h being even integers; the eigenvalue of Casimir C is $-\frac{1}{2}-\lambda^2$ for $\lambda \in \mathbb{R}^+$ or $\lambda \in (0, \frac{1}{2})$.
(2) Non-spherical principal series $\sigma_{i\lambda,-}$; for $\lambda \in \mathbb{R}^+$ with all weights of h are odd integers; the eigenvalue of Casimir C is $-\frac{1}{2} - \lambda^2$ for $\lambda \in \mathbb{R}^+$.
(3) Highest weight representations $\sigma_{-\nu}$ with negative integral weights $-\nu$, $\nu \geq 1$; the eigenvalue of Casimir C is $\frac{1}{2}\nu^2 - \nu$.
(4) Lowest weight representations σ_ν with positive integral weights ν, $\nu \geq 1$; the eigenvalue of Casimir C is $\frac{1}{2}\nu^2 - \nu$.

The non-integral highest or lowest weights ν correspond representations of the universal covering of $SU(1,1)$, which will not concern us here.

1.4. **Continuous dual Hahn polynomials and spectra of multiplication operators.** We recall the orthogonality relation of continuous dual Hahn polynomials. They are special cases of Wilson hypergeometric polynomials [13], and will be used in Section 3 to find the spectra of Casimir element.

Proposition 1.2 ([13, pp. 196-199])**.** *Let $a \ge 0, b > 0, c > 0$, $\mu = \mu_{a,b,c}$ be the measure*

$$d\mu_{a,b,c}(x) = \frac{1}{2\pi}\frac{1}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)}\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 dx$$

on $\mathbb{R}^+ = [0,\infty)$ and $L^2(\mathbb{R}^+,\mu)$ the corresponding L^2-space. Let

$$\omega_m(x^2) = \omega_{m;a,b,c}(x^2) = {}_3F_2(-m, a+ix, a-ix; a+b, a+c, 1), \quad m = 0, 1, \dots,$$

be the continuous dual Hahn polynomials. Then $\{\omega_m(x^2)\}$ form an orthogonal basis of $L^2(\mathbb{R}^+,\mu)$,

$$\langle \omega_m, \omega_l \rangle = \frac{m!(b+c)_m}{(a+b)_m(a+c)_m}\delta_{m,l},$$

where

$$(a)_m = a(a+1)\cdots(a+m-1)$$

is the Pochhammer symbol. The multiplication operator by $-x^2$ has the following 3-term recursion on the basis,

$$-x^2\omega_m(x^2) = A_m\omega_{m+1}(x^2) - (A_m + C_m - a^2)\omega_m(x^2) + C_m\omega_{m-1}(x^2),$$

where

$$A_m = (m+a+b)(m+a+c),\ C_m = m(m+b+c-1).$$

In particular the spectrum of the multiplication x^2 on $L^2(\mathbb{R}^+,\mu)$ is the positive half line $\mathbb{R}^+ = [0,\infty)$.

We shall need rescaled Wilson polynomials $\widetilde{\omega}_m(x^2) := \widetilde{\omega}_m((\frac{x}{3})^2)$. The factor of 3 in our case is closely related to our imbedding of $SL(2,\mathbb{R})$ in $Sp(2,\mathbb{R})$ via triple symmetric tensor product $S^3\mathbb{R}^2 = \mathbb{R}^4$. We denote the corresponding measure by

$$d\widetilde{\mu}(x) = d\mu_{a,b,c}(\frac{x}{3}).$$

Corollary 1.3. *Let $a \ge 0, b > 0, c > 0$, A_m, C_m be as above. The polynomials $\widetilde{\omega}_m(x^2)$ form an orthogonal basis, $L^2(\mathbb{R}^+,\widetilde{\mu})$,*

$$\|\widetilde{\omega}_m\|^2 = \frac{m!(b+c)_m}{(a+b)_m(a+c)_m}, \tag{1.8}$$

and satisfy the following recurrence relation

$$-\frac{1}{3^2}(x^2+d)\widetilde{\omega}_m(x^2) = A_m\widetilde{\omega}_{m+1}(x^2) - (A_m + C_m - a^2 + \frac{d}{3^2})\widetilde{\omega}_m(x^2) + C_m\widetilde{\omega}_{m-1}(x^2), \tag{1.9}$$

for any $d \in \mathbb{R}$.

The introduction of d is purely for practical convenience for identifying the spectrum of some abstract operator with the multiplication operator $x^2 + d$ in Section 3.

2. THE GROUP HOMOMORPHISM $\iota : G_0 = SU(1,1) \to G$ AND THE RESTRICTION OF REPRESENTATIONS OF G TO G_0

We find the explicit realization of the principal $G_0 = SU(1,1)$ ($= SL(2,\mathbb{R})$) subgroup in the symplectic group G and find explicit formulas for pull-back of the metaplectic representation to G_0.

2.1. **The homomorphism $\iota : G_0 \to G$.** The defining action of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ preserves the complex symplectic form $dz_1 \wedge dz_2$. Its symmetric power representation $\iota : SL(2,\mathbb{C}) \to SL(2n,\mathbb{C})$ on $S^{2n-1}\mathbb{C}^2 = \mathbb{C}^{2n}$ preserves the symplectic form $S^{2n-1}(dz_1 \wedge dz_2)$. This defines a group homomorphism $\iota : SL(2,\mathbb{C}) \to Sp(n,\mathbb{C})$.

It is immediate that $\iota(G_0)$, $G_0 = SU(1,1)$, preserves also the Hermitian form $\langle\cdot,\cdot\rangle_{n,n} = S^{2n-1}\langle\cdot,\cdot\rangle_{1,1}$ of signature (n,n). Thus we have the group homomorphism

$$\iota : G_0 = SU(1,1) \to U(n,n) \cap Sp(n,\mathbb{C}) = Sp(n,\mathbb{R}) = G. \tag{2.1}$$

We can find explicitly the image of ι of $SU(1,1)$ and of the Lie algebra $sl(2,\mathbb{C})$. Let $m = 2n-1$ and realize $V = S^m\mathbb{C}^2$ as the space homogeneous polynomials $f(x,y)$ degree of m on $\mathbb{C}^2$, $g \in GL(2,\mathbb{C}) : f(x,y) \mapsto f((x,y)g)$. The space V is equipped with the dual of Hermitian form $S^{2n-1}\langle\cdot,\cdot\rangle_{1,1}$ and dual of the symplectic form $S^{2n-1}(dx \wedge dy)$, namely $\Omega(p,q) = (\partial_{x_1}\partial_{y_2} - \partial_{y_1}\partial_{x_2})^m(p(x_1,y_1)q(x_2,y_2))$. We fix an $\langle\cdot,\cdot\rangle_{n,n}$-orthonormal and $\Omega(\cdot,\cdot)$-symplectic basis of V,

$$p_k = \sqrt{\binom{m}{2(k-1)}}x^{m-2(k-1)}y^{2(k-1)} = \sqrt{\binom{m}{2(n-k)+1}}x^{m-2(k-1)}y^{2(k-1)},$$
$$q_k = \sqrt{\binom{m}{2(k-1)}}y^{m-2(k-1)}x^{2(k-1)} = \sqrt{\binom{m}{2(n-k)+1}}y^{m-2(k-1)}x^{2(k-1)},$$
$$1 \le k \le n;$$

namely they satisfy

$$\langle p_k, p_j\rangle_{n,n} = -\langle q_k, q_j\rangle_{n,n} = \delta_{kj}, \quad \Omega(p_k,p_j) = \Omega(q_k,q_j) = 0,$$
$$\Omega(p_k,q_j) = \delta_{kj},\ 1 \le k,j \le n.$$

Proposition 2.1. *Let k_θ be the diagonal matrix*

$$k_\theta = \mathrm{diag}(e^{(2n-1)i\theta}, e^{(2n-1-4)i\theta}, \cdots, e^{-(2n-3)i\theta}).$$

Then $\iota(u_\theta) = k_\theta$ and the Lie algebra $\mathfrak{g}^{\mathbb{C}}$-elements

$$H := \iota(h), E^{\pm} := \iota(e^{\pm}), E = \iota(e) = E^+ + E^-$$

are given by

$$H = \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix}, \quad E^+ = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}, \quad E^- = (E^+)^t, \quad E = \begin{bmatrix} 0 & B+C \\ B+C & 0 \end{bmatrix}$$

where

$$D = \mathrm{diag}(2n-1, 2n-5, \cdots, -(2n-3)),$$

B is skew diagonal and symmetric and C lower skew diagonal and symmetric,

$$B = \begin{bmatrix} 0 & \dots & 0 & \beta_1 \\ 0 & \dots & \beta_2 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ \beta_n & \dots & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \gamma_2 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & \gamma_n & 0 & 0 \end{bmatrix},$$

with

$$\beta_k = \sqrt{(2k-1)(2(n-k)+1)}, \qquad 1 \le k \le n, \gamma_k = 2\sqrt{(k-1)(n-k+1)}, \quad 2 \le k \le n;$$

that is $B = (b_{jk})$, $C = (c_{jk})$ *with*

$$b_{jk} = \beta_k \delta_{j,n-k+1}, \quad c_{jk} = \gamma_k \delta_{j,n-k+2}, \, 1 \le k, j \le n$$

in term of the Kronecker symbol δ_{pq}.

Proof. The action of $k_\theta = \iota(u_\theta)$ is diagonal and is found immediately, so is $H = \iota(h)$. Performing $\frac{d}{dt}|_{t=0}$ on the action $\iota(\exp(te^+)) = \iota(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}) : f(x,y) \mapsto f(x, tx+y)$ we find

$$\begin{aligned} E^+ p_1 &= 0, \\ E^+ p_k = \pi(e^+) p_k &= (2k-2)\sqrt{\binom{m}{2(k-1)}} x^{m-2(k-1)+1} y^{2(k-1)-1} \\ &= \frac{(2k-2)\sqrt{\binom{m}{2(k-1)}}}{\sqrt{\binom{m}{2(k-1)-1}}} q_{n-(k-2)} \\ &= 2\sqrt{(k-1)(n-k+1)} q_{n-(k-2)} \end{aligned}$$

for $k \ge 2$, and

$$E^+ q_k = \sqrt{(2k-1)(m-2k+2)} p_{n-k+1}, \quad 1 \le k \le n.$$

The element $E^- = \pi(\iota(e^-)) = (E^+)^t$ by our choice of the basis. $\square$

We remark that the homomorphism of $SL(2,\mathbb{C})$ into $Sp(n,\mathbb{C})$ realizes also $SL(2,\mathbb{R})$ as a subgroup in a different real form $Sp(n,\mathbb{R})$ in $Sp(n,\mathbb{C})$, and it is the principle $SL(2,\mathbb{R})$-subgroup of $G = Sp(n,\mathbb{R})$; see [14] for the general study of principle $SL(2,\mathbb{R})$-subgroup in real split groups. This specific case of $SL(2,\mathbb{R})$ in $G = Sp(n,\mathbb{R})$ is also of interests in topology [1].

2.2. **The induced action of Lie algebra of $\mathfrak{sl}(2)$ on $\mathcal{F}$ for $n = 2$.** We shall find polynomials p that are annihilated by the differential operator $(B\partial, \partial)$, and we shall call them $(B\partial,\partial)$-*harmonic*, i.e.

$$B(\partial, \partial) p(z) = 0. \tag{2.2}$$

When $n = 2$ the matrices k_θ, B, C, D are of the form

$$k_\theta = \operatorname{diag}(e^{3i\theta}, e^{-i\theta}), \; B = \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \; D = \operatorname{diag}(3, -1). \tag{2.3}$$

This combined with Lemma 1.1 gives the explicit form for the action $\Lambda \circ \iota$ of $\mathfrak{sl}(2)$. With some abuse of notation we denote it also by Λ, and we find
(2.4)

$$\Lambda(H) = -1 - (3z_1\partial_1 - z_2\partial_2), \; \Lambda(E^+) = -\pi^{-1}\sqrt{3}\partial_1\partial_2 + \pi z_2^2, \; \Lambda(E^-) = -\pi^{-1}\partial_2^2 + \pi\sqrt{3} z_1 z_2.$$

We can also prove independently that this defines a representation of $\mathfrak{sl}(2)$ on the Fock space without using the metaplectic representation. Indeed we have for any f,

$$\begin{aligned}[\Lambda(E^+),\Lambda(E^-)]f &= [-\pi^{-1}\sqrt{3}\partial_1\partial_2+\pi z_2^2, -\pi^{-1}\partial_2^2+\pi\sqrt{3}z_1z_2]f\\ &= -3[\partial_1\partial_2, z_1z_2]f-[z_2^2,\partial_2^2]f\\ &= -3\Big(z_1\partial_1 f+z_2\partial_2 f+f\Big)+2f+4z_2\partial_2 f=-3z_1\partial_1 f+z_2\partial_2 f-f\\ &= \Lambda(H)f,\end{aligned}$$

$$\begin{aligned}[\Lambda(H),\Lambda(E^+)]f &= [-3z_1\partial_1+z_2\partial_2, -\pi^{-1}\sqrt{3}\partial_1\partial_2+\pi z_2^2]f\\ &= \pi^{-1}3\sqrt{3}[z_1\partial_1,\partial_1\partial_2]f-\pi^{-1}\sqrt{3}[z_2\partial_2,\partial_1\partial_2]+\pi[z_2\partial_2,z_2^2]\\ &= -\pi^{-1}3\sqrt{3}\partial_1\partial_2 f+\pi^{-1}\sqrt{3}\partial_1\partial_2 f+2\pi z_2^2 f=-\pi^{-1}2\sqrt{3}\partial_1\partial_2 f+2\pi z_2^2 f\\ &= 2(-\pi^{-1}\sqrt{3}\partial_1\partial_2 f+\pi z_2^2)=2\Lambda(E^+)f,\end{aligned}$$

and

$$\begin{aligned}[\Lambda(H),\Lambda(E^-)]f &= [-3z_1\partial_1+z_2\partial_2, -\pi^{-1}\partial_2^2+\pi\sqrt{3}z_1z_2]f\\ &= -3\pi\sqrt{3}[z_1\partial_1,z_1z_2]f-\pi^{-1}[z_2\partial_2,\partial_2^2]f+\pi\sqrt{3}[z_2\partial_2,z_1z_2]f\\ &= -3\pi\sqrt{3}z_1z_2 f+2\pi^{-1}\partial_2^2 f+\pi\sqrt{3}z_1z_2 f=-2\Lambda(E^-)f.\end{aligned}$$

Remark 2.2. It was pointed out to me by the anonymous referee that $\mathfrak{sp}(2,\mathbb{R})$ has two principal nilpotent orbits under the adjoint action of $Sp(2,\mathbb{R})$. In terms of the simple roots $\{\epsilon_1-\epsilon, 2\epsilon_2\}$ for the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$, the complex principal nilpotent orbit is determined, up to scalars, by the element $H=2\epsilon_1+(\epsilon_1+\epsilon_2)=3\epsilon_1+\epsilon_2$ in a standard $\mathfrak{sl}(2)$-triple $\{H,E^\pm\}$, namely the diagonal matrix diag$(3,1,-3,1)$. The nilpotent element E^+ has two possible forms,

$$E^+: q_1\to p_2\to q_2\to p_1\to 0,\quad E^+: p_1\to q_2\to p_2\to q_1\to 0.$$

Our principal $\mathfrak{sl}(2,\mathbb{R})$ subalgebra is conjugated to the first one; the two nilpotent matrices are conjugated by a matrix exchanging the symplectic forms Ω and $-\Omega$.

3. The complete decomposition for $(\Lambda, Mp(2,\mathbb{R}))$ under $SU(1,1)$

3.1. **Orthogonal basis for $U(1)$-weight vectors.** Let $n=2$. We shall find orthogonal polynomials in $\mathcal{F}=\mathcal{F}(\mathbb{C}^2)$ of a fixed $U(1)=\iota(U(1))$-weight. Let $\mathcal{P}\subset\mathcal{F}$ be the polynomial space. All weights refer to the metaplectic representations under $U(1)$ unless otherwise explicitly stated.

Denote

$$I(z)=\frac{\pi^2}{3\sqrt{3}}z_1z_2^3. \tag{3.1}$$

(The coefficient is chosen to simplify the expression for the solutions of the equation (3.8).) Then I generates all invariants in the polynomial space $\mathcal{P}$ of the defining action of $\iota(U(1))$. Recall $(\alpha)_m=\alpha(\alpha+1)\cdots(\alpha+m-1)$, the Pochhammer symbol.

The orthogonal basis vectors $\{z_1^{m_1}z_2^{m_2}\}$ of $\mathcal{F}$ are weight vectors of $U(1)$ of weights $-1-3m_1+m_2$, and modulo 3 they are of the form $\mu=-3k-1,-3k,-3k+1$. We denote $\mathcal{F}|_{U(1)}^{\mu}$ the subspace of all weight vectors of weight μ. Then if $k \ge 0$ we have

$$\mathcal{F}|_{U(1)}^{-3k-1} = \sum_{l\ge 0}^{\oplus} \mathbb{C}(I^l(z) z_1^k), \tag{3.2}$$

$$\mathcal{F}|_{U(1)}^{-3k} = \sum_{l\ge 0}^{\oplus} \mathbb{C}(I^l(z) z_1^k z_2), \tag{3.3}$$

and

$$\mathcal{F}|_{U(1)}^{-3k+1} = \sum_{l\ge 0}^{\oplus} \mathbb{C}(I^l(z) z_1^k z_2^2); \tag{3.4}$$

if $k < 0$ we then replace z_1^k by z_2^{-3k} in the above formulas.

We compute the norm of weight vectors.

Lemma 3.1. *The square norms $\|I^m z_1^k\|_{\mathcal{F}}^2$ and $\|I^m z_2^k\|_{\mathcal{F}}^2$ are given by*

$$\|I^m z_1^k\|^2 = \frac{1}{\pi^k}(m!)^2 (m+1)_k (\frac{2}{3})_m (\frac{1}{3})_m,$$
$$\|I^m z_2^k\|^2 = \frac{1}{\pi^k}(m!)^2 (3m+1)_k (\frac{2}{3})_m (\frac{1}{3})_m.$$

Proof. Using (1.4) we see that

$$\begin{aligned}\|I^m z_1^k\|^2 &= (\frac{\pi^2}{3\sqrt{3}})^{2m} \|z_1^{m+k} z_2^{3m}\|^2 \\ &= \frac{\pi^{4m}}{3^{3m}} \frac{1}{\pi^{4m+k}} (m+k)!(3m)! \\ &= \frac{\pi^{4m}}{3^{3m}} \frac{1}{\pi^{4m+k}} m!(m+1)_k (3m)!.\end{aligned}$$

Now

$$\begin{aligned}(3m)! &= (3m)(3m-1)(3m-2)\cdots 3\cdot 2\cdot 1 \\ &= 3^{3m} m(m-\frac{1}{3})(m-\frac{2}{3})\cdots 1\cdot\frac{2}{3}\cdot\frac{1}{3} \\ &= 3^{3m} m! (\frac{2}{3})_m (\frac{1}{3})_m,\end{aligned}$$

and this proves the first formula.

Similarly $\|I^m z_2^k\|_{\mathcal{F}}^2$ is

$$\frac{\pi^{4m}}{3^{3m}} \frac{1}{\pi^{4m+k}} m!(3m+k)! = \frac{1}{\pi^k}\frac{1}{3^{3m}} m!(3m+1)_k (3m)!$$

with $(3m)!$ being computed as above. □

3.2. **Irreducible decomposition of** $(\Lambda, Sp(n,\mathbb{R}))$**.** Our main result is the following

Theorem 3.2. *The decomposition of $(\Lambda, \mathcal{F}, Mp(2,R))$ under $SU(1,1)$ is given by*

$$(\Lambda, Mp(2,R))|_{SU(1,1)} \cong \int_0^\infty \sigma_{i\lambda,+} d\tilde{\mu}_{\frac{1}{2},\frac{1}{6},\frac{5}{6}}(\lambda) \oplus \int_0^\infty \sigma_{i\lambda,-} d\tilde{\mu}_{0,\frac{1}{3},\frac{2}{3}}(\lambda) \oplus \bigoplus_{k=1}^{\infty} \sigma_{-3k-1}, \tag{3.5}$$

where $\tilde{\mu}_{a,b,c}$ is the orthogonality measure for the continuous dual Hahn polynomials in Corollary 1.3.

Remark 3.3. The measure $\tilde{\mu}_{a,b,c}$ can be written as $|c_l(\lambda)|^{-2}b(\lambda)^{-1}$ where $c_l(\lambda)$ is the Harish-Chandra c-function for line bundles with parameter l [18] and $b(\lambda)$ is the symbol of a Berezin transform [20]. We can follow the method of Berezin transform to study the decomposition above, however it requires different realization of the metaplectic representation and it is less effective in finding the discrete components; see Remark 3.8.

We shall find the spectral decomposition of the Casimir element C on $\mathcal{F}$. We have $\iota(C) = E^+E^- + E^-E^+ + \frac{H^2}{2}$ as element in the enveloping algebra of $\mathfrak{sp}(n,\mathbb{C})$, and

$$\Lambda(\iota(C)) = \Lambda(E^+)\Lambda(E^-) + \Lambda(E^-)\Lambda(E^+) + \frac{\Lambda(H)^2}{2}$$

as operator on $\mathcal{F}$. To ease notation we write it just as C. The decomposition is done in several steps.

By general abstract theory the operator C has a well-defined self-adjoint extension on $\mathcal{F}$ and on any subspace $\mathcal{F}|^k_{U(1)}$ of fixed weight k.

Lemma 3.4. *The Casimir element $-C$ on $\mathcal{F}^{-1}|_{U(1)}$ has continuous spectrum $[\frac{1}{2},\infty)$.*

Proof. The Casimir operator $C = E^+E^- + E^-E^+ + \frac{H^2}{2} = 2E^+E^- - H + \frac{H^2}{2}$ acts on any element $f \in \mathcal{P}|^{-1}_{U(1)}$ of weight -1, $\Lambda(H)f = -f$, as

$$Cf = 2\Lambda(E^+)\Lambda(E^-)f + f + \frac{1}{2}f = 2\Lambda(E^+)\Lambda(E^-)f + \frac{3}{2}f.$$

The subspace $\mathcal{F}|^{-1}_{U(1)}$ has an orthogonal basis $W_m(z) = \frac{I^m(z)}{(\frac{1}{3})_m(\frac{2}{3})_m}$ with

$$\|W_m\|^2 = \frac{1}{\pi}\frac{(m!)^2}{(\frac{1}{3})_m(\frac{2}{3})_m} \tag{3.6}$$

by Lemma 3.1. We compute the action of $\Lambda(E^+)\Lambda(E^-)$ on $\{W_m\}$. We have, $\pi\Lambda(E^-) = -\partial_2^2 + \pi^2\sqrt{3}z_1z_2$, and by straightforward computations,

$$\begin{aligned}\pi\Lambda(E^-)I_m(z) &= (-\partial_2^2 + \pi^2\sqrt{3}z_1z_2)\left(\frac{\pi^2}{3\sqrt{3}}z_1z_2^3\right)^m \\ &= -(3m)(3m-1)I^m(z)z_2^{-2} + 3^2I^{m+1}(z)z_2^{-2} \\ &= -3^2m(m-\frac{1}{3})I^m z_2^{-2} + 3^2I^{m+1}(z)z_2^{-2}.\end{aligned} \tag{3.7}$$

Acting by $\pi\Lambda(E^+) = -\sqrt{3}\partial_1\partial_2 + \pi^2 z_2^2$ we find, writing $I(z)^m = I(z)I^{m-1}(z) = \frac{\pi^2 z_1 z_2^3}{3\sqrt{3}} I^{m-1}$, that

$$
\begin{aligned}
&\pi^2\Lambda(E^+)\Lambda(E^-)I_m \\
&= -\sqrt{3}\partial_1\partial_2\left(-3^2 m(m-\frac{1}{3})I^m z_2^{-2} + 3^2 I^{m+1}(z)z_2^{-2}\right) \\
&\quad + \pi^2 z_2^2\left(-3^2 m(m-\frac{1}{3})I^m z_2^{-2} + 3^2 I^{m+1}(z)z_2^{-2}\right) \\
&= -\sqrt{3}\left(-3^3 m(m-\frac{1}{3})m(m-\frac{2}{3})\frac{\pi^2}{3\sqrt{3}}I^{m-1} + 3^3(m+1)(m+\frac{1}{3})\frac{\pi^2}{3\sqrt{3}}I^m\right) \\
&\quad + \pi^2 3^2\left(-m(m-\frac{1}{3})I^m + I^{m+1}\right) \\
&= 3^2\pi^2(\alpha_m I^{m+1} + \beta_m I^m + \gamma_m I^{m-1})
\end{aligned}
$$

with the leading coefficient $\alpha_m = 1$, the last coefficient

$$\gamma_m = m^2(m-\frac{1}{3})(m-\frac{2}{3})$$

and the middle

$$\beta_m = -(m+1)(m+\frac{1}{3}) - m(m-\frac{1}{3}) = -(2m+m+\frac{1}{3}).$$

Thus

$$\frac{1}{3^2}\Lambda(E^+)\Lambda(E^-)I^m = I^{m+1} - (2m^2+m+\frac{1}{3})I^m + m^2(m-\frac{1}{3})(m-\frac{2}{3})I^{m-1}.$$

Writing in terms of $W_m(z) = \frac{I^m(z)}{(\frac{1}{3})_m(\frac{2}{3})_m}$ and using $(\frac{1}{3})_{k+1}(\frac{2}{3})_{k+1} = (\frac{1}{3})_k(\frac{2}{3})_k(k+\frac{1}{3})(k+\frac{2}{3})$ for $k = m, m+1$, this becomes

$$\frac{1}{3^2}\pi(E^+)\pi(E^-)W_m = (m+\frac{1}{3})(m+\frac{2}{3})W_{m+1} - (2m^2+m+\frac{1}{3})W_m + m^2 W_{m-1}.$$

This is exact the same recursion relation as (1.9) for the continuous dual Hahn polynomials for

$$a = 0,\ b = \frac{1}{3},\ c = \frac{2}{3},\ d = 1;$$

moreover square norms (3.6) and (1.8) are same. In other words, $W_m \to \widetilde{\omega}_m$ is a unitary operator intertwining $-\Lambda(E^+)\Lambda(E^-)$ with the multiplication operator by $1+x^2$ on the space $L^2(\mathbb{R}^+, \widetilde{\mu})$. Thus $-\Lambda(E^+)\Lambda(E^-)$ has continuous spectrum $[1,\infty)$. But $-Cf = -2\Lambda(E^+)\Lambda(E^-)f - \frac{3}{2}f$ so $-C$ has spectrum $[\frac{1}{2},\infty)$. This finishes the proof. □

Note that the recursion formula (3.7) can also be obtained by just finding the leading term $3^2 I^{m+1}z_2^{-2}$ and by using the unitarity of Λ.

Lemma 3.5. *The Casimir element $-C$ on $\mathcal{F}^0|_{U(1)}$ has continuous spectrum $[\frac{1}{2},\infty)$.*

Proof. This is proved by similar computations as the above lemma. We consider the action of C on $\mathcal{P}^0|_{U(1)} \subset \mathcal{F}^0|_{U(1)}$, the subspace of polynomials f of weight 0. If $f \in \mathcal{P}|_{U(1)}^0$ then $\Lambda(H)f = 0$ and

$$Cf = 2\Lambda(E^+)\Lambda(E^-)f.$$

The space $\mathcal{F}|^0_{U(1)}$ has an orthogonal basis given by the polynomials $W_m(z) = \frac{I^m(z)z_2}{(\frac{2}{3})_m(\frac{4}{3})_m}$, with square norms computed in Lemma 3.1,

$$\|W_m\|^2 = \frac{1}{\pi}\frac{(m!)^2}{(\frac{2}{3})_m(\frac{4}{3})_m}.$$

We compute the action of C on $I^m z_2$ and find

$$\frac{1}{3^2}\Lambda(E^+)\Lambda(E^-)(I^m z_2)$$
$$= I^{m+1}(z)z_2 - (2m^2+2m+\frac{2}{3})(I^m z_2) + m^2(m-\frac{1}{3})(m+\frac{1}{3})(I^{m-1}z_2).$$

Written in terms of W_m this is

$$\frac{1}{3^2}\Lambda(E^+)\Lambda(E^-)W_m = (\frac{2}{3}+m)(\frac{4}{3}+m)W_{m+1} - (2m^2+2m+\frac{2}{3})W_m + m^2W_{m-1}.$$

This is the same recursion formula for continuous Hahn polynomials in for

$$a=\frac{1}{2},\ b=\frac{1}{6},\ c=\frac{5}{6},\ d=\frac{1}{4},$$

and it follows that the $\Lambda(E^+)\Lambda(E^-)$ with the multiplication operator by $-(d+x^2) = -(\frac{1}{4}+x^2)$ on $L^2(\mathbb{R}^+,\widetilde{\mu})$. This gives the spectrum of $\Lambda(E^+)\Lambda(E^-)$ and the Casimir element as claimed. □

We study then the appearance of lowest or highest weight representations.

Lemma 3.6. *There exists no lowest weight unitary representation of $SU(1,1)$ in $(\Lambda,\mathcal{F})$. The only highest weight unitary representations of $SU(1,1)$ in $(\Lambda,\mathcal{F})$ are of the form σ_{-3k-1}, $k>0$, and they all appear of multiplicity one.*

Proof. Let f be a lowest weight vector, i.e., f is a weight vector and $\Lambda(E^-)f=0$. Now

$$E^- = (E^+)^t = \begin{bmatrix} 0 & C \\ B & 0\end{bmatrix}$$

with B and C as in (2.3). Thus the lowest weight vector condition for f becomes

$$-\frac{1}{\pi}(\partial_2^2 f)(z) + \pi\sqrt{3}z_1z_2f = 0.$$

Then the lowest degree term f_0 of f satisfies $\partial_2 f_o = 0$ and must be of the form $f_0 = z_1^k$ or $f_0 = z_1^k z_2$ up to non-zero constants. But these vectors have negative weights under $\Lambda(H)$ and any unitary lowest weight irreducible representation of $\mathfrak{su}(1,1)$ has positive lowest weight, consequently lowest weight representations do not appear.

Next we shall find all highest weight vectors in the space $\mathcal{F}$. So let σ_ν be an irreducible representation for the Lie algebra $su(1,1)$ appearing in Λ with highest weight vector f of weight $-\nu$, $\nu>0$.

Expand f as sum of homogeneous polynomials

$$f = \sum_{m=0}^{\infty} p_{m+k},$$

with p_{m+k} of homogeneity $m+k$ and k being the lowest degree. The highest weight condition $\Lambda(E^+)f = 0$ becomes

$$\frac{1}{2\pi}(B\partial,\partial)f = \frac{\pi}{2}(Cz,z)f, \tag{3.8}$$

i.e.

$$\frac{1}{2\pi}(B\partial,\partial)p_{m+k} = \frac{\pi}{2}(Cz,z)p_{m+k-4}, m = 0,1,\dots.$$

Thus p_{m+k-4} determines p_{m+k} up to B-harmonic polynomials of degree $m+k$. Hence the sum $\sum_{l=0}^{\infty} p_{k+4l}$ in f satisfies the equation (3.8) and it starts with the lowest degree p_k of f. But since we are finding all solutions we can assume $f = \sum_{l=0}^{\infty} p_{k+4l}$. To ease notation we write this as $f = \sum_{l=0}^{\infty} f_l$, $f_l = p_{k+4l}$.

We have then

$$\frac{\sqrt{3}}{\pi^2}\partial_1\partial_2 f_{l+1} = z_2^2 f_l, l = 0,1,\cdots, \tag{3.9}$$

and the leading term f_0 satisfies $\partial_1\partial_2 f_0 = 0$. So $f_0 = z_1^k$, $k \ge 0$ or $f_0 = z_2^k, k > 0$.

If $f_0 = z_2^k$ then $\Lambda(H)f_0 = (-1+k)f_0$, it is of nonnegative weight $-1+k$ and this contradicts our assumption that f is of negative weight, so this case is excluded.

Let $f_0 = z_1^k$, $k \ge 0$. The weight of f is $\nu = -1-3k$. We claim that f is uniquely solved by

$$f(z) = k! z_1^k \sum_{l=0}^{\infty} \frac{I(z)^l}{l!(l+k)!}. \tag{3.10}$$

Now the action of $U(1) := \iota(U(1))$ on the space of $\mathcal{P}_m$ of homogeneous polynomials of degree m is multiplicity free for any m. Thus that f_l is of weight $-k-1$ and of degree $k+4l$ implies that

$$f_l = a_l I(z)^l z_1^k.$$

Note that

$$\begin{aligned}
\frac{\sqrt{3}}{\pi^2}\partial_1\partial_2(I(z)^{l+1}z_1^k) &= \frac{\sqrt{3}}{\pi^2}(\frac{\pi^2}{3\sqrt{3}})^{l+1}\partial_1\partial_2(z_1^{l+k+1}z_2^{3(l+1)}) \\
&= \frac{\sqrt{3}}{\pi^2}(\frac{\pi^2}{3\sqrt{3}})^{l+1}(l+k+1)3(l+1)(z_1^{l+k}z_2^{3l+2}) \\
&= (\frac{\pi^2}{3\sqrt{3}}z_1 z_2^3)^l(l+k+1)(l+1)z_2^2 \\
&= (l+k+1)(l+1)z_2^2 I(z)^l.
\end{aligned}$$

The recursive relation (3.9) becomes

$$a_{l+1} = \frac{1}{(l+k+1)(l+1)}a_l$$

and is solved by

$$a_l = \frac{k!}{l!(k+l)!}$$

This proves our claim.

Finally we prove that $f \in \mathcal{F}$ if $k > 0$. The square norm of f is

$$
\begin{aligned}
(k!)^2 \sum_l \frac{1}{(l!^2((k+l)!)^2} \|I^l z_1^k\|^2 &= \sum_l \frac{1}{l!^2((k+l)!)^2}(l!)(l+k)!(\frac{2}{3})_l(\frac{1}{3})_l \\
&= \sum_l \frac{(l+k)!(\frac{2}{3})_l(\frac{1}{3})_l}{l!((k+l)!)^2} \\
&= k! \sum_l \frac{(\frac{2}{3})_l(\frac{1}{3})_l}{(1)_l(k+l)!} \\
&= k! \frac{\Gamma(k+1)}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})} \sum_l \frac{\Gamma(\frac{2}{3}+l)\Gamma(\frac{1}{3}+l)}{\Gamma(l+1)\Gamma(k+l+1)} \\
&\asymp \sum_l \frac{1}{l^{k+1}}
\end{aligned}
$$

which is convergent iff $k \geq 1$. Here we have used the known fact that

$$\frac{\Gamma(a+l)}{\Gamma(b+l)} \asymp \frac{1}{l^{b-a}}.$$

This finished the proof. □

We can now prove Theorem 3.2.

Proof. It follows from Lemmas 3.4, 3.5 and 3.6 that right-hand side in (3.5) are subrepresentation of $(\Lambda, \mathcal{F}, SU(1,1))$. We have to prove that it gives full decomposition. The proof relies on some general and abstract arguments. Let $\mathcal{D} \subset \mathcal{F}$ the the subspace of $SU(1,1)$-smooth vectors in $\mathcal{F}$ and $\mathcal{D}^\mu \subset \mathcal{F}^\mu$ be the subspace of all vectors of weight $\mu = -3k-1, -3k, -3k+1$. The decomposition of $\mathcal{F}$ is obtained by the spectral decomposition of the Casimir operator C on $\mathcal{D}^\mu$ and its self-adjoint extension. The operator E^- has $\operatorname{Ker} E^-|_{\mathcal{D}} = 0$, and all elements in $\operatorname{Ker} E^+|_{\mathcal{D}}$ are found in Lemma 3.6. Moreover the space $\operatorname{Ker} E^+|_{\mathcal{D}}$ generates the discrete series representation σ_{-3k-1} in $\mathcal{F}$. Let $\mu = -6k-1$, $k > 0$. The operator $(E^+)^{3k}$ defines an injective operator $\mathcal{D}^\mu \cap (\operatorname{Ker}(E^+)^{3k})^\perp \to \mathcal{D}^{-1} \subset \mathcal{F}^{-1}$ and it intertwines the Casimir operator. The Casimir operator C on $\mathcal{F}^{-1}$ has only continuous spectrum and thus it has only continuous spectrum on $\mathcal{F}^\mu \cap (\operatorname{Ker}(E^+)^{3k})^\perp$. Similar considerations apply to the subspaces $\mathcal{F}^\mu$ of all weights. This give the spectral decomposition of C on $\mathcal{F}^\mu$ and then on $\mathcal{F}$, the spectral measure being given in Lemma 3.4 and 3.5. This completes the proof. □

The even part $(Mp(2,R), \Lambda^+, \mathcal{F}_+)$ of the metaplectic representation consists of even holomorphic functions, the corresponding discrete components correspond to the the highest weight vector (3.10) with $k = 2l$ even. We have thus

Corollary 3.7. *The discrete components appearing in the restriction of the even part of the metaplectic representation $(\Lambda^+, \mathcal{F}_+, Mp(2,R))$ to $SU(1,1)$ are precisely the highest weight representations of weights $-6l-1$, $l \geq 1$.*

Remark 3.8. The continuous part of the decomposition of the even part (as holomorphic functions) of the metaplectic representation Λ^+ can also be found using its realization as reproducing kernel space. We give a very brief description. The representation Λ^+ can be realize [11] on the Siegel bounded symmetric domain $D = G/K = Sp(2,\mathbb{R})/U(2)$ as a Hilbert space $\mathcal{H}$ of holomorphic function with reproducing kernel $\det(1 - z\bar{w})^{-\frac{1}{2}}$. It is a subspace of holomorphic sections of a line bundle over the Siegel domain $Sp(2,\mathbb{R})/U(2)$. Consider the the hyperbolic disc $\Delta = SU(1,1)/U(1) = \{z \in \mathbb{C}; |z| < 1\}$ and the Hilbert space $L^2(D,(1-|z|^2)^{-1}dm(z))$ with the natural $G_0 = SU(1,1)$ action as

$$g \in G_0 : f(z) \to f((az+b)(cz+d))(cz+d)^{-1}.$$

This is also the L^2-space of the half-bundle $\mathbb{C}(dz)^{\frac{1}{2}}$ of the cotangent bundle. The pull-back of ι, (2.1),

$$R = \iota^* : \mathcal{D} \subset \mathcal{H} \to L^2(D,(1-|z|^2)^{-1}dm(z))$$

defines an intertwining map R from a dense subspace of H to

$$L^2(\Delta,(1-|z|^2)^{-1}dm(z)).$$

The operator RR^* is a Berezin transform on Δ and its spectral symbol can be computed using the method in [20]. However the restriction map R here is not injective and it will produce only the continuous part, and other discrete components can be constructed using covariant differentiation; this requires rather subtle and detailed computations.

It is interesting to notice that the solution (3.10) for $k = 0$ still defines a holomorphic function, which is a highest weight vector but not in the Fock space. This can also be explained by another fundamental fact about the Hardy space of holomorphic functions on the unit disc. This space $(L^2(\Delta,(1-|z|^2)^{-1}dm(z)), G_0)$ has no discrete component. However the Hardy on the unit disc is invariant under the same action but it is not a subspace $L^2(D,(1-|z|^2)^{-1}dm(z))$. So the solution (3.10) plays the role of a Hardy space function whereas the restriction of the reproducing kernel space to D plays the role of $L^2(D,(1-|z|^2)^{-1}dm(z))$.

We observe that the proof of Lemmas 3.4 and 3.5 actually also provides generalized eigenfunctions for the Casimir operator as a sum of bi-orthogonal polynomials

$$\Psi_x(z) = \sum_{m=0}^{\infty} e_m(z)\widetilde{e}_m(x^2)$$

where $e_m(z) = \frac{W_m(z)}{\|W_m\|}$, $\widetilde{e}_m = \frac{\widetilde{\omega}_m(x^2)}{\|\widetilde{\omega}_m\|}$. It is easy to prove that this series is convergent point-wise for all $x \in \mathbb{R}$ and $z \in \mathbb{C}^2$. We have not been able to compute the sum. There exist several formulas [13, pp. 196-199] for generating functions $\sum_m c_m\widetilde{\omega}_m(x^2)t^m$ of the polynomials $c_m\widetilde{\omega}_m(x^2)$ however they are not related to the eigenfunctions $\Psi_x(z)$.

It might be possible to find all discrete components for $(\Lambda, \mathcal{F}, Mp(n,\mathbb{R}))$ for general $n \geq 3$ by refining the techniques in Lemma 3.6. It would also be an interesting problem to study the branching problems of minimal representations for other split real simple Lie groups, such as quaternionic representations of G_2, under its principal $SL(2,\mathbb{R})$-subgroup.

Acknowledgments

I would like to thank Wee Teck Gan for posing the question studied in this paper and for some stimulating discussions. I'm grateful to the anonymous referee for a careful reading of an earlier version of this paper and for several expert suggestions.

References

[1] Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Tight homomorphisms and Hermitian symmetric spaces*, Geom. Funct. Anal. **19** (2009), no. 3, 678–721, DOI 10.1007/s00039-009-0020-8. MR2563767 ↑499, 504

[2] David H. Collingwood and William M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993. MR1251060 ↑499

[3] Michel Duflo, Esther Galina, and Jorge A. Vargas, *Square integrable representations of reductive Lie groups with admissible restriction to* $\mathrm{SL}_2(\mathbb{R})$, J. Lie Theory **27** (2017), no. 4, 1033–1056. MR3646030 ↑499

[4] Jacques Faraut and Adam Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1994. Oxford Science Publications. MR1446489 ↑499

[5] J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), no. 1, 64–89, DOI 10.1016/0022-1236(90)90119-6. MR1033914 ↑499

[6] Gerald B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, vol. 122, Princeton University Press, Princeton, NJ, 1989, DOI 10.1515/9781400882427. MR983366 ↑499, 500

[7] J. Frahm, *Conformally invariant differential operators on Heisenberg groups and minimal representations*, Preprint, `arXiv:2012.05952v1`, 2020. 499

[8] J. Frahm, C. Weiske, and G. Zhang, *Principal series for Hermitian Lie groups induced from Heisenberg parabolic subgroups and intertwining operators*, in preparation. 499

[9] Roger Howe and Eng-Chye Tan, *Nonabelian harmonic analysis*, Universitext, Springer-Verlag, New York, 1992. Applications of $\mathrm{SL}(2,\mathbf{R})$, DOI 10.1007/978-1-4613-9200-2. MR1151617 ↑501

[10] Roger Howe, *On some results of Strichartz and Rallis and Schiffman*, J. Functional Analysis **32** (1979), no. 3, 297–303, DOI 10.1016/0022-1236(79)90041-7. MR538856 ↑499

[11] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), no. 1, 1–47, DOI 10.1007/BF01389900. MR463359 ↑511

[12] Toshiyuki Kobayashi, *Restrictions of unitary representations of real reductive groups*, Lie theory, Progr. Math., vol. 229, Birkhäuser Boston, Boston, MA, 2005, pp. 139–207, DOI 10.1007/0-8176-4430-X_3. MR2126642 ↑499

[13] Roelof Koekoek, Peter A. Lesky, and René F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010. With a foreword by Tom H. Koornwinder, DOI 10.1007/978-3-642-05014-5. MR2656096 ↑502, 512

[14] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032, DOI 10.2307/2372999. MR114875 ↑499, 504

[15] Ottmar Loos, *Jordan pairs*, Lecture Notes in Mathematics, Vol. 460, Springer-Verlag, Berlin-New York, 1975. MR0444721 ↑499

[16] Bent Ørsted and Jorge A. Vargas, *Branching problems for semisimple Lie groups and reproducing kernels* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **357** (2019), no. 9, 697–707, DOI 10.1016/j.crma.2019.09.004. MR4018081 ↑499

[17] Joe Repka, *Tensor products of unitary representations of* $\mathrm{SL}_2(\mathbf{R})$, Amer. J. Math. **100** (1978), no. 4, 747–774, DOI 10.2307/2373909. MR509073 ↑499

[18] Henrik Schlichtkrull, *One-dimensional K-types in finite-dimensional representations of semisimple Lie groups: a generalization of Helgason's theorem*, Math. Scand. **54** (1984), no. 2, 279–294, DOI 10.7146/math.scand.a-12059. MR757468 ↑507

[19] Genkai Zhang, *Branching coefficients of holomorphic representations and Segal-Bargmann transform*, J. Funct. Anal. **195** (2002), no. 2, 306–349, DOI 10.1006/jfan.2002.3957. MR1940358 ↑498

[20] Genkai Zhang, *Berezin transform on real bounded symmetric domains*, Trans. Amer. Math. Soc. **353** (2001), no. 9, 3769–3787, DOI 10.1090/S0002-9947-01-02832-X. MR1837258 ↑498, 507, 512

[21] Genkai Zhang, *Tensor products of minimal holomorphic representations*, Represent. Theory **5** (2001), 164–190, DOI 10.1090/S1088-4165-01-00103-0. MR1835004 ↑498

[22] Genkai Zhang, *Principal series of Hermitian Lie groups induced from Heisenberg parabolic subgroups*, J. Funct. Anal. **282** (2022), no. 8, Paper No. 109399, 39, DOI 10.1016/j.jfa.2022.109399. MR4377991 ↑499

Mathematical Sciences, Chalmers University of Technology and Mathematical Sciences, Göteborg University, SE-412 96 Göteborg, Sweden

Email address: `genkai@chalmers.se`