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SOLID LOCALLY ANALYTIC REPRESENTATIONS OF p-ADIC LIE GROUPS

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Abstract. We develop the theory of locally analytic representations of compact p-adic Lie groups from the perspective of the theory of condensed mathematics of Clausen and Scholze. As an application, we generalise Lazard's isomorphisms between continuous, locally analytic and Lie algebra cohomology to solid representations. We also prove a comparison result between the group cohomology of a solid representation and of its analytic vectors.

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1. Introduction

The theory of p-adic representations of p-adic groups has a long history and it has played a key role in the field of Number Theory during the last decades, as witnessed, e.g., in the study of the p-adic Langlands correspondence [Col10], [CDP14]. In this article, we intend to reformulate the theory of locally analytic representations of p-adic Lie groups as developed in [ST02b], [STP01], [ST03], [ST05], [Eme17], using the theory of condensed mathematics of Clausen and Scholze.

More precisely, we define and study the notions of analytic and locally analytic representations of p-adic Lie groups on solid modules. One of our main new results, which was the departing point of our investigations, is a generalisation of Lazard's comparison theorems [Laz65, Théorèmes V.2.3.10 et V.2.4.10] between continuous, locally analytic and Lie algebra cohomology of a finite dimensional representation of a compact p-adic Lie group over $\mathbf{Q}_p$ to arbitrary solid locally analytic representations.[1] Generalisations of Lazard's comparison results have already been considered

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[1]In particular, to any complete compactly generated locally convex vector space, e.g. metrizable.

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in [HKN11], [Lec12], [Tam15]. Our second main new result is a comparison between continuous cohomology of a solid representation and the continuous cohomology of its locally analytic vectors. These results can be seen as a p-adic analogue of theorems of G. D. Mostow [Mos61] and P. Blanc [Bla79].

1.1. **Background.** Let G be a topological group. One usual way of studying continuous G-representations is through their cohomology. In a favorable situation, the cohomology groups are Ext-groups in certain abelian categories. For instance, this is the case for the category of continuous representations of G on discrete modules. On the other hand, since the category of continuous G-representations on topological abelian groups is not abelian, it is not clear that one can define cohomology groups as Ext-groups in this setting. The continuous group cohomology of G with values in a topological representation is usually defined via a complex of continuous cochains. This definition lacks some conceptual advantages. For example: it is not clear whether the cohomology groups carry a natural topology, short exact sequences do not necessarily induce long exact sequences in cohomology, and certain basic results such as Hochschild-Serre need to be proved by hand (cf. [Ser94, §2]).

These inconveniences would be overcome if one is able to find a suitable abelian category where topological representations can be embedded. The theory of condensed mathematics developed by Clausen and Scholze [CS19], [CS20], [CS] provides a natural approach to making this work. Very vaguely, the condensed objects in a category $\mathscr{C}$ can be defined as sheaves on the proétale site of a point with values in $\mathscr{C}$. It is shown in [CS19, Theorem 2.2] that the category of condensed abelian groups is an abelian category satisfying Grothendieck's axioms. Hence, one is on a good footing for doing homological algebra.

Let p be a prime number. In this paper we will be interested in the case where G is a compact p-adic Lie group. The classical theory of p-adic representations of G comes in different flavours: there exist the notions of continuous, analytic and locally analytic representations V of G, according to whether the orbit maps $o_v : G \to V$, $g \mapsto g \cdot v$ (for all $v \in V$) are continuous, resp. analytic, resp. locally analytic functions of G. As an example, in order to study locally analytic representations from an algebraic perspective, in [ST03], Schneider and Teitelbaum defined the notion of an admissible locally analytic representation, and they showed that these objects form an abelian category. Even with this result at hand, there are still various obstacles to define a satisfactory cohomological theory for admissible locally analytic representations. For instance, it is not clear how to show that the Ext-groups in this abelian category coincide with the cohomology groups defined via locally analytic cochain complexes; this is one of the main results of [Koh11].

1.2. **Statement of the main results.** Let us now describe in some more detail what is carried out in this article.

1.2.1. *Solid non-archimedean functional analysis.* Let K be a finite extension of $\mathbf{Q}_p$. The field K naturally defines a condensed ring which, moreover, has an analytic ring structure $K_\blacksquare$ in the sense of [CS19, Definition 7.4], usually called the solid ring structure on K. We let $\mathrm{Mod}_K^{\mathrm{solid}}$ be the category of solid K-vector spaces. This category is stable under limits, colimits and extensions, it has a tensor product $\otimes_{K_\blacksquare}$ and an internal Hom denoted by $\underline{\mathrm{Hom}}_K(-,-)$ [CS19, Proposition 7.5]. Let us point out that all the important spaces in the classical theory of non-archimedean functional analysis [Sch02] live naturally in $\mathrm{Mod}_K^{\mathrm{solid}}$. Indeed, there is a natural functor

$$\mathcal{LC}_K \to \mathrm{Mod}_K^{\mathrm{solid}} \tag{1}$$

from the category of complete locally convex K-vector spaces to solid K-vector spaces, as any complete locally convex K-vector space can be written as a cofiltered limit of Banach spaces. Moreover, it is fully faithful on a very large class of complete locally convex K-vector spaces, e.g. all compactly generated ones, e.g. all metrizable ones. The main notions of the theory of condensed non-archimedean functional analysis we use are due to Clausen and Scholze [CS19], [CS], [Bos21]. All the vector spaces considered in this text are solid K-vector spaces, unless otherwise specified.

Our first result is an anti-equivalence between two special families of solid K-vector spaces. Let us first give some definitions. A Smith space is a K-vector space of the form $\underline{\mathrm{Hom}}_K(V, K)$, where V is a Banach space. In classical terms, a Smith space is the dual of a Banach space equipped with the compact-open topology. An *LS* space is a countable filtered inductive limit of Smith spaces with injective transition maps. We then have the following result.

Theorem 1.1 (Theorem 3.40)**.** *The functor $V \mapsto V^\vee := \underline{\mathrm{Hom}}_K(V, K)$ induces an anti-equivalence between Fréchet and LS spaces such that*

$$\underline{\mathrm{Hom}}_K(V, V') = \underline{\mathrm{Hom}}_K(V'^\vee, V^\vee).$$

Remark 1.2. Theorem 1.1 restricts in particular to an anti-equivalence between classical nuclear Fréchet spaces and *LB* spaces of compact type (see, e.g., [ST02b, Theorem 1.3]).

1.2.2. *Representation theory.* Let G be a compact p-adic Lie group. A representation of G on a solid K-vector space V is a map of condensed sets $G \times V \to V$ satisfying the usual axioms. Define the Iwasawa algebra of G with coefficients in K as $K_\blacksquare[G]$; explicitly,

$$K_\blacksquare[G] = \big(\varprojlim_N \mathcal{O}_K[G/N]\big)[\frac{1}{p}],$$

where N runs over all the open normal subgroups of G. This is the solid algebra defined by the classical Iwasawa algebra endowed with the weak topology. The category of G-representations on solid K-vector spaces is equivalent to the category $\mathrm{Mod}^{\mathrm{solid}}_{K_\blacksquare[G]}$ of solid $K_\blacksquare[G]$-modules. Observe that the category of continuous representations of G on complete locally convex K-vector spaces lives naturally in $\mathrm{Mod}^{\mathrm{solid}}_{K_\blacksquare[G]}$ via the functor (1).

Inspired by Emerton's treatment [Eme17], we define analytic and locally analytic vectors of solid representations of G. Roughly speaking, they are defined as those vectors whose induced orbit map is analytic or locally analytic. One advantage of our approach is that definitions make sense at the level of derived categories, so one can speak about derived (locally) analytic vectors of complexes $C \in D(K_\blacksquare[G])$ in the derived category of $K_\blacksquare[G]$-modules. The derived functors of the locally analytic vectors (for Lie groups over finite extensions of $\mathbf{Q}_p$) for admissible representations have been considered in [Sch09].

More precisely, let $\mathbb{G}$ be an affinoid group neighbourhood of G, i.e., an analytic affinoid group over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ such that $G = \mathbb{G}(\mathbf{Q}_p, \mathbf{Z}_p)$. Suppose in addition that $\mathbb{G}$ is isomorphic to a finite disjoint union of polydiscs. In practice, the group $\mathbb{G}$ will be constructed using some local charts of G, see Remark 1.4 for a more detailed description. Let $C(\mathbb{G}, K) := \mathscr{O}(\mathbb{G}) \otimes_{\mathbf{Q}_p} K$ be the algebra of functions of $\mathbb{G}$. The affinoid algebra $C(\mathbb{G}, K)$ has a natural analytic ring structure denoted by $C(\mathbb{G}, K)_{\blacksquare}$, see [And21, Theorem 3.27]. We denote by $\mathcal{D}(\mathbb{G}, K) = \mathscr{O}(\mathbb{G})^{\vee} \otimes_{\mathbf{Q}_p} K$ the distribution algebra of the affinoid group $\mathbb{G}$. We define the derived $\mathbb{G}$-analytic vectors of an object $C \in D(K_{\blacksquare}[G])$ to be the complex[2]

$$C^{R\mathbb{G}-an} := R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C \otimes^L_{K_{\blacksquare}} C(\mathbb{G}, K)_{\blacksquare}), \tag{2}$$

where K is the trivial representation, and the G-action on $C \otimes^L_{K_{\blacksquare}} C(\mathbb{G}, K)_{\blacksquare}$ is the diagonal one induced by the action on C and the left regular action on $C(\mathbb{G}, K)$. We endow $C^{R\mathbb{G}-an}$ with the right regular action of G. It turns out that there is a natural map $C^{\mathbb{G}-an} \to C$, and we say that C is derived $\mathbb{G}$-analytic if this map is a quasi-isomorphism. If V is a Banach G-representation, then $V \otimes^L_{K_{\blacksquare}} C(\mathbb{G}, K)_{\blacksquare} = V \otimes_{K_{\blacksquare}} C(\mathbb{G}, K)$ coincides with the projective tensor product of Banach spaces. Thus, our definition of derived $\mathbb{G}$-analytic vectors is the derived extension of [Eme17, Definition 3.3.13].

Now let $\mathring{\mathbb{G}}$ be a Stein group neighbourhood of G, i.e. an analytic group over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ which is written as a strict increasing union of affinoid group neighbourhoods $\mathbb{G}^{(h)}$ of G. In practice, $\mathring{\mathbb{G}}$ will be as in Remark 1.4. We also denote by $\mathcal{D}(\mathring{\mathbb{G}}, K) := \mathscr{O}(\mathring{\mathbb{G}})^{\vee} \otimes_{\mathbf{Q}_p} K = \varinjlim_h \mathcal{D}(\mathbb{G}^{(h^+)}, K)$ the distribution algebra of $\mathring{\mathbb{G}}$. Then the derived $\mathring{\mathbb{G}}$-analytic vectors of $C \in D(K_{\blacksquare}[G])$ are defined as the complex

$$C^{R\mathring{\mathbb{G}}-an} := R\varprojlim_h C^{R\mathbb{G}^{(h)}-an}, \tag{3}$$

and we say that C is derived $\mathring{\mathbb{G}}$-analytic if the natural map $C^{R\mathring{\mathbb{G}}-an} \to C$ is a quasi-isomorphism. Again, if V is a Banach G-representation, this definition is compatible with the $\mathring{\mathbb{G}}$-analytic vectors of [Eme17, Definition 3.4.1]. The main theorem is the following:

Theorem 1.3 (Theorem 4.36)**.** *A complex $C \in D(K_{\blacksquare}[G])$ is derived $\mathring{\mathbb{G}}$-analytic if and only if it is a module over $\mathcal{D}(\mathring{\mathbb{G}}, K)$.*

This theorem is a generalisation of the integration map constructed by Schneider and Teitelbaum to solid G-modules, cf. [ST02b, Theorem 2.2]. It will serve us as a bridge between solid analytic representations and solid modules over the distribution algebras.

Remark 1.4. Suppose that G is a uniform pro-p-group and $\phi : \mathbf{Z}_p^d \to G$ is an analytic chart given by a basis of G (see, e.g., Example 4.5). Using this chart, one can define for any $h \in \mathbf{Q}_{>0}$ affinoid group neighbourhoods $\mathbb{G}^{(h)}$ of G whose underlying adic spaces are isomorphic to $\mathbf{Z}_p^d + p^h \mathbb{D}^d_{\mathbf{Q}_p} \subset \mathbb{D}^d_{\mathbf{Q}_p}$, where $\mathbb{D}_{\mathbf{Q}_p} = \mathrm{Spa}(\mathbf{Q}_p\langle T \rangle, \mathbf{Z}_p\langle T \rangle)$ is the closed unit disc. The p^h-analytic functions on G with respect to the chart ϕ (or simply the $\mathbb{G}^{(h)}$-analytic functions) are defined as the rigid functions of $\mathbb{G}^{(h)}$. In this situation, the $\mathbb{G}^{(h)}$-analytic vectors are those whose orbit map is p^h-analytic. Given $h > 0$, the group $\mathbb{G}^{(h^+)} := \bigcup_{h' > h} \mathbb{G}^{(h')}$ is a Stein group neighbourhood of G.

[2]The tensor product $\otimes_{K_{\blacksquare}} C(\mathbb{G}, K)_{\blacksquare}$ is the derived base change of modules over analytic rings, see [CS19, Proposition 7.7].

In the definition of derived analytic vectors of equations (2) and (3) we will take $\mathbb{G} = \mathbb{G}^{(h)}$ and $\mathring{\mathbb{G}} = \mathbb{G}^{(h^+)}$. The locally analytic functions of G are

$$C^{la}(G,K) = \varinjlim_{h\to+\infty} C(\mathbb{G}^{(h)},K) = \varinjlim_{h\to+\infty} C(\mathbb{G}^{(h^+)},K).$$

Now let $\mathcal{D}^{la}(G,K) = C^{la}(G,K)^\vee$ be the algebra of locally analytic distributions. We point out that the analogous statement of Theorem 1.3 does not hold in general for locally analytic representations and $\mathcal{D}^{la}(G,K)$-modules. In the particular case of locally analytic representations on LB spaces of compact type, this is nevertheless true, and was already known by [ST02b, Theorem 2.2].

1.2.3. *Comparison theorems in cohomology.* We finish this section by describing the main applications of Theorems 1.1 and 1.3 to the study of the cohomology of continuous representations.

For $C \in D(K_\blacksquare[G])$, we define the solid group cohomology of C to be the complex

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K,C).$$

Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g})$ its universal enveloping algebra. Let $\mathring{\mathbb{G}}$ be a Stein group neighbourhood of G as in Theorem 1.3 and $\mathcal{D}(\mathring{\mathbb{G}},K)$ the distribution algebra of $\mathring{\mathbb{G}}$. If in addition C is $\mathring{\mathbb{G}}$-analytic, then it is equipped with an action of $\mathfrak{g}$ by derivations and we define the $\mathring{\mathbb{G}}$-analytic cohomology (resp. the Lie algebra cohomology) of C as $R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathring{\mathbb{G}},K)}(K,C)$ (resp. $R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K,C)$). Using bar resolutions and Theorem 1.3, one verifies that these definitions recover the usual continuous, analytic, and Lie algebra cohomology groups.

Our first new result compares continuous cohomology of a solid representation and the continuous cohomology of its locally analytic vectors. This result can be seen as a p-adic analogue of a theorem of P. Blanc [Bla79] and G. D. Mostow [Mos61] in the archimedean setting, which compares continuous and differentiable cohomology of a real Lie group G.

Let $C \in D(K_\blacksquare[G])$, we define the derived locally analytic vectors of C as the homotopy colimit

$$C^{Rla} = \operatorname*{hocolim}_{\mathbb{G}} C^{R\mathbb{G}-an},$$

where $\mathbb{G}$ runs over all the affinoid neighbourhoods of G. We say that C is derived locally analytic if $C^{Rla} = C$. If V is a Banach representation, then $H^0(V^{Rla})$ coincides with the locally analytic vectors of V in the sense of [Eme17, Definition 3.5.3]. We have Theorem 1.5.

Theorem 1.5 (Theorem 5.3)**.** *Let $C \in D(K_\blacksquare[G])$ and let C^{Rla} be the complex of derived locally analytic vectors of C. Then*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K,C) \cong R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K,C^{Rla}).$$

In particular, if $V \in \mathrm{Mod}^{\mathrm{solid}}_{K_\blacksquare[G]}$ then, setting $V^{R^i la} := H^i(V^{Rla})$ for $i \geq 0$, there is a spectral sequence of solid K-vector spaces

$$E_2^{i,j} := \underline{\mathrm{Ext}}^i_{K_\blacksquare[G]}(K, V^{R^j la}) \implies \underline{\mathrm{Ext}}^{i+j}_{K_\blacksquare[G]}(K,V).$$

We give an application in the classical context. Let V be a continuous representation of G on a complete locally convex K-vector space. Denote $H^i_{cont}(G,V)$ the usual continuous cohomology groups. These coincide with the underlying sets of the solid cohomology groups $\underline{\mathrm{Ext}}^i_{K_\blacksquare[G]}(K,V)$. We say that V has no higher locally analytic vectors if $V^{R^i la} = 0$ for all $i > 0$. This is the case for admissible representations (cf. [Pan22, Theorem 2.2.3] or Proposition 4.48). One deduces Corollary 1.6.

Corollary 1.6. *If V has no higher locally analytic vectors, then for all $i \geq 0$,*

$$H^i_{cont}(G, V) = H^i_{cont}(G, V^{la}).$$

Our last result concerns a generalisation of Lazard's comparison between continuous, (locally) analytic and Lie algebra cohomology from finite dimensional representations V to arbitrary solid derived (locally) analytic representations.

Theorem 1.7 (Continuous vs. analytic vs. Lie algebra cohomology, Theorem 5.5)**.** *Let $C \in D(K_\blacksquare[G])$ be a derived $\mathring{\mathbb{G}}$-analytic complex. Then*[3]

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) \cong R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathring{\mathbb{G}},K)}(K, C) \cong (R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C))^G.$$

1.3. **Organisation of the paper.** In §2 we review very briefly the theory of condensed mathematics and solid abelian groups. We recall the notion of analytic ring and give some examples that will be used throughout the text.

In §3 we develop the theory of solid K-vector spaces following the appendix of [Bos21]. Most of the results exposed in this section will be presented in the forthcoming work of Clausen and Scholze [CS]. We review in particular the main properties of classical vector spaces: Banach, Fréchet, LB and LF spaces. Our main original result is Theorem 1.1 generalising the classical anti-equivalence between LB spaces of compact type and nuclear Fréchet vector spaces.

In §4 we introduce the different analytic neighbourhoods $\mathbb{G}^{(h)}$ of our p-adic Lie group G. We begin in §4.1 by introducing spaces of analytic functions, following [Eme17, §2] closely. Then, we define the algebras of distributions as the duals of the spaces of analytic functions. We also introduce another class of distribution algebras, used already in [ST03, §4], which are more adapted to the coordinates of the Iwasawa algebra. In §4.3, we define the notion of analytic and derived analytic representation, we prove Theorem 1.3, except for a technical lemma whose proof is postponed to §5. We finish the section with some applications to locally analytic and admissible representations.

Finally, in §5, we recall (Theorem 5.7) a lemma of Serre used by Lazard to construct finite free resolutions of the trivial representation when G is a uniform pro-p-group. We use this result, as well as its enhancement due to Kohlhaase (Theorem 5.8) to prove the technical lemma necessary for Theorem 1.3. We state Theorems 1.5 and 1.7 in §5.2 and give a proof in §5.4. We conclude with some formal consequences, namely by showing a solid version of Hochschild-Serre and proving a duality between group homology and cohomology.

2. Recollections of condensed mathematics

First, we review some elementary notions in condensed mathematics: we recall the definitions of condensed sets, solid abelian groups and analytic rings. In the future we will be only interested in the categories of solid modules, and modules of analytic rings over $\mathbf{Z}_\blacksquare$.

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[3]The Lie algebra cohomology $R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C)$ lands naturally in the derived category of smooth representations of G on solid K-vector spaces. Since K is of characteristic 0, taking G-invariants in this category is exact and the superscript G means the composition with this functor, cf. Remark 5.6.

2.1. **Condensed objects.** In their recent work [CS19] and [CS20], Clausen and Scholze have introduced the new world of condensed mathematics, which aims to be the good framework where algebra and topology live together. Let $*_{\text{proet}}$ be the pro-étale site of a geometric point, equivalently, the category of profinite sets with continuous maps and coverings given by finitely many jointly surjective maps.

Definition 2.1 ([CS19, Definitions 2.1 and 2.11])**.** A condensed set/group/ring/$\ldots$ $\mathcal{F}$ is a sheaf over $*_{\text{proet}}$ with values in *Sets*, *Groups*, *Rings*, $\ldots$. We denote by Cond the category of condensed sets and CondAb that of condensed abelian groups. If R is a condensed ring, we denote by $\mathrm{Mod}_R^{\mathrm{cond}}$ the category of condensed R-modules.

Remark 2.2. There are set theoretical subtleties with this definition as $*_{\text{proet}}$ is not small. What Clausen and Scholze do (cf. [CS19, Lecture II]) is to cut off by an uncountable strong limit cardinal κ, considering the category of sheaves of κ-small profinite sets $*_{\text{proet}}$ and take the direct limit. Hence the category of condensed sets is not the category of sheaves on a site. In this article we proceed in the same manner and we subsequently avoid any further mention to κ.

Let Top denote the category of $T1$ topological spaces, we define the functor $(\underline{\ \ })$: Top $\to$ Cond mapping a topological space T to the condensed set

$$T : S \mapsto \underline{T}(S) = \mathrm{Cont}(S, T),$$

where $\mathrm{Cont}(S,T)$ is the set of continuous functions from S to T. Recall that a topological space X is compactly generated if a map $X \to Y$ to another topological space Y is continuous if and only if the composition $S \to X \to Y$ is continuous for all maps $S \to X$ from a compact Hausdorff space. Then the restriction of the functor $T \mapsto \underline{T}$ to the category of compactly generated Hausdorff topological spaces is fully faithful.

A compact Hausdorff space can be written as a quotient of a profinite set. Indeed, if S is a compact Hausdorff space, let S_{dis} denote the underlying set with the discrete topology, and βS_{dis} its Stone-Čech compactification. Then βS_{dis} is profinite and the natural map $\beta S_{dis} \to S$ is a surjective map of compact Hausdorff spaces. In particular, we can test if a topological space is compactly generated by restricting to profinite sets.

Recall that a condensed set X is called quasicompact if there is a profinite set S and a surjective map $\underline{S} \to X$. Similarly, a condensed set X is quasiseparated if for any pair of profinite sets S and S' over X the fiber product $\underline{S} \times_X \underline{S'}$ is quasicompact. More generally, a map $Y \to X$ of condensed sets is quasicompact if for any profinite set S and any map $S \to X$, the fiber product $S \times_X Y$ is quasicompact, the map $Y \to X$ is quasiseparated if the diagonal $Y \to Y \times_X Y$ is quasicompact.

From now on we identify a profinite set S with the condensed set $\underline{S}$. Given a condensed set X and a profinite set S we denote by $\mathrm{Cont}(S,X) = X(S)$ the maps from S to X as condensed sets, we also define $\underline{\mathrm{Cont}}(S,X)$ to be the condensed set whose value at a profinite set S' is

$$\underline{\mathrm{Cont}}(S,X)(S') = \mathrm{Cont}(S \times S', X) = X(S \times S').$$

We have the following result.

Proposition 2.3 ([CS20, Proposition 1.2])**.** *Consider the functor* $T \mapsto \underline{T}$ *from* Top *to* Cond

(1) *The functor has a left adjoint* $X \mapsto X(*)_{\text{top}}$ *sending any condensed set* X *to the set* $X(*)$ *equipped with the quotient topology arising from the map*

$$\bigsqcup_{S, a \in X(S)} S \to X(*)$$

with S *profinite.*

(2) *Restricted to compactly generated topological spaces, the functor is fully faithful.*

(3) *The functor induces an equivalence between the category of compact Hausdorff spaces and qcqs condensed sets.*

(4) *The functor induces a fully faithful functor from the category of compactly generated weak Hausdorff spaces to quasiseparated condensed sets. The category of quasiseparated condensed sets is equivalent to the category of ind-compact Hausdorff spaces "*$\varinjlim_i T_i$*" where all transition maps* $T_i \to T_j$ *are closed immersions. If* $X_0 \to X_1 \to \cdots$ *is a sequence of compact Hausdorff spaces with closed immersions and* $X = \varinjlim_n X_n$ *as topological spaces, then the map*

$$\varinjlim_n \underline{X}_n \to \underline{X}$$

is an isomorphism of condensed sets.

Among the class of profinite sets there is the special class of extremally disconnected sets, which are the projective objects in the category $*_{\text{proet}}$. Moreover, all of them are retractions of a Stone-Čech compactification of a discrete set. Let Extdis denote the full subcategory of extremally disconnected sets. The condensed sets can be defined using only this kind of profinite sets.

Proposition 2.4 ([CS19, Proposition 2.7])**.** *Consider the site of extremally disconnected sets with covers given by finite families of jointly surjective maps. Its category of sheaves is equivalent to the category of condensed sets via the restriction from profinite sets. Hence, a condensed set is a functor* $X : \text{Extdis} \to \text{Sets}$ *such that* $X(\emptyset) = *$ *and* $X(S_1 \bigsqcup S_2) = X(S_1) \times X(S_2)$*.*

Extremally disconnected sets play a similar role as points do for locally ringed spaces. Namely, if $F \to G$ is a map of condensed sets, it is injective (resp. surjective) if and only if it is so after evaluating at all extremally disconnected sets.

The inclusion CondAb $\to$ Cond admits a left adjoint $T \mapsto \mathbf{Z}[T]$, where $\mathbf{Z}[T]$ is the sheafification of $S \mapsto \mathbf{Z}[T(S)]$. Let S be an extremally disconnected set, Proposition 2.4 implies that the object $\mathbf{Z}[S]$ is projective in the category CondAb. We have a tensor product in CondAb given by the sheafification of the usual tensor product at the level of points. We also have an internal $\underline{\text{Hom}}$ in CondAb defined as

$$\underline{\text{Hom}}_{\mathbf{Z}}(M, N)(S) = \text{Hom}_{\mathbf{Z}}(M \otimes \mathbf{Z}[S], N)$$

for any profinite set S. If R is a condensed ring we write $R[S] := R \otimes \mathbf{Z}[S]$. They form a family of compact projective generators of $\text{Mod}_R^{\text{cond}}$. Then, if X is a condensed abelian group and S is a profinite set we have $\underline{\text{Cont}}(S, X) = \underline{\text{Hom}}_{\mathbf{Z}}(\mathbf{Z}[S], X)$. All the nice properties of the category of condensed abelian groups are summarised in Theorem 2.5.

Theorem 2.5 ([CS19, Theorem 2.2])**.** *The category of condensed abelian groups is an abelian category which satisfies the Grothendieck axioms* ($AB3$)*,* ($AB4$)*,* ($AB5$)*,* *(AB6), (AB3*) and (AB4*): all limits (AB3*) and colimits (AB3) exist, arbitrary products (AB4*), arbitrary direct sums (AB4) and filtered colimits (AB5) are exact, and (AB6): for all index sets J and filtered categories I_j, $j \in J$, with functor $i \mapsto M_i$, from I_j to condensed abelian groups, the natural map*

$$\varinjlim_{(i_j \in I_j)_{j\in J}} \prod_{j\in J} M_{i_j} \to \prod_{j\in J} \varinjlim_{i_j\in I_j} M_{i_j}$$

is an isomorphism. Moreover, the category of condensed abelian groups is generated by compact projective objects given by $\mathbf{Z}[S]$ with S an extremally disconnected set.

2.2. **Quasiseparated condensed sets.** We collect here some basic facts on quasiseparated condensed sets that will be needed later.

Proposition 2.6 ([CS20, Proposition 4.13])**.** *Let X be a quasiseparated condensed set. Then quasicompact injections $\iota : Z \to X$ are equivalent to closed subspaces $W \subseteq X(*)_{\mathrm{top}}$ via $Z \mapsto Z(*)_{\mathrm{top}}$, resp. sending a closed subspace $W \subseteq X(*)_{\mathrm{top}}$ to the subspace $Z \subseteq X$ given by*

$$Z(S) = X(S) \times_{\mathrm{Cont}(S, X(*)_{\mathrm{top}})} \mathrm{Cont}(S, W).$$

From now on, we will refer to quasicompact injections $\iota : Z \to X$ as closed subspaces of X.

Lemma 2.7 ([CS20, Lemma 4.14])**.** *Let X be a condensed set. The inclusion of the category of quasiseparated condensed sets into the category of all condensed sets admits a left adjoint $X \mapsto X^{qs}$, with the unit $X \to X^{qs}$ being a surjection of condensed sets. The functor $X \mapsto X^{qs}$ preserves finite products. Moreover, for any quasiseparated condensed ring A, this functor defines a similar left adjoint for the inclusion of quasiseparated condensed A-modules into all condensed A-modules.*

2.3. **Analytic rings.** Next, we recall the notion of analytic ring

Definition 2.8 ([CS19, Definitions 7.1, 7.4] and [CS20, Definition 6.12])**.** A pre-analytic ring $(\mathcal{A}, \mathcal{M})$ is the data of a condensed ring $\mathcal{A}$ (called the underlying condensed ring of the analytic ring) equipped with a functor

$$\mathrm{Extdis} \to \mathrm{Mod}_{\mathcal{A}}^{\mathrm{cond}}, \quad S \mapsto \mathcal{M}[S],$$

called the functor of measures of $(\mathcal{A}, \mathcal{M})$, that sends finite disjoint union into products, and a natural transformation of functors $S \to \mathcal{M}[S]$.

A pre-analytic ring is said to be analytic if for any complex $C : \ldots \to C_i \to \ldots \to C_1 \to C_0 \to 0$ of $\mathcal{A}$-modules such that each C_i is a direct sum of objects of the form $\mathcal{M}[T]$ for varying extremally disconnected sets T, the map

$$R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S], C) \to R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}[S], C)$$

is an isomorphism for all extremally disconnected sets S. An analytic ring $(\mathcal{A}, \mathcal{M})$ is normalised if $\mathcal{A} \to \mathcal{M}[*]$ is an isomorphism.

Example 2.9.

(1) ([CS19, Theorem 5.8]). We define the pre-analytic ring $\mathbf{Z}_{\blacksquare}$ with underlying ring $\mathbf{Z}$ and functor of measures mapping an extremally disconnected $S = \varprojlim_i S_i$, written as an inverse limit of finite sets, to the condensed abelian group $\mathbf{Z}_{\blacksquare}[S] := \varprojlim_i \mathbf{Z}[S_i]$. Then $\mathbf{Z}_{\blacksquare}$ is an analytic ring.

(2) ([CS19, Theorem 8.1]). More generally, let A be a discrete commutative algebra, and let $S = \varprojlim_i S_i$ be an extremally disconnected set. We have an analytic ring $A_\blacksquare$ with underlying condensed ring A, and with functor of measures given by

$$A_\blacksquare[S] = \varinjlim_{B\subset A} \varprojlim_i B[S_i],$$

where B runs over all the $\mathbf{Z}$-algebras of finite type in A.

(3) ([CS19, Proposition 7.9]) Let p be a prime number, K a finite extension of $\mathbf{Q}_p$ and $\mathcal{O}_K$ its valuation ring. Let $S = \varprojlim_i S_i$ be an extremally disconnected set. We have analytic structures for the rings $\mathcal{O}_K$ and K, denoted $\mathcal{O}_{K,\blacksquare}$ and $K_\blacksquare$ respectively, given by

$$\mathcal{O}_{K,\blacksquare}[S] := \varprojlim_i \mathcal{O}_K[S_i] \text{ and } K_\blacksquare[S] := \mathcal{O}_{K,\blacksquare}[S][\frac{1}{p}] = K \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\blacksquare}[S].$$

This analytic ring structure is induced from the analytic ring structure of $\mathbf{Z}_\blacksquare$ by base change to $\mathcal{O}_K$.

(4) ([And21, Theorem 1.5]) Let (A, A^+) be a complete Huber pair. Andreychev defines an analytic ring $(A, A^+)_\blacksquare$ associated to (A, A^+), whose underlying ring is $\underline{A}$, and with functor of measures

$$(A, A^+)_\blacksquare[S] = \varinjlim_{B\to A^+, M} \varprojlim_i \underline{M}[S_i],$$

where the colimit is taken over all the finitely generated subrings $B \subset A^+$ and all the quasifinitely generated B-submodules M of A (cf. [And21, Definition 3.4]). If $A^+ = A^\circ$ is the ring of power bounded elements, we simply write $A_\blacksquare$ for $(A, A^\circ)_\blacksquare$.

(5) ([And21, Lemma 4.7]) A particular example of the previous case is the analytic ring associated to the Tate algebra $(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$, namely, the analytic ring $K\langle T\rangle_\blacksquare$ whose functor of measures is given by

$$K\langle T\rangle_\blacksquare[S] = K[S] \otimes_{\mathbf{Z}_\blacksquare} \mathbf{Z}[T]_\blacksquare.$$

Theorem 2.10 explains the importance of the analytic rings

Theorem 2.10 ([CS19, Proposition 7.5]). *Let $(\mathcal{A}, \mathcal{M})$ be an analytic ring.*

(1) *The full subcategory*

$$\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})} \subset \mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}$$

of all $\mathcal{A}$-modules M such that for all extremally disconnected set S, the map

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}[S], M) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}[S], M)$$

is an isomorphism, is an abelian subcategory stable under all limits, colimits and extensions. Objects of the form $\mathcal{M}[S]$, where S is an extremally disconnected profinite set, constitute a family of compact projective generators of $\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})}$. The inclusion functor admits a left adjoint

$$\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}} \to \mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})}, \quad M \mapsto M \otimes_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$$

which is the unique colimit preserving extension of the functor given by $\mathcal{A}[S] \mapsto \mathcal{M}[S]$. Finally, if $\mathcal{A}$ is commutative, there is a unique symmetric monoidal tensor product $- \otimes_{(\mathcal{A},\mathcal{M})} -$ on $\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})}$ making the functor

$-\otimes_{\mathcal{A}} (\mathcal{A},\mathcal{M})$ symmetric monoidal. We call $\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})}$ the category of $(\mathcal{A},\mathcal{M})$-modules.

(2) *The functor of derived categories*
$$D(\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})}) \to D(\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}})$$
is fully faithful and its essential image is stable under all limits and colimits and given by those $C \in D(\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}})$ for which the map
$$R\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}[S], C) \to R\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}[S], C)$$
is an isomorphism for all extremally disconnecteds sets S. In that case, the map
$$R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S], C) \to R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}[S], C)$$
is also an isomorphism.

An object $C \in D(\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}})$ lies in $D(\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})})$ if and only if for each $n \in \mathbf{Z}$, the cohomology group $H^n(C)$ lies in $\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})}$. The inclusion functor $D(\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})}) \subset D(\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}})$ admits a left adjoint
$$D(\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}) \to D(\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})}), \quad C \mapsto C \otimes^L_{\mathcal{A}} (\mathcal{A},\mathcal{M})$$
which is the left derived functor of $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{A}} (\mathcal{A},\mathcal{M})$. Finally, if $\mathcal{A}$ is commutative, there is a unique symmetric monoidal tensor product $-\otimes^L_{(\mathcal{A},\mathcal{M})} -$ on $D(\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})})$ making the functor $-\otimes^L_{\mathcal{A}} (\mathcal{A},\mathcal{M})$ symmetric monoidal.

Remark 2.11. The functor $-\otimes^L_{(\mathcal{A},\mathcal{M})} -$ is the derived functor of $-\otimes_{(\mathcal{A},\mathcal{M})} -$ if and only if $\mathcal{M}[S'] \otimes^L_{(\mathcal{A},\mathcal{M})} \mathcal{M}[S''] = \mathcal{A}[S \times S'] \otimes^L_{\mathcal{A}} (\mathcal{A},\mathcal{M})$ sits in degree 0, cf. [CS19, Warning 7.6]. This is the case for the analytic rings of Example 2.9, and in fact for all the analytic rings over $\mathbb{Z}_{\blacksquare}$.

Remark 2.12. The terminology for the functor of measures is justified as follows. If $(\mathcal{A},\mathcal{M})$ is an analytic ring, P is an $(\mathcal{A},\mathcal{M})$-module, S is an extremally disconnected set, $f \in \mathrm{Cont}(S,P)$ and $\mu \in \mathcal{M}[S]$ then, using the isomorphism $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}[S], P) = \mathrm{Cont}(S,P)$, one can evaluate f at μ and define $\int f \cdot \mu = f(\mu) \in P$, which allows to see μ as a function on the space of functions $f \in \mathrm{Cont}(S,P)$.

In the following we shall write $D(\mathcal{A},\mathcal{M})$ for the derived category $D(\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})})$. The functor $-\otimes^L_{\mathcal{A}} (\mathcal{A},\mathcal{M})$ should be thought of as a completion with respect to the measures $\mathcal{M}$.

One of the main theorems of [CS19] is the proof that $\mathbf{Z}_{\blacksquare}$ is an analytic ring, it has as input several non-trivial computations of Ext-groups of locally compact abelian groups. The category $\mathrm{Mod}^{\mathrm{solid}}_{\mathbf{Z}} := \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}_{\blacksquare}}$ is called the category of solid abelian groups.

Let p be a prime number, then $\mathbf{Z}_p$ is a solid abelian group (being an inverse limit of discrete abelian groups) and for S extremally disconnected we have (cf. [CS19, Proposition 7.9])
$$\mathbf{Z}_{p,\blacksquare}[S] = \mathbf{Z}_p \otimes^L_{\mathbf{Z}_{\blacksquare}} \mathbf{Z}_{\blacksquare}[S] = \mathbf{Z}_p \otimes_{\mathbf{Z}_{\blacksquare}} \mathbf{Z}_{\blacksquare}[S]. \tag{4}$$

Definition 2.13. Let K be a finite extension of $\mathbf{Q}_p$ and $\mathcal{O}_K$ its ring of integers, we denote by $\mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_K}$ (resp. $\mathrm{Mod}^{\mathrm{solid}}_{K}$) the category of solid $\mathcal{O}_K$-modules (resp. the category of solid K-vector spaces) and $D(\mathcal{O}_{K,\blacksquare})$, $D(K_{\blacksquare})$ their respective derived categories.

Remark 2.14. An object in $\mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_K}$ (resp. $\mathrm{Mod}^{\mathrm{solid}}_{K}$) is the same as an object in $\mathrm{Mod}^{\mathrm{solid}}_{\mathbf{Z}}$ endowed with an action of $\mathcal{O}_K$ (resp. K). Indeed, this follows directly from Theorem 2.10 and (4).

2.4. **Analytic rings attached to Tate power series algebras.** Let A be a reduced Tate algebra of finite type over $\mathbf{Q}_p$, recall that $A_{\blacksquare} = (A, A^{\circ})_{\blacksquare}$ where $A^{\circ} \subset A$ is the subring of power bounded elements.

Definition 2.15. We define the ring $A\langle\langle T^{-1}\rangle\rangle$ to be

$$A\langle\langle T^{-1}\rangle\rangle = \Big(\varprojlim_{s\to\infty} A^{\circ}/\varpi^s((T^{-1}))\Big)[\frac{1}{\varpi}],$$

where $A^{\circ}/\varpi^s((T^{-1})) = (A^{\circ}/\varpi^s)[[T^{-1}]][T]$.

Lemma 2.16. *The residue pairing* $\mathrm{res}_0 : A\langle\langle T^{-1}\rangle\rangle \times A\langle T\rangle \to A$ *defined as*

$$(\sum_{n\in\mathbf{Z}} a_n T^n, \sum_{m\in\mathbf{N}} b_m T^m) \mapsto \mathrm{res}_0((\sum_{n\in\mathbf{Z}} a_n T^n)(\sum_{m\in\mathbf{N}} b_m T^m)) = \sum_{n+m=-1} a_n b_m$$

induces an isomorphism of $A\langle T\rangle$*-modules*

$$\frac{A\langle\langle T^{-1}\rangle\rangle}{A\langle T\rangle} \xrightarrow{\sim} \underline{\mathrm{Hom}}_A(A\langle T\rangle, A).$$

Proof. We first observe that the pairing is well defined as a map of condensed sets. Indeed, the ring $A\langle\langle T^{-1}\rangle\rangle$ can be written as $A\langle\langle T^{-1}\rangle\rangle = (T^{-1}A^{\circ}[[T^{-1}]]\oplus A^{\circ}\langle T\rangle)[\frac{1}{\varpi}]$ and, in particular, it is the condensed set associated to its underlying topological space. Since the pairing is well defined at the level of topological spaces, taking associated condensed sets we are done.

It is clear from the definition of res_0 that the induced map

$$A\langle\langle T^{-1}\rangle\rangle \to \underline{\mathrm{Hom}}_A(A\langle T\rangle, A)$$

is $A\langle T\rangle$ linear, and that it factors through $\frac{A\langle\langle T^{-1}\rangle\rangle}{A\langle T\rangle}$. On the other hand, we have

$$\begin{aligned}\underline{\mathrm{Hom}}_A(A\langle T\rangle, A) &= \underline{\mathrm{Hom}}_{A^{\circ}}(A^{\circ}\langle T\rangle, A^{\circ})[\frac{1}{\varpi}]\\ &= \varprojlim_{s\in\mathbf{N}} \underline{\mathrm{Hom}}_{A^{\circ}/\varpi^s}(A^{\circ}/\varpi^s[T], A/\varpi^s)[\frac{1}{\varpi}]\\ &= \varprojlim_{s\in\mathbf{N}}\Big(\prod_{n\in\mathbf{N}}(A^{\circ}/\varpi^s)T^{n,\vee}\Big)[\frac{1}{\varpi}]\\ &= \Big(\prod_{n\in\mathbf{N}} A^{\circ}T^{n,\vee}\Big)[\frac{1}{\varpi}],\end{aligned}$$

where $\{T^{n,\vee}\}_{n\in\mathbf{N}}$ is the dual basis of $\{T^n\}_{n\in\mathbf{N}}$. But the map res_0 sends T^{-n-1} to $T^{n,\vee}$. The lemma follows. □

Remark 2.17. Let $a_1, \dots, a_k \in A^{\circ}$ be a finite set of topological generators over $\mathbf{Z}_p$, then A° is quasifinitely generated $\mathbf{Z}[a_1, \dots, a_k]$-module, namely, A°/ϖ^s is a finite $\mathbf{Z}[a_1, \dots, a_k]$-module for all $s \geq 1$, cf. [And21, Definition 3.4]. By [And21, Lemma 3.26], this implies that

$$\prod_I \mathbf{Z}\otimes_{\mathbf{Z}_{\blacksquare}} A_{\blacksquare} = \Big(\prod_I A^{\circ}\Big)[\frac{1}{\varpi}].$$

In particular, since $\frac{A\langle\langle T^{-1}\rangle\rangle}{A\langle T\rangle} = \frac{A^\circ((T^{-1}))}{A^\circ[T]}[\frac{1}{\varpi}]$ as A-module, we get

$$\frac{A\langle\langle T^{-1}\rangle\rangle}{A\langle T\rangle} = \frac{\mathbf{Z}((T^{-1}))}{\mathbf{Z}[T]} \otimes_{\mathbf{Z}_\blacksquare} A_\blacksquare.$$

Proposition 2.18 provides a description of the solidification of $Z[T]_\blacksquare$-modules and will be crucial.

Proposition 2.18 ([And21, Proposition 3.13])**.** *Let $C \in D(\mathbf{Z}_\blacksquare)$, then*

$$C \otimes^L_{\mathbf{Z}_\blacksquare} \mathbf{Z}[T]_\blacksquare = R\underline{\mathrm{Hom}}_{\mathbf{Z}}(\frac{\mathbf{Z}((T^{-1}))}{\mathbf{Z}[T]}, C).$$

Corollary 2.19. *Let A be a reduced Tate algebra of finite type over $\mathbf{Q}_p$. Let $A\langle \underline{T}\rangle = A\langle T_1, \ldots, T_d\rangle$ be the Tate power series algebra in d variables, and $A\langle\underline{T}\rangle^\vee = \underline{\mathrm{Hom}}_A(A\langle\underline{T}\rangle, A)$. Then for any $W, C \in D(A_\blacksquare)$ we have a functorial quasi-isomorphism*

$$R\underline{\mathrm{Hom}}_A(W, C \otimes^L_{A_\blacksquare} A\langle\underline{T}\rangle_\blacksquare) = R\underline{\mathrm{Hom}}_A(W \otimes^L_{A_\blacksquare} A\langle\underline{T}\rangle^\vee, C).$$

Proof. Let us first prove the case of one variable, which follows essentially from Proposition 2.18. Indeed, by [And21, Lemma 4.7] we have $A\langle T\rangle_\blacksquare = A_\blacksquare \otimes^L_{\mathbf{Z}_\blacksquare} \mathbf{Z}[T]_\blacksquare$, so that $C \otimes^L_{A_\blacksquare} A\langle T\rangle_\blacksquare = C \otimes^L_{\mathbf{Z}_\blacksquare} \mathbf{Z}[T]_\blacksquare$. Thus, we get

$$\begin{aligned} R\underline{\mathrm{Hom}}_A(W, C \otimes^L_{A_\blacksquare} A\langle\underline{T}\rangle_\blacksquare) &= R\underline{\mathrm{Hom}}_A(W, C \otimes^L_{\mathbf{Z}_\blacksquare} \mathbf{Z}[T]_\blacksquare) \\ &= R\underline{\mathrm{Hom}}_A\left(W \otimes^L_{\mathbf{Z}_\blacksquare} \frac{\mathbf{Z}((T^{-1}))}{\mathbf{Z}[T]}, C\right) \\ &= R\underline{\mathrm{Hom}}_A(W \otimes^L_{A_\blacksquare} \frac{A\langle\langle T^{-1}\rangle\rangle}{A\langle T\rangle}, C) \\ &= R\underline{\mathrm{Hom}}_A(W \otimes_{A_\blacksquare} A\langle T\rangle^\vee, C). \end{aligned}$$

Assume that the result holds for $d-1$ variables and let $A' = A\langle T_1, \ldots, T_{d-1}\rangle$, and set $W' = W \otimes^L_{A_\blacksquare} A'_\blacksquare$ and $C' = C \otimes^L_{A_\blacksquare} A'_\blacksquare$. Then, by the case of one variable we have

$$R\underline{\mathrm{Hom}}_{A'}(W', C' \otimes^L_{A'_\blacksquare} A'_\blacksquare\langle T_d\rangle) = R\underline{\mathrm{Hom}}_{A'}(W' \otimes^L_{A'} \underline{\mathrm{Hom}}_{A'}(A'\langle T_d\rangle, A'), C'). \tag{5}$$

Observe also that

$$\underline{\mathrm{Hom}}_{A'}(A'\langle T_d\rangle, A') = A\langle T_d\rangle^\vee \otimes^L_{A_\blacksquare} A'_\blacksquare$$

and $(A')^\vee \otimes^L_{A_\blacksquare} A\langle T_d\rangle^\vee = A\langle T_1, \ldots, T_d\rangle^\vee$. Then,

$$\begin{aligned} R\underline{\mathrm{Hom}}_A(W, C \otimes^L_{A_\blacksquare} A\langle\underline{T}\rangle_\blacksquare) &= R\underline{\mathrm{Hom}}_{A'}(W', C' \otimes^L_{A'_\blacksquare} A'\langle T_d\rangle_\blacksquare) \\ &= R\underline{\mathrm{Hom}}_{A'}(W \otimes^L_{A_\blacksquare} A\langle T_d\rangle^\vee \otimes^L_{A_\blacksquare} A'_\blacksquare, C') \\ &= R\underline{\mathrm{Hom}}_A(W \otimes^L_{A_\blacksquare} A\langle T_d\rangle^\vee, C \otimes^L_{A_\blacksquare} A'_\blacksquare) \\ &= R\underline{\mathrm{Hom}}_A(W \otimes^L_{A_\blacksquare} A\langle T_d\rangle^\vee \otimes^L_{A_\blacksquare} (A')^\vee, C) \\ &= R\underline{\mathrm{Hom}}_A(W \otimes^L_{A_\blacksquare} A\langle\underline{T}\rangle^\vee, C), \end{aligned}$$

where the second equality follows from equation (5), the third equality is a base change from A to A' and the fourth equality uses the induction step for $d-1$ variables. This finishes the proof. $\square$

Remark 2.20. Corollary 2.19 can be seen as an instance of the six functor formalism of [CS19, Theorem 8.2]. Indeed, let $f : \mathrm{Spa}(A\langle T\rangle, A^\circ\langle T\rangle) \to \mathrm{Spa}(A, A^\circ)$. By a slight extension of the results of [CS19, Lecture VIII], one can define pairs of adjoint functors $f^* : D(A_\blacksquare) \rightleftharpoons D(A\langle T\rangle_\blacksquare) : f_*$ and $f_! : D(A\langle T\rangle_\blacksquare) \rightleftharpoons D(A_\blacksquare) : f^!$, which are the basic cases of the six functor formalism. The functor f_* is the forgetful functor and f^* is given by the solidification functor $-\otimes^L_{A_\blacksquare} A\langle T\rangle_\blacksquare$. The other two functors are given by $f_! M = M \otimes^L_{(A\langle T\rangle, A^\circ)_\blacksquare} \frac{A\langle\langle T^{-1}\rangle\rangle}{A\langle T\rangle}[-1]$ and $f^! N = f^* N \otimes_{A\langle T\rangle_\blacksquare} f^! A$, where $f^! A \cong A\langle T\rangle[1]$. Then, Corollary 2.19 reflects the adjunction $R\underline{\mathrm{Hom}}_{A_\blacksquare}(f_! M, N) = R\underline{\mathrm{Hom}}_{A\langle T\rangle_\blacksquare}(M, f^! N)$.

3. Non-archimedean condensed functional analysis

The main purpose of this section is to state a duality between two classes of solid vector spaces over a finite extension of $\mathbf{Q}_p$, namely Fréchet spaces and LS spaces (to be defined below), generalising the duality between Banach spaces and Smith spaces (cf. [CS20, Theorem 3.8]), and the duality between nuclear Fréchet spaces and LB spaces of compact type (see, e.g., [ST02b, Theorem 1.3]). For doing so, we will use many results on the theory of condensed non-archimedean functional analysis developed by Clausen and Scholze. Since these results haven't yet appeared in the literature, we give a detailed account with proofs included. The reader should be aware that most of the theory in this section must be attributed to Clausen and Scholze [CS], and the only original work is that concerning duality, principally Theorem 3.40.

3.1. **Banach and Smith spaces.** Let K and $\mathcal{O}_K$ be as in the previous section, and let ϖ be a uniformiser of $\mathcal{O}_K$. In this paragraph we focus our attention on the category of solid K-vector spaces (or $K_\blacksquare$-vector spaces), i.e. the category $\mathrm{Mod}_K^{\mathrm{solid}}$. We start with some basic concepts

Definition 3.1.

(1) A condensed set T is said to be discrete if $T = \underline{X}$ where X is a discrete topological space.

(2) A solid $\mathcal{O}_K$-module M is said to be ϖ-adically complete if

$$M = \varprojlim_{s\in\mathbf{N}} M/\varpi^s M$$

as solid $\mathcal{O}_K$-modules.

Next we introduce Banach and Smith spaces in the category of solid K-vector spaces.

Definition 3.2.

(1) A solid K-Banach space is a solid K-vector space V admitting a ϖ-adically complete $\mathcal{O}_{K,\blacksquare}$-module $V^0 \subset V$ such that:
   (a) $V = V^0 \otimes_{\mathcal{O}_{K,\blacksquare}} K$.
   (b) $V^0/\varpi^s V^0$ is discrete for all $s \in \mathbf{N}$.
   We say that V^0 is a lattice of V.

(2) A solid K-Smith space is a solid K-vector space W admitting a profinite $\mathcal{O}_{K,\blacksquare}$-submodule W^0 such that $W = W^0 \otimes_{\mathcal{O}_{K,\blacksquare}} K = W^0[\frac{1}{\varpi}]$. We say that W^0 is a lattice of W.

Remark 3.3. Over $\mathbf{R}$, a Smith space [CS20, Definition 3.6] is a complete locally convex topological $\mathbf{R}$-vector space W containing a compact absolutely convex subset $C \subset V$ such that $V = \bigcup_{a>0} aC$. As its p-adic analogue, we define a classical K-Smith space to be a topological K-vector space W containing a profinite $\mathcal{O}_K$-module $W^0 \subset W$ such that $W = W^0[\frac{1}{\varpi}]$.

Example 3.4. One of the first examples of Banach algebras over K is the Tate algebra $K\langle T\rangle$, it is the ring of global sections of an affinoid disc $\mathbb{D}^1_K = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$. An orthonormal basis for $K\langle T\rangle$ is provided by the monomials $\{T^n\}_{n\in\mathbf{N}}$. Let $K\langle S\rangle$ be another Tate algebra, one has that $K\langle T\rangle \otimes_{K_{\blacksquare}} K\langle S\rangle = K\langle T, S\rangle$ (cf. Lemma 3.13) is the Tate algebra over K in 2 variables, i.e. the global sections of $\mathbb{D}^2_K = \mathbb{D}^1_K \times \mathbb{D}^1_K$. By Corollary 2.19, the dual $K\langle T\rangle^\vee$ is isomorphic to $\frac{K\langle\langle T^{-1}\rangle\rangle}{K\langle T\rangle} \cong \prod_{n\geq 1} \mathcal{O}_K T^{-n}[\frac{1}{p}]$ which is a Smith space.

Proposition 3.5. *The functor $V \mapsto V(*)_{\mathrm{top}}$ induces an exact equivalence of categories between solid and classical K-Banach spaces (resp. solid and classical K-Smith spaces).*

Proof. Let V be a Banach space over K in the classical sense, and let $V^0 \subset V$ be the unit ball. Then V^0 is endowed with the ϖ-adic topology, i.e. it is the inverse limit $V^0 = \varprojlim_s V^0/\varpi^s V^0$, with $V^0/\varpi^s V^0$ discrete for all $s \in \mathbf{N}$. By Proposition 2.3 we know that

$$\underline{V} = \varinjlim_{n\in\mathbf{N}} (\varprojlim_{s\in\mathbf{N}} \underline{V^0/\varpi^s V^0})\varpi^{-n} = \underline{V^0} \otimes_{\mathcal{O}_{K,\blacksquare}} K,$$

so that $\underline{V}$ is a solid K-Banach space. Conversely, if V is a solid Banach space then $V(*)_{\mathrm{top}}$ is clearly a K-Banach space in the classical sense.

Let W be a (classical) K-Smith space and $W^0 \subset W$ a lattice. As W^0 is compact, it is profinite and Proposition 2.3 implies that $\underline{W} = \underline{W^0} \otimes_{\mathcal{O}_{K,\blacksquare}} K$ is a solid Smith space as in our previous definition. The converse is clear.

Let $0 \to V' \to V \to V'' \to 0$ be a strict exact sequence of classical Banach (resp. Smith) spaces. Since V' is identified with a closed subspace of V and V'' is identified with the topological quotient V/V', one easily verifies that $0 \to \underline{V'} \to \underline{V} \to \underline{V''} \to 0$ is an exact sequence of solid K-vector spaces. Conversely, let $0 \to V' \to V \to V'' \to 0$ be an exact sequence of solid Banach or Smith spaces. Since the functor $T \mapsto T(*)$ from condensed abelian groups to abelian groups is exact, one has a short exact sequence of K-vector spaces

$$0 \to V'(*) \to V(*) \to V''(*) \to 0. \tag{6}$$

The fact that $V' \to V$ is a quasicompact immersion (being the pullback of $0 \to V''$ by $V \to V''$) implies that $V'(*)_{\mathrm{top}} \subset V(*)_{\mathrm{top}}$ is a closed subspace, cf. Proposition 2.6. Finally, since $V'' \cong V/V'$ as condensed K-vector spaces, $V''(*)_{\mathrm{top}}$ is isomorphic to the topological quotient $V(*)_{\mathrm{top}}/V'(*)_{\mathrm{top}}$ proving that (6) is strict exact. □

Convention 3.6. From now on we will refer to solid Banach and Smith spaces simply as Banach and Smith spaces respectively. The Banach and Smith spaces considered as topological spaces will be called classical Banach and Smith spaces.

Proposition 3.7. *There is a natural functor*

$$\mathcal{LC}_K \to \mathrm{Mod}_K^{\mathrm{solid}} \ : \ V \mapsto \underline{V}$$

from the category of complete locally convex K-vector spaces to solid K-vector spaces. Moreover, the restriction of $\mathcal{LC}_K$ to the subcategory of compactly generated complete locally convex K-vector spaces is fully faithful.

Proof. This follows from the fact that any complete locally convex K-vector space can be written as a cofiltered limit of classical Banach spaces [Sch02, Chapter I, §4], and the adjunction of Proposition 2.3. For the fully faithfulness, let V_1 and V_2 be two objects in $\mathcal{LC}_K$ which are compactly generated as topological spaces, then, by [CS20, Proposition 1.2 (2)] the natural map $\mathrm{Hom}_{\mathrm{Top}}(V_1, V_2) \to \mathrm{Hom}_{\mathrm{Cond}}(\underline{V_1}, \underline{V_2})$ is a bijection. Since a continuous map $f : V_1 \to V_2$ (resp. a morphism of condensed sets $f : \underline{V_1} \to \underline{V_2}$) is a continuous K-linear map (resp. a morphism of condensed K-vector spaces) if and only if it makes the obvious diagrams commute, the proposition follows. $\square$

From now, all the complete locally convex K-vector spaces will be considered as solid K-vector spaces unless otherwise specified. The following result shows that solid Banach and Smith spaces over K have orthonormal basis.

Lemma 3.8.

(1) *A solid K-vector space is Banach if and only if it is of the form*

$$\widehat{\bigoplus}_{i\in I} K := \big(\varprojlim_s (\bigoplus_{i\in I} \mathcal{O}_K/\varpi^s)\big)[\frac{1}{\varpi}],$$

for some index set I.

(2) *A solid K-vector space is Smith if and only if it is of the form*

$$(\prod_{i\in I} \mathcal{O}_K)[\frac{1}{\varpi}],$$

for some index set I. In particular, by [CS19, Corollary 5.5] *and Theorem* 2.10, *Smith spaces form a family of compact projective generators of the category* $\mathrm{Mod}_K^{\mathrm{solid}}$ *of solid K-vector spaces.*

Proof. By definition, an object of the form $\widehat{\bigoplus}_{i\in I} K$ is a solid K-Banach space. Conversely, let V be a solid K-Banach space and $V^0 \subset V$ a lattice. By Proposition 3.5 $V(*)_{\mathrm{top}}$ is a classical Banach space with unit ball $V^0(*)_{\mathrm{top}} \subset V(*)_{\mathrm{top}}$ and $\underline{V(*)_{\mathrm{top}}} = V$. But then $V^0(*)_{\mathrm{top}}$ has an orthonormal $\mathcal{O}_K$-basis by taking any lift of a $\mathcal{O}_K/\varpi\mathcal{O}_K$-basis of V^0/ϖ. This proves (1).

To prove (2), let W be a solid Smith space and $W^0 \subset W$ a lattice. Since W^0 is a profinite $\mathcal{O}_{K,\blacksquare}$-module, $W^0 = \underline{W^0(*)_{\mathrm{top}}}$ with $W^0(*)_{\mathrm{top}}$ a profinite $\mathcal{O}_K$-module in the usual sense. Moreover, $W^0(*)_{\mathrm{top}}$ is $\mathcal{O}_K$-flat and the topological Nakayama's lemma implies that $W^0(*)_{\mathrm{top}}$ must be of the form $\prod_{i\in I} \mathcal{O}_K$ [70, Exposé VII$_B$, 0.3.8]. $\square$

We will need the following useful proposition

Proposition 3.9 ([CS])**.** *Let V be a solid K-vector space. The following statements are equivalent*

(1) *V is a Smith space*
(2) *V is quasiseparated, and there is a quasicompact $\mathcal{O}_K$-submodule $M \subset V$ such that $V = M[\frac{1}{\varpi}]$.*

Moreover, the class of Smith spaces is stable under extensions, closed subobjects and quotients by closed subobjects.

Proof. The fact that (1) implies (2) follows immediately from the definition. The other implication follows from the fact that any qsqc $\mathcal{O}_K$-module M is profinite. Indeed, as M is quasicompact there is a profinite S and an epimorphism $f : \mathcal{O}_{K,\blacksquare}[S] \to M$. Since M is quasiseparated, the kernel of f is a closed subspace of $\mathcal{O}_{K,\blacksquare}[S]$ which is profinite. This implies that $M \cong \mathcal{O}_{K,\blacksquare}[S]/\ker f = \underline{\mathcal{O}_{K,\blacksquare}[S](*)_{\mathrm{top}}/\ker f(*)_{\mathrm{top}}}$ is profinite.

Since Smith spaces are projective (cf. Lemma 3.8(2)), every extension splits and in particular they are stable under extensions. The stability under closed subobjects and quotients follows from the description of a Smith space as in (2) of the equivalence. □

Lemma 3.10 provides an anti-equivalence between Banach and Smith spaces as solid K-vector spaces, cf. [Smi52] (or also [CS20, Theorem 3.8]) for the analogous statement over the real or complex numbers.

Lemma 3.10 ([CS]). *The assignment $V \mapsto V^\vee$ induces an anti-equivalence between K-Banach spaces and K-Smith spaces. More precisely, the following hold.*

(1) $\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\widehat{\bigoplus}_i \mathcal{O}_K, \mathcal{O}_K) = \prod_i \mathcal{O}_K$ *and* $\underline{\mathrm{Hom}}_K(\widehat{\bigoplus}_{i\in I} K, K) = (\prod_{i\in I}\mathcal{O}_K)[\frac{1}{\varpi}]$.

(2) $\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\prod_{i\in I}\mathcal{O}_K, \mathcal{O}_K) = \widehat{\bigoplus}_{i\in I}\mathcal{O}_K$ *and* $\underline{\mathrm{Hom}}_K((\prod_{i\in I}\mathcal{O}_K)[\frac{1}{\varpi}], K) = \widehat{\bigoplus}_{i\in I} K$.

Proof. To prove (1), notice that

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\widehat{\bigoplus_i}\mathcal{O}_K, \mathcal{O}_K) &= \varprojlim_s \underline{\mathrm{Hom}}_{\mathcal{O}_K/\varpi^s}(\bigoplus_i \mathcal{O}_K/\varpi^s, \mathcal{O}_K/\varpi^s)\\ &= \varprojlim_s \prod_i \mathcal{O}_K/\varpi^s\\ &= \prod_i \mathcal{O}_K.\end{aligned}$$

To prove the second equality it is enough to show that

$$\underline{\mathrm{Hom}}_K(\widehat{\bigoplus_i} K, K) = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\widehat{\bigoplus_i}\mathcal{O}_K, \mathcal{O}_K)[\frac{1}{\varpi}].$$

Let S be an extremally disconnected set, by adjunction it is enough to show that

$$\mathrm{Hom}_K(\widehat{\bigoplus_i} K, \underline{\mathrm{Cont}}(S, K)) = \mathrm{Hom}_{\mathcal{O}_K}(\widehat{\bigoplus_i}\mathcal{O}_K, \underline{\mathrm{Cont}}(S, \mathcal{O}_K))[\frac{1}{\varpi}].$$

But all the solid spaces involved arise as the condensed set associated to a compactly generated Hausdorff topological space, the claim follows from the fact that lattices are mapped to lattices for continuous maps of classical Banach spaces.

Part (2) follows from the fact that an object of the form $\prod_{i\in I}\mathcal{O}_K$ is a retraction of a compact projective generator $\mathcal{O}_{K,\blacksquare}[S]$ for S extremally disconnected, and the fact that

$$\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[S], \mathcal{O}_K) = \underline{\mathrm{Cont}}(S, \mathcal{O}_K).$$

This finishes the proof of the Lemma. □

Remark 3.11. Part (2) of Lemma 3.10 also holds with $R\underline{\mathrm{Hom}}$ since $\prod_{i\in I}\mathcal{O}_K$ is a projective $\mathcal{O}_{K,\blacksquare}$-module. The same proof of the first assertion of (1) can also be adapted to show that $R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\widehat{\bigoplus}_i\mathcal{O}_K,\mathcal{O}_K)=\prod_i\mathcal{O}_K$. Indeed,

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\widehat{\bigoplus_i}\mathcal{O}_K,\mathcal{O}_K) &= R\varprojlim_{s\in\mathbf{N}} R\underline{\mathrm{Hom}}_{\mathcal{O}_K/\varpi^s}(\bigoplus_i \mathcal{O}_K/\varpi^s,\mathcal{O}_K/\varpi^s)\\ &= R\varprojlim_{s\in\mathbf{N}}\prod_i\mathcal{O}_K/\varpi^s\\ &= \prod_i\mathcal{O}_K. \end{aligned}$$

The authors ignore how to calculate $R\underline{\mathrm{Hom}}_K(\widehat{\bigoplus}_{i\in I}K,K)$.

We now study the behaviour of the tensor product.

Proposition 3.12. *Let* $V=\prod_{i\in I}\mathcal{O}_K$ *and* $W=\prod_{j\in J}\mathcal{O}_K$. *Then*

$$V\otimes^L_{\mathcal{O}_{K,\blacksquare}}W=\prod_{(i,j)\in I\times J}\mathcal{O}_K.$$

Proof. See [CS19, Proposition 6.3]. □

The following useful result shows that the solid tensor product of solid K-Banach spaces coincides with the projective tensor product of classical K-Banach spaces.

Lemma 3.13 ([CS]). *Let* V *and* V' *be classical Banach spaces over* K, *and let* $V\widehat{\otimes}_K V'$ *denote its projective tensor product. Then* $\underline{V\widehat{\otimes}_K V'}=\underline{V}\otimes_{K_\blacksquare}\underline{V'}$.

Proof. Fix an isomorphism $V=\widehat{\bigoplus}_{i\in I}K$. As a convergent series $\sum_i a_i$ has only countably many terms different from 0, we can write

$$\underline{V}=\varinjlim_{I'\subset I}\widehat{\bigoplus}_{i\in I'}K,$$

where I' runs over all the countable subsets of I. Therefore, we can assume that $V\cong V'\cong\widehat{\bigoplus}_{n\in\mathbf{N}}K$. Let $\mathscr{S}$ denote the direct set of functions $f:\mathbf{N}\to\mathbf{Z}$ such that $f(n)\to+\infty$ as $n\to+\infty$, endowed with the order $f\preceq g$ iff $f(n)\geq g(n)$ for all $n\in\mathbf{N}$. Thus, we can write

$$\widehat{\bigoplus}_{n\in\mathbf{N}}K=\varinjlim_{f\in\mathscr{S}}\prod_{n\in\mathbf{N}}\mathcal{O}_K\varpi^{f(n)}. \tag{7}$$

Indeed, by evaluating at an extremally disconnected set S, $\underline{V}(S)=\mathrm{Cont}(S,\underline{V})=\widehat{\bigoplus}_{n\in\mathbf{N}}\mathrm{Cont}(S,K)$ has a natural (classical) Banach space structure, for which a function $\phi:S\to\underline{V}$ can be written in a unique way as a sum $\phi=\sum_n\phi_n$ with $\phi_n\in\mathrm{Cont}(S,K)$, such that $|\phi_n|\to0$ as $n\to\infty$. Then, from (7) and Proposition 3.12 we deduce that

$$\begin{aligned} \underline{V}\otimes_{K_\blacksquare}\underline{V'} &= \varinjlim_{f,g\in\mathscr{S}}\prod_{n\in\mathbf{N}}\mathcal{O}_K\varpi^{f(n)}\otimes_{\mathcal{O}_{K,\blacksquare}}\prod_{m\in\mathbf{N}}\mathcal{O}_K\varpi^{g(m)}\\ &= \varinjlim_{f,g\in\mathscr{S}}\prod_{n,m\in\mathbf{N}\times\mathbf{N}}\mathcal{O}_K\varpi^{f(n)+g(m)}. \end{aligned}$$

Given $f,g\in\mathscr{S}$ define the function $h_{f,g}:\mathbf{N}\times\mathbf{N}\to\mathbf{Z}$ as $h_{f,g}(n,m)=f(n)+g(m)$. Let $\mathscr{S}'$ be the direct set of functions $h:\mathbf{N}\times\mathbf{N}\to\mathbf{Z}$ such that $h(n,m)\to$

$+\infty$ as $\max\{n,m\} \to +\infty$. Then the set $\{h_{f,g}\}_{f,g\in\mathscr{S}}$ is a cofinal family in $\mathscr{S}'$. Indeed, given $h : \mathbf{N}\times\mathbf{N} \to \mathbf{Z}$, if we define $f(n) = \frac{1}{2}\min_m h(n,m)$ and $g(m) = \frac{1}{2}\min_n h(n,m)$, then $h \preceq h_{f,g}$. Therefore,

$$\varinjlim_{f,g\in\mathscr{S}} \prod_{n,m\in\mathbf{N}\times\mathbf{N}} \mathcal{O}_K \varpi^{h_{f,g}(n,m)} = \varinjlim_{h\in\mathscr{S}'} \prod_{n,m} \mathcal{O}_K\varpi^{h(n,m)} = \widehat{\bigoplus}_{n,m} K.$$

□

Let us recall the concept of a nuclear solid K-vector space.

Definition 3.14 ([CS20, Definition 13.10])**.** Let $V \in \mathrm{Mod}_K^{\mathrm{solid}}$. We say that V is nuclear if, for all extremally disconnected sets S, the natural map

$$\underline{\mathrm{Hom}}_K(K_{\blacksquare}[S], K) \otimes_{K_{\blacksquare}} V \to \underline{\mathrm{Hom}}_K(K_{\blacksquare}[S], V)$$

is an isomorphism.

Remark 3.15. We warn the reader that this notion of nuclearity differs from the classical one, say in [Sch02, §19]. Indeed, if a classical Banach space is nuclear in the classical sense then it is finite dimensional (cf. *loc. cit.* §19). On the other hand, solid K-Banach spaces are always nuclear in the condensed sense.

Corollary 3.16 ([CS])**.** *Let V be a Banach space over K, then V is a nuclear $K_{\blacksquare}$-vector space.*

Proof. Let S be an extremally disconnected set. By taking basis we can write $K_{\blacksquare}[S] = \prod_J \mathcal{O}_K[\frac{1}{\varpi}]$ and $V = \widehat{\bigoplus}_I K$. Then

$$\begin{aligned} \underline{\mathrm{Hom}}_K(K_{\blacksquare}[S], V) &= \underline{\mathrm{Hom}}_K(K_{\blacksquare}[S]\otimes_{K_{\blacksquare}} V^{\vee}, K) \\ &= \underline{\mathrm{Hom}}_K\Big(\prod_{J\times I}\mathcal{O}_K[\frac{1}{\varpi}], K\Big) \\ &= \widehat{\bigoplus}_{J\times I} K \\ &= \underline{\mathrm{Hom}}_K(K_{\blacksquare}[S], K)\otimes_{K_{\blacksquare}} V, \end{aligned}$$

where the first and third equalities follow from the duality between Banach and Smith spaces of Lemma 3.10 and the ⊗-Hom adjunction, the second equality follows from Proposition 3.12, and the last equality is a consequence of Lemma 3.13. □

Corollary 3.17. *Let V and W be a Banach and a Smith space respectively. Then*

$$\underline{\mathrm{Hom}}_K(V, W) = V^{\vee}\otimes_{K_{\blacksquare}} W,$$
$$\underline{\mathrm{Hom}}_K(W, V) = W^{\vee}\otimes_{K_{\blacksquare}} V.$$

Proof. The second equality follows from nuclearity of V (Corollary 3.16). For the first equality, tensor-Hom adjunction gives

$$\underline{\mathrm{Hom}}_K(V, W) = \underline{\mathrm{Hom}}_K(V\otimes_{K_{\blacksquare}} W^{\vee}, K),$$

and the result follows immediately from the description of the tensor product of two Banach spaces (Lemma 3.13) and duality between Banach and Smith spaces (Lemma 3.10). □

We finish this section with some elementary lemmas that will be needed later.

Lemma 3.18. *The functor $V \mapsto V^{qs}$ defines a left adjoint to the inclusion of the category of quasiseparated solid K-vector spaces into the category of all solid K-vector spaces.*

Proof. By Lemma 2.7 we only need to show that V^{qs} is a solid K-vector space. Let S be an extremally disconnected set, we have a commutative diagram with exact arrows

$$\begin{tikzcd} \mathrm{Hom}_K(K[S], V) \arrow[r] & \mathrm{Hom}_K(K[S], V^{qs}) \arrow[r] & 0 \\ \mathrm{Hom}_K(K_\blacksquare[S], V) \arrow[r] \arrow[u, "\wr"] & \mathrm{Hom}_K(K_\blacksquare[S], V^{qs}) \arrow[u] & \end{tikzcd} \tag{8}$$

where the left vertical arrow is an isomorphism since V is solid. This implies that $\mathrm{Hom}_K(K_\blacksquare[S], V^{qs}) \to \mathrm{Hom}_K(K[S], V^{qs})$ is surjective. To prove injectivity, it is enough to show that $(K_\blacksquare[S]/K[S])^{qs} = 0$, for this, it is enough to show that $(K[S])(*)_{\mathrm{top}}$ is dense in $(K_\blacksquare[S])(*)_{\mathrm{top}}$ which is clear by definition of $K_\blacksquare[S]$. Therefore, the right vertical arrow of (8) is a bijection, proving that V^{qs} is a solid K-vector space. $\square$

We say that a map of quasiseparated condensed sets has dense image if the map of underlying topological spaces does. One has Lemma 3.19.

Lemma 3.19.

(1) *Let V be a solid K-vector space such that the maximal quasiseparated quotient V^{qs} is zero. Then $\underline{\mathrm{Hom}}(V, K) = 0$.*
(2) *A map of Banach spaces $V \to V'$ is injective (resp. with dense image) if and only if its dual $V'^{\vee} \to V^{\vee}$ has dense image (resp. is injective).*

Proof. Let S be an extremally disconnected set, then

$$\underline{\mathrm{Hom}}_K(V, K)(S) = \mathrm{Hom}_K(V \otimes_{K_\blacksquare} K_\blacksquare[S], K) = \mathrm{Hom}_K(V, \underline{\mathrm{Cont}}(S, K)).$$

But $\underline{\mathrm{Cont}}(S, K)$ is a Banach space. Then, by adjunction (Proposition 2.3), we get

$$\underline{\mathrm{Hom}}_K(V, K)(S) = \mathrm{Hom}_K(V(*)_{\mathrm{top}}, \mathrm{Cont}(S, K)).$$

Since $V^{qs} = 0$, the maximal Hausdorff quotient of $V(*)_{\mathrm{top}}$ is zero. This implies that

$$\mathrm{Hom}_K(V(*)_{\mathrm{top}}, \mathrm{Cont}(S, K)) = 0$$

proving (1).

To prove (2), let $f : V \to V'$ be a map of Banach spaces. Suppose that f has not dense image and let $\overline{f(V)} \subset V'$ be the closure of its image (i.e. the solid K-Banach space associated to the closure of $f(V(*)_{\mathrm{top}})$ in. $V'(*)_{\mathrm{top}}$). Then $V'/\overline{f(V)}$ is a non-zero Banach space and we have a short exact sequence

$$0 \to \overline{f(V)} \to V' \to V'/\overline{f(V)} \to 0$$

which splits as any Banach space over a local field is orthonormalisable. Taking duals we get a short exact sequence

$$0 \to (V'/\overline{f(V)})^{\vee} \to V'^{\vee} \to \overline{f(V)}^{\vee} \to 0.$$

Since $f^{\vee} : V'^{\vee} \to V^{\vee}$ factors through $\overline{f(V)}^{\vee}$, the map $f^{\vee}$ is not injective. Conversely, suppose that the map $f^{\vee}$ is not injective, then its kernel $\ker f^{\vee}$ is a closed subspace of $V'^{\vee}$ which is a Smith space by Proposition 3.9. Since the quotient

$V'^{\vee}/\ker(f^{\vee})$ is also a Smith space, there is a retraction $r: V'^{\vee} \to \ker(f^{\vee})$. Taking duals one sees that the composition $V \to V' \to \ker(f^{\vee})^{\vee}$ is zero and that the last map is surjective (because of r), this implies that f has not dense image.

Finally, suppose that $f: V \to V'$ is injective. If $f^{\vee}: V'^{\vee} \to V^{\vee}$ does not have dense image, the closure $\overline{f(V'^{\vee})} \subset V^{\vee}$ is a closed Smith subspace and its quotient $V^{\vee}/\overline{f(V'^{\vee})}$ is a non-zero Smith space. Taking duals we get a short exact sequence

$$0 \to (V^{\vee}/\overline{f(V'^{\vee})})^{\vee} \to V \to \overline{f(V'^{\vee})}^{\vee} \to 0.$$

But $f: V \to V'$ factors through $\overline{f(V'^{\vee})}^{\vee}$, this is a contradiction with the injectivity of f. Conversely, suppose that $f^{\vee}: V'^{\vee} \to V^{\vee}$ has dense image, consider the quotient

$$0 \to V'^{\vee} \to V^{\vee} \to Q \to 0.$$

Taking duals one obtains an exact sequence

$$0 \to \underline{\mathrm{Hom}}_K(Q, K) \to V \to V'.$$

But part (1) implies that $\underline{\mathrm{Hom}}_K(Q, K) = 0$, proving that f is injective. □

3.2. **Quasiseparated solid K-vector spaces.** We shall use the following results throughout the text without explicit mention. They are due to Clausen and Scholze, and explained to us by Guido Bosco in the study group of La Tourette.

Proposition 3.20 ([CS]). *Let V be a solid K-vector space. The following are equivalent,*

(1) *V is quasiseparated.*
(2) *V is equal to the filtered colimit of its Smith subspaces.*

Proof. Let V be a quasiseparated $K_{\blacksquare}$-vector space, let W_1, W_2 be Smith subspaces of V. As V is quasiseparated, $W_1 \cap W_2$ is a closed Smith subspace of W_1, and the sum $W_1 + W_2 \subset V$ is isomorphic to $(W_1 \oplus W_2)/W_1 \cap W_2$. This shows that the Smith subspaces of V form a direct system, let V_0 denote their colimit. We claim that $V/V_0 = 0$, let W' be a Smith space and $f: W' \to V/V_0$ a map of solid $K_{\blacksquare}$-vector spaces. As W' is projective, there is a lift $f': W' \to V$. But $\ker f' \subset W'$ is a closed Smith subspace since V is quasiseparated. This implies that f' factors through V_0 and that $f = 0$. Since the Smith spaces form a family of compact projective generators of $K_{\blacksquare}$-vector spaces, one must have $V/V_0 = 0$ proving (1) $\Rightarrow$ (2).

Conversely, let $V = \varinjlim_{i \in I} V_i$ be a vector space written as a filtered colimit of Smith spaces by injective transition maps. Let S_1, S_2 be two profinite sets and $f_j: S_j \to V$ be two maps for $j = 1, 2$. As the S_i are profinite, there exists $i \in I$ such that f_j factors through V_i for $j = 1, 2$. Then, as the map $V_i \to V$ is injective, one has

$$S_1 \times_V S_2 = S_1 \times_{V_i} S_2.$$

The implication (2) $\Rightarrow$ (1) follows as a Smith space is quasiseparated. □

Lemma 3.21 ([CS]). *A quasiseparated K-solid space is flat. In other words, if V is a quasiseparated $K_{\blacksquare}$-vector space, then $- \otimes_{K_{\blacksquare}} V = - \otimes^L_{K_{\blacksquare}} V$.*

Proof. Let V be a quasiseparated $K_{\blacksquare}$-vector space. Since filtered colimits are exact in the category of condensed abelian groups, and the solid tensor product commutes with colimits, by Proposition 3.20 it is enough to prove the lemma for $V = \prod_I \mathcal{O}_K[\frac{1}{p}]$ a Smith space. Let $W \in \mathrm{Mod}_K^{\mathrm{solid}}$. We want to show that $V \otimes^L_{K_{\blacksquare}} W$ is concentrated in degree zero. As the Smith spaces are compact projective generators, W can be written as a quotient $0 \to W'' \to W' \to W \to 0$ where W' is a direct sum of Smith spaces. Then we are reduced to showing that $W'' \otimes_{K_\blacksquare} V \to W' \otimes_{K_\blacksquare} V$ is injective. Since W'' is quasiseparated, by Proposition 3.20, it can be written as filtered colimit of its Smith subspaces. Therefore, by compacity of Smith spaces, the arrow $W'' \to W'$ is a filtered colimit of injections of Smith spaces. It is hence enough to show that if $\prod_{J_1} \mathcal{O}_K[\frac{1}{p}] \to \prod_{J_2} \mathcal{O}_K[\frac{1}{p}]$ is an injective map, then

$$\prod_{J_1} \mathcal{O}_K[\frac{1}{p}] \otimes_{K_\blacksquare} V \to \prod_{J_2} \mathcal{O}_K[\frac{1}{p}] \otimes_{K_\blacksquare} V$$

is injective. This follows immediately from the following formula for the tensor product of two Smith spaces given by Proposition 3.12

$$(\prod_I \mathcal{O}_K[\frac{1}{p}]) \otimes_{K_\blacksquare} (\prod_J \mathcal{O}_K[\frac{1}{p}]) = \prod_{I\times J} \mathcal{O}_K[\frac{1}{p}].$$

This finishes the proof. □

3.3. **Fréchet and *LS* spaces.** Our next goal is to extend the duality between Banach and Smith spaces to a larger class of solid K-vector spaces. We need a definition

Definition 3.22.

(1) A solid Fréchet space is a solid K-vector space which can be written as a countable cofiltered limit of Banach spaces.
(2) A solid LS (resp. LB, resp. LF) space is a solid K-vector space which can be written as a countable filtered colimit of Smith (resp. Banach, resp. solid Fréchet) spaces with injective transition maps.

Example 3.23. Let $\mathring{\mathbb{D}}^1_K = \bigcup_{r<1} \mathbb{D}^1_K(r)$ be the rigid analytic open unit disc written as the increasing union of the closed affinoid discs of radius $r < 1$. The global section $\mathcal{O}(\mathring{\mathbb{D}}^1_K) = \varprojlim_{r<1} \mathcal{O}(\mathbb{D}^1_K(r))$ is a Fréchet space and we will see that its dual $\mathcal{O}(\mathring{\mathbb{D}}^1_K)^\vee = \varinjlim_{r<1} \mathcal{O}(\mathbb{D}^1_K(r))^\vee$ is a solid LS space (cf. Theorem 3.40). The tensor product $\mathcal{O}(\mathring{\mathbb{D}}^1_K) \otimes_{K_\blacksquare} \mathcal{O}(\mathring{\mathbb{D}}^1_K)$ is isomorphic to $\mathcal{O}(\mathring{\mathbb{D}}^2_K)$ (cf. Lemma 3.28).

Lemma 3.24.

(1) *The functor $V \mapsto V(*)_{\mathrm{top}}$ induces an equivalence of categories between solid and classical Fréchet spaces such that $V = \underline{V(*)_{\mathrm{top}}}$.*
(2) *An LS space is quasiseparated. Conversely, a quasiseparated $K_\blacksquare$-vector space W is an LS space if and only if it is countably compactly generated, i.e. for every surjection $\bigoplus_{i\in I} P_i \to W$ by direct sums of Smith spaces, there is a countable index subset $I_0 \subset I$ such that $\bigoplus_{i\in I_0} P_i \to W$ is surjective.*

Proof. Part (1) follows from the fact that a classical Fréchet space is complete for a countable family of seminorms (i.e. it can be written as a countable cofiltered limit of classical Banach spaces), Proposition 2.3(1), and Proposition 3.5. For part (2), the fact that an LS space is quasiseparated follows from Proposition 3.20. Let W be quasiseparated $K_\blacksquare$-vector space. Assume it is countably compactly generated. Write $W = \varinjlim_{W'\subset W} W'$ as a the colimit of its Smith subspaces. As W is quasiseparated, the sum of two Smith subspaces is Smith, so the colimit is filtered. By hypothesis, there are countably many W' such that $W = \varinjlim_{s\in \mathbf{N}} W'_s$.

Moreover, we can assume that $W_0 \subset W_1 \subset \cdots$. This proves that W is an LS space. Conversely, let W be an LS space and let $\bigoplus_{i\in I} P_i \to W$ be a surjective map with P_i Smith. The image P_i' of P_i in W is a Smith space since W is quasiseparated, hence $W = \sum_i P_i'$. Thus, without loss of generality we can assume that $\varinjlim_{i\in I} P_i'$ is filtered with injective transition maps and equal to W. Let $W = \varinjlim_{s\in \mathbf{N}} W_s$ be a presentation as a countable colimit of Smith spaces by injective transition maps. By compactness of the Smith spaces, for all s there exists i_s such that $W_n \subset P_{i_s}' \subset W$. We can assume that $P_{i_s}' \subset P_{i_{s+1}}'$ for all $s \in \mathbf{N}$. Thus, $\bigoplus_{s\in\mathbf{N}} P_{i_s} \to W$ is surjective, this finishes the lemma. ☐

Lemma 3.25 says that we can always choose a presentation of a solid Fréchet space as an inverse limit of Banach spaces with dense transition maps.

Lemma 3.25. *Let V be a solid Fréchet space, then we can write $V = \varprojlim_{n\in\mathbf{N}} V_n$ with V_n Banach spaces such that $V(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ has dense image for all $n \in \mathbf{N}$. Conversely, let $\{V_n\}_{n\in\mathbf{N}}$ be a cofiltered limit of Banach spaces such that $V_{n+1}(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ has dense image, and let $V = \varprojlim V_n$ be its inverse limit. Then $V(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ has dense image for all $n \in \mathbf{N}$.*

Proof. Let $V = \varprojlim_{n\in\mathbf{N}} V_n$ be a presentation of the solid Fréchet space as an inverse limit of Banach spaces. Changing V_n by the closure of the image of V (i.e. the solid Banach space corresponding to the closure of the image of $V(*)_{\mathrm{top}}$ in $V_n(*)_{\mathrm{top}}$), we obtain a desired presentation with $V(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ of dense image. Conversely, let $\{V_n\}_{n\in\mathbf{N}}$ be an inverse system of Banach spaces with maps $V_{n+1}(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ of dense image, and let $V = \varprojlim_n V_n$ be a solid Fréchet space, we want to show that the image of $V(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ is dense for all $n \in \mathbf{N}$. Fix $n_0 \in \mathbf{N}$, let $w \in V_{n_0}(*)_{\mathrm{top}}$ and $0 < \varepsilon < 1$. Let $|\cdot|_n$ denote the norm of $V_n(*)_{\mathrm{top}}$. Without loss of generality we assume that $|\cdot|_n \leq |\cdot|_{n+1}$ on $V_{n+1}(*)_{\mathrm{top}}$, where we identify $|\cdot|_n$ with the seminorm on $V_{n+1}(*)_{\mathrm{top}}$ defined by composing with the map $V_{n+1}(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$. By density of the transition maps $\phi_n^{n+1} : V_{n+1}(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$, there exists $v_{n_0+1} \in V_{n+1}(*)_{\mathrm{top}}$ such that $|\phi_{n_0}^{n_0+1}(v_{n_0+1}) - w| \leq \varepsilon$. By induction, for all $n \geq n_0+1$ we can find $v_n \in V_n(*)_{\mathrm{top}}$ such that $|\phi_n^{n+1}(v_{n+1} - v_n)| \leq \varepsilon^n$. Let $n \geq n_0+1$ be fixed and let $k \geq 0$, then by construction the sequence $\{\phi_n^{n+k}(v_{n+k})\}$ converges in $V_n(*)_{\mathrm{top}}$ to an element v_n'. Moreover, it is immediate to check that $\phi_n^{n+1}(v_{n+1}') = v_n'$ so that $v' = (v_n') \in V(*)_{\mathrm{top}}$, and $|\phi_{n_0}^{n_0+1}(v_{n_0+1}' - w)| \leq \varepsilon$. This proves the lemma. ☐

Convention 3.26. From now on we refer to solid Fréchet spaces simply as Fréchet spaces, we call the underlying topological space a classical Fréchet space.

3.4. Properties of Fréchet spaces.

We now present some basic properties of Fréchet spaces, most of the results in the context of condensed mathematics are due to Clausen and Scholze [CS].

Lemma 3.27 (Topological Mittag-Leffler [CS])**.** *Let $V = \varprojlim_n V_n$ be a Fréchet space written as an inverse limit of Banach spaces with dense transition maps. Then*

$$R^j \varprojlim_n V_n = 0$$

for all $j > 0$. In particular, $V = R\underline{\mathrm{Hom}}_K(V^\vee, K)$.

Proof. See [Bos21, Lemma A.18]. ☐

Lemma 3.28 ([CS]). *Let $(V_n)_{n\in\mathbf{N}}$ and $(W_m)_{m\in\mathbf{N}}$ be countable families of Banach spaces.*

(1) *We have*

$$\Big(\prod_n V_n\Big) \otimes^L_{K_\blacksquare} \Big(\prod_m W_m\Big) = \prod_{n,m} V_n \otimes_{K_\blacksquare} W_m.$$

(2) *More generally, if $V = \varprojlim_n V_n$ and $W = \varprojlim_m W_m$ are Fréchet spaces written as inverse limits of Banach spaces by dense transition maps (which is always possible thanks to Lemma* 3.25*), one has*

$$V \otimes^L_{K_\blacksquare} W = \varprojlim_{n,m} V_n \otimes_{K_\blacksquare} W_m.$$

Proof. Property (AB6) of Theorem 2.5 and Proposition 3.20 imply that products of quasiseparated solid K-vector spaces are quasiseparated. Then, by Lemma 3.21, all the derived tensor products in the statements are already concentrated in degree 0. By Lemma 3.27 we have a short exact sequences

$$0 \to \varprojlim_n V_n \to \prod_n V_n \to \prod_n V_n \to 0,$$
$$0 \to \varprojlim_m W_m \to \prod_m W_m \to \prod_m W_m \to 0.$$

Then (2) follows from (1) by taking the tensor product of the above sequences.

For (1), suppose that the statement is true for all V_n and W_n possessing a countable orthonormal basis. Let $V_n \cong \widehat{\bigoplus}_{I_n} K$ and $W_m \cong \widehat{\bigoplus}_{J_m} K$ for all $n, m \in N$. We write $V_n = \varinjlim_{I'_n\subset I_n} \widehat{\bigoplus}_{I'_n} K$ and $W_m = \varinjlim_{J'_m\subset J_m} \widehat{\bigoplus}_{J'_m} K$ with I'_n and J'_m running among all the countable subsets of I_n and J_m respectively. Then

$$\begin{aligned}
\Big(\prod_n V_n\Big) \otimes_{K_\blacksquare} \Big(\prod_m W_m\Big) &= \Big(\prod_n\big(\varinjlim_{I'_n\subset I_n} \widehat{\bigoplus}_{I'_n} K\big)\Big) \otimes_{K_\blacksquare} \Big(\prod_m\big(\varinjlim_{J'_m\subset J_m} \widehat{\bigoplus}_{J'_m} K\big)\Big)\\
&= \varinjlim_{\substack{\forall (n,m)\in\mathbf{N}\times\mathbf{N}\\ I'_n\times J'_m\subset I_n\times J_m}} \Big(\prod_n\big(\widehat{\bigoplus}_{I'_n} K\big)\Big) \otimes_{K_\blacksquare} \Big(\prod_m\big(\widehat{\bigoplus}_{J'_m} K\big)\Big)\\
&= \varinjlim_{\substack{\forall (n,m)\in\mathbf{N}\times\mathbf{N}\\ I'_n\times J'_m\subset I_n\times J_m}} \prod_{n,m}\big(\widehat{\bigoplus}_{I'_n\times J'_m} K\big)\\
&= \prod_{n,m}\big(\varinjlim_{I'_n\times J'_m\subset I_n\times J_m} \widehat{\bigoplus}_{I'_n\times J'_m} K\big)\\
&= \prod_{n,m}(V_n \otimes_{K_\blacksquare} W_m).
\end{aligned}$$

Hence, we are left to prove (1) for $W_m = V_n = \widehat{\bigoplus}_{\mathbf{N}} K$ for all $n, m \in \mathbf{N}$. Let $\mathscr{S}$ be the filtered set of functions $f : \mathbf{N} \to \mathbf{Z}$ such that $f(k) \to +\infty$ as $k \to +\infty$. For all

$n,m\in \mathbf{N}$ we can write

$$V_n=\widehat{\bigoplus}_{\mathbf{N}}K=\varinjlim_{f_n\in\mathscr{S}}\prod_{k\in\mathbf{N}}\mathcal{O}_K\varpi^{f_n(k)},$$
$$W_m=\widehat{\bigoplus}_{\mathbf{N}}K=\varinjlim_{g_m\in\mathscr{S}}\prod_{s\in\mathbf{N}}\mathcal{O}_K\varpi^{g_m(s)}.$$

Therefore we get

$$\begin{aligned}(\prod_n V_n)&\otimes_{K_\blacksquare}(\prod_m W_m)\\ &=\Big(\varinjlim_{\forall n,\ f_n\in\mathscr{S}}\prod_n\prod_k\mathcal{O}_K\varpi^{f_n(k)}\Big)\otimes_{K_\blacksquare}\Big(\varinjlim_{\forall m,\ g_m\in\mathscr{S}}\prod_m\prod_s\mathcal{O}_K\varpi^{g_m(s)}\Big)\\ &=\varinjlim_{\substack{\forall(n,m)\\ f_n,g_m\in\mathscr{S}}}\Big(\prod_{n,m}(\prod_k\mathcal{O}_K\varpi^{f_n(k)})\otimes_{K_\blacksquare}(\prod_s\mathcal{O}_K\varpi^{g_m(s)})\Big).\end{aligned}$$

Given $f,g\in\mathscr{S}$ we define $h_{f,g}:\mathbf{N}\times\mathbf{N}\to\mathbf{Z}$ as $h_{f,g}(k,s)=f(k)+g(s)$. Let $\mathscr{S}'$ be the set of functions $h:\mathbf{N}\times\mathbf{N}\to\mathbf{Z}$ such that $h(n,m)\to\infty$ as $\min\{n,m\}\to\infty$. Then the family of functions $\{h_{f,g}\}_{f,g\in\mathscr{S}}$ is cofinal in $\mathscr{S}'$ (see the proof of Lemma 3.13). One obtains

$$\begin{aligned}(\prod_n V_n)\otimes_{K_\blacksquare}(\prod_m W_m)&=\varinjlim_{\substack{\forall(n,m)\\ h_{n,m}\in\mathscr{S}'}}\prod_{n,m}\prod_{k,s}\mathcal{O}_K\varpi^{h_{n,m}(k,s)}\\ &=\prod_{n,m}\Big(\varinjlim_{h_{n,m}\in\mathscr{S}'}\prod_{k,s}\mathcal{O}_K\varpi^{h_{n,m}(k,s)}\Big)\\ &=\prod_{n,m}\widehat{\bigoplus}_{k,s}K\\ &=\prod_{n,m}(V_n\otimes_{K_\blacksquare}W_m),\end{aligned}$$

this finishes the proof. □

Proposition 3.29 ([CS]). *A Fréchet space is a nuclear $K_\blacksquare$-vector space.*

Proof. Let $V=\varprojlim_n V_n$ be a Fréchet space written as an inverse limit of Banach spaces with dense transition maps. Let S be an extremally disconnected set, then by Corollary 3.16

$$\begin{aligned}\underline{\mathrm{Hom}}_K(K_\blacksquare[S],V)&=\varprojlim_n\underline{\mathrm{Hom}}_K(K_\blacksquare[S],V_n)\\ &=\varprojlim_n(\underline{\mathrm{Hom}}_K(K_\blacksquare[S],K)\otimes_{K_\blacksquare}V_n)\\ &=\underline{\mathrm{Hom}}_K(K_\blacksquare[S],K)\otimes_{K_\blacksquare}(\varprojlim_n V_n)\\ &=\underline{\mathrm{Hom}}_K(K_\blacksquare[S],K)\otimes_{K_\blacksquare}V,\end{aligned}$$

this finishes the proof. □

Remark 3.30. Let S be an extremally disconnected set and T a condensed set, by definition $T(S)$ can be written as $\underline{\mathrm{Cont}}(S,T)(*)$, in particular, it carries a natural compactly generated topology by Proposition 2.3. If $V = \varprojlim V_n$ is a solid Fréchet space written as an inverse limit of Banach spaces V_n, then $V(S) = \varprojlim_n V_n(S)$. By Proposition 3.29 we have that

$$V_n(S) = \underline{\mathrm{Hom}}_K(K_\blacksquare[S], V_n)(*) = (\underline{\mathrm{Hom}}_K(K_\blacksquare[S], K) \otimes_{K_\blacksquare} V_n)(*),$$

which shows that the topological spaces $V_n(S)_{\mathrm{top}}$ are naturally classical Banach spaces and hence that $V(S)_{\mathrm{top}}$ is a classical Fréchet space.

Lemmas 3.31 and 3.32 describe the maps between LF, Fréchet and Banach spaces

Lemma 3.31. *Let $V = \varprojlim_n V_n$ be a Fréchet space written as a countable cofiltered limit of Banach spaces with projection maps $V \to V_n$ of dense image. Let W be a Banach space. Then any continuous linear map $f : V \to W$ factors through some V_n. More generally, we have $\underline{\mathrm{Hom}}_K(V, W) = \varinjlim_n \underline{\mathrm{Hom}}_K(V_n, W)$.*

Proof. First, evaluating $\underline{\mathrm{Hom}}_K(V, W)$ at an extremally disconnected set, using adjunction and the nuclearity of W, one reduces to showing that $\mathrm{Hom}_K(V, W) = \varinjlim_n \mathrm{Hom}_K(V_n, W)$. Since all the spaces involved come from compactly generated topological K-vector spaces, we might assume that V and W are classical Fréchet and Banach spaces respectively. Let $|\cdot|_n$ be the seminorm of V given by V_n, without loss of generality we may assume that $|\cdot|_n \leq |\cdot|_{n+1}$. We denote the norm of W by $\|\cdot\|$. The map f factors through V_n if and only if it is continuous with respect to the seminorm $|\cdot|_n$. Suppose that f does not factor through any n, then there exist sequences of vectors $(v_{n,m})_m$ in V for all n such that

$$|v_{n,m}|_n \xrightarrow{m\to\infty} 0 \text{ and } \|f(v_{n,m})\| \geq 1 \ \ \forall m.$$

Moreover, we may assume that $|v_{n,n}|_n < \frac{1}{n}$. Then the sequence $(v_{n,n})_n$ converges to 0 in V but $\|f(v_{n,n})\| \geq 1$ for all n, which is a contradiction with the continuity of f. ☐

Lemma 3.32. *Let $W = \varinjlim_n W_n$ and $W' = \varinjlim_m W'_m$ be LF spaces presented as a filtered colimit of Fréchet spaces by injective transition maps. Then*

$$\underline{\mathrm{Hom}}_K(W, W') = \varprojlim_n \varinjlim_m \underline{\mathrm{Hom}}_K(W_n, W'_m).$$

Proof. First observe that formally

$$\underline{\mathrm{Hom}}_K(W, W') = \varprojlim_n \underline{\mathrm{Hom}}_K(W_n, W'),$$

so we can assume that $W = W_0$ is a Fréchet space. Let S be an extremally disconnected set, then by nuclearity of W'

$$\begin{aligned}\underline{\mathrm{Hom}}_K(W, \varinjlim_m W'_m)(S) &= \mathrm{Hom}_K(W \otimes_K K[S], \varinjlim_m W'_m) \\ &= \mathrm{Hom}_K(W, \varinjlim_m W'_m \otimes_{K_\blacksquare} \underline{\mathrm{Cont}}(S, K)),\end{aligned}$$

which shows that one can reduce to proving

$$\mathrm{Hom}_K(W, \varinjlim_m W'_m) = \varinjlim_m \mathrm{Hom}_K(W, W'_m).$$

So let

$$W \to \varinjlim_m W'_m$$

be a map of solid K-vector spaces. Evaluating at an extremally disconnected set S, by Remark 3.30 we get a map

$$f : W(S)_{\mathrm{top}} \to \varinjlim_{m} W'_m(S)_{\mathrm{top}}$$

between a (classical) Fréchet space and a (classical) LF space. We claim that this map factors through some m. Indeed, this follows from [Sch02, Corollary 8.9], but we also give a direct argument. Assume not. Then there exists a sequence $(x_m)_{m\geq 1}$ in $W(S)_{\mathrm{top}}$ such that $f(x_m) \notin W'_m(S)_{\mathrm{top}}$. Multiplying x_m by big powers of p, we can assume $x_m \to 0$ as $m \to +\infty$. This translates into the existence of a map

$$\mathbf{N} \cup \{\infty\} \to \varinjlim_{m} W'_m(S)_{\mathrm{top}}, \quad m \mapsto f(m), \infty \mapsto 0$$

from the profinite set $\mathbf{N} \cup \{\infty\}$ into an LF-space. Since $\mathbf{N} \cup \{\infty\}$ is profinite, this maps must factorise through some m, which is a contradiction. This shows that, for each extremally disconnected set S, there exists a smallest $n(S) \in \mathbf{N}$ such that the map

$$W(S)_{\mathrm{top}} \to \varinjlim_{m} W'_m(S)_{\mathrm{top}}$$

factors through $W(S)_{\mathrm{top}} \to W'_{n(S)}(S)_{\mathrm{top}} \to \varinjlim_m W'_m(S)_{\mathrm{top}}$. We conclude the proof by showing that the $n(S)'s$ are uniformly bounded. We argue again by contradiction. Assume that there are extremally disconnected sets $S_1, S_2, \dots$ such that $n(S_i) \to +\infty$ as $i \to +\infty$. Let $S = \prod_i S_i$, which is a profinite set. Let $\widetilde{S}$ be an extremally disconnected set surjecting to S. Let $i \in \mathbf{N}$ be such that $n(S_i) > n(\widetilde{S})$ and let $S_i \to \widetilde{S}$ be a section of the surjection $\widetilde{S} \to S \to S_i$. Then the map $W(S_i)_{\mathrm{top}} \to \varinjlim_m W'_m(S_i)_{\mathrm{top}}$ factors through

$$W(S_i)_{\mathrm{top}} \to W(\widetilde{S})_{\mathrm{top}} \to W'_{n(\widetilde{S})}(\widetilde{S})_{\mathrm{top}} \to W'_{n(\widetilde{S})}(S_i)_{\mathrm{top}},$$

which is a contradiction. This finishes the proof. $\square$

3.5. **Spaces of compact type.** Before proving the duality between Fréchet and LS spaces let us recall the definition of (classical) nuclear Fréchet space and an LB space of compact type. Due to the fact that Fréchet spaces are always nuclear in the world of solid K-vector spaces (Proposition 3.29), we will say that a classical nuclear Fréchet space is a Fréchet space of compact type. We recall the definition of trace class maps and compact maps for $K_{\blacksquare}$-vector spaces.

Definition 3.33 ([CS20, Definition 13.11])**.**

(1) A trace class map of Smith spaces is a K-linear map $f : Q_1 \to Q_2$ such that there is a map $g : K \to Q_1^{\vee} \otimes_{K_{\blacksquare}} Q_2$ such that f is the composition $Q_1 \xrightarrow{1\otimes g} Q_1 \otimes Q_1^{\vee} \otimes Q_2 \to Q_2$.
(2) A map of Banach spaces is compact if its dual is a trace class map.

Definition 3.34.

(1) A Fréchet space V is of compact type if it has a presentation $V = \varprojlim_n V_n$ as an inverse limit of Banach spaces where the maps $V_{n+1} \to V_n$ are compact.
(2) An LS space is of compact type if it admits a presentation $W = \varinjlim_{n\in\mathbf{N}} W_n$ by injective trace class maps of Smith spaces.

(3) Let W be a Smith space with lattice $W^0 \subset W$. We denote by W^B the Banach space whose underlying space is

$$W^B = \varprojlim_{s\in \mathbf{N}} (\underline{W^0(*)_{\mathrm{dis}}}/\varpi^s)[\frac{1}{\varpi}].$$

In other words, W^B is the Banach space attached to the underlying set $W(*)$ with unit ball $W^0(*)$. Note that there is a natural injective map with dense image $W^B \to W$.

(4) Let V be a Banach space, we denote $V^S = (V^{\vee,B})^\vee$ and call this space the "Smith completion of V". Note that there is an injective map with dense image $V \to V^S$.

Remark 3.35. Let W be a Smith space, the formation of W^B is independent of the lattice $W^0 \subset W$, namely, if W'^0 is a second lattice we can find $s \in \mathbf{N}$ big enough such that $\varpi^s W^0 \subset W'^0 \subset \varpi^{-s}W^0$. On the other hand, note that $W^B(*) = W(*)$ as sets but not as topological spaces, this follows from the fact that $W^0(*)_{\mathrm{top}}$ is ϖ-adically complete.

The following two results will be very useful later when studying spaces of compact type. The reader can compare them to the ideas appearing in [Sch02, §16] (see, e.g., the discussion after [Sch02, Proposition 16.5]).

Lemma 3.36.

(1) *A map of Smith spaces $W \to W'$ is trace class if and only if it factors as $W \to W'^B \to W'$. Dually, a map of Banach spaces $V \to V'$ is of compact type if and only of it can be extended to $V \to V^S \to V'$.*

(2) *Let $f : V \to W$ be a map from a Banach space to a Smith space, then f extends uniquely to a commutative diagram*

$$\begin{array}{ccc} V & \longrightarrow & V^S \\ \downarrow & & \downarrow \\ W^B & \longrightarrow & W. \end{array}$$

Proof. (1) The factorisation for a morphism of Banach spaces follows from the one of Smith spaces. Let $f : W \to W'$ be a map of Smith spaces and suppose that it factors through W'^B. Then f belongs to $\mathrm{Hom}(W, W'^B) = W^\vee \otimes_{K_\blacksquare} W'^B(*)$ as W'^B is Banach. This shows that f is a trace class map. Conversely, let $f : W \to W'$ be trace class, then there is $g : K \to W^\vee \otimes_{K_\blacksquare} W'$ such that f factors as a composition $W \xrightarrow{1\otimes g} W \otimes_{K_\blacksquare} W^\vee \otimes_{K_\blacksquare} W' \to W'$. Since K is ϖ-adically complete, the map g factors through the ϖ-completion of $(W^\vee \otimes_{K_\blacksquare} W')_{dis}$ (i.e. the condensed object associated to the ϖ-adic completion of the underlying discrete objects), which we denote by $(W^\vee \otimes_{K_\blacksquare} W')^B$. We claim that $(W^\vee \otimes_{K_\blacksquare} W')^B = W^\vee \otimes_{K_\blacksquare} W'^B$, this shows part (1) since f factors as

$$W \xrightarrow{1\otimes g} W \otimes_{K_\blacksquare} W^\vee \otimes_{K_\blacksquare} W'^B \to W'^B \to W'.$$

Let us prove the claim, by piking basis we can write $W^\vee = \widehat{\bigoplus}_I K$ and $W' = \prod_J \mathcal{O}_K[\frac{1}{\varpi}]$. One computes that $W^\vee \otimes_{K_\blacksquare} W' = \widehat{\bigoplus}_I \prod_J \mathcal{O}_K[\frac{1}{\varpi}]$, so that $(W^\vee \otimes$

$W')^B = \widehat{\bigoplus}_I (\prod_J \mathcal{O}_K)^B[\frac{1}{\varpi}]$, where $(\prod_J \mathcal{O}_K)^B$ is the ϖ-adic completion of the discrete space $(\prod_J \mathcal{O}_K)_{dis}$. On the other hand, one has

$$\begin{aligned} W^\vee \otimes_{K_\blacksquare} W'^B &= (\widehat{\bigoplus}_I \mathcal{O}_K \otimes_{\mathcal{O}_{K,\blacksquare}} (\prod_J \mathcal{O}_K)^B)[\frac{1}{\varpi}] \\ &= \varprojlim_s (\bigoplus_I \mathcal{O}_K/\varpi_s \otimes_{\mathcal{O}_K} (\prod_J \mathcal{O}_K/\varpi^s)_{dis})[\frac{1}{\varpi}] \\ &= \widehat{\bigoplus}_I (\prod_J \mathcal{O}_K)^B[\frac{1}{\varpi}]. \end{aligned}$$

(2) Let W^0 be a lattice of W. Since $W^0(*)_{\text{top}}$ is open in $W(*)_{\text{top}}$, there is a lattice $V^0 \subset V$ such that $f(V^0) \subset W^0$. Moreover, $W^0(*)_{\text{top}}$ is ϖ-adically complete, so that the restriction of f to V^0 factors through $V^0 \to \varprojlim_s \underline{(W^0(*)_{dis}/\varpi^s)} \to W^0$. One gets the factorisation $V \to W^B \to W$ by inverting ϖ. Taking duals we see that $f^\vee : W^\vee \to V^\vee$ factors as $W^\vee \to V^{\vee,B} \to V$, taking duals again one gets the factorisation $f : V \to V^S \to W$. The uniqueness follows from the density of V_{top} in $V^S(*)_{\text{top}}$ (resp. $W^B(*)_{\text{top}}$ in $W(*)_{\text{top}}$) and the fact that all the condensed sets involved are quasiseparated. □

Remark 3.37. Part (2) of Lemma 3.36 shows that W^B is the final object in the category of all Banach spaces V equipped with an arrow $V \to W$ and morphisms given by maps of $K_\blacksquare$-vector spaces making the obvious diagram commute. Dually, V^S is the initial object in the category of all Smith spaces W equipped with an arrow $V \to W$ and morphisms given by maps commuting with the obvious diagram.

Corollary 3.38.

(1) *Let $V = \varprojlim_n V_n$ be a Fréchet space of compact type. Then we can write $V = \varprojlim_{n\in N} V_n^S$ as an inverse limit of Smith spaces with trace class transition maps. Conversely, any such vector space is a Fréchet space of compact type.*

(2) *Let $W = \varinjlim_n W_n$ be an LS space of compact type, then $W = \varinjlim_n W_n^B$ can be written as a filtered colimit of Banach spaces with injective compact maps. Conversely, a colimit of Banach spaces by injective compact maps is an LS space of compact type. In particular, being an LB or an LS space of compact type is equivalent.*

Lemma 3.39. *Let*

$$0 \to V' \to V \to V'' \to 0 \tag{9}$$

be a short exact sequence of Fréchet (resp. LS) spaces. Then (9) can be written as an inverse limit (resp. direct limit) of short exact sequences of Banach (resp. Smith) spaces.

Proof. Let us suppose that (9) is a short exact sequence of LS spaces, let $n \in \mathbf{N}$ and write $V'' = \varinjlim_n V_n''$ as a colimit of Smith spaces by injective transition maps. The pullback of (9) by $V_n'' \to V''$ is a short exact sequence

$$0 \to V' \to V \times_{V''} V_n'' \to V_n'' \to 0,$$

where $V \times_{V''} V_n''$ is still an LS space. In fact, by writing $V = \varinjlim_n \widetilde{V}_n$ as a colimit of Smith spaces one has $V \times_{V''} V_n'' = \varinjlim_n \widetilde{V}_n \times_{V''} V_n''$ and each $\widetilde{V}_n \times_{V''} V_n''$ is a Smith space. By compacity of V_n'', there exists a Smith subspace $V_n \subset V \times_{V''} V_n''$

such that the induced arrow $V_n \to V_n''$ is surjective. By Proposition 3.9 the kernel $V_n' := \ker(V_n \to V_n'')$ is a Smith subspace of V'. Moreover, we can take V_n such that $V = \varinjlim_n V_n$. Then (9) can be written as the colimit of the short exact sequences

$$\varinjlim_n [0 \to V_n' \to V_n \to V_n'' \to 0].$$

For the Fréchet case, it is enough to show that the map $V \to V'' \to 0$ can be written as an inverse limit of surjective maps of Banach spaces $V_n \to V_n'' \to 0$. In fact, by setting $V_n' := \ker(V_n \to V_n'')$ one has $V' = \varprojlim_n V_n'$. Let us write $V = \varprojlim_n V_n$ as an inverse limit of Banach spaces with dense transition maps. The space V' is identified with the kernel of $V \to V''$. Let $V_n' \subset V_n$ be the closure of the image of $V'(*)_{\mathrm{top}}$ in $V_n(*)_{\mathrm{top}}$ and set $V_n'' := V_n/V_n'$. Then $V \to V_n \to V_n''$ factors through V''. Moreover, the map $V_n \to V_n''$ is surjective by construction and one easily checks that $V'' = \varprojlim_n V_n''$. □

3.6. **Duality.** We conclude with the main result of this section.

Theorem 3.40.

(1) *The functor $V \mapsto \underline{\mathrm{Hom}}_K(V, K)$ induces an exact anti-equivalence between Fréchet and LS spaces such that $\underline{\mathrm{Hom}}_K(V, V') = \underline{\mathrm{Hom}}_K(V'^\vee, V^\vee)$, extending the one between Banach and Smith spaces. Moreover, V is Fréchet of compact type if and only if $V^\vee$ is an LS space of compact type.*

(2) *Let $V = \varprojlim_{n\in N} V_n$ be a Fréchet space and $W = \varinjlim_{n\in \mathbf{N}} W_n$ an LS space. Then*

$$\underline{\mathrm{Hom}}_K(W, V) = W^\vee \otimes_{K_\blacksquare} V \text{ and } \underline{\mathrm{Hom}}_K(V, W) = V^\vee \otimes_{K_\blacksquare} W.$$

In particular, if V and V' are Fréchet spaces (resp. LS spaces) then

$$(V \otimes_{K_\blacksquare} V')^\vee = V^\vee \otimes_{K_\blacksquare} V'^\vee.$$

Proof. (1) Let V be a Fréchet space and let $V = \varprojlim_n V_n$ be a presentation as an inverse limit of Banach spaces with transition maps of dense image. Let S by an extremally disconnected set, we want to compute

$$\underline{\mathrm{Hom}}_K(V, K)(S) = \mathrm{Hom}_K(V \otimes_{\mathcal{O}_{K,\blacksquare}} \mathcal{O}_{K,\blacksquare}[S], K) = \mathrm{Hom}_K(\varprojlim_n V_n, \underline{\mathrm{Cont}}(S, K)).$$

By Lemma 3.31 we have

$$\begin{aligned}\mathrm{Hom}_K(\varprojlim_n V_n, \underline{\mathrm{Cont}}(S, K)) &= \varinjlim_n \mathrm{Hom}_K(V_n, \underline{\mathrm{Cont}}(S, K)) \\ &= \varinjlim_n \mathrm{Hom}_K(V_n \otimes_{\mathcal{O}_{K,\blacksquare}} \mathcal{O}_{K,\blacksquare}[S], K).\end{aligned}$$

In other words, we have a natural isomorphism

$$\underline{\mathrm{Hom}}_K(V, K) = \varinjlim_n V_n^\vee.$$

By Lemma 3.19(2), the transition maps $V_n^\vee \to V_{n+1}^\vee$ are injective, proving that $V^\vee$ is an LS space.

Conversely, let W be an LS space and $W = \varinjlim_n W_n$ a presentation as a colimit of Smith spaces with injective transition maps. Then it follows formally that

$$\underline{\mathrm{Hom}}_K(W, K) = \varprojlim_n W_n^\vee.$$

By Lemma 3.19(2) again, the transition maps $W_{n+1}^{\vee} \to W_n^{\vee}$ have dense image. It is clear from the construction that $(V^{\vee})^{\vee} = V$ and $(W^{\vee})^{\vee} = W$ for V Fréchet and W an LS space. This implies formally that $\mathrm{Hom}_K(V, V') = \mathrm{Hom}_K(V^{'\vee}, V^{\vee})$, which gives the anti-equivalence between Fréchet and LS spaces. Finally, by Corollary 3.38 and the previous computation, the duality restricts to Fréchet and LS spaces of compact type.

We now extend the equality of homomorphisms to the internal $\underline{\mathrm{Hom}}$. Let V and V' be Fréchet spaces and S an extremally disconnected set. We have

$$\begin{aligned} \underline{\mathrm{Hom}}_K(V, V')(S) &= \mathrm{Hom}_K(V \otimes_{K_\blacksquare} K_\blacksquare[S], V') \\ &= \mathrm{Hom}_K(V, \underline{\mathrm{Cont}}(S, K) \otimes_{K_\blacksquare} V') \\ &= \mathrm{Hom}_K((\underline{\mathrm{Cont}}(S, K) \otimes_{K_\blacksquare} V')^{\vee}, V^{\vee}) \\ &= \mathrm{Hom}_K(V^{'\vee} \otimes_{K_\blacksquare} K_\blacksquare[S], V^{\vee}) \\ &= \underline{\mathrm{Hom}}_K(V^{'\vee}, V^{\vee})(S). \end{aligned}$$

The second equality follows from adjunction and nuclearity of V'. The third equality is the duality between Fréchet and LS spaces. The fourth equality follows from the compatibility of the tensor product and duality between Fréchet and LS spaces of part (2).

To prove exactness of the functor $V \mapsto V^{\vee}$, it is enough to see that it sends short exact sequences

$$0 \to V' \xrightarrow{f} V \xrightarrow{f} V'' \to 0 \tag{10}$$

of Fréchet spaces (resp. LS spaces) to short exact sequences of LS spaces (resp. Fréchet spaces). Note that we are taking exact sequences in the category of solid K-vector spaces, in particular, $V''(*)_{\mathrm{top}}$ is the topological quotient $V(*)_{\mathrm{top}}/V'(*)_{\mathrm{top}}$ and the sequence

$$0 \to V'(*)_{\mathrm{top}} \xrightarrow{f(*)} V(*)_{\mathrm{top}} \xrightarrow{g(*)} V''(*)_{\mathrm{top}} \to 0$$

is strict exact in classical terms, i.e., it is exact as a sequence of K-vector spaces and the map $V'(*)_{\mathrm{top}} \to \ker(g(*))$ is an isomorphism of topological K-vector spaces. We first prove the result for Smith and Banach spaces. Given a short exact sequence (10) of Smith spaces, by projectivity of V'' one has a section $s : V'' \to V$ splitting the short exact sequence, and hence the claim follows. Dually, if (10) is a short exact sequence of Banach spaces, one can construct a section $s : V'' \to V$ by taking orthonormalisable basis. We finally reduce the general case to Smith and Banach spaces to conclude the proof of the statement. Indeed, if

$$0 \to V' \to V \to V'' \to 0$$

is an exact sequence of LS spaces, then by Lemma 3.39 it can be written as a direct limit

$$\varinjlim_{n \in \mathbf{N}} [0 \to V'_n \to V_n \to V''_n \to 0]$$

of short exact sequences of Smith spaces. Taking duals and using exactness for the case of Smith spaces, we get an inverse limit of short exact sequences

$$0 \to (V''_n)^{\vee} \to V_n^{\vee} \to (V''_n)^{\vee} \to 0.$$

Since $R^1 \varprojlim_n (W'_n)^\vee = 0$ by topological Mittag-Leffler, one has a short exact sequence

$$0 \to (V')^\vee \to V^\vee \to (V'')^\vee \to 0$$

of Fréchet spaces as desired. The reduction for exact sequences of Fréchet spaces to the case of Banach spaces is deduced similarly.

(2) Let $V = \varprojlim_n V_n$ and $W = \varinjlim_m W_m$ be a Fréchet and an LS space respectively. We can write

$$\begin{aligned} \underline{\mathrm{Hom}}_K(W,V) &= \varprojlim_m \underline{\mathrm{Hom}}_K(W_m,V) \\ &= \varprojlim_m W_m^\vee \otimes_{K_\blacksquare} V \\ &= W^\vee \otimes_{K_\blacksquare} V, \end{aligned}$$

where the first equality is formal, the second equality follows from nuclearity of Fréchet spaces, and the third equality from the tensor product of Fréchet spaces (Lemma 3.28). Dually, we have

$$\begin{aligned} \underline{\mathrm{Hom}}_K(V,W) &= \underline{\mathrm{Hom}}_K(V \otimes_{K_\blacksquare} W^\vee, K) \\ &= \underline{\mathrm{Hom}}_K(\varprojlim_{n,m} V_n \otimes_{K_\blacksquare} W_m^\vee, K) \\ &= \varinjlim_{n,m} \underline{\mathrm{Hom}}_K(V_n \otimes_{K_\blacksquare} W_m^\vee, K) \\ &= \varinjlim_{n,m} V_n^\vee \otimes_{K_\blacksquare} W_m \\ &= V^\vee \otimes_{K_\blacksquare} W, \end{aligned}$$

where the first equality follows from self-duality of LS spaces and the tensor-Hom adjunction, the second equality from the tensor product of Fréchet spaces, the third equality from Lemma 3.31, and the last two equalities from Corollary 3.17 and the commutativity between tensor product and colimits. □

We conclude this section with a conjecture which is a derived enhancement of Lemma 3.31. This will only be used in Proposition 4.43 and the reader is invited to skip it in a first lecture.

Conjecture 3.41. *Let $V = \varprojlim_n V_n$ be a Fréchet space of compact type written as a limit of Banach spaces with compact and dense transition maps, then*

$$R\underline{\mathrm{Hom}}_K(V,K) = \operatorname*{hocolim}_n R\underline{\mathrm{Hom}}_K(V_n,K).$$

Corollary 3.42. *Suppose that Conjecture 3.41 is true. Let $V = \varprojlim_n V_n$ be a Fréchet space of compact type and B a Banach space, then*

(1) $R\underline{\mathrm{Hom}}_K(V,B) = \mathrm{hocolim}_n R\underline{\mathrm{Hom}}_K(V_n,B)$.
(2) *We have*

$$R\underline{\mathrm{Hom}}_K(V,B) = V^\vee \otimes_{K_\blacksquare} B.$$

In particular $V^\vee$ is the derived dual of V.
(3) *The duality Theorem 3.40 gives a derived duality between bounded complexes of Fréchet spaces of compact type and bounded complexes of LS spaces of compact type.*

(4) *Let C be a bounded complex of Fréchet spaces of compact type and D a bounded complex of LS spaces of compact type. Then*

$$R\underline{\mathrm{Hom}}_K(C,D) = R\underline{\mathrm{Hom}}_K(C,K) \otimes^L_{K_\blacksquare} D$$

and

$$R\underline{\mathrm{Hom}}_K(D,C) = R\underline{\mathrm{Hom}}_K(D,K) \otimes^L_{K_\blacksquare} C.$$

(5) *Let C and C' be bounded complexes of Fréchet spaces of compact type. Then*

$$R\underline{\mathrm{Hom}}_K(C,C') = R\underline{\mathrm{Hom}}_K(C'^{\vee}, C^{\vee}).$$

Proof. Part (1) follows from Conjecture 3.41 using the fact that $K_\blacksquare[S] \otimes^L_{K_\blacksquare} K_\blacksquare[S'] = K_\blacksquare[S\times S']$ for S and S' profinite sets, that

$$R\underline{\mathrm{Hom}}_K(V,K)(S) = R\mathrm{Hom}_K(V\otimes_{K_\blacksquare} K_\blacksquare[S], K) = R\mathrm{Hom}_K(V, \underline{\mathrm{Cont}}(S,K))$$

and that any Banach space over K is a direct summand of a space of the form $\underline{\mathrm{Cont}}(S,K)$ for S a profinite set.

If V is Fréchet of compact type, we can write $V = \varprojlim_n V_n^S$ where V_n^S is the Smith completion of V_n (cf. Corollary 3.38). Then

$$\begin{aligned} R\underline{\mathrm{Hom}}_K(V,B) &= \operatorname*{hocolim}_n R\underline{\mathrm{Hom}}_K(V_n,B) \\ &= \operatorname*{hocolim}_n R\underline{\mathrm{Hom}}_K(V_n^S,B) \\ &= \operatorname*{hocolim}_n (V_n^S)^{\vee} \otimes_{K_\blacksquare} B \\ &= \varinjlim_n V_n^{\vee} \otimes_{K_\blacksquare} B = V^{\vee} \otimes_{K_\blacksquare} B. \end{aligned}$$

Part (3) follows immediately from part (2) and an easy induction via the stupid truncation. For part (4), notice that if C is a bounded Fréchet complex of compact type, and D is a bounded LS complex of compact type, then

$$\begin{aligned} R\underline{\mathrm{Hom}}_K(C,D) &= R\underline{\mathrm{Hom}}_K(C\otimes^L_{K_\blacksquare} D^{\vee}, K) \\ &= C^{\vee} \otimes^L_{K_\blacksquare} D \end{aligned}$$

since $C\otimes^L_{K_\blacksquare} D^{\vee}$ is a bounded Fréchet complex of compact type. Similarly one shows $R\underline{\mathrm{Hom}}_K(D,C) = D^{\vee} \otimes^L_K C$.

Finally, by a dévissage using the stupid filtration, (5) is reduced to showing that if V and V' are Fréchet spaces of compact type then

$$R\underline{\mathrm{Hom}}_K(V,V') = R\underline{\mathrm{Hom}}_K(V'^{\vee}, V^{\vee}).$$

Writing $V' = \varprojlim_n V'_n$ one gets, by (2),

$$R\underline{\mathrm{Hom}}_K(V,V') = R\varprojlim_n R\underline{\mathrm{Hom}}_K(V,V'_n) = R\varprojlim_n (V^{\vee} \otimes_{K_\blacksquare} V'_n).$$

Dually, we have that

$$\begin{aligned} R\underline{\mathrm{Hom}}_K(V'^{\vee}, V^{\vee}) &= R\varprojlim_n R\underline{\mathrm{Hom}}_K(V_n'^{,\vee}, V^{\vee}) \\ &= R\varprojlim_n (V'_n \otimes_{K_\blacksquare} V^{\vee}) \end{aligned}$$

which gives the desired equality. $\square$

4. Representation theory

Let G be a compact p-adic Lie group. In this section we translate the theory of analytic and locally analytic representations of G from the classical framework to condensed mathematics. We hope that this new point of view will simplify some proofs and provide a better understanding of the theory. Our main sources of inspiration are the works of Lazard [Laz65], Schneider-Teitelbaum [ST02b, ST03] and Emerton [Eme17].

We begin with the introduction of various algebras of distributions, each one serving to a particular purpose in the theory. Namely, there are distribution algebras arising from affinoid groups which appear naturally in the analytification functor of Corollary 2.19, distribution algebras which are localizations of the Iwasawa algebra, and distribution algebras attached to Stein analytic groups relating the previous two.

Next, we introduce the category of solid G-modules, it is a generalisation of the category of non-archimedean topological spaces endowed with a continuous action of G (Banach, Fréchet, LB, LF, $\ldots$).

We continue with the definition of analytic representations for the affinoid and Stein analytic groups of Definition 4.4. We recall how the analytic vectors for a Banach representation are defined, and how it serves as motivation for the derived analytic vectors of solid G-modules. We prove the main Theorem 4.36 which, roughly speaking, says that being analytic for a Stein analytic group as in Definition 4.4 is the same as being a module over its distribution algebra.

We end with an application to locally analytic representations, reproving a theorem of Schneider-Teitelbaum describing a duality between locally analytic representations on LB spaces of compact type and Fréchet modules of compact type over the algebra of locally analytic distributions, cf. Proposition 4.41. Under the assumption of Conjecture 3.41, we state a generalisation to a duality between locally analytic bounded complexes of LB spaces of compact type and bounded Fréchet complexes of compact type endowed with an action of the algebra of locally analytic distributions.

4.1. **Function spaces and distribution algebras.** In the following subsections we define different classes of spaces of functions and distributions that will be used throughout this text. These are algebras already appearing in the literature ([ST03, §4], [Eme17, §5]) that we introduce in a way adapted to our interests and purposes.

4.1.1. *Rigid analytic neighbourhoods.* Let G be a compact p-adic Lie group of dimension d.

Definition 4.1. The Iwasawa algebra of G is defined as the solid ring

$$\mathcal{O}_{K,\blacksquare}[G] = \varprojlim_{H \trianglelefteq G} \mathcal{O}_K[G/H] \in \mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_K},$$

$$K_\blacksquare[G] = \mathcal{O}_{K,\blacksquare}[G][1/p] = (\varprojlim_{H \trianglelefteq G} \mathcal{O}_K[G/H])[1/p] \in \mathrm{Mod}^{\mathrm{solid}}_K,$$

where H runs over all the open and normal subgroups of G.

Remark 4.2. Classically, the Iwasawa algebra is denoted by $\Lambda(G, \mathcal{O}_K)$ and $\Lambda(G, K)$. We decide to adopt the new notations from the theory of condensed mathematics to best fit our results in this language.

Remark 4.3. According to the notations of Example 2.9, the Iwasawa algebra corresponds precisely to the evaluation at G of the functor of measures of the analytic rings $(\mathcal{O}_K, \mathcal{O}_K)_{\blacksquare}$ and $(K, \mathcal{O}_K)_{\blacksquare}$ which, in turn, are commonly denoted by $\mathcal{O}_{K,\blacksquare}$ and $K_{\blacksquare}$ respectively.

In order to define the space of locally analytic functions of G, we need to work with coordinates locally around the identity. By [DdSMS99, Corollary 8.34], there exists an open normal subgroup G_0 of G which is a uniform pro-p group.[4] Such a group can be equipped with a p-valuation w and an ordered basis $g_1, \dots, g_d \in G_0$ in the sense of [Laz65, III.2] (or cf. [ST03, §4]). By [Laz65, Proposition III.3.1.3], after shrinking G_0 if necessary, we can assume that $w(g_1) = \ldots = w(g_d) = w_0 > 1$ is an integer. This basis induces charts

$$\phi : \mathbf{Z}_p^d \to G_0, \quad (x_1, \dots, x_d) \mapsto g_1^{x_1} \cdot \ldots \cdot g_d^{x_d}$$

such that ϕ is a homeomorphism between G_0 and $\mathbf{Z}_p^d$ with $w(g_1^{x_1} \cdot \ldots \cdot g_d^{x_d}) = w_0 + \min_{1 \leq i \leq d} v_p(x_i)$, and such that the map $\psi : G_0 \times G_0 \to G_0$, $(g, h) \mapsto gh^{-1}$, defining the group structure of G_0, is given by an analytic function $\psi : \mathbf{Z}_p^d \times \mathbf{Z}_p^d \to \mathbf{Z}_p^d$ with coefficients in $\mathbf{Z}_p$. In other words, the function $\psi : (x, y) \mapsto \phi^{-1}(\phi(x)\phi(y)^{-1})$ can be written as a tuple of power series with coefficients in $\mathbf{Z}_p$ converging to 0. By further shrinking G_0 if necessary we can also assume that conjugation on G_0 by any element of $g \in G$ is given by a family of power series with bounded coefficients.

Let $r = p^{-s} > 0$ for $s \in \mathbf{Q}$, and $\mathbb{D}^d_{\mathbf{Q}_p}(r) \subset \mathbb{A}^d_{\mathbf{Q}_p}$ the affinoid polydisc of radius r. If $s \in \mathbf{Z}$ then $\mathbb{D}^d_{\mathbf{Q}_p}(r)$ is the affinoid space defined by the algebra $\mathbf{Q}_p\langle \frac{T_1}{p^s}, \dots \frac{T_d}{p^s} \rangle$, where $T_1, \dots, T_d$ are the coordinates of $\mathbb{A}^d_{\mathbf{Q}_p}$. We let $\mathring{\mathbb{D}}^d_{\mathbf{Q}_p}(r) = \bigcup_{r' < r} \mathbb{D}^d_{\mathbf{Q}_p}(r')$ denote the open polydisc of radius r.

Definition 4.4. We define the following rigid analytic groups, see Lemma 4.6.

(1) $\mathbb{G}_0 = (\mathbb{D}^d_{\mathbf{Q}_p}(1), \psi)$; the affinoid group defined by the group law ψ of G_0.
(2) For any $h \in \mathbf{Q}_{\geq 0}$, the affinoid groups $\mathbb{G}_h = (\mathbb{D}^d_{\mathbf{Q}_p}(p^{-h}), \psi)$ of radius p^{-h}. We also denote $\mathbb{G}_0^{(h)} = G_0 \mathbb{G}_h \subset \mathbb{G}_0$.
(3) For any $h \in \mathbf{Q}_{\geq 0}$ the Stein groups $\mathbb{G}_{h^+} = \cup_{h' > h} \mathbb{G}_{h'}$ and $\mathbb{G}_0^{(h^+)} = G_0 \mathbb{G}_{h^+} = \cup_{h' > h} \mathbb{G}_0^{(h')}$.

Example 4.5. If $G = \mathbf{GL}_2(\mathbf{Z}_p)$, an example of a uniform pro-p-subgroup G_0 is

$$G_0 = \begin{pmatrix} 1 + p^n \mathbf{Z}_p & p^n \mathbf{Z}_p \\ p^n \mathbf{Z}_p & 1 + p^n \mathbf{Z}_p \end{pmatrix}$$

for $n \geq 2$ if $p = 2$ and $n \geq 1$ if $p > 2$. Let's take $p > 2$ and $n = 1$. In this case $\mathbb{G}_h$ is the rigid analytic group

$$\begin{aligned} \mathbb{G}_h &= \begin{pmatrix} 1 + \mathbb{D}^1_{\mathbf{Q}_p}(p^{-h-1}) & \mathbb{D}^1_{\mathbf{Q}_p}(p^{-h-1}) \\ \mathbb{D}^1_{\mathbf{Q}_p}(p^{-h-1}) & 1 + \mathbb{D}^1_{\mathbf{Q}_p}(p^{-h-1}) \end{pmatrix} \\ &= \begin{pmatrix} 1 + p^{h+1}\mathbb{D}^1_{\mathbf{Q}_p}(1) & p^{h+1}\mathbb{D}^1_{\mathbf{Q}_p}(1) \\ p^{h+1}\mathbb{D}^1_{\mathbf{Q}_p}(1) & 1 + p^{h+1}\mathbb{D}^1_{\mathbf{Q}_p}(1) \end{pmatrix}, \end{aligned}$$

[4] Recall that a uniform pro-p group H is a pro-p group which is finitely generated, torsion free and powerful, i.e., $[H, H] \subseteq H^p$ if $p > 2$ or $[H, H] \subseteq H^4$ if $p = 2$.

whereas the Stein group $\mathbb{G}_{h^+}$ is equal to

$$\mathbb{G}_{h^+} = \begin{pmatrix} 1+p^{h+1}\mathring{\mathbb{D}}^1_{\mathbf{Q}_p}(1) & p^{h+1}\mathring{\mathbb{D}}^1_{\mathbf{Q}_p}(1) \\ p^{h+1}\mathring{\mathbb{D}}^1_{\mathbf{Q}_p}(1) & 1+p^{h+1}\mathring{\mathbb{D}}^1_{\mathbf{Q}_p}(1) \end{pmatrix}.$$

Lemma 4.6 says that the rigid spaces of Definition 4.4 are indeed rigid analytic groups.

Lemma 4.6. *The affinoid $\mathbb{G}_h$ is an open normal subgroup of $\mathbb{G}_0$ stable by conjugation of G.*

Proof. By simplicity we suppose that $h \in \mathbb{N}$. The map

$$\psi : \mathbb{G}_0 \times \mathbb{G}_0 \to \mathbb{G}_0 \quad : \quad (x,y) \mapsto xy^{-1}$$

is defined by a family of power series $(Q_1(X,Y), \ldots, Q_d(X,Y))$ with integral coefficients satisfying the group axioms. In particular, one has $Q_i(0,0) = 0$. The inclusion $\mathbb{G}_h \to \mathbb{G}_0$ is given by the map $\mathbf{Q}_p\langle T_1, \ldots, T_d\rangle \to \mathbf{Q}_p\langle \frac{T_1}{p^h}, \ldots, \frac{T_d}{p^h}\rangle$. Thus, the image of $\frac{T_i}{p^h}$ by the multiplication map ψ is equal to

$$\frac{1}{p^h}Q_i(X,Y) = \frac{1}{p^h}\sum_{(\alpha,\beta)\neq 0} a_{\alpha,\beta}X^\alpha Y^\beta = \sum_{(\alpha,\beta)\neq 0} a_{\alpha,\beta}p^{h(|\alpha|+|\beta|-1)}\left(\frac{X}{p^h}\right)^\alpha \left(\frac{Y}{p^h}\right)^\beta.$$

This shows that ψ restricts to a map $\mathbb{G}_h \times \mathbb{G}_h \to \mathbb{G}_h$, proving that $\mathbb{G}_h$ is a subgroup of $\mathbb{G}_0$. A similar argument shows that $\mathbb{G}_h$ is normal in $\mathbb{G}_0$ and that it is stable by the conjugation of elements of G. $\square$

4.1.2. *Analytic distributions.* Classically, the analytic vectors of Banach representations are defined via the affinoid algebras of the analytic groups of Definition 4.4, see [Eme17, §3]. In order to develop properly this theory for solid K-vector spaces we shall need to introduce some notation for the algebras of the analytic groups, as well as for their analytic distributions. Recall once more that we see all complete locally convex K-vector spaces as solid objects.

Definition 4.7.

(1) We consider the following spaces of functions
   (i) $C(\mathbb{G}_0^{(h)}, \mathcal{O}_K) := \mathcal{O}^+(\mathbb{G}_0^{(h)}) \otimes_{\mathbf{Z}_p} \mathcal{O}_K$; the power bounded analytic functions of the affinoid group.
   (ii) $C(\mathbb{G}_0^{(h)}, K) := \mathcal{O}(\mathbb{G}_0^{(h)}) \otimes_{\mathbf{Q}_p} K$; the regular functions of the affinoid group.
   (iii) $C(\mathbb{G}_0^{(h^+)}, K) := \varprojlim_{h'>h^+} C(\mathbb{G}_0^{(h')}, K)$; the regular functions of the Stein group.

(2) We define the following spaces of distributions
   (i) $\mathcal{D}^{(h)}(G_0, \mathcal{O}_K) := \underline{\mathrm{Hom}}_{\mathcal{O}_K}(C(\mathbb{G}_0^{(h)}, \mathcal{O}_K), \mathcal{O}_K)$.
   (ii) $\mathcal{D}^{(h)}(G_0, K) := \underline{\mathrm{Hom}}_K(C(\mathbb{G}_0^{(h)}, K), K)$.
   (iii) $\mathcal{D}^{(h^+)}(G_0, K) := \underline{\mathrm{Hom}}_K(C(\mathbb{G}_0^{(h^+)}, K), K)$.

Remark 4.8. Let $h \in \mathbb{N}$, the algebra $C(\mathbb{G}_0^{(h)}, K)$ is the space of functions on G_0 with values in K which are analytic of radius p^{-h}. More precisely, using the coordinates $g_1, \ldots, g_d \in G_0$, and identifying G_0 with $\mathbf{Z}_p^d$, $C(\mathbb{G}_0^{(h)}, K)$ is the space of functions $f : \mathbf{Z}_p^d \to K$ whose restriction at cosets $x + p^h\mathbf{Z}_p^d$ is given by a convergent power series with coefficients in K. Thus all the function spaces and distributions of Definition 4.7 depend on the choice of the ordered basis $g_1, \ldots, g_d$.

Remark 4.9. Observe that, by Theorem 3.40, one has $C(\mathbb{G}_0^{(h)}, K) = \mathcal{D}^{(h)}(G_0, K)^\vee$ as well as $\mathcal{D}^{(h^+)}(G_0, K) = \varinjlim_{h'>h} \mathcal{D}^{(h')}(G_0, K)$. In addition, for $h' > h$, the inclusions $\mathbb{G}_0^{(h'^+)} \subset \mathbb{G}_0^{(h')} \subset \mathbb{G}_0^{(h^+)}$ induce maps of distributions $\mathcal{D}^{(h'^+)}(G_0, K) \to \mathcal{D}^{(h')}(G_0, K) \to \mathcal{D}^{(h^+)}(G_0, K)$. Moreover, since the inclusion $\mathbb{G}_0^{(h')} \subset \mathbb{G}_0^{(h)}$ is strict, the restriction map $C(\mathbb{G}_0^{(h)}, K) \to C(\mathbb{G}_0^{(h')}, K)$ is compact and hence factors through $C(\mathbb{G}_0^{(h)}, K)^S$. Dually, one has a factorisation $\mathcal{D}^{(h')}(G_0, K) \to \mathcal{D}^{(h)}(G_0, K)^B \to \mathcal{D}^{(h)}(G_0, K)$. In particular, $\mathcal{D}^{(h^+)}(G_0, K) = \varinjlim_{h'>h} \mathcal{D}^{(h')}(G_0, K)^B$ is an LS space of compact type. Finally, observe that these algebras are all either solid Fréchet or LS spaces, so they are the condensed set associated to their underlying topological spaces, which are nothing but the classical distribution algebras endowed with the compact-open topology.

Lemma 4.10. *The solid $\mathcal{O}_K$-module $\mathcal{D}^{(h)}(G_0, \mathcal{O}_K)$ has a natural structure of associative unital $\mathcal{O}_K$-algebra induced by the multiplication map $\mathbb{G}_0^{(h)} \times \mathbb{G}_0^{(h)} \to \mathbb{G}_0^{(h)}$. In particular, $\mathcal{D}^{(h)}(G_0, K)$ and $\mathcal{D}^{(h^+)}(G_0, K)$ are associative unital $\mathcal{O}_K$-algebras.*

Proof. The multiplication $\mathbb{G}_0^{(h)} \times \mathbb{G}_0^{(h)} \to \mathbb{G}_0^{(h)}$ defines a comultiplication map

$$\nabla : C(\mathbb{G}_0^{(h)}, \mathcal{O}_K) \to C(\mathbb{G}_0^{(h)}, \mathcal{O}_K) \otimes_{\mathcal{O}_{K,\blacksquare}} C(\mathbb{G}_0^{(h)}, \mathcal{O}_K).$$

As $C(\mathbb{G}_0^{(h)}, \mathcal{O}_K)$ is an orthonormalisable Banach $\mathcal{O}_K$-module, taking the dual of ∇ one obtains a map $\mathcal{D}^{(h)}(G_0, \mathcal{O}_K) \otimes_{\mathcal{O}_{K,\blacksquare}} \mathcal{D}^{(h)}(G_0, \mathcal{O}_K) \to \mathcal{D}^{(h)}(G_0, \mathcal{O}_K)$ which is easily seen to be the convolution product. $\square$

4.1.3. *Localizations of the Iwasawa algebra.* Recall that we have fixed an open normal subgroup G_0 which is a uniform pro-p-group with basis $g_1, \ldots, g_d$ of constant valuation $w_0 > 1$. Let $\mathbf{b}_i = [g_i] - 1 \in \mathcal{O}_{K,\blacksquare}[G_0](*)$, one has [ST03, §4]

$$\mathcal{O}_{K,\blacksquare}[G_0](*)_{\mathrm{top}} = \prod_{\alpha \in \mathbb{N}^d} \mathcal{O}_K \mathbf{b}^\alpha,$$

where $\mathbf{b}^\alpha = \mathbf{b}_1^{\alpha_1} \cdots \mathbf{b}_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, and the right hand side is a profinite $\mathcal{O}_K$-module. After taking the associated condensed sets we will simply write

$$\mathcal{O}_{K,\blacksquare}[G_0] = \prod_{\alpha \in \mathbb{N}^d} \mathcal{O}_K \mathbf{b}^\alpha.$$

Remark 4.11. Taking Mahler expansion of continuous functions on $\phi : G_0 \cong \mathbf{Z}_p^d$, an explicit computation of finite differences shows that the elements $\mathbf{b}^\alpha$ correspond to the dual basis of the Mahler basis $\binom{x}{\alpha}$ i.e., $\mathbf{b}^\alpha(f) = c_\alpha$ for any continuous function $f \in \mathrm{Cont}(G_0, K)$ such that $\phi^*(f) = \sum_{\alpha \in \mathbf{N}^d} c_\alpha \binom{x}{\alpha}$.

We now introduce a second family of distribution algebras using the basis $(\mathbf{b}_i)_{1 \leq i \leq d}$, these can be thought of as localizations of the Iwasawa algebra $K_\blacksquare[G_0]$.

Definition 4.12. Let $h > 0$ be rational. We define the condensed rings $\mathcal{D}_{(h)}(G_0, \mathcal{O}_K)$ and $\mathcal{D}_{(h)}(G_0, K)$ so that, for any extremally disconnected set S, one has

- $\mathcal{D}_{(h)}(G_0, \mathcal{O}_K)(S) = \{\sum_{\alpha \in \mathbf{N}^d} a_\alpha \mathbf{b}^\alpha \ : \ \sup_\alpha \{|a_\alpha| p^{-\frac{p-h}{p-1}|\alpha|}\} \leq 1, \quad a_\alpha \in \mathrm{Cont}(S, \mathcal{O}_K)\}$,

- $\mathcal{D}_{(h)}(G_0, K)(S) = \{\sum_{\alpha\in\mathbf{N}^d} a_\alpha \mathbf{b}^\alpha \;:\; \sup_\alpha\{|a_\alpha|p^{-\frac{p-h}{p-1}|\alpha|}\} < +\infty, \quad a_\alpha \in \mathrm{Cont}(S,K)\}$.

Remark 4.13. The condensed module $\mathcal{D}_{(h)}(G_0, \mathcal{O}_K)$ is in fact a profinite $\mathcal{O}_K$-module provided $b(h) = \frac{p-h}{p-1} \in v_p(K)\otimes_{\mathbf{Z}} \mathbf{Q}$. Indeed, if $b(h)$ is the valuation of an element of K, we have an isomorphism of profinite $\mathcal{O}_K$-modules

$$\mathcal{D}_{(h)}(G_0, \mathcal{O}_K) = \prod_{\alpha\in\mathbf{N}^d} \mathcal{O}_K\left(\frac{\mathbf{b}_1}{p^{b(h)}}\right)^{\alpha_1}\cdots\left(\frac{\mathbf{b}_d}{p^{b(h)}}\right)^{\alpha_d}.$$

Remark 4.14. The distribution algebras $\mathcal{D}_{(h)}(G_0, K)$ are variations of the (p-adic completions of the) rings $\hat{A}^{(m)}$ of [Eme17, §5.2], adapted to the Iwasawa algebra instead of the enveloping algebra of $\mathrm{Lie}\, G$.

Lemma 4.15. *The multiplication map of $K_\blacksquare[G_0]$ extends uniquely to a multiplication map of $\mathcal{D}_{(h)}(G_0, \mathcal{O}_K)$.*

Proof. This follows directly from [ST03, Proposition 4.2] after taking the associated condensed sets of the corresponding topological spaces. □

One can describe the analytic distributions of Definition 4.7 in terms of the basis $\mathbf{b}_i$ of the Iwasawa algebra.

Proposition 4.16. *Let S be an extremally disconnected set, then*

$$\mathcal{D}^{(h)}(G_0,K)(S)=\left\{\sum_{\alpha\in\mathbf{N}^d} a_\alpha\mathbf{b}^\alpha \;:\; \sup_\alpha\{|a_\alpha|p^{-\frac{p^{-h}|\alpha|-s(\alpha)}{p-1}}\}<+\infty,\;\; a_\alpha\in\mathrm{Cont}(S,K)\right\},$$

where $s(\alpha) = \sum_{1\le i\le d} s(\alpha_i)$ and $s(\alpha_i)$ is the sum of the p-adic digits of α_i. In particular, one has

$$\mathcal{D}^{(h^+)}(G_0,K)(S) = \left\{\sum_{\alpha\in\mathbf{N}^d} a_\alpha\mathbf{b}^\alpha \;:\; \sup_\alpha\{|a_\alpha|p^{-\frac{p^{-h'}|\alpha|-s(\alpha)}{p-1}}\} < +\infty \right.$$
$$\left. \text{for some } h' > h,\;\; a_\alpha\in\mathrm{Cont}(S,K)\right\}.$$

Proof. Let $\phi : \mathbf{Z}_p^d \to G_0$ be the chart defined by the basis $g_1, \dots, g_d$. By a theorem of Amice (cf. [Laz65, III.1.3.8]), a continuous function $f : G_0 \to K$ is h-analytic (i.e. belongs to $C(\mathbb{G}_0^{(h)}, K)$) if and only if

$$v(c_\alpha) - \frac{p^{-h}|\alpha| - s(\alpha)}{p-1} \to +\infty$$

whenever $|\alpha| \to +\infty$, where $\phi^* f(g) = \sum_{\alpha\in\mathbf{N}^d} c_\alpha\binom{x}{\alpha}$. After dualizing this gives the claimed result as the algebra of analytic distributions is attached to its underlying topological space. □

Remark 4.17. Using the formula

$$v_p(\alpha!) = \frac{|\alpha| - s(\alpha)}{p-1},$$

one can rewrite the above condition on the valuation of the coefficients as

$$\sup_\alpha\{|a_\alpha \alpha!|\, p^{-a(h)|\alpha|}\} < +\infty,$$

where $a(h) = \frac{p^{-h}-1}{p-1}$.

Corollary 4.18. *There is an isomorphism of solid $\mathcal{O}_K$-algebras*

$$\mathcal{D}^{(h^+)}(G_0, K) \cong \varinjlim_{h'>h} \mathcal{D}_{(h')}(G_0, K).$$

Proof. Let $h'' > h'$, then one can write

$$p^{-\frac{p^{-h'}|\alpha|-s(\alpha)}{p-1}} = p^{-\frac{p^{-h''}}{p-1}|\alpha|} p^{-\frac{(p^{-h'}-p^{-h''})|\alpha|-s(\alpha)}{p-1}}.$$

Since $p^{-h'} - p^{-h''} > 0$, we see that $(p^{-h'} - p^{-h''})|\alpha| - s(\alpha) \to +\infty$ as $|\alpha| \to +\infty$. This implies that for any $h'' > h'$ we have

$$\mathcal{D}_{(h'')}(G_0, K) \subseteq \mathcal{D}^{(h')}(G_0, K) \subset \mathcal{D}_{(h')}(G_0, K).$$

Taking limits as $h' \to h^+$ and $h'' \to h^+$ one obtains the corollary. $\square$

4.1.4. *Distribution algebras over G.* The algebras we have defined can be extended to the whole compact group G in an obvious way. Indeed, by Lemma 4.6 the spaces $\mathcal{D}^{(h)}(G_0, K)$ and $\mathcal{D}_{(h)}(G_0, K)$ admit an action of G extending the inner action on $K_\blacksquare[G_0]$. Let us define the distributions

$$\mathcal{D}^{(h)}(G, K) = K_\blacksquare[G] \otimes_{K_\blacksquare[G_0]} \mathcal{D}^{(h)}(G_0, K),$$
$$\mathcal{D}^{(h^+)}(G, K) = \varinjlim_{h'>h} \mathcal{D}^{(h')}(G, K),$$
$$\mathcal{D}_{(h)}(G, K) = K_\blacksquare[G] \otimes_{K_\blacksquare[G_0]} \mathcal{D}_{(h)}(G_0, K),$$

where in the tensor products we see $\mathcal{D}_{(h)}(G_0, K)$ as a left $K_\blacksquare[G_0]$-module. They are unital associative algebras admitting $K_\blacksquare[G]$ as a dense subspace. Notice that even though the distribution algebras over G depend on the choice of the open normal subgroup $G_0 \subset G$, the projective systems $\{\mathcal{D}^{(h)}(G, K)\}_{h>0}$, $\{\mathcal{D}^{(h^+)}(G, K)\}_{h>0}$ and $\{\mathcal{D}_{(h)}(G, K)\}_{h>0}$ do not. Moreover, by Remark 4.9 and Corollary 4.18 these inverse systems are isomorphic. We define the algebra of locally analytic distributions of G as the Fréchet algebra

$$\mathcal{D}^{la}(G, K) = \varprojlim_{h\to\infty} \mathcal{D}^{(h^+)}(G, K) = \varprojlim_{h\to\infty} \mathcal{D}^{(h)}(G, K),$$

by Remark 4.9 it is a Fréchet space of compact type.

For future reference let us introduce algebras of analytic functions over G. Let $h \geq 0$ be a rational number and $\mathbb{G}_0^{(h)}$ the rigid analytic groups of Definition 4.4, recall that they are stable under conjugation by G. Let $\mathbb{G}^{(h)}$ be the rigid analytic group given by $G\mathbb{G}_0^{(h)}$, i.e. if $s_1, \ldots, s_n$ are representatives of the cosets G/G_0 then

$$\mathbb{G}^{(h)} = \bigsqcup_{i=1}^{n} s_i \mathbb{G}_0^{(h)}.$$

We also define

$$\mathbb{G}^{(h^+)} = \bigcup_{h'>h} \mathbb{G}^{(h')}.$$

Let $C(\mathbb{G}^{(h)}, K) = \mathscr{O}(\mathbb{G}^{(h)}) \otimes_{\mathbf{Q}_p} K$ be the affinoid algebra of analytic functions of G over K of radius p^{-h}. It is immediate to check that

$$\mathcal{D}^{(h)}(G, K) = \underline{\mathrm{Hom}}_K(C(\mathbb{G}^{(h)}, K), K).$$

We define $C(\mathbb{G}^{(h)}, \mathcal{O}_K)$ and $C(\mathbb{G}^{(h^+)}, K)$ in the obvious way. The space of K-valued locally analytic functions of G is the LB space $C^{la}(G, K) = \varinjlim_{h\to\infty} C(\mathbb{G}^{(h)}, K)$. Note that, since the inclusion $\mathbb{G}^{(h')} \subset \mathbb{G}^{(h)}$ is a strict immersion for $h' > h$, $C^{la}(G, K)$ is an LB space of compact type, or equivalently, an LS space of compact type (Corollary 3.38). By Theorem 3.40, $\mathcal{D}^{la}(G, K) = \underline{\mathrm{Hom}}_K(C^{la}(G, K), K)$ is a Fréchet space of compact type. Summarizing, for $h' > h$, we have the following maps of distribution algebras and their corresponding dual spaces of analytic functions

(11)

$$K_{\blacksquare}[G] \to \mathcal{D}^{la}(G, K) \to \cdots \to \mathcal{D}^{(h')}(G, K) \to \mathcal{D}^{(h^+)}(G, K) \to \mathcal{D}^{(h)}(G, K),$$
$$\underline{\mathrm{Cont}}(G, K) \leftarrow C^{la}(G, K) \leftarrow \cdots \leftarrow C(\mathbb{G}^{(h')}, K) \leftarrow C(\mathbb{G}^{(h^+)}, K) \leftarrow C(\mathbb{G}^{(h)}, K).$$

4.2. **Solid G-modules.** Let G be a profinite group and $\mathcal{O}_{K,\blacksquare}[G]$ its Iwasawa algebra over $\mathcal{O}_K$.

Lemma 4.19. *Let V be a solid $\mathcal{O}_K$-module. An $\mathcal{O}_{K,\blacksquare}[G]$-module structure on V is equivalent to the following data: for all extremally disconnected sets S, an $\mathcal{O}_K$-linear action* $\mathrm{Cont}(S, G) \times V(S) \to V(S)$ *which is functorial on S.*

Proof. Let V be an $\mathcal{O}_{K,\blacksquare}[G]$-module and S an extremally disconnected set. There is a natural map of condensed sets $[\cdot] : G \to \mathcal{O}_{K,\blacksquare}[G]$. The action of $\mathcal{O}_{K,\blacksquare}[G]$ over V is provided by a linear map $\mathcal{O}_{K,\blacksquare}[G] \otimes_{\mathcal{O}_{K,\blacksquare}} V \to V$ satisfying the usual axioms. Composing with $[\cdot]$ and evaluating at S we obtain a map $\mathrm{Cont}(S, G) \times V(S) \to V(S)$, it is easy to check that this provides the desired $\mathcal{O}_K$-linear action functorial on S.

Conversely, to have such a functorial action is equivalent to having a map of condensed sets $G \times V \to V$ making the usual diagrams commutative. By adjunction of the functor $X \to \mathbf{Z}[X]$, this provides a linear map $\mathbf{Z}[G] \otimes_{\mathbf{Z}} V \to V$. As V is a solid $\mathcal{O}_K$-module it extends uniquely to a map $\mathcal{O}_{K,\blacksquare}[G] \otimes_{\mathcal{O}_{K,\blacksquare}} V \to V$ which is easily seen to satisfy the obvious diagrams of an $\mathcal{O}_{K,\blacksquare}[G]$-module. ☐

Definition 4.20. A solid G-module over $\mathcal{O}_K$ (or a solid $\mathcal{O}_{K,\blacksquare}[G]$-module) is a solid abelian group endowed with an $\mathcal{O}_{K,\blacksquare}[G]$-module structure. We denote the category of solid $\mathcal{O}_{K,\blacksquare}[G]$-modules by $\mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_{K,\blacksquare}[G]}$ and its derived category by $D(\mathcal{O}_{K,\blacksquare}[G])$.

Let us extend the definition of the condensed set of "continuous functions" to complexes:

Definition 4.21. Let V be a solid $\mathcal{O}_K$-module and S a profinite set, we denote by $\underline{\mathrm{Cont}}(S, V)$ the $\mathcal{O}_K$-module $\underline{\mathrm{Hom}}(\mathbf{Z}[S], V) = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[S], V)$. More generally, for $C \in D(\mathcal{O}_{K,\blacksquare})$, we denote $\underline{\mathrm{Cont}}(S, C) := R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[S], C)$, which is consistent with the previous definition as $\mathcal{O}_{K,\blacksquare}[S]$ is a projective module.

Proposition 4.22. *The functor $V \mapsto \underline{\mathrm{Cont}}(G, V)$ is exact and factors through a functor* $\mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_K} \to \mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_{K,\blacksquare}[G^2]}$ *induced by the left and right regular actions respectively. Moreover, it extends to an exact functor of derived categories $D(\mathcal{O}_{K,\blacksquare}) \to D(\mathcal{O}_{K,\blacksquare}[G^2])$.*

Proof. As G is profinite, $\mathcal{O}_{K,\blacksquare}[G]$ is a compact projective $\mathcal{O}_{K,\blacksquare}$-module. This makes the functor $\underline{\mathrm{Cont}}(G, V)$ exact. Thus, the second statement reduces to the first one. We prove the first statement. Let S be an extremally disconnected set, then $\underline{\mathrm{Cont}}(G,V)(S)=\mathrm{Cont}(G\times S,V)=V(G\times S)$. Let us define a map

$$\mathrm{Cont}(S,G^2)\times V(G\times S)\to V(G\times S)$$

as in (2) of Lemma 4.19. Let $f=(f_1,f_2):S\to G^2$ and $v:G\times S\to V$ be objects in $\mathrm{Cont}(S,G^2)$ and $V(G\times S)$ respectively. We define the product $f\cdot v$ to be the composition

$$\begin{array}{ccccc} S\times G & \longrightarrow & S\times G & \longrightarrow & V, \\ (s,g) & \longmapsto & (s,f_1(s)^{-1}gf_2(s)) & \longmapsto & v(s,f_1(s)^{-1}gf_2(s)). \end{array}$$

It is immediate so check that this endows $\underline{\mathrm{Cont}}(G,V)$ with an action of G^2 which is the left and right regular actions on the first and second components respectively. $\square$

Remark 4.23. If we suppose also that V is an $\mathcal{O}_{K,\blacksquare}[G]$-module, then $\underline{\mathrm{Cont}}(G,V)$ is naturally an $\mathcal{O}_{K,\blacksquare}[G^3]$-module. Namely, for S be an extremally disconnected set, $f_1,f_2,f_3\in\mathrm{Cont}(S,G)$ and $v\in V(G\times S)$, we have the action

$$[(f_1,f_2,f_3)\cdot v](g,s)=f_3(s)v(f_1(s)^{-1}gf_2(s),s).$$

This action induces an exact functor of derived categories

$$D(\mathcal{O}_{K,\blacksquare}[G])\to D(\mathcal{O}_{K,\blacksquare}[G^3]).$$

Definition 4.24. Let V be a solid $G^n=\overbrace{G\times\cdots\times G}^{n}$-representation over $\mathcal{O}_K$. Given $I\subset\{1,2,\dots,n\}$ a non-empty subset, we denote by $\star_I$ the diagonal action of G on V induced by the embedding $\iota_I:G\to G^n$ in the components of I. We denote by $V_{\star_I}$ the module V endowed with the action $\star_I$. If $I=\emptyset$ we write $V_0:=V_\emptyset$ for the $\mathcal{O}_{K,\blacksquare}$-module V endowed with the trivial action of G.

Proposition 4.25 below basically says that any action on a solid module is continuous (compare it with [Eme17, Definition 3.2.8]).

Proposition 4.25. *Let C be an object in $D(\mathcal{O}_{K,\blacksquare}[G])$. Then there is a natural quasi-isomorphism of $\mathcal{O}_{K,\blacksquare}[G]$-modules*

$$R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K,\underline{\mathrm{Cont}}(G,C)_{\star_{1,3}})\xrightarrow{\sim}C, \tag{12}$$

where the action of $\mathcal{O}_{K,\blacksquare}[G]$ on the left-hand-side is via the $\star_2$-action. The inverse of this map is called the orbit map of C.

Proof. First, we claim that there exists a natural quasi-isomorphism $\underline{\mathrm{Cont}}(G,C)_{\star_{1,3}}\simeq\underline{\mathrm{Cont}}(G,C)_{\star_1}$ for $C\in D(\mathcal{O}_{K,\blacksquare}[G])$. Suppose that the previous is true, then we have

$$\begin{aligned} &R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K,\underline{\mathrm{Cont}}(G,C)_{\star_{1,3}})\\ &\quad\simeq R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K,\underline{\mathrm{Cont}}(G,C)_{\star_1})\\ &\quad=R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K,R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[G],C)_{\star_1})\\ &\quad=R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_K\otimes^L_{\mathcal{O}_{K,\blacksquare}[G]}\mathcal{O}_{K,\blacksquare}[G],C)\\ &\quad=R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_K,C)=C.\end{aligned}$$

To prove the claim, it is enough to define a natural isomorphism $\underline{\mathrm{Cont}}(G,V)_{\star_{1,3}} \to \underline{\mathrm{Cont}}(G,V)_{\star_1}$ for $V \in \mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_{K,\blacksquare}[G]}$. Let S be an extremally disconnected set, and take $v \in V(G\times S)$. Consider the inverse map $u : G \to G \;\; g \mapsto g^{-1}$ and the multiplication map $m_V : G \times V \to V$. Define $\psi_V(v)$ to be the composition

$$\psi_V(v) : G\times S \xrightarrow{u\times v} G\times V \xrightarrow{m_V} V.$$

The application

$$\begin{aligned}\psi_V : V(G\times S) &\to V(G\times S),\\ v &\mapsto \psi_V(v)\end{aligned}$$

induces an isomorphism of solid $\mathcal{O}_K$-modules $\psi_V : \underline{\mathrm{Cont}}(G,V) \to \underline{\mathrm{Cont}}(G,V)$. It is easy to check that it transfers the $\star_{1,3}$-action to the $\star_1$-action and the $\star_2$-action to the $\star_{2,3}$-action. This proves the claim and that the isomorphism (12) is G-equivariant. $\square$

Remark 4.26. The previous proof shows that if V is an $\mathcal{O}_{K,\blacksquare}[G]$-module arising from a topological space, the isomorphism $V \to (\underline{\mathrm{Cont}}(G,V)_{\star_{1,3}})^G$ is given by the usual orbit map $v \mapsto (g\mapsto gv)$.

4.3. **Analytic representations.** Let $h\geq 0$ and $\mathbb{G}^{(h)}$ be the rigid analytic group of §4.1 extending the group law of G. We recall that $\mathbb{G}^{(h)}$ depends on the choice of an open normal uniform pro-p-subgroup $G_0 \subset G$. To motivate the forthcoming definitions of analytic vectors let us first recall how this works for Banach spaces, where we follow [Eme17, §3].

Let V be a K-Banach space endowed with a continuous action of G, the space of V-valued $\mathbb{G}^{(h)}$-analytic functions is by definition the projective tensor product $C(\mathbb{G}^{(h)},V) := C(\mathbb{G}^{(h)},K)\widehat{\otimes}_K V$. As V and $C(\mathbb{G}^{(h)},K)$ are Banach spaces, the projective tensor product coincides with the solid tensor product $C(\mathbb{G}^{(h)},K)\otimes_{K_\blacksquare} V$ (Lemma 3.13). This space has an action of G^2 given by the left and right regular actions of G, and an extra action of G induced by the one of V. Following the notation of Definition 4.24, the $\mathbb{G}^{(h)}$-analytic vectors of V are the Banach space

$$V^{\mathbb{G}^{(h)}-an} := (C(\mathbb{G}^{(h)},V)_{\star_{1,3}})^G. \tag{13}$$

There is a natural map $V^{\mathbb{G}^{(h)}-an} \to V$ given by evaluating at $1\in\mathbb{G}^{(h)}$, and V is $\mathbb{G}^{(h)}$-analytic if this arrow is an isomorphism.

To generalise the previous construction of analytic vectors to solid $K_\blacksquare[G]$-modules we need to rewrite (13) in a slightly different way. Consider the affinoid ring $(C(\mathbb{G}^{(h)},K), C(\mathbb{G}^{(h)},\mathcal{O}_K))$, it is a finite product of Tate power series rings in d-variables. In Example 2.9(4) and (5), we saw how this affinoid ring provides a natural analytic ring that we denote as $C(\mathbb{G}^{(h)},K)_\blacksquare$. We also denote by $C(\mathbb{G}^{(h)},\mathcal{O}_K)_\blacksquare$ the analytic ring attached to its subalgebra of power-bounded elements. Now, as V is a K-Banach vector space, one has

$$C(\mathbb{G}^{(h)},V) = C(\mathbb{G}^{(h)},K)\otimes_{K_\blacksquare} V = C(\mathbb{G}^{(h)},K)_\blacksquare \otimes_{K_\blacksquare} V,$$

where the last tensor product is the completion functor with respect to the measures of $C(\mathbb{G}^{(h)},K)_\blacksquare$. The last equality follows by Corollary 2.19; one has that $C(\mathbb{G}^{(h)},K)_\blacksquare\otimes_{K_\blacksquare} V = \underline{\mathrm{Hom}}_K(\mathcal{D}^{(h)}(G,K),V) = C(\mathbb{G}^{(h)},K)\otimes_{K_\blacksquare} V$, where in the last equality we use the nuclearity of V (cf. Corollary 3.16). Hence, we can write the $\mathbb{G}^{(h)}$-analytic vectors of V in the form

$$V^{\mathbb{G}^{(h)}-an} = \underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, (C(\mathbb{G}^{(h)}, K)_{\blacksquare} \otimes_{K_{\blacksquare}} V)_{\star_{1,3}}).$$

In order to generalise the construction of analytic vectors we need some basic properties of the tensor $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_{K,\blacksquare}} -$.

Proposition 4.27. *Consider the functor* $V \mapsto C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_{K,\blacksquare}} V$ *for* $V \in \mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_K}$. *The following statements hold.*

(1) *The functor is exact.*
(2) *It induces an exact functor of derived categories* $D(\mathcal{O}_{K_{\blacksquare}}) \to D(\mathcal{O}_{K,\blacksquare}[G^2])$ *given by the left and right regular actions.*
(3) *There is a functorial map* $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes^L_{\mathcal{O}_K} C \to \underline{\mathrm{Cont}}(G, C)$ *for* $C \in D(\mathcal{O}_{K_{\blacksquare}})$ *compatible with the left and right regular actions.*

Proof. Exactness follows from Corollary 2.19. Indeed, as $\mathbb{G}^{(h)}$ is a finite disjoint union of polydiscs one has

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes^L_{\mathcal{O}_{K,\blacksquare}} V = R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{D}^{(h)}(G, \mathcal{O}_K), V). \tag{14}$$

But $\mathcal{D}^{(h)}(G, \mathcal{O}_K)$ is a projective $\mathcal{O}_K$-module, this implies that

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes^L_{\mathcal{O}_{K,\blacksquare}} V = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{D}^{(h)}(G, \mathcal{O}_K), V) = C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_{K,\blacksquare}} V$$

is exact.

To prove (2), it is enough to show that $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_{K,\blacksquare}} V$ has natural left and right regular actions for $V \in \mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_K}$. Writing V as a quotient $P_1 \to P_0 \to V$ of objects of the form $P_i = \bigoplus_{I_i} \prod_{J_i} \mathcal{O}_K$, we have an exact sequence

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_K} P_1 \xrightarrow{f} C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_K} P_0 \to C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_K} V \to 0.$$

The functor $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_K} -$ commutes with colimits and, by equation (14), it also commutes with products. Hence

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes P_i = \bigoplus_{I_i} \prod_{J_i} C(\mathbb{G}^{(h)}, \mathcal{O}_K)$$

and these modules are equipped with the natural left and right regular actions of G. Moreover, the map f is equivariant for these actions. We endow $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_K} V$ with the action induced by the quotient map. It is easy to check that this action is independent of the presentation of V, and that it is functorial.

For the last statement, it is enough to construct a functorial equivariant map

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_K} V \to \underline{\mathrm{Cont}}(G, V)$$

for $V \in \mathrm{Mod}^{\mathrm{solid}}_{\mathcal{O}_K}$. Recall that by definition $\underline{\mathrm{Cont}}(G, V) = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[G], V)$. Similarly as before, we are reduced to constructing the map for an object of the form $P = \bigoplus_I \prod_J \mathcal{O}_K$. As both functors commute with colimits and products, one reduces to treat the case $P = \mathcal{O}_K$, for which we have the natural inclusion $C(\mathbb{G}^{(h)}, \mathcal{O}_K) \to \underline{\mathrm{Cont}}(G, \mathcal{O}_K)$ provided by $G \subset \mathbb{G}^{(h)}$, which is equivariant for the left and right regular actions of G. This ends the proof. $\square$

Remark 4.28. It is clear that if V is a solid G-module then $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_{\blacksquare} \otimes_{\mathcal{O}_{K,\blacksquare}} V$ can be endowed with an action of G^3.

Definition 4.29. Let $h \geq 0$.

(1) Let $V \in \mathrm{Mod}^{\mathrm{solid}}_{K_{\blacksquare}[G]}$, the space of $\mathbb{G}^{(h)}$-analytic vectors of V is the solid $K_{\blacksquare}[G]$-module

$$V^{\mathbb{G}^{(h)}-an} := \underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, (C(\mathbb{G}^{(h)}, K)_{\blacksquare} \otimes_{K_{\blacksquare}} V)_{\star_{1,3}}),$$

where the action of G is induced by the $\star_2$-action. Similarly, we define the $\mathbb{G}^{(h^+)}$-analytic vectors of V to be

$$V^{\mathbb{G}^{(h^+)}-an} := \varprojlim_{h'>h} V^{\mathbb{G}^{(h')}-an},$$

where the transition maps are induced by the base change of analytic rings $C(\mathbb{G}^{(h')}, K)_{\blacksquare} \otimes_{K_{\blacksquare}} V \to C(\mathbb{G}^{(h'')}, K)_{\blacksquare} \otimes_{K_{\blacksquare}} V$ for $h'' > h' > h$.

(2) Given a complex $C \in D(K_{\blacksquare}[G])$ we define the derived $\mathbb{G}^{(h)}$-analytic vectors of C as the complex in $D(K_{\blacksquare}[G])$

$$C^{R\mathbb{G}^{(h)}-an} := R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, (C(\mathbb{G}^{(h)}, K)_{\blacksquare} \otimes^L_{K_{\blacksquare}} C)_{\star_{1,3}}),$$

where the action of G is induced by the $\star_2$-action. Similarly, we define the derived $\mathbb{G}^{(h^+)}$-analytic vectors of C to be

$$C^{R\mathbb{G}^{(h^+)}-an} := R \varprojlim_{h'>h} C^{R\mathbb{G}^{(h')}-an}.$$

Remark 4.30. As $C(\mathbb{G}^{(h)}, K)_{\blacksquare} \otimes_{K_{\blacksquare}} -$ is exact, $C \mapsto C^{R\mathbb{G}^{(h)}-an}$ is the right derived functor of $V \mapsto V^{\mathbb{G}^{(h)}-an}$. Similarly, $C \mapsto C^{R\mathbb{G}^{(h^+)}-an}$ is the right derived functor of $V \mapsto V^{\mathbb{G}^{(h^+)}-an}$.

Lemma 4.31. *Let $h \geq 0$ and $C \in D(K_{\blacksquare}[G])$. There is a natural morphism of objects in $D(K_{\blacksquare}[G])$*

$$C^{R\mathbb{G}^{(h)}-an} \to C.$$

Proof. Notice that we have a natural map

$$C(\mathbb{G}^{(h)}, K)_{\blacksquare} \otimes^L_K C \to \underline{\mathrm{Cont}}(G, C)$$

which commutes with the three actions of G. Taking $\star_{1,3}$-invariant one gets the lemma by Proposition 4.25. □

Definition 4.32. Let $h \geq 0$.

(1) A solid $K_{\blacksquare}[G]$-module V is called $\mathbb{G}^{(h)}$-analytic if the natural map $V^{\mathbb{G}^{(h)}-an} \to V$ is an isomorphism. Similarly, it is called $\mathbb{G}^{(h^+)}$-analytic if $V^{\mathbb{G}^{(h^+)}-an} \to V$ is an isomorphism.

(2) A complex $C \in D(K_{\blacksquare}[G])$ is called derived $\mathbb{G}^{(h)}$-analytic if the natural map $C^{R\mathbb{G}^{(h)}-an} \to C$ is a quasi-isomorphism. Similarly, it is derived $\mathbb{G}^{(h^+)}$-analytic if the map $C^{R\mathbb{G}^{(h^+)}-an} \to C$ is a quasi-isomorphism.

So far we have introduced two definitions of analytic vectors depending on whether we choose the radius to be closed or open. It turns out that to have a theory in terms of modules over distribution algebras, we need to work with the Stein analytic groups $\mathbb{G}^{(h^+)}$. The main reason is that these algebras are localizations of the Iwasawa algebra, as it is reflected in Lemma 4.33 (to be proved in §5.3, see Corollary 5.11).

Lemma 4.33. *One has* $\mathcal{D}^{(h^+)}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}^{(h^+)}(G,K) = \mathcal{D}^{(h^+)}(G,K)$.

An immediate consequence of the previous result is the following fully-faithfulness property.

Corollary 4.34. *The category* $\mathrm{Mod}^{\mathrm{solid}}_{\mathcal{D}^{(h^+)}(G,K)}$ *(resp.* $D(\mathcal{D}^{(h^+)}(G,K))$*) is a full subcategory of* $\mathrm{Mod}^{\mathrm{solid}}_{K_\blacksquare[G]}$ *(resp.* $D(K_\blacksquare[G])$*). In other words, if* $V, V' \in \mathrm{Mod}^{\mathrm{solid}}_{\mathcal{D}^{(h^+)}(G,K)}$ *and* $C, C' \in D(\mathcal{D}^{(h^+)}(G,K))$ *then*

$$\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(V,V') = \underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(V,V'),$$
$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(C,C') = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(C,C').$$

Proof. We first address the second statement, the first one follows similarly. It follows from the usual extension of scalars:

$$\begin{aligned}
&R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(C,C')\\
&\quad = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{D}^{(h^+)}(G,K) \otimes^L_{K_\blacksquare[G]} C, C')\\
&\quad = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{D}^{(h^+)}(G,K) \otimes^L_{K_\blacksquare[G]} (\mathcal{D}^{(h^+)}(G,K) \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C), C')\\
&\quad = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}((\mathcal{D}^{(h^+)}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}^{(h^+)}(G,K)) \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C, C')\\
&\quad = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{D}^{(h^+)}(G,K) \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C, C')\\
&\quad = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(C,C').
\end{aligned}$$

□

Remark 4.35. As it will be seen in Corollary 5.11, the same holds true for the distribution algebra $\mathcal{D}^{la}(G,K)$. In particular, $D(\mathcal{D}^{la}(G,K))$ is a full subcategory of $D(K_\blacksquare[G])$.

We can now state the main theorem of this section.

Theorem 4.36. *Let* $W \in D(K_\blacksquare)$ *and* $C \in D(K_\blacksquare[G])$*. The following hold.*

(1) *There are natural isomorphisms of* $K_\blacksquare[G]$*-modules*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h)}(G,K) \otimes^L_{K_\blacksquare} W, C) = R\underline{\mathrm{Hom}}_K(W, C^{R\mathbb{G}^{(h)}-an}),$$
$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G,K) \otimes^L_{K_\blacksquare} W, C) = R\underline{\mathrm{Hom}}_K(W, C^{R\mathbb{G}^{(h^+)}-an}).$$

The $K_\blacksquare[G]$*-module structure of the terms inside the* $R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}$ *in the LHS are the left multiplication on the distribution algebras and the module structure on* C*. The* G*-action on the LHS* $R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(-,-)$ *is induced by the right multiplication on the distribution algebras.*

(2) *The category of* $\mathbb{G}^{(h^+)}$*-analytic representations of* G *is equal to* $\mathrm{Mod}^{\mathrm{solid}}_{\mathcal{D}^{(h^+)}(G,K)}$. *In other words, a* $K_\blacksquare[G]$*-module* V *is* $\mathbb{G}^{(h^+)}$*-analytic if and only if the action of* $K_\blacksquare[G]$ *extends to an action of* $\mathcal{D}^{(h^+)}(G,K)$*.*

(3) *Furthermore, a complex* $C \in D(K_\blacksquare[G])$ *is derived* $\mathbb{G}^{(h^+)}$*-analytic if and only if for all* $n \in \mathbf{Z}$ *the cohomology groups* $H^n(C)$ *are* $\mathbb{G}^{(h^+)}$*-analytic. Equivalently,* C *is derived* $\mathbb{G}^{(h^+)}$*-analytic if and only if it belongs to the essential image of* $D(\mathcal{D}^{(h^+)}(G,K))$*.*

Proof. (1) Let $W \in D(K_{\blacksquare})$ and $C \in D(K_{\blacksquare}[G])$, by Corollary 2.19 there is a natural quasi-isomorphism

$$R\underline{\mathrm{Hom}}_K(\mathcal{D}^{(h)}(G,K) \otimes^L_{K_{\blacksquare}} W, C) = R\underline{\mathrm{Hom}}_K(W, C(\mathbb{G}^{(h)},K)_{\blacksquare} \otimes^L_K C). \tag{15}$$

It is easy to verify that the left and right regular actions of the RHS are translated in the left and right multiplication of the distributions in the LHS. Indeed, one can reduce to $W = \prod_i \mathcal{O}_K[\frac{1}{p}]$ and $C = K_{\blacksquare}[G]$ in which case it is straightforward. Then, the $\star_{1,3}$-action in the RHS translates in the left multiplication on the distributions and the action on C in the LHS. Taking $R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K,-)$ in (15) one gets

$$R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(\mathcal{D}^{(h)}(G,K)\otimes^L_{K_{\blacksquare}} W, C) = R\underline{\mathrm{Hom}}_K(W, C^{R\mathbb{G}^{(h)}-an}).$$

Taking derived inverse limits and using that $R\underline{\mathrm{Hom}}$ commutes with colimits in the first factor and limits in the second factor, one gets

$$R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(\mathcal{D}^{(h^+)}(G,K)\otimes^L_{K_{\blacksquare}} W, C) = R\underline{\mathrm{Hom}}_K(W, C^{R\mathbb{G}^{(h^+)}-an}).$$

(2) Consider the pre-analytic ring $(K_{\blacksquare}[G], \mathcal{M}^{(h^+)})$ such that for any extremally disconnected S one has

$$\mathcal{M}^{(h^+)}(S) = \mathcal{D}^{(h^+)}(G,K) \otimes_{K_{\blacksquare}} K_{\blacksquare}[S].$$

Corollary 4.34 implies that it is in fact an analytic ring. Indeed, let $P^{\bullet}$ be a complex of $K_{\blacksquare}[G]$-modules concentrated in positive homological degrees whose terms are direct sums of $\mathcal{M}^{(h^+)}[S_i]$ for $\{S_i\}_{i\in I}$ a family of profinite sets. Let S be a profinite set, then

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(\mathcal{M}^{(h^+)}[S], P^{\bullet}) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{M}^{(h^+)}[S], P^{\bullet}) \\ &= R\underline{\mathrm{Hom}}_K(K_{\blacksquare}[S], P^{\bullet}). \end{aligned}$$

Moreover, the category of solid $(K_{\blacksquare}[G], \mathcal{M}^{(h^+)})$-modules is equal to the category of solid $\mathcal{D}^{(h^+)}(G,K)$-modules. More precisely, by Theorem 2.10, a family of compact projective generators of $\mathrm{Mod}^{\mathrm{solid}}_{(K_{\blacksquare}[G],\mathcal{M}^{(h^+)})}$ is given by $\mathcal{M}^{(h^+)}(S)$ for S extremally disconnected, which are naturally $\mathcal{D}^{(h^+)}(G,K)$-modules, and any $K_{\blacksquare}[G]$-linear map between these objects is automatically $\mathcal{D}^{(h^+)}(G,K)$-linear again by Corollary 4.34 again. Hence, part (1) and Theorem 2.10 imply that a $K_{\blacksquare}[G]$-module V is $\mathbb{G}^{(h^+)}$-analytic if and only if it is a $\mathcal{D}^{(h^+)}(G,K)$-module.

(3) Theorem 2.10 and the same argument as before tell us that a complex $C \in D(K_{\blacksquare}[G])$ is derived $\mathbb{G}^{(h^+)}$-analytic if and only if it belongs to the essential image of $D(\mathcal{D}^{(h^+)}(G,K))$, if and only if for all $n \in \mathbb{Z}$ the module $H^n(C)$ is a $\mathcal{D}^{(h^+)}(G,K)$-module, finishing the proof. $\square$

Remark 4.37. Let us highlight the importance of the compactness assumption on G in Theorem 4.36. Let G be a locally profinite p-adic Lie group, and let $G_0 \subset G$ be an open compact subgroup. One can construct, as we did above, rigid analytic neighbourhoods $\mathbb{G}_0^{(h^+)}$ of G_0 and corresponding distribution algebras $\mathcal{D}^{(h^+)}(G_0,K)$. One may ask if there are analogous of these (analytic) distribution algebras $\mathcal{D}^{(h^+)}(G,K)$ over G in such a way that the G-representations whose restriction to G_0 is $\mathbb{G}_0^{(h^+)}$-analytic are the same as $\mathcal{D}^{(h^+)}(G,K)$-modules, but this turns out to be false in general. Indeed, let W be a solid G-representation, by part (1) of Theorem 4.36 the derived $\mathbb{G}_0^{(h^+)}$-analytic vectors of W are equal to

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G_0]}(\mathcal{D}^{(h)}(G_0,K),W) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G]\otimes_{K^L_\blacksquare[G_0]}\mathcal{D}^{(h)}(G_0,K),W),$$

so that the natural candidate is $\mathcal{D}^{(h^+)}(G,K) := K_\blacksquare[G]\otimes_{K_\blacksquare[G_0]}\mathcal{D}^{(h^+)}(G_0,K)$, but this G-representation is not an algebra unless G normalises G_0.

Another reason why the distribution algebras $\mathcal{D}^{(h^+)}(G,K)$ do not exist in general is that the action of G might change the radius of analyticity of a locally analytic vector. For example, take $G = \mathbf{GL}_2(\mathbf{Q}_p)$, $G_0 = \begin{pmatrix} 1+p^2\mathbf{Z}_p & p^2\mathbf{Z}_p \\ p^2\mathbf{Z}_p & 1+p^2\mathbf{Z}_p \end{pmatrix}$ and $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. Take $\mathbb{G}_0 := \begin{pmatrix} 1+p^2\mathbb{D}^1_{\mathbf{Q}_p} & p^2\mathbb{D}^1_{\mathbf{Q}_p} \\ p^2\mathbb{D}^1_{\mathbf{Q}_p} & 1+p^2\mathbb{D}^1_{\mathbf{Q}_p} \end{pmatrix}$, then

$$g\mathbb{G}_0g^{-1} = \begin{pmatrix} 1+p^2\mathbb{D}^1_{\mathbf{Q}_p} & p^3\mathbb{D}^1_{\mathbf{Q}_p} \\ p\mathbb{D}^1_{\mathbf{Q}_p} & 1+p^2\mathbb{D}^1_{\mathbf{Q}_p} \end{pmatrix}.$$

Therefore, if V is a locally analytic representation of G and $v\in V$ is a $\mathbb{G}_0$-analytic vector, gv is a $g\mathbb{G}_0g^{-1}$-analytic vector, but $\mathbb{G}_0 \neq g\mathbb{G}_0g^{-1}$.

Notice however that the tensor product $\mathcal{D}^{la}(G,K) := K_\blacksquare[G]\otimes_{K_\blacksquare[G_0]}\mathcal{D}^{la}(G_0,K)$ is an algebra since $\mathcal{D}^{la}(G_0,K)$ for varying $G_0\subset G$ admit an action of G by conjugation, this is nothing but the algebra of compactly supported locally analytic distributions of G.

Remark 4.38. Notice that Theorem 4.36 implies that the tensor product over K of two $\mathbb{G}^{(h^+)}$-analytic representations is still $\mathbb{G}^{(h^+)}$-analytic. Indeed, if V and W are $\mathbb{G}^{(h^+)}$-analytic, then they are modules over $\mathcal{D}^{(h^+)}(G,K)$ and so it is its tensor product $V\otimes^L_{K_\blacksquare}W$, which in turn implies that $V\otimes^L_{K_\blacksquare}W$ is $\mathbb{G}^{(h^+)}$-analytic. A similar result holds for locally analytic representations by writing them as colimits of their $\mathbb{G}^{(h^+)}$-analytic vectors.

The diagram (11) relating the distribution algebras gives rise to the following fully faithful forgetful functors between derived categories for $h'>h$

$$D(K_\blacksquare[G]) \leftarrow D(\mathcal{D}^{la}(G,K)) \leftarrow D(\mathcal{D}^{(h'^+)}(G,K)) \leftarrow D(\mathcal{D}^{(h^+)}(G,K)).$$

Since the forgetful functors preserve limits and colimits, they admit left and right adjoints. For example, consider the forgetful functor $F : D(\mathcal{D}^{(h^+)}(G,K)) \to D(K_\blacksquare[G])$, and let $V\in D(K_\blacksquare[G])$ and $W\in D(\mathcal{D}^{(h^+)}(G,K))$, then we have that

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(V,F(W)) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{D}^{(h^+)}(G,K)\otimes^L_{K_\blacksquare[G]}V,W), \\ R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(F(W),V) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(W,R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G,K),V)) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(W,V^{R\mathbb{G}^{(h^+)}-la}). \end{aligned}$$

Remark 4.39. By definition $D(K_\blacksquare[G])$ is the derived category of solid representations of G. By Theorem 4.36, the category $D(\mathcal{D}^{(h^+)}(G,K))$ is the derived category of $\mathbb{G}^{(h^+)}$-analytic representations of G. We point out that $D(\mathcal{D}^{la}(G,K))$ is not the derived category of locally analytic representations to G. Indeed, one has for example that $\mathcal{D}^{la}(G,K)$ is not a locally analytic representation (cf. the discussion just after Definition 4.40).

4.4. **Locally analytic representations.** We finish this section with some applications of Theorem 4.36 to the theory of locally analytic representations. Let us begin with the definition of the locally analytic vectors.

Definition 4.40.

(1) Let V be a solid $K_{\blacksquare}[G]$-module, the space of locally analytic vectors of V is the solid $K_{\blacksquare}[G]$-module
$$V^{la} := \varinjlim_{h\to\infty} V^{\mathbb{G}^{(h)}-an} = \varinjlim_{h\to\infty} V^{\mathbb{G}^{(h^+)}-an}.$$
We say that V is locally analytic if the natural map $V^{la} \to V$ is an isomorphism.

(2) Let $C \in D(K_{\blacksquare}[G])$, the derived locally analytic vectors of C are the complex
$$C^{Rla} := \operatorname*{hocolim}_{h\to\infty} C^{R\mathbb{G}^{(h)}-an} = \operatorname*{hocolim}_{h\to\infty} C^{R\mathbb{G}^{(h^+)}-an}.$$
We say that C is derived locally analytic if the natural map $C^{Rla} \to C$ is a quasi-isomorphism.

In §4.1.4 we defined the algebra of locally analytic distributions of G as the Fréchet algebra
$$\mathcal{D}^{la}(G,K) = \varprojlim_{h\to\infty} \mathcal{D}^{(h)}(G,K) = \varprojlim_{h\to\infty} \mathcal{D}^{(h^+)}(G,K).$$
Since a locally analytic representation V is a (homotopy) colimit of $\mathbb{G}^{(h^+)}-$analytic representations, Theorem 4.36 implies that V is naturally a $\mathcal{D}^{la}(G,K)$-module. Furthermore, this structure is unique as $\mathcal{D}^{la}(G,K)\otimes^L_{K_{\blacksquare}[G]}\mathcal{D}^{la}(G,K) = \mathcal{D}^{la}(G,K)$, see Corollary 5.11. Nevertheless, not all the $\mathcal{D}^{la}(G,K)$-modules are locally analytic representations of G, e.g. $\mathcal{D}^{la}(G,K)$ is not a locally analytic representation as it cannot be written as a colimit of $\mathcal{D}^{(h^+)}(G,K)$-modules. Indeed, if $\mathcal{D}^{la}(G,K)$ was locally analytic then $1 \in \mathcal{D}^{la}(G,K)$ would be analytic for certain group $\mathbb{G}^{(h^+)}$, this would provide a section of the map $\mathcal{D}^{la}(G,K) \to \mathcal{D}^{(h^+)}(G,K)$ which is a contradiction: the map $\mathcal{D}^{la}(G,K) \to \mathcal{D}^{(h^+)}(G,K)$ is injective by (a variant of) Lemma 3.19 so it would be an isomorphism, by taking duals we would have that $C^{la}(G,K) \cong C(\mathbb{G}^{(h^+)},K)$ which is impossible since there are locally analytic functions which are not $\mathbb{G}^{(h^+)}$-analytic. In Propositions 4.41-4.43 we give some conditions for a $\mathcal{D}^{la}(G,K)$-module to be a locally analytic representation of G.

Proposition 4.41. *Let V be a Banach $K_{\blacksquare}[G]$-module. The following are equivalent.*

(1) *the $K_{\blacksquare}[G]$-module structure of V extends to $\mathcal{D}^{la}(G,K)$.*
(2) *the $K_{\blacksquare}[G]$-module structure of $V^{\vee}$ extends to $\mathcal{D}^{la}(G,K)$.*
(3) *V is $\mathbb{G}^{(h^+)}$-analytic for some $h \geq 0$.*

Proof. Formally, (1) and (2) are equivalent as the dual of a solid $\mathcal{D}^{la}(G,K)$ module is naturally a solid $\mathcal{D}^{la}(G,K)$-module, (3) implies (1) is clear from the previous discussion. Let us show that (1) implies (3). Suppose that V is a $\mathcal{D}^{la}(G,K)$-module. Consider the multiplication map $m_V : \mathcal{D}^{la}(G,K)\otimes_{K_{\blacksquare}}V \to V$. As V is

Banach and $\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V$ is a Fréchet space, Lemma 3.31 and Corollary 3.38 imply that there exists $h\geq 0$ such that m_V factors as

$$\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V\to \mathcal{D}^{(h)}(G,K)^B\otimes_{K_\blacksquare}V\to V.$$

It is immediate to check that this endows V with a structure of $\mathcal{D}^{(h^+)}(G,K)$-module for $h'>h$. By Theorem 4.36 V is $\mathbb{G}^{(h'^+)}$-analytic. □

Proposition 4.42 (Schneider-Teitelbaum)**.** *Let V be an LS space of compact type, cf. Definition* 3.34*. The following are equivalent*

(1) *the $K_\blacksquare[G]$-module structure of V extends to $\mathcal{D}^{la}(G,K)$-module.*
(2) *the $K_\blacksquare[G]$-module structure of $V^\vee$ extends to $\mathcal{D}^{la}(G,K)$-module.*
(3) *V is a locally analytic representation of G.*

Proof. By Theorem 3.40, (1) and (2) are equivalent. It is also clear that (3) implies (1). Suppose that V is a $\mathcal{D}^{la}(G,K)$-module which is an LS space of compact type. The multiplication map $m_V:\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V\to V$ gives an element of $\mathrm{Hom}(\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V,V)$. Let $V=\varinjlim_n V_n$ be a presentation as a colimit of Smith spaces by injective transition maps, and let V_n^B be the underlying Banach space of V_n (cf. Definition 3.34). As V is of compact type we have $V=\varinjlim_n V_n^B$. Therefore

$$\begin{aligned}\underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V,V) &= \varprojlim_n \underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V_n^B,V)\\ &= \varprojlim_n \underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K),(V_n^B)^\vee\otimes_{K_\blacksquare}V)\\ &= \varprojlim_n C^{la}(G,K)\otimes_{K_\blacksquare}(V_n^B)^\vee\otimes_{K_\blacksquare}V\\ &= \varprojlim_n\varinjlim_m C^{la}(G,K)\otimes_{K_\blacksquare}(V_n^B)^\vee\otimes_{K_\blacksquare}V_m^B,\end{aligned}$$

where the first equality is formal, the second follows from the fact that V is nuclear, and the third equality follows from Theorem 3.40. This shows that given $n\in\mathbb{N}$ there is $m\in\mathbb{N}$ such that the map $m_V:\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V_n^B\to V$ factors through $m_V:\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V_n^B\to V_m^B$. By Lemma 3.31 there exists $h\geq 0$ such that m_V factors as $\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V_n^B\to\mathcal{D}^{(h)}(G,K)\otimes_{K_\blacksquare}V_n^B\to V_m^B$. Equivalently, there is $h\geq 0$ (maybe different) such that m_V factors as $\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V_n\to\mathcal{D}^{(h)}(G,K)\otimes_{K_\blacksquare}V_n\to V_m$. Let V_n' be the image of $\mathcal{D}^{(h)}(G,K)\otimes_{K_\blacksquare}V_n\to V_m$, it is a Smith space endowed with an action of $\mathcal{D}^{(h)}(G,K)$ extending the one of $\mathcal{D}^{la}(G,K)$. It is immediate to see that $V=\varinjlim_n V_n'$, this shows that V is written as a colimit of $\mathcal{D}^{(h'^+)}(G,K)$-modules (for $h'>h$), which implies that it is a locally analytic representation of G by Theorem 4.36. □

One may wonder whether Propositions 4.41-4.42 can be extended to the bounded derived category. Under the assumption of Conjecture 3.41 we can improve Propositions 4.41 and 4.42 as follows:

Proposition 4.43. *Suppose that Conjecture* 3.41 *holds. Let $C\in D(\mathcal{D}^{la}(G,K)_\blacksquare)^b$ be a bounded solid $\mathcal{D}^{la}(G,K)$-module. Assume that either*

(a) *C is quasi-isomorphic to a bounded complex of K-Banach spaces as a $K_\blacksquare$-complex.*
(b) *C is quasi-isomorphic to a bounded complex of LS spaces of compact type as a $K_\blacksquare$-complex.*

Then C is a derived locally analytic representation of G. In particular there is a derived duality between locally analytic complexes quasi-isomorphic to bounded complexes of LS spaces of compact type, and $\mathcal{D}^{la}(G,K)$-complexes quasi-isomorphic to bounded complexes of Frechét spaces of compact type.

Proof. We will prove in Corollary 5.11 that

$$\mathcal{D}^{la}(G,K)\otimes^L_{K_\blacksquare[G]}\mathcal{D}^{la}(G,K)=\mathcal{D}^{la}(G,K). \tag{16}$$

Let C be a complex in $D(\mathcal{D}^{la}(G,K))$ which is quasi-isomorphic to a bounded complex of Banach spaces as $K_\blacksquare$-complex. By Corollary 3.42(1) we have

$$R\underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K),C)=\operatorname*{hocolim}_{h\to\infty}R\underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G,K),C).$$

Taking $K_\blacksquare[G]$-invariants, using that K is a compact $K_\blacksquare[G]$-module (cf. Theorem 5.7) and hence that it commutes with filtered colimits, one gets that

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{la}(G,K),C)=\operatorname*{hocolim}_{h\to\infty}R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G,K),C).$$

But (16) implies that the LHS is equal to C while Theorem 4.36 implies that the RHS is equal to $\operatorname{hocolim}_h C^{R\mathbb{G}^{(h^+)}}$. This shows that C is derived locally analytic.

Now suppose that C is quasi-isomorphic to a bounded complex of LS spaces of compact type. Then from Corollary 3.42(3) and (4) one has

$$R\underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K),C)=C^{la}(G,K)\otimes^L_{K_\blacksquare}C=\operatorname*{hocolim}_{h\to\infty}C(\mathbb{G}^{(h)},K)_\blacksquare\otimes^L_{K_\blacksquare}C.$$

Taking $K_\blacksquare[G]$-invariants one gets again by (16) that $C=C^{Rla}$, i.e. C is a derived locally analytic representation of G. □

Remark 4.44. In the situation (a) of Proposition 4.43, one can show in addition that C is derived $\mathbb{G}^{(h^+)}$-analytic for some $h>0$. Indeed, it is enough to prove that for all $n\in\mathbf{Z}$ there is $h>0$ such that $H^n(C)$ is derived $\mathbb{G}^{(h^+)}$-analytic. As C is bounded, we are left to show that if V is a $\mathcal{D}^{la}(G,K)$-module which is a quotient of two K-Banach spaces (not necessarily $\mathcal{D}^{la}(G,K)$-modules), then V is already a $\mathcal{D}^{(h^+)}(G,K)$-module for some $h>0$. Let $m_V:\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V\to V$ be the multiplication map, by Corollary 3.42(4) one has that

$$R\underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare}V,V)=\operatorname*{hocolim}_{h\to\infty}R\underline{\mathrm{Hom}}_K(\mathcal{D}(\mathbb{G}^{(h)},K)\otimes_{K_\blacksquare}V,V).$$

Thus, m_V factors as $m_V:\mathcal{D}(\mathbb{G}^{(h)},K)\otimes_{K_\blacksquare}V\to V$ for some $h>0$. After taking $h'>h$ one shows that the map $\mathcal{D}(\mathbb{G}^{(h'^+)},K)\otimes_{K_\blacksquare}V\to V$ is in fact an action of $\mathcal{D}(\mathbb{G}^{(h'^+)},K)$, proving that V is derived $\mathbb{G}^{(h^+)}$-analytic as desired.

4.5. **Admissible representations.** We conclude this chapter by showing how the theory developed until now together with some algebraic facts about the Iwasawa and the distribution algebras, provide a description of the locally analytic vectors of an admissible representation in terms of its dual. All results in this sections were already known ([ST03], [Pan22]).

We recall the following important results of the classical Iwasawa and distribution algebras

Theorem 4.45 (Lazard, Schneider-Teitelbaum)**.** *The following hold*

(1) *The (classical) Iwasawa algebra $K_\blacksquare[G](*)$ is a (left and right) noetherian integral domain.*

(2) *The (classical) distribution algebras* $\mathcal{D}^{(h^+)}(G,K)(*)$ *are flat over* $\mathcal{O}_{K,\blacksquare}[G](*)$ *algebraically.*
(3) *Let* $h' > h$*, then* $\mathcal{D}^{(h'^+)}(G,K)(*)$ *is a flat algebra over* $\mathcal{D}^{(h^+)}(G,K)(*)$ *algebraically.*
(4) *The (classical) locally analytic distribution algebra* $\mathcal{D}^{la}(G,K)(*)$ *is faithfully flat over* $K_\blacksquare[G](*)$.

Proof. Part (1) is [ST03, Remark 4.6], part (2) is [ST03, Proposition 4.7], part (3) is [ST03, Theorem 4.9] and part (4) is [ST03, Theorem 4.11]. □

Definition 4.46. A solid Banach representation V of G is admissible if its dual is a finite module over the Iwasawa algebra. Equivalently, if V admits an equivariant closed immersion into a finite direct sum of $\underline{\mathrm{Cont}}(G,K)$.

Remark 4.47. By (1) of Theorem 4.45, any finite $K_\blacksquare[G](*)$-module W is of finite presentation. The module W has a natural Hausdorff topology given by the quotient topology of any presentation of W as a quotient of finite free $K_\blacksquare[G](*)$-modules (equipped with their canonical topology induced from $K_\blacksquare[G](*)_{\mathrm{top}}$), cf. [ST02a, Proposition 3.1]. Moreover, any $K_\blacksquare[G](*)$-equivariant map between finite $K_\blacksquare[G](*)$-modules is continuous for this topology. If $P_1 \to P_0 \to W \to 0$ is a presentation of W, one has an exact sequence of solid K-vector spaces $\underline{P_1} \to \underline{P_0} \to \underline{W} \to 0$, where the $\underline{P_i}$ are finite free $K_\blacksquare[G]$-modules. In other words, the category of finite $K_\blacksquare[G](*)$-modules embeds fully faithfully in the category of finite $K_\blacksquare[G]$-modules. This shows that a solid Banach representation of G is admissible if and only if $V(*)_{\mathrm{top}}$ is an admissible representation in the classical sense.

Proposition 4.48. *Let* V *be a Banach* G*-representation, then*

$$V^{R\mathbb{G}^{(h^+)}-an} = R\underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G,K)\otimes^L_{K_\blacksquare[G]} V^\vee, K).$$

In particular, if V *is admissible then*

$$V^{R\mathbb{G}^{(h^+)}-an} = V^{\mathbb{G}^{(h^+)}-an} = \underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G,K)\otimes_{K_\blacksquare[G]} V^\vee, K).$$

Furthermore, $V^{la} = \underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare[G]} V^\vee, K)$.

Proof. By Theorem 4.36 we have

$$\begin{aligned} V^{R\mathbb{G}^{(h^+)}-an} &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G,K), V) \\ &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G,K), R\underline{\mathrm{Hom}}_K(V^\vee, K)) \\ &= R\underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G,K)\otimes^L_{K_\blacksquare[G]} V^\vee, K), \end{aligned}$$

this proves the first statement. Moreover, if V is admissible then $V^\vee$ is a finite $K_\blacksquare[G]$-module. By flatness of the distribution algebra one gets that

$$\mathcal{D}^{(h^+)}(G,K)\otimes^L_{K_\blacksquare[G]} V^\vee = \mathcal{D}^{(h^+)}(G,K)\otimes_{K_\blacksquare[G]} V^\vee$$

is concentrated in degree 0 (we warn that the flatness is only algebraic, and that we use the fact that $V^\vee$ is a finite module over the Iwasawa algebra), this implies the second claim.

We now prove the last statement. Writing $V^\vee$ as a quotient $K_\blacksquare[G]^n \xrightarrow{f} K_\blacksquare[G]^m \to V^\vee \to 0$, we have an exact sequence

$$\mathcal{D}^{(h^+)}(G,K)^n \xrightarrow{1\otimes f} \mathcal{D}^{(h^+)}(G,K)^m \to \mathcal{D}^{(h^+)}(G,K)\otimes_{K_\blacksquare[G]} V^\vee \to 0. \tag{17}$$

Taking inverse limits in (17) one obtains an exact sequence (by topological Mittag-Leffler, cf. Lemma 3.27)

$$\mathcal{D}^{la}(G,K)^n \xrightarrow{1\otimes f} \mathcal{D}^{la}(G,K)^m \to \varprojlim_{h\to\infty} (\mathcal{D}^{(h^+)}(G,K) \otimes_{K_\blacksquare[G]} V^\vee) \to 0, \tag{18}$$

proving that $\varprojlim_{h\to\infty}(\mathcal{D}^{(h^+)}(G,K) \otimes_{K_\blacksquare[G]} V^\vee) = \mathcal{D}^{la}(G,K) \otimes_{K_\blacksquare[G]} V^\vee$. In particular, it also follows from (18) that $\mathcal{D}^{la}(G,K) \otimes_{K_\blacksquare[G]} V^\vee$ is a Fréchet space of compact type since it is a quotient of such. Moreover,

$$\mathcal{D}^{la}(G,K) \otimes_{K_\blacksquare[G]} V^\vee = \varprojlim_{h\to\infty} (\mathcal{D}^{(h)}(G,K)^B \otimes_{K_\blacksquare[G]} V^\vee)$$

(recall that we have arrows $\mathcal{D}^{(h')}(G,K) \to \mathcal{D}^{(h^+)}(G,K) \to \mathcal{D}^{(h)}(G,K)$ for $h' > h$, which induce arrows $\mathcal{D}^{(h')}(G,K)^B \to \mathcal{D}^{(h^+)}(G,K) \to \mathcal{D}^{(h)}(G,K)^B$). Using all this, one obtains

$$\begin{aligned}
\underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K) \otimes_{K_\blacksquare[G]} V^\vee, K) &= \underline{\mathrm{Hom}}_K(\varprojlim_{h\to\infty} (\mathcal{D}^{(h)}(G,K)^B \otimes_{K_\blacksquare[G]} V^\vee), K) \\
&= \varinjlim_{h\to\infty} \underline{\mathrm{Hom}}_K(\mathcal{D}^{(h)}(G,K)^B \otimes_{K_\blacksquare[G]} V^\vee, K) \\
&= \varinjlim_{h\to\infty} \underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G,K) \otimes_{K_\blacksquare[G]} V^\vee, K) \\
&= \varinjlim_{h\to\infty} V^{\mathbb{G}^{(h^+)}-an} \\
&= V^{la},
\end{aligned}$$

where the second equality follows from Lemma 3.31. □

Corollary 4.49. *The functor $V \mapsto V^{la}$ is exact on the category of admissible Banach G-representations. Moreover $V^{la}(*)_{\mathrm{top}} \subset V(*)_{\mathrm{top}}$ is a dense subspace.*

Proof. First, note that the category of admissible representations is an abelian category (being the dual of the category of finite $K_\blacksquare[G]$-modules, cf. Remark 4.47), and that a closed subrepresentation of an admissible representation is admissible (being the dual of a quotient of a finite $K_\blacksquare[G]$-module).

Let us prove that the functor $V \mapsto V^{la}$ is exact for admissible Banach G-representations. Indeed, by Proposition 4.48 we have

$$V^{la} = \underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K) \otimes_{K_\blacksquare[G]} V^\vee, K).$$

Then, the exactness follows by the anti-equivalence $V \mapsto V^\vee$ between admissible representations and finite $K_\blacksquare[G]$-modules, the faithful flatness of $\mathcal{D}^{la}(G,K)$ over $K_\blacksquare[G]$, and the duality Theorem 3.40. Furthermore, this also shows that $V^{la} \neq 0$ provided $V \neq 0$.

Finally, let $V' \subseteq V$ be the closure of V^{la} (i.e. the condensed object associated to the closure of $V^{la}(*)_{\mathrm{top}}$ in $V(*)_{\mathrm{top}}$). Then V' and $Q = V/V'$ are admissible representations and we have an exact sequence $0 \to (V')^{la} \to V^{la} \to Q^{la} \to 0$. The density of $V^{la}(*)_{\mathrm{top}}$ in $V(*)_{\mathrm{top}}$ follows since the functor $V \mapsto V^{la}$ is non-zero for non-zero V. □

5. Cohomology

In this last section we present our main applications to group cohomology. We obtain in particular

(1) An isomorphism between the continuous cohomology of a solid representation and that of its derived analytic vectors. This can be seen as a p-adic version of theorems of P. Blanc and G. D. Mostow for real Lie groups, cf. [Bla79, Mos61].
(2) A comparison theorem, for a locally analytic G-representation V, between its continuous and its locally analytic cohomology. This generalises the classical result of Lazard [Laz65, Théorème V.2.3.10] for finite dimensional $\mathbf{Q}_p$-representations to arbitrary solid K-vector spaces.
(3) A comparison theorem between locally analytic cohomology and Lie algebra cohomology. This recovers and generalises a result of Tamme [Tam15] (cf. also [HKN11] and [Lec12]), which in turns was a generalisation of the other main result of Lazard [Laz65, Théorème V.2.4.9] for finite dimensional representations.

The proof of (1) is an immediate consequence of our main Theorem 4.36 and a result of Kohlhaase (Theorem 5.8).

The key input for the comparison result in (2) is the existence of a finite free resolution of the trivial representation, as a module over the Iwasawa algebra of a small neighbourhood of 1 in G. This is an application of a lemma of Serre used by Lazard in [Laz65, Definition V.2.2.2]. We shall also need a version of the lemma proved by Koohlhase for distribution algebras, cf. [Koh11, Theorem 4.4]. Once one has this lemma at hand, the proof of (2) is rather formal using the machinery developed throughout this text.

Finally, to show (3) we follow the proof of Tamme constructing a resolution of the trivial representation in terms of the de Rham complex of the analytic groups. Then, we apply the same formal computation as before.

5.1. **Cohomology theories.** In this section we define different cohomology groups, which correspond to continuous, analytic and locally analytic cohomology in the literature. Indeed, using the bar resolutions and Corollary 2.19, one verifies that whenever V is a solid representation coming from a "classical space" (i.e. a Banach, Fréchet, LB or LF space) our definitions coincide with the usual ones.

In the following we will use the conventions of §4.1. In particular, we fix a compact p-adic Lie group G and an open normal uniform pro-p-group $G_0 \subset G$, we denote the h-analytic neighbourhood of G as $\mathbb{G}^{(h)}$, and define its open h-analytic neighbourhood as $\mathbb{G}^{(h^+)} = \bigcup_{h'>h} \mathbb{G}^{(h')}$. Recall from Lemma 4.19 that a solid $\mathcal{O}_K$-module V is an $\mathcal{O}_K$-linear G-representation if and only if it is a module over the Iwasawa algebra $\mathcal{O}_{K,\blacksquare}[G]$. If V is in addition a $\mathbb{G}^{(h^+)}$-analytic representation, by the main Theorem 4.36, V is naturally equipped with a $\mathcal{D}^{(h^+)}(G,K)$-module structure. Let $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of G and $U(\mathfrak{g})$ its universal enveloping algebra. There is a natural map of solid K-algebras $U(\mathfrak{g}) \to \mathcal{D}^{(h^+)}(G,K)$ given by derivations of $\mathfrak{g}$ on $C(\mathbb{G}^{(h^+)}, K)$ (cf. the discussion after [ST02b, Proposition 2.3]). In particular, a solid (locally) analytic representation of G has a natural action of $U(\mathfrak{g})$. We let $D(U(\mathfrak{g})_\blacksquare)$ denote the derived category of solid $U(\mathfrak{g})$-modules.

Definition 5.1.

(1) Let $C \in D(\mathcal{O}_{K,\blacksquare}[G])$. We define the continuous group cohomology of C as $R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K, C)$.
(2) Let $C \in D(\mathcal{O}_{K,\blacksquare}[G])$ be a derived $\mathbb{G}^{(h^+)}$-analytic representation. We define the $\mathbb{G}^{(h^+)}$-analytic cohomology of C as $R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C)$.
(3) Let $C \in D(\mathcal{O}_{K,\blacksquare}[G])$ be a derived locally analytic representation. We define the locally analytic cohomology of C as

$$\operatorname*{hocolim}_{h\to\infty} R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C^{R\mathbb{G}^{(h^+)}-an}).$$

(4) Let $C \in D(U(\mathfrak{g})_\blacksquare)$. We define the Lie algebra cohomology of C as

$$R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C).$$

Lemma 5.2.

(1) *There is a solid projective resolution of the trivial representation*

$$\cdots \to K_\blacksquare[G^{n+1}] \xrightarrow{d_n} K_\blacksquare[G^n] \to \cdots \to K_\blacksquare[G] \to K \to 0, \tag{19}$$

with differentials induced by the formula

$$d_n((g_0, g_1, \dots, g_n)) = \sum_{i=0}^{n} (-1)^i (g_0, \dots, \widehat{g}_i, \dots, g_{n+1}).$$

In particular, if V is a complete locally convex G-representation, one has that

$$\mathrm{Ext}^i_{K_\blacksquare[G]}(K, \underline{V}) = H^i_{cont}(G, V).$$

(2) *Let $h > 0$. The complex* (19) *has a unique extension to a projective resolution of $\mathcal{D}^{(h)}(G,K)$-modules*

$$\cdots \to \mathcal{D}^{(h)}(G^{n+1}, K) \to \cdots \to \mathcal{D}^{(h)}(G,K) \to K \to 0, \tag{20}$$

where $\mathcal{D}^{(h)}(G^{n+1},K) = \mathcal{D}^{(h)}(G,K) \otimes_{K_\blacksquare} \cdots \otimes_{K_\blacksquare} \mathcal{D}^{(h)}(G,K)$ ($(n+1)$-times). In particular, if V is a classical LB locally analytic representation, we have that

$$\varinjlim_{h\to\infty} \mathrm{Ext}^i_{\mathcal{D}^{(h^+)}(G,K)}(K, \underline{V}^{\mathbb{G}^{(h^+)}-an}) = \varinjlim_{h\to\infty} \mathrm{Ext}^i_{\mathcal{D}^{(h)}(G,K)}(K, \underline{V}^{\mathbb{G}^{(h)}-an}) = H^i_{la}(G, V),$$

where $H^i_{la}(G,V)$ is the cohomology of locally analytic cochains of G with values in V.

Proof. Let $(G^{n+1})_{n\in\mathbf{N}} \xrightarrow{\varepsilon} *$ be the augmented simplicial space with boundary maps $d_i^n : G^{n+1} \to G^n$ given by $d_i^n(g_0, \dots, g_n) = (g_0, \dots, \widehat{g}_i, \dots, g_n)$ for $n \geq 1$ and $0 \leq i \leq n$, degeneracy maps $s_i^n : G^{n+1} \to G^{n+2}$ given by $s_i^n(g_0, \dots, g_n) = (g_0, \dots, g_{i-1}, 1, g_{i+1}, \dots, g_n)$ for $n \geq 0$ and $0 \leq i \leq n+1$, and augmentation $\varepsilon : G \to *$. Taking the total complex of the augmented simplicial solid K-vector space $(K_\blacksquare[G^{n+1}])_{n\in\mathbf{N}} \xrightarrow{\varepsilon} K$ one obtains the bar resolution with homogeneous cochains of K

$$\dots \xrightarrow{d_2} K_\blacksquare[G^2] \xrightarrow{d_1} K_\blacksquare[G] \xrightarrow{\varepsilon} K \to 0, \tag{21}$$

where d_n is the unique map extending $d_n(g_0, \dots, g_n) = \sum_{i=0}^n (-1)^i (g_0, \cdots, \widehat{g}_i, \cdots, g_n)$, as well as homotopies $h_n : K_\blacksquare[G^{n+1}] \to K_\blacksquare[G^{n+2}]$ (for $i \geq -1$), defined by $h_n(g_0, \dots, g_n) \mapsto (1, g_0, \dots, g_n)$, between the identity of (21) and 0. Note in addition that, since G is profinite, the terms $K_\blacksquare[G^{n+1}]$ are compact projective

$K_{\blacksquare}[G]$-modules, where G acts diagonally. Moreover, the map $g_0 \otimes (g_1, \ldots, g_n) \mapsto (g_0, g_0 g_1, \ldots, g_0 g_1 \cdots g_n)$ induces an isomorphism $K_{\blacksquare}[G] \otimes_{K_{\blacksquare}} (K_{\blacksquare}[G^n])_0 \cong K_{\blacksquare}[G^{n+1}]$, where the second term in the tensor product is equipped with the trivial action of G. The second statement then follows from the fact that $\underline{V}$ is a solid K-vector space if V is complete locally convex, and that

$$\mathrm{Hom}_{K_{\blacksquare}[G]}(K_{\blacksquare}[G^{n+1}], \underline{V}) \cong \mathrm{Hom}_K((K_{\blacksquare}[G^n])_0, \underline{V}) = \mathrm{Cont}(G^n, V),$$

where the last equality follows from adjunction (cf. Proposition 2.3(1)). Note that the differential on the right hand side translates into $d_n f(g_1, \ldots, g_n) \mapsto g_1 f(g_2, \ldots g_n) + \sum_{i=1}^{n-1} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^n f(g_1, \ldots, g_{n-1})$, for $f : G^n \to V$, as usual.

For the second point we reason similarly. Consider the augmented simplicial space $(\mathbb{G}^{(h),n+1})_{n\in\mathbf{N}} \to *$ with boundary maps and degeneracy maps defined as above. Taking the total complex of the associated distribution spaces $(\mathcal{D}^{(h)}(G^{n+1}, K))_{n\in\mathbf{N}} \to K$ of $(\mathbb{G}^{(h),n+1})_{n\in\mathbf{N}} \to *$, one obtains that (20) is a resolution of K. Notice that the term $\mathcal{D}^{(h)}(G^{n+1}, K)$ is a projective $\mathcal{D}^{(h)}(G, K)$-module since it is isomorphic to $\mathcal{D}^{(h)}(G, K) \otimes_{K_{\blacksquare}} (\mathcal{D}^{(h)}(G^n, K))_0$ where the second term in the tensor product has the trivial action of G, and $\mathcal{D}^{(h)}(G^n, K)$ is a Smith K-vector space. Furthermore, Corollary 2.19 implies that for any solid K-vector space W,

$$C(\mathbb{G}^{(h),n}, W) = C(\mathbb{G}^{(h),n}, K)_{\blacksquare} \otimes_{K_{\blacksquare}} W = \underline{\mathrm{Hom}}_K(\mathcal{D}^{(h)}(G^n, K), W).$$

This implies that

$$\mathrm{Ext}^i_{\mathcal{D}^{(h)}(G,K)}(K, V^{\mathbb{G}^{(h)}-an}) = H^i(C(\mathbb{G}^{(h),\bullet}, V^{\mathbb{G}^{(h)}-an})),$$

where $C(\mathbb{G}^{(h),\bullet}, V^{\mathbb{G}^{(h)}-an}))$ is the complex of $\mathbb{G}^{(h)}$-analytic cochains of $V^{\mathbb{G}^{(h)}-an}$. Finally, taking colimits as $h \to \infty$ one obtains

$$\varinjlim_{h\to\infty} \mathrm{Ext}^i_{\mathcal{D}^{(h^+)}(G,K)}(K, V^{\mathbb{G}^{(h^+)}-an}) = \varinjlim_{h\to\infty} \mathrm{Ext}^i_{\mathcal{D}^{(h)}(G,K)}(K, V^{\mathbb{G}^{(h)}-an}) = H^i_{la}(G, V)$$

as wanted. $\square$

5.2. **Comparison results.** Next we state the main theorems of this section, which will be proved in §5.4.

Theorem 5.3. *Let $C \in D(K_{\blacksquare}[G])$, then*

$$R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C) = R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C^{Rla}).$$

Remark 5.4. More concretely, we will show that

$$R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C) = R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C^{R\mathbb{G}^{(h^+)}-an})$$

for $h \gg 0$.

Theorem 5.5 (Continuous vs. analytic vs. Lie algebra cohomology)**.** *Let $C \in D(K_{\blacksquare}[G])$ be a derived $\mathbb{G}^{(h^+)}$-analytic representation. Then*

$$R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C) \cong R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) \cong (R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C))^G.$$

Remark 5.6. The RHS term of the equation above means the following: if C is a derived $\mathbb{G}^{(h^+)}$-analytic complex, then we will show that there is an open normal subgroup $G_h \subset G$ such that

$$R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C) = R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G_h]}(K, C),$$

and the group G/G_h acts on the previous cohomology complex. As we are working in characteristic 0 and G/G_h is a finite group, taking invariants in the category of solid $K[G/G_h]$-modules is exact and one can form the complex

$$R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K,C)^G := R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K,C)^{G/G_h}.$$

5.3. **Key lemmas.** In the following we will work with complexes with equal terms but different differential maps. To make explicit the differentials we use the following notation: let C be a (homological) complex of $\mathcal{O}_{K,\blacksquare}$-modules with i-th term C_i and i-th differential $d_i : C_i \to C_{i-1}$, we note

$$C = [\cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots ; d_\bullet].$$

5.3.1. *Iwasawa and distribution algebras.* The following result is the main input for our calculations.

Theorem 5.7 (Lazard-Serre)**.** *Let G_0 be a uniform pro-p group of dimension d. Then there exists a projective resolution of the trivial module $\mathbf{Z}_p$ of the form*

$$P := [0 \to \mathbf{Z}_{p,\blacksquare}[G_0]^{\binom{d}{d}} \to \cdots \to \mathbf{Z}_{p,\blacksquare}[G_0]^{\binom{d}{i}} \to \cdots \to \mathbf{Z}_{p,\blacksquare}[G_0]^{\binom{d}{0}}; \alpha_\bullet].$$

Proof. We briefly sketch how the complex P is constructed from [Laz65, Définition V.2.2.2.1, Lemme V.2.1.1]. Let $g_1, \ldots, g_d \in G_0$ be a basis of the group and $b_i = [g_i] - 1 \in \mathbf{Z}_{p,\blacksquare}[G_0](*)$. The valuation of G_0 defines a filtration in $\mathbf{Z}_{p,\blacksquare}[G_0](*)$ whose graded algebra $\mathrm{gr}^\bullet(\mathbf{Z}_{p,\blacksquare}[G_0](*))$ is isomorphic to $\mathbb{F}_p[\pi][\overline{b}_1, \ldots, \overline{b}_d]$, where $\mathbb{F}_p[\pi] = \mathrm{gr}^\bullet(\mathbf{Z}_p)$ is the graduation of $\mathbf{Z}_p$ for the filtration induced by (p). Then, the Koszul complex of $\mathbb{F}_p[\pi][\overline{b}_1, \ldots, \overline{b}_d]$ with respect to the regular sequence $(\overline{b}_1, \ldots, \overline{b}_d)$ can be lifted by approximations to the complex P of the theorem. Furthermore, the proof also lifts a chain homotopy $\tilde{s}_\bullet$ between the identity and the augmentation map $\varepsilon : K[\overline{b}_1, \ldots, \overline{b}_d] \to \mathbb{F}_p[\pi]$, to a chain homotopy $s_\bullet$ between the identity and the augmentation map $\varepsilon : \mathbf{Z}_{p,\blacksquare}[G_0] \to \mathbf{Z}_p$. □

Theorem 5.8 (Kohlhaase)**.** *Let G_0 a uniform pro-p group of dimension d and $h > 0$. Then*

$$\mathcal{O}_K \otimes^L_{\mathcal{O}_{K,\blacksquare}[G_0]} \mathcal{D}_{(h)}(G_0, K) = K.$$

More precisely, the differentials $\alpha_i : \mathcal{O}_{K,\blacksquare}[G_0]^{\binom{d}{i}} \to \mathcal{O}_{K,\blacksquare}[G_0]^{\binom{d}{i-1}}$ of the resolution given by Theorem 5.7 extend to maps $\alpha_i : \mathcal{D}_{(h)}(G_0,K)^{\binom{d}{i}} \to \mathcal{D}_{(h)}(G_0,K)^{\binom{d}{i-1}}$, inducing a resolution of the trivial module K of the form

$$P_{(h)} := [0 \to \mathcal{D}_{(h)}(G_0,K)^{\binom{d}{d}} \to \cdots \to \mathcal{D}_{(h)}(G_0,K)^{\binom{d}{i}} \to \cdots \to \mathcal{D}_{(h)}(G_0,K)^{\binom{d}{0}}; \alpha_\bullet].$$

Proof. This is essentially [Koh11, Theorem 4.4]. Let $g_1, \ldots, g_d \in G_0$ be a basis and $b_i \in \mathbf{Z}_{p,\blacksquare}[G_0](*)$. The idea of the proof is to show that the differentials $\alpha_\bullet$ and the chain homotopy $s_\bullet$ of Theorem 5.7 are continuous with respect to the norms $|\sum_\alpha a_\alpha \mathbf{b}^\alpha|_r = \sup_\alpha |a_\alpha| r^{|\alpha|}$ for $\frac{1}{p} < r < 1$. Thus, the differentials $\alpha_\bullet$ and the chain homotopy $s_\bullet$ extend to the weak completion of these norms (i.e. the completion with respect to a radius r seen as a subspace in the completion of a slightly bigger radius $r < r' < 1$). Note that the distribution algebras constructed in this way are precisely the algebras $\mathcal{D}_{(h)}(G_0,K)$ of Definition 4.12. □

Remark 5.9. By definition $\mathcal{D}_{(h)}(G,K) := K_\blacksquare[G] \otimes_{K_\blacksquare[G_0]} \mathcal{D}_{(h)}(G_0,K)$. Therefore $\mathcal{D}_{(h)}(G,K) \otimes_{K_\blacksquare[G]} K = \mathcal{D}_{(h)}(G_0,K) \otimes_{K_\blacksquare[G_0]} K = K$. This implies that

$$\mathcal{D}_{(h)}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}_{(h)}(G,K) = K_\blacksquare[G] \otimes^L_{K_\blacksquare[G_0]} (\mathcal{D}_{(h)}(G_0,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}_{(h)}(G_0,K)).$$

With the help of Theorem 5.8 we can compute the following derived tensor product

Proposition 5.10. *We have*

$$\mathcal{D}_{(h)}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}_{(h)}(G,K) = \mathcal{D}_{(h)}(G,K).$$

Proof. First, we reduce to the case when G is a uniform pro-p-group by Remark 5.9. By Theorem 5.7 we can write

$$[0 \to K_\blacksquare[G] \to \cdots \to K_\blacksquare[G]^d \to K_\blacksquare[G]; \alpha] \simeq K.$$

Tensoring with $\mathcal{D}_{(h)}(G,K)$ over K we get

$$\begin{aligned}[0 \to K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K) &\to \cdots K_\blacksquare[G]^d \otimes_K \mathcal{D}_{(h)}(G,K) \\ &\to K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K); \alpha\otimes 1] \simeq \mathcal{D}_{(h)}(G,K).\end{aligned}$$

The quasi-isomorphism above is of $K_\blacksquare[G]$-modules for the diagonal action of G in the terms of the complex.

Let $\iota : K_\blacksquare[G] \to K_\blacksquare[G]$ be the antipode, i.e. the map induced by the inverse of the group, and denote in the same way its extension to the distribution algebra $\mathcal{D}_{(h)}(G,K)$. Consider the composition

$$\begin{aligned}K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K) &\xrightarrow{(1\otimes\iota)\otimes 1} K_\blacksquare[G] \otimes_K K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K) \\ &\xrightarrow{1\otimes m} K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K),\end{aligned}$$

where m is the left multiplication of $K_\blacksquare[G]$ on the distribution algebra. This map defines a G-equivariant isomorphism

$$\phi : K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K) \cong K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K)_0,$$

where the action of G in the image is left multiplication on $K_\blacksquare[G]$ and trivial on $\mathcal{D}_K(G,K)_0$. Notice that ϕ can be extended naturally to a G-equivariant isomorphism

$$\phi : \mathcal{D}_{(h)}(G,K) \otimes_K \mathcal{D}_{(h)}(G,K) \cong \mathcal{D}_{(h)}(G,K) \otimes_K \mathcal{D}_{(h)}(G,K)_0.$$

We define the complex

$$[0 \to K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K)_0 \to \cdots \to K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K)_0; \beta_\bullet] \tag{22}$$

to be the complex whose differentials are given by $\beta_\bullet = \phi \circ \alpha_\bullet \circ \phi^{-1}$. Notice that the maps β extend to respective complex with terms direct sums of $\mathcal{D}_{(h)}(G,K) \otimes_K \mathcal{D}_{(h)}(G,K)_0$.

Using this complex one can easily compute the derived tensor product by replacing the right $\mathcal{D}_{(h)}(G,K)$ with (22):

$$\begin{aligned}&\mathcal{D}_{(h)}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}_{(h)}(G,K) \\ &\quad\simeq [\cdots \to \mathcal{D}_{(h)}(G,K) \otimes_{K_\blacksquare[G]} (K_\blacksquare[G]^{\binom{d}{i}} \otimes_K \mathcal{D}_{(h)}(G,K)_0) \to \cdots; 1\otimes\beta] \\ &\quad= [\ldots \to \mathcal{D}_{(h)}(G,K)^{\binom{d}{i}} \otimes_K \mathcal{D}_{(h)}(G,K)_0 \to \cdots; \beta] \\ &\quad\simeq [\cdots \to \mathcal{D}_{(h)}(G,K)^{\binom{d}{i}} \otimes_K \mathcal{D}_{(h)}(G,K) \to \cdots; \alpha\otimes 1] \\ &\quad\simeq \mathcal{D}_{(h)}(G,K).\end{aligned}$$

In the above sequence of isomorphisms, the first quasi-isomorphism follows from the observation that the action of G on the complex (22) representing $\mathcal{D}_{(h)}(G,K)$ is trivial on the factor $\mathcal{D}_{(h)}(G,K)_0$. The second step is trivial. The third one follows by applying ϕ^{-1}. The fourth quasi-isomorphism follows from Theorem 5.8. This finishes the proof. $\square$

Corollary 5.11. *We have*

$$\begin{aligned}\mathcal{D}^{(h^+)}(G,K)\otimes^L_{K_{\blacksquare}[G]}\mathcal{D}^{(h^+)}(G,K) &= \mathcal{D}^{(h^+)}(G,K),\\ \mathcal{D}^{la}(G,K)\otimes^L_{K_{\blacksquare}[G]}\mathcal{D}^{la}(G,K) &= \mathcal{D}^{la}(G,K).\end{aligned}$$

Proof. This follows from Proposition 5.10 and the fact that $\mathcal{D}^{(h^+)}(G,K)$ can be written as a colimit of distribution algebras $\mathcal{D}_{(h')}(G,K)$, cf. Corollary 4.18. The case of $\mathcal{D}^{la}(G,K)$ follows from the same proof of Proposition 5.10 knowing that the complex of Theorem 5.8 extends to $\mathcal{D}^{la}(G,K)$. $\square$

5.3.2. *Enveloping and distribution algebras.* Let $\mathbb{G}_h$ be a rigid analytic group of Definition 4.4 and let $G_h=\mathbb{G}_h(\mathbf{Q}_p)$ be its rational points. Recall that $\mathbb{G}_h$ is a polydisc centered in $1\in G$ of radius p^{-h}, and that $\mathbb{G}^{(h)}=G\mathbb{G}_h$ is a finite disjoint union of copies of $\mathbb{G}_h$. Note that $G_h\subset G$ is an open compact subgroup. We denote by $\mathcal{D}(\mathbb{G}_h,K)$ the distribution algebra of $\mathbb{G}_{(h)}$-analytic functions, i.e. the dual of $C(\mathbb{G}_h,K)$. We also denote $\mathcal{D}(\mathbb{G}_{h^+},K)=\varinjlim_{h'>h}\mathcal{D}(\mathbb{G}_{h'},K)$, in other words, the distribution algebra of the rigid analytic group defined by the open unit polydisc $\mathbb{G}_{h^+}=\bigcup_{h'>h}\mathbb{G}_{h'}$. We assume that $\mathbb{G}_{h^+}(\mathbf{Q}_p)=G_h$.

Proposition 5.12 (Tamme)**.** *Keep the above notation, and let*

$$CE(\mathfrak{g}):=[0\to U(\mathfrak{g})\otimes\wedge^d\mathfrak{g}\to\cdots\to U(\mathfrak{g})\otimes\mathfrak{g}\to U(\mathfrak{g});d]$$

be the Chevalley-Eilenberg complex resolving the trivial representation K. Then $\mathcal{D}(\mathbb{G}_{h^+},K)\otimes^L_{U(\mathfrak{g})}K=K$. Moreover, the complex $\mathcal{D}(\mathbb{G}_{h^+},K)\otimes^L_{U(\mathfrak{g})}CE(\mathfrak{g})$ is the dual of the global sections of the de Rham complex of $\mathbb{G}_{h^+}$.

Proof. Let $[\Omega^\bullet_{\mathbb{G}_{h^+}},d]$ be the de Rham complex of $\mathbb{G}_{h^+}$, notice that the global sections of $\Omega^i_{\mathbb{G}_{h^+}}$ are equal to $C(\mathbb{G}_{h^+},K)\otimes_K\bigwedge^i(\mathfrak{g}^\vee)$. As $\mathbb{G}_{h^+}$ is an open polydisc, the Poincaré lemma holds (cf. [Tam15, Lemma 26]) and the global sections of the de Rham complex are

$$C(\mathbb{G}_{h^+},K)\xrightarrow{d}C(\mathbb{G}_{h^+},K)\otimes_K\mathfrak{g}^\vee\xrightarrow{d}\cdots\xrightarrow{d}C(\mathbb{G}_{h^+},K)\otimes_K\bigwedge^d\mathfrak{g}^\vee\to 0,\tag{23}$$

which is quasi-isomorphic to K via the inclusion of the constant functions $K\subset C(\mathbb{G}_{h^+},K)$. It is easy to show that the dual of (23) is equal to $\mathcal{D}(\mathbb{G}_{h^+},K)\otimes_{U(\mathfrak{g})}CE(\mathfrak{g})$. Finally, the fact that $\mathcal{D}(\mathbb{G}_{h^+},K)\otimes_{U(\mathfrak{g})}CE(\mathfrak{g})$ is a projective resolution of K as $\mathcal{D}(\mathbb{G}_{h^+},K)$-module follows from the exactness of (23) and the duality Theorem 3.40. $\square$

Lemma 5.13. *We have*

$$\mathcal{D}(\mathbb{G}_{h^+},K)\otimes^L_{U(\mathfrak{g})}\mathcal{D}(\mathbb{G}_{h^+},K)=\mathcal{D}(\mathbb{G}_{h^+},K).$$

In particular $\mathrm{Mod}^{\mathrm{solid}}_{\mathcal{D}(\mathbb{G}_{h^+},K)_{\blacksquare}}$ is a full subcategory of the category of solid $U(\mathfrak{g})$-modules.

Proof. The proof follows identically as that of Proposition 5.10 by replacing the Lazard-Serre resolution by the Chevalley-Eilenberg resolution $CE(\mathfrak{g})$ of Proposition 5.12. $\square$

5.4. **Proofs.**

Proof of Theorem 5.3. Let $C \in D(K_{\blacksquare}[G])$. By Theorem 4.36 we have

$$C^{R\mathbb{G}^{(h^+)}-an} = R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(\mathcal{D}^{(h^+)}(G,K), C).$$

We now compute

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C^{R\mathbb{G}^{(h^+)}-an}) &= R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(\mathcal{D}^{(h^+)}(G,K), C)) \\ &= R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K \otimes^L_{K_{\blacksquare}[G]} \mathcal{D}^{(h^+)}(G,K), C) \\ &= R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C), \end{aligned}$$

where the second equality is the tensor-Hom adjunction, and the third one follows from Theorem 5.8. □

Proof of Theorem 5.5. Let C be a derived $\mathbb{G}^{(h^+)}$-analytic representation of G. Theorem 4.36 says that C is a $\mathcal{D}^{(h^+)}(G,K)$-module. By Theorem 5.8 one has

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{D}^{(h^+)}(G,K) \otimes^L_{K_{\blacksquare}[G]} K, C) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C). \end{aligned}$$

On the other hand, since $\mathcal{D}^{(h^+)}(G,K) = \mathcal{O}_{K,\blacksquare}[G] \otimes_{\mathcal{O}_{K,\blacksquare}[G_h]} \mathcal{D}(\mathbb{G}_{h^+}, K)$, one has

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) = R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{G}_{h^+},K)}(K, C)^{G/G_h}.$$

By Proposition 5.12 we get

$$\begin{aligned} R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C) &= R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{G}_{h^+},K)}(\mathcal{D}(\mathbb{G}_{h^+}, K) \otimes^L_{U(\mathfrak{g})} K, C) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{G}_{h^+},K)}(K, C). \end{aligned}$$

Putting all together we obtain

$$R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) = R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C)^G$$

as we wanted. □

5.5. **Shapiro's lemma and Hochschild-Serre.** Let G be a compact p-adic Lie group of dimension d and H a closed subgroup of dimension e. One can find an open uniform pro-p-group $G_0 \subset G$ satisfying the following conditions:

(1) $H_0 := H \cap G_0$ is a uniform pro-p-group.
(2) There are charts $\phi_{G_0} : \mathbf{Z}_p^d \to G_0$ and $\phi_{H_0} : \mathbf{Z}_p^e \to H_0$ such that $\phi_{G_0} \circ \iota_e = \phi_{H_0}$, where $\iota_e : \mathbf{Z}_p^e \to \mathbf{Z}_p^d$ is the inclusion in the last e-components.

Indeed, taking $\mathfrak{g}_0 \subset \mathfrak{g}$ a small enough lattice as in [Eme17, §5.2] and $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0$, one can take $G_0 := \exp(\mathfrak{g}_0)$ and $H_0 := \exp(\mathfrak{h}_0)$. The profinite groups H_0 and G_0 allow us to define compatible rigid analytic neighbourhoods $\mathbb{H}^{(h^+)}$ and $\mathbb{G}^{(h^+)}$ of H and G respectively, with $\mathbb{G}^{(h^+)}/\mathbb{H}^{(h^+)}$ a finite disjoint union of open polydiscs of dimension $d-e$, and such that

$$C(\mathbb{G}^{(h^+)}, K) = C(\mathbb{G}^{(h^+)}/\mathbb{H}^{(h^+)}, K) \otimes_{K_{\blacksquare}} C(\mathbb{H}^{(h^+)}, K).$$

In other words, if $\mathcal{D}^{(h^+)}(G/H, K)$ denotes the dual of $C(\mathbb{G}^{(h^+)}/\mathbb{H}^{(h)}, K)$, we have an isomorphism of right $\mathcal{D}^{(h^+)}(H,K)$-modules

$$\mathcal{D}^{(h^+)}(G,K) = \mathcal{D}^{(h^+)}(G/H,K) \otimes_{K_{\blacksquare}} \mathcal{D}^{(h^+)}(H,K).$$

One has a similar description as left $\mathcal{D}^{(h^+)}(H,K)$-modules.

Definition 5.14. For $C \in D(K_\blacksquare[H])$ we define the solid induction and coinduction of C from H to G as

$$\mathrm{ind}_H^G(C) := K_\blacksquare[G] \otimes^L_{K_\blacksquare[H]} C,$$

$$\mathrm{coind}_H^G(C) := R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(K_\blacksquare[G], C),$$

where the action of G is given by left multiplication on $K_\blacksquare[G]$ for the induction, and by right multiplication on $K_\blacksquare[G]$ for the coinduction. If C is derived $\mathbb{H}^{(h^+)}$-analytic, define the analytic induction and coinduction as

$$h\text{-}\mathrm{ind}_H^G(C) := \mathcal{D}^{(h^+)}(G,K) \otimes^L_{\mathcal{D}^{(h^+)}(H,K)} C,$$

$$h\text{-}\mathrm{coind}_H^G(C) := R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(H,K)}(\mathcal{D}^{(h^+)}(G,K), C).$$

Proposition 5.15 (Shapiro's lemma)**.** *Let $C \in D(K_\blacksquare[G])$, $C' \in D(K_\blacksquare[H])$. Then ind_H^G (resp. coind_H^G) is the left (resp. right) adjoint of the restriction map $D(K_\blacksquare[G]) \to D(K_\blacksquare[H])$. In other words,*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathrm{ind}_H^G(C'), C) = R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(C', C),$$

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(C, \mathrm{coind}_H^G(C')) = R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(C, C').$$

Analogously, if C and C' are derived $\mathbb{G}^{(h^+)}$-analytic and $\mathbb{H}^{(h^+)}$-analytic representations, then

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(h\text{-}\mathrm{ind}_H^G(C'), C) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(H,K)}(C', C),$$

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(C, h\text{-}\mathrm{coind}_H^G(C')) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(H,K)}(C, C').$$

Proof. The first statement follows formally:

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G] \otimes^L_{K_\blacksquare[H]} C, C') &= R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(C, R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G], C')) \\ &= R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(C, C'). \end{aligned}$$

The rest of the statements are proved in a similar way. $\square$

Proposition 5.16 (Hochschild-Serre)**.** *Let $H \subset G$ be a normal closed subgroup and $C \in D(K_\blacksquare[G])$. Then*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G/H]}(K, R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(K, C)).$$

If C is derived $\mathbb{G}^{(h^+)}$-analytic, then

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G/H,K)}(K, R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(H,K)}(K, C)).$$

Proof. By Shapiro's lemma we have

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(K, C) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G] \otimes^L_{K_\blacksquare[H]} K, C).$$

Applying the functor $R\underline{\mathrm{Hom}}_{K_\blacksquare[G/H]}(K, -)$ to both sides, using $K_\blacksquare[G] \otimes^L_{K_\blacksquare[H]} K = K_\blacksquare[G/H]$ and the usual adjunction one obtains

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G/H]}(K, R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(K, C)) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C),$$

as desired. The rest of the statements are proved in a similar way. $\square$

5.6. **Homology and duality.**

Definition 5.17. Let $C \in D(K_{\blacksquare}[G])$. We define the solid group homology of C as

$$K \otimes^L_{K_{\blacksquare}[G]} C.$$

Analogously, if C is derived $\mathbb{G}^{(h^+)}$-analytic, define its $\mathbb{G}^{(h^+)}$-analytic homology as

$$K \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C.$$

We have the following formal duality between homology and cohomology.

Lemma 5.18. *Let $C \in D(K_{\blacksquare}[G])$. Then*

$$R\underline{\mathrm{Hom}}_K(K \otimes^L_{K_{\blacksquare}[G]} C, K) = R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, R\underline{\mathrm{Hom}}_K(C, K)).$$

If C is $\mathbb{G}^{(h^+)}$-analytic, then

$$R\underline{\mathrm{Hom}}_K(K \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C, K) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, R\underline{\mathrm{Hom}}_K(C, K)).$$

Let $K(\chi) = \bigwedge^d \mathfrak{g}^\vee$ denote the determinant of the dual adjoint representation of G. Using Lazard-Serre's Theorem 5.7 one easily deduces that $R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, K_{\blacksquare}[G])$, endowed with the right multiplication of G, is a character concentrated in degree $-d$. Moreover, using the de Rham complex of $\mathbb{G}^{(h^+)}$ one can even prove that

$$R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, K_{\blacksquare}[G]) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, \mathcal{D}^{(h^+)}(G,K)) = K(\chi)[-d]. \tag{24}$$

Theorem 5.19 relates cohomology and homology in a more interesting way.

Theorem 5.19. *Let $C \in D(K_{\blacksquare}[G])$. Then there is a natural quasi-isomorphism*

$$R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K, C) = K(\chi)[-d] \otimes^L_{K_{\blacksquare}[G]} C.$$

Furthermore, if C is derived $\mathbb{G}^{(h^+)}$-analytic, we have

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) = K(\chi)[-d] \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C.$$

Proof. First observe that, given any G-equivariant map $\alpha : K_{\blacksquare}[G]_{\star_1} \to K_{\blacksquare}[G]_{\star_1}$, one has a commutative diagram

$$\begin{tikzcd} R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K_{\blacksquare}[G]_{\star_1}, C) \arrow[d, "\alpha^*"] & \underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K_{\blacksquare}[G]_{\star_1}, K_{\blacksquare}[G]) \otimes^L_{K_{\blacksquare}[G]} C \arrow[l, "\sim"'] \arrow[d, "\alpha^*\otimes 1"] \\ R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K_{\blacksquare}[G]_{\star_1}, C) & \underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K_{\blacksquare}[G]_{\star_1}, K_{\blacksquare}[G]) \otimes^L_{K_{\blacksquare}[G]} C, \arrow[l, "\sim"'] \end{tikzcd} \tag{25}$$

where the rows are G-equivariant. The G-actions on the different terms of (25) are given as follows: on $R\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K_{\blacksquare}[G]_{\star_1}, C)$ the group G acts via the right regular action $\star_2$ on $K_{\blacksquare}[G]$. Concerning the term $\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K_{\blacksquare}[G]_{\star_1}, K_{\blacksquare}[G]) \otimes^L_{K_{\blacksquare}[G]} C$ we have: the Hom space $\underline{\mathrm{Hom}}_{K_{\blacksquare}[G]}(K_{\blacksquare}[G]_{\star_1}, K_{\blacksquare}[G])$ is taken with respect to the left regular action $\star_1$ on each term. It is endowed with an action of $G \times G$ given by the right regular actions. The tensor product $\otimes^L_{K_{\blacksquare}[G]} C$ is taken with respect to the action given by $\{1\} \times G$ (i.e. the $\star_2$-action of G on the target of the Hom space). Finally, the G-action on the whole term is the one induced by the $\star_2$-action of G on the source of the Hom space.

Notice that there is a natural identification of right $K_\blacksquare[G]$-modules

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G]_{\star_1}, K_\blacksquare[G]) = K_\blacksquare[G]_{\star_2}.$$

Recall that, by Theorem 5.7, we have a projective resolution $K \simeq [K_\blacksquare[G]_{\star_1}^{\binom{d}{\bullet}}; \alpha_\bullet]$. We obtain

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}([K_\blacksquare[G]_{\star_1}^{\binom{d}{\bullet}}; \alpha_\bullet], C) \\ &= [\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G]_{\star_1}, K_\blacksquare[G])^{\binom{d}{\bullet}}; \alpha_\bullet^* \otimes 1] \otimes^L_{K_\blacksquare[G]} C \\ &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}([K_\blacksquare[G]_{\star_1}^{\binom{d}{\bullet}}; \alpha_\bullet], K_\blacksquare[G]) \otimes^L_{K_\blacksquare[G]} C \\ &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, K_\blacksquare[G]) \otimes^L_{K_\blacksquare[G]} C \\ &= K(\chi)[-d] \otimes^L_{K_\blacksquare[G]} C, \end{aligned}$$

where the second equality follows by (25), and the last one by (24). The statement for $\mathbb{G}^{(h^+)}$-analytic cohomology is proven in the same way. $\square$

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