REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 26, Pages 1025–1046 (October 4, 2022)
https://doi.org/10.1090/ert/628

TOTAL POSITIVITY IN SYMMETRIC SPACES

G. LUSZTIG

ABSTRACT. In this paper we extend the theory of total positivity for reductive groups to the case of symmetric spaces.

0.1. Let K be a real closed field, that is, a field of characteristic zero such that K is not algebraically closed but $\overline{K} := K[\sqrt{-1}]$ is algebraically closed. Let G be a connected reductive algebraic group over $\overline{K}$ with a pinning $(B^+, B^-, T, x_i, y_i (i \in I))$ as in [Lus94, 1.1]. Let U^+, U^- be the unipotent radicals of B^+, B^-. Recall that $x_i : \overline{K} \to U^+, y_i : \overline{K} \to U^-$ are certain imbeddings of algebraic groups. We identify G with its group of $\overline{K}$-points.

Let $\sigma : G \to G$ be the antiautomorphism of G such that

$$\sigma(x_i(a)) = x_i(a), \sigma(y_i(a)) = y_i(a)$$

for all $i \in I, a \in \overline{K}$ and $\sigma(t) = t^{-1}$ for all $t \in T$. Let $\omega : G \to G$ be an involutive automorphism of G preserving T and such that for some involution $i \mapsto i^*$ of I we have

$$\omega(x_i(a)) = x_{i^*}(a), \omega(y_i(a)) = y_{i^*}(a)$$

for all $i \in I, a \in \overline{K}$. We have $\sigma^2 = 1$, $\sigma\omega = \omega\sigma$. We set $\tau = \sigma\omega = \omega\sigma$; this is an involutive antiautomorphism of G preserving T and such that

$$\tau(x_i(a)) = x_{i^*}(a), \tau(y_i(a)) = y_{i^*}(a)$$

for all $i \in I, a \in \overline{K}$. Hence $g \mapsto \tau(g^{-1})$ is an involutive automorphism of G. Let $H = \{g \in G; \tau(g^{-1}) = g\}$ be the fixed point set of this automorphism; then G/H is a symmetric space (of a special type). Let $G^\tau = \{g \in G; \tau(g) = g\}$ (a closed smooth subvariety of G).

Now G acts on G by

(a) $g : g_1 \mapsto g g_1 \tau(g)$.

The stabilizer of 1 in this action is H so that G/H can be identified with the G-orbit $(G^\tau)^0$ of 1 under this action via $gH \mapsto g\tau(g)$. We have $(G^\tau)^0 \subset G^\tau$. (This containment can be strict.)

0.2. Since U^+, U^- are τ-stable, $U^{+\tau} = U^+ \cap G^\tau$ is a closed smooth subvariety of U^+ on which U^+ acts by 0.1(a) and $U^{-\tau} = U^- \cap G^\tau$ is a closed smooth subvariety of U^- on which U^- acts by 0.1(a).

Received by the editors January 23, 2022, and, in revised form, May 30, 2022, and August 4, 2022.

2020 *Mathematics Subject Classification.* Primary 20G05, 20G99.

Key words and phrases. Reductive group, symmetric space, total positivity.

This work was supported by NSF grant DMS-1855773 and by a Simons Fellowship.

0.3. Let $G(K)$ be the group of K-points of G, a subgroup of G. Let $G^\tau(K) = G^\tau \cap G(K)$, $(G^\tau)^0(K) = (G^\tau)^0 \cap G(K)$, $U^{+\tau}(K) = U^{+\tau} \cap G(K)$, $U^{-\tau}(K) = U^{-\tau} \cap G(K)$, $T(K) = T \cap G(K)$. Let $(G/H)(K)$ be the subset of G/H corresponding to $(G^\tau)^0(K)$ under the identification $(G^\tau)^0 = G/H$.

0.4. Let $K_{>0} = \{a^2; a \in K - \{0\}\} \subset K - \{0\}$. Then $K_{>0}$ is closed under addition, multiplication and division (it is a semifield). Let $K_{\geq 0} = K_{>0} \cup \{0\} \subset K$. For a, b in K we write $a > b$ (resp. $a \geq b$) whenever $a - b \in K_{>0}$ (resp. $a - b \in K_{\geq 0}$); this is a total order on K.

Let $T_{>0} = \{t^2; t \in T(K)\}$, a subgroup of $T(K)$. Following [Lus94] let

$U^+_{\geq 0}$ be the submonoid of U^+ generated by $x_i(a)$ with $i \in I, a \in K_{\geq 0}$;

$U^-_{\geq 0}$ be the submonoid of U^- generated by $y_i(a)$ with $i \in I, a \in K_{\geq 0}$;

$G_{\geq 0}$ be the submonoid of G generated by $x_i(a), y_i(a)$ with $i \in I, a \in K_{\geq 0}$ and by $T_{>0}$.

We set $G^\tau_{\geq 0} = G^\tau \cap G_{\geq 0}$, $U^{+\tau}_{\geq 0} = U^{+\tau} \cap U^+_{\geq 0}$, $U^{-\tau}_{\geq 0} = U^{-\tau} \cap U^-_{\geq 0}$.

The subsets $G^\tau_{\geq 0}, U^{+\tau}_{\geq 0}, U^{-\tau}_{\geq 0}$ are not monoids but instead are sets with action of the monoid $G_{\geq 0}, U^+_{\geq 0}, U^-_{\geq 0}$ (respectively) which act by 0.1(a).

In §3.7 we show that $G^\tau_{\geq 0}$ is actually a subset of $(G^\tau)^0(K)$; hence it can be viewed as a subset of $(G/H)(K)$ said to be the totally positive part of $(G/H)(K)$.

In this paper we show that several properties of the totally positive monoids $G_{\geq 0}, U^+_{\geq 0}, U^-_{\geq 0}$ discussed in [Lus94], such as existence of natural "cell" decompositions (1.2(d), 1.2(j)), have analogues for the totally positive subsets $G^\tau_{\geq 0}, U^{+\tau}_{\geq 0}, U^{-\tau}_{\geq 0}$ of $G^\tau, U^{+\tau}, U^{-\tau}$. (See 2.4(b), 2.5(c), 3.8(b). We will write "cell" for something that is really a cell when $K = \mathbf{R}$.) Actually most of the results in this paper are not only analogous to results in [Lus94] but contain those results in [Lus94] as a special case (see §5.3). But there is one property of cells in [Lus94] which needs to be modified in the present case.

Let W be the Weyl group of G. In [Lus94] to each $w \in W$ we associate a cell parametrized in several ways by $K^n_{>0}$ (with $K = \mathbf{R}$) in such a way that one obtains a positive structure on that cell, that is, any two parametrizations are related by a transformation given by substraction free rational functions. One of our results is that for any twisted involution (see 2.1) $w \in W$ one can attach a cell parametrized in several ways by $K^{n'}_{>0}$ (with $K = \mathbf{R}$) in such a way that (with some restriction on $*$) one obtains something more general than a positive structure in the sense that any two parametrizations are related by a transformation given by substraction free rational functions combined with extraction of square roots. See §6. We regard this as the main contribution of this paper.

0.5. Many results of this paper extend (with essentially the same proof) to the case where G is replaced by a group attached to a crystallographic, possibly not positive definite, symmetric Cartan matrix.

Notation. For any $c \in K_{>0}$ we denote by $\sqrt{c}$ the > 0 square root of c.

1. Preliminaries on $G_{\geq 0}, U^+_{\geq 0}, U^-_{\geq 0}$

1.1. For $i \in I$ let $s_i \in W$ be the simple reflection corresponding to i. Let $S = \{s_i; i \in I\}$. Let $w \mapsto w^*$ be the involution of W such that $(s_i)^* = s_{i^*}$ for any $i \in I$.

Let $w \mapsto |w|$ be the length function on W. There is a well defined monoid structure $w, w' \mapsto w \bullet w'$ on W characterized by

$$s_i \bullet w = s_i w \text{ if } i \in I, w \in W, |s_i w| = |w| + 1,$$
$$s_i \bullet w = w \text{ if } i \in I, w \in W, |s_i w| = |w| - 1.$$

For $w \in W$ let $\mathcal{I}(w)$ be the set of sequences $(i_1, i_2 \dots, i_k)$ in I such that $k = |w|$ and $w = s_{i_1} s_{i_2} \dots s_{i_k}$.

Let w_0 be the longest element of W.

1.2. We now recall some definitions and results from [Lus94]. Let $w \in W$. For $\mathbf{i} = (i_1, i_2, \dots, i_k) \in \mathcal{I}(w)$ we define $\Psi^+_{\mathbf{i}} : (K - \{0\})^k \to U^+$ by

$$(a_1, a_2, \dots, a_k) \mapsto x_{i_1}(a_1) x_{i_2}(a_2) \dots x_{i_k}(a_k)$$

and $\Psi^-_{\mathbf{i}} : (K - \{0\})^k \to U^-$ by

$$(a_1, a_2, \dots, a_k) \mapsto y_{i_1}(a_1) y_{i_2}(a_2) \dots y_{i_k}(a_k).$$

By the proof of [Lus94, 2.7(a)],

(a) $\Psi^+_{\mathbf{i}}, \Psi^-_{\mathbf{i}}$ are injective.

Let $\psi^+_{\mathbf{i}} : K^k_{>0} \to U^+$, $\psi^-_{\mathbf{i}} : K^k_{>0} \to U^-$ be the restrictions of $\Psi^+_{\mathbf{i}}, \Psi^-_{\mathbf{i}}$. By [Lus94],

(b) $\psi^+_{\mathbf{i}}$ defines a bijection of $K^k_{>0}$ onto a subspace $U^+_{\geq 0,w}$ of $U^+_{\geq 0}$ independent of the choice of $\mathbf{i}$ in $\mathcal{I}(w)$;

(c) $\psi^-_{\mathbf{i}}$ defines a bijection of $K^k_{>0}$ onto a subspace $U^-_{\geq 0,w}$ of $U^-_{\geq 0}$ independent of the choice of $\mathbf{i}$ in $\mathcal{I}(w)$;

(d) we have a partition $U^+_{\geq 0} = \sqcup_{w \in W} U^+_{\geq 0,w}$;

we have a partition
$U^-_{\geq 0} = \sqcup_{w \in W} U^-_{\geq 0,w}$.

The subsets $U^+_{\geq 0,w}$ of $U^+_{\geq 0}$ are said to be the "cells" of $U^+_{\geq 0}$; the subsets $U^-_{\geq 0,w}$ of $U^-_{\geq 0}$ are said to be the "cells" of $U^-_{\geq 0}$.

For any w, w' in W we set

(e) $G_{\geq 0,w,w'} = U^-_{\geq 0,w} T_{>0} U^+_{\geq 0,w'} = U^+_{\geq 0,w'} T_{>0} U^-_{\geq 0,w} \subset G_{\geq 0}$.

By [Lus94], we have bijections

(f) $U^-_{\geq 0,w} \times T_{>0} \times U^+_{\geq 0,w'} \xrightarrow{\sim} G_{\geq 0,w,w'}$,

(g) $U^+_{\geq 0,w'} \times T_{>0} \times U^-_{\geq 0,w} \xrightarrow{\sim} G_{\geq 0,w,w'}$,

given by multiplication in G. By [Lus94],

(i) we have a partition $G_{\geq 0} = \sqcup_{(w,w') \in W \times W} G_{\geq 0,w,w'}$.

The subsets $G_{\geq 0,w,w'}$ of $G_{\geq 0}$ are said to be the "cells" of $G_{\geq 0}$.

1.3. By [Lus94], for w, w' in W, multiplication in G satisfies

(a) $U^+_{\geq 0,w} U^+_{\geq 0,w'} \subset U^+_{\geq 0,w \bullet w'}$,

(b) $U^-_{\geq 0,w} U^-_{\geq 0,w'} \subset U^-_{\geq 0,w \bullet w'}$.

It follows that for w_1, w_2, w'_1, w'_2 in W, multiplication in G satisfies

(c) $G_{\geq 0,w_1,w'_1} G_{\geq 0,w_2,w'_2} \subset G_{\geq 0,w_1 \bullet w_2,w'_1 \bullet w'_2}$.

1.4. In this subsection we fix $w \in W, i \in I$ such that setting $s_i = s$, we have $s^*w = ws, |w| = |s^*w| + 1$. Let $\mathbf{i}' = (i_1, i_2, \dots, i_k) \in \mathcal{I}(s^*w)$, $\mathbf{i} = (i^*, i_1, i_2, \dots, i_k) \in \mathcal{I}(w)$, $\tilde{\mathbf{i}} = (i_1, i_2, \dots, i_k, i) \in \mathcal{I}(w)$. We define

(a) $\alpha : U^+_{\ge 0, s^*w} \times K_{>0} \to U^+_{\ge 0, w}$

by $(u, a) \mapsto x_{i^*}(a)ux_i(a)$. This is well defined since $s^* \bullet (s^*w) \bullet s = w$ (see 1.3(a)). We show:

(b) *The map α is a bijection.*

We first show that α is injective. Assume that u, u' in $U^+_{\ge 0, s^*w}$ and a, a' in $K_{>0}$ satisfy $x_{i^*}(a)ux_i(a) = x_{i^*}(a')u'x_i(a')$. Then $x_{i^*}(a - a')u = u'x_i(a' - a)$.

If $a - a' > 0$, then by 1.2(b) we have

$$x_{i^*}(a - a')u \in \psi^+_{\mathbf{i}}(K^{k+1}_{>0}) = \psi^+_{\tilde{\mathbf{i}}}(K^{k+1}_{>0}),$$

hence $x_{i^*}(a - a')u = u_1 x_i(b)$ where $u_1 \in U^+_{\ge 0, s^*w}$ and $b \in K_{>0}$. Thus we have $u_1 x_i(b) = u'x_i(a' - a)$. Using 1.2(a) (recall that $a' - a \ne 0$) we see that $b = a' - a$. Thus, $a' - a > 0$, a contradiction.

If $a' - a > 0$, then by 1.2(b) we have

$$u'x_i(a' - a) \in \psi^+_{\tilde{\mathbf{i}}}(K^{k+1}_{>0}) = \psi^+_{\mathbf{i}}(K^{k+1}_{>0}),$$

hence $u'x_i(a' - a) = x_{i^*}(b)u_1$ where $u_1 \in U^+_{\ge 0, s^*w}$ and $b \in K_{>0}$. Thus we have $x_{i^*}(b)u_1 = x_{i^*}(a - a')u$. Using 1.2(a) (recall that $a - a' \ne 0$) we see that $b = a - a'$. Thus $a - a' > 0$, a contradiction.

We see that we must have $a = a'$. It follows that $u = u'$ and the injectivity of α is proved.

We now prove that α is surjective. It is enough to show that for any $u \in U^+_{\ge 0, s^*w}$ and any $b \in K_{>0}$ we have $x_{i^*}(b)u = x_{i^*}(b')u'x_i(b')$ for some $u' \in U^+_{\ge 0, s^*w}$ and some $b' \in K_{>0}$. This is proved by the following argument, inspired by one which I have learned from Xuhua He.[1]

We can write uniquely $u = x_{i_1}(a_1)x_{i_2}(a_2) \dots x_{i_k}(a_k)$ with $a_1, a_2 \dots, a_k$ in $K_{>0}$. For $c \in K_{>0}$ we have $x_{i^*}(c)u = x_{i^*}(c)x_{i_1}(a_1)x_{i_2}(a_2) \dots x_{i_k}(a_k)$. Using 1.2(b) we see that we have also $x_{i^*}(c)u = x_{i_1}(b_1)x_{i_2}(b_2) \dots x_{i_k}(b_k)x_i(b_{k+1})$ where $b_1, b_2, \dots, b_{k+1}$ are uniquely determined in $K_{>0}$. We shall regard $b_1, b_2, \dots, b_{k+1}$ as functions of c (here $a_1, a_2, \dots, a_k$ are fixed). By results in [Lus94](see also [Lus19]), each of $b_1, b_2, \dots, b_{k+1}$ can be expressed as a substraction free rational function in $c, a_1, a_2, \dots, a_k$. In particular, there exist non-zero polynomials $P(\epsilon), Q(\epsilon)$ in $K[\epsilon]$ (ϵ is an indeterminate) with all coefficients in $K_{\ge 0}$ such that $b_{k+1} = P(c)/Q(c)$ when $c \in K_{>0}$; note that $Q(c) > 0$ when $c \in K_{>0}$. (Similar expressions exist for the other b_i but we do not need them). We can assume that not both $P(\epsilon), Q(\epsilon)$ are divisible by ϵ.

Now one checks easily that the map $(\overline{K} - \{0\})^k \times \overline{K} \to U^+$ given by

$$(d_1, d_2, \dots, d_k, d_{k+1}) \mapsto x_{i_1}(d_1)x_{i_2}(d_2) \dots x_{i_k}(d_k)x_i(d_{k+1})$$

is an isomorphism of $(\overline{K} - \{0\})^k \times \overline{K}$ onto a subvariety Z of U^+. Let $\gamma : Z \to (\overline{K} - \{0\})^k \times \overline{K}$ be the inverse of this isomorphism. Note that $c \mapsto x_{i^*}(c)u$ is a morphism $\delta : \overline{K} \to U^+$ such that $\delta(K_{\ge 0}) \subset Z$. (We have $\delta(0) = u \in Z$.) It follows that $\delta^{-1}(Z)$ is a subvariety of $\overline{K}$ containing $K_{\ge 0}$, hence there exists an

[1] X. He showed, in connection with a different problem, that α is surjective when $K = \mathbf{R}$, W is of type B_2 and $|w| = 4$.

open subset $\mathcal{U}$ of $\overline{K}$ such that $K_{\geq 0} \subset \mathcal{U}$ and δ restricts to a morphism $\delta' : \mathcal{U} \to Z$. Then the composition of $\gamma\delta' : \mathcal{U} \to (\overline{K} - \{0\})^k \times \overline{K}$ with the last projection $(\overline{K} - \{0\})^k \times \overline{K} \to \overline{K}$ is a morphism $\rho : \mathcal{U} \to \overline{K}$. In other words $c \mapsto b_{k+1}$ from $K_{>0}$ to $K_{>0}$ is the restriction of the morphism $\rho : \mathcal{U} \to \overline{K}$. We have $\rho(0) = 0$. (Indeed, we have $\delta(0) = u = \gamma(a_1, \dots, a_k, 0)$.) We see that $c \mapsto P(c)/Q(c)$ from $K_{>0}$ to $K_{>0}$ is the restriction of $\rho : \mathcal{U} \to \overline{K}$ and $\rho(0) = 0$. It follows that $P(0) = 0, Q(0) \neq 0$ and that $\rho(c) = P(c)/Q(c)$ for $c \in K_{\geq 0}$. Since all coefficients of $Q(\epsilon)$ are in $K_{\geq 0}$ it follows that $Q(0) > 0$.

For $c \in K_{\geq 0}$ we have $x_{i^*}(c)u = u'(c)x_i(P(c)/Q(c))$ where $u'(c) \in U^+_{\geq 0, s^*w}$.

Let $[0, b] = \{b' \in K; 0 \leq b' \leq b\}$. If $b' \in [0, b]$ we have

$$\begin{aligned} x_{i^*}(b)u = x_{i^*}(b')x_{i^*}(b - b')u &= x_{i^*}(b')u'(b - b')x_i(P(b - b')/Q(b - b')) \\ &= x_{i^*}(b')u'(b - b')x_i(\frac{P(b - b')}{Q(b - b')} - b')x_i(b'). \end{aligned}$$

Next we note that

(c) the function $b' \mapsto r(b') := P(b - b') - b'Q(b - b')$ from $[0, b]$ to K takes the value 0 for some $b' \in K, 0 < b' < b$.

This follows from the intermediate value theorem (known to hold for our K) applied to the polynomial function $r : [0, b] \to K$. This function changes sign: we have $r(0) = P(b) > 0$ and $r(b) = P(0) - bQ(0) = -bQ(0) < 0$.

For b' as in (c) we have

$$x_{i^*}(b)u = x_{i^*}(b')u'(b - b')x_i(b').$$

This completes the proof of (b).

1.5. In the setup of §1.4 recall that

$$\mathbf{i}' = (i_1, i_2, \dots, i_k) \in \mathcal{I}(s^*w), \tilde{\mathbf{i}} = (i_1, i_2, \dots, i_k, i) \in \mathcal{I}(w).$$

We identify $U^+_{\geq 0, s^*w} = K^k_{>0}$ via $\psi_{\mathbf{i}'}$ and $U^+_{\geq 0, w} = K^{k+1}_{>0}$ via $\psi_{\tilde{\mathbf{i}}}$. Then α in §1.4 becomes a bijection $K^k_{>0} \times K_{>0} \to K^{k+1}_{>0}$. One can show that the inverse of this bijection is not in general given by rational functions. For example, if W is of type A_3, $w = w_0$, $* = 1$ and $s \in S$ is such that $sw = ws$ then the formula for this inverse involves taking roots of quadratic equations.

1.6. In this subsection we fix $w \in W, w' \in W, i \in I$ such that, setting $s_i = s$, we have $s^*w = ws, |w| = |s^*w| + 1$. We define

(a) $\alpha : G_{\geq 0, s^*w, w'} \times K_{>0} \to G_{\geq 0, w, w'}$

by $(g, a) \mapsto y_{i^*}(a)gy_i(a)$. This is well defined since $s^* \bullet (s^*w) \bullet s = w$ (see 1.3(c)). We show:

(b) *the map α is a bijection.*

The proof is almost a copy of that of §1.4. We first show that α is injective. Assume that g, g' in $G_{\geq 0, s^*w, w'}$ and a, a' in $K_{>0}$ satisfy $y_{i^*}(a)gy_i(a) = y_{i^*}(a')g'y_i(a')$. Then $y_{i^*}(a - a')g = g'y_i(a' - a)$.

If $a - a' > 0$ then using 1.2(c),(e) we see that $y_{i^*}(a - a')g = g_1y_i(b)$ where $g_1 \in G_{\geq 0, s^*w, w'}$ and $b \in K_{>0}$. Thus we have $g_1y_i(b) = g'y_i(a' - a)$. We can write $g_1 = u_1t\tilde{u}_1$, $g' = u't'\tilde{u}'$ with u_1, u' in $U^+_{\geq 0, w'}$, $\tilde{u}_1, \tilde{u}'$ in $U^-_{\geq 0, s^*w}$, t, t' in $T_{>0}$. From $u_1t\tilde{u}_1y_i(b) = u't'\tilde{u}'y_i(a' - a)$ we deduce $u_1 = u', t = t'$, $\tilde{u}_1y_i(b) = \tilde{u}'y_i(a' - a)$

(uniqueness of Bruhat decomposition). From the last equality we see using 1.2(b) and $a'-a\ne 0$ that $b=a'-a$. Thus $a'-a>0$, a contradiction.

If $a'-a>0$ then using 1.2(c),(e) we see that $g'y_i(a'-a)=y_{i^*}(b)g_1$ where $g_1\in G_{\ge 0,s^*w,w'}$ and $b\in K_{>0}$. Thus we have $y_{i^*}(b)g_1=y_{i^*}(a-a')g$. We can write $g_1=\tilde u_1tu_1$, $g=\tilde u't'u'$ with u_1,u' in $U^+_{\ge 0,w'}$, $\tilde u_1,\tilde u'$ in $U^-_{\ge 0,s^*w}$, t,t' in $T_{>0}$. From $y_{i^*}(b)u_1t\tilde u_1=y_{i^*}(a-a')u't'\tilde u'$ we deduce $u_1=u',t=t'$, $y_{i^*}(b)u_1=y_{i^*}(a-a')u'$ (uniqueness of Bruhat decomposition). From the last equality we see using 1.2(b) and $a-a'\ne 0$ that $b=a-a'$. Thus $a-a'>0$, a contradiction.

We see that we must have $a=a'$. It follows that $g=g'$ and the injectivity of α is proved.

We now prove that α is surjective. It is enough to show that for any $g\in G_{\ge 0,sw,w'}$ and any $b\in K_{>0}$ we have $y_{i^*}(b)g=y_{i^*}(b')g'y_i(b')$ for some $g'\in G_{\ge 0,s^*w}$ and some $b'\in K_{>0}$.

Using 1.2(e) we see that there are well defined maps $c\to z(c),K_{\ge 0}\to K_{\ge 0}$ and $c\to g'(c)$, $K_{\ge 0}\to G_{\ge 0,s^*w,w'}$ such that $y_{i^*}(c)g=g'(c)y_i(z(c))$ for all $c\in K_{>0}$. We have $z(0)=0$, $g'(0)=g$. As in §1.4 we see that there exist non-zero polynomials $P(\epsilon),Q(\epsilon)$ in $K[\epsilon]$ with all coefficients in $K_{\ge 0}$ such that $Q(c)\ne 0, z(c)=P(c)/Q(c)$ when $c\in K_{>0}$. As in §1.4 we see that $c\mapsto z(c)$ is the restriction to $K_{>0}$ of a morphism ρ from an open subset of $\overline{K}$ containing $K_{\ge 0}$ to $\overline{K}$ and that $\rho(0)=0$. As in §1.4 we deduce that we can assume $P(0)=0,Q(0)>0$. Let $[0,b]$ be as in §1.4. If $b'\in[0,b]$ we have

$$y_{i^*}(b)g=y_{i^*}(b')y_i(b-b')g=y_{i^*}(b')g'(b-b')y_i(z(b-b')),$$
$$y_{i^*}(b')g'(b-b')y_i(\frac{P(b-b')}{Q(b-b')}-b')y_i(b').$$

As in §1.4 we see using the intermediate value theorem for K that for some $b'\in K, 0<b'<b$ we have $\frac{P(b-b')}{Q(b-b')}-b'=0$, hence

$$y_{i^*}(b)g=y_{i^*}(b')g'(b-b')y_i(b').$$

This completes the proof of (b).

1.7. In this subsection we fix w,w' in W, $i\in I$ such that, setting $s_i=s$, we have $s^*w'=w's, |w'|=|s^*w'|+1$. We define

(a) $\alpha: G_{\ge 0,w,s^*w'}\times K_{>0}\to G_{\ge 0,w,w'}$

by $(g,a)\mapsto x_{i^*}(a)gx_i(a)$. This is well defined since $s^*\bullet(s^*w')\bullet s=w'$ (see 1.3(c)). The proof of the following result is entirely similar to that of 1.6(b):

(b) *the map α is a bijection.*

1.8. Let w,w' in W and $i\in I$ be such that setting $s=s_i\in S$ we have $|s^*ws|=|w|-2$. We show:

(a) *If g,g' in $G_{\ge 0,s^*ws,w'}$ and a,a',c,c' in $K_{>0}$ satisfy*

$$y_{i^*}(a)gy_i(c)=y_{i^*}(a')g'y_i(c'),$$

then $a=a'$.

We have $g=ut\tilde u$, $g'=u't'\tilde u'$ where u,u' are in $U^-_{\ge 0,s^*ws}$, $\tilde u,\tilde u'$ are in $U^+_{\ge 0,w'}$ and t,t' are in $T_{>0}$. We have

$$y_{i^*}(a)gy_i(c)=y_{i^*}(a)uy_i(b)t_1\tilde u_1,$$
$$y_{i^*}(a')g'y_i(c')=y_{i^*}(a')u'y_i(b')t'_1\tilde u'_1,$$

where $\tilde{u}_1, \tilde{u}'_1$ are in $U^+_{\geq 0,w'}$, t_1, t'_1 are in $T_{>0}$ and b, b' are in $K_{>0}$. We have

$$y_{i^*}(a)uy_i(b)t_1\tilde{u}_1 = y_{i^*}(a')u'y_i(b')t'_1\tilde{u}'_1.$$

By the uniqueness in Bruhat decomposition we deduce

$$y_{i^*}(a)uy_i(b) = y_{i^*}(a')u'y_i(b').$$

Using this together with 1.2(a) and with $|w| = |s^*ws| + 2$ we deduce that $a = a'$ proving (a).

1.9. Let $w \in W$ with $|w| = l$. For any $\mathbf{i}, \mathbf{i}'$ in $\mathcal{I}(w)$ we set $u_{\mathbf{i},\mathbf{i}'} = \psi_{\mathbf{i}'}^{-1}\psi_{\mathbf{i}} : K^l_{>0} \xrightarrow{\sim} K^l_{>0}$. From [Lus94] it is known that $u_{\mathbf{i},\mathbf{i}'}$ is given by substraction free rational functions, so that we can substitute $K_{>0}$ by any semifield F and get a bijection $u_{\mathbf{i},\mathbf{i}',F} : F^l \xrightarrow{\sim} F^l$. Let $U^+_w(F)$ be the set of all $(\mathbf{c_i})_{\mathbf{i}\in\mathcal{I}(w)} \in \prod_{\mathbf{i}\in\mathcal{I}(w)} F^l$ such that $u_{\mathbf{i},\mathbf{i}',F}(\mathbf{c_i}) = \mathbf{c_{i'}}$ for any $\mathbf{i}, \mathbf{i}'$ in $\mathcal{I}(w)$. Then $U^+(F) = \sqcup_{w\in W} U^+_w(F)$ has a natural monoid structure and for $F = K_{>0}$ one can identify $U^+(F) = U^+_{\geq 0}$ as monoids; for $F = \{1\}$ we have $U^+(F) = W$.

2. "Cell" decomposition of $U^{+\tau}_{\geq 0}$

2.1. Let $\mathbf{I}_* = \{w \in W; ww^* = 1\}$ be the set of twisted involutions of W. For any $w \in W$, τ restricts to a bijection

(a) $U^+_{\geq 0,w} \xrightarrow{\sim} U^+_{\geq 0,(w^*)^{-1}}$.

To see this we use that for $\mathbf{i} = (i_1, i_2, \dots, i_k) \in \mathcal{I}(w)$ and $(a_1, a_2, \dots, a_k) \in K^k_{>0}$ we have

$$\tau(x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_k}(a_k)) = x_{i^*_k}(a_k)x_{i^*_{k-1}}(a_{k-1})\dots x_{i^*_1}(a_1).$$

In particular, for $w \in \mathbf{I}_*$, τ defines a bijection $U^+_{\geq 0,w} \xrightarrow{\sim} U^+_{\geq 0,w}$ whose fixed point is denoted by $U^{+\tau}_{\geq 0,w}$. We see that we have a partition

(b) $U^{+\tau}_{\geq 0} = \sqcup_{w\in\mathbf{I}_*} U^{+\tau}_{\geq 0,w}$.

The subsets $U^{+\tau}_{\geq 0,w}$ of $U^{+\tau}_{\geq 0}$ are said to be the "cells" of $U^{+\tau}_{\geq 0}$.

2.2. Let $w \in \mathbf{I}_*, i \in I$ be such that setting $s = s_i \in S$ we have $s^*w = ws, |w| = |s^*w| + 1$. Note that $s^*w \in \mathbf{I}_*$. The map in 1.4(a) restricts to a map

(a) $U^{+\tau}_{\geq 0,sw} \times K_{>0} \to U^{+\tau}_{\geq 0,w}$.

We show:

(b) *the map* (a) *is a bijection.*

The fact that (a) is injective follows from 1.4(b). We prove that (a) is surjective. Let $u' \in U^{+\tau}_{\geq 0,w}$. By 1.4(b) we have $u' = x_{i^*}(a)ux_i(a)$ with $u \in U^+_{\geq 0,sw}, a \in K_{>0}$. Since $\tau(u') = u'$, we have $x_{i^*}(a)\tau(u)x_i(a) = x_{i^*}(a)ux_i(a)$; we deduce that $\tau(u) = u$. Thus (a) is surjective, hence a bijection.

2.3. Let $w \in \mathbf{I}_*$, $i \in I$ be such that, setting $s = s_i$, we have $s^*w \neq ws, |w| = |s^*w| + 1$. It follows that $s^*ws \in \mathbf{I}_*, |s^*ws| = |w| - 2$. We define

(a) $U^{+\tau}_{\geq 0,s^*ws} \times K_{>0} \to U^{+\tau}_{\geq 0,w}$

by $(u, a) \mapsto x_{i^*}(a)ux_i(a)$. This is well defined since $s^* \bullet (s^*ws) \bullet s = w$ (see 1.3(a)). We show:

(b) *the map* (a) *is a bijection.*

The fact that (a) is injective follows immediately from 1.2(a) since $|w| = |s^* ws| + 2$. We show that (a) is surjective. Let $u' \in U^{+\tau}_{\geq 0,w}$. Since $u' \in U^{+}_{\geq 0,w}$ we have $u' = x_{i^*}(a) u x_i(b)$ for some $u \in U^{+}_{\geq 0,s^* ws}$ and a, b in $K_{>0}$. Since $\tau(u') = u'$ we have $x_{i^*}(b)\tau(u)x_i(a) = x_{i^*}(a)ux_i(b)$. We have $\tau(u) \in U^{+}_{\geq 0,s^* ws}$ since $s^* ws \in \mathbf{I}_*$; using 1.2(a) we deduce that $a = b$ and $\tau(u) = u$. Thus (a) is surjective, hence a bijection.

2.4. Let $w \in \mathbf{I}_*$. Following [RS90] let $\mathcal{J}(w)$ be the set of all sequences $\mathbf{i} = (i_1, i_2, \dots, i_k)$ in I such that

$$w = s_{i_k^*} \bullet s_{i_{k-1}^*} \bullet \dots \bullet s_{i_1^*} \bullet s_{i_1} \bullet s_{i_2} \bullet \dots \bullet s_{i_k}$$

with k minimum possible. This set is non-empty; the minimum value of k is denoted by $\|w\|$. It is known that

$$\|w\| = (1/2)(|w| + \phi(w)) \in \mathbf{N},$$

where $\phi : \mathbf{I}_* \to \mathbf{N}$ is the unique function such that $\phi(1) = 0$ and such that for any $s \in S, w \in \mathbf{I}_*$ with $|s^* w| = |w| - 1$ we have $\phi(w) = \phi(s^* w) + 1$ if $s^* w = ws$ and $\phi(w) = \phi(s^* ws)$ if $s^* w \neq ws$. (If $* = 1$, $\phi(w)$ is the dimension of the (-1)-eigenspace of w on the reflection representation of W.)

For $\mathbf{i} \in \mathcal{J}(w), k$ as above, we define

$$\kappa_{\mathbf{i}} : K^k_{>0} \to U^{+\tau}_{\geq 0}$$

by

$$(a_1, a_2, \dots, a_k) \mapsto x_{i_k^*}(a_k) x_{i_{k-1}^*}(a_{k-1}) \dots x_{i_1^*}(a_1) x_{i_1}(a_1) x_{i_2}(a_2) \dots x_{i_k}(a_k).$$

We show:

(a) *$\kappa_{\mathbf{i}}$ defines a bijection $K^k_{>0} \to U^{+\tau}_{\geq 0,w}$.*

We argue by induction on $|w|$. If $w = 1$ we have $U^{+}_{\geq 0,w} = U^{+\tau}_{\geq 0,w} = \{1\}$ so that (a) is obvious. Assume now that $w \neq 1$. We set $s = s_{i_k}$, $\mathbf{i}' = (i_1, i_2, \dots, i_{k-1})$. We set $w' = s^* w$ if $s^* w = ws$ and $w' = s^* ws$ if $s^* w \neq ws$. Then $w' \in \mathbf{I}_*$, $|w'| < |w|$, $\|w'\| = \|w\| - 1$ and $\mathbf{i}' \in \mathcal{J}(w')$. It is enough to show that the map $U^{+\tau}_{\geq 0,w'} \times K_{>0} \to U^{+\tau}_{\geq 0,w}$ given by $(u, a) \mapsto x_{i_k^*}(a) u x_{i_k}(a)$ is a well defined bijection. When $s^* w = ws$ this follows from §2.2; when $s^* w \neq ws$ this follows from §2.3. This proves (a).

2.5. For $w \in \mathbf{I}_*$, we set $U^{-\tau}_{\geq 0,w} = \{u \in U^{-}_{\geq 0,w}; \tau(u) = u\}$. We have a partition

(a) $U^{-\tau}_{\geq 0} = \sqcup_{w \in \mathbf{I}_*} U^{-\tau}_{\geq 0,w}$.

For $w \in \mathbf{I}_*$, $\mathbf{i} = (i_1, i_2, \dots, i_k) \in \mathcal{J}(w)$ we define

$$\kappa^-_{\mathbf{i}} : K^k_{>0} \to U^{-\tau}_{\geq 0}$$

by

$$(a_1, a_2, \dots, a_k) \mapsto y_{i_k^*}(a_k) y_{i_{k-1}^*}(a_{k-1}) \dots y_{i_1^*}(a_1) y_{i_1}(a_1) y_{i_2}(a_2) \dots y_{i_k}(a_k).$$

Then the following analogue of 2.4(a) holds (with a similar proof):

(b) *$\kappa^-_{\mathbf{i}}$ defines a bijection $K^k_{>0} \to U^{-\tau}_{\geq 0,w}$.*

2.6. Consider the action 0.1(a) of G on G. When an element of the "cell" $U^+_{\geq 0,w}$ ($w \in W$) of $U^+_{\geq 0}$ is applied to an element of the "cell" $U^{+\tau}_{\geq 0,w'}$ ($w' \in \mathbf{I}_*$) of $U^{+\tau}_{\geq 0}$, the result is an element of the "cell" $U^{+\tau}_{\geq 0,w\bullet w'\bullet w^*}$ of $U^{+\tau}_{\geq 0}$. (This follows by applying 1.3(a) twice.) We see that the action 0.1(a) induces an action of the monoid W (viewed as the indexing set of the "cells" of $U^+_{\geq 0}$) on the set $\mathbf{I}_*$ (viewed as the indexing set of the "cells" of $U^{+\tau}_{\geq 0}$). This action is given by:

(a) $w : w' \mapsto w \bullet w' \bullet w^*$.

It is remarkable that the same action appears in [LV12b] as the limit when the parameter is specialized to 0 of the action [LV12a] of an Iwahori-Hecke algebra on a module with basis indexed by $\mathbf{I}_*$,

2.7. It is clear that $U^{+\tau}_{\geq 0} \subset U^{+\tau}(K)$. In the remainder of this subsection we assume that $K = \mathbf{R}$. We show:

(a) *$U^{+\tau}_{\geq 0,w_0}$ is an open subset of $U^{+\tau}(K)$.*

Let $\mathbf{i} \in \mathcal{J}(w_0)$. Then $\kappa_{\mathbf{i}}$ is an injective continuous map from $K^{\|w_0\|}_{>0}$ to $U^{+\tau}(K)$ which can be shown to be homeomorphic to $K^{\|w_0\|}$. Hence $\kappa_{\mathbf{i}}$ is a homeomorphism on an open subset of $U^{+\tau}(K)$ which must be $U^{+\tau}_{\geq 0,w_0}$. (We have used Brouwer's theorem on invariance of domain, see [Dol72, IV, 7.4].) This proves (a).

Similarly,

(b) *$U^{-\tau}_{\geq 0,w_0}$ is an open subset of $U^{-\tau}(K)$.*

3. "Cell" decomposition of $G^\tau_{\geq 0}$

3.1. For any w, w' in W, τ restricts to a bijection

(a) $G_{\geq 0,w,w'} \to G_{\geq 0,(w^*)^{-1},(w'^*)^{-1}}$.

Indeed if $u \in U^-_{\geq 0,w}, u' \in U^+_{\geq 0,w'}, t \in T_{>0}$ then $\tau(utu') = \tau(u')\tau(t)\tau(u)$ where $\tau(u') \in U^+_{\geq 0,(w'^*)^{-1}}$ (see §2.1), and similarly $\tau(u) \in U^-_{\geq 0,(w^*)^{-1}}$. In particular, for w, w' in $\mathbf{I}_*$, τ defines a bijection $G_{\geq 0,w,w'} \to G_{\geq 0,w,w'}$ whose fixed point is denoted by $G^\tau_{\geq 0,w,w'}$. We see that we have a partition

(b) $G^\tau_{\geq 0} = \sqcup_{w,w' \text{ in } \mathbf{I}_*} G^\tau_{\geq 0,w,w'}$.

The subsets $G^\tau_{\geq 0,w,w'}$ of $G^\tau_{\geq 0}$ are said to be the "cells" of $G^\tau_{\geq 0}$.

3.2. Let w, w' in $\mathbf{I}_*$ and $i \in I$ be such that, setting $s = s_i \in S$, we have $s^*w = ws, |w| = |s^*w| + 1$. Note that $s^*w \in \mathbf{I}_*$. The map in 1.6(a) restricts to a map

(a) $G^\tau_{\geq 0,s^*w,w'} \times K_{>0} \to G^\tau_{\geq 0,w,w'}$.

We show:

(b) *the map* (a) *is a bijection.*

The fact that (a) is injective follows from 1.6(b). We prove that (a) is surjective. Let $g' \in G^\tau_{\geq 0,w,w'}$. By 1.6(b) we have $g' = y_{i^*}(a)gy_i(a)$ with $g \in G_{\geq 0,s^*w}, a \in K_{>0}$. Since $\tau(g') = g'$ we have $y_{i^*}(a)\tau(g)y_i(a) = y_{i^*}(a)gy_i(a)$; we deduce that $\tau(g) = g$. Thus (a) is surjective, hence it is a bijection.

3.3. Let w, w' in $\mathbf{I}_*$ and $i \in I$ be such that setting $s = s_i \in S$ we have $s^*w \neq ws, |w| = |s^*w| + 1$. Note that $s^*ws \in \mathbf{I}_*$, $|s^*ws| = |w| - 2$. We define

(a) $G^\tau_{\geq 0, s^*ws, w'} \times K_{>0} \to G^\tau_{\geq 0, w, w'}$

by $(g, a) \mapsto y_{i^*}(a)gy_i(a)$. This is well defined since $s^* \bullet (s^*ws) \bullet s = w$ (see 1.3(c)). We show:

(b) *the map* (a) *is a bijection.*

To prove injectivity of (a) we consider g, g' in $G^\tau_{\geq 0, s^*ws, w'}$ and a, a' in $K_{>0}$ such that $y_{i^*}(a)gy_i(a) = y_{i^*}(a')g'y_i(a')$. By 1.8(a) we have $a = a'$, hence $g = g'$. Thus (a) is injective.

We show that (a) is surjective. Let $g' \in G^\tau_{\geq 0, w, w'}$. Since $g \in G_{\geq 0, w, w'}$ we have $g = y_{i^*}(a)g'y_i(b)$ for some $g \in G_{\geq 0, s^*ws, w'}$ and a, b in $K_{>0}$. Since $\tau(g') = g'$ we have $y_{i^*}(b)\tau(g)y_i(a) = y_{i^*}(a)gy_i(b)$. We have $\tau(g) \in G_{\geq 0, s^*ws, w'}$ since $s^*ws \in \mathbf{I}_*$. Using again 1.8(a) we see that the last equality implies $a = b$. It follows that $\tau(g) = g$. Thus (a) is a bijection.

3.4. Let w, w' in $\mathbf{I}_*$ and $i \in I$ be such that, setting $s = s_i \in S$, we have $s^*w' = w's, |w'| = |s^*w'| + 1$. Note that $s^*w' \in \mathbf{I}$. The map in 1.7(a) restricts to a map

(a) $G^\tau_{\geq 0, w, s^*w'} \times K_{>0} \to G^\tau_{\geq 0, w, w'}$.

A proof entirely similar to that in §3.2 shows:

(b) *the map* (a) *is a bijection.*

3.5. Let w, w' in $\mathbf{I}_*$ and $i \in I$ be such that, setting $s = s_i \in S$, we have $s^*w' \neq w's, |w'| = |s^*w'| + 1$. Note that $s^*w's \in \mathbf{I}_*$. We define

(a) $G^\tau_{\geq 0, w, s^*w's} \times K_{>0} \to G^\tau_{\geq 0, w, w'}$

by $(g, a) \mapsto x_{i^*}(a)gx_i(a)$. This is well defined since $s^* \bullet (s^*w's) \bullet s = w'$; (see 1.3(c)). A proof entirely similar to that in §3.3 shows:

(b) *the map* (a) *is a bijection.*

3.6. We consider two copies $I^1 = \{i^1; i \in I\}$, $I^{-1} = \{i^{-1}; i \in I\}$ of I. For w, w' in $\mathbf{I}_*$ with $\|w\| = k, \|w'\| = k'$ let $\mathcal{J}(w, w')$ be the set of all sequences $\mathbf{j} = (i_1^{\epsilon_1}, i_2^{\epsilon_2}, \dots, i_{k+k'}^{\epsilon_{k+k'}})$ in $I^1 \sqcup I^{-1}$ (here $\epsilon_1, \dots, \epsilon_{k+k'}$ are 1 or -1) such that the subsequence consisting of terms with exponent -1 is obtained from a sequence in $\mathcal{J}(w)$ (by attaching the exponent -1) and the subsequence consisting of terms with exponent 1 is obtained from a sequence in $\mathcal{J}(w')$ (by attaching the exponent 1).

Let $T^\tau_{>0} = \{t \in T_{>0}; \tau(t) = t\} \subset G^\tau_{\geq 0}$.

For $\mathbf{j} \in \mathcal{J}(w, w')$ as above we define $\kappa_{\mathbf{j}} : K_{>0}^{k+k'} \times T^\tau_{>0} \to G^\tau_{\geq 0}$ by

$$(a_1, a_2, \dots, a_{k+k'}, t) \mapsto$$
$$x^{\epsilon_{k+k'}}_{i^*_{k+k'}}(a_{k+k'})x^{\epsilon_{k+k'-1}}_{i^*_{k+k'-1}}(a_{k+k'-1}) \dots x^{\epsilon_1}_{i^*_1}(a_1)tx^{\epsilon_1}_{i_1}(a_1)x^{\epsilon_2}_{i_2}(a_2) \dots x^{\epsilon_{k+k'}}_{i_{k+k'}}(a_{k+k'}),$$

where we set $x^\epsilon_i(a) = x_i(a)$ if $\epsilon = 1$, $x^\epsilon_i(a) = y_i(a)$ if $\epsilon = -1$. We show:

(a) $\kappa_{\mathbf{j}}$ *defines a bijection* $K_{>0}^{k+k'} \times T^\tau_{>0} \xrightarrow{\sim} G^\tau_{\geq 0, w, w'}$.

We argue by induction on $|w| + |w'|$. If $w = w' = 1$ we have $G^\tau_{\geq 0, w, w'} = T^\tau_{>0}$ so that (a) is obvious.

We now assume that $(w, w') \neq (1, 1)$ so that $k + k' \geq 1$. We set $s = s_{i_{k+k'}} \in S, \epsilon = \epsilon_{k+k'}$. We set $\mathbf{j}' = (i_1^{\epsilon_1}, i_2^{\epsilon_2}, \dots, i_{k+k'-1}^{\epsilon_{k+k'-1}})$.

Assume first that $\epsilon = 1$. Then $w' \neq 1$. We set $w'_1 = s^*w'$ if $s^*w' = w's$ and $w'_1 = s^*w's$ if $s^*w' \neq w's$. Then $w'_1 \in \mathbf{I}_*$, $|w'_1| < |w'|$, $\|w'_1\| = \|w'\| - 1$ and $\mathbf{j}' \in \mathcal{J}(w, w'_1)$. It is enough to show that the map $G^\tau_{\geq 0,w,w'_1} \times K_{>0} \to G^\tau_{\geq 0,w,w'}$ given by $(g, a) \mapsto x_{i^*_{k+k'}}(a) g x_{i_{k+k'}}(a)$ is a well defined bijection. When $s^*w' = ws'$ this follows from §3.4; when $s^*w' \neq ws'$ this follows from §3.5.

Next we assume that $\epsilon = -1$. Then $w \neq 1$. We set $w_1 = s^*w$ if $s^*w = ws$ and $w_1 = s^*ws$ if $s^*w \neq ws$. Then $w_1 \in \mathbf{I}_*$, $|w_1| < |w|$, $\|w_1\| = \|w\| - 1$ and $\mathbf{j}' \in \mathcal{J}(w_1, w')$. It is enough to show that the map $G^\tau_{\geq 0,w_1,w'} \times K_{>0} \to G^\tau_{\geq 0,w,w'}$ given by $(g, a) \mapsto y_{i^*_{k+k'}}(a) g y_{i_{k+k'}}(a)$ is a well defined bijection. When $s^*w = ws$ this follows from §3.2; when $s^*w \neq ws$ this follows from §3.3.

This completes the proof of (a).

3.7. We show:

(a) $G^\tau_{\geq 0} \subset (G^\tau)^0(K)$

(notation of §0.1). By 3.1(a) it is enough to show that $G^\tau_{\geq 0,w,w'} \subset (G^\tau)^0$ for any w, w' in $\mathbf{I}_*$. (The inclusion $G^\tau_{\geq 0} \subset G(K)$ is obvious.)

Using 3.6(a) we see that it is enough to show that $T^\tau_{>0} \subset (G^\tau)^0$. It is enough to show that any element $t_1 \in T^\tau_{>0}$ is contained in the image of $c : T_{>0} \to T_{>0}$, $t \mapsto t\tau(t)$ (the converse is obvious). We shall use the following fact:

(b) the map $T_{>0} \to T_{>0}$, $e \mapsto e^2$ is an isomorphism.

Assuming (b) we write $t_1 = t_2^2$ where $t_2 \in T_{>0}$. We have $(t_2\tau(t_2))^2 = t_2^2\tau(t_2^2) = t_1\tau(t_1) = t_1t_1 = t_1^2$. Using the injectivity of the map in (b) we deduce $t_2\tau(t_2) = t_1$ completing the proof of (a).

To prove (b) it is enough to show that

(c) the homomorphism $K_{>0} \to K_{>0}$, $a \mapsto a^2$ is an isomorphism.

Let $a \in K_{>0}$. From the definition we have $a = b^2 = (-b)^2$ for some $b \in K - \{0\}$. Since either $b > 0$ or $-b > 0$, the map (c) is surjective. If $a \in K_{>0}$ satisfies $a^2 = 1$ then $a = 1$ or $a = -1$. But a cannot be -1 since -1 is not a square in K. Thus $a = 1$ and the map (c) is injective, hence an isomorphism.

In the remainder of this subsection (and the next one) we assume that $K = \mathbf{R}$. We have

(d) $\dim(G/H) = 2\|w_0\| + \dim(T^\tau)$,

where dim is complex dimension and $T^\tau = \{t \in T; \tau(t) = t\}$. An equivalent formula is $\dim(H) = |w_0| - \phi(w_0) + \dim T - \dim(T^\tau)$ where $\phi(w_0)$ is as in §2.4. This follows from the definitions. (If G is almost simple, w_0 is central in W and $* = 1$ then the last equality becomes $\dim(H) = |w_0|$.)

We show:

(e) $G^\tau_{\geq 0,w_0,w_0}$ is an open subset of $(G^\tau)^0(K)$.

Let $\mathbf{j} \in \mathcal{J}(w_0, w_0)$. Then $\kappa_{\mathbf{j}}$ is an injective continuous map from $K_{>0}^{2\|w_0\|} \times T^\tau_{>0}$ to $(G^\tau)^0(K)$ which is a manifold of real dimension $2\|w_0\| + \dim(T^\tau_{>0})$ (see (d)). Hence $\kappa_{\mathbf{j}}$ is a homeomorphism on an open subset of $(G^\tau)^0(K)$ which must be $G^\tau_{\geq 0,w_0,w_0}$. (We have used Brouwer's theorem on invariance of domain.) This proves (e).

3.8. By the exponential map $T_{>0}, \tau$ become an $\mathbf{R}$-vector space and a linear involution on it. It follows that $T^\tau_{>0}$ is a cell. Hence from §3.6 we see that:

(a) *for any* w, w' *in* $\mathbf{I}_*$, $G^{\tau}_{\geq 0,w,w'}$ *is a cell of dimension* $\|w\| + \|w'\| + \dim(T^{\tau}_{>0})$. and that

(b) *the partition* 3.1(a) *of* $G^{\tau}_{\geq 0}$ *is a cell decomposition.*

3.9. As in §2.6 we see that the action 0.1(a) induces an action of the monoid $W \times W$ (viewed as the indexing set of the "cells" of $G_{\geq 0}$) on the set $\mathbf{I}_* \times \mathbf{I}_*$ (viewed as the indexing set of the "cells" of $G^{\tau}_{\geq 0}$). This action is given by:

(a) $(w_1, w_2) : (w, w') \mapsto (w_1 \bullet w \bullet w_1^*, w_2 \bullet w' \bullet w_2^*)$.

4. Transition maps

4.1. Now let $w \in \mathbf{I}_*$ with $\|w\| = k$. For any $\mathbf{i}, \mathbf{i}'$ in $\mathcal{J}(w)$ we set $v_{\mathbf{i},\mathbf{i}'} = \kappa_{\mathbf{i}'}^{-1}\kappa_{\mathbf{i}} : K^k_{>0} \xrightarrow{\sim} K^k_{>0}$ (a transition map).

4.2. Hu and Zhang [HZ16], [HZ17] have shown (at least when W is of classical type and $* = 1$) that $\mathcal{J}(w)$ can be viewed as the set of vertices of a (connected) graph in which $\mathbf{i}, \mathbf{i}'$ are joined when they are related by certain elementary moves which only involve a small number of indices; these elementary moves include the standard braid moves but there are also a small number of non-standard moves. The results of [HZ16],[HZ17] were extended to W of type F_4 in [HZW17] and to a general $(W, *)$ in [M17] (with the use of a computer) and in [HH19] (without the use of a computer).

Note that, by the connectedness of the graph $\mathcal{J}(w)$, any transition map $v_{\mathbf{i},\mathbf{i}'}$ in §4.1 is a composition of transition maps corresponding to pairs $\mathbf{i}, \mathbf{i}'$ which form an edge of the graph. When the edge corresponds to a standard braid move, the corresponding transition map is of the type appearing in [Lus94]; it involves only (substraction free) rational functions. When the edge corresponds to a non-standard braid move, the corresponding transition map is of a type appearing in the next subsection.

4.3. Here are examples of transition maps associated to non-standard braid moves. (This is actually the complete list of examples associated to "irreducible" non-standard braid moves.) In each of the examples below we have $w = w_0$.

(i) Assume first that W is of type A_2, $* = 1$ and $I = \{1, 2\}$. There is a unique pair of inverse bijections $K^2_{>0} \leftrightarrow K^2_{>0}$, $(a, b) \leftrightarrow (a', b')$ such that

$$x_1(a)x_2(b)x_2(b)x_1(a) = x_2(a')x_1(b')x_1(b')x_2(a').$$

This is a pair of inverse transition maps associated to a non-standard move from $\mathbf{i} = (1, 2)$ to $\mathbf{i}' = (2, 1)$ in Hu-Zhang [HZ16]. Note that we must have $a' = b, b' = a$.

(ii) Assume next that W is of type A_2, $* \neq 1$ and $I = \{1, 2\}$. There is a unique pair of inverse bijections $K^2_{>0} \leftrightarrow K^2_{>0}$, $(a, b) \leftrightarrow (a', b')$ such that

$$x_1(a)x_2(b)x_1(b)x_2(a) = x_2(a')x_1(b')x_2(b')x_1(a').$$

This is a pair of transition maps associated to a non-standard move from $\mathbf{i} = (1, 2)$ to $\mathbf{i}' = (2, 1)$ in Marberg [M17].

(iii) Assume next that W is of type B_2, $* = 1$ and $I = \{1, 2\}$. There is a unique pair of inverse bijections $K^3_{>0} \leftrightarrow K^3_{>0}$, $(a, b, c) \leftrightarrow (a', b', c')$ such that

$$x_1(a)x_2(b)x_1(c)x_1(c)x_2(b)x_1(a) = x_2(a')x_1(b')x_2(c')x_2(c')x_1(b')x_2(a').$$

This is a pair of transition maps associated to a non-standard move from $\mathbf{i} = (1, 2, 1)$ to $\mathbf{i}' = (2, 1, 2)$ in Hu-Zhang [HZ17].

(iv) Assume next that W is of type A_3, $* \neq 1$ and $I = \{1,2,3\}$ with $1^* = 3, 2^* = 2, 3^* = 1$. There is a unique pair of inverse bijections $K^4_{>0} \leftrightarrow K^4_{>0}$,

$$(a_1, a_2, a_3, a_2') \leftrightarrow (b_1, b_2, b_3, b_2')$$

such that

$$\begin{aligned} &x_2(a_2)x_3(a_3)x_1(a_1)x_2(a_2')x_2(a_2')x_3(a_1)x_1(a_3)x_2(a_2) \\ &\quad = x_3(b_3)x_2(b_2)x_1(b_1)x_2(b_2')x_2(b_2')x_3(b_1)x_2(b_2)x_1(b_3). \end{aligned}$$

This is a pair of transition maps associated to a non-standard move from $\mathbf{i} = (2,3,1,2)$ to $\mathbf{i}' = (2,3,2,1)$ in Marberg [M17].

(v) Assume next that W is of type B_3, $* = 1$ and $I = \{1,2,3\}$ with $s_1 s_2$ of order 4 and $s_1 s_3 = s_3 s_1$. There is a unique pair of inverse bijections $K^6_{>0} \leftrightarrow K^6_{>0}$,

$$(a,b,c,d,e,f) \leftrightarrow (a',b',c',d',e',f')$$

such that

$$\begin{aligned} &x_2(a)x_1(b)x_2(c)x_3(d)x_2(e)x_1(f)x_1(f)x_2(e)x_3(d)x_2(c)x_1(b)x_2(a) \\ &x_1(a')x_2(b')x_1(c')x_3(d')x_2(e')x_1(f')x_1(f')x_2(e')x_3(d')x_1(c')x_2(b')x_1(a'). \end{aligned}$$

This is a pair of transition maps associated to a non-standard move from $\mathbf{i} = (1,2,3,2,1,2)$ to $\mathbf{i}' = (1,2,3,1,2,1)$ in Hu-Zhang [HZ17].

(vi) Assume next that W is of type D_4, $* = 1$ and $I = \{0,1,2,3\}$ with s_1, s_2, s_3 commuting with each other but not with s_0. There is a unique pair of inverse bijections $K^8_{>0} \leftrightarrow K^8_{>0}$,

$$(a_1, a_2, a_3, a_0, b_1, b_2, b_3, b_0) \leftrightarrow (a_0', a_1', a_2', a_3', b_0', b_1', b_2', b_3')$$

such that

$$\begin{aligned} &x_3(a_3)x_2(a_2)x_1(a_1)x_0(a_0)x_2(b_2)x_1(b_1)x_3(b_3)x_0(b_0)x_0(b_0)x_3(b_3)x_1(b_1)x_2(b_2)x_0(a_0) \\ &x_1(a_1)x_2(a_2)x_3(a_3) = \\ &x_0(a_0')x_3(a_3')x_1(a_1')x_2(a_2')x_0(b_0')x_1(b_1')x_2(b_2')x_3(b_3')x_3(b_3')x_2(b_2')x_1(b_1') \\ &x_0(b_0')x_2(a_2')x_1(a_1')x_3(a_3')x_0(a_0'). \end{aligned}$$

This is a pair of inverse transition maps associated to a non-standard move from

$$\mathbf{i} = (0,3,1,2,0,1,2,3) \text{ to } \mathbf{i}' = (3,2,1,0,2,1,3,0)$$

in Hu-Zhang [HZ17].

(vii) Assume next that W is of type G_2, $* = 1$ and $I = \{1,2\}$. There is a unique pair of inverse bijections $K^5_{>0} \leftrightarrow K^5_{>0}$,

$$(a,b,c,d,e) \leftrightarrow (a',b',c',d',e')$$

such that

$$\begin{aligned} &x_1(a)x_2(b)x_1(c)x_2(d)x_1(e)x_1(e)x_2(d)x_1(c)x_2(b)x_1(a) \\ &\quad = x_2(a')x_1(b')x_2(c')x_1(d')x_2(e')x_2(e')x_1(d')x_2(c')x_1(b')x_2(a'). \end{aligned}$$

This is a pair of inverse transition maps associated to a non-standard move from $\mathbf{i} = (1,2,1,2,1)$ to $\mathbf{i}' = (2,1,2,1,2)$ in Marberg [M17].

In the following subsections we will describe the bijections $K^n_{>0} \leftrightarrow K^n_{>0}$ in (ii)-(vi) more explicitly (in case (i) the bijection is already explicit). The deduction of 4.7(a), 4.8(a), 4.9(a) from the corresponding equalities in (v), (vi) was done by rewriting those equalities as products of matrices in a standard representation of

G and then using a computer to multiply those matrices. (I thank Gongqin Li for help with programming.)

4.4. In the setup of 4.3(ii), we see by calculation that $a+b=a'+b'$, $a^2+2ab=b'^2$. Hence

$$a'=b^2/(a+b+\sqrt{\delta}), b'=\sqrt{\delta},$$

where $\delta=a^2+2ab>0$ and

$$a=b'^2/(a'+b'+\sqrt{\delta'}), b=\sqrt{\delta'},$$

where $\delta'=a'^2+2a'b'>0$.

4.5. In the setup of 4.3(iii), assuming that the value of the root corresponding to 2 on the coroot corresponding to 1 is -2, we see by calculation that

$$b'=a+c, a'=bc^2/(a+c)^2, c'=ab(a+2c)/(a+c)^2,$$

so that a',b',c' can be expressed in terms of a,b,c using only rational functions. On the other hand a,b,c can be expressed in terms of a',b',c' as follows:

$$a=b'^2c'/(b'(a'+c')+\sqrt{\delta}),$$

$$b=a'+c',$$

$$c=\sqrt{\delta}/(a'+c'),$$

where $\delta=a'b'^2(a'+c')>0$.

4.6. In the setup of 4.3(iv), we see by calculation that

(a) $a_1+a_3=b_1+b_3, a_2+a'_2=b_2+b'_2, a_1^2a'_2=b_1^2b'_2, (a_1+a_3)a_2+2a_1a'_2=b_1(b_2+2b'_2)$.

We try to express (a_1,a_2,a_3,a'_2) in terms of (b_1,b_2,b_3,b'_2). Substituting $a_1=\alpha$, $a_3=b_1+b_3-\alpha$, $a'_2=b_1^2b'_2\alpha^{-2}, a_2=b_2+b'_2-b_1^2b'_2\alpha^{-2}$ in the last equation in (a) we obtain

$$(b_1+b_3)b_1^2b'_2\alpha^{-2}-2b_1^2b'_2\alpha^{-1}+b_1b'_2-b_3(b_2+b'_2)=0$$

so that

$$\alpha^{-1}=(b_1+\epsilon\sqrt{\delta})/b_1(b_1+b_3) \text{ and } \alpha=b_1(b_1+b_3)/(b_1+\epsilon\sqrt{\delta}),$$

where $\delta=b_3^2+b_2b_2'^{-1}b_3(b_1+b_3)>0$ and $\epsilon=\pm1$. We have

$$a_3=b_1+b_3-b_1(b_1+b_3)/(b_1+\epsilon\sqrt{\delta})=(b_1+b_3)\epsilon\sqrt{\delta}/(b_1+\epsilon\sqrt{\delta}).$$

Since $a_3>0$, it follows that $\epsilon=1$. We have

$$a'_2=b_1^2b'_2(b_1+\sqrt{\delta})^2/b_1^2(b_1+b_3)^2=b'_2(b_1+\sqrt{\delta})^2/(b_1+b_3)^2.$$

We have

$$\begin{aligned}
a_2 &= b_2 + b_2' - b_2'(b_1 + \sqrt{\delta})^2/(b_1 + b_3)^2 \\
&= ((b_2 + b_2')(b_1 + b_3)^2 - b_2'(b_1^2 + 2b_1\sqrt{\delta} + \delta))/(b_1 + b_3)^2 \\
&= (b_2 b_1^2 + b_2 b_1 b_3 + 2b_2' b_1 b_3 - 2b_1 b_2' \sqrt{\delta})/(b_1 + b_3)^2 \\
&= \frac{(b_2 b_1^2 + b_2 b_1 b_3 + 2b_2' b_1 b_3)^2 - 4b_1^2 {b_2'}^2 \delta}{(b_2 b_1^2 + b_2 b_1 b_3 + 2b_2' b_1 b_3 + 2b_1 b_2' \sqrt{\delta})(b_1 + b_3)^2} \\
&= \frac{b_2^2 b_1^4 + b_2^2 b_1^2 b_3^2 + 2b_2^2 b_1^3 b_3}{(b_2 b_1^2 + b_2 b_1 b_3 + 2b_2' b_1 b_3 + 2b_1 b_2' \sqrt{\delta})(b_1 + b_3)^2} \\
&= b_2^2 b_1^2/(b_2 b_1^2 + b_2 b_1 b_3 + 2b_2' b_1 b_3 + 2b_1 b_2' \sqrt{\delta}) \\
&= b_2^2 b_1/(b_2 b_1 + b_2 b_3 + 2b_2' b_3 + 2b_2' \sqrt{\delta}).
\end{aligned}$$

We now try to express (b_1, b_2, b_3, b_2') in terms of (a_1, a_2, a_3, a_2'). Substituting $b_1 = \beta, b_3 = a_1 + a_3 - \beta, b_2' = a_1^2 a_2' \beta^{-2}$ in the last equation in (a) we obtain

$$(a_2 + a_2')\beta^2 - ((a_1 + a_3)a_2 + 2a_1 a_2')\beta + a_1^2 a_2' = 0,$$

that is

$$\beta = ((a_1 + a_3)a_2 + 2a_1 a_2' + \epsilon' \sqrt{\delta'})/2(a_2 + a_2'),$$

where $\delta' = a_2^2(a_1 + a_3)^2 + 4a_1 a_2 a_2' a_3 > 0$, $\epsilon' = \pm 1$. We have

$$\begin{aligned}
b_3 &= a_1 + a_3 - b_1 = a_1 + a_3 - ((a_1 + a_3)a_2 + 2a_1 a_2' + \epsilon' \sqrt{\delta'})/2(a_2 + a_2') \\
&= (2(a_1 + a_3)(a_2 + a_2') - (a_1 + a_3)a_2 - 2a_1 a_2' + \epsilon' \sqrt{\delta'})/2(a_2 + a_2') \\
&= (a_1 a_2 + a_2 a_3 + 2a_2' a_3 + \epsilon' \sqrt{\delta'})/2(a_2 + a_2') \\
&= \frac{(a_1 a_2 + a_2 a_3 + 2a_2' a_3)^2 - \delta'}{2(a_2 + a_2')(a_1 a_2 + a_2 a_3 + 2a_2' a_3 - \epsilon' \sqrt{\delta'})} \\
&= \frac{4{a_2'}^2 a_3^2 + 4a_2 a_2' a_3^2}{2(a_2 + a_2')(a_1 a_2 + a_2 a_3 + 2a_2' a_3 - \epsilon' \sqrt{\delta'})} \\
&= 2a_2' a_3^2/(a_1 a_2 + a_2 a_3 + 2a_2' a_3 - \epsilon' \sqrt{\delta'}) \\
&= 2a_2' a_3^2(a_1 a_2 + a_2 a_3 + 2a_2' a_3 + \epsilon' \sqrt{\delta'})/((a_1 a_2 + a_2 a_3 + 2a_2' a_3)^2 - \delta') \\
&= 2a_2' a_3^2(a_1 a_2 + a_2 a_3 + 2a_2' a_3 + \epsilon' \sqrt{\delta'})/(4{a_2'}^2 a_3^2 + 4a_2 a_2' a_3^2) \\
&= (a_1 a_2 + a_2 a_3 + 2a_2' a_3 + \epsilon' \sqrt{\delta'})/2(a_2 + a_2').
\end{aligned}$$

We have

$$b_2' = 4a_1^2 a_2'(a_2 + a_2')^2((a_1 + a_3)a_2 + 2a_1 a_2' + \epsilon' \sqrt{\delta'})^{-2}.$$

We have

$$\begin{aligned}
b_2 &= a_2 + a_2' - b_2' = a_2 + a_2' - 4a_1^2 a_2'(a_2 + a_2')^2((a_1 + a_3)a_2 + 2a_1 a_2' + \epsilon' \sqrt{\delta'})^{-2} \\
&= (a_2 + a_2') \frac{((a_1 + a_3)a_2 + 2a_1 a_2' + \epsilon' \sqrt{\delta'})^2 - 4a_1^2 a_2'(a_2 + a_2')}{((a_1 + a_3)a_2 + 2a_1 a_2' + \epsilon' \sqrt{\delta'})^2} \\
&= (a_2 + a_2') \\
&\times \frac{a_1^2 a_2^2 + a_3^2 a_2^2 + 2a_1 a_3 a_2^2 + 4a_1 a_2' a_2 a_3 + 2\epsilon'((a_1 + a_3)a_2 + 2a_1 a_2')\sqrt{\delta'} + \delta'}{((a_1 + a_3)a_2 + 2a_1 a_2' + \epsilon' \sqrt{\delta'})^2}.
\end{aligned}$$

When $\epsilon' = 1$ the values of b_1, b_2, b'_2, b_3 (expressed in terms of a_1, a_2, a'_2, a_3) are > 0. It follows that ϵ' must be equal to 1.

4.7. Assume that we are in the setup of 4.3(v) with the underlying reductive group of type C_3. We see by calculation that

$$d = d', a + c + e = b' + e', b + f = a' + c' + f',$$
$$e^2 f = e'^2 f', bc + be + ef = a'b' + a'e' + c'e' + e'f',$$
$$\text{(a)} \qquad abc + abe + aef + cef + e^2 f = e'(b'e' + b'f' + e'f').$$

We try to express a', b', c', d', e', f' in terms of a, b, c, d, e, f. Setting $\alpha = e'$ we have

$$b' = a + c + e - \alpha, a'b' = bc + be + ef - (b + f)\alpha.$$

Setting $A = abc + abe + aef + cef$, we have $\alpha b'(\alpha + f') = A$, hence $\alpha(a + c + e - \alpha)(\alpha + e^2 f \alpha^{-2}) = A$, that is $(a + c + e - \alpha)(\alpha^3 + e^2 f) = A\alpha$. Thus $e' = \alpha$ is a root of a polynomial of degree 4 with coefficients rational functions in a, b, c, d, e, f. Then a', b', f', c', d' are rational functions in a, b, c, d, e, f, e'.

We now try to express a, b, c, d, e, f in terms of a', b', c', d', e', f'. Let $u' = a' + c' + f'$, $z = e'^2 f'$, $w' = b' + e'$.

Setting $\beta = e$ we have $f = z\beta^{-2}$, $b + f = u'$ hence $b = u' - z\beta^{-2}$, $bc + \beta u' = A'$ where $A' = a'b' + e'u'$, hence $c = (A' - \beta u'))/(u' - z\beta^{-2})$. We have

$$\begin{aligned} a &= w' - \beta - (A' - \beta u')/(u' - z\beta^{-2}) \\ &= ((w' - \beta)(u' - z\beta^{-2}) - (A' - \beta u'))/(u' - z\beta^{-2}) \\ &= (-zw'\beta^{-2} + z\beta^{-1} + A'')/(u' - z\beta^{-2}), \end{aligned}$$

where $A'' = w'u' - A'$.

Substituting these values of a, b, c, e, f in the last equation (a) we see that $e = \beta$ is a root of a polynomial of degree 2 in a', b', c', e', f'. Then a, b, c, d, f are rational functions in $a', b', c', d', e', f', e$.

4.8. Assume that we are in the setup of 4.3(v) with the underlying reductive group of type B_3. We see by calculation that

$$d = d', a + c + e = b' + e', b + f = a' + c' + f', ef = e'f',$$
$$bc + be + ef = a'b' + a'e' + c'e' + e'f',$$
$$\text{(a)} \qquad 2abef + 2aef^2 + 2cef^2 + ab^2c + ab^2e = e'b'((c' + f')^2 + f'^2).$$

We try to express a', b', c', d', e', f' in terms of a, b, c, d, e, f. Setting $\alpha = e'$ we have

$$b' = a + c + e - \alpha, a'b' = -(b + f)\alpha + bc + be + ef.$$

Hence

$$\begin{aligned} c' + f' &= b + f - a' = b + f - (-(b + f)\alpha + bc + be + ef)/(a + c + e - \alpha)) \\ &= ((b + f)(a + c + e - \alpha) - (-(b + f)\alpha + bc + be + ef))/(a + c + e - \alpha) \\ &= ((b + f)(a + c + e) - (bc + be + ef))/(a + c + e - \alpha) \\ &= (ab + af + cf)/(a + c + e - \alpha). \end{aligned}$$

Setting $A = 2abef + 2aef^2 + 2cef^2 + ab^2c + ab^2e$ we have

$$A = \alpha(a + c + e - \alpha)((ab + af + cf)^2/(a + c + e - \alpha)^2 + e^2 f^2 \alpha^{-2}),$$

that is

$$A(a+c+e-\alpha)\alpha=\alpha^2(ab+af+cf)^2+(a+c+e-\alpha)^2e^2f^2.$$

Thus $e'=\alpha$ is a root of a polynomial of degree 2 with coefficients rational functions in a,b,c,d,e,f. Then a',b',f',c',d' are rational functions in a,b,c,d,e,f,e'.

We try to express a,b,c,d,e,f in terms of a',b',c',d',e',f'. Let $u'=a'+c'+f'$, $z=e'f'$, $w'=b'+e'$. Setting $\beta=e$ we have $f=z\beta^{-1}$, $b+f=u'$, hence $b=u'-z\beta^{-1}$, $bc+\beta u'=A'$ where $A'=a'b'+e'u'$, hence $c=(A'-\beta u'))/(u'-z\beta^{-1})$. We have

$$\begin{aligned}a&=w'-\beta-(A'-\beta u')/(u'-z\beta^{-1})\\&=((w'-\beta)(u'-z\beta^{-1})-(A'-\beta u'))/(u'-z\beta^{-1})\\&=(-zw'\beta^{-1}+A'')/(u'-z\beta^{-1}),\end{aligned}$$

where $A''=w'u'-A'+z$. Substituting these values of a,b,c,e,f in the last equation (a) we see that $e=\beta$ is a root of a polynomial of degree 2 in a',b',c',e',f'. Then a,b,c,d,f are rational functions in a',b',c',d',e',f',e.

4.9. Assume that we are in the setup of 4.3(vi). By calculation we see that

$a_i+b_i=a'_i+b'_i$ for $i=0,1,2,3$;

$(a_ia_j+a_ib_j+b_ia_j)(a_0+b_0)+b_ib_jb_0=(a'_ia'_j+a'_ib'_j+b'_ia'_j)b'_0$ for $i\neq j$ in $\{1,2,3\}$;

$2a_3b_1b_2a_0(a_0+b_0)+b_1b_2b_3a_0b_0$

(a)

$=(a'_1a'_2a'_3+a'_1a'_3b'_2+a'_1a'_2b'_3+a'_2a'_3b'_1)a'_0b'_0+2a'_3b'_1b'_2b'_0(a'_0+b'_0).$

We set $\beta=b'_0,\alpha=a_0$, $m_i=a_i+b_i=a'_i+b'_i=m'_i$. From the first two equalities in (a) we deduce for $i\neq j$ in $\{1,2,3\}$:

(b) $\qquad b'_ib'_j=d_{ij}\beta^{-1}+m_im_j$ where $d_{ij}=b_ib_ja_0-m_im_jm_0$,

(c) $\qquad b_ib_j=d'_{ij}\alpha^{-1}$ where $d'_{ij}=b'_ib'_jb'_0+m'_im'_ja'_0$.

Taking product over (i,j) in $(1,2),(2,3),(1,3)$ we obtain

$$\begin{aligned}&{b'_1}^2{b'_2}^2{b'_3}^2=(d_{12}\beta^{-1}+m_1m_2)(d_{23}\beta^{-1}+m_2m_3)(d_{13}\beta^{-1}+m_1m_3)\\&:=\Delta=\delta_0\beta^{-3}+\delta_1\beta^{-2}+\delta_2\beta^{-1}+\delta'_3,\end{aligned}$$

where δ_t are polynomials in a_i,b_i;

$$b_1^2b_2^2b_3^2=d'_{12}d'_{23}d'_{13}\alpha^{-3}:=\Delta'=\delta'\alpha^{-3},$$

where $\delta'=d'_{12}d'_{23}d'_{13}$. It follows that

$$b'_1b'_2b'_3=\sqrt{\Delta},b_1b_2b_3=\sqrt{\Delta'}.$$

Combining this with (b), (c) we deduce

(d) $\qquad b'_k=\sqrt{\Delta}/(d_{ij}\beta^{-1}+m_im_j)$, $a'_k=m'_k-\sqrt{\Delta}/(d_{ij}\beta^{-1}+m_im_j)$,

(e) $\qquad b_k=\sqrt{\Delta'}/(d'_{ij}\alpha^{-1})$, $a_k=m_k-\sqrt{\Delta'}/(d'_{ij}\alpha^{-1})$,

where i,j,k is any permutation of $1,2,3$.

In the last equality in (a) the left hand side is $C:=2a_3b_1b_2a_0m_0+b_1b_2b_3a_0b_0$; the right hand side is equal to

$(m_1m_2m_3-m_1b'_2b'_3-m_2b'_1b'_3-m_3b'_1b'_2+2b'_1b'_2b'_3)\beta(m_0-\beta)+2(m_3-b'_3)b'_1b'_2\beta m_0;$

we substitute in it $b'_ib'_j = d_{ij}\beta^{-1} + m_im_j$, $b'_1b'_2b'_3 = \sqrt{\Delta}$. We obtain

$$m_1m_2m_3\beta(m_0-\beta) - (d_{12}\beta^{-1}+m_1m_2)m_3\beta(m_0-\beta)$$
$$-(d_{23}\beta^{-1}+m_2m_3)m_1\beta(m_0-\beta) - (d_{13}\beta^{-1}+m_1m_2)m_3\beta(m_0-\beta)$$
$$+2(d_{12}\beta^{-1}+m_1m_2)m_3\beta m_0 - 2\sqrt{\Delta}\beta^2 = C.$$

We see that β is a root of a polynomial of degree 4 with coefficients rational functions in a_i, b_i. For $k \in \{1,2,3\}$ we have ${b'_k}^2 = \Delta/(d_{ij}\beta^{-1}+m_im_j)^2$, where $\{i,j\} = \{1,2,3\} - \{k\}$, hence b'_k is a square root from a rational function in a_i, b_i, b'_0. For $k \in \{0,1,2,3\}$ we have $a'_k = m_k - b'_k$, hence a'_k is a linear function in a_k, b_k, b'_k.

In the last equality in (a) the right side is denoted by C' and in the left hand side we substitute $a_3 = m'_3 - \sqrt{\Delta'}/(d'_{12}\alpha^{-1})$ $b_1b_2 = d'_{12}\alpha^{-1}$, $b_1b_2b_3 = \sqrt{\Delta'}$, $b_0 = m'_0 - \alpha$. We obtain

$$2(m'_3d'_{12}\alpha^{-1} - \sqrt{\Delta'})\alpha m'_0 + \sqrt{\Delta'}\alpha(m'_0-\alpha) = C',$$

that is

$$2m'_3m'_0d'_{12} - \sqrt{\delta'\alpha^{-3}}(2\alpha m'_0+\alpha^2) = C',$$
$$\sqrt{\delta'\alpha^{-3}}(2\alpha m'_0+\alpha^2) = 2m'_3m'_0d'_{12} - C',$$
$$\delta'\alpha^{-3}(2\alpha m'_0+\alpha^2)^2 = (2m'_3m'_0d'_{12}-C')^2,$$
$$\alpha + 4m'_0 + 4{m'_0}^2\alpha^{-1} = (2m'_3m'_0d'_{12}-C')^2/\delta'.$$

We see that α is a root of a polynomial of degree 2 with coefficients rational functions in a'_i, b'_i.

For $k \in \{1,2,3\}$ we have $b_k^2 = \Delta'/(d_{ij}\alpha^{-1})^2$, where $\{i,j\} = \{1,2,3\} - \{k\}$, hence b_k is a square root from a rational function in a'_i, b'_i, α. For $k \in \{0,1,2,3\}$ we have $a_k = m'_k - b_k$, hence a_k is a linear function in a'_k, b'_k, b_k.

4.10. By the computations above we see that each of the bijections $\zeta : K^n_{>0} \to K^n_{>0}$ in 4.3(i)-(vi) is of the form $(a_1,a_2,\dots,a_n) \mapsto (a'_1,a'_2,\dots,a'_n)$ (up to possibly reordering the coordinates a_i) where

a'_1 is a root of a polynomial equation of degree N_1 with coefficients rational functions in $a_1,a_2,\dots,a_n$;

a'_2 is a root of a polynomial equation of degree $N_2 \le N_1$ with coefficients rational functions in $a_1,a_2,\dots,a_n,a'_1$;

a'_3 is a root of a polynomial equation of degree $N_3 \le N_2$ with coefficients rational functions in $a_1,a_2,\dots,a_n,a'_1,a'_2$,

etc. Moreover, either all N_i for ζ are ≤ 2 or all N_i for ζ^{-1} are ≤ 2. Thus ζ has in some sense a triangular form. We expect that the same holds in case 4.3(vii).

5. Examples

5.1. Assume that $G = GL_n(\overline{K})$ with the usual pinning and $* = 1$. Now $\sigma = \tau$ maps the matrix $(a_{ij})_{i,j\in[1,n]}$ to $((-1)^{i+j}a_{ij})^{-1}_{i,j\in[1,n]}$. Thus G^τ is the set of all $(a_{ij}) \in G$ such that $\sum_j (-1)^{j+k}a_{ij}a_{jk} = \delta_{ik}$ for all i,k and H is the set of all $(a_{ij}) \in G$ such that $a_{ij} = 0$ whenever $i+j$ is odd (a subgroup of G isomorphic to $GL_{n/2}(\overline{K}) \times GL_{n/2}(\overline{K})$ if n is even or to $GL_{(n+1)/2}(\overline{K}) \times GL_{(n-1)/2}(\overline{K})$ if n is odd).

If in addition we have $n = 2$ then $(G^\tau)^0$ consists of all 2×2 matrices in G with equal diagonal entries and with determinant 1; $G^\tau - (G^\tau)^0$ consists of the two diagonal matrices with entries $1, -1$. In this case $G^{\tau}_{\geq 0}$ consists of all 2×2 matrices with entries in $K_{\geq 0}$ with equal diagonal entries and with determinant 1.

5.2. Assume that G is almost simple of rank > 1, that w_0 is in the centre of W and that $* = 1$, hence $\tau = \sigma$. Let $\mathcal{R} \subset \mathrm{Hom}(T, \mathbf{C}^*)$ be the set of roots and let $ht : \mathcal{R} \to \mathbf{Z}$ be the height function so that for $\alpha \in \mathcal{R}$ written as $\mathbf{Z}$-linear combination of simple roots, $ht(\alpha)$ is the sum of coefficients in this linear combination. Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{t}$ be the Lie algebras of G, H, T. Then $\mathfrak{h}$ is the subspace of $\mathfrak{g}$ spanned by $\mathfrak{t}$ and by the root subspaces corresponding to roots of even height. This is a simple Lie algebra for which the set of simple roots with respect to $\mathfrak{t}$ consists of the $\dim \mathfrak{t} - 1$ roots of height 2 in $\mathcal{R}$ and the unique root of height $-(h-2)$ in $\mathcal{R}$. (Here h is the Coxeter number.) For example, if G is of type D_{2n}, E_7, E_8, then H is of type $D_n \times D_n, A_7, D_8$.

5.3. In this subsection we replace G by $G \times G$ with the pinning induced from that of G; we define $\omega : G \times G \to G \times G$ by $(g, g') \mapsto (g', g)$. Then $\tau(g, g') = (\sigma(g'), \sigma(g))$ where σ refers to G. We have $(G \times G)^{\tau} = \{(g, g') \in G \times G; g' = \sigma(g)\}$. We have $(G \times G)_{\geq 0} = G_{\geq 0} \times G_{\geq 0}$ and $(G \times G)^{\tau}_{\geq 0} = \{(g, g') \in G_{\geq 0} \times G_{\geq 0}; g' = \sigma(g)$. This can be identified with $G_{\geq 0}$ of 0.4 by $(g, g') \mapsto g$. In this way the theory of total positivity in [Lus94] becomes a special case of the theory in this paper.

6. Passage to zones

6.1. Let F be a semifield. Now τ acts naturally as an involutive antiautomorphism of the monoid $U^+(F)$ in §1.9 and one could define $U^{+\tau}(F)$ as the fixed point set of $\tau : U^+(F) \to U^+(F)$. (One could give a similar definition for $G^{\tau}(F)$.) But with this definition it is not clear how to parametrize $U^{+\tau}(F)$ or $G^{\tau}(F)$ by a union of pieces of the form F^n when F is other than $K_{>0}$ or $\{1\}$.

In the remainder of this section $U^{+\tau}(F)$ will not refer to the above definition. Instead we will try to give another definition of it (possibly different from the previous one) for certain F and certain G using the method of passage to zones in [Lus94].

We now assume that K is the field of Puiseux series in a variable ξ with real coefficients (by a theorem of Newton and Puiseux, this field is real closed). Any $x \in K - \{0\}$ is of the form $\sum_{m \in e(x)} a_m \xi^m$ where $a_m \in \mathbf{R} - \{0\}$ and $e(x)$ is a non-empty subset of $\mathbf{Q}$ such that $ne(x) \subset \mathbf{Z}$ for some $n \in \{1, 2, \dots\}$ and $a + e(x) \subset \mathbf{Q}_{\geq 0}$ for some $a \in \mathbf{Q}$; note that $e(x)$ has a well defined smallest element $v(x) \in \mathbf{Q}$. Note that $K_{>0}$ is the set of all $x \in K - \{0\}$ such that $a_{v(x)}(x) \in \mathbf{R}_{>0}$. We regard $\mathbf{Q}$ as a semifield with the product of a, b being $a + b$ and the sum of a, b being $\min(a, b)$. Then $v : K_{>0} \to \mathbf{Q}$ is a semifield homomorphism.

Let $k \in \mathbf{N}$. We define a *zone* of $K^k_{>0}$ to be any fibre of the map $v^k : K^k_{>0} \to \mathbf{Q}^k$ given by $(x_1, x_2, \dots, x_k) \mapsto (v(x_1), v(x_2), \dots, v(x_k))$.

6.2. In this section (until the end of §6.6) we assume that G is almost simple and that

(i) $* = 1$ and G is of type A, or
(ii) $* = 1$ and G is of type B_2, or
(iii) $* \neq 1$ (so that G is of type A, D or E_6).

Now let $w \in \mathbf{I}_*$ with $\|w\| = k$. Let ${}'U^{+\tau}_{\geq 0, w}$ be the set of all $(\mathbf{c_i})_{\mathbf{i} \in \mathcal{J}(w)} \in \prod_{\mathbf{i} \in \mathcal{J}(w)} K^k_{>0}$ such that $v_{\mathbf{i}, \mathbf{i}'}(\mathbf{c_i}) = \mathbf{c_{i'}}$ (notation of §4.1) for any $\mathbf{i}, \mathbf{i}'$ in $\mathcal{J}(w)$.

We define $\kappa_w : {}'U^{+\tau}_{\geq 0,w} \to U^{+\tau}_{\geq 0,w}$ by

$$\kappa_w((\mathbf{c_i})_{\mathbf{i}\in\mathcal{J}(w)}) = \kappa_{\mathbf{i}}(\mathbf{c_i}) \text{ for some/any } \mathbf{i} \in \mathcal{J}(w) \text{ (notation of §2.4).}$$

From §2.4(a) it follows that

(a) κ_w is a bijection.

For $\mathbf{i}, \mathbf{i}'$ in $\mathcal{J}(w)$, the following holds:

(b) $v_{\mathbf{i},\mathbf{i}'} : K^k_{>0} \to K^k_{>0}$ maps any zone to a zone.

We can assume that $\mathbf{i}, \mathbf{i}'$ are related by an elementary move (see §4.2). If this elementary move involves a standard braid move then (b) follows from the results in [Lus94]; if it involves a non-standard move then we use the formulas in 4.3(i) (in case (i)), 4.5 (in case (ii)) or §§4.4, 4.6 (in case (iii)).

From (b) we deduce that $v_{\mathbf{i},\mathbf{i}'} : K^k_{>0} \to K^k_{>0}$ induces a map from the set of zones in $K^n_{>0}$ to itself that is a map (necessarily a bijection) $\bar{v}_{\mathbf{i},\mathbf{i}'} : \mathbf{Q}^k \to \mathbf{Q}^k$.

We define $U^{+\tau}_w(\mathbf{Q})$ to be the set of all $(\mathbf{c_i})_{\mathbf{i}\in\mathcal{J}(w)} \in \prod_{\mathbf{i}\in\mathcal{J}(w)} \mathbf{Q}^k$ such that $\bar{v}_{\mathbf{i},\mathbf{i}'}(\mathbf{c_i}) = \mathbf{c_{i'}}$ for any $\mathbf{i}, \mathbf{i}'$ in $\mathcal{J}(w)$.

We define a map

$$U^{+\tau}_{\geq 0,w} \to U^{+\tau}_w(\mathbf{Q})$$

by $u \mapsto \bar{u}$ where the $\mathbf{i}$-coordinate of $\bar{u}$ is v^k applied to the $\mathbf{i}$-th coordinate of $\kappa_w^{-1}(u)$. The fibres of this map are called the zones of $U^{+\tau}_{\geq 0,w}$. Taking disjoint union over $w \in \mathbf{I}_*$ we obtain a (surjective) map

$$U^{+\tau}_{\geq 0} \to U^{+\tau}(\mathbf{Q}) := \sqcup_{w\in\mathbf{I}_*} U^{+\tau}_w(\mathbf{Q}).$$

In [Lus94, §9] a definition of zones of $U^+_{\geq 0}$ is given, so that the set of zones which may be denoted $U^+(\mathbf{Q})$ is defined. (Actually in *loc. cit.* a subfield of K on which v has integer values is used instead of K.) From the definition we see that the natural action of $U^+_{\geq 0}$ on $U^{+\tau}_{\geq 0}$ induces by passage to zones an action of $U^+(\mathbf{Q})$ on $U^{+\tau}(\mathbf{Q})$.

6.3. Similarly for any $(w, w') \in \mathbf{I}_* \times \mathbf{I}_*$ with $\|w\| = k, \|w'\| = k'$ we can define a partition of $G^\tau_{\geq 0,w,w'}$ into zones. For this we use the various parametrizations $\kappa_{\mathbf{j}}$ ($\mathbf{j} \in \mathcal{J}(w, w')$ of $G^\tau_{\geq 0,w,w'}$ in §3.6). We must show that any $\mathbf{j}, \tilde{\mathbf{j}}$ in $\mathcal{J}(w, w')$ can be joined by a sequence $\mathbf{j} = \mathbf{j}_1, \mathbf{j}_2, \dots, \mathbf{j}_u = \tilde{\mathbf{j}}$ in $\mathcal{J}(w, w')$ so that for any two consecutive terms $\mathbf{j}_h, \mathbf{j}_{h+1}$ in this sequence the composition $\kappa^{-1}_{\mathbf{j}_{h+1}} \kappa_{\mathbf{j}_h} : K^{k+k'}_{>0} \times T^\tau_{>0} \to K^{k+k'}_{>0} \times T^\tau_{>0}$ is of the type considered in [Lus94, 1.3] or of the type considered for $U^{+\tau}_{\geq 0}$ in §6.2 or of the analogous type for $U^{-\tau}_{\geq 0}$. It follows that $\kappa^{-1}_{h+1}\kappa_h : K^{k+k'}_{>0} \times T^\tau_{>0} \to K^{k+k'}_{>0} \times T^\tau_{>0}$ maps any zone of $K^{k+k'}_{>0}$ times $T^\tau_{>0}$ to a zone of $K^{k+k'}_{>0}$ times $T^\tau_{>0}$.

Let $G^\tau_{w,w'}(\mathbf{Q})$ be the set of zones of $G^\tau_{\geq 0,w,w'}$. We set

$$G^\tau(\mathbf{Q}) = \sqcup_{(w,w')\in\mathbf{I}_*\times\mathbf{I}_*} G^\tau_{w,w'}(\mathbf{Q}).$$

Now the zones of $G_{\geq 0}$ are defined as in [Lus94]. (Actually in *loc. cit.* a subfield of K on which v has integer values is used instead of K.) The set of zones of $G_{\geq 0}$ is denoted by $G(\mathbf{Q})$; it inherits a monoid structure from that of $G_{\geq 0}$. The action of the monoid $G_{\geq 0}$ on $G^\tau_{\geq 0}$ induces an action of the monoid $G(\mathbf{Q})$ on $G^\tau(\mathbf{Q})$.

6.4. Let $w \in \mathbf{I}_*$ with $\|w\| = k$. The collection of parametrizations $\kappa_{\mathbf{i}} : K^k_{>0} \to U^{+\tau}_{\geq 0,w}$ for various $\mathbf{i} \in \mathcal{J}(w)$ is something slightly more general than what in [Lus19] (and going back to [Lus94]) was called a "positive structure". Namely the compositions $\kappa_{\mathbf{i}'}^{-1}\kappa_{\mathbf{i}} : K^k_{>0} \to K^k_{>0}$ with $\mathbf{i}, \mathbf{i}'$ in $\mathcal{J}(w)$ are of the form $(a_1, a_2, \dots, a_k) \mapsto (a'_1, a'_2, \dots, a'_k)$ where each a'_i is obtained from $a_1, a_2, \dots, a_k$ by using a succession of the following operations: addition, multiplication, division and extracting of a square root, not necessarily in this order. (This last operation was not allowed in [Lus19].)

However in the case 6.2(i) we have just the old type of positive structure.

6.5. We will now define $U^{+\tau}(F)$ for F as in (i), (ii) below:

(i) $F = K_{2,>0} = K_2 \cap K_{>0}$ where $K_2 = \{x \in K - \{0\}; 2^c e(x) \subset \mathbf{Z}$ for some $c \in \mathbf{N}\} \sqcup \{0\}$;

(ii) $F = \mathbf{Z}[1/2] = \{q \in \mathbf{Q}; 2^c q \in \mathbf{Z}$ for some $c \in \mathbf{N}\}$.

Here $2^c q$ is the usual product in $\mathbf{Q}$. Note that $K_{2,>0}$ (resp. $\mathbf{Z}[1/2]$) is a sub-semifield of $K_{>0}$ (resp. of $\mathbf{Q}$).

Let $w \in \mathbf{I}_*$ with $\|w\| = k$. Let $U^{+\tau}_w(K_{2,>0})$ be the set of all $(\mathbf{c}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{J}(w)} \in \prod_{\mathbf{i} \in \mathcal{J}(w)} K^k_{2,>0}$ such that $v_{\mathbf{i},\mathbf{i}'}(\mathbf{c}_{\mathbf{i}}) = \mathbf{c}_{\mathbf{i}'}$ (notation of §4.1) for any $\mathbf{i}, \mathbf{i}'$ in $\mathcal{J}(w)$. This is well defined since $v_{\mathbf{i},\mathbf{i}'}$ restricts to a bijection $K^k_{2,>0} \to K^k_{2,>0}$ (by results in §§4.4–4.6). We define $U^{+\tau}(K_{2,>0}) = \sqcup_{w \in \mathbf{I}_*} U^{+\tau}_w(K_{2,>0})$; it has a natural action of the monoid $U^+(K_{2,>0})$.

We define $U^{+\tau}_w(\mathbf{Z}[1/2])$ to be the set of all $(\mathbf{c}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{J}(w)} \in \prod_{\mathbf{i} \in \mathcal{J}(w)} (\mathbf{Z}[1/2])^k$ such that $\bar{v}_{\mathbf{i},\mathbf{i}'}(\mathbf{c}_{\mathbf{i}}) = \mathbf{c}_{\mathbf{i}'}$ for any $\mathbf{i}, \mathbf{i}'$ in $\mathcal{J}(w)$. (Note that $\bar{v}_{\mathbf{i},\mathbf{i}'} : \mathbf{Q}^k \to \mathbf{Q}^k$ restricts to a map $(\mathbf{Z}[1/2])^k \to (\mathbf{Z}[1/2])^k$.) We define $U^{+\tau}(\mathbf{Z}[1/2]) = \sqcup_{w \in \mathbf{I}_*} U^{+\tau}_w(\mathbf{Z}[1/2])$; it has a natural action of the monoid $U^+(\mathbf{Z}[1/2])$.

We can define in a similar way $G^\tau(F)$ for F as in (i) or (ii).

6.6. We regard $\mathbf{Z}$ as a sub-semifield of $\mathbf{Q}$. It is known that the set $U^+(\mathbf{Z})$ is closely related to the parametrization of the canonical basis in [Lus90]. It would be interesting to find an analogous interpretation of $U^{+\tau}(\mathbf{Z}[1/2])$.

6.7. We expect that $U^{+\tau}(\mathbf{Q}), G^\tau(\mathbf{Q})$ can be defined by the method of §6.2, §6.3 without the assumption in §6.2. This should follow from a better understanding of the transition maps in 4.3(v)-(vii).

Acknowledgment

The author thanks Xuhua He for useful discussions.

References

[Dol72] A. Dold, *Lectures on algebraic topology* (German), Die Grundlehren der mathematischen Wissenschaften, Band 200, Springer-Verlag, New York-Berlin, 1972. MR0415602

[HH19] Mikael Hansson and Axel Hultman, *A word property for twisted involutions in Coxeter groups*, J. Combin. Theory Ser. A **161** (2019), 220–235, DOI 10.1016/j.jcta.2018.07.006. MR3861777

[HZ16] Jun Hu and Jing Zhang, *On involutions in symmetric groups and a conjecture of Lusztig*, Adv. Math. **287** (2016), 1–30, DOI 10.1016/j.aim.2015.10.003. MR3422674

[HZ17] Jun Hu and Jing Zhang, *On involutions in Weyl groups*, J. Lie Theory **27** (2017), no. 3, 671–706. MR3592481

[HZW17] Jun Hu, Jing Zhang, and Yabo Wu, *Involutions in Weyl group of type F_4*, Front. Math. China **12** (2017), no. 4, 891–906, DOI 10.1007/s11464-017-0646-z. MR3663749

[Lus90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498, DOI 10.2307/1990961. MR1035415

[Lus94] G. Lusztig, *Total positivity in reductive groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568, DOI 10.1007/978-1-4612-0261-5_20. MR1327548

[Lus19] G. Lusztig, *Total positivity in reductive groups, II*, Bull. Inst. Math. Acad. Sin. (N.S.) **14** (2019), no. 4, 403–459, DOI 10.21915/bimas.2019402. MR4054343

[LV12a] George Lusztig and David A. Vogan Jr., *Hecke algebras and involutions in Weyl groups*, Bull. Inst. Math. Acad. Sin. (N.S.) **7** (2012), no. 3, 323–354. MR3051317

[LV12b] G. Lusztig and D. Vogan, *Involutions in Weyl groups and nil-Hecke algebras*, `arXiv:2107.10754`, 2012.

[M17] E. Marberg, *Braid relations for involution words in affine Coxeter groups*, `arXiv:1703.10437`, 2017.

[RS90] R. W. Richardson and T. A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata **35** (1990), no. 1-3, 389–436, DOI 10.1007/BF00147354. MR1066573

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139