

TOWARDS COMBINATORIAL INVARIANCE FOR KAZHDAN-LUSZTIG POLYNOMIALS

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ABSTRACT. Kazhdan-Lusztig polynomials are important and mysterious objects in representation theory. Here we present a new formula for their computation for symmetric groups based on the Bruhat graph. Our approach suggests a solution to the combinatorial invariance conjecture for symmetric groups, a well-known conjecture formulated by Lusztig and Dyer in the 1980s.

1. INTRODUCTION

Kazhdan-Lusztig polynomials are important polynomials associated to pairs of elements x, y in Coxeter groups. They appear throughout representation theory and related fields. Lusztig (ca. 1983) and Dyer (1987) [Dye87] independently conjectured that Kazhdan-Lusztig polynomials depend only on the poset of elements between x and y in Bruhat order. This is a fascinating conjecture. For example, it suggests that Kazhdan-Lusztig polynomials are providing subtle invariants of Bruhat order. This conjecture is known in several special cases ([Dye87, Bre04, Inc06, BCM06, Pat21, BLP21], see [Bre02] for an overview).

In this work we prove a new formula for Kazhdan-Lusztig polynomials for symmetric groups. Our formula was discovered whilst trying to understand certain machine learning models trained to predict Kazhdan-Lusztig polynomials from Bruhat graphs (see [DVB⁺21]). The new formula suggests an approach to the combinatorial invariance conjecture for symmetric groups.

Traditionally, Kazhdan-Lusztig polynomials are computed inductively based on the length. This means that in order to compute $P_{x,y}$ one might need to know $P_{u,v}$ where u, v lies anywhere in $[\text{id}, y]$. Our new formula allows inductive calculation using only polynomials $P_{u,v}$ where u, v belong to the Bruhat interval $[x, y]$ —thus it “stays in the Bruhat interval”. Roughly speaking, we compute Kazhdan-Lusztig polynomials via induction over the *rank* of the symmetric group, rather than the *length* of a permutation.

Our new formula uses only information present in the Bruhat interval $[x, y]$ and thus might be useful for approaching the combinatorial invariance conjecture. However, the formula requires the knowledge of the intersection of $[x, y]$ with a coset $S_{n-1}x$, which is not combinatorially invariant information. It is not difficult to see that the intersection

$$[x, y] \cap S_{n-1}x$$

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forms a subinterval $[x, c] \subset [x, y]$. We observe that the embedding $[x, c] \subset [x, y]$ has remarkable properties: the edges connecting any node $u \in [x, c]$ to a node $v \notin [x, c]$ possess a lattice-like quality which we call a *hypercube cluster*.

Abstracting the notion of hypercube cluster leads to the general notion of *hypercube decomposition*, of which our subinterval $[x, c] \subset [x, y]$ is an example. Hypercube decompositions appear to provide a powerful way to understand Bruhat intervals in symmetric groups. We expect them to play a role in the solutions of other problems.

In general, there may be many more hypercube decompositions than come from intersections with cosets of S_{n-1} . Remarkably, our formula appears to give the right answer for any hypercube decomposition. We conjecture that this is always the case. We have checked our conjecture on all Bruhat intervals and all hypercube decompositions up to S_7 , and over a million intervals in S_8 and S_9 (again over all hypercube decompositions). Our conjecture implies the combinatorial invariance conjecture in more precise form.

1.1. Structure of this paper. We begin in §2 with an overview of Kazhdan-Lusztig polynomials and the combinatorial invariance conjecture. This is introductory, and can be skipped over by an experienced reader. In §3 we introduce hypercube clusters, and state our formula and conjecture. This section is purely combinatorial. In §4 and §5 we prove our formula. Here geometric machinery (Schubert varieties, intersection cohomology, torus actions, weights . . .) is needed. In §6 we outline a route to a proof of our conjecture in general, and explain that our conjecture would follow from the purity of a certain cohomology group, or the surjectivity of a restriction map in cohomology.

2. KAZHDAN-LUSZTIG POLYNOMIALS AND COMBINATORIAL INVARIANCE

In this background section, we give some background on Coxeter groups, Bruhat graphs, Kazhdan-Lusztig polynomials and the combinatorial invariance conjecture. Excellent references for the following include [Hum90, Bre02, BB05, Soe97, EMTW20].

2.1. Coxeter groups and Kazhdan-Lusztig polynomials. *Coxeter groups* are an important class of groups, which arose out of H. S. M. Coxeter's study of finite reflection groups in the 1930s. They are characterised by a presentation via generators and relations. In this presentation the elements in the generating set are called the *simple reflections*. An important example of a Coxeter group is the symmetric group S_{n+1} consisting of all permutations of $0, 1, \dots, n$, with simple reflections consisting of the set $S = \{(i, i + 1)\}$ of adjacent transpositions.

In a seminal paper [KL79], Kazhdan and Lusztig associated to any pair of elements x, y in a Coxeter group a polynomial with integer coefficients

$$x, y \in W \mapsto P_{x,y} \in \mathbb{Z}[q]$$

known as the *Kazhdan-Lusztig polynomial*. All that we say here concerning their definition is that it is highly inductive: one works one's way "out" in the group, starting at the identity and applying generators from S . At each step in the calculation one might need any of the previously computed polynomials. Thus, they are rather cumbersome to calculate by hand, but it is not difficult to compute billions

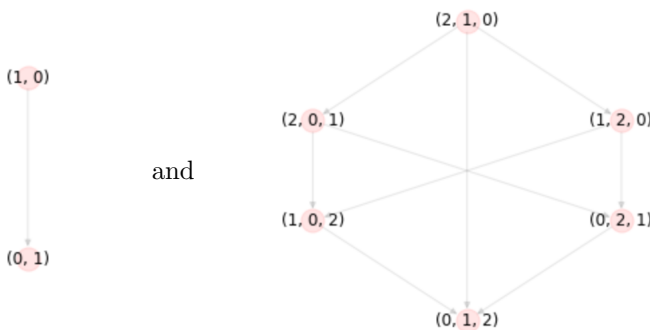


FIGURE 1. Ordered Bruhat graphs for $n = 2, 3$

of them on a computer with enough memory. (This is useful for machine learning, as one often needs access to large data sets.¹)

2.2. The Bruhat graph. To any Coxeter group one may associate its *Bruhat graph*. For the symmetric group, this is the graph with vertices corresponding to all elements of S_{n+1} , and an edge joining x and y if and only if they differ by multiplication by a transposition. (In other words, x and y are connected in the Bruhat graph if they agree on all but two elements of $0, 1, \dots, n$.) Below, we will denote the transposition that exchanges i and j by $t_{(i,j)}$. The transpositions are precisely the conjugates of the simple reflections in S_{n+1} .

The symmetric group has a natural *length function* given by the number of inversions:

$$\ell(x) = \#\{i < j \mid x(i) > x(j)\}.$$

We regard the length function as giving us a notion of “height” on the Bruhat graph. This allows us to orient the edges of the Bruhat graph via decreasing length. Figures 1 and 2 give pictures of the Bruhat graph for $n = 2, 3$ and 4.²

2.3. Bruhat order. The Bruhat graph allows us to define the *Bruhat order*, which is a partial order on W . It is defined as follows:

$$(1) \quad x \leq y \iff \begin{array}{l} \text{there exists a downward path} \\ \text{from } y \text{ to } x \text{ in the Bruhat graph.} \end{array}$$

(We include paths of length zero, so $x \leq x$ always holds.) For example, in the symmetric group S_{n+1} the minimal element is always the identity permutation, and the maximal element is the permutation w_0 which interchanges 0 and n , 1 and $n - 1$, etc.

The Bruhat order is remarkably complex, and it has long been suspected that Kazhdan-Lusztig polynomials reflect subtle properties of the Bruhat order. An

¹The reader who wants to get a feeling for Kazhdan-Lusztig polynomials is encouraged to experiment with Joel Gibson’s LieVis software. For example, Kazhdan-Lusztig polynomials for an affine Weyl group of type A_2 are computed live [Gib22].

²Throughout this paper we use *string notation* for permutations. Thus $(2, 0, 3, 1)$ (or often simply 2031) denotes the permutation of 0, 1, 2 and 3 that sends $0 \mapsto 2, 1 \mapsto 0, 2 \mapsto 3$ and $3 \mapsto 1$.

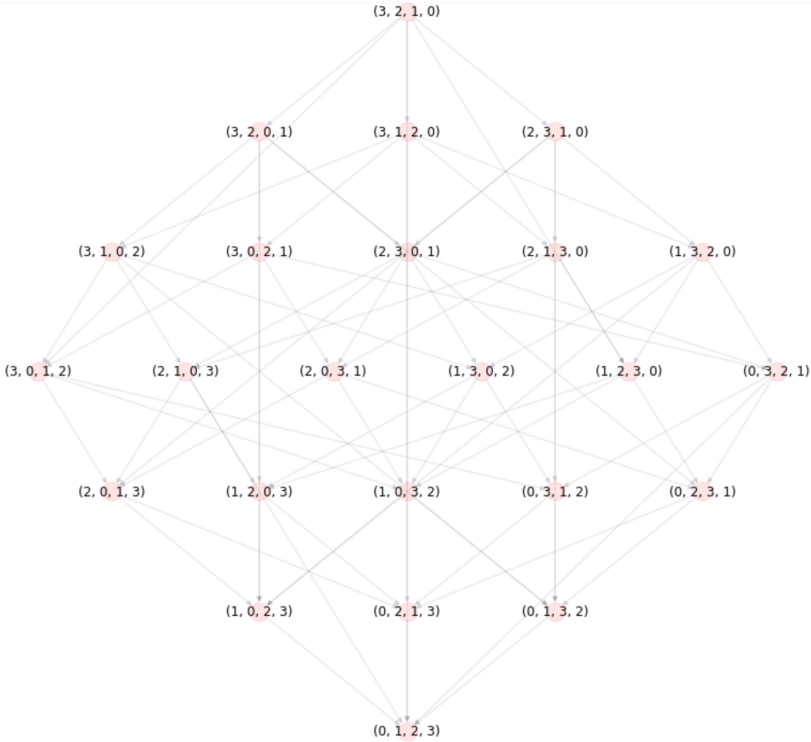


FIGURE 2. Ordered Bruhat graph for $n = 4$

elementary manifestation of this phenomenon (easy to prove) is that:

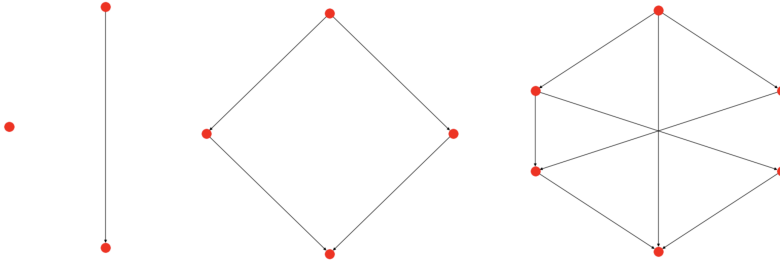
$$P_{x,y} \neq 0 \Leftrightarrow x \leq y.$$

Less elementary connections tend to involve the interval $[x, y]$ consisting of the full subgraph of the Bruhat graph between x and y . (That is, this consists of all edges and vertices which may be reached from y on the way to x , whilst progressing downwards.) For example, a much less obvious fact [Car94, Dye93] is that

$$P_{x,y} = 1 \Leftrightarrow \begin{array}{l} \text{the graph obtained from } [x, y] \\ \text{by forgetting edge orientations is regular} \\ \text{(i.e. all vertices have the same degree).} \end{array}$$

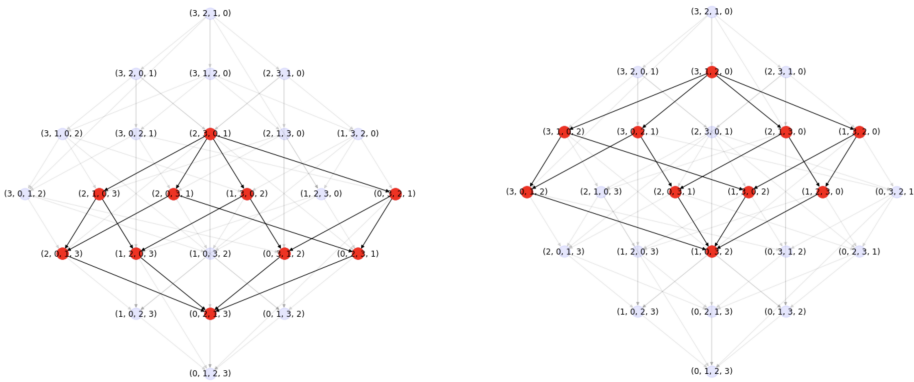
(For example, it is easy to see that the full Bruhat graph is regular, and in particular $P_{\text{id}, w_0} = 1$.)

2.4. **First examples.** In S_3 all proper intervals are isomorphic to the following posets³ (as the reader may check easily, using Figure 1):

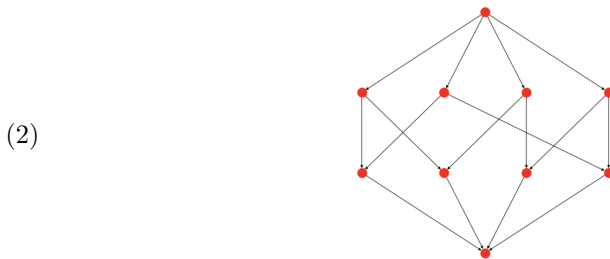


All these graphs are regular (after forgetting edge orientations), and thus all Kazhdan-Lusztig polynomials are 1.

In S_4 , almost all intervals $[x, y]$ are regular, and hence almost all Kazhdan-Lusztig polynomials $P_{x,y}$ are 1. There are four intervals which are not regular. Here we depict two intervals which are not: those between 0213 and 2301, and between 1032 and 3120:



In both cases, the interval is isomorphic to the following directed graph, known as the “4 crown”:



In both these cases the Kazhdan-Lusztig polynomials are equal:

(3)
$$P_{0213,2301} = P_{1032,3120} = 1 + q.$$

³Poset = partially ordered set

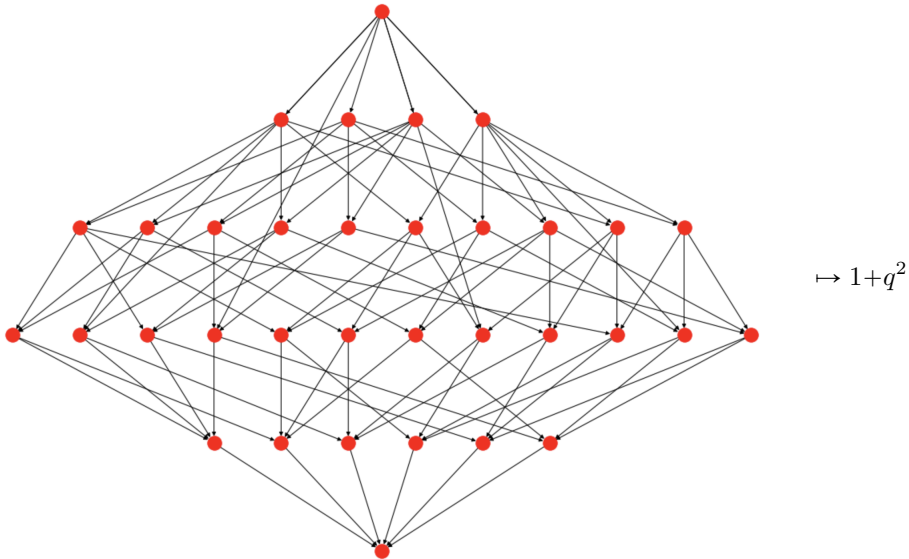


FIGURE 3. Interval and Kazhdan-Lusztig polynomial for $x = 03214$ and $y = 34201$

2.5. The combinatorial invariance conjecture. Conjecture 2.1, formulated independently by Lusztig and Dyer in the 1980s, was a major motivation for the current work:

Conjecture 2.1. *The Kazhdan-Lusztig polynomial $P_{x,y}$ depends only on the isomorphism type of Bruhat graph of the interval $[x, y]$.*

For example, given only the ordered graph (2) (and not the labelling of its vertices) we should be able to predict the Kazhdan-Lusztig polynomial $1 + q$. The fact that (2) occurs in two different ways in the Bruhat graph of S_4 with equal Kazhdan-Lusztig polynomials can be seen as an instance of this conjecture. Figures 3 and 4 show two more examples of the assignment of Kazhdan-Lusztig polynomial to Bruhat intervals.

Remark 2.2. The combinatorial invariance conjecture is a central conjecture in the study of Bruhat intervals. The reader is referred to [Bre02] for more detail on known cases (see also [Dye87, Bre04, Inc06, Pat21, BLP21]). We do not discuss the various partial results towards the conjecture here, except to mention that it is known to hold for intervals starting at the identity [BCM06].

3. THE NEW FORMULA

In this section we describe our new formula. Before going into detail, let us give a rough idea of what the formula looks like. Recall that our goal is to compute the Kazhdan-Lusztig polynomial starting from the Bruhat graph. By induction, we can assume that we can do this for any smaller graph. In particular, we can

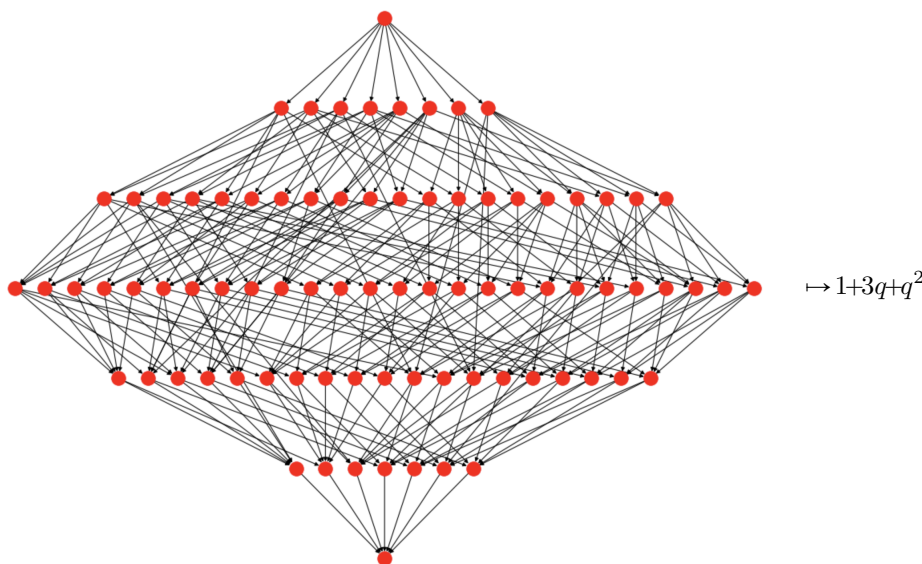


FIGURE 4. Interval and Kazhdan-Lusztig polynomial for $x = 021435$ and $y = 234501$

assume that all intermediate polynomials $P_{u,v}$ are known, for all $u, v \in [x, y]$ with $(u, v) \neq (x, y)$.

Our formula depends on the choice of an auxiliary structure on our graph, called a *hypercube decomposition*. Such a decomposition amounts to the choice of a subinterval $J \subset [x, y]$ satisfying certain concrete combinatorial conditions. (The reader is encouraged to skip ahead a few pages to Figure 6, where a typical hypercube decomposition is illustrated.) There always exists at least one hypercube decomposition, but in general there will be many. Any choice of hypercube decomposition determines two polynomials in q , the *inductive piece* and *hypercube piece*. Our formula is:

$$(4) \quad P_{x,y}^\partial = \text{inductive piece} + \text{hypercube piece}.$$

The left hand side is the “ q -derivative of the Kazhdan-Lusztig polynomial”, from which the Kazhdan-Lusztig polynomial can be recovered. The calculation of the *inductive piece* (resp. *hypercube piece*) uses only the part of the graph which lies (resp. does not lie) in J . (Again, the reader is encouraged to glance at Figure 6. The nodes necessary for the computation of the hypercube and inductive piece are in blue (resp. red).)

3.1. The q -derivative of Kazhdan-Lusztig polynomials. We now introduce a new polynomial, whose knowledge is equivalent to knowledge of the Kazhdan-Lusztig polynomial, but which is easier to handle. Define

$$P_{x,y}^\partial(q) = \frac{P_{x,y}(q) - q^{\ell(y)-\ell(x)}P_{x,y}(q^{-1})}{1 - q}.$$

(One checks easily that the denominator always divides the numerator, so $P_{x,y}^\partial$ is always an integer valued polynomial.) We refer to $P_{x,y}^\partial$ as the q -derivative of the Kazhdan-Lusztig polynomial.⁴ Defining properties of Kazhdan-Lusztig polynomials⁵ ensure that $P_{x,y}^\partial$ determines $P_{x,y}$.

Example 3.1. We give three examples of Kazhdan-Lusztig polynomials, and their corresponding q -derivatives:

- (1) $P_{x,y} = 1, \ell(y) - \ell(x) = 3, P_{x,y}^\partial = 1 + q + q^2, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} .$
- (2) $P_{x,y} = 1 + q, \ell(y) - \ell(x) = 3, P_{x,y}^\partial = 1 + 2q + q^2, \begin{array}{|c|c|c|} \hline & \square & \\ \hline \square & \square & \square \\ \hline \end{array} .$
- (3) $P_{x,y} = 1 + q^2, \ell(y) - \ell(x) = 6, P_{x,y}^\partial = 1 + q + 2q^2 + 2q^3 + q^4 + q^5, \begin{array}{|c|c|c|c|c|c|} \hline & & \square & \square & & \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array} .$

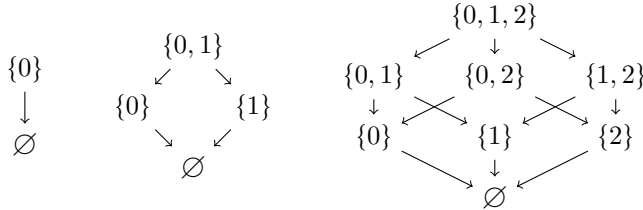
(We leave it up to the reader to determine the meaning of the box diagrams, as well as how to recover the Kazhdan-Lusztig polynomial from them.)

3.2. Hypercube clusters. We work in the setting of directed acyclic graphs. Any such graph is a poset in natural way, where we declare $x \leq y$ if there exists a directed path from y to x .

For any finite set E , the E -hypercube H_E is the directed acyclic graph with:

- (1) vertices consisting of subsets of E ;
- (2) an edge $I \rightarrow J$ if J is obtained from I by removing one element.

Example 3.2. E -hypercubes H_E for $E = \{0\}, \{0, 1\}$ and $\{0, 1, 2\}$:

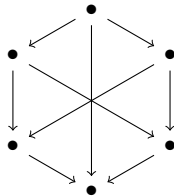


Now suppose given a directed acyclic graph X (in our applications X will be a Bruhat graph) and a node $x \in X$. We say that a subset E of edges with target x spans a hypercube if there exists a unique embedding (i.e. injection) of directed graphs

$$\vartheta : H_E \rightarrow X$$

sending the edge $(\{\alpha\} \rightarrow \emptyset)$ in H_E to α , for all edges α in E . If E spans a hypercube, its crown is $\vartheta(E)$, a node of X .

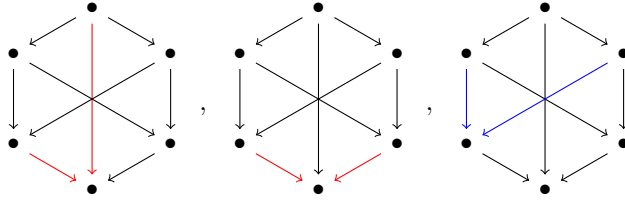
Example 3.3. Consider the following directed graph (isomorphic to the Bruhat graph of S_3):



⁴In the related theory of Kazhdan-Lusztig-Stanley polynomials, this polynomial is often called the “ H -polynomial”.

⁵More precisely, the fact that their degree is bounded above by $(\ell(y) - \ell(x) - 1)/2$

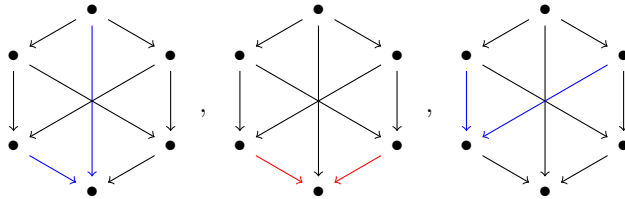
In the following we show some pairs of edges with common target with a colour. The pairs of red marked edges do not span hypercubes, whereas the blue edges do:



It should be clear that the first pair of red edges does not span a hypercube. In the second example, the issue is that there is not a *unique* way to map in a hypercube, with these specified base vertices.

We now come to a key definition. Suppose that E is a set of arrows with target x as above. We say that E spans a hypercube cluster if every subset $E' \subseteq E$ consisting of arrows with pairwise incomparable⁶ sources spans a hypercube.

Example 3.4. Continuing Example 3.3, the pairs of blue arrows span hypercube clusters, whereas the red arrows do not:



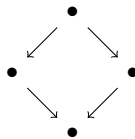
In the first diagram, the sources of the blue arrows are comparable in X , so the condition to span a hypercube cluster reduces to each singleton spanning a hypercube, which is trivially the case.

Suppose that E spans a hypercube cluster. Then the edges in E form a partially ordered set by declaring that $\alpha \leq \beta$ if there exists a directed path from the source of β to that of α . Given a subset $F \subset E$, we define F_{\max} to be the subset of maximal elements with respect to this poset structure. Thus $F_{\max} \subset F$ consists of incomparable elements. Define the *hypercube map*

$$\theta : F \mapsto \text{crown of the hypercube spanned by } F_{\max}.$$

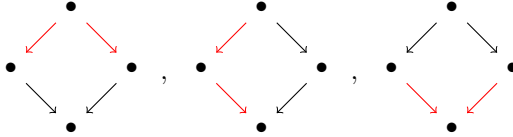
(Later we will see another rather different looking map arising from geometry, and finally see that both are equal. We will sometimes refer to θ as defined above as the *combinatorial hypercube map*.)

3.3. Diamonds. Given a directed graph X , a *diamond* in X is a subgraph isomorphic to:



⁶Elements a and b in a poset are *incomparable* if neither $a \leq b$ nor $b \leq a$ holds.

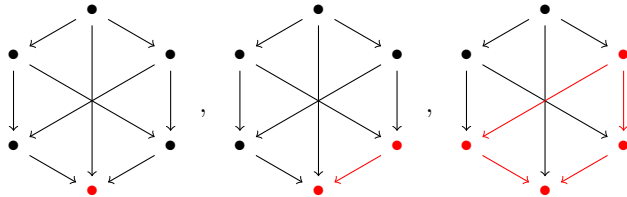
A full subgraph $J \subset X$ is *diamond complete*⁷ if whenever it contains two edges sharing a node, it contains the entire diamond. In other words, for all diamonds in X , if J contains the red edges in any of the diagrams below, it necessarily contains the black edges as well:



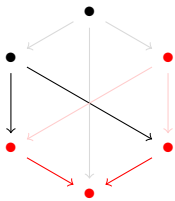
3.4. Hypercube decompositions. Recall that $X = [x, y]$ denotes the Bruhat graph of the interval between x and y . The following is the most important definition of this work. We say that a full subgraph $J \subset X$ is a *hypercube decomposition* if

- (1) $J = \{v \in X \mid v \leq z\}$ for some $y \neq z \in X$, and J is diamond complete;
- (2) for all $v \in J$, the set $E = \{\alpha : u \rightarrow v \mid u \notin J\}$ spans a hypercube cluster.

Example 3.5. Continuing Example 3.3, here are some possible choices of J (indicated by red):



Only the middle choice of J constitutes a hypercube decomposition. In the example on the left, the edges arriving at the base vertex do not span a hypercube cluster. In the example on the right, J is not diamond complete. Here is the incomplete diamond:



3.5. The hypercube piece. We are now ready to define the hypercube piece and inductive piece in our formula. We assume that X is the Bruhat graph corresponding to the interval $[x, y]$ and that we have fixed a hypercube decomposition $J \subset X$. The set

$$E = \{\alpha : v \rightarrow x \mid v \notin J\}$$

spans a hypercube cluster by definition. In particular we have a hypercube map:

$$\theta : \text{subsets of } E \rightarrow X.$$

⁷This notion is due to Patimo [Pat21].

We consider the polynomial:

$$\tilde{Q}_{x,y,J} = \sum_{\emptyset \neq I \subset E} (q-1)^{|I|-1} P_{\theta(I),y} \in \mathbb{Z}[q].$$

(Note that we may assume that all terms on the right hand side are known by induction.) We define the hypercube piece as follows:

$$Q_{x,y,J} = q^{\ell(y)-\ell(x)-1} \tilde{Q}_{x,y,J}(q^{-1}).$$

3.6. The inductive piece. Recall that we have fixed $J \subset X$. Consider the free $\mathbb{Z}[q]$ -module:

$$M_J = \bigoplus_{x \neq v \in J} \mathbb{Z}[q]\delta_v.$$

This has a *standard basis* $\{\delta_v \mid x \neq v \in J\}$. If we define

$$b_v = \sum_{x \neq w \in J} P_{w,v} \cdot \delta_w,$$

then $\{b_v \mid x \neq v \in J\}$ is also a basis for M , which we call the *Kazhdan-Lusztig basis*. This basis is known by induction.

We may now define the inductive piece. Recall that X is a Bruhat interval, with top node y . Define

$$r_{x,y,J} = \sum_{x \neq v \in J} P_{v,y} \cdot \delta_v \in M.$$

(In other words we consider some inductively computed Kazhdan-Lusztig and “restrict” them to J .) Now expand r in the Kazhdan-Lusztig basis:

$$r_{x,y,J} = \sum_{x \neq v \in J} \gamma_v \cdot b_v.$$

The inductive piece is defined as follows:

$$I_{x,y,J} = \sum_{x \neq v \in J} \gamma_v \cdot P_{x,v}^\partial.$$

3.7. A theorem. As above, $X = [x, y]$ denotes the Bruhat graph of the interval between x and y .

Suppose for a moment that we know the labelling of the nodes of X by permutations. (Note that this is forbidden information in the combinatorial invariance conjecture.) In this case, consider the full subgraph

$$L = \{v \in X \mid v^{-1}(0) = x^{-1}(0)\} \subset X.$$

Remark 3.6. For any node $v \in L$, the “hypercube edges” (i.e. those edges with target v and source $\notin L$) are those edges corresponding to swapping 0 and i in v . These are precisely the edges which saliency analysis tells us are most important in our machine learning models (see Figure 3(a) in [DVB⁺21]). This was our initial motivation for considering L .

We have:

Theorem 3.7. $L \subset X$ is a hypercube decomposition, and we have:

$$P_{x,y}^\partial = I_{x,y,L} + Q_{x,y,L}.$$

We will prove this formula in §4 and §5, using geometric techniques. This is a powerful new formula for Kazhdan-Lusztig polynomials for symmetric groups, and should have other applications. However, it does not solve the combinatorial invariance conjecture, as the node labellings are needed to define L .

3.8. A conjecture. Theorem 3.7 motivated us to consider more general hypercube decompositions. Remarkably, it seems that this combinatorial notion is exactly what is needed to make Theorem 3.7 hold:

Conjecture 3.8. *For any hypercube decomposition $J \subset X$ we have*

$$P_{x,y}^{\hat{c}} = I_{x,y,J} + Q_{x,y,J}.$$

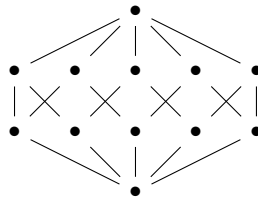
Some remarks on this conjecture:

- (1) We have just seen that any interval admits a hypercube decomposition, and thus the conjecture implies the combinatorial invariance conjecture for symmetric groups.
- (2) The conjecture is equivalent (via Theorem 3.7) to the statement that $I_{x,y,J} + Q_{x,y,J}$ is independent of the choice of hypercube decomposition.
- (3) We have considerable computational evidence for this conjecture. It has been checked for all hypercube decompositions of all Bruhat intervals up to S_7 , and over a million non-isomorphic intervals in S_8 and S_9 .

Positivity plays an important role in Kazhdan-Lusztig theory. Remarkably, both pieces $I_{x,y,J}$ and $Q_{x,y,J}$ in our formula should have positive coefficients:

- (1) The polynomials $Q_{x,y,J}$ have positive coefficients. (This can be shown directly, using the “monotonicity of Kazhdan-Lusztig polynomials” [Irv88, BM01].)
- (2) Conjecturally, the polynomials γ_v involved in the computation of the inductive piece have positive coefficients. This has also been checked in all the cases mentioned above, and is true for the hypercube decomposition L discussed above.

Remark 3.9. It is interesting to ask whether our formula might solve the combinatorial invariance conjecture for Coxeter groups other than symmetric groups. It does not, as one can see by inspecting the 5-crown:



This occurs as a Bruhat graph of an interval in the group H_3 of symmetries of the icosahedron (see [BB05, §2.8]). One can check directly that it does not admit a hypercube decomposition.

3.9. Two worked examples. We give the reader two examples of our formula in action. These examples are illustrated in Figures 5 and 6.

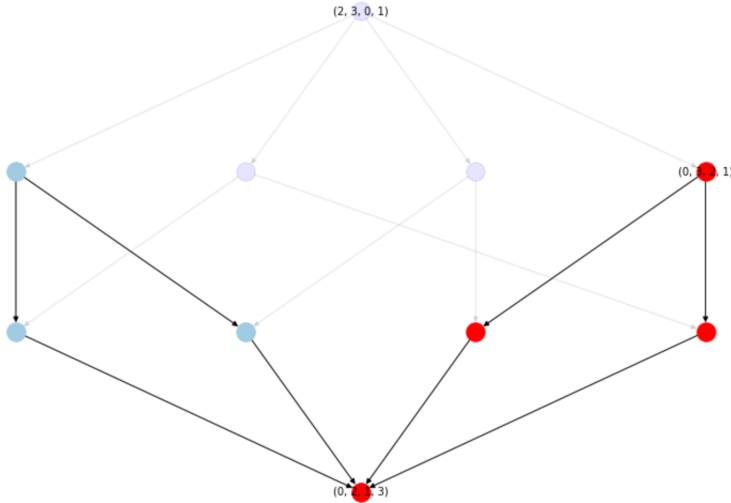


FIGURE 5. The Bruhat graph for the interval between $x = (0, 2, 1, 3)$ and $y = (2, 3, 0, 1)$. The image of the hypercube map at x has blue vertices, and the inductive piece has red vertices. All Kazhdan-Lusztig polynomials $P_{z,y}$ for $z \neq x$ are 1. All hypercube decompositions of this interval are isomorphic to this one.

3.9.1. *The interval between $x = (0, 2, 1, 3)$ and $y = (2, 3, 0, 1)$.* This is the first non-trivial example of a Kazhdan-Lusztig polynomial. In several respects this example is “too simple”, but we discuss it anyway. The interval together with a choice of hypercube decomposition is illustrated in Figure 5. (The reader may check that in this example all hypercube decompositions are isomorphic.)

We have

$$P_{x,y} = 1 + q \quad \text{and} \quad P_{x,y}^\partial = 1 + 2q + q^2.$$

In this case the polynomial $\tilde{Q}_{x,y,J}$ is

$$1 + 1 + (q - 1) = 1 + q$$

and hence the hypercube piece is

$$q^2(1 + q^{-1}) = q + q^2.$$

The inductive piece is

$$1 + q$$

and we indeed have

$$P_{x,y}^\partial = (1 + q) + (q + q^2).$$

3.9.2. *The interval between $x = (1, 0, 2, 4, 3)$ and $y = (4, 1, 2, 3, 0)$.* This is a more interesting case, and illustrates several features of the general case. The interval together with a choice of hypercube decomposition J is illustrated in Figure 6.

The reader may check with a little work that we have

$$\tilde{Q}_{x,y,J} = 1 + 2q + q^2$$

and hence

$$Q_{x,y,J} = q^4(1 + 2q^{-1} + q^{-2}) = q^2 + 2q^3 + q^4.$$

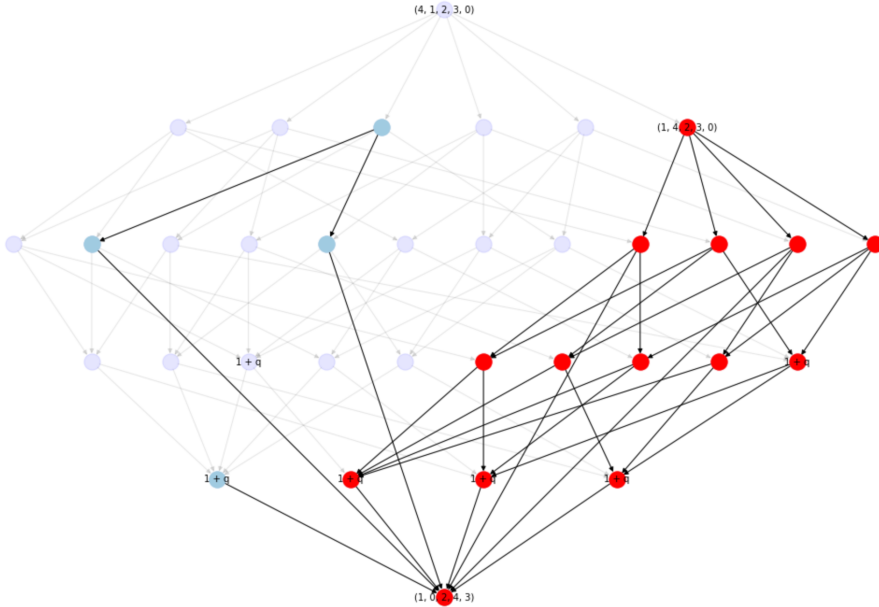


FIGURE 6. The Bruhat graph for the interval between $x = (1, 0, 2, 4, 3)$ and $y = (4, 1, 2, 3, 0)$ with a choice of hypercube decomposition. The image of the hypercube map at x is shaded blue, and the inductive piece is shaded red. All Kazhdan-Lusztig polynomials $P_{z,y}$ for $z \neq x$ are 1 unless indicated.

We now turn to the inductive piece. We have

$$r_{x,y,J} = b_{14230} + q \cdot b_{10432}.$$

(In Figure 6, 10432 is the only node of length $\ell(x) + 2$ in the inductive piece with non-trivial Kazhdan-Lusztig polynomial.) In particular,

$$\begin{aligned} I_{x,y,J} &= P_{x,14230}^\partial + q \cdot P_{x,10432}^\partial = 1 + 2q + 2q^2 + q^3 + q(1 + q) \\ &= 1 + 3q + 3q^2 + q^3 \end{aligned}$$

and we deduce correctly that

$$P_{x,y}^\partial = (1 + 3q + 3q^2 + q^3) + (q^2 + 2q^3 + q^4) = 1 + 3q + 4q^2 + 3q^3 + q^4.$$

Or in other words

$$P_{x,y} = 1 + 2q + q^2.$$

(The fact that $Q_{x,y,J}$ and $I_{x,y,J}$ are “shifted hard Lefschetz polynomials” is a relic of this example, and not a general phenomenon.)

4. GEOMETRY OF THE NEW FORMULA

In this section we explain the proof of Theorem 3.7. This is a consequence of geometric techniques. We first recall fundamental results of Kazhdan and Lusztig and Bernstein and Lunts which connect Kazhdan-Lusztig polynomials to the geometry of Schubert varieties, and then proceed to the proof.

4.1. Geometric background. Given a complex d -dimensional complex algebraic variety X , we denote by $\mathbf{IC}(X, \mathbb{Q})$ the intersection cohomology complex on X , with coefficients in \mathbb{Q} , normalized so that its restriction to the smooth locus of X is isomorphic to the constant sheaf *without shift*. Given a complex of sheaves \mathcal{F} of \mathbb{Q} vector spaces on X we denote by $H^\bullet(X, \mathcal{F})$ its hypercohomology (a graded \mathbb{Q} -vector space). The intersection cohomology of X is by definition

$$IH^\bullet(X, \mathbb{Q}) = H^\bullet(X, \mathbf{IC}(X, \mathbb{Q})).$$

Because of our conventions, the intersection cohomology of a complex d -dimensional variety is concentrated in degrees between 0 and $2d$, as opposed to the more usual convention where it is concentrated in degrees between $-d$ and d .

Given a finite-dimensional graded \mathbb{Q} -vector space H we define its *Poincaré polynomial* to be

$$\sum_{i \in \mathbb{Z}} (\dim H^i) q^{i/2} \in \mathbb{Z}[q^{\pm 1/2}].$$

Given a complex of \mathbb{Q} -vector spaces with finite-dimensional total cohomology we define its Poincaré polynomial to be that of its cohomology groups. In this paper, all Poincaré polynomials we consider are of vector spaces concentrated in positive, even degrees. In this case the Poincaré polynomial is a polynomial in q .

Throughout this section, we will work in the equivariant derived category (see [BL94]). Throughout T will denote an algebraic torus acting algebraically on a variety X . By an equivariant sheaf we mean an object of the equivariant derived category $D_T^b(X, \mathbb{Q})$. All sheaves considered will be constructible. On (equivariant) constructible categories we have the usual menagerie of functors $f^*, f_*, f^!, f_!$ for equivariant maps $f : X \rightarrow Y$ which all commute with the forgetful functor.

We will also need some arguments with weights. We will exploit the fact that $\mathbf{IC}(X, \mathbb{Q})$ is a constructible complex underlying a mixed Hodge module, and hence so are all complexes obtained from $\mathbf{IC}(X, \mathbb{Q})$ via standard functors (see e.g. [Sai89, Sch14]). In particular, each cohomology group $IH^i(X, \mathbb{Q})$ underlies a mixed Hodge structure, and we can talk about its weights. Our normalizations are such that $\mathbf{IC}(X, \mathbb{Q})$ is pure of weight zero. When we say that a cohomology group H^\bullet is pure of weight w , we mean that H^i is pure of weight $w + i$ for all i . We can also talk about weights in equivariant cohomology, by taking appropriate resolutions in the category of algebraic varieties. Our demands on the theory of weights are modest, and the reader who prefers to work with étale cohomology and Frobenius weights will have no difficulty adapting the arguments.

4.2. Tools of the trade. The following techniques will be used repeatedly:

- (1) (“Attractive Proposition”) Suppose that there exists a \mathbb{C}^* -action on X which *retracts X equivariantly onto its fixed points $X^{\mathbb{C}^*}$* . In other words, there exists a morphism

$$a : \mathbb{C} \times X \rightarrow X$$

such that the restriction of a to $\mathbb{C}^* \times X \rightarrow X$ agrees with the action map. Then $\{0\} \times X$ maps into $X^{\mathbb{C}^*}$.⁸ Thus the \mathbb{C}^* -action extends over 0,

⁸One can either see this analytically by writing $0 \cdot x = \lim_{z \rightarrow 0} z \cdot x$ or algebraically via a comodule computation.

attracting X onto $X^{\mathbb{C}^*}$. We can consider the inclusion and projection maps

$$\begin{array}{ccc} & i & \\ & \curvearrowright & \\ X^{\mathbb{C}^*} & & X \\ & \curvearrowleft & \\ & p & \end{array}$$

where $p : X = \{0\} \times X \rightarrow X^{\mathbb{C}^*}$ is induced by a . Given any \mathbb{C}^* -equivariant constructible sheaf \mathcal{F} on X , we have canonical isomorphisms

$$p_*\mathcal{F} \xrightarrow{\sim} i^*\mathcal{F} \quad \text{and} \quad i^!\mathcal{F} \xrightarrow{\sim} p_!\mathcal{F}.$$

We have similar isomorphisms if X is a T -space, and \mathbb{C}^* commutes with the T -action. This technique is fundamental to Geometric Representation Theory, for proofs and more detail see [Spr84], [Soe89, Proposition 1] and [FW14, §2.6].

- (2) (“Attractive Weight Argument”) Let us stay in the setting of (1). We have seen that we have an isomorphism

$$p_*\mathcal{F} \xrightarrow{\sim} i^*\mathcal{F}.$$

Now let us assume that \mathcal{F} is pure (of weight 0, say). Then $i^*\mathcal{F}$ is of weights ≤ 0 , and $p_*\mathcal{F}$ is of weight ≥ 0 because i^* (resp. p_*) can only decrease (resp. increase) weights [Sai89, Proposition 1.7]. Thus $i^*\mathcal{F}$ is pure of weight 0. An identical argument establishes that $i^!\mathcal{F}$ is also pure of weight 0.

- (3) (“Finite Stabilizer Argument”) Suppose that \mathbb{C}^* acts on X with finite stabilizers with (geometric) quotient X/\mathbb{C}^* . Then we have a full embedding

$$D^b(X/\mathbb{C}^*, \mathbb{Q}) \rightarrow D_{\mathbb{C}^*}^b(X, \mathbb{Q}).$$

If $\tilde{\mathcal{F}}$ and \mathcal{F} correspond under this embedding, we have a canonical isomorphism

$$H_{\mathbb{C}^*}^\bullet(X, \mathcal{F}) = H^\bullet(X/\mathbb{C}^*, \tilde{\mathcal{F}}).$$

We have analogous statements for equivariant cohomology and equivariant derived categories (i.e. when X is a T -space). If we assume a free action, these facts are true very generally, and are known as the quotient equivalence [BL94, Proposition 2.2.5]. When we allow finite stabilizers the situation is a little more subtle, and coefficients of characteristic zero are essential [BL94, §9].

4.3. Geometric interpretation of Kazhdan-Lusztig polynomials. Let $G = GL_{n+1}(\mathbb{C})$, $B \subset G$ the Borel subgroup of upper triangular matrices and T the maximal torus of diagonal matrices. We identify the Weyl group with permutation matrices, which is canonically isomorphic to S_{n+1} . We have the flag variety G/B and its Bruhat decomposition into Schubert cells

$$G/B = \bigsqcup_{y \in S_{n+1}} ByB/B.$$

For any $y \in S_{n+1}$ we can consider the Schubert variety $\overline{ByB/B} \subset G/B$. On $\overline{ByB/B}$ we can consider the intersection cohomology complex $\mathbf{IC}(\overline{ByB/B}, \mathbb{Q})$ (see §4.1 for conventions). We have the following fundamental result of Kazhdan and Lusztig [KL80]:

Theorem 4.1. *The cohomology sheaves $H^i(\mathbf{IC}(\overline{ByB/B}, \mathbb{Q})_{xB/B})$ vanish in odd degree, moreover their Poincaré polynomial is given by the Kazhdan-Lusztig polynomial:*

$$P_{x,y} = \sum_{i \geq 0} H^{2i}(\mathbf{IC}(\overline{ByB/B}, \mathbb{Q})_{xB/B})q^i.$$

Let U^- denote lower unitriangular matrices. For any $x \in S_{n+1}$ the product

$$BxB/B \times U^-xB/B$$

is isomorphic to a T -stable affine neighbourhood of x in G/B . Similarly,

$$BxB/B \times (U^-xB/B \cap \overline{ByB/B})$$

is isomorphic to a T -stable affine neighbourhood of x in $\overline{ByB/B}$. Fundamental to everything below will be the space

$$S_{x,y} = U^-xB/B \cap \overline{ByB/B}$$

which is a normal slice to the B -orbit through xB/B in $\overline{ByB/B}$. (As U^- is preserved under conjugation by T , and xB/B is fixed, it follows that $S_{x,y}$ is T -stable.) We denote the point xB/B in $S_{x,y}$ simply by x . Because $S_{x,y}$ is a normal slice, we deduce the following from Theorem 4.1:

Proposition 4.2. *The cohomology sheaves $H^i(\mathbf{IC}(S_{x,y}, \mathbb{Q})_x)$ vanish in odd degree, moreover their Poincaré polynomial is given by the Kazhdan-Lusztig polynomial:*

$$P_{x,y} = \sum_{i \geq 0} H^{2i}(\mathbf{IC}(S_{x,y}, \mathbb{Q})_x)q^i.$$

Remark 4.3. There is no shift in the gradings between Theorem 4.1 and Proposition 4.2 because of our normalizations of \mathbf{IC} .

4.4. Equivariance and the fundamental example. Recall that T denotes the maximal torus inside GL_{n+1} consisting of diagonal matrices. Every T -fixed point xB/B on G/B is *attractive* in the sense that there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow T$ in T such that

$$\lim_{z \rightarrow 0} \lambda(z) \cdot v = xB/B$$

for all v in some neighbourhood of xB/B . (Equivalently and more algebraically: there exists λ as above which acts with positive weights on functions on a T -stable affine neighbourhood of xB/B .) It follows that $x \in S_{x,y}$ is attractive and that (with λ as above)

$$\lim_{z \rightarrow 0} \lambda(z) \cdot v = x$$

for all $v \in S_{x,y}$.

Throughout, set

$$\dot{S}_{x,y} := S_{x,y} - \{x\}.$$

Because λ acts with positive weights the geometric quotient

$$\mathbb{P}_\lambda \dot{S}_{x,y} := \dot{S}_{x,y} /_\lambda \mathbb{C}^*$$

exists and is a projective variety. Here “/ λ ” means that we consider the quotient by the \mathbb{C}^* -action given by λ . (One may establish the existence of this quotient by choosing a linear embedding of $S_{x,y}$ inside an affine space \mathbb{C}^n with linear T -action. Then $(\mathbb{C}^n \setminus \{0\}) /_\lambda \mathbb{C}^*$ exists, and is isomorphic to a weighted projective space. The chosen embedding $S_{x,y} \subset \mathbb{C}^n$ provides a closed embedding of $\mathbb{P}_\lambda \dot{S}_{x,y}$ inside a weighted projective space.)

Proposition 4.4. *The intersection cohomology groups $IH^i(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbb{Q})$ vanish in odd degree, moreover their Poincaré polynomial is given by the q -derivative of the Kazhdan-Lusztig polynomial:*

$$P_{x,y}^{\partial} := \sum_{i \geq 0} IH^{2i}(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbb{Q})q^i.$$

Remark 4.5. The following proof is standard in Geometric Representation Theory. For a detailed discussion of related ideas, the reader might enjoy [BL94, §14].

Proof. Note that $\mathbf{IC}(S_{x,y}, \mathbb{Q})$ is equivariant for T , and hence may be regarded as an object in the equivariant derived category $D_{\mathbb{C}^*}^b(S_{x,y}, \mathbb{Q})$. In this proof we always regard $S_{x,y}$ as a \mathbb{C}^* -variety via the action induced from $\lambda : \mathbb{C}^* \rightarrow T$.

Consider the closed/open decomposition

$$\{x\} \xrightarrow{i} S_{x,y} \xleftarrow{j} \dot{S}_{x,y}.$$

This gives a distinguished triangle

$$i_!i^!\mathbf{IC}(S_{x,y}, \mathbb{Q}) \rightarrow \mathbf{IC}(S_{x,y}, \mathbb{Q}) \rightarrow j_*j^*\mathbf{IC}(S_{x,y}, \mathbb{Q}) \xrightarrow{[1]}.$$

Taking global sections (i.e. hypercohomology) we get a long exact sequence. We claim that all terms are pure of the same weight, and hence that the connecting homomorphism is zero. Indeed:

- (1) Consider the projection $p : S_{x,y} \rightarrow \{x\}$. We have

$$H_{\mathbb{C}^*}^\bullet(\{x\}, p_*\mathbf{IC}(S_{x,y}, \mathbb{Q})) = H_{\mathbb{C}^*}^\bullet(S_{x,y}, \mathbf{IC}(S_{x,y}, \mathbb{Q}))$$

by definition, and

$$H_{\mathbb{C}^*}^\bullet(\{x\}, p_*\mathbf{IC}(S_{x,y}, \mathbb{Q})) = H_{\mathbb{C}^*}^\bullet(\{x\}, i^*\mathbf{IC}(S_{x,y}, \mathbb{Q}))$$

by the Attractive Proposition. By the Attractive Weight Argument,

$$H_{\mathbb{C}^*}^\bullet(S_{x,y}, \mathbf{IC}(S_{x,y}, \mathbb{Q}))$$

is pure of weight zero as claimed.

- (2) By the Attractive Proposition again we have

$$H_{\mathbb{C}^*}^\bullet(\{x\}, p_!\mathbf{IC}(S_{x,y}, \mathbb{Q})) = H_{\mathbb{C}^*}^\bullet(\{x\}, i^!\mathbf{IC}(S_{x,y}, \mathbb{Q}))$$

and the purity of both sides follows by the Attractive Weight Argument again.

- (3) Because j is an open inclusion we have $j^*\mathbf{IC}(S_{x,y}, \mathbb{Q}) = \mathbf{IC}(\dot{S}_{x,y}, \mathbb{Q})$. Because \mathbb{C}^* acts with finite stabilizers and our coefficients are \mathbb{Q} we have

$$H_{\mathbb{C}^*}^\bullet(\dot{S}_{x,y}, \mathbf{IC}(\dot{S}_{x,y}, \mathbb{Q})) = IH^\bullet(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbb{Q}).$$

The latter is pure of weight zero, as the intersection cohomology of a projective variety.

Thus we have a short exact sequence

$$(5) \quad H_{\mathbb{C}^*}^\bullet(\{x\}, i^!\mathbf{IC}(S_{x,y}, \mathbb{Q})) \hookrightarrow H_{\mathbb{C}^*}^\bullet(\{x\}, i^*\mathbf{IC}(S_{x,y}, \mathbb{Q})) \twoheadrightarrow IH^\bullet(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbb{Q}).$$

A similar argument to the one given above using the attractive proposition shows that the non-equivariant cohomology groups

$$H^\bullet(\{x\}, i^!\mathbf{IC}(S_{x,y}, \mathbb{Q})) \quad (\text{resp.} \quad H^\bullet(\{x\}, i^*\mathbf{IC}(S_{x,y}, \mathbb{Q})))$$

are pure. Moreover, their Poincaré polynomials are given by

$$q^{\ell(y)-\ell(x)}P_{x,y}(q^{-1}) \quad (\text{resp.} \quad P_{x,y}).$$

(The second is simply a restatement of Proposition 4.2, the first follows from it by Verdier duality.) It follows that the equivariant cohomology groups are free over $H_{\mathbb{C}^*}^*(pt, \mathbb{Q})$ of graded ranks given by

$$q^{\ell(y)-\ell(x)} P_{x,y}(q^{-1})/(1-q) \quad (\text{resp. } P_{x,y}/(1-q)).$$

We deduce that the Poincaré polynomial of the last term in (5) is given by

$$P_{x,y}/(1-q) - q^{\ell(y)-\ell(x)} P_{x,y}(q^{-1})/(1-q) = P_{x,y}^\partial$$

which is what we wanted to show. □

Example 4.6. Consider the first singular Schubert variety $y = (2, 3, 0, 1)$ and $x = (0, 2, 1, 3)$. We will keep this as a running example throughout. One can compute directly (or wait until we discuss this example in more detail in Example 4.8) that $S_{x,y}$ may be identified with the following subvariety of 4×4 -matrices:

$$S_{x,y} = \left\{ \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 \\ y & 1 & 0 & 0 \\ 0 & z & w & 1 \end{array} \right) \middle| xw + yz = 0 \right\}.$$

We can choose λ such that the attractive \mathbb{C}^* -action simply scales x, y, z and w . With this choice of λ the quotient $\dot{S}_{x,y}/\lambda\mathbb{C}^*$ is simply

$$\{[x : y : z : w] \in \mathbb{P}^3 \mid xz + yw = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

This is smooth, thus its intersection cohomology and cohomology agree, and its Poincaré polynomial is $1 + 2q + q^2 = P_{x,y}^\partial$ (so $P_{x,y} = 1 + q$).

4.5. Where does the formula come from? We are now in a position to describe our formula in more detail. Here we give an outline of the argument.

In the previous section we saw that the q -derivative of the Kazhdan-Lusztig polynomial $P_{x,y}^\partial$ computes the Poincaré polynomial of the intersection cohomology of the projective variety $\mathbb{P}_\lambda \dot{S}_{x,y}$. Now $S_{x,y}$ has a natural stratification via intersections with Schubert cells:

$$S_{x,y} = \bigsqcup_{u \in [x,y]} S_{x,y}^u \quad \text{where } S_{x,y}^u = S_{x,y} \cap BuB/B.$$

We will refer to this stratification as the *Schubert stratification* of $S_{x,y}$. The Schubert stratification is T -invariant and induces a stratification of the quotient

$$(6) \quad \mathbb{P}_\lambda \dot{S}_{x,y} = \dot{S}_{x,y}/\lambda\mathbb{C}^* = \bigsqcup_{u \in (x,y)} S_{x,y}^u/\lambda\mathbb{C}^*.$$

(Note that $[x, y]$ has been replaced by $(x, y]$ because the point $\{x\} = S_{x,y}^x = S_{x,y} \cap BxB/B$ has been removed in $\dot{S}_{x,y}$.)

The pullback of the IC sheaf on $\mathbb{P}_\lambda \dot{S}_{x,y}$ to $\dot{S}_{x,y}$ agrees with the IC sheaf on $\dot{S}_{x,y}$, which in turn agrees with the restriction of the IC sheaf on $\overline{ByB/B}$, because $S_{x,y}$ is a normal slice. In particular, all the Poincaré polynomials of the stalks of the IC sheaf on $\mathbb{P}_\lambda \dot{S}_{x,y}$ are known inductively. (To spell things out: the Poincaré polynomial of $\mathbf{IC}(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbb{Q})$ along the stratum $S_{x,y}^u/\lambda\mathbb{C}^*$ is given by $P_{u,y}$.) Thus it is reasonable, perhaps, that we can compute the Poincaré polynomial of the global sections of $\mathbf{IC}(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbb{Q})$ via some kind of long exact sequence.

We will define an open/closed decomposition

$$(7) \quad Z \xrightarrow{i} \mathbb{P}_\lambda \dot{S}_{x,y} \xleftarrow{j} U$$

with U open and Z closed. We have an associated long exact sequence:

$$\rightarrow H^\bullet(Z, i^! \mathbf{IC}) \rightarrow H^\bullet(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbf{IC}) \rightarrow H^\bullet(U, j^* \mathbf{IC}) \rightarrow$$

(where we have abbreviated $\mathbf{IC} := \mathbf{IC}(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbb{Q})$). The decomposition (7) has several favourable properties:

- (1) Z is isomorphic to a weighted projective space, the stratification induced by (6) coarsens its toric stratification,⁹ and $i^! \mathbf{IC}$ is pure;
- (2) U retracts T -equivariantly onto a space Z' isomorphic to $\mathbb{P}_\lambda \dot{S}_{x,c}$, where c denotes the maximal element in $[x, y]$ in the S_n -coset of x , and the restriction of \mathbf{IC} to Z' is pure.

By the Attractive Proposition our long exact sequence can be rewritten

$$(8) \quad \rightarrow H^\bullet(Z, i^! \mathbf{IC}) \rightarrow H^\bullet(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbf{IC}) \rightarrow H^\bullet(Z', (i')^* \mathbf{IC}) \rightarrow$$

where $i' : Z' \hookrightarrow \mathbb{P}_\lambda \dot{S}_{x,y}$ denotes the inclusion. Because everything is pure of weight zero, we conclude that our sequence is in fact short exact. Thus, taking Poincaré polynomials (and using that the q -derivative of the KL polynomial is the Poincaré polynomial of the middle term) we deduce

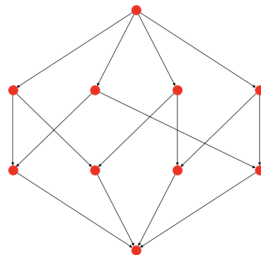
$$P_{x,y}^\partial = Q + I,$$

where Q (resp. I) is the Poincaré polynomial of the left-most (resp. right-most) term in (8). We can compute Q via a simple formula (yielding the hypercube piece), and I is computed by induction (yielding the inductive piece).

Example 4.7. We continue our running example. We have seen that $\dot{S}_{x,y}/\lambda \mathbb{C}^*$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The torus $T/\lambda(\mathbb{C}^*) \cong \mathbb{C}^* \times \mathbb{C}^*$ still acts on this quotient, and one may check that this is the standard action, i.e.

$$(z, z') \cdot ([u : v], [u' : v']) = ([u : zv], [u' : z'v']).$$

There are $4 + 4 + 1 = 9$ orbits of $(\mathbb{C}^*)^2$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and these orbits coincide with the image of the Schubert stratification. It is pleasant to notice that the closure patterns of these orbits nicely match the vertices $\neq x$ in the interval $[x, y]$, as indeed it must:



Thus the four vertices at height 1 correspond to the 4-fixed points on $\mathbb{P}^1 \times \mathbb{P}^1$, the four vertices at height 2 correspond to the 4 invariant \mathbb{C}^* 's, and the unique vertex at height 3 corresponds to the open orbit.

⁹See §4.11 for the definition of “toric stratification”.

Our decomposition has Z equal to a torus invariant $\mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$, and U its complement, which retracts equivariantly onto the “opposite” \mathbb{P}^1 which yields Z' . The short exact sequence

$$0 \rightarrow H^*(Z, i^! \mathbb{Q}_{\mathbb{P}^1 \times \mathbb{P}^1}) \rightarrow H^\bullet(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q}) \rightarrow H^*(Z', i^* \mathbb{Q}_{\mathbb{P}^1 \times \mathbb{P}^1}) \rightarrow 0$$

gives the unsurprising identity

$$1 + 2q + q^2 = (q + q^2) + (1 + q)$$

coming from the Poincaré polynomial of the left hand term (resp. right hand term) in the short exact sequence.

4.6. Slices and their subvarieties in the flag variety. Our eventual goal is to make the sketch provided in the previous section precise. We will need an explicit description of the slice $S_{x,y}$ in the case of GL_{n+1} . Most of this is rather standard, for more information, see [Ful92], [WY08, §3.2] and [WY12, §2.2]. The experienced reader can probably skim this subsection and the next.

Recall that $B \subset GL_{n+1}(\mathbb{C})$ denotes the subgroup of upper triangular matrices, U^- denotes the subgroup of lower uni-triangular matrices and that we identify permutations in S_{n+1} with the corresponding permutation matrices in $GL_{n+1}(\mathbb{C})$.

Because the multiplication $U^- \times B \rightarrow GL_{n+1}(\mathbb{C})$ is an open immersion, U^- provides an open neighbourhood of B/B in G/B . It follows that xU^- provides an open neighbourhood of the point xB/B in G/B . We have

$$xU^- = \{(\gamma_{i,j}) \mid \gamma_{x(i),i} = 1, \gamma_{x(i),j} = 0 \text{ if } j > x(i)\}.$$

That is, “zero if right of a 1”. Here is a picture for $x = (1, 0, 3, 2) \in S_4$:

$$xU^- = \begin{pmatrix} * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ * & * & * & 1 \\ * & * & 1 & 0 \end{pmatrix}.$$

In this chart, the B -orbit through xB/B consists of “zero if not above a 1”.

In particular, a normal slice to the B -orbit through xB/B is given by

$$S_x = \{(\gamma_{i,j}) \mid \gamma_{x(i),i} = 1, \gamma_{i,j} = 0 \text{ if } j > x(i) \text{ or } i < x^{-1}(j)\}.$$

That is, “zero if right or above a 1”. A picture for $x = (1, 0, 3, 2) \in S_4$:

$$S_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ * & * & 0 & 1 \\ * & * & 1 & 0 \end{pmatrix}.$$

Throughout an important role will be played by

$$m := x^{-1}(0).$$

This is the column in which the 1 occurs in the top row in x . Consider the subvarieties

$$S_x^H = \{\gamma \in S_x \mid \gamma_{i,j} = 0 \text{ unless } (i,j) = (x(j),j) \text{ or } j = m\},$$

$$S_x^I = \{\gamma \in S_x \mid \gamma_{i,j} = 0 \text{ unless } (i,j) = (x(j),j) \text{ or } j \neq m\}.$$

That is, “zero if not in column number m ” and “zero if in column number m ” respectively. A picture for $x = (1, 0, 3, 2) \in S_4$ (so $m = 1$):

$$S_x^H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & * & 0 & 1 \\ 0 & * & 1 & 0 \end{pmatrix} \quad \text{and} \quad S_x^I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ * & 0 & 0 & 1 \\ * & 0 & 1 & 0 \end{pmatrix}.$$

4.7. Equations defining slices to Schubert varieties. Let $g \in G = \text{GL}_{n+1}(\mathbb{C})$ be a matrix. By a *lower left corner* we mean the submatrix:

$$g^{\leq(p,q)} = \{(g_{ij})_{i \geq p, j \leq q}\}.$$

Given a matrix, its *corner rank matrix* is the matrix which at position (p, q) records the rank of $g^{\leq(p,q)}$. These matrices will be particularly important for permutation matrices. For example, if $y = (1, 2, 3, 0)$ then

$$y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \text{corner rank matrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Suppose that g' is obtained from g by a scaling rows or columns, upwards row operation, or a rightwards column operation, then the rank of any lower left corner remains unchanged. In particular, any matrix $g \in ByB$ satisfies

$$\text{rank}(g^{\leq(p,q)}) = \text{rank}(y^{\leq(p,q)}) = (p, q)^{\text{th}} \text{ entry in corner rank matrix of } y$$

for all p, q . It is not difficult to see that $g \in ByB$ if and only if these conditions are met.

In fact, $\overline{ByB/B}$ is cut out by the equations:

$$(9) \quad \text{rank}(g^{\leq(p,q)}) \leq \text{rank}(y^{\leq(p,q)}) \quad \text{for all } p, q.$$

Intersecting with the slices S_x, S_x^H and S_x^I defines subvarieties

$$\begin{aligned} S_{x,y} &:= S_x \cap \overline{ByB/B}, \\ S_{x,y}^H &:= S_x^H \cap \overline{ByB/B}, \\ S_{x,y}^I &:= S_x^I \cap \overline{ByB/B} \end{aligned}$$

which are cut out of the respective affine spaces by rank conditions.

Example 4.8. We continue our running example (see Example 4.6 and Example 4.7). The corner rank matrix $y = (2, 3, 0, 1)$ is

$$\dot{y} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \text{corner rank matrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Recall that $x = (0, 2, 1, 3)$ and hence the slice S_x is:

$$S_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 0 & 1 & 0 \\ b & 1 & 0 & 0 \\ c & d & e & 1 \end{pmatrix}.$$

In this case, the equations cutting out $S_{x,y} \subset S_x$ reduce to $\text{rank } g^{\leq(3,0)} \leq 0$, i.e. $c = 0$, and $\text{rank } g^{\leq(1,2)} = 1$, i.e. $ae + bd = 0$. This recovers the description claimed in Example 4.6.

4.8. Decomposition of the projectivized slice. In this section we make the outline in §4.5 precise. In particular, we define the subvarieties U, Z and Z' . As in §4.5 we denote

$$\mathbf{IC} := \mathbf{IC}(\mathbb{P}_\lambda \dot{S}_{x,y}, \mathbb{Q}).$$

Given a T -stable closed subvariety $X \subset \dot{S}_{x,y}$ we denote by $\mathbb{P}_\lambda \dot{X}$ the closed subvariety obtained as the image of $\dot{X} := X - \{x\}$ in $\mathbb{P}_\lambda \dot{S}_{x,y}$. It is a closed subvariety. Set

$$Z := \mathbb{P}_\lambda \dot{S}_{x,y}^H, \quad U := \mathbb{P}_\lambda \dot{S}_{x,y} - \mathbb{P}_\lambda \dot{S}_{x,y}^H, \quad Z' := \mathbb{P}_\lambda \dot{S}_{x,y}^I$$

and denote by i, j, i' the inclusions of Z, U, Z' into $\mathbb{P}_\lambda \dot{S}_{x,y}$.

Let

$$L = \{z \in [x, y] \mid z^{-1}(0) = x^{-1}(0)\} \subset [x, y].$$

In order to carry out the program outlined in §4.5 we need to check the following statements:

Theorem 4.9. *U retracts T -equivariantly onto Z' , $(i')^*\mathbf{IC}$ is pure, and the Poincaré polynomial of $H^\bullet(Z', (i')^*\mathbf{IC})$ is given by $I_{x,y,L}$.*

We prove this in §4.9 and §4.10.

Theorem 4.10. *$i^!\mathbf{IC}$ is pure, and the Poincaré polynomial of $H^\bullet(Z, i^!\mathbf{IC})$ is given by $Q_{x,y,L}$.*

We prove this in §4.14, using an alternative (geometric) definition of the hypercube map. In §5 we check that these two definitions agree.

Example 4.11. In the setting of our running example (see Examples 4.6, 4.7 and 4.8) we have

$$S_x^H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S_x^I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix}$$

which produce two T -stable “opposite \mathbb{P}^1 ’s” in $\mathbb{P}^1 \times \mathbb{P}^1$ discussed in Example 4.7.

4.9. The inductive piece: Geometry. The results for the inductive piece follow essentially because everything is nicely compatible with restriction to the fixed points of a particular choice of $\gamma : \mathbb{C}^* \rightarrow T$ whose centralizer is $\text{GL}_1 \times \text{GL}_n$. This is a common theme in Geometric Representation Theory: much information can be gained from Levi subgroups via torus localization. Now we give the details.

Recall from above that we have fixed a one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow T$ which acts attractively on $S_{x,y}$. It is this λ that we used to form the quotient

$$\mathbb{P}_\lambda \dot{S}_{x,y} = \dot{S}_{x,y} / \lambda \mathbb{C}^*.$$

Remark 4.12. It will not be important to fix λ . However, the reader might like to check that we could take

$$\lambda : z \mapsto \text{diag}(1, z, z^2, \dots, z^n).$$

Below another one-parameter subgroup will be important. Consider

$$\begin{aligned} \gamma : \mathbb{C}^* &\rightarrow \mathrm{GL}_{n+1}, \\ z &\mapsto \mathrm{diag}(1, z, z, \dots, z). \end{aligned}$$

Because we have two \mathbb{C}^* 's at play, we will use λ and γ to distinguish them. For example, if we refer to the \mathbb{C}^*_λ -action we mean the \mathbb{C}^* -action provided by λ , and $S^\gamma_{x,y}$ denotes the \mathbb{C}^* -fixed points on $S_{x,y}$, with \mathbb{C}^* -action given by γ .

Firstly, note that \mathbb{C}^*_γ acts naturally on $(n+1) \times (n+1)$ matrices via conjugation. The action is via scaling block matrices as follows (here $G = \mathrm{GL}_{n+1}$):

$$\left(\begin{array}{c|c} 1 & z^{-1} \\ \hline z & 1 \end{array} \right).$$

In particular,

$$G^\gamma = \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}).$$

This subgroup will play an important role below. We give it its own notation:

$$G_n = \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \subset G.$$

This is a Levi subgroup of G . We denote by S_n its Weyl group. It consists of the subgroup of S_{n+1} consisting of those permutations which fix 0. We let $B_n = G_n \cap B$ denote the Borel subgroup of upper triangular matrices in G_n .

Remark 4.13. We caution the reader that the inclusion of $G_n \hookrightarrow G$ is via block $1 \times n$ -matrices, not $n \times 1$ as it is more common. Similarly, our inclusion $S_n \hookrightarrow S_{n+1}$ via permutations fixing 0 is slightly non-standard.

Proposition 4.14. $(G/B)^\gamma$ is a disjoint union of flag varieties for G_n . More precisely,

$$(G/B)^\gamma = \bigsqcup_{x \in S_n \backslash S_{n+1}} G_n \cdot x$$

and each $G_n \cdot x$ orbit is isomorphic to G_n/B_n .

Proof. This is a standard result, so we will be brief. Firstly, $G_n = G^\gamma$ certainly acts on the $(G/B)^\gamma$. If z denotes a minimal coset representative of $S_n \backslash S_{n+1}$ (i.e. $z(l) = 0$ for some $0 \leq l \leq n$ and z maps the remainder $0, 1, \dots, l-1, l+1, \dots, n$ monotonically to $1, \dots, n$) then the stabilizer of G_n acting on zB/B is

$$z^{-1}G_n z \cap B = B_n$$

as follows from an easy calculation. Thus action on the points given by minimal coset representatives gives an inclusion

$$(10) \quad \bigsqcup_{x \in S_n \backslash S_{n+1}} G_n/B_n \subset (G/B)^\gamma.$$

A similar computation to that establishing $G^\gamma = G_n$ above shows that on each of the xU^- for G/B , the γ -action is via scaling the free variables in 0^{th} -row by z^{-1} , and scaling the entries in the $x^{-1}(0)^{th}$ -column by z . Thus the fixed points in this chart have zeroes in the 0^{th} -row and $x^{-1}(0)^{th}$ -column, except the 1 in position $(0, x^{-1}(0))$. It follows that

$$(xU)^\gamma \subset G_n \cdot xB/B.$$

As the charts xU^- cover G/B , we deduce that the inclusion in (10) is an equality. \square

Consider a T -fixed point xB/B . We can consider the slice $S_x \subset G/B$ to the B -orbit through xB/B . On the other hand, if $z \in S_n x$ denotes the minimal coset representative then we have seen that $g \mapsto g \cdot zB/B$ yields an embedding

$$\phi : G_n/B_n \hookrightarrow G/B$$

of flag varieties. If we set $x' := xz^{-1}$ then $x' \in S_n$, and $x'B_n/B_n$ maps to xB/B under this inclusion. We denote by $S_{x,n}$ the image under this inclusion of the slice $S_{x'}$ to the B_n -orbit through x' in G_n/B_n .

Proposition 4.15. *We have $S_x^\gamma = S_{x,n}$ as subvarieties of G/B .*

Proof. Recall the subvariety S_x^I defined in §4.6. A simple computation shows

$$S_x^I = S_x^\gamma.$$

On the other hand, the slice to x' in G_n/B_n has the form:

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_{x'} \end{array} \right).$$

Right multiplication by z produces exactly S_x^I , and the result follows. \square

Finally, one checks easily that γ has positive weights on the slice S_x . The following is immediate:

Lemma 4.16. *For any $p \in S_x - S_x^H$ we have*

$$\lim_{z \rightarrow 0} \gamma(z) \cdot p \in S_x^I.$$

4.10. The inductive piece: Sheaves. In this section we prove Theorem 4.9. We keep the notation of previous sections, in particular §4.8. Note first that Lemma 4.16 implies that $U = \mathbb{P}_\lambda \dot{S}_{x,y} - \mathbb{P}_\lambda \dot{S}_{x,y}^H$ retracts equivariantly onto $\mathbb{P}_\lambda \dot{S}_{x,y}^I$. Hence, by the Attractive Proposition and the Attractive Weight Argument, we conclude:

- $(i')^* \mathbf{IC}$ is pure;
- $H^\bullet(U, j^* \mathbf{IC}) = H^\bullet(Z', (i')^* \mathbf{IC})$, and both are pure.

All that remains is to check that the Poincaré polynomial of $H^\bullet(Z', (i')^* \mathbf{IC})$ is given by $I_{x,y,L}$. This will take a little more work, and uses in a more substantial way the results of the previous section.

Consider the embedding

$$\phi : G_n/B_n \hookrightarrow G/B$$

of flag varieties considered in the previous section, whose image is the G_n -orbit through $xB/B \in G/B$.

Proposition 4.17. *$\phi^* \mathbf{IC}(\overline{ByB/B}, \mathbb{Q})$ is pure.*

Proof. Consider the Białynicki-Birula decomposition with respect to $-\gamma$ (i.e. $z \mapsto \text{diag}(1, z^{-1}, \dots, z^{-1})$). Because $-\gamma$ has positive weights on B , the attracting sets

are unions of Bruhat cells. In particular, the attracting set $Z^+ \subset G/B$ for the component $Z = \phi(G_n/B_n)$ of $(G/B)^\gamma$ is given by

$$Z^+ = \bigsqcup_{u \in S_n x} BuB/B$$

and the attracting map π realizes Z^+ as an affine bundle over Z . We have a diagram:

$$\begin{array}{ccc} & Z^+ & \xrightarrow{\phi'} G/B \\ \iota \uparrow & \searrow \pi & \uparrow \phi \\ & G_n/B_n & \end{array}$$

By Braden’s theorem on hyperbolic localization [Bra03, Theorems 1 and 2] we know that

$$\pi_!(\phi')^* \mathbf{IC}(\overline{ByB/B}, \mathbb{Q})$$

is pure. (It is a summand of the hyperbolic localization of $\mathbf{IC}(\overline{ByB/B}, \mathbb{Q})$ to the fixed points of $-\gamma$.) On the other hand, we know that this is equal to

$$\iota^!(\phi')^* \mathbf{IC}(\overline{ByB/B}, \mathbb{Q})$$

by the Attractive Proposition. Finally, Z^+ is stratified by Bruhat cells, and ι is a normally non-singular inclusion for this stratification. Hence $\iota^! = \iota^*[-2m]$, where m is the complex dimension of the fibres. We deduce that the above complexes are equal (up to a shift) to

$$\iota^*(\phi')^* \mathbf{IC}(\overline{ByB/B}, \mathbb{Q}) \cong \phi^* \mathbf{IC}(\overline{ByB/B}, \mathbb{Q}).$$

The latter complex is the one for which we wished to establish purity. □

As $\phi^* \mathbf{IC}(\overline{ByB/B}, \mathbb{Q})$ is pure, it decomposes as a direct sum of shifts of intersection cohomology complexes. We now describe how to compute some of these multiplicities.

First some notation: given a polynomial $p = \sum a_i q^i$ with positive coefficients and an object \mathcal{F} in a derived category, write

$$p \cdot \mathcal{F} := \bigoplus \mathcal{F}^{\oplus a_i}[-2i].$$

Because $\phi^* \mathbf{IC}(\overline{ByB/B}, \mathbb{Q})$ is pure and its stalks have no cohomology in odd or negative degree, we can certainly write

$$(11) \quad \phi^* \mathbf{IC}(\overline{ByB/B}, \mathbb{Q}) \cong \bigoplus_{v \in S_n x} \gamma'_v \cdot \mathbf{IC}(\overline{BvB/B}),$$

for certain $\gamma'_v \in \mathbb{Z}_{\geq 0}[q]$.

Recall that in §3.6 we defined certain polynomials γ_v as part of the definition of the inductive piece:

Proposition 4.18. *For $x \neq v \in L$ we have $\gamma_v = \gamma'_v$.*

Proof. Taking Poincaré polynomials of stalks on both sides of (11) for any $x \neq v \in L$. We deduce the relations:

$$P_{v,y} = \sum_{x \neq u \in L} \gamma'_v P_{u,v}.$$

These are (a repackaging of) the relations that determine the γ_v in §3.6. The proposition follows. \square

We now reach the goal of this section:

Theorem 4.19. *The Poincaré polynomial of $H^\bullet(Z', (i')^*\mathbf{IC})$ is given by $I_{x,y,L}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
 G_n/B_n & \xrightarrow{\phi} & G/B \\
 \uparrow k_n & & \uparrow k \\
 \dot{S}_{x,n} & \xrightarrow{i''} & \dot{S}_x \\
 \downarrow & & \downarrow \\
 \mathbb{P}_\lambda \dot{S}_{x,n} & \xrightarrow{i'} & \mathbb{P}_\lambda \dot{S}_x
 \end{array}$$

where we have used Proposition 4.15 to identify S_x^I with $S_{x,n}$. Because \mathbb{C}^* acts (via λ) on \dot{S}_x with finite stabilizers we can apply the Finite Stabilizer Argument. We have

$$H^\bullet(Z', (i')\mathbf{IC}) = H_{\mathbb{C}^*}^\bullet(\dot{S}_{x,n}, (i'')^*\mathbf{IC}(\dot{S}_{x,y}, \mathbb{Q})) = H_{\mathbb{C}^*}^\bullet(\dot{S}_{x,n}, (i'')^*k^*\mathbf{IC}(\overline{ByB/B}, \mathbb{Q})),$$

where for the second equality we have used that k is a normally non-singular inclusion. Using the above commutative diagram, we can rewrite this as:

$$\begin{aligned}
 H_{\mathbb{C}^*}^\bullet(\dot{S}_{x,n}, k_n^*\phi^*\mathbf{IC}(\overline{ByB/B}, \mathbb{Q})) &\cong \bigoplus_{x \neq v \in L} H_{\mathbb{C}^*}^\bullet(\dot{S}_{x,n}, \gamma_v \cdot k_n^*\mathbf{IC}(\overline{BvB/B}, \mathbb{Q})) \\
 &\cong \bigoplus_{x \neq v \in L} H^\bullet(\mathbb{P}_\lambda \dot{S}_{x,n}, \gamma_v \cdot \mathbf{IC}(\mathbb{P}_\lambda \dot{S}_{x,v}, \mathbb{Q})).
 \end{aligned}$$

We used Proposition 4.18 for the first step, and the normal non-singularity of $S_{x,n} \hookrightarrow G_n/B_n$ and the Finite Stabilizer Argument again for the second step. Taking Poincaré polynomials (and using Proposition 4.4) we deduce that its Poincaré polynomial is

$$\sum_{x \neq v \in L} \gamma_v \cdot P_{x,v}^\partial = I_{x,y,L}. \quad \square$$

4.11. Geometry of weighted projective space. We now turn to the hypercube piece, which comes from a sheaf on $S_x^H/\lambda\mathbb{C}^*$. We will see that this space is isomorphic to a weighted projective space. In this section we discuss some preliminaries on weighted projective spaces with torus action.

Suppose that a torus T acts diagonally on $V = \bigoplus_{i=1}^p \mathbb{C}e_i$. Assume moreover:

- (1) the weights of T on V are linearly independent;
- (2) we have fixed a cocharacter $\lambda : \mathbb{C}^* \rightarrow T$, all of whose weights on V (with respect to the action induced by T) are positive.

Note that (1) implies:

- (3) T has finitely many orbits on V . Moreover, T -orbits on V are classified by subsets of $\{1, \dots, p\}$; given such a subset I the corresponding T -orbit is

$$V_I = \left\{ \sum_{i \in I} \mu_i e_i \mid \mu_i \in \mathbb{C}^* \right\}.$$

The stratification by the $\{V_I\}$ will be called the *toric stratification* of V .

- (4) For any subset $I \subset \{1, \dots, n\}$ we can find a cocharacter $\lambda_I : \mathbb{C}^* \rightarrow T$ such that the induced action fixes each e_i for $i \in I$ and has positive weights on the rest. In particular

$$\lim_{z \rightarrow 0} \lambda_I(z) \cdot v \in \bigoplus_{i \in I} \mathbb{C}e_i$$

for all $v \in V$.

Denote the weighted projective space obtained as the quotient by

$$\mathbb{P}_\lambda \dot{V} := (V - \{0\}) / \lambda \mathbb{C}^*.$$

Statements (3) and (4) for V imply the following for $\mathbb{P}_\lambda \dot{V}$:

- (5) T has finitely many orbits on $\mathbb{P}_\lambda \dot{V}$. Moreover, these orbits are classified by non-empty subsets of $\{1, \dots, p\}$. Given such a subset I the corresponding T -orbit is \mathcal{O}_I , given by the image of V_I in $\mathbb{P}_\lambda \dot{V}$. We will refer to this stratification as the *toric stratification* of $\mathbb{P}_\lambda \dot{V}$.
- (6) For any $\emptyset \neq I \subset \{1, \dots, p\}$ we set

$$U_I = \bigcup_{I \subset J} \mathcal{O}_J = \{[\lambda] \mid \lambda = \sum \lambda_i e_i, \lambda_i \neq 0 \text{ if } i \in I\}.$$

(Note that $U_{\{1\}}, \dots, U_{\{p\}}$ form an open covering of $\mathbb{P}_\lambda \dot{V}$ by quotients of \mathbb{C}^{p-1} with finite stabilizers.) Also, (4) above implies that, for all $\emptyset \neq I \subset \{1, \dots, p\}$,

$$(12) \quad U_I \text{ retracts equivariantly onto } \mathcal{O}_I.$$

4.12. Equivariant sheaves on vector spaces. Let T and $V = \bigoplus \mathbb{C}e_i$ be as above. (In applications, V will be a slice to a stratum in weighted projective space. Hence, it will not be the “same V ” as in the previous section. We hope that this does not lead to too much confusion.) In particular, the weights of T on V are linearly independent and we have the stratification V_I by subsets of $\{1, \dots, p\}$ (so $V_\emptyset = \{0\}$ and $V_{\{1, \dots, p\}}$ consists of vectors all of whose coordinates are non-zero.)

The aim of this section is to prove the following:

Proposition 4.20. *Consider $\mathcal{F} \in D_T^b(V, \mathbb{Q})$. Suppose that:*

- (1) *The stalks of \mathcal{F} are pure of weight zero.*
- (2) *The restriction of \mathcal{F} to each stratum is isomorphic to a direct sum of even shifts of constant sheaves.*
- (3) *For any I the restriction map*

$$H_T^\bullet(V, \mathcal{F}) \rightarrow H_T^\bullet(V_I, \mathcal{F}|_{V_I})$$

is surjective.

Then \mathcal{F} is pure of weight zero.

Remark 4.21. This is close to several results in the literature. As the reader will see, it uses in a crucial way in the proof that all subvarieties \overline{V}_I are smooth. The analogue of this theorem for a general attractive fixed point is false.

Proof. Consider the truncation functors $\tau_{\leq k}$ and $\tau_{> k}$ associated to the standard (i.e. not the perverse!) t -structure on $D_T^b(V, \mathbb{Q})$. We first argue that each sheaf in the distinguished triangle

$$(13) \quad \tau_{\leq k} \mathcal{F} \rightarrow \mathcal{F} \rightarrow \tau_{> k} \mathcal{F} \xrightarrow{[1]}$$

satisfies the conditions (1), (2) and (3) of the proposition. (1) and (2) are clear, as $\tau_{\leq k}$ and $\tau_{> k}$ have a predictable effect on stalks. For (3), note that, from the existence of an attractive cocharacter, we know by the Attractive Proposition that the restriction map

$$H_T^\bullet(V, \mathcal{F}) \rightarrow H_T^\bullet(\{0\}, \mathcal{F}_0)$$

is an isomorphism, and similarly for $\tau_{\leq k}\mathcal{F}$ and $\tau_{> k}\mathcal{F} \xrightarrow{[1]}$. Hence we have a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_T^\bullet(\{0\}, (\tau_{\leq k}\mathcal{F})_0) & \longrightarrow & H_T^\bullet(\{0\}, \mathcal{F}_0) & \longrightarrow & H_T^\bullet(\{0\}, (\tau_{> k}\mathcal{F})_0) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_T^\bullet(\{0\}, (\tau_{\leq k}\mathcal{F})_{V_I}) & \longrightarrow & H_T^\bullet(\{0\}, \mathcal{F}_{V_I}) & \longrightarrow & H_T^\bullet(\{0\}, (\tau_{> k}\mathcal{F})_{V_I}) & \longrightarrow & 0 \end{array}$$

(The rows are short exact by parity vanishing of T -equivariant cohomology of T -orbits). The middle arrow is a surjection, hence so are the left and right arrows by the 5-lemma.¹⁰ Thus (1), (2) and (3) are true for each term in (13).

If we know the theorem for the left and right terms in (13), then each term is isomorphic to a direct sum of even shifts of constant sheaves on the closure of the strata V_I , by purity. In particular, $\text{Hom}(\tau_{> k}\mathcal{F}, \tau_{\leq k}\mathcal{F}[1]) = 0$ and hence (13) splits, and the theorem is true for \mathcal{F} too. In particular, it is enough to show the proposition for the left and right terms. By induction, we may (and do) assume that \mathcal{F} is concentrated in degree zero.

Fix a stratum $\mathcal{O} := V_I$ which is maximal (i.e. open) in the support of \mathcal{F} . (Of course, \mathcal{O} may not be unique, but this won't worry us.) We will argue that a choice of summand $\mathbb{Q}_{\mathcal{O}} \subset \mathcal{F}_{\mathcal{O}}$ leads to a summand $\mathbb{Q}_{\overline{\mathcal{O}}} \subset \mathcal{F}$.

By the Attractive Proposition we have

$$\text{Hom}(\mathbb{Q}_{\overline{\mathcal{O}}}, \mathcal{F}) = H_T^0(\{0\}, \mathcal{F}_0) = H_T^0(V, \mathcal{F}).$$

Because we have a surjection

$$H_T^0(V, \mathcal{F}) = \text{Hom}(\mathbb{Q}_{\overline{\mathcal{O}}}, \mathcal{F}) \rightarrow \text{Hom}(\mathbb{Q}_{\mathcal{O}}, \mathcal{F}_{\mathcal{O}}) = H_T^0(\mathcal{O}, \mathcal{F}_{\mathcal{O}}),$$

we may find a map $\beta : \mathbb{Q}_{\overline{\mathcal{O}}} \rightarrow \mathcal{F}$ whose restriction to \mathcal{O} is the inclusion of our summand.

On the other hand, our chosen projection $\mathcal{F}_{\mathcal{O}} = j^*\mathcal{F} \rightarrow \mathbb{Q}_{\mathcal{O}}$ corresponds (via adjunction) to a morphism

$$\mathcal{F} \rightarrow j_*\mathbb{Q}_{\mathcal{O}}$$

which restricts to our chosen map on the open stratum. Applying $\tau_{\leq 0}$ we get a morphism

$$\alpha : \tau_{\leq 0}\mathcal{F} = \mathcal{F} \rightarrow \tau_{\leq 0}j_*\mathbb{Q}_{\mathcal{O}} = \mathbb{Q}_{\overline{\mathcal{O}}}$$

which still agrees with our chosen projection on the open stratum. Composing we get $\alpha \circ \beta : \mathbb{Q}_{\overline{\mathcal{O}}} \rightarrow \mathbb{Q}_{\overline{\mathcal{O}}}$ which is the identity on the open stratum. As

$$\text{Hom}(\mathbb{Q}_{\overline{\mathcal{O}}}, \mathbb{Q}_{\overline{\mathcal{O}}}) \rightarrow \text{Hom}(\mathbb{Q}_{\mathcal{O}}, \mathbb{Q}_{\mathcal{O}})$$

is an isomorphism, we deduce that $\alpha \circ \beta$ is an isomorphism everywhere. In other words, we have found $\mathbb{Q}_{\overline{\mathcal{O}}}$ as a summand of \mathcal{F} . Continuing in this way, we deduce

¹⁰Or more precisely the 4-lemma...

that we can find an isomorphism

$$\mathcal{F} = \bigoplus_{I \subset \{1, \dots, p\}} \mathbb{Q}_{\overline{V}_I}^{\oplus m_I}.$$

As each \overline{V}_I is smooth, $\mathbb{Q}_{\overline{V}_I}$ is pure of weight zero, hence so is \mathcal{F} and we are done. \square

Recall the notation of the previous subsection on weighted projective spaces. The previous statement has Corollary 4.22, which will be important below:

Corollary 4.22. *Suppose that \mathcal{F} is a sheaf on weighted projective space $\mathbb{P}_\lambda \dot{V}$. Assume:*

- (1) *The stalks of \mathcal{F} are pure.*
- (2) *The restriction of \mathcal{F} to each toric stratum \mathcal{O}_I is isomorphic to a direct sum of even shifts of constant sheaves.*
- (3) *For $i \in \{1, \dots, p\}$ and any $J \subset \{1, \dots, p\}$ containing i the restriction map*

$$H_T^\bullet(U_{\{i\}}, \mathcal{F}_{U_{\{i\}}}) \rightarrow H_T^\bullet(\mathcal{O}_J, \mathcal{F}_{\mathcal{O}_J})$$

is surjective.

Then \mathcal{F} is pure.

Proof. It is enough to prove that the restriction of \mathcal{F} to each $U_{\{i\}}$ is pure. This then follows (using the Finite Stabilizer Argument) from Proposition 4.20. \square

The following allows us to compute Poincaré polynomials of global sections from Poincaré polynomials of stalks:

Proposition 4.23. *Suppose that $\mathcal{F} \in D^b(\mathbb{P}_\lambda \dot{V}, \mathbb{Q})$ is a pure sheaf whose restriction to each toric stratum is a direct sum of constant sheaves in even degrees. Then the Poincaré polynomial of $H^\bullet(\mathbb{P}_\lambda \dot{V}, \mathcal{F})$ is given by*

$$\sum_{\emptyset \neq I \subset \{1, \dots, p\}} (q-1)^{|I|-1} P(\mathcal{F}_{x_I}),$$

where x_I denotes a point in the toric stratum \mathcal{O}_I , and $P(\mathcal{F}_{x_I})$ denotes the Poincaré polynomial of the stalk.

Proof. If the formula is true for \mathcal{F}_1 and \mathcal{F}_2 , then it is true for their sum. Hence (using our purity assumption on \mathcal{F} to deduce that \mathcal{F} is a direct sum of IC-sheaves, and then our assumptions on the restriction to each stratum to deduce that only constant sheaves show up) it is enough to check the formula when

$$\mathcal{F} = \mathbf{IC}(\overline{\mathcal{O}_I}, \mathbb{Q}) = \mathbb{Q}_{\overline{\mathcal{O}_I}}.$$

In this case, $\overline{\mathcal{O}_I}$ is a weighted projective space of dimension $|I| - 1$ and its Poincaré polynomial is

$$1 + q + \dots + q^{|I|-1}.$$

Our formula predicts that this should equal

$$\sum_{\emptyset \neq J \subset I} (q-1)^{|J|-1} = \sum_{0 \neq k \leq |I|} \binom{|I|}{k} (q-1)^{k-1}.$$

This is easily checked.¹¹ \square

¹¹For example, by (a) multiplying both terms by $(q-1)$, adding 1, and using the binomial theorem, or (b) considering the toric stratification of projective space and counting points.

4.13. The hypercube piece: Geometry. In §4.14 and §5 we establish Theorem 4.10, which describes the Poincaré polynomial for the hypercube piece. Here we begin by describing the (rather simple) geometry involved in the hypercube piece.

Remark 4.24. The simple geometry of the hypercube piece reminds one of the importance of the “mirabolic subgroup” in type A . This suggests that generalization of these results to other types might need new ideas.

Recall that S_x^H denotes the subvariety of S_x where all coordinates are zero, except for those below the 1 in the m^{th} column. (See §4.6 for the definition of S_x^H and m .) For example, if $x = 210345$ then

$$S_x^H = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 & 0 \\ 0 & 0 & * & 0 & 1 & 0 \\ 0 & 0 & * & 0 & 0 & 1 \end{pmatrix}.$$

It will be useful for the reader to keep the explicit form of S_x^H in mind below.

Recall that in §4.7 we gave equations cutting out $S_{x,y}$ inside the affine space S_x . In general these equations are complicated. However in simple situations one can be lucky:

Proposition 4.25. $S_{x,y}^H \cong \mathbb{C}^p$ for some p .

Proof. Consider an $(l \times l)$ -matrix of the form

$$\begin{pmatrix} & x_1 \\ P & \vdots \\ & x_l \end{pmatrix},$$

where P is an $(l-1) \times l$ -matrix of 0's and 1's having at most one 1 in each row and column. Then its determinant is either 0 or $\pm x_i$ for some $1 \leq i \leq l$, as one sees easily by expanding the determinant from left to right. Applying this observation to the above equations implies that $S_{x,y}^H \subset S_x^H$ is cut out by the vanishing of certain coordinates in the m^{th} -column. The proposition follows. \square

Remark 4.26. We can calculate what p is explicitly. In §5 we will see that it agrees with the number of edges in the hypercube piece which have target x . (We do not need this fact below.)

Denote by ε_i the character of the torus given by $\text{diag}(\lambda_0, \lambda_1, \dots, \lambda_n) \mapsto \lambda_i$. The following is an easy direct calculation:

Lemma 4.27. *The T -weights on S_x^H are all distinct, and belong to $\{\varepsilon_i - \varepsilon_0\}_{i=1}^n$. In particular, T -has finitely many orbits on S_x^H .*

Recall that

$$Z = \mathbb{P}_\lambda \dot{S}_{x,y}^H,$$

where λ is a cocharacter acting attractively on the slice $S_{x,y}$. We deduce from the above that Z is a weighted projective space, and hence results of §4.11 and §4.12 apply.

Proposition 4.28. $Z = \mathbb{P}_\lambda S_{x,y}^H$ is isomorphic to a weighted projective space, and T has finitely many orbits on Z . Moreover, each toric stratum in Z is the intersection with a unique Schubert stratum $\mathbb{P}_\lambda \dot{S}_{x,y}^u$ for $u \in (x, y]$.

Proof. The only statement not clear from the above is the one starting “Moreover, ...”. However, note that the intersection of Z with the image of any Schubert stratum is T -stable, and hence a union of T -orbits. The statement follows because T has finitely many orbits on Z . □

Let \mathcal{X} denote the set of T -weights occurring in $S_{x,y}^H$. (Note that the weights in \mathcal{X} are linearly independent and $S_{x,y}^H \cong \mathbb{C}^{|\mathcal{X}|}$.) It follows that we obtain a map

$$\theta_{\text{geom}} : \text{subsets of } \mathcal{X} \rightarrow [x, y]$$

uniquely characterised by the fact that for any subset I of \mathcal{X} , $S_{x,y} \cap S_{x,y}^{\theta(I)}$ is the toric stratum determined by the non-vanishing of the coordinates in I . This is the (geometric) hypercube map and will play an important role below. We will eventually see (in §5.2) that it agrees with the combinatorial hypercube map (defined in §3.5). Until we know their equality, we keep the subscript to distinguish the two.

4.14. The hypercube piece: Sheaves. In this section, we prove that $i^*\mathbf{IC}$ is pure, and that its Poincaré polynomial is given by the hypercube piece in our formula, where we use the geometric hypercube map in place of its combinatorial version. All that then remains is to check that the geometric θ_{geom} (defined above) agrees with the combinatorial hypercube map θ (defined in §3.5). This we do in the subsequent section.

Proposition 4.29. $i^*\mathbf{IC}$ is pure.

Proof. We will check the conditions of Corollary 4.22. As $i^*\mathbf{IC}$ is the restriction of a pointwise pure sheaf, its stalks are pure, and so (1) is satisfied. Moreover, the restriction of $\mathbf{IC}(\overline{ByB/B}, \mathbb{Q})$ to each Bruhat stratum is a direct sum of shifts of constant sheaves in even degrees. As the stratification of Z is via intersections with Bruhat cells, it follows that (2) is satisfied. We now turn to (3) which is the most subtle. Denote by $T' = T/\lambda(\mathbb{C}^*)$ the torus which acts on $\mathbb{P}_\lambda \dot{S}_{x,y}$. We want to check that for each i , the restriction map

$$H_{T'}^\bullet(U_{\{i\}}, (i^*\mathbf{IC})_{U_{\{i\}}}) \rightarrow H_{T'}^\bullet(Z_J, (i^*\mathbf{IC})_J)$$

is surjective, for all subsets J containing i . Now, via the Attractive Proposition we can rewrite this as the restriction map

$$H_{T'}^\bullet(Z_{\{i\}}, (i^*\mathbf{IC})_{Z_{\{i\}}}) \rightarrow H_{T'}^\bullet(Z_J, (i^*\mathbf{IC})_J).$$

We already know that both are isomorphic to direct sums of (even) shifts of $H_{T'}^\bullet(Z_{\{i\}}, \mathbb{Q})$ and $H_{T'}^\bullet(Z_J, \mathbb{Q})$ respectively. In particular, when we reduce modulo the maximal ideal of $H_{T'}^\bullet(\text{pt}, \mathbb{Q})$ (i.e. elements of positive degree) we obtain a map

$$H^\bullet(\{p\}, (i^*\mathbf{IC})_p) \rightarrow H^\bullet(\{p'\}, (i^*\mathbf{IC})_{p'}),$$

where p (resp. p') denotes the unique (resp. a choice of) point in the stratum $Z_{\{i\}}$ (resp. Z_J). By Nakayama’s lemma we are done if we know that this map is surjective. However, by definition of the geometric hypercube map, this map may be identified with the map

$$H^\bullet(\{u\}, \mathbf{IC}(\overline{ByB/B})_u) \rightarrow H^\bullet(\{v\}, \mathbf{IC}(\overline{ByB/B})_v)$$

considered in [BM01, §3.5]. This map is surjective by [BM01, Theorem 3.6]. \square

Remark 4.30. We briefly recall why [BM01, Theorem 3.6] holds. The key point is that in the case of flag varieties the natural restriction map

$$(14) \quad IH^\bullet(\overline{ByB/B}, \mathbb{Q}) \rightarrow H^\bullet(\{u\}, \mathbf{IC}(\overline{ByB/B}, \mathbb{Q})_u)$$

is surjective, for any point $u \in G/B$. Indeed, by equivariance it is enough to check this for T -fixed points, and then this follows because all fixed points on G/B are attractive. (Once one has the surjectivity in (14) it is easy to deduce that one has a surjection $H^\bullet(\{u\}, \mathbf{IC}(\overline{ByB/B}, \mathbb{Q})_u) \rightarrow H^\bullet(\{v\}, \mathbf{IC}(\overline{ByB/B}, \mathbb{Q})_v)$ whenever v 's stratum contains u in its closure.) Braden and MacPherson use [BM01, Theorem 3.6] to deduce the monotonicity of Kazhdan-Lusztig polynomials.

By definition of the hypercube map, the Poincaré polynomial of $i^*\mathbf{IC}$ on the toric stratum $\mathcal{O}_I \subset Z$ agrees with the Kazhdan-Lusztig polynomial $P_{\theta_{\text{geom}}(I), y}$. Hence we can use Propositions 4.29 and 4.23 to deduce the following:

Proposition 4.31. *The Poincaré polynomial of $H^\bullet(Z, i^*\mathbf{IC})$ is given by*

$$\tilde{Q}_{\text{geom}} := \sum_{\emptyset \neq I \subset \{1, \dots, p\}} (q-1)^{|I|-1} P_{\theta_{\text{geom}}(I), z}$$

Corollary 4.32. *The Poincaré polynomial of $H^\bullet(Z, i^!\mathbf{IC})$ is given by*

$$Q_{\text{geom}} = q^{\ell(y)-\ell(x)-1} \tilde{Q}_{\text{geom}}(q^{-1}).$$

Proof. The Poincaré polynomial of the dual of $H^\bullet(Z, i^*\mathbf{IC})$ is given by $\tilde{Q}_{\text{geom}}(q^{-1})$. Because Verdier duality interchanges $i^!$ and i^* we deduce that $\tilde{Q}_{\text{geom}}(q^{-1})$ agrees with the Poincaré polynomial of

$$H^\bullet(Z, i^!\mathbb{D}\mathbf{IC}) = H^\bullet(Z, i^!\mathbf{IC}[2(\ell(y) - \ell(x) - 1)]).$$

We use that $\mathbb{D}\mathbf{IC}(X, \mathbb{Q}) = \mathbf{IC}(X, \mathbb{Q})[2d]$, where $d = \dim_{\mathbb{C}} X$, and the fact that $\mathbb{P}_{\lambda} \dot{S}_{x,y}$ has dimension $\ell(y) - \ell(x) - 1$. Hence the Poincaré polynomial of $H^\bullet(Z, i^!\mathbf{IC})$ is $q^{\ell(y)-\ell(x)-1} \tilde{Q}_{\text{geom}}(q^{-1})$ as claimed. \square

Remark 4.33. The formula for the Poincaré polynomial of $H^\bullet(Z, i^!\mathbf{IC})$ agrees with the polynomial of hypercube piece, with the only difference being that the former involves the geometric hypercube map, whereas the latter involves the combinatorial hypercube map. Thus to prove our formula it is enough to show that the two hypercube maps agree.

5. COMBINATORICS OF THE HYPERCUBE MAP

In the previous section we almost finished the proof of our formula. The only missing piece is Theorem 5.1 that compares the “geometric” and “combinatorial” hypercube maps. Here the arguments are of a different flavour, hence the new section. The goal is the following:

Theorem 5.1. *The set*

$$L = \{z \in [x, y] \mid z^{-1}(0) = x^{-1}(0)\}$$

provides a hypercube decomposition of the Bruhat interval $[x, y]$. Moreover, we have $\theta = \theta_{\text{geom}}$, thus the geometric and combinatorial hypercube maps agree.

5.1. Explicit description of the hypercube map. Recall that S_x^H denotes sub-variety of S_x where all coordinates are zero, except for those below the 1 in the 0^{th} row (and m^{th} column). For example, if $x = 210345$ then

$$S_x^H = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 & 0 \\ 0 & 0 & * & 0 & 1 & 0 \\ 0 & 0 & * & 0 & 0 & 1 \end{pmatrix}.$$

We can compute the geometric hypercube map explicitly by taking any subset of the $*$ -variables, setting the remaining $*$ -variables to zero, and then describing in which Schubert cell we lie. (It is a consequence of Proposition 4.28 that the resulting Schubert cell only depends on which $*$ -variables are non-zero.) In any specific example this is easily decided by performing row and column operations.

Example 5.2. Here we give two examples when $n = 3$:

- (1) Suppose that $x = (0, 1, 2, 3)$ is the identity. Consider a general element of S_x^H :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{pmatrix}.$$

If $c \neq 0$, then by rescaling the last row and performing row operations upwards followed by column operations rightwards we see that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & -c^{-1} \\ 0 & 1 & 0 & -ac^{-1} \\ 0 & 0 & 1 & -bc^{-1} \\ 1 & 0 & 0 & c^{-1} \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

all lie in the same Bruhat cell. In particular $\theta(I) = t_{0,3}$ if $3 \in I$ (i.e. if $c \neq 0$). Similar arguments show that when $x = 0$ the hypercube map only depends on the last non-zero element in the first column. In other words

$$\theta(I) = t_{(0, \max I)} \quad \text{for all } I.$$

- (2) Consider the case $x = (0, 3, 2, 1)$. Again, by upwards row and rightwards column operations we see:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ c & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -c^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ c & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $\theta(\{3\}) = (3, 0, 2, 1) = t_{(0,3)}x$. On the other and if a, b and c are all non-zero then

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 \\ b & 0 & 1 & 0 \\ c & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

One can check this by hand. A more efficient way is to notice that the corner rank matrices of both matrices are given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and hence both belong to the same Schubert cell by the results of §4.7. Thus $\theta(\{1, 2, 3\}) = w_0$.

We now give an explicit general description of the hypercube map. We will make the following simplifications:

- (1) The subvariety $S_{x,y}^H$ can be viewed as a coordinate subspace inside $S_x^H = S_{x,w_0}^H$. In particular, the geometric hypercube map for $S_{x,y}^H$ is the restriction of that for S_x^H . Thus it is enough to describe the hypercube map for S_x^H .
- (2) Recall that $m = x^{-1}(0)$. The free variables in S_x^H can be identified with the set $x(m+1), x(m+2), \dots, x(n)$. (In this indexing, the variable i corresponds to the unique free variable in S_x^H which occurs in the i^{th} -row. These are exactly the free variables which have a 1 to their right in S_x^H .)

Thus, from now on we regard the hypercube map as a map

$$\theta_{\text{geom}} : \left\{ \begin{array}{c} \text{subsequences of} \\ x(m+1), x(m+2), \dots, x(n) \end{array} \right\} \rightarrow [x, w_0]$$

which maps the empty subsequence to x . Note as the elements of the set $x(m+1), x(m+2), \dots, x(n)$ are distinct, subsequences and subsets are the same thing. However, in the following the order in which entries occur in the subsequence will play an important role in determining the map.

Consider a non-empty subsequence I of $x(m+1), x(m+2), \dots, x(n)$. Define a sequence i'_1, i'_2, \dots inductively as follows:

$$\begin{aligned} i_1 &= \text{maximum of the entries in } I, \\ i_{k+1} &= \text{maximum of the entries right of } i'_k \text{ in } I. \end{aligned}$$

For some l , the set of entries right of i_l is empty, in which case i_{l+1} is undefined. We let

$$I_{\text{decr}} = (i_1, i_2, \dots, i_l)$$

denote the resulting subsequence.

Remark 5.3. One may consider $I \mapsto I_{\text{decr}}$ as the extraction of a “greedy decreasing subsequence”: first choose the maximal element of I , then choose a maximal element right of it, etc.

Now consider the $k+1$ -cycle σ_I which sends:

$$i_l \mapsto i_{l-1} \mapsto \dots \mapsto i_2 \mapsto i_1 \mapsto 0 \mapsto i_l.$$

We define

$$\theta'_{\text{geom}}(I) := \sigma_I x.$$

If I is empty, we set $\theta'_{\text{geom}}(I) := x$. We obtain in this way a map

$$\theta'_{\text{geom}} : \left\{ \begin{array}{c} \text{subsequences of} \\ x(m+1), x(m+2), \dots, x(n) \end{array} \right\} \rightarrow S_{n+1}.$$

The following shows that this explicit description does indeed describe the geometric hypercube map.

Theorem 5.4. $\theta_{\text{geom}} = \theta'_{\text{geom}}$.

Proof. In the discussion below we make use of the description of Schubert cells in terms of the corner rank matrix recalled in §4.7. In particular, we will use the notation $g^{\leq(p,q)}$ introduced there.¹²

Consider the permutation matrix x . Its corner rank matrix is given by

$$\text{rank } x^{\leq(p,q)} = \#\{\text{elements in } x(0), x(1), \dots, x(q) \text{ which are } \geq p\}.$$

On the other hand, if g denotes an element of the stratum $(S_x^H)_I$ (so that I is a subset of $\{x(m+1), \dots, x(n)\}$), its corner rank matrix is computed as follows:

$$(15) \quad \text{rank } g^{\leq(p,q)} = \begin{cases} \text{rank } x^{\leq(p,q)} & \text{if } q < m, \\ \text{rank } x^{\leq(p,q)} & \text{if } I \cap \{\geq p\} \subset \{x(0), x(1), \dots, x(q)\}, \\ \text{rank } x^{\leq(p,q)} + 1 & \text{otherwise.} \end{cases}$$

(Indeed, x and g differ only at the m^{th} -column. This column is not contained in $g^{\leq(p,q)}$ if $q < m$. Otherwise, the condition $I \cap \{\geq p\} \subset \dots$ expresses whether this new column is in the span of the others in $g^{\leq(p,q)}$.)

Now consider the permutations x and $\sigma_I x$ in string notation (i.e. as the sequences $(x(0), x(1), \dots, x(n))$ and $(\sigma_I x(0), \sigma_I x(1), \dots, \sigma_I x(n))$). In the passage from x to $\sigma_I x$ the elements I_{decr} move left, and 0 moves all the way to the right, to the position occupied by the final element of I_{decr} . In particular, if we consider the elements which are larger than p , then those belonging to I_{decr} move to the left. It follows that:

$$(16) \quad \text{rank } (\sigma_I x)^{\leq(p,q)} = \begin{cases} \text{rank } x^{\leq(p,q)} & \text{if } q < m, \\ \text{rank } x^{\leq(p,q)} & \text{if } I_{\text{decr}} \cap \{\geq p\} \subset \{x(0), x(1), \dots, x(q)\}, \\ \text{rank } x^{\leq(p,q)} + 1 & \text{otherwise.} \end{cases}$$

We claim that if $q \geq m$, then

$$(17) \quad I \cap \{\geq p\} \subset \{x(0), x(1), \dots, x(q)\} \Leftrightarrow I_{\text{decr}} \cap \{\geq p\} \subset \{x(0), x(1), \dots, x(q)\}.$$

The implication \Rightarrow is obvious, as I_{decr} is a subset of I . For the other direction, assume that $I \cap \{\geq p\} \not\subset \{x(0), x(1), \dots, x(q)\}$ and let j belong to $I \cap \{\geq p\}$ but not to $\{x(0), x(1), \dots, x(q)\}$. Then, by definition of I_{decr} , there exists some element j' in I_{decr} which occurs to the right of j in x , and is larger than j . Thus j' belongs to $I_{\text{decr}} \cap \{\geq p\}$ but not to $\{x(0), x(1), \dots, x(q)\}$. Thus (17) holds.

Thus the corner rank matrices described by (15) and (16) coincide. We deduce that g and $\sigma_I x$ belong to the same Schubert cell. Thus

$$\theta_{\text{geom}}(I) = \theta'_{\text{geom}}(I)$$

which proves the theorem. □

Example 5.5. The extreme examples of the hypercube map are the following:

¹²We are grateful to A. Henderson for suggesting the proof below. Any inaccuracies are ours!

- (1) If $x = (0, 1, \dots, n)$ is the identity then $m = 0$ and for any sequence of $(1, \dots, n)$, $I = (\max I)$ because x contains no descending sequences whatsoever. Hence

$$\theta_{\text{geom}}(I) = t_{0, \max I}.$$

In this case the image of θ_{geom} is isomorphic to the totally ordered set

$$t_{(0,1)} \leq t_{(0,2)} \leq \dots \leq t_{(0,n)}.$$

The image of θ_{geom} is illustrated (for $n = 3$) in Figure 7. This example matches the explicit calculation in Example 5.2(1).

- (2) If $x = (0, n, \dots, 2, 1)$ then $m = 0$ and for any subsequence I of $(n, \dots, 2, 1)$, we have $I_{\text{decr}} = I$ in descending order. Hence each σ_I is distinct and

$$\theta : \text{subsequences of } (n, \dots, 1, 0) \rightarrow S_{n+1}$$

is injective. It identifies the lattice of subsets $\{1, \dots, n\}$ with the lattice between x and w_0 in S_{n+1} . The image of θ is illustrated (for $n = 3$) in Figure 8. (This example generalizes the explicit calculations in Example 5.2(2).)

That such a nice lattice is present in Bruhat order might come as a surprise to the reader. It is less surprising when one realizes that $c = w_0x = (n, 0, 1, \dots, n - 1)$ is a Coxeter element, and hence w_0 gives an order reversing bijection between the interval $[x, w_0]$ and the interval $[\text{id}, c]$. The interval $[\text{id}, c]$ is easily seen to be isomorphic to a $\{0, \dots, n\}$ -hypercube.

5.2. The hypercube map and Bruhat graph. In this section we prove that the geometric hypercube map is determined by what it does to subsequences of $(x(m + 1), \dots, x(n))$ of length 0 and 1. In particular, we will see that the edges

$$\{\theta(\{i\}) \rightarrow x \mid i \in (x(m + 1), \dots, x(n))\}$$

in the Bruhat graph of the interval $[x, y]$ span a hypercube cluster, and that the geometrical hypercube map and combinatorial hypercube maps (defined in §3.5) agree.

In the following, the following standard fact on Bruhat order for the symmetric group will be indispensable (see e.g. [Bil98, Lemma 2.1]):

Proposition 5.6 (“Subword Condition”). *Suppose that u and v are permutations in S_{n+1} such that $u(i) = v(i)$ except in positions $i_1 < i_2 < \dots < i_m$. Consider the permutations u' (resp. v') of the set $\{u(i_j) \mid 1 \leq j \leq m\}$ (with its natural increasing ordering) given in string notation by $(u(i_1), u(i_2), \dots, u(i_m))$ (resp. $(v(i_1), v(i_2), \dots, v(i_m))$). Then $u \leq v$ if and only if $u' \leq v'$.*

Recall that x and y are fixed permutations in S_{n+1} , and that

$$\theta_{\text{geom}} : \left\{ \begin{array}{c} \text{subsequences of} \\ x(m + 1), x(m + 2), \dots, x(n) \end{array} \right\} \rightarrow [x, w_0]$$

denotes the geometric hypercube map. The combinatorial characterisation of the hypercube map discussed above will be a consequence of two simple propositions:

Proposition 5.7. *Suppose that i occurs to the left of j in I .*

- (1) *If $i > j$ then $\theta_{\text{geom}}(\{i\})$ and $\theta_{\text{geom}}(\{j\})$ are incomparable.*
- (2) *If $i < j$ then $\theta_{\text{geom}}(\{i\}) < \theta_{\text{geom}}(\{j\})$.*

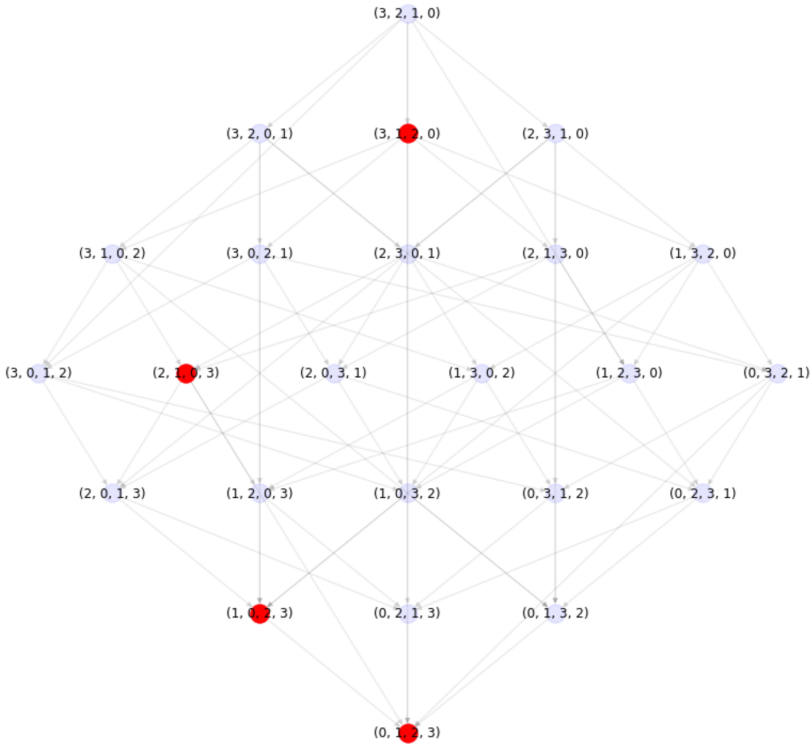


FIGURE 7. The image of the hypercube map based at $x = 0123$. The reader can check that the red dots form a subposet isomorphic to a chain $3 \rightarrow 2 \rightarrow 1 \rightarrow 0$.

Proof. The hypercube map in question only disturbs $0, i$ and j . Hence, by the Subword Condition, we can assume $x = (0, i, j)$ viewed as a permutation of $\{0, i, j\}$. In both cases we have

$$\theta_{\text{geom}}(\{i\}) = (i, 0, j) \quad \text{and} \quad \theta_{\text{geom}}(\{j\}) = (j, i, 0).$$

If $i < j$ then $(j, i, 0) < (i, 0, j)$ establishing (2). If $i > j$ then $(i, 0, j)$ and $(j, i, 0)$ are distinct and both of length 2, and hence incomparable, establishing (1). \square

Remark 5.8. We could have also deduced Proposition 5.7 from the $n = 2$ case of Example 5.5.

Proposition 5.9. *Suppose that I is a subsequence of $(x(m+1), \dots, x(n))$ such that $I_{\text{decr}} = I$. (That is, I_{decr} is decreasing.) Then θ_{geom} gives the unique embedding of directed graphs*

$$H_I \rightarrow [x, w_0]$$

such that $\emptyset \mapsto x$ and $\{i\} \mapsto \theta_{\text{geom}}(\{i\})$ for all $i \in I$.

Proof. Let $I = (i_1, \dots, i_l)$. By the Subword Condition we can work with permutations of the (ordered) set $0, i_l, \dots, i_0$, and assume that

$$x = (0, i_0, \dots, i_l).$$

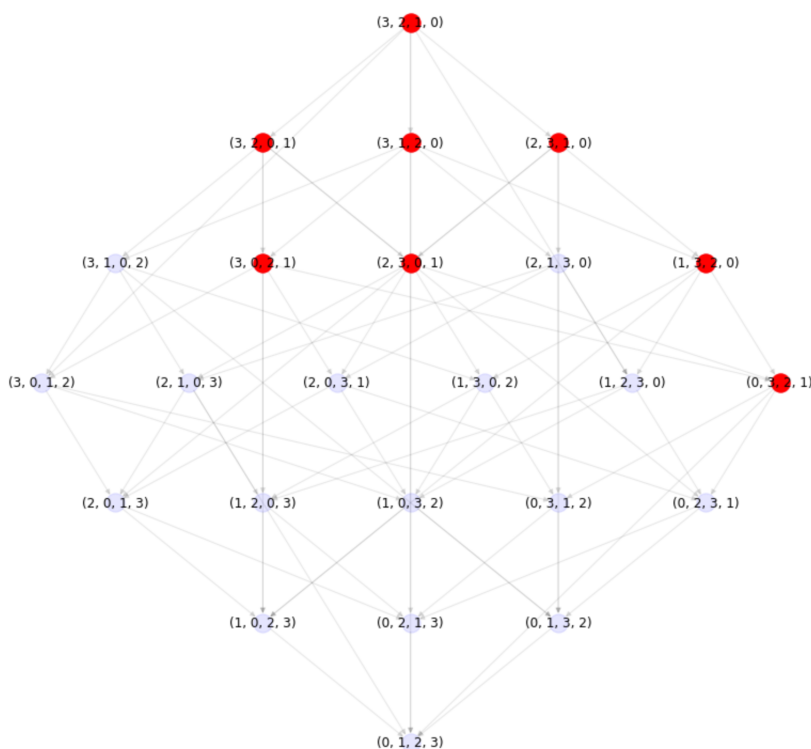


FIGURE 8. The image of the hypercube map based at $x = 0321$. The reader can check that the red dots form a subposet isomorphic to the 1-skeleton of a $\{0, 1, 2, 3\}$ -hypercube.

However we have seen in Example 5.5(2) that the interval between $\theta_{\text{geom}}(\emptyset) = x$ and $\theta(I) = (i_0, \dots, i_l, 0)$ is isomorphic to an I -hypercube. The proposition follows, as any morphism between I -hypercubes is uniquely determined by what it does on the base vertex and vertices of height 1. \square

We are now ready to give:

Proof of Theorem 5.1. We first need to show that the edges $\{z \rightarrow x \mid z \notin L\}$ span a hypercube cluster, and that the combinatorial and geometric hypercube maps agree. First note that $tx \in L$ if and only if $t(0) \neq 0$, in other words that $t = (0, l)$ for some l . In order for $tx > x$ we must have $x^{-1}(l) > x^{-1}(0)$. Thus the edges incident to x in both the combinatorial and geometric hypercubes may be identified with the set

$$E = \{tx \rightarrow x \mid t = (0, l) \text{ for some } l \in \{x(m+1), \dots, x(n)\}, \text{ such that } x < tx \leq y\}.$$

It will be convenient to consider

$$E' = \{l \in \{x(m+1), \dots, x(n)\} \mid x < t_{(0,l)}x \leq y\}.$$

Of course, E and E' are obviously isomorphic, and we let $\phi : E \rightarrow E'$ denote the obvious isomorphism.

The reason to distinguish these two sets is to compare two partial orders on them. Consider a partial order on E given by $\alpha \leq \beta$ if they are \leq in Bruhat order. On the other hand, say $i < j$ in E' if i occurs to the left of j and $i < j$ (in the usual sense, i.e. as numbers!). Now Proposition 5.7 tells us that

$$\alpha < \beta \Leftrightarrow \phi(\alpha) < \phi(\beta).$$

In other words, E and E' are isomorphic as posets.

Now consider a subset F of E . To compute the image of F under the combinatorial hypercube map we first form the subset $F_{\max} \subset F$ consisting of maximal elements, and then define

$$\theta(F) = \text{crown of hypercube spanned by } F_{\max}.$$

On the other hand, consider a subset $I \subset E'$. To compute the image of I under the geometric hypercube map we first form the subset $I_{\text{decr}} \subset I$ consisting of a greedy decreasing subsequence. Thanks to Proposition 5.9 we may describe the image of I as

$$\theta_{\text{geom}}(I) = \text{crown of hypercube spanned by } \{\theta_{\text{geom}}(\{i\}) \rightarrow x \mid i \in I'\}.$$

Via the isomorphism ϕ the formation of $F_{\max} \subset F$ and $I_{\text{decr}} \subset I$ correspond to each other. We conclude that the combinatorial and geometric hypercube maps agree. It also follows from Proposition 5.9 that the edges E span a hypercube cluster.

Finally, note that there was nothing special about x in the above argument. In particular, for any $z \in L$ the set

$$\{u \rightarrow z \mid u \notin L\}$$

spans a hypercube cluster. In particular, $L \subset [x, y]$ is a hypercube decomposition, and the theorem is proved. □

6. GEOMETRY OF THE CONJECTURE

Despite considerable effort we are unable to prove Conjecture 3.8. Our conjecture is motivated by Theorem 3.7, and hence it is tempting to imitate its proof. In this section we discuss the obstacles remaining in doing so.

6.1. Subvarieties of slices. In this section, we explain the natural generalizations (for any hypercube decomposition) of the subvarieties $S_{x,y}^H$ and $S_{x,y}^I$ of the slice $S_{x,y}$ (see §4.6).

Recall the coordinate characters ε_i of our torus defined in §4.13 and let $R^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$ denote positive roots with respect to B . Recall that we have a bijection between reflections $t_{(i,j)}$ in S_{n+1} and *negative* roots $\varepsilon_j - \varepsilon_i$. We use this to relabel the edges of our Bruhat graph with negative roots.

As above, let $S_{x,y}$ denote the standard slice through xB/B inside the Schubert variety $\overline{ByB/B}$. Because we are in type A , we have a T -equivariant embedding

$$(18) \quad \iota : S_{x,y} \hookrightarrow T_x(S_{x,y}) = \bigoplus_{u \xrightarrow{\alpha} x \in E} \mathbb{C}_{\alpha},$$

where T_x denotes the tangent space at x , E denotes the edges with target x in the Bruhat graph $[x, y]$, and \mathbb{C}_α denotes the one-dimensional T -module on which T acts via the character α .¹³

Suppose fixed a choice of hypercube decomposition $J \subset [x, y]$. This breaks the set E into two “inductive” and “hypercube” pieces $E = E_{\text{ind}} \sqcup E_{\text{hyp}}$ where

$$E_{\text{ind}} = \{u \rightarrow x \in E \mid u \in J\} \quad \text{and} \quad E_{\text{hyp}} = \{v \rightarrow x \in E \mid v \notin J\}.$$

To simplify notation we denote by

$$V = \bigoplus_{u \xrightarrow{\alpha} x \in E} \mathbb{C}_\alpha$$

the vector space with T -action into which $S_{x,y}$ embeds. Our decomposition of E leads to a direct sum decomposition

$$V = V_{\text{ind}} \oplus V_{\text{hyp}}$$

and it is tempting to consider the subvarieties

$$S_{x,y}^I := S_{x,y} \cap V_{\text{ind}} \quad \text{and} \quad S_{x,y}^H := S_{x,y} \cap V_{\text{hyp}}.$$

Recall that $J = \{\leq z\} \cap [x, y]$ for some $z \in [x, y]$. The following is the analogue of Proposition 4.15:

Proposition 6.1. *We have $S_{x,y}^I = S_{x,z}$.*

The proof is satisfying, and neatly explains the “diamond complete” condition in the definition of a hypercube decomposition. We give it in §6.3.

The following should be the analogue of Proposition 4.28:

Conjecture 6.2. *We have $S_{x,y}^H = V_{\text{hyp}}$. Moreover, each toric stratum (corresponding to a subset $I \subset E_{\text{hyp}}$) is the intersection of $S_{x,y}^H$ with a unique Schubert cell BuB/B and we have*

$$\theta(I) = u,$$

where θ is the hypercube map associated to our hypercube decomposition J .

We believe we have a proof of this conjecture. However the argument is unsatisfying and long. We are willing to try to write it down if the obstacle to be discussed in §6.2 can be overcome.

Remark 6.3. Two remarks of a hygienic nature:

- (1) The (T -equivariant) embedding ι inside the tangent space involves a choice of (T -equivariant) sections of the quotient map

$$\mathfrak{m}_x \twoheadrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2,$$

where \mathfrak{m}_x denotes the maximal ideal at x . Thus, although the right-hand side of (18) is canonical, the embedding isn’t in general. Thus we need to be careful with constructions based on ι . For example, it is not clear a priori that $S_{x,y}^H$ and $S_{x,y}^I$ are well-defined. It turns out that they are, as follows from some considerations in equivariant geometry which we omit.

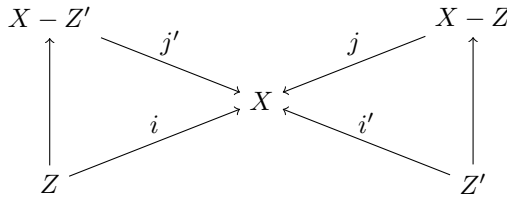
¹³Indeed, any attractive fixed point z has an affine neighbourhood that may be embedded inside the tangent space at x , see [Bri98, Proof of Theorem 17]. In type A it is known [LS84] that this tangent space is spanned by its T -invariant curves. Hence the statement.

- (2) The embedding ι is not the same as the embedding of $S_{x,y}$ considered in §4.6. One can get a choice of embedding ι by composing the embedding considered in §4.6, with a projection to those coordinates which span one-dimensional orbits in $S_{x,y}$. This gives a canonical ι which is useful in examples.

6.2. The main obstacle. In this section we explain the main difficulty in generalizing our proof. We assume that Conjecture 6.2 is known. To simplify notation, let

$$X = \mathbb{P}_\lambda \dot{S}_{x,y}$$

and let Z and Z' denote closed subvarieties of X given by the images of $\dot{S}_{x,y}^H$ and $\dot{S}_{x,y}^I$ respectively. We have the following diagram of inclusions:



As above, we denote by \mathbf{IC} the intersection cohomology complex on X , and by $T' = T/\lambda(\mathbb{C}^*)$ the torus acting on X .

We have a commutative diagram of cohomology groups:

$$(19) \quad \begin{array}{ccccc}
 H_{T',c}^\bullet(X - Z', (j')^! \mathbf{IC}) & & & & H_{T'}^\bullet(X - Z, j^* \mathbf{IC}) \\
 \uparrow \xi & \searrow & & \nearrow & \downarrow \zeta \\
 & & H_{T'}^\bullet(X, \mathbf{IC}) & & \\
 & \nearrow \alpha & & \searrow \beta & \\
 H_{T'}^\bullet(Z, i^! \mathbf{IC}) & & & & H_{T'}^\bullet(Z', (i')^* \mathbf{IC})
 \end{array}$$

(Here $H_{T',c}^\bullet(X - Z', (j')^! \mathbf{IC}) = H_{T',c}^\bullet(X - Z', (j')_!(j')^! \mathbf{IC})$ denotes compactly supported equivariant cohomology.) We know:

- (1) The diagonal

$$H_{T'}^\bullet(Z, i^* \mathbf{IC}) \xrightarrow{\alpha} H_{T'}^\bullet(X, \mathbf{IC}) \longrightarrow H_{T'}^\bullet(X - Z, j^* \mathbf{IC})$$

and anti-diagonal

$$H_{T',c}^\bullet(X - Z, (j')^! \mathbf{IC}) \longrightarrow H_{T'}^\bullet(X, \mathbf{IC}) \xrightarrow{\beta} H_{T'}^\bullet(Z', (i')^* \mathbf{IC})$$

are part of the long exact sequence for the pair $(X, X - Z)$ (resp. (X, Z')).

- (2) The groups $H_{T'}^\bullet(Z, i^* \mathbf{IC})$ and $H_{T'}^\bullet(Z', (i')^* \mathbf{IC})$ are pure. (For the first group, this follows from Conjecture 6.2, and for the second this may be assumed by induction.)

It will be useful to recall the magic that happened where our theorem holds. In that case $X - Z$ retracts equivariantly onto Z' (using the character γ in §4.9) and hence: ζ is an isomorphism, $H_{T'}^\bullet(X - Z, i^* \mathbf{IC})$ is pure, and

$$H_{T'}^\bullet(Z, i^* \mathbf{IC}) \xrightarrow{\alpha} H_{T'}^\bullet(X, \mathbf{IC}) \xrightarrow{\beta} H_{T'}^\bullet(Z', (i')^* \mathbf{IC})$$

is a short exact sequence. Thus, both ξ and ζ are isomorphisms in this case and (19) collapses to a single short exact sequence.

We do not know whether ζ and ξ are isomorphisms in general. However, if one is, then so is the other one, and our conjecture holds. Recall that we say that $H_{T'}^\bullet(X, \mathcal{F})$ is equivariantly formal if it is free as a $H_{T'}^\bullet(\text{pt}, \mathbb{Q})$ -module. The following are easily seen to be equivalent:

- (1) β is surjective,
- (2) $H_{T',c}^\bullet(X - Z, (j')^! \mathbf{IC})$ is pure and equivariantly formal.

Similarly, we have equivalences between the following statements:

- (3) α is split injective (or equivalently, its non-equivariant analogue is injective),
- (4) $H_{T',c}^\bullet(X - Z', j^* \mathbf{IC})$ is pure and equivariantly formal.

It seems that one could use equivariant localization to reduce to rank 2 and deduce equivalences between:

- (5) $H_{T',c}^\bullet(X - Z', (j')^! \mathbf{IC})$ is pure and equivariantly formal,
- (6) ξ is an isomorphism.

Similarly, localization techniques should show equivalences between:

- (7) $H_{T'}^\bullet(X - Z, j^* \mathbf{IC})$ is pure and equivariantly formal,
- (8) ζ is an isomorphism.

Thus it seems likely that (1)–(8) are all equivalent, and any one of them implies our conjecture.

6.3. Bruhat polytopes and the inductive piece. In this final section we prove Proposition 6.1. We include this proof because we think it is enlightening, and explains neatly why the “diamond complete” condition in the definition of a hypercube decomposition arises. Recall that z denotes the crown of the inductive piece, i.e. $J = \{\leq z\}$. We want to show:

Proposition 6.4. $S_{x,y} \cap V_{\text{ind}} = S_{x,z}$.

We will need some results from the beautiful paper [GS87] before turning to the proof. Let us fix throughout a regular dominant weight λ . In the following we will identify W with its image in the λ orbit, i.e. x is identified with $x(\lambda) \in \mathfrak{h}^*$, where \mathfrak{h}^* denotes the realification of the character lattice of T . We also consider G/B as embedded by the Plücker embedding:

$$\text{Plücker} : G/B \hookrightarrow \mathbb{P}(V_\lambda).$$

Take a point $p \in G/B$. Consider its T -orbit closure $\overline{T \cdot p}$. Gelfand and Seganova prove the following:

- (1) The set $\{u \mid u \in (\overline{T \cdot p})^T\} \subset \mathfrak{h}^*$ are the vertices of a convex polytope $\text{Poly}(p) \subset \mathfrak{h}^*$.
- (2) $\text{Poly}(p)$ agrees with the moment map image of $\overline{T \cdot p}$;
- (3) $x \in \text{Poly}(p)$ if and only if the x -coordinate of the Plücker coordinate is non-zero. (The $x(\lambda)$ -weight space in V_λ is one-dimensional, so this statement makes sense.)

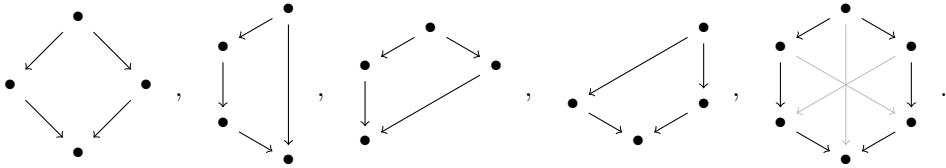
Let us call a polytope that occurs in this way (with λ fixed as always) a *flag polytope*. The theory of toric varieties also produces the important fact (also proved in [GS87]) that:

- (20) Any face of a flag polytope is a flag polytope.

From (20) we deduce (by considering one-dimensional faces) that:

$$(21) \quad \text{Any edge of } \text{Poly}(p) \text{ is parallel to a root.}$$

We also need the classification of 2-dimensional flag polytopes. They are the following, together with their horizontal flips:



(The gray edges in the last diagram are edges in the Bruhat graph which are not edges of the polytope.) Note that these pictures represent conformal equivalence classes of polytopes: we can stretch edge lengths, but angles must be preserved. For example, there is no need for the hexagon above to be regular, but all angles between edges (including gray edges) must be as displayed.

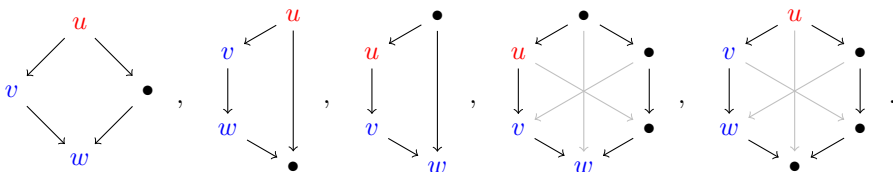
Proof of Proposition 6.4. The inclusion $S_{x,y} \cap I \supset S_{x,z}$ is immediate. The tricky bit is to establish that $S_{x,y} \cap I \subset S_{x,z}$.

Choose $p \in S_{x,y} \cap I$. It is enough to show that if $u \in \text{Poly}(p)$ then $u \leq z$. (Note $\lim_{z \rightarrow \infty} \lambda(z) \cdot p = u'$ for some $u' \in \text{Poly}(p)$, where λ is our anti-dominant cocharacter as always. Thus p belongs to $Bu'B/B \cap U^-xB/B$ which is contained in $S_{x,z}$ if $u' \leq z$.)

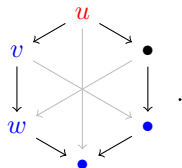
Consider the vertices of $\text{Poly}(p)$. We can divide these into those edges contained in inductive piece (which we will call *blue* vertices) and those which are not (which we will call *red* vertices). We will show the contrapositive to the statement of the last paragraph: namely we will show that if the set of red vertices in $\text{Poly}(p)$ is non-zero, then $p \notin S_{x,y} \cap I$.

So let us assume that red vertices exist. We can certainly find a red vertex u which is connected to a blue vertex v by an edge in $\text{Poly}(p)$. Note that the edges in $\text{Poly}(p)$ inherit a direction from the Bruhat graph (equivalently, from λ). Because $v \rightarrow v'$ implies that v' is also blue, we know that the orientation of the edge joining u to v has to be $u \rightarrow v$.

If $v = x$, then the u -coordinate of Plücker(p) is non-zero, and hence $p \notin S_{x,y} \cap I$ and we are done. Otherwise, there exists an edge $v \rightarrow w$ in the polytope. Now consider the 2-dimensional face F of $\text{Poly}(p)$ containing u, v and w . From our knowledge of the 2-dimensional flag polytopes, and forgetting conformal structure (i.e. just remembering isomorphism type of directed graph) we see that there are 5 possibilities:

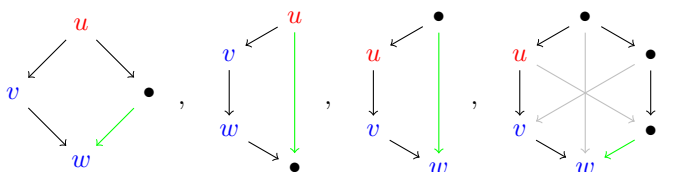


We claim the last possibility is impossible. Indeed, because the blue nodes are closed under the Bruhat order, the following nodes must be blue:



Now diamond completeness forces all six nodes to be blue, which is a contradiction.

In the remaining four cases, consider the green edge below:



We claim that the source of each green arrow is red: in the first and last cases this follows from diamond completeness of the blue vertices; in the second case this is clear; in the third case this follows because the blue nodes are closed under Bruhat order.

Now the green arrow gives us a new edge $u' \rightarrow v'$ of our polytope, with u' blue, v' red, and $v' < v$. Continuing in this way we eventually produce an edge $u'' \rightarrow x$ with u'' red, and we deduce that $p \notin S_{x,y} \cap I$ as above (i.e. because then the u'' -Plücker coordinate is non-zero), and we are done. \square

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REFERENCES

- [BB05] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR2133266 ↑1146, 1156
- [BCM06] Francesco Brenti, Fabrizio Caselli, and Mario Marietti, *Special matchings and Kazhdan-Lusztig polynomials*, Adv. Math. **202** (2006), no. 2, 555–601, DOI 10.1016/j.aim.2005.01.011. MR2222360 ↑1145, 1150
- [Bil98] Sara C. Billey, *Pattern avoidance and rational smoothness of Schubert varieties*, Adv. Math. **139** (1998), no. 1, 141–156, DOI 10.1006/aima.1998.1744. MR1652522 ↑1181
- [BL94] Joseph Bernstein and Valery Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics, vol. 1578, Springer-Verlag, Berlin, 1994, DOI 10.1007/BFb0073549. MR1299527 ↑1159, 1160, 1162
- [BLP21] G. Burrull, N. Libedinsky, and D. Plaza, *Combinatorial invariance conjecture for \tilde{A}_2* , arXiv:2105.04609, 2021. 1145, 1150
- [BM01] Tom Braden and Robert MacPherson, *From moment graphs to intersection cohomology*, Math. Ann. **321** (2001), no. 3, 533–551, DOI 10.1007/s002080100232. MR1871967 ↑1156, 1177
- [Bra03] Tom Braden, *Hyperbolic localization of intersection cohomology*, Transform. Groups **8** (2003), no. 3, 209–216, DOI 10.1007/s00031-003-0606-4. MR1996415 ↑1170

- [Bre02] Francesco Brenti, *Kazhdan-Lusztig polynomials: history problems, and combinatorial invariance*, Sém. Lothar. Combin. **49** (2002/04), Art. B49b, 30. MR2006565 ↑1145, 1146, 1150
- [Bre04] Francesco Brenti, *The intersection cohomology of Schubert varieties is a combinatorial invariant*, European J. Combin. **25** (2004), no. 8, 1151–1167, DOI 10.1016/j.ejc.2003.10.011. MR2095476 ↑1145, 1150
- [Bri98] Michel Brion, *Equivariant cohomology and equivariant intersection theory*, Representation theories and algebraic geometry (Montreal, PQ, 1997), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., Dordrecht, 1998, pp. 1–37. Notes by Alvaro Rittatore. MR1649623 ↑1185
- [Car94] James B. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, pp. 53–61. MR1278700 ↑1148
- [DVB⁺21] A. Davies, P. Veličković, L. Buesing, S. Blackwell, D. Zheng, N. Tomašev, R. Tanburn, P. Battaglia, C. Blundell, A. Juhász, M. Lackenby, G. Williamson, D. Hassabis, and P. Kohli, *Advancing mathematics by guiding human intuition with AI*, Nature **600** (2021), no. 7887, 70–74 1145, 1155
- [Dye87] M. Dyer, *Hecke algebras and reflections in Coxeter groups*, Ph.D. Thesis, University of Sydney, 1987. 1145, 1150
- [Dye93] M. J. Dyer, *The nil Hecke ring and Deodhar’s conjecture on Bruhat intervals*, Invent. Math. **111** (1993), no. 3, 571–574, DOI 10.1007/BF01231299. MR1202136 ↑1148
- [EMTW20] Ben Elias, Shotaro Makisumi, Ulrich Thiel, and Geordie Williamson, *Introduction to Soergel bimodules*, RSMC Springer Series, vol. 5, Springer, Cham, [2020] ©2020, DOI 10.1007/978-3-030-48826-0. MR4220642 ↑1146
- [Ful92] William Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), no. 3, 381–420, DOI 10.1215/S0012-7094-92-06516-1. MR1154177 ↑1165
- [FW14] Peter Fiebig and Geordie Williamson, *Parity sheaves, moment graphs and the p-smooth locus of Schubert varieties* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **64** (2014), no. 2, 489–536. MR3330913 ↑1160
- [GS87] I. M. Gel’fand and V. V. Serganova, *Combinatorial geometries and the strata of a torus on homogeneous compact manifolds* (Russian), Uspekhi Mat. Nauk **42** (1987), no. 2(254), 107–134, 287. MR898623 ↑1187
- [Gib22] Joel Gibson, *LieVis: Interactive visualisations in Lie theory*, in preparation (2022), software available at <https://www.jgibson.id.au/lievvis/>. 1147
- [Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990, DOI 10.1017/CBO9780511623646. MR1066460 ↑1146
- [Inc06] Federico Incitti, *On the combinatorial invariance of Kazhdan-Lusztig polynomials*, J. Combin. Theory Ser. A **113** (2006), no. 7, 1332–1350, DOI 10.1016/j.jcta.2005.12.003. MR2259064 ↑1145, 1150
- [Irv88] Ronald S. Irving, *The socle filtration of a Verma module*, Ann. Sci. École Norm. Sup. (4) **21** (1988), no. 1, 47–65. MR944101 ↑1156
- [KL79] David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184, DOI 10.1007/BF01390031. MR560412 ↑1146
- [KL80] David Kazhdan and George Lusztig, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 185–203. MR573434 ↑1160
- [LS84] V. Lakshmibai and C. S. Seshadri, *Singular locus of a Schubert variety*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), no. 2, 363–366, DOI 10.1090/S0273-0979-1984-15309-6. MR752799 ↑1185
- [Pat21] Leonardo Patimo, *A combinatorial formula for the coefficient of q in Kazhdan-Lusztig polynomials*, Int. Math. Res. Not. IMRN **5** (2021), 3203–3223, DOI 10.1093/imrn/rnz255. MR4227568 ↑1145, 1150, 1154

- [Sai89] Morihiko Saito, *Introduction to mixed Hodge modules*, Astérisque **179-180** (1989), 10, 145–162. Actes du Colloque de Théorie de Hodge (Luminy, 1987). MR1042805 ↑1159, 1160
- [Sch14] Christian Schnell, *An overview of Morihiko Saito’s theory of mixed Hodge modules*, Representation theory, automorphic forms & complex geometry, Int. Press, Somerville, MA, [2019] ©2019, pp. 27–80. MR4337393 ↑1159
- [Soe89] W. Soergel, *n-cohomology of simple highest weight modules on walls and purity*, Invent. Math. **98** (1989), no. 3, 565–580, DOI 10.1007/BF01393837. MR1022307 ↑1160
- [Soe97] Wolfgang Soergel, *Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules*, Represent. Theory **1** (1997), 83–114, DOI 10.1090/S1088-4165-97-00021-6. MR1444322 ↑1146
- [Spr84] T. A. Springer, *A purity result for fixed point varieties in flag manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), no. 2, 271–282. MR763421 ↑1160
- [WY08] Alexander Woo and Alexander Yong, *Governing singularities of Schubert varieties*, J. Algebra **320** (2008), no. 2, 495–520, DOI 10.1016/j.jalgebra.2007.12.016. MR2422304 ↑1165
- [WY12] Alexander Woo and Alexander Yong, *A Gröbner basis for Kazhdan-Lusztig ideals*, Amer. J. Math. **134** (2012), no. 4, 1089–1137, DOI 10.1353/ajm.2012.0031. MR2956258 ↑1165

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