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FORMAL DEGREES AND THE LOCAL THETA CORRESPONDENCE: THE QUATERNIONIC CASE

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Abstract. In this paper, we determine a constant occurring in a local analogue of the Siegel-Weil formula, and describe the behavior of the formal degrees under the local theta correspondence for a quaternionic dual pair of almost equal rank over a non-Archimedean local field of characteristic 0. As an application, we prove the formal degree conjecture of Hiraga, Ichino and Ikeda for the non-split inner forms of Sp_4 and GSp_4.

Contents

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1. Introduction

The principal aim of this paper is to describe the behavior of the formal degrees under the local theta correspondence. This is related to two important topics in the representation theory of p-adic reductive groups. One is the formal degree conjecture of Hiraga, Ichino, and Ikeda [HII08], which is an explicit formula of the formal degree in terms of the Langlands parameters. Here, by the Langlands parameter we mean a pair (ϕ, η) where ϕ is an L-parameter and η is an irreducible representation of the S-group (see §1.4). The other is the behavior of the Langlands parameters under the local theta correspondence. This has not been formulated yet, but we observe how $\dim \eta$ changes under the theta correspondence associated with a quaternionic dual pair of almost equal rank (Proposition 20.5). Moreover, by admitting conjectural properties on Langlands parameters containing the formal degree conjectural, we infer the behavior of the formal degrees under the local theta correspondence (§20). Although the local Langlands correspondence is assumed in these two topics, Gan and Ichino pointed out that only analytic invariants are needed to describe the behavior of the formal degrees under the local theta correspondence associated with a non-quaternionic dual pair of almost equal rank, and computed it [GI14]. In this paper, we extend their formula to a quaternionic dual pair and prove it unconditionally (Theorem 11.2). This agrees with the observation in §20. As an application, we prove the formal degree conjecture for the non-split inner forms of Sp_4 and GSp_4.

We prove Theorem 11.2 by induction. As in [GI14], the induction steps are attained by using a formula of Heiermann [Hei04]. However, it is difficult to prove the base case by case-by-case discussions similar to [GI14]. More precisely, it seems difficult to find enough examples of quaternionic dual pairs (H, G) and square-integrable irreducible representations π of G such that we can know the formal degree $\deg \pi$ of π, the formal degree $\deg \sigma$ of the theta correspondence σ of π, and the standard local γ-factor $\gamma(s, \sigma \boxtimes \chi, \psi)$ with a quadratic character χ at the same time even in low-rank cases. To avoid the difficulty, we analyze the local analogue of the Siegel-Weil formula, and we obtain a relation between the constant in the local Siegel-Weil formula and the local zeta value for enough cases. Here, the constant in the local Siegel-Weil formula appears in an expression of the ratio of the formal degrees of irreducible representations corresponding to each other by the local theta correspondence. Hence, to establish the description of the behavior of the formal degrees under the local theta correspondence, we compute some local zeta values. On the other hand, a general formula of the local zeta value is obtained by reversing the above discussion. For a quaternionic dual pair $(G(V), G(W))$ of almost equal rank, we denote by $\alpha_1(W)$ the local zeta value, by $\alpha_2(V, W)$ the constant in the local Siegel-Weil formula, and by $\alpha_3(V, W)$ the constant appearing in the behavior of the formal degree under the theta correspondence. The results obtained in this paper are summarized as follows.

1.1. The constant $\alpha_1(W)$.

Let F be a non-Archimedean local field of characteristic 0, let $\epsilon = \pm 1$, let W be an n-dimensional right $(-\epsilon)$-Hermitian space equipped with a $(-\epsilon)$-Hermitian form $\langle\ ,\ \rangle$ (see §3) over a division quaternion algebra D over F, and let $G(W)$ be the unitary group of W. We denote by $W^\square$ the doubled space which is the vector space $W \oplus W$ equipped with the $(-\epsilon)$-Hermitian form $\langle\ ,\ \rangle^\square = \langle\ ,\rangle \oplus (-\langle\ ,\ \rangle)$, by $W^\triangle$ the diagonal subset of $W^\square$, and by $P(W^\triangle)$ the parabolic subgroup preserving $W^{\triangle}$. For a character ω of $F^{\times}$, we denote by $I(s,\omega)$ the representation of $G(W^{\square})$ induced by the character $\omega_s \circ \Delta$ of $P(W^{\triangle})$, which is given by $\omega_s(\Delta(p)) = \omega(N(p|_{W^{\triangle}}))^{-1}|N(p|_{W^{\triangle}})|^{-s}$. (Here, we denote by $N(x)$ the reduced norm of $x \in \mathrm{End}(W^{\triangle})$.) We denote by $1_{F^{\times}}$ the trivial character of $F^{\times}$. Then we define

$$\alpha_1(W) = Z^W(f^{\circ}_{\rho}, \xi^{\circ}).$$

Here

- $Z^W(\ ,\ )$ is the doubling zeta integral (see §7.1),
- f°_s is the $K(\underline{e'}^{\square})$-invariant section of $I(s, 1_{F^{\times}})$ so that $f^{\circ}_s(1) = 1$ where $K(\underline{e'}^{\square})$ is a special maximal compact subgroup of the unitary group $G(W^{\square})$ of $W^{\square}$, which depends on the choice of a basis $\underline{e}$ for W (see §7.1),
- ξ° is the coefficient of the trivial representation of $G(W)$ so that $\xi^{\circ}(1) = 1$, and
- $\rho = n - \frac{\epsilon}{2}$.

This invariant is technically important because it appears in a certain local functional equation, which relates the zeta integral with the intertwining operator (see Lemma 7.8). In this paper, we first compute $\alpha_1(W)$ directly for some W (Proposition 7.6), and finally, we complete the formula for the remaining cases as a corollary of Theorem 1.2 (Proposition 19.4). We also note that by determining $\alpha_1(W)$, we can compute the constant given by the scalar multiplication appearing in a formula of the zeta integral for a certain section (see Appendix A), which has not been computed yet.

1.2. **The constant** $\alpha_2(V,W)$**.** Let V be an m-dimensional ϵ-Hermitian space, and let $(\ ,\ )$ be an ϵ-Hermitian form on V, let $\psi\colon F \to \mathbb{C}^{\times}$ be an additive non-trivial unitary character, and let $\omega^{\square}_{\psi}$ be the Weil representation of $G(V) \times G(W^{\square})$. We realize $\omega^{\square}_{\psi}$ on the Schwartz space $\mathcal{S}(V \otimes W^{\triangledown})$ where $W^{\triangledown}$ is the anti-diagonal subset of $W^{\square}$. We assume that $2n - 2m = 1 + \epsilon$. Then, we define the local theta integral

$$\mathcal{I}(\phi, \phi') = \int_{G(V)} (\omega^{\square}_{\psi}(h,1)\phi, \phi')\, dh$$

for $\phi, \phi' \in \mathcal{S}(V \otimes W^{\triangledown})$. Here, we denote by $(\ ,\ )$ the normalized L^2-inner product on $\mathcal{S}(V \otimes W^{\triangledown})$ (as in Proposition 8.2). Moreover, we define another map $\mathcal{E}\colon \mathcal{S}(V \otimes W^{\triangledown})^2 \to \mathbb{C}$ as follows. For $\phi \in \mathcal{S}(V \otimes W^{\triangledown})$, we define $F_{\phi} \in I(-\frac{1}{2}, \chi_V)$ by $F_{\phi}(g) = [\omega^{\square}_{\psi}(g)\phi](0)$, and we choose $F^{\dagger}_{\phi} \in I(\frac{1}{2}, \chi_V)$ so that $M(\frac{1}{2}, \chi_V)F^{\dagger} = F_{\phi}$ where $M(s, \chi_V)$ is an intertwining operator (see §7.1). Then, the map $\mathcal{E}$ is defined by

$$\mathcal{E}(\phi, \phi') = \int_{G(W)} F^{\dagger}_{\phi}(\iota(g,1))\overline{F_{\phi'}(\iota(g,1))}\, dg$$

for $\phi, \phi' \in \mathcal{S}(V \otimes W^{\triangledown})$. Here, $\iota\colon G(W)^2 \to G(W^{\square})$ is given by the natural action of $G(W)^2$ on $W^{\square}$. Then, the constant $\alpha_2(V,W)$ is defined as a non-zero constant satisfying $\mathcal{I} = \alpha_2(V,W) \cdot \mathcal{E}$ (see Lemma 10.2). Then, we have

Theorem 1.1. *Choose the basis $\underline{e} = (e_1, \dots, e_m)$ for V as in §7.1. Then,*

$$\alpha_2(V,W) = e(G(W)) \cdot |2|^{2n\rho + n(n-\frac{1}{2})} \cdot |N(R(\underline{e})|^{\rho} \cdot \prod_{i=1}^{n-1} \frac{\zeta_F(1-2i)}{\zeta_F(2i)}$$
$$\times \begin{cases} 2\chi_V(-1)^n \gamma(1-n, \chi_V, \psi)^{-1} \epsilon(\frac{1}{2}, \chi_V, \psi) & (-\epsilon = 1), \\ 1 & (-\epsilon = -1). \end{cases}$$

Here, $R(\underline{e}) = ((e_i, e_j))_{i,j} \in \mathrm{GL}_m(D)$.

To prove Theorem 1.1, we will first prove it in the case where either V or W is non-zero anisotropic (§§12–13). In this case, we can express $\alpha_2(V,W)$ using $\alpha_1(W)$, and thus Theorem 1.1 is derived from the formula of $\alpha_1(W)$. The remaining cases will be proved after completing the proof of Theorem 1.2 (§19).

1.3. **The constant** $\alpha_3(V,W)$**.** Let G be a reductive group over F, let A be the maximal F-split torus of the center G, and π be a square-integrable irreducible representation of G. We choose a canonical Haar measure dg on $G(F)/A(F)$ depending only on G and a fixed non-trivial additive character ψ of F as in [GG99, §8]. If $G = G(W)$, the measure is given in §6.1. Then, we define the formal degree of π as the positive real number $\deg \pi$ satisfying

$$\int_{G(F)/A(F)} (\pi(g)x_1, x_2)\overline{(\pi(g)x_3, x_4)}\, dg = \frac{1}{\deg \pi}(x_1, x_3)\overline{(x_2, x_4)}$$

for $x_1, x_2, x_3, x_4 \in \pi$. Here, $(\ ,\ )$ is a non-zero $G(F)$-invariant Hermitian form. Suppose that $\theta_\psi(\pi, V)$ is non-zero and is square-integrable. We denote its central character by $c_{\theta_\psi(\pi,V)}$. Then, as in [GI14, p. 597], we can prove that

$$\frac{\deg \pi}{\deg \theta_\psi(\pi, V)} \cdot c_{\theta_\psi(\pi,V)}(-1) \cdot \gamma^V(0, \theta_\psi(\pi, V) \times \chi_W, \psi)^{-1}$$

does not depend on π whenever π is square irreducible and $\theta_\psi(\pi, V) \neq 0$. We denote it by $\alpha_3(V,W)$. Then, our main theorem is stated as follows:

Theorem 1.2. *We have*

$$\alpha_3(V,W) = \begin{cases} \epsilon(\frac{1}{2}, \chi_V, \psi)^{-1} & (-\epsilon = 1), \\ \frac{1}{2}\chi_W(-1)^m \epsilon(\frac{1}{2}, \chi_W, \psi)^{-1} & (-\epsilon = -1). \end{cases}$$

When either V or W is anisotropic, we prove Theorem 1.2 by expressing $\alpha_3(V,W)$ using $\alpha_2(V,W)$ (more precisely, see Proposition 15.1). In general, we use induction on $\dim W$ to compute $\alpha_3(V,W)$ (§18).

1.4. **Langlands parameters and the local theta correspondence.** Let G be a reductive group over F, and let π be an irreducible representation of G, let Γ be the Galois group of F^s/F where F^s denotes the separate closure of F, let W_F be a Weil group of F, let $L_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Langlands group of F, let $\widehat{G}$ be the Langlands dual group of G, let $Z(\widehat{G})$ be the center of $\widehat{G}$, let $\widehat{G}_{\mathrm{ad}}$ be the adjoint group $\widehat{G}$, let $\widehat{G}_{\mathrm{sc}}$ be the simply connected covering of $\widehat{G}_{\mathrm{ad}}$, and let ${}^L G$ be the L-group of G. The Langlands parameter of π is given by a pair (ϕ, η) where

- $\phi\colon L_F \to {}^L G$ is the L-parameter of π,
- η is an irreducible representation of the component group $\widetilde{\mathcal{S}}_\phi = \pi_0(\widetilde{S}_\phi)$ of $\widetilde{S}_\phi$ where $\widetilde{S}_\phi$ is the preimage of $S_\phi := C_\phi / Z(\widehat{G})^\Gamma \subset \widehat{G}_{\mathrm{ad}}$ in $\widehat{G}_{\mathrm{sc}}$.

Here, we denote by C_ϕ the centralizer of $\operatorname{Im}\phi$ in $\widehat{G}$. The group $\widetilde{\mathcal{S}}_\phi$ is called the S-group of ϕ. See §20 for the discussion on how we define η from π. Now we consider the pair $(G(W), G(V))$ with $2n-2m=1+\epsilon$ again. Then, we have the following:

Proposition 1.3. *Assume Hypothesis* 20.1 *and Conjecture* 20.2 *hold. Let π be a tempered irreducible representation of $G(W)$ and let (ϕ,η) be its Langlands parameter. Then $\theta_\psi(\pi,V)$ is non-zero if and only if* $\mathrm{std}\circ\phi$ *contains* $\chi_V\boxtimes 1_{\mathrm{SL}_2(\mathbb{C})}$ *as representations of* $W_F\times\mathrm{SL}_2(\mathbb{C})$. *Here,* std *is the standard embedding of ${}^LG$ into* $\mathrm{GL}_N(\mathbb{C})$ *and* $1_{\mathrm{SL}_2(\mathbb{C})}$ *is the trivial representation of* $\mathrm{SL}_2(\mathbb{C})$. *Suppose that $\theta_\psi(\pi,V)$ is non-zero, and we denote it by σ. We denote by (ϕ',η') its Langlands parameter. Then we have*

$$\mathrm{std}\circ\phi\cong(\mathrm{std}\circ\phi'\otimes\chi_V\chi_W^{-1})\oplus(\chi_V\boxtimes 1_{\mathrm{SL}_2(\mathbb{C})})$$

as representations of $W_F\times\mathrm{SL}_2(\mathbb{C})$, *and we have*

$$\frac{\dim\eta}{\dim\eta'}=\begin{cases}1 & (\epsilon=1),\\ 1 & (\epsilon=-1,\phi'^{\varepsilon}=\phi'),\\ 2 & (\epsilon=-1,\phi'^{\varepsilon}\neq\phi').\end{cases}$$

Here, ε is the generator of $\mathrm{Out}(\widehat{G(V)})$.

Using Proposition 1.3, we verify that our main theorem (Theorem 1.2) is consistent with Hypothesis 20.1 and Conjecture 20.2 (§20).

1.5. **Formal degree conjecture for non-split inner forms of Sp_4 and GSp_4.** Let G be a reductive group over F, let π be a square-integrable irreducible representation of $G(W)$, and let (ϕ,η) be its Langlands parameter. We denote by A the maximal F-split torus of the center of G, and we put $C'_\phi=\widehat{G/A}\cap C_\phi$. Then, the formal degree conjecture of Hiraga, Ichino, and Ikeda asserts that

$$\deg\pi=\frac{\dim\eta}{\#C'_\phi}\cdot|\gamma(0,\mathrm{ad}\circ\phi,\psi)|,$$

where ad is the adjoint representation of ${}^LG$ on $\mathrm{Lie}(\widehat{G}_{\mathrm{ad}})$ [HII08]. This conjecture has been proved for reductive groups over Archimedean local fields and for the inner forms of GL_n by themselves [HII08]. It has also been proved for SO_{2n+1}, Mp_{2n}, U_n, and Sp_4 ([ILM17], [BP21], [GI14]). Moreover, Gross and Reeder reformulated it by using the Eular-Poincáre measure on G [GR10]. Note that the absolute value does not appear in their reformulation. For the non-split inner forms of GSp_4 and Sp_4, the Langlands correspondence is constructed by Gan and Tantono [GT14] and Choiy [Cho17] respectively. We prove the conjecture for these groups by using Theorem 1.2:

Theorem 1.4. *Let F be a local field of characteristic* 0. *Then, the formal degree conjecture holds for the non-split inner forms of* Sp_4 *and* GSp_4.

1.6. **Structure of this paper.** Now, we explain the structure of this paper. In §§2–3, we set up the notations for fields, quaternion algebras, and $\pm\epsilon$-Hermitian spaces. In §4, we define some symbols which are referred to when we take bases for $\pm\epsilon$-Hermitian spaces. In §5, we recall the Bruhat-Tits theory for quaternionic unitary groups and define the Iwahori subgroup. In §6, we explain the normalization of Haar measures on reductive groups and certain nilpotent groups, and we give some volume formulas. In §7, we explain the doubling method and recall the definition of the doubling γ-factor. Moreover, we compute the constant $\alpha_1(W)$ for some cases. In §8, we set up and explain the doubling method and the Weil representations. In §9, we set up the theta correspondence. In §§10–11, 19–21, we state our main results, and in §§12–18, we prove these results. More precisely, §§12–13 are devoted to the computation of $\alpha_2(V, W)$ when either V or W is anisotropic, §14 is a preliminary for §15 which associates $\alpha_2(V, W)$ with $\alpha_3(V, W)$, and §§16–17 are preliminaries for §18 in which we verify the commutativity of $\alpha_3(V, W)$ with the parabolic inductions. Finally, in Appendix A, we give a formula for doubling zeta integrals of certain sections as an application of the formula of $\alpha_1(W)$. Note that this corrects the errors in [Kak20, Proposition 8.3].

2. Quaternion algebras over local fields

Let F be a non-Archimedean local field of characteristic 0, and let D be a quaternion algebra over F. In this paper except §§14 and 16.5, we assume that D is division. We denote by $\mathcal{O}_F$ the valuation ring of F, by ϖ_F a uniformizer of $\mathcal{O}_F$, by $\mathrm{ord}_F\colon F^\times \to \mathbb{Z}$ the additive valuation normalized so that $\mathrm{ord}_F(\varpi_D) = 1$, by q the cardinality of $\mathcal{O}_F/\varpi_F$, by $|\ |_F$ the absolute value normalized so that $|\varpi_F|_F = q^{-1}$, by $*\colon D \to D$ the canonical involution of D, by $N_D\colon D \to F$ the reduced norm, by $T_D\colon D \to F$ the reduced trace, by $\mathrm{ord}_D = \mathrm{ord}_F \circ N_D$ the additive valuation of D, by $|\ |_D = |\ |_F \circ N_D$ the absolute value of D, and by $\mathcal{O}_D$ the valuation ring of D.

Lemma 2.1. *There exist two elements δ and ϖ_D of D so that the subfield $F(\delta)$ is unramified over F, $T_D(\delta) = T_D(\varpi_D) = 0$, $\mathrm{ord}_D(\delta) = 0$, $\mathrm{ord}_D(\varpi_D) = 1$, and $\delta\varpi_D + \varpi_D\delta = 0$.*

Proof. Take $d \in F$ so that $\mathrm{ord}_F(d) = 0$ and $F(\sqrt{d})$ is unramified over F. To prove the claim, it suffices to show that the quaternion algebra $(d, \varpi_F/F)$ is isomorphic to D. Since the 2-torsion subgroup of the Brauer group of F is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, it remains to show that $(d, \varpi_F/F)$ is division. This is obtained by the fact that ϖ_F is not contained in the image of the norm map of the unramified extension $F(\sqrt{d})/F$. Hence we have the lemma. ∎

3. ϵ-Hermitian spaces and their unitary groups

Let $\epsilon \in \{\pm 1\}$. Now, we consider the following:

- a pair $(W, \langle\ ,\ \rangle)$ where W is a left free D-module of rank n, and $\langle\ ,\ \rangle$ is a map $W \times W \to D$ satisfying
$$\langle ax, by\rangle = a\langle x, y\rangle b^*,\ \langle y, x\rangle = -\epsilon\langle x, y\rangle$$
for $x, y \in W$ and $a, b \in D$,
- a pair $(V, (\ ,\ ))$ where V is a right free D-module of rank m, and $(\ ,\ )$ is a map $V \times V \to D$ satisfying
$$(v_1 a, v_2 b) = a^*(x, y)b,\ (y, x) = \epsilon(x, y)^*$$
for $x, y \in V$ and $a, b \in D$.

We call them an n-dimensional right ϵ-Hermitian space and an m-dimensional left $(-\epsilon)$-Hermitian space respectively if they are non-degenerate. Then, we define $G(W)$ by the group of the left D-linear automorphisms g of W and
$$\langle x \cdot g, y \cdot g\rangle = \langle x, y\rangle$$

for all $x, y \in W$. We also define $G(V)$ by the group of the right D-linear automorphisms g of V and

$$(g \cdot x, g \cdot y) = (x, y)$$

for all $x, y \in V$. Put $\mathbb{W} = V \otimes_F W$ and define $\langle\langle\ ,\ \rangle\rangle$ by

$$\langle\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle\rangle = T_D((x_1, y_1)\langle x_2, y_2\rangle^*)$$

for $x_1, y_1 \in V$ and $x_2, y_2 \in W$. Then, $\langle\langle\ ,\ \rangle\rangle$ is a symplectic form on $\mathbb{W}$, and the $(G(W), G(V))$ is a reductive dual pair in $\mathrm{Sp}(\mathbb{W})$. We define

$$l = l_{V,W} = \begin{cases} 2n - 2m - 1 & (\epsilon = 1), \\ 2n - 2m + 1 & (\epsilon = -1). \end{cases}$$

We define the characters χ_V and χ_W of $F^\times$ by

$$\chi_V(a) = \begin{cases} 1_{F^\times} & (\epsilon = 1), \\ (a, \mathfrak{d}(W))_F & (\epsilon = 1) \end{cases} \text{ and } \chi_W(a) = \begin{cases} (a, \mathfrak{d}(V))_F & (\epsilon = -1), \\ 1_{F^\times} & (\epsilon = -1) \end{cases}$$

for $a \in F^\times$.

4. Bases for W and V

In this section, we discuss bases for W, which we will consider in this paper. The discussion for V goes the same line as that of W. For a basis $\underline{e} = \{e_1, \ldots, e_n\}$ for W, we define

$$R(\underline{e}) := (\langle e_i, e_j \rangle)_{ij} \in \mathrm{GL}_n(D).$$

Denote by W_0 the anisotropic kernel of W, and put $n_0 = \dim_D W_0$, $r = \frac{1}{2}(n - n_0)$. We assume that

$$W_0 = \sum_{i=r+1}^{r+n_0} e_i D,$$

both

$$X = \sum_{i=1}^{r} e_i D \text{ and } \sum_{i=r+n_0+1}^{n} e_i X^*$$

are totally isotropic subspaces of W, and

$$R(\underline{e}) = \begin{pmatrix} 0 & 0 & J_r \\ 0 & R_0 & 0 \\ -\epsilon J_r & 0 & 0 \end{pmatrix}, \tag{4.1}$$

where

$$J_r = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

and $R_0 \in \mathrm{GL}_{n_0}(D)$. By this basis, we regard $G(W)$ as a subgroup of $\mathrm{GL}_n(D)$. It is known that

$$\begin{cases} n_0 \leq 1 & (-\epsilon = 1), \\ n_0 \leq 3 & (-\epsilon = -1). \end{cases}$$

Moreover, in the case $-\epsilon = -1$, it is known that $n_0 = 2$ if and only if n is even and $\chi_V \neq 1_{F^\times}$, $n_0 = 3$ if and only if n is odd and $\chi_V = 1_{F^\times}$. (Cf. [Sch85, §10, Example 1.8 (ii) and Theorem 3 .6].)

5. Bruhat-Tits theory

In this section, we recall the definition and construction of the Iwahori subgroups of quaternionic unitary groups. Before giving the definition, we discuss the apartments.

5.1. **Apartments.** Take a basis $\underline{e}$ as in §4. Put $I = \{e_1, \dots, e_r\}$, $I_0 = \{e_{r+1}, \dots, e_{n-r}\}$, and $I^* = \{e_{n-r+1}, \dots, e_n\}$. We denote by S the maximal F-split torus

$$\{\mathrm{diag}(x_1, \dots, x_r, 1, \dots, 1, x_r^{-1}, \dots, x_1^{-1}) \mid x_1, \dots, x_r \in F^\times\}$$

of $G(W)$. We denote by $Z_{G(W)}(S)$ the centralizer of S in $G(W)$, by $N_{G(W)}(S)$ the normalizer of S in $G(W)$, by $\mathcal{W} = N_{G(W)}(S)/Z_{G(W)}(S)$ the relative Weyl group with respect to S, by Φ the relative root system of $G(W)$ with respect to S, by $X^*(S)$ the group of algebraic characters of S, by $E^\vee$ the vector space $X^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$, and by E the $\mathbb{R}$ dual space of $E^\vee$. Moreover, we define the bilinear map $\langle\ ,\ \rangle \colon E \times E^\vee \to \mathbb{R}$ by $\langle y, \eta \rangle = \eta(y)$ for $y \in E^\vee$ and $\eta \in E$. Then, we can define the map $\mu \colon Z_{G(W)}(S) \to E$ by

$$[\mu(z)](a') = -\mathrm{ord}_F(a'(z))$$

for $a' \in X^*(S)$. Then, there is a unique morphism $\nu \colon N_{G(W)}(S) \to \mathrm{Aff}(E)$ so that the following diagram is commutative:

$$\begin{tikzcd}
1 \arrow[r] & Z_{G(W)}(S) \arrow[r] \arrow[d, "\mu"'] & N_{G(W)}(S) \arrow[r] \arrow[d, "\nu"'] & \mathcal{W} \arrow[r] \arrow[d] & 1 . \\
0 \arrow[r] & E \arrow[r] & \mathrm{Aff}(E) \arrow[r] & \mathrm{Aut}(E) \arrow[r] & 1
\end{tikzcd}$$

For $a \in \Phi$, we denote by X_a the root subgroup in $G(W)$. Let $u \in X_a \setminus \{1\}$. Then one can prove that $X_{-a} \cdot u \cdot X_{-a} \cap N_{G(W)}(S)$ consists of an unique element. We denote it by $m_a(u)$. We define a map $\varphi_a \colon X_a \setminus \{1\} \to \mathbb{R}$ by

$$m_a(u)(\eta) = \eta - (\langle a, \eta \rangle + \varphi_a(u))a^\vee$$

for all $\eta \in E$. We put Φ_{aff} the affine root system

$$\{(a,t) \mid a \in \Phi,\ t = \varphi_a(u) \text{ for some } u \in X_a \setminus \{1\}\} \subset \Phi \times \mathbb{R},$$

and by $E_{a,t}$ the subset $\{\eta \in E \mid [m_a(u)](\eta) = \eta\}$ where $u \in X_a$ so that $\varphi_a(u) = t$. We call a connected component of

$$E \setminus \bigcup_{(a,t) \in \Phi_{\mathrm{aff}}} E_{a,t}$$

a chamber of E. For $i \in I$ (resp. $i \in I^*$), we define $a_i \in X^*(S) \subset E^\vee$ by $a_i(x) = N_D(x_i)$ (resp. $a_i(x) = N_D(x_{n+1-i})^{-1}$) for

$$x = \mathrm{diag}(x_1, \dots, x_r, 1, \dots, 1, x_r^{-1}, \dots, x_1^{-1}) \in S.$$

Now we describe φ_a explicitly following [BT72, §10]. The root system of $G(W)$ with respect to S is divided into

$$\Phi = \Phi_1^+ \cup \Phi_1^- \cup \Phi_2^+ \cup \Phi_2^- \cup \Phi_3^+ \cup \Phi_3^- \cup \Phi_4^+ \cup \Phi_4^-,$$

where

$$\begin{aligned}
\Phi_1^+ &= \{a_i - a_j \mid 1 \leq j < i \leq r\},\\
\Phi_2^+ &= \{a_i \mid i = 1, \dots, r\},\\
\Phi_3^+ &= \{a_i + a_j \mid 1 \leq j < i \leq r\},\\
\Phi_4^+ &= \{2a_i \mid i = 1, \dots, r\},
\end{aligned}$$

and $\Phi_k^- = -\Phi_k^+$ for $k = 1, 2, 3, 4$. Let $a = a_i - a_j \in \Phi_1^+ \cup \Phi_1^-$. For $x \in D$, we define $u_a(x) \in X_a$ by

$$e_k \cdot u_a(x) = \begin{cases} e_k & (k \neq i, n-i),\\ e_i + x \cdot e_j & (k = i),\\ e_{n-i} + x^* \cdot e_{n-j} & (k = n-i). \end{cases}$$

Let $a = a_i \in \Phi_2^+$. For $c = (c_1, \dots, c_{n_0}) \in W_0 = D^{n_0}$ and $d \in D$ with $(d^* - \epsilon d) + \langle c, c \rangle = 0$, we define $u_a(c, d) \in X_a$ by

$$e_k \cdot u_a(c,d) = \begin{cases} e_k & (k \neq i, r+1, \dots, r+n_0),\\ e_i + \sum_{t=1}^{n_0} c_t e_{r+t} + d e_{n-i} & (k = i),\\ e_k + \alpha_{k-r} c_{k-r}^* e_{n-i} & (k = r+1, \dots, r+n_0). \end{cases}$$

Let $a = -a_i \in \Phi_2^-$. For $c = (c_1, \dots, c_{n_0}) \in W_0 = D^{n_0}$ and $d \in D$ with $(d - \epsilon d^*) + \langle c, c \rangle = 0$, we define $u_a(c, d) \in X_a$ by

$$e_k \cdot u_a(c,d) = \begin{cases} e_k & (k \neq r+1, \dots, r+n_0, n-i),\\ e_k - \alpha_{k-r} c_{k-r}^* e_i & (k = r+1, \dots, r+n_0),\\ d e_i + \sum_{t=1}^{n_0} c_t e_{r+t} + e_{n-i} & (k = n-i). \end{cases}$$

Let $a = (a_i + a_j) \in \Phi_3^+$. For $x \in D$, we define $u_a(x) \in X_a$ by

$$e_k \cdot u_a(x) = \begin{cases} e_k & (k \neq i, j),\\ e_i + x \cdot e_{n-i} & (k = i),\\ e_j + \epsilon x^* e_{n-j} & (k = j). \end{cases}$$

Let $a \in \Phi_3^-$. For $x \in D$, we define $u_a(x) \coloneqq {}^t u_{-a}(x)^* \in X_a$. Finally, let $a = \pm 2a_i \in \Phi_4^\pm$. For $d \in D$ with $d^* - \epsilon d = 0$, we define $u_a(d) \coloneqq u_{\pm a_i}(0, d) \in X_{2a}$.

Lemma 5.1. *For $a \in \Phi$, we have*

- $\varphi_a(u_a(x)) = \mathrm{ord}_D(x)$ *for $x \in D$ if $a \in \Phi_1^+ \cup \Phi_1^- \cup \Phi_3^+ \cup \Phi_3^-$,*
- $\varphi_a(u_a(c, d)) = \frac{1}{2}\mathrm{ord}_D(d)$ *for $c \in D^{n_0}$ and $d \in D$ with $(d^* - \epsilon d) \pm \langle c, c \rangle = 0$ if $a \in \Phi_2^\pm$,*
- $\varphi_a(u_a(d)) = \mathrm{ord}_D(d)$ *for $d \in D$ with $d^* - \epsilon d = 0$ if $a \in \Phi_4^+ \cup \Phi_4^-$.*

5.2. **Iwahori subgroups.** Before stating the definition of the Iwahori subgroup, we explain a map of Kottwitz. Let F^{ur} be the maximal unramified extension of F, let F^s be the separable closure of F, let $I = \mathrm{Gal}(F^s/F^{\mathrm{ur}})$ be the inertia group of F, and let Fr be a Frobenius element. Then, Kottwitz defined a surjective map

$$\kappa_W \colon G(W) \to \mathrm{Hom}(Z(\widehat{G(W)})^I, \mathbb{C}^\times)^{\mathrm{Fr}}$$

(see [Kot97, §7.4]). Here, we denote by $\widehat{G(W)}$ the Langlands dual group of $G(W)$, by $Z(\widehat{G(W)})^I$ the I-invariant subgroup of the center of $\widehat{G(W)}$, and by

$\mathrm{Hom}(Z(\widehat{G(W)})^I, \mathbb{C}^\times)^{\mathrm{Fr}}$ the Fr-invariant subgroup of $\mathrm{Hom}(Z(\widehat{G(W)})^I, \mathbb{C}^\times)$. Then, an Iwahori subgroup of $G(W)$ is defined to be a subgroup consisting of the elements g of $G(W)$ which preserves each point of a chamber of the building and $\kappa_W(g) = 1$. Now we describe an Iwahori subgroup of $G(W)$. Let $\mathcal{C}$ be a chamber in E so that

- for any root $a \in \Phi(S, G(W))$ with $X_a \subset B$, $\langle a, \mathcal{C}\rangle \subset \mathbb{R}_{>0}$,
- the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ contains the origin $0 \in E$.

Then, the Iwahori subgroup associated with the chamber $\mathcal{C}$ is given by

$$\mathcal{B} := \{g \in G(W) \mid \kappa_W(g) = 1 \text{ and } g \cdot p = p \text{ for all } p \in \mathcal{C}\}.$$

By the construction of the map κ_W, the following diagram is commutative:

$$\begin{array}{ccc} Z_{G(W)}(S) & \xrightarrow{\kappa_{Z_{G(W)}(S)}} & \mathrm{Hom}(Z(\widehat{Z_{G(W)}(S)})^I, \mathbb{C}^\times)^{\mathrm{Fr}} \\ \downarrow & & \downarrow \\ G(W) & \xrightarrow{\kappa_W} & \mathrm{Hom}(Z(\widehat{G(W)})^I, \mathbb{C}^\times)^{\mathrm{Fr}} \end{array}$$

where the vertical maps are (induced from) the natural embeddings. Hence, we have:

Lemma 5.2.

$$\mathcal{B} = Z_{G(W)}(S)_1 \cdot \prod_{a\in\Phi^+} X_{a,0} \cdot \prod_{a\in\Phi^-} X_{a,\frac{1}{2}},$$

where $Z_{G(W)}(S)_1$ is the set of matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & g_0 & 0 \\ 0 & 0 & a^{*-1} \end{pmatrix} \quad (a = \mathrm{diag}(a_1, \ldots, a_r), g_0 \in G(W_0))$$

such that $a_i \in \mathcal{O}_D^\times$ for $i = 1, \ldots, r$, and $\kappa_{W_0}(g_0) = 1$. Here, we denote by $X_{a,t}$ the subset

$$\{u \in X_a \mid \varphi_a(u) \geq t\}$$

of X_a for $t \in \mathbb{R}$.

6. Haar measures

In this section, we explain how we choose Haar measures in this paper for reductive groups and unipotent groups. Let $\psi\colon F \to \mathbb{C}^\times$ be a non-trivial additive character of F. For a reductive group, Gross and Gan constructed a Haar measure dg depending only on the group G and the non-trivial additive character ψ [GG99, §8]. (In [GG99], it is denoted by μ_G.) For a unipotent group, it is useful to consider the "self-dual measures" du with respect to ψ. In both cases, we denote by $|X|$ the volume of X for a measurable set X.

6.1. **Measures on reductive groups.** Let G be a connected reductive group, and let G' be the quasi-split inner form of G. Moreover, let S be a maximal F-split torus of G, let S' be a maximal F-split torus of G', let T' be the centralizer of S' in G' (it becomes a torus over F^s), and let $\mathcal{W}(T', G')$ be the Weyl group of G' with respect to T'. Put $E' := X^*(T') \otimes \mathbb{Q}$. Then the space E' can be regarded as a graded $\mathbb{Q}[\Gamma]$-module

$$E' = \oplus_{d\geq 1} E'_d$$

as follows: consider a $\mathcal{W}(T', G')$-invariant subalgebra $R = \mathrm{Sym}^\bullet(E')^{\mathcal{W}(T',G')}$ of symmetric algebra $\mathrm{Sym}^\bullet(E')$. We denote by R_+ the ideal consisting of the elements of positive degrees. Then, there is a $\mathbb{Q}[\Gamma]$-isomorphism $E' \cong R_+/R_+^2$. Then, the grading of E' is the one deduced from the natural grading of R_+/R_+^2.

Let $\Psi\colon G' \to G$ be an inner isomorphism defined over F^{ur}. We may assume that the torus $\Psi(S')$ is a maximal F^{ur} split torus containing S. Then the automorphism $\Psi^{-1} \circ \mathrm{Fr}(\Psi)$ preserves the torus T' and the action agrees with that by a Weyl element $w_G \in W(T', G')^I$. We denote by $\mathfrak{M}$ the motive

$$\oplus_{d \geq 1} E'_d(d-1)$$

of G (see [Gro97]), and by $a(\mathfrak{M})$ the Artin invariant

$$\sum_{d \geq 1} (2d-1) \cdot a(E'_d)$$

of $\mathfrak{M}$ (see [GG99]). Then, the Haar measure dg is normalized so that the volume of the Iwahori subgroup $\mathcal{B}$ is given by

$$|\mathcal{B}| = q^{-\mathfrak{N} - \frac{1}{2}a(\mathfrak{M})} \cdot \det(1 - \mathrm{Fr} \circ w_G; E'(1)^I). \tag{6.1}$$

Here, we put

$$\mathfrak{N} = \sum_{d \geq 1} (d-1) \dim_{\mathbb{Q}} {E'_d}^I.$$

Now, consider the case $G = G(W)$ where W is an n-dimensional $(-\epsilon)$-Hermitian space over D. Then, we have the following:

Proposition 6.1.

(1) *Suppose that $-\epsilon = 1$. Then, we have*

$$|\mathcal{B}| = (1 - q^{-1})^{\lfloor \frac{n}{2} \rfloor} \cdot (1 + q^{-1})^{\lceil \frac{n}{2} \rceil} \cdot q^{-n^2},$$

where $\mathcal{B}$ is an Iwahori subgroup of $G(W)$.

(2) *Suppose that $-\epsilon = -1$. Then, we have*

$$|\mathcal{B}| = \begin{cases} (1-q^{-2})^{\frac{n}{2}} \cdot q^{-n^2+n} & (n_0 = 0), \\ (1-q^{-2})^{\frac{n-1}{2}} \cdot q^{-n^2+(2n-1)(1-\frac{\mathfrak{f}(\chi_W)}{2})} & (n_0 = 1, \chi_W \textit{ is ramified}), \\ (1-q^{-2})^{\frac{n-1}{2}} \cdot (1+q^{-1}) \cdot q^{-n^2+n} & (n_0 = 1, \chi_W \textit{ is unramified}), \\ (1-q^{-2})^{\frac{n-2}{2}} \cdot (1+q^{-1}) q^{-n^2+(2n-1)(1-\frac{\mathfrak{f}(\chi_W)}{2})} & (n_0 = 2, \chi_W \textit{ is ramified}), \\ (1-q^{-2})^{\frac{n-2}{2}} (1+q^{-2}) q^{-n^2+n} & (n_0 = 2, \chi_W \textit{ is unramified}), \\ (1-q^{-2})^{\frac{n-3}{2}} (1+q^{-1}+q^{-2}+q^{-3}) q^{-n^2+n} & (n_0 = 3). \end{cases}$$

where $\mathfrak{f}(\chi_W)$ is the conductor of χ_W.

Proof. Let $\mathcal{A}$ be a $\mathcal{O}_F$-scheme so that the fibered product $\mathcal{A} \times_{\mathrm{Spec}\mathcal{O}_F} \mathrm{Spec}F$ is isomorphic to $\Psi(S')$ over F. Then, we have

$$\det(1 - \mathrm{Fr} \circ w_G; E'(1)^I) = q^{-\dim_F S'} \#\mathcal{A}(\mathcal{O}_F/\varpi_F). \tag{6.2}$$

In our case, we may assume that the torus $\Psi(S')$ is isomorphic to

$$\mathrm{Res}_{L_2/F}(\mathbb{G}_m)^{\frac{n-n_0}{2}} \times \begin{cases} 1 & (n_0=0, n_0=1 \text{ with } \chi_W \text{ ramified}), \\ \ker N_{L_2/F} & (n_0=1 \text{ with } \chi_W \text{ unramified}, n_0=2 \text{ with } \chi_W \text{ ramified}), \\ \ker N_{L_4/L_2} & (n_0=2 \text{ with } \chi_W \text{ unramified}), \\ \ker N_{L_4/F} & (n_0=3), \end{cases}$$

where L_d denotes the unramified extension field of F of $[L_d : F] = d$, and $N_{L/K}$ denotes the norm map $\mathrm{Res}_{L/F}\,\mathbb{G}_m^\times \to \mathrm{Res}_{K/F}\,\mathbb{G}_m^\times$ associated with a field extension L/K. Hence, by (6.2), we have

$$\det(1-\mathrm{Fr}\circ w_G; E'(1)^I) = (1-q^{-2})^{\frac{n-n_0}{2}} \times \begin{cases} 1 & (n_0{=}0, n_0{=}1 \text{ with } \chi_W \text{ ramified}), \\ (1{+}q^{-1}) & (n_0{=}1, \text{ with } \chi_W \text{ unramified}, n_0{=}1 \text{ with } \chi_W \text{ ramified}), \\ (1{+}q^{-2}) & (n_0{=}2, \text{ with } \chi_W \text{ unramified}), \\ (1{+}q^{-1}{+}q^{-2}{+}q^{-3}) & (n_0{=}3). \end{cases}$$

We define a grading and a Γ-action on the polynomial ring $\mathbb{Q}[X,Y]$ by

$$\begin{aligned} &\deg X^k = k,\ \deg Y^l = nl\ (k,l=0,1,\ldots),\ \text{and} \\ &\sigma\cdot f(X,Y) = f(X,\eta_W(\sigma)Y) \text{ for } f(X,Y)\in\mathbb{Q}[X,Y], \sigma\in\Gamma. \end{aligned}$$

Here, η_W is a character on Γ associated with χ_W via the local class field theory. Then we have that E' is isomorphic to

$$\begin{cases} \mathbb{Q}X^2+\mathbb{Q}X^4+\cdots+\mathbb{Q}X^{2n} & (-\epsilon=1), \\ \mathbb{Q}X^2+\mathbb{Q}X^4+\cdots+\mathbb{Q}X^{2n-2}+\mathbb{Q}Y & (-\epsilon=-1) \end{cases}$$

as a graded $\mathbb{Q}[\Gamma]$-module. Hence, we have

$$\mathfrak{N} = \begin{cases} n^2 & (-\epsilon=1), \\ n^2-n & (-\epsilon=-1 \text{ with } \chi_W \text{ unramified}), \\ n^2-2n+1 & (-\epsilon=-1 \text{ with } \chi_W \text{ ramified}), \end{cases}$$

and

$$a(\mathfrak{M}) = \begin{cases} 0 & (\chi_W \text{ is unramified}), \\ (2n-1)\cdot\mathfrak{f}(\chi_W) & (\chi_W \text{ is ramified}). \end{cases}$$

By computing the right-hand side of (6.1), we have the claim. □

If $G(W)$ is anisotropic, then $\mathcal{B} = \ker\kappa_W$ (see §5.2). Hence, its total volume is given by Corollary 6.2:

Corollary 6.2. *Suppose that W is anisotropic.*

(1) *If $-\epsilon = 1$ and $n=1$, then we have $|G(W)| = q^{-1}(1+q^{-1})$.*

(2) *If* $-\epsilon = -1$, *then we have*

$$|G(W)| = \begin{cases} 1{+}q^{-1} & (n{=}1 \textit{ with } \chi_W \textit{ unramified}), \\ 2q^{-\frac{\mathfrak{f}(\chi_W)}{2}} & (n{=}1 \textit{ with } \chi_W \textit{ ramified}), \\ 2{\cdot}q^{-2}(1{+}q^{-2}) & (n{=}2 \textit{ with } \chi_W \textit{ non-trivial and unramified}), \\ 2(1{+}q^{-1})q^{-\frac{3}{2}\mathfrak{f}(\chi_W)-1} & (n{=}2 \textit{ with } \chi_W \textit{ ramified}), \\ 2q^{-6}(1{+}q^{-1})(1{+}q^{-2}) & (n{=}3). \end{cases}$$

Proof. Since the Kottwitz map κ_W is surjective,

$$\begin{aligned} [G(W) : \mathcal{B}] &= \#(X^*(Z(\widehat{G})^I)^{\mathrm{Fr}}) \\ &= \begin{cases} 1 & (n = 1 \text{ with } \chi_W \text{ unramified}), \\ 2 & (\text{ otherwise}), \end{cases} \end{aligned}$$

where I is the inertia group of F, and Fr is a Frobenius element of F. Hence we have the claim. □

6.2. **Measures on unipotent groups.** Take a basis $\underline{e}$ and regard $G(W)$ as a subgroup of $\mathrm{GL}_n(D)$ as in §4. Let

$$\mathfrak{f} : 0 = X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = X$$

be a flag consisting of totally isotropic subspaces. We put $r_i = \dim_D X_i/X_{i-1}$ for $i = 1, \ldots, k$. Moreover, we put

$$\mathfrak{u}_{r'} = \{z \in \mathrm{M}_{r'}(D) \mid {}^t z^* - \epsilon z = 0\}$$

for a positive integer r'. We denote by P the parabolic subgroup of all $p \in G(W)$ satisfying $X_i \cdot p \subset X_i$ for $i = 0, \ldots, k$, and by $U(P)$ the unipotent radical of P. Moreover, we denote by $U_i(P)$ the subgroup

$$\{u \in U(P) \mid X \cdot (u - 1) \subset X_i\}$$

for $i = 1, \ldots, k$. Then, for $i = 1, \ldots, k$, we have the exact sequence

$$1 \to U_{i-1}(P) \to U_i(P) \to \prod_{j=(i+2)/2}^{i} \mathrm{M}_{r_j, r_{i+1-j}}(D) \to 0 \tag{6.3}$$

if i is even, and the exact sequence

$$1 \to U_{i-1}(P) \to U_i(P) \to \mathfrak{u}_{r_{(i+1)/2}} \times \prod_{j=(i+3)/2}^{i} \mathrm{M}_{r_{i+1-j}, r_j}(D) \to 0 \tag{6.4}$$

if i is odd. Here, the first maps are the inclusions and the second maps are given by

$$u = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ z_1 & 0 & 1 & & & \\ * & z_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & 0 & 1 & \\ * & \cdots & * & z_i & 0 & 1 \end{pmatrix} \mapsto (z_{\lceil (i+1)/2 \rceil} J_{\lceil (i+1)/2 \rceil}, \ldots, z_i J_i),$$

for $u \in U_i(P)$. We define a measure dz on $\mathfrak{u}_{r'}$ to be the self-dual Haar measure with respect to a pairing

$$\mathfrak{u}_r \times \mathfrak{u}_r \to \mathbb{C}\colon (z, z') \mapsto \psi(T_D(z \cdot {}^t z'^*)), \tag{6.5}$$

and we define a measure dx on $\mathrm{M}_{r',r''}(D)$ to be the self-dual Haar measure with respect to a pairing

$$\mathrm{M}_{r',r''}(D) \times \mathrm{M}_{r',r''}(D) \to \mathbb{C}^\times \colon (x, x') \mapsto \psi(T_D(x \cdot {}^t x'^*)). \tag{6.6}$$

Then, the Haar measure du on $U_i(P)$ is defined inductively by the exact sequences (6.3) and (6.4) for $i = 1, \ldots, k$.

In the rest of this section, we compute the volumes $|\mathfrak{u}_{r'} \cap \mathrm{M}_{r'}(\mathcal{O}_D)|$ and $|\mathrm{M}'_{r',r''}(\mathcal{O}_D)|$ with respect to the self-dual measures above. To compute them, we observe lattices of $\mathfrak{u}_r$ and $\mathrm{M}_{r',r''}(D)$ with $r = r' = r'' = 1$.

Lemma 6.3.

(1) *Suppose that $r' = r'' = 1$. Then, the dual lattice $\mathcal{O}_D^*$ of $\mathcal{O}_D$ with respect to the pairing* (6.6) *is given by $\varpi_D^{-1}\mathcal{O}_D$.*
(2) *Suppose that $\epsilon = 1$ and $r = 1$. Then, the dual lattice of $\mathcal{O}_D \cap \mathfrak{u}_1$ with respect to the pairing* (6.5) *is given by $\frac{1}{2}\delta\mathcal{O}_F + \varpi_D^{-1}\mathcal{O}_{F(\delta)}$.*
(3) *Suppose that $\epsilon = -1$ and $r = 1$. Then, the dual lattice of $\mathcal{O}_D \cap \mathfrak{u}_1$ with respect to the pairing* (6.5) *is given by $\frac{1}{2}\mathcal{O}_F$.*

Proof. Since the order of ψ is zero, there exists $a \in \mathcal{O}_F^\times$ such that $\psi(\varpi_F^{-1}a) \neq 1$. The assertion (3) is well-known, thus we only prove (1) and (2). We admit the existence of an element $b \in \mathcal{O}_{F(\delta)}^\times$ satisfying $b + b^* = a$ at once. We take two elements δ, ϖ_D as in Lemma 2.1. If $x \in D^\times$ satisfies $\mathrm{ord}_D(x) < -1$, then $x^{-1}\varpi_F^{-1}b \in \mathcal{O}_D$, and we have

$$\psi(T_D(x \cdot x^{-1}\varpi_F^{-1}b)) = \psi(\varpi_F^{-1}a) \neq 1.$$

Thus we have that $\mathcal{O}_F^*$ is contained in $\varpi_D^{-1}\mathcal{O}_D$. On the other hand, $\psi(T_D(\varpi_D^{-1}\mathcal{O}_D)) = 1$ since $T_D(\varpi_D^{-1}\mathcal{O}_D) \subset \mathcal{O}_F$. Hence we have (1). Suppose that $\epsilon = 1$ and $r = 1$. An element x of $\mathfrak{u}_1$ can be written in the form $x = \delta \cdot x_1 + \varpi_F \cdot x_2$ where $x_1 \in F$ and $x_2 \in F(\delta)$. If $x_1 \notin \frac{1}{2}\mathcal{O}_F$, then $\delta^{-1}(2x_1)^{-1}\varpi_F^{-1}a \in \mathcal{O}_D$, and

$$\psi(T_D(\delta x \cdot \delta^{-1}(2x_1)^{-1}\varpi_F^{-1}a)) = \psi(\varpi_F^{-1}a) \neq 1.$$

If $x_2 \notin \mathcal{O}_{F(\delta)}$, then $x_2^{-1}b \in \mathcal{O}_D$, and

$$\psi(T_D(\varpi_D^{-1}x_2 \cdot x_2^{-1}b)) = \psi(\varpi_F^{-1}a) \neq 1.$$

Thus we have that the dual lattice of $\mathcal{O}_D \cap \mathfrak{u}_1$ is contained in $\frac{1}{2}\delta\mathcal{O}_F + \varpi_D^{-1}\mathcal{O}_{F(\delta)}$. On the other hand, the subset $(\frac{1}{2}\delta\mathcal{O}_F + \varpi_D^{-1}\mathcal{O}_{F(\delta)}) \cdot \mathcal{O}_D$ is contained in the subset $\frac{1}{2}\mathcal{O}_F + \varpi_D^{-1}\mathcal{O}_D$ on which $\psi \circ T_D$ vanishes. Hence we have (2).

It remains to show that there exists an element $b \in \mathcal{O}_{F(\delta)}^\times$ satisfying $b + b^* = a$. Put

$$\mathcal{X} = \{T^2 - uT + v \mid u, v \in (\mathcal{O}_F/\varpi_F)^\times\}, \text{ and}$$
$$\mathcal{Y} = \{(T - x)(T - y) \mid x, y \in (\mathcal{O}_F/\varpi_F)^\times, x + y \neq 0\}.$$

Then we have $\mathcal{X} \supset \mathcal{Y}$ and

$$\#\mathcal{X} = (q-1)^2 > \frac{1}{2}q(q-1) - (q-1) = \#\mathcal{Y}.$$

This inequation implies that $\mathcal{X}$ possesses at least one irreducible polynomial $h(T)$. Take $c \in \mathcal{O}_{F(\delta)}^{\times}$ so that its image $\overline{c} \in \mathcal{O}_{F(\delta)}/\varpi_F$ satisfies $h(\overline{c}) = 0$. Then, by the definition of $\mathcal{X}$, we have $c + c^* \in \mathcal{O}_F^{\times}$. Thus, putting $b = c(c+c^*)^{-1}a$, we have $b + b^* = a$. This completes the proof of Lemma 6.3. □

Let r, r' and r'' be arbitrary positive integers again. Then, by Lemma 6.3, we have the following:

Corollary 6.4.

(1) *We have*

$$|\mathfrak{u}_r \cap \mathrm{M}_r(\mathcal{O})| = \begin{cases} |2|^{\frac{1}{4}r(r+1)} q^{-\frac{1}{2}r(r+1)} & (\epsilon = 1), \\ |2|^{\frac{1}{4}r(r+1)} q^{-\frac{1}{2}r(r-1)} & (\epsilon = -1). \end{cases}$$

(2) *We have* $|\,\mathrm{M}_{r',r''}(\mathcal{O}_D)| = q^{-r'r''}$.

7. Doubling method and local γ-factors

In this section, we explain the doubling method, and we recall the analytic definition of the local standard γ-factor (§7.2). The doubling method also appears in the formulation of the local Siegel-Weil formula (§10) and the local analogue of the Rallis inner product formula (§15). Let W be a $(-\epsilon)$-Hermitian space over D. In this section, we also define the local zeta value $\alpha_1(W)$, which depends on W and its basis $\underline{e}$. In §7.3, we compute $\alpha_1(W)$ for a $(-\epsilon)$-Hermitian space and for a basis $\underline{e}$ for W under some assumptions. As explained in §1, this computation of the constant $\alpha_1(W)$ will play an important role in the computation of the constant in the local Siegel-Weil formula (§10).

7.1. **Doubling method.** Let $(W^\square, \langle\ ,\ \rangle^\square)$ be the pair where $W^\square = W \oplus W$ and $\langle\ ,\ \rangle^\square$ is the map $W^\square \times W^\square \to D$ defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle^\square = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle$$

for $x_1, x_2, y_1, y_2 \in W$. Let $G(W^\square)$ be the isometric group of $W^\square$. Then, the natural action

$$G(W) \times G(W) \curvearrowright W \oplus W \colon (x_1, x_2) \cdot (g_1, g_2) = (x_1 \cdot g_1, x_2 \cdot g_2)$$

induces an embedding $\iota \colon G(W) \times G(W) \to G(W^\square)$. Consider maximal totally isotropic subspaces

$$W^\triangle = \{(x, x) \in W^\square \mid x \in W\}, \text{ and}$$
$$W^\triangledown = \{(x, -x) \in W^\square \mid x \in W\}.$$

Then we have a polar decomposition $W^\square = W^\triangle \oplus W^\triangledown$. We denote by $P(W^\triangle)$ the maximal parabolic subgroup of $G(W^\square)$ which preserves $W^\triangle$. Then, a Levi subgroup of $P(W^\triangle)$ is isomorphic to $\mathrm{GL}(W^\triangle)$. We denote by Δ the character of $P(W^\triangle)$ given by

$$\Delta(x) = N_{W^\triangle}(x)^{-1}.$$

Here $N_{W^\triangle}(x)$ is the reduced norm of the image of x in $\mathrm{End}_D(W^\triangle)$. Let $\omega \colon F^\times \to \mathbb{C}^\times$ be a character. For $s \in \mathbb{C}$, put $\omega_s = \omega \cdot |-|^s$. Let $\underline{e}$ be a basis for W. Then we define a basis $\underline{e}'^{\square} = (e'_1, \ldots, e'_{2n})$ for $W^{\square}$ by

$$e'_i = (e_i, e_i),\ e'_{n+i} = \sum_{k=1}^{n} a_{jk}(e_i, -e_i)$$

for $i = 1, \ldots, n$, where $(a_{jk})_{j,k} = R(\underline{e})^{-1}$. Then we have

$$(\langle e'_i, e'_j \rangle)_{i,j} = \begin{pmatrix} 0 & 2 \cdot I_n \\ -2\epsilon \cdot I_n & 0 \end{pmatrix}.$$

We choose a maximal compact subgroup $K(\underline{e}'^{\square})$ of $G(W^{\square})$ which preserves the lattice

$$\mathcal{O}_{W^{\square}} = \sum_{i=1}^{2n} \mathcal{O}_D e'_i$$

of $W^{\square}$. Then, we have $P(W^{\triangle})K(\underline{e}'^{\square}) = G(W^{\square})$. Denote by $I(s, \omega)$ the degenerate principal series representation

$$\mathrm{Ind}_{P(W^{\triangle})}^{G(W^{\square})}(\omega_s \circ \Delta)$$

consisting of the smooth right $K(\underline{e}'^{\square})$-finite functions $f \colon G(W^{\square}) \to \mathbb{C}$ satisfying

$$f(pg) = \delta_{P(W^{\triangle})}^{\frac{1}{2}}(p) \cdot \omega_s(\Delta(p)) \cdot f(g)$$

for $p \in P(W^{\triangle})$ and $g \in G(W^{\square})$, where $\delta_{P(W^{\triangle})}$ is the modular function of $P(W^{\triangle})$. We may extend $|\Delta|$ to a right $K(\underline{e}'^{\square})$-invariant function on $G(W^{\square})$ uniquely. We denote by $U(W^{\triangle})$ the unipotent radicals of $P(W^{\triangle})$. For $f \in I(0, \omega)$, put $f_s = f \cdot |\Delta|^s \in I(s, \omega)$. Then, we define an intertwining operator $M(s, \omega) \colon I(s, \omega) \to I(-s, \omega^{-1})$ by

$$[M(s, \omega) f_s](g) = \int_{U(W^{\triangle})} f_s(\tau u g)\, du,$$

where τ is the Weyl element of $G(W^{\square})$ given by

$$\begin{cases} \tau(e'_i) = e'_{n+i} & (i = 1, \ldots, n), \\ \tau(e'_i) = -\epsilon e'_{i-n} & (i = n+1, \ldots, 2n). \end{cases}$$

This integral converges absolutely for $\Re s > 0$ and admits a meromorphic continuation to $\mathbb{C}$. Let π be a representation of $G(W)$ of finite length. For a matrix coefficient ξ of π, and for $f \in I(0, \omega)$, we define the doubling zeta integral by

$$Z^W(f_s, \xi) = \int_{G(W)} f_s(\iota(g, 1)) \xi(g)\, dg.$$

Then the zeta integral satisfies the following properties, which are stated in [Yam14, Theorem 4.1]. This gives a generalization of [LR05, Theorem 3].

Proposition 7.1.

(1) *The integral $Z^W(f_s, \xi)$ converges absolutely for $\Re s \geq n - \epsilon$ and has an analytic continuation to a rational function of q^{-s}.*

(2) *There is a meromorphic function $\Gamma^W(s, \pi, \omega)$ such that*

$$Z^W(M(s, \omega) f_s, \xi) = \Gamma^W(s, \pi, \omega) Z^W(f_s, \xi)$$

for all matrix coefficients ξ of π and $f_s \in I(s, \omega)$.

7.2. **Local γ-factors.** Fix a non-trivial additive character $\psi\colon F \to \mathbb{C}^\times$ and $A \in \mathrm{End}_D(W^\square)$ so that $\operatorname{rank} A = n$ and $1 + A \in U(W^\triangle)$. We use the Haar measures on $U(W^\triangledown)$ and $U(W^\triangle)$ by identifying them with $\mathfrak{u}_n$ by the basis $\underline{e'}^\square$ (see §6.2). We define

$$\psi_A\colon U(W^\triangledown) \to \mathbb{C}^\times\colon u \mapsto \psi(\mathrm{T}_{W^\square}(uA)),$$

where $\mathrm{T}_{W^\square}$ denotes the reduced trace of $\mathrm{End}_D(V^\square)$. Moreover, we define the character χ_A of $F^\times$ by $\chi_A(x) = (x, \mathfrak{d}(A))$ for $x \in F^\times$ where $\mathfrak{d}(A)$ denotes the element of $F^\times/{F^\times}^2$ defined as in [Kak20, §5.1]. For $f \in I(0, \omega)$ we define

$$l_{\psi_A}(f_s) = \int_{U(W^\triangledown)} f_s(u)\psi_A(u)\, du.$$

Then, this integral defining l_{ψ_A} converges for $\Re s \gg 0$ and admits a holomorphic continuation to $\mathbb{C}$ [Kar79, §3.2]. Let $A_0 \in \mathrm{GL}_n(D)$ be the matrix representation of the linear map $A\colon W^\triangledown \to W^\triangle$ with respect to the bases $e'_{n+1}, \ldots, e'_{2n}$ for $W^\triangledown$ and $e'_1, \ldots, e'_n$ for $W^\triangle$. We denote by $e(G(W))$ the Kottwitz sign of $G(W)$, which is given by

$$e(G(W)) = \begin{cases} (-1)^{\frac{1}{2}n(n+1)} & (-\epsilon = 1), \\ (-1)^{\frac{1}{2}n(n-1)} & (-\epsilon = -1). \end{cases}$$

Then, as in [Kak20, Proposition 4.2], we have the following:

Proposition 7.2. *We have*

$$l_{\psi_A} \circ M(s, \omega) = c(s, \omega, A, \psi) \cdot l_{\psi_A},$$

where $c(s, \omega, A, \psi)$ is the meromorphic function of s given by

$$\begin{aligned} c(s, \omega, A, \psi) = {} & e(G(W)) \cdot \omega_s(N(A_0))^{-1} \cdot |2|^{-2ns+n(n-\frac{1}{2})} \cdot \omega^{-1}(4) \cdot \gamma(s - n + \frac{1}{2}, \omega, \psi)^{-1} \\ & \times \prod_{i=0}^{n-1} \gamma(2s - 2i, \omega^2, \psi)^{-1} \cdot \gamma(s + \frac{1}{2}, \omega\chi_{A_0}, \psi) \cdot \epsilon(\frac{1}{2}, \chi_{A_0}, \psi)^{-1} \end{aligned}$$

in the case $-\epsilon = 1$, and

$$c(s, \omega, A, \psi) = e(G(W)) \cdot \omega_s(N(A_0))^{-1} \cdot |2|^{-2ns+n(n-\frac{1}{2})} \cdot \omega^{-1}(4) \cdot \prod_{i=0}^{n-1} \gamma(2s - 2i, \omega^2, \psi)^{-1}$$

in the case $-\epsilon = -1$.

Remark 7.3. These formulas differ from those in [Kak20, Proposition 4.2]. This is caused by a typo where $\omega_{n\pm\frac{1}{2}}(N(R))$ should be replaced by $|N(R)|^{-(n\pm\frac{1}{2})}$ in [Kak20, Proposition 4.2].

Now we define the doubling γ-factor as in [Kak20]. Note that the above error has no effect on the definition in [Kak20].

Definition 7.4. Let π be an irreducible representation of $G(W)$, let ω be a character of $F^\times$, and let ψ be a non-trivial character of F. Then we define the γ-factor by

$$\gamma^W(s + \frac{1}{2}, \pi \times \omega, \psi) = c(s, \omega, A, \psi)^{-1} \cdot \Gamma^W(s, \pi, \omega) \cdot c_\pi(-1) \cdot R(s, \omega, A, \psi),$$

where c_π is the central character of π, and

$$R(s,\omega,A,\psi)=\begin{cases}\omega_s(N(R(\underline{e})A_0)^{-1}\gamma(s+\frac{1}{2},\omega\chi_A,\psi)\epsilon(\frac{1}{2},\chi_A,\psi)^{-1} & (-\epsilon=1),\\ \omega_s(N(R(\underline{e})A_0)^{-1}\epsilon(\frac{1}{2},\chi_W,\psi) & (-\epsilon=-1).\end{cases}$$

The doubling γ-factor $\gamma^W(s+\frac{1}{2},\pi\boxtimes\omega,\psi)$ is expected to coincide with the standard γ-factor $\gamma(s+\frac{1}{2},\pi\boxtimes\omega,\mathrm{std},\psi)$ where std is the standard embedding of ${}^L(G(W)\times\mathrm{GL}_1)$. Another notable property is the commutativity with parabolic inductions, which is useful in the computation. For example, the doubling γ-factor of the trivial representation is given by Lemma 7.5, which we use in the computation of the doubling zeta integral (§7.3 and Appendix A).

Lemma 7.5. *Denote by $1_{G(W)}$ the trivial representation of $G(W)$. Then we have*

$$\begin{aligned}&\gamma^W(s+\frac{1}{2},1_{G(W)}\times 1_{F^\times},\psi)\\ &\quad=\begin{cases}\prod_{i=-n}^{n}\gamma_F(s+\frac{1}{2}+i,1,\psi) & (-\epsilon=1),\\ \gamma_F(s+\frac{1}{2},\chi_W,\psi)\prod_{i=-n+1}^{n-1}\gamma_F(s+\frac{1}{2}+i,1,\psi) & (-\epsilon=-1).\end{cases}\end{aligned}$$

Proof. [Kak20, Proposition 7.1]. □

7.3. **Local zeta values.** We use the same setting and notation of §7.1. Let $f_s^\circ\in I(s,1_{F^\times})$ be the unique $K(\underline{e}'^\square)$-fixed section with $f_s^\circ(1)=1$, and let ξ° be the matrix coefficient of the trivial representation of $G(W)$ with $\xi^\circ(1)=1$. Put $\rho=n-\frac{\epsilon}{2}$. Then, we define

$$\alpha_1(W)\coloneqq Z^W(f_\rho^\circ,\xi^\circ),$$

which is the first constant we are interested in. The integral defining $\alpha_1(W)$ converges absolutely by Proposition 7.1. The purpose of this subsection is to obtain a formula of $\alpha_1(W)$ in the case where either $R(\underline{e})\in\mathrm{GL}_n(\mathcal{O}_D)$ or W is anisotropic. The general formula of $\alpha_1(W)$ will be obtained in §19.

Proposition 7.6.

(1) *In the case $-\epsilon=1$ and $R(\underline{e})\in\mathrm{GL}_n(\mathcal{O}_D)$, we have*

$$\alpha_1(W)=|2|^{n(2n+1)}\cdot q^{-n_0^2-(2n_0+1)r-2r^2}\cdot\prod_{i=1}^{n}(1+q^{-(2i-1)}).$$

(2) *In the case $-\epsilon=-1$ and $R(\underline{e})\in\mathrm{GL}_n(\mathcal{O}_D)$, we have*

$$\alpha_1(W)=|2|^{n(2n-1)}\cdot q^{-2rn_0-2r^2+r}\cdot\prod_{i=1}^{n}(1+q^{-(2i-1)}).$$

(3) *In the case $-\epsilon=-1$ and W is anisotropic, we have*

$$\alpha_1(W)=|N(R(\underline{e}))|^{-n+\frac{1}{2}}\times\begin{cases}|2|_F\cdot(1+q^{-1}) & (n=1),\\ |2|_F^{6}\cdot q^{-1}\cdot(1+q^{-1})(1+q^{-3}) & (n=2),\\ |2|_F^{15}\cdot q^{-3}\cdot(1+q^{-1})(1+q^{-3})(1+q^{-5}) & (n=3).\end{cases}$$

Unless q is a power of 2 , the assertions (1) and (2) are conclusions of [Kak20, Proposition 8.3] and the volume formula of a maximal compact subgroup containing $\mathcal{B}$ which can be obtained by a generalization of the Bruhat decomposition [PR08, Appendix, Proposition 8]. However, to contain the case $2|q$, we prove them in another way. Before proving Proposition 7.6, we observe the following two important lemmas:

Lemma 7.7.

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(I(\rho, 1_{F^\times}), \mathbb{C}) = 1.$$

Proof. By Proposition 7.1, the integral

$$\int_{G(W)} f((g,1))\,dg$$

converges absolutely for $f \in I(\rho, 1_{F^\times})$. We denote it by $Z(f)$, and we obtain a non-zero map $Z \in \mathrm{Hom}_{G(W)\times G(W)}(I(\rho, 1_{F^\times}), \mathbb{C})$. To prove the lemma, it suffices to show that $\ker Z$ is spanned by the set

$$\{h - R(g)h \mid h \in I(\rho, 1_{F^\times}), g \in G(W)\times G(W)\}.$$

Here, we denote by $R(g)$ the right translation by g. Let $f \in \ker Z$. Take a compact open subgroup K' of $K(\underline{e}'^{\square})$, complex numbers $a_i \in \mathbb{C}$ and elements $g_i \in G(W) \times G(W)$ for $i = 1, \ldots, t$ so that

$$f = \sum_{i=1}^{t} a_i R(g_i)\mathfrak{c},$$

where $\mathfrak{c} \in I(\rho, 1_{F^\times})$ is the section defined by

$$\mathfrak{c}(g) \coloneqq \begin{cases} \delta_{P(W^\triangle)}(p) & g = pk' \ (p \in P(W^\triangle), k' \in K'), \\ 0 & g \notin P(W^\triangle)K'. \end{cases}$$

Then, we have

$$a_1 + \cdots + a_t = \frac{Z(f)}{Z(\mathfrak{c})} = 0$$

and we have

$$\sum_{i=1}^{t-1} b_i(R(g_i)\mathfrak{c} - R(g_{i+1})\mathfrak{c}) = f,$$

where $b_i \coloneqq a_1 + \cdots + a_i$ for $i = 1, \ldots, t-1$. Hence we have the lemma. $\square$

Lemma 7.8. *For $f \in I(\rho, 1_{F^\times})$, we have*

$$\int_{G(W)} f((g,1))\,dg = m^\circ(\rho)^{-1} \cdot \alpha_1(W) \cdot \int_{U(W^\triangle)} f(\tau u)\,du,$$

where

$$m^\circ(s) = \begin{cases} |2|_F^{n(n-\frac{1}{2})} q^{-\frac{1}{2}n(n+1)} \frac{\zeta_F(s-n+\frac{1}{2})}{\zeta_F(s+n+\frac{1}{2})} \prod_{i=0}^{n-1} \frac{\zeta_F(2s-2i)}{\zeta_F(2s+2n-4i-3)} & (-\epsilon = 1), \\ |2|_F^{n(n-\frac{1}{2})} q^{-\frac{1}{2}n(n-1)} \prod_{i=0}^{n-1} \frac{\zeta_F(2s-2i)}{\zeta_F(2s+2n-4i-1)} & (-\epsilon = -1). \end{cases}$$

Proof. Define a map $\mathfrak{A}\colon \mathcal{S}(G(W^\square)) \to I(\rho, 1_{F^\times})$ by

$$[\mathfrak{A}\varphi](g) = \int_{P(W^\triangle)} \delta_{P(W^\triangle)}(p)^{-1}\varphi(pg)\,dp.$$

Then $\mathfrak{A}$ is surjective. Moreover, we have

$$\begin{aligned}\int_{U(W^\triangle)} [\mathfrak{A}\varphi](\tau u)\,du &= \int_{U(W^\triangle)}\int_{M(W^\triangle)}\int_{U(W^\triangle)} \delta_{P(W^\triangle)}^{-1}(m)\varphi(xm\tau y)\,dydmdx \\ &= \gamma(G(W^\square)/P(W^\triangle))\int_{G(W^\square)} \varphi(g)\,dg.\end{aligned}$$

Here, $\gamma(G(W^\square)/P(W^\triangle))$ is the constant defined by

$$\gamma(G(W^\square)/P(W^\triangle)) = \int_{U(W^\triangle)} f^\circ(\tau u)\,du,$$

where $f^\circ \in I(\rho, 1_{F^\times})$ is a unique $K(\underline{e}'^\square)$-invariant section with $f^\circ(1) = 1$. Hence we conclude that the map

$$I(\rho, 1_{F^\times}) \to \mathbb{C}\colon f \mapsto \int_{U(W^\triangle)} f(\tau u)\,du$$

is $G(W^\square)$-invariant, in particular, it is $G(W)\times G(W)$-invariant. Hence, by Lemma 7.7, we conclude that there is a constant $\alpha' \in \mathbb{C}$ such that

$$\int_{G(W)} f((g,1))\,dg = \alpha' \int_{U(W^\triangle)} f(\tau u)\,du$$

for all $f \in I(\rho, 1_{F^\times})$. To determine the constant α', we use f° as a test function. By Gindikin-Karperevich formula [Cas80, Theorem 3.1] or Shimura's computation [Shi99, Proposition 3.5], we have

$$\int_{U(W^\triangle)} f^\circ(\tau u)\,du = m^\circ(\rho).$$

Moreover, comparing this to Proposition A.2, we have the claim. $\square$

Now we prove Proposition 7.6. As a consequence of Lemma 7.8, we use another section $f(s, 1_{\varpi_F\mathcal{O}_\mathfrak{u}}, -) \in I(s,1)$ to compute the ratio $\alpha_1(W)m^\circ(\rho)^{-1}$. Here, we denote the set $\mathfrak{u}\cap \mathrm{M}_n(\mathcal{O})$ by $\mathcal{O}_\mathfrak{u}$, and we define a section $f(s,\Phi,-) \in I(s,\omega)$ by

$$f(s,\Phi,g) := \begin{cases} 0 & g \notin P(W^\triangle)\tau U(W^\triangle), \\ \omega_{s+\rho}(\Delta(p))\Phi(X) & g = p\tau\begin{pmatrix}1 & 0\\ X & 1\end{pmatrix}\ (p\in P(W^\triangle),\ X\in\mathfrak{u})\end{cases}$$

for a character ω of $F^\times$ and $\Phi \in \mathcal{S}(\mathfrak{u})$. Let $g \in G(W)$ with $\iota(g) \in P(W^\triangle)\tau U(W^\triangle)$. Then,

$$\begin{pmatrix}\frac{g+1}{2} & \frac{g-1}{2}R(\underline{e})^{-1}\\ R(\underline{e})\frac{g-1}{2} & R(\underline{e})\frac{g+1}{2}R(\underline{e})^{-1}\end{pmatrix} = \begin{pmatrix}a & 0\\ b & {}^t a^{*-1}\end{pmatrix}\tau\begin{pmatrix}1 & 0\\ X & 1\end{pmatrix}$$

for some $a \in \mathrm{GL}_n(D), b\in \mathrm{M}_n(D)$ and $X \in \mathfrak{u}$. If $X \in \varpi_F\mathcal{O}_\mathfrak{u}$, then a, g are given by

$$a = (X - R(\underline{e}))^{-1},\ g = a(X + R(\underline{e})) = 2aX - 1,$$

and thus $a \in \mathrm{GL}_n(\mathcal{O}_D)$ and $g \in -K^+_{2\varpi_F}$. Here we denote the set

$$\{g \in G(W)\cap \mathrm{GL}_n(\mathcal{O}_D) \mid g-1 \in 2\varpi_F\,\mathrm{M}_n(\mathcal{O}_D)\}$$

by $K^+_{2\varpi_F}$. Conversely, if $g \in -K^+_{2\varpi_F}$, then a, X are given by

$$aI_n = \frac{g-1}{2} R(\underline{e})^{-1},\ aX = \frac{g+1}{2},$$

and thus $a \in \mathrm{GL}_n(\mathcal{O}_D)$ and $X \in \varpi_F \mathcal{O}_{\mathfrak{u}}$. Summarizing the above discussions, we have

$$f(s, 1_{\varpi_F \mathfrak{u}}, \iota(g,1)) = 1_{-K^+_{2\varpi_F}}(g)$$

for $g \in G(W)$. Put

$$m'(s) := \int_{U(W^\triangle)} f(s, 1_{2\varpi_F \mathcal{O}_{\mathfrak{u}}}, \tau u)\, du.$$

Then, we have

$$\begin{aligned}\frac{\alpha_1(W)}{m^\circ(\rho)} &= \frac{Z(f(\rho, 1_{2\varpi_F \mathcal{O}_{\mathfrak{u}}}, -))}{m'(\rho)} \\ &= \frac{|K^+_{2\varpi_F}|}{|\varpi_F \mathcal{O}_{\mathfrak{u}}|} \\ &= |2|_F^{2n\rho - n(n-\frac{1}{2})} q^{\frac{1}{2}n(n-\epsilon)} q^{n(2n-\epsilon)} |K^+_{\varpi_F}|.\end{aligned}$$

Since

$$\log_q [\mathcal{B}^+ : K^+_{\varpi_F}] = 6(n_0 r + r(r-1)) + 5r + n_0 - (2r + n_0)\epsilon$$

and

$$\log_q |\mathcal{B}^+| = \begin{cases} -n^2 - n & (-\epsilon = 1), \\ -n^2 & (-\epsilon = -1), \end{cases}$$

we have

$$\begin{aligned}&\log_q (q^{\frac{1}{2}n(n-\epsilon)} q^{n(2n-\epsilon)} |K^+_{\varpi_F}|) \\ &= \frac{1}{2} n(n-\epsilon) + n(2n-\epsilon) - 6(n_0 r + r(r-1)) - 5r - n_0 + (2r+n_0)\epsilon \\ &\quad - \begin{cases} -n^2 - n & (-\epsilon = 1), \\ -n^2 & (-\epsilon = -1) \end{cases} \\ &= \frac{1}{2} n(n-\epsilon) - \begin{cases} 2r^2 + (2n_0+1)r + n_0^2 & (-\epsilon = 1), \\ 2r^2 + 2n_0 r - r & (-\epsilon = -1). \end{cases}\end{aligned}$$

Hence we have

$$\begin{aligned}\alpha_1(W) &= m^\circ(\rho) \cdot \frac{\alpha_1(W)}{m^\circ(\rho)} \\ &= \begin{cases} |2|^{n(2n+1)} \cdot q^{-n_0^2 - (2n_0+1)r - 2r^2} \cdot \prod_{i=1}^n (1 + q^{-(2i-1)}) & (-\epsilon = 1), \\ |2|^{n(2n-1)} \cdot q^{-2rn_0 - 2r^2 + r} \cdot \prod_{i=1}^n (1 + q^{-(2i-1)}) & (-\epsilon = -1). \end{cases}\end{aligned}$$

This proves (1) and (2) of Proposition 7.6.

Finally, we prove (3). By the definition of the γ-factor, we have the following (local) functional equation of the zeta integral:

$$\begin{aligned}\frac{Z^W(f_s^\circ, \xi^\circ)}{m^\circ(s)} =& e(G(W)) \frac{Z^W(f_{-s}^\circ, \xi^\circ)}{\gamma^W(s + \frac{1}{2}, 1_{G(W)} \times 1_{F^\times}, \psi)} \prod_{i=0}^{n-1} \gamma(2s - 2i, 1_{F^\times}, \psi) \\ & \times |2|_F^{2ns - n(n-\frac{1}{2})} |N(R(\underline{e}))|_F^{-s} \cdot \epsilon(\frac{1}{2}, \chi_W, \psi).\end{aligned}$$

Since $f^{\circ}_{-\rho}$ is the constant function with value 1 on $G(W^{\square})$, we have $Z^W(f^{\circ}_{\rho}, \xi^{\circ}) = |G(W)|$. Hence, by Lemma 7.5, we have

$$\frac{Z^W(f^{\circ}_{-\rho}, \xi^{\circ})}{m^{\circ}(\rho)} = |N(R(\underline{e}))|^{-n+\frac{1}{2}} \times \begin{cases} |2|_F^{\frac{1}{2}} \cdot e(G(W)) & (n=1), \\ -|2|_F^{3} \cdot e(G(W)) & (n=2), \\ -|2|_F^{\frac{15}{2}} \cdot e(G(W)) & (n=3). \end{cases}$$

Therefore, we have

$$\alpha_1(W) = |N(R(\underline{e}))|^{-n+\frac{1}{2}} \times \begin{cases} |2|_F \cdot (1+q^{-1}) & (n=1), \\ |2|_F^{6} \cdot q^{-1} \cdot (1+q^{-1})(1+q^{-3}) & (n=2), \\ |2|_F^{15} \cdot q^{-3} \cdot (1+q^{-1})(1+q^{-3})(1+q^{-5}) & (n=3). \end{cases}$$

Thus, we complete the proof of Proposition 7.6.

8. Local Weil representations

In this paper, we consider the two reductive dual pairs: $(G(V), G(W^{\square}))$ and $(G(V), G(W))$. Here, we use the word "reductive dual pair" in the sense of [GS12]. (See [GS12, Remarks (a)] for the discussion for this.) The purpose of this section is to describe the Schrödinger models of the local Weil representations on $(G(V), G(W^{\square}))$ and $(G(V), G(W))$.

8.1. **The metaplectic group and the Weil representation.** First, we recall the definition of the Metaplectic group and the Schrödinger model of the Weil representation. Let U be a symplectic space over F, let $\langle\ ,\ \rangle_U$ be the symplectic form on U, and let K, L be maximal totally isotropic subspaces so that $U = K + L$. We fix a non-trivial additive character $\psi\colon F \to \mathbb{C}^1$. We denote by $r_{\psi,L}$ the Segal-Shale-Weil projective representation which is given by

$$[r_{\psi,L}(g)\phi](x) = \int_{\mathcal{Y}_c} \psi(\frac{1}{2}\langle xa, xb\rangle_U + \langle yc, xb\rangle_U + \frac{1}{2}\langle yc, yd\rangle_U)\phi(xa+yc)\, d\mu_g(y)$$

for $\phi \in \mathcal{S}(K)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(U)$. Here, we consider a basis $(u_1, \ldots, u_{2t})$ with $u_1, \ldots, u_t \in L$ and $u_{t+1}, \ldots, u_{2t} \in K$ to give the matrix representation of g, we denote by $\mathcal{Y}_c$ the quotient $\ker(c) \cap L \backslash L$, and we denote by $\mu_g(y)$ the Haar measure on $\mathcal{Y}_c$ so that $r_L(g)$ keeps the L^2-norm of $\mathcal{S}(K)$. Then, there is a 2-cocycle $c_{\psi,L}\colon \mathrm{Sp}(U) \times \mathrm{Sp}(U) \to \mathbb{C}^1$ so that

$$r_{\psi,L}(g_1) r_{\psi,L}(g_2) = c_{\psi,L}(g_1, g_2) r_{\psi,L}(g_1 g_2)$$

for $g_1, g_2 \in \mathrm{Sp}(U)$. For the discussion of the definition, see [RR93, Theorem 3.5]. The explicit formula of $c_{\psi,L}$ has been already established ([Per81], [RR93]), but we do not discuss it. By $\mathrm{Mp}(U, c_{\psi,L})$ we mean the group $\mathrm{Sp}(U) \times \mathbb{C}^1$ together with the binary operation

$$(g_1, z_1) \cdot (g_2, z_2) = (g_1 g_2, z_1 z_2 c_{\psi,L}(g_1, g_2))$$

for $g_1, g_2 \in \mathrm{Sp}(U)$ and $z_1, z_2 \in \mathbb{C}^1$, and we call it the metaplectic group associated to c_L. Then, the Weil representation $\omega_{\psi,L}$ of $\mathrm{Mp}(U, c_{\psi,L})$ is realized on the space $\mathcal{S}(K)$ of Schwartz-Bruhat functions on K by

$$[\omega_{\psi,L}(g, z)\phi](x) = z \cdot [r_{\psi,L}(g)\phi](x)$$

for $g \in \mathrm{Sp}(U)$ and $z \in \mathbb{C}^1$.

Note that the Segal-Shale-Weil projective representation $r_{\psi,L}$ and the 2-cocycle $c_{\psi,L}$ depend on the symplectic form $\langle\ ,\ \rangle$: if we consider the symplectic form $-\langle\ ,\ \rangle_U$ instead of $\langle\ ,\ \rangle_U$, then the associated Segal-Shele-Weil representation and 2-cocycle are the unitary dual $\overline{r_{\psi,L}}$ of $r_{\psi,L}$ and complex conjugation $\overline{c_{\psi,L}}$ of $c_{\psi,L}$ respectively. We denote by $\overline{\omega_{\psi,L}}$ the Weil representation of $\mathrm{Mp}(U, \overline{c_{\psi,L}})$ induced by $\overline{r_{\psi,L}}$.

8.2. **For the pair** $(G(V), G(W^\square))$**.** In this subsection, we recall the explicit definition of Weil representation for the reductive dual pair $(G(V), G(W^\square))$, which is given by Kudla [Kud94].

We fix a basis $\underline{e}$ for W, and we take a basis $\underline{e}'^{\square}$ of $W^\square$ as in §7.1. In this subsection, we identify $G(W)$ (resp. $G(W^\square)$) with a subgroup of $\mathrm{GL}_n(D)$ (resp. $\mathrm{GL}_{2n}(D)$) by the basis $\underline{e}$ (resp. $\underline{e}'^{\square}$). Moreover, we identify $G(V)$ with a subgroup of $\mathrm{GL}_m(D)$ by some fixed basis of V. Let $\mathbb{W}^\square = V \otimes_D W^\square$, and let $\langle\langle\ ,\ \rangle\rangle^\square$ be the pairing on $\mathbb{W}^\square$ defined by

$$\langle\langle x \otimes (y_1, y_2), x' \otimes (y_1', y_2')\rangle\rangle^\square = T_D((x, x') \cdot (\langle y_1, y_1'\rangle^* - \langle y_2, y_2'\rangle^*))$$

for $x_1, x_2 \in V$ and $y_1, y_2, y_1', y_2' \in W$. Then, $(G(V), G(W^\square))$ is a reductive dual pair in $\mathrm{Sp}(\mathbb{W}^\square)$. We consider a polar decomposition $\mathbb{W}^\square = (V \otimes W^\nabla) \oplus (V \otimes W^\triangle)$. Then we denote by $r_{\psi, V\otimes W^\triangle}$ the Segal-Shele-Weil representation of $\mathrm{Sp}(\mathbb{W}^\square)$ with respect to the symplectic form $\frac{1}{2}\langle\langle\ ,\ \rangle\rangle^\square$. Kudla defined a function $\beta_V\colon G(W^\square) \to \mathbb{C}^1$ [Kud94, p. 378], and gave an explicit embedding

$$\widetilde{j^\square}\colon G(V) \times G(W^\square) \to \mathrm{Mp}(\mathbb{W}^\square, c_{\psi, V\otimes W^\triangle})$$

by $\widetilde{j^\square}(h, g) = (h^{-1} \otimes g, \beta_V(g))$. From now on, we denote by $\omega_\psi^\square$ the pull-back $\widetilde{j^\square}^* \omega_\psi^\square$ of the Weil representation $\omega_{\psi, V\otimes W^\triangle}$ of $\mathrm{Mp}(\mathbb{W}^\square, c_{\psi, V\otimes W^\triangle})$. We will describe it explicitly following Kudla [Kud94, p. 400]. Recall that τ denotes the certain Weyl element (for the definition, see §7.1). Moreover, for $a \in \mathrm{GL}(W^\triangle)$, we denote by $m(a)$ the unique element of $G(W^\square)$ such that $m(a)|_{W^\triangle} = a$. Then, we have $\beta_V(b) = 1$ for $b \in U(W^\triangle)$, $\beta_V(m(a)) = \chi_V(N(a))$ for $a \in \mathrm{GL}(W^\nabla)$, and $\beta_V(\tau) = (-1)^{mn}\chi_V(-1)^n$. Here, we denote by N the reduced norm of $\mathrm{End}_D(W^\nabla)$ over F. Thus, we have the following:

Proposition 8.1. *Let $\phi \in \mathcal{S}(V \otimes W^\nabla)$. Then, $\omega_\psi^\square(h, g)\phi = \beta_V(g) r(g)(\phi \circ h^{-1})$. More precisely,*

- $[\omega_\psi^\square(h, 1)\phi](x) = \phi(h^{-1}x)$ *for* $h \in G(V)$,
- $[\omega_\psi^\square(1, m(a))\phi](x) = \chi_V(N(a))|N(a)|^{-m}\phi(x \cdot {}^t a^{*-1})$ *for* $a \in \mathrm{GL}(W^\triangle)$,
- $[\omega_\psi^\square(1, b)\phi](x) = \psi(\frac{1}{4}\langle\langle x, x \cdot b\rangle\rangle^\square)\phi(x)$ *for* $b \in U(W^\triangle)$,
- *the action of τ is given by*

$$[\omega_\psi^\square(1, \tau)\phi](x) = \beta_V(\tau) \cdot \int_{V\otimes W^\nabla} \psi(\frac{1}{2}\langle\langle y, x\tau\rangle\rangle^\square)\phi(y)\, dy,$$

where dy is the self-dual measure of $V \otimes W^\nabla$ with respect to the pairing

$$V \otimes W^\nabla \times V \otimes W^\nabla \to \mathbb{C}\colon x, y \mapsto \psi(\frac{1}{2}\langle\langle y, x\tau\rangle\rangle^\square).$$

8.3. **For the pair** $(G(V), G(W))$**.** Now we consider the dual pair $(G(V), G(W))$. In this case, the splitting of metaplectic cover is defined via the "doubled" pair $(G(V), G(W^\square))$. Let $\mathbb{W} = V \otimes_D W$, and let $\langle\langle\ ,\ \rangle\rangle$ be the pairing on $\mathbb{W}$ defined by

$$\langle\langle x \otimes y, x' \otimes y'\rangle\rangle = T_D((x, x') \cdot \langle y, y'\rangle^*)$$

for $x, x' \in V$ and $y, y' \in W$. Fix a polar decomposition $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ where $\mathbb{X}$ and $\mathbb{Y}$ are certain totally isotropic subspaces. We denote by $r_{\mathbb{Y}}$ the Segal-Shele-Weil representation of $\mathrm{Sp}(\mathbb{W})$ with respect to the symplectic form $\frac{1}{2}\langle\langle\ ,\ \rangle\rangle$. Since $\mathbb{Y}^\square$ is a totally isotropic subspace of $\mathbb{W}^\square$, there is $\alpha \in \mathrm{Sp}(\mathbb{W}^\square)$ so that $\mathbb{Y}^\square \cdot \alpha = V \otimes W^\triangle$. Put

$$\lambda(g) = c_{\mathbb{Y}^\square}(\alpha, g\alpha^{-1}) \cdot c_{\mathbb{Y}^\square}(g, \alpha^{-1})$$

for $g \in \mathrm{Sp}(\mathbb{W}^\square)$, whose coboundary realizes the ratio of the 2-cocycles $c_{\mathbb{Y}^\square}(-,-)$ and $c_{V\otimes W^\triangle}(-,-)$ [Kud94, (4.4)]. Then we define the function $\beta_{\mathbb{Y}}^V : G(W) \to \mathbb{C}^1$ by $\beta_{\mathbb{Y}}^V(g) = \lambda(1\otimes g)^{-1}\beta_V(g)$ for $g \in G(W)$. Then, the map

$$g \mapsto (1 \otimes g, \lambda(1\otimes g)^{-1}\beta_V(g))$$

defines the embedding $G(W^\square) \to \mathrm{Mp}(\mathbb{W}^\square, c_{\mathbb{Y}^\square})$. We also define $\beta_{\mathbb{Y}}^W : G(V) \to \mathbb{C}^1$ by the same way using the doubled space $V^\square$ of V. Then, we define the embedding

$$\widetilde{j} \colon G(V) \times G(W) \to \mathrm{Sp}(\mathbb{W})$$

by

$$\widetilde{j}(h, g) = (h^{-1} \otimes g, \beta_{\mathbb{Y}}^W(h^{-1})\beta_{\mathbb{Y}}^V(g)c_{\mathbb{Y}}(h^{-1}\otimes 1, 1\otimes g))$$

for $h \in G(V)$ and $g \in G(W)$. From now on, we denote by ω_ψ the pull-back $\widetilde{j}^*\omega_{\psi,\mathbb{Y}}$. An important property of ω_ψ is the relation with $\omega_\psi^\square$. We fix Haar measures dx and dy of $\mathbb{X}$ and $\mathbb{Y}$ so that they are dual of each other with respect to the pairing

$$\mathbb{X} \times \mathbb{Y} \to \mathbb{C}^\times \colon (x, y) \mapsto \psi(\langle\langle x, y\rangle\rangle).$$

Moreover, we define

$$\mathbb{X}^\triangle = (\mathbb{X} \oplus \mathbb{X}) \cap W^\triangle,\ \mathbb{X}^\nabla = (\mathbb{X} \oplus \mathbb{X}) \cap W^\nabla$$

and

$$\mathbb{Y}^\triangle = (\mathbb{Y} \oplus \mathbb{Y}) \cap W^\triangle,\ \mathbb{Y}^\nabla = (\mathbb{Y} \oplus \mathbb{Y}) \cap W^\nabla.$$

Then the vector space $V \otimes W^\nabla$ decomposes into the direct sum

$$\mathbb{X}^\nabla \oplus \mathbb{Y}^\nabla.$$

For $z \in V \otimes W^\nabla$, we denote by z_x (resp. z_y) the $\mathbb{X}^\nabla$-component (resp. the $\mathbb{Y}^\nabla$-component) of z. We define the Haar measure $dx^\triangle$ on $\mathbb{X}^\triangle$ by the push out measure $p_*(dx)$ where $p \colon \mathbb{X}^\triangle \to \mathbb{X}$ is the first projection. We define the Haar measures dx^∇, $dy^\triangle$, dy^∇ in the same way. Then, the map

$$\delta \colon \mathcal{S}(\mathbb{X}) \otimes \overline{\mathcal{S}(\mathbb{X})} = \mathcal{S}(\mathbb{X} \oplus \mathbb{X}) \to \mathcal{S}(V \otimes W^\nabla)$$

given by the partial Fourier transform

$$[\delta(\phi_1 \otimes \overline{\phi_2})](z) = \int_{\mathbb{X}^\triangle} (\phi_1 \otimes \overline{\phi_2})(x^\triangle + z_x) \cdot \psi(\frac{1}{2}\langle\langle x^\triangle, z\rangle\rangle)\, dx^\triangle$$

is known to be compatible with the embedding $\iota \colon G(W) \times G(W) \to G(W^\square)$. Hence, we have

$$F_{\delta(\phi_1\otimes\overline{\phi_2})}(\iota(g,1)) = (\omega_\psi(g)\phi_1, \phi_2)_{\mathbb{X}}$$

for $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{X})$ where $(\ ,\ )_{\mathbb{X}}$ is the L^2-inner product on $\mathbb{X}$ defined by the measure dx.

Finally, we prove the Plancherel formula for δ:

Proposition 8.2. *Let dz be the self-dual Haar measure on $V \otimes W^{\triangledown}$ with respect to the pairing*

$$V \otimes W^{\triangledown} \times V \otimes W^{\triangledown} \to \mathbb{C}^{\times} : (x, y) \mapsto \psi(\frac{1}{2}\langle\langle y, x\tau\rangle\rangle) \tag{8.1}$$

and let $(\ ,\ )$ be the L^2-inner product on $V \otimes W^{\triangledown}$ defined by dz. Then, we have

$$(\delta(\phi_1 \otimes \overline{\phi_2}), \delta(\phi_3 \otimes \overline{\phi_4})) = |2|_F^{-2mn} \cdot |N(R(\underline{e}))|^m \cdot (\phi_1, \phi_3)_{\mathbb{X}} \cdot \overline{(\phi_2, \phi_4)_{\mathbb{X}}}$$

for $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{S}(\mathbb{X})$.

Proof. First, one can prove that $dz = |N(R(\underline{e}))|^m \cdot dz_x^{\triangledown} \otimes dz_y^{\triangledown}$. Hence, we have

$$\begin{aligned}
&(\delta(\phi_1 \otimes \overline{\phi_2}), \delta(\phi_3 \otimes \overline{\phi_4}))\\
&= \int_{V\otimes W^{\triangledown}} \delta(\phi_1 \otimes \overline{\phi_2})(z) \cdot \overline{\delta(\phi_3 \otimes \overline{\phi_4})(z)}\, dz\\
&= |N(R(\underline{e}))|^m \int_{\mathbb{X}^{\triangledown}} \int_{\mathbb{Y}^{\triangledown}} \delta(\phi_1 \otimes \overline{\phi_2})(z_x + z_y) \cdot \overline{\delta(\phi_3 \otimes \overline{\phi_4})(z_x + z_y)}\, dz_x^{\triangledown} dz_y^{\triangledown}\\
&= |N(R(\underline{e}))|^m \int_{\mathbb{X}^{\triangledown}} \int_{\mathbb{Y}^{\triangledown}} \int_{\mathbb{X}^{\triangle}} (\phi_1 \otimes \overline{\phi_2})(z_x + x^{\triangle})\psi(\frac{1}{2}\langle\langle x^{\triangle}, z_y\rangle\rangle^{\square})\\
&\quad \cdot \overline{\delta(\phi_3 \otimes \overline{\phi_4})(z_x + z_y)}\, dx^{\triangledown} dz_x^{\triangledown} dz_y^{\triangledown}\\
&= |N(R(\underline{e}))|^m \int_{\mathbb{X}^{\triangledown}} \int_{\mathbb{X}^{\triangle}} (\phi_1 \otimes \overline{\phi_2})(z_x + x^{\triangle}) \cdot \overline{(\phi_3 \otimes \overline{\phi_4})(z_x + x^{\triangle})}\, dx^{\triangle} dz_x^{\triangledown}\\
&= |2|^{-2mn} \cdot |N(R(\underline{e}))|^m \int_{\mathbb{X}} \int_{\mathbb{X}} (\phi_1 \otimes \overline{\phi_2})(x, x') \cdot \overline{(\phi_3 \otimes \overline{\phi_4})(x, x')}\, dx dx'\\
&= |2|^{-2mn} \cdot |N(R(\underline{e}))|^m \cdot (\phi_1, \phi_3)_{\mathbb{X}} \cdot \overline{(\phi_2, \phi_4)_{\mathbb{X}}}.
\end{aligned}$$

Thus, we have the proposition. ☐

9. Local theta correspondence

In this section, we recall notations and properties of local theta correspondence for quaternionic dual pairs.

9.1. **Definition.** Fix a non-trivial additive character ψ of F. Let ω_ψ be the Weil representation of $G(V) \times G(W)$ (see §8.3). For an irreducible representation π of $G(W)$, we define $\Theta_\psi(\pi, V)$ as the largest quotient module

$$(\omega_\psi \otimes \pi^\vee)_{G(W)}$$

of $\omega_\psi \otimes \pi^\vee$ on which $G(W)$ acts trivially. This is a representation of $G(V)$. We define the theta correspondence $\theta_\psi(\pi, V)$ of π by

$$\theta_\psi(\pi, V) = \begin{cases} 0 & (\Theta_\psi(\pi, V) = 0),\\ \text{the maximal semisimple quotient of } \Theta_\psi(\pi, V) & (\Theta_\psi(\pi, V) \neq 0). \end{cases}$$

If we consider the pair $(G(V), G(W^{\square}))$, we use $\omega_\psi^{\square}$ instead of ω_ψ to define the theta correspondence.

Theorem 9.1 is a fundamental result in the study of theta correspondence. In particular, the properties (1) and (2) are called the Howe duality, which was proved by Waldspurger [Wal90] when the residual characteristic of F is not 2, and was completely proved by Gan and Takeda [GT16] (for the non-quaternionic dual pairs) and Gan and Sun [GS17] (for the quaternionic dual pairs).

Theorem 9.1. *For irreducible representations π_1, π_2 of $G(W)$, we have*

(1) $\theta_\psi(\pi_1, V)$ *is irreducible if it is non-zero,*
(2) $\pi_1 \cong \pi_2$ *if* $\theta_\psi(\pi_1, V) \cong \theta_\psi(\pi_2, V) \neq 0$,
(3) $\theta_\psi(\pi_1, V)^\vee \cong \theta_{\overline{\psi}}(\pi_1^\vee, V)$.

Proof. [GS17, Theorem 1.3]. □

For an irreducible representation ρ of a group H, we denote by ω_ρ the central character of ρ.

Proposition 9.2. *Let π be an irreducible representation of $G(W)$, and suppose that $\theta_\psi(\pi, V)$ is non-zero. We denote by σ the representation $\theta_\psi(\pi, V)$. Then, we have*

$$c_\pi(-1)c_\sigma(-1) = \chi_V(-1)^n \chi_W(-1)^m.$$

Proof. Let $\mathbb{W} = \mathbb{X} + \mathbb{Y}$ be a polar decomposition as in §8.3. It suffices to show that $\omega_\psi(-1,-1)$ acts on $\mathcal{S}(\mathbb{X})$ by the scalar multiplication by $\chi_V(-1)^n\chi_W(-1)^m$. Since $r_{\psi,\mathbb{Y}}(-1,-1)$ is the identity operator on $\mathcal{S}(\mathbb{X})$, we have the action of $\omega_\psi(-1,-1)$ is the scalar multiplication by $\beta_{\mathbb{Y}}^V(-1)\beta_{\mathbb{Y}}^W(-1)c_{\psi,\mathbb{Y}}(-1,-1)$. One can show that $c_{\psi,\mathbb{Y}}(-1,-1) = 1$. Besides, by the definition of $\beta_{\mathbb{Y}}^V$, we have

$$\beta_{\mathbb{Y}}^V(-1) = \beta_V(\iota(-1,1)) = \beta_V(\tau) = (-1)^{mn}\chi_V(-1)^n.$$

By the same way, we have $\beta_{\mathbb{Y}}^W(-1) = (-1)^{mn}\chi_W(-1)^m$. These imply the proposition. □

9.2. **Square integrability.** In this subsection, we explain the preservation of the square integrability under the theta correspondence, which is necessary for the setup of the main result. Let π be an irreducible square-integrable representation of $G(W)$, and let $\sigma := \theta_\psi(\pi, V)$. In this subsection, we assume that $l = 1$ and $\sigma \neq 0$. We denote by θ the $G(V)$-equivalent and $G(W)$-invariant natural quotient map

$$\omega_\psi \otimes \pi \to \sigma.$$

Let $(\ ,\ )_\pi \colon \pi \times \pi \to \mathbb{C}$ be a non-zero $G(W)$-invariant Hermitian pairing on π. We define a non-zero $G(V)$-invariant Hermitian pairing $(\ ,\ )_\sigma \colon \sigma \times \sigma \to \mathbb{C}$ by

$$(\theta(\phi_1, v_1), \theta(\phi_2, v_2))_\sigma := \int_{G(W)} (\omega_\psi(g)\phi_1, \phi_2) \cdot \overline{(\pi(g)v_1, v_2)_\pi}\, dg. \tag{9.1}$$

Lemma 9.3. *The integral of the right-hand side of* (9.1) *converges absolutely, and the map yielded by the integral factors through the natural quotient map*

$$\omega_\psi \otimes \pi^\vee \times \omega_\psi \otimes \pi^\vee \to \sigma \times \sigma.$$

Moreover, we have that σ is a square-integrable representation.

Proof. One can construct an integrable dominating function of the function

$$g \mapsto (\omega_\psi(g)\phi_1, \phi_2) \cdot \overline{(\pi(g)v_1, v_2)_\pi}$$

on $G(W)$, which implies that the integral converges absolutely (see [GI14, Lemma 9.5]). Similar to [GI14, Appendix D, Lemma D1]. □

9.3. **Tower properties.** In this subsection, we discuss some properties related to Witt towers. Let V_0 be a right anisotropic ϵ-Hermitian space. Put $m_0 \coloneqq \dim_D V_0$. For a non-negative integer t, we define

$$V_t = X_t \oplus V_0 \oplus X_t^*,$$

where X_t and X_t^* are t-dimensional right D-vector spaces. Fix a basis $\lambda_1, \ldots, \lambda_t$ for X_t and fix a basis $\lambda_{-1}, \ldots, \lambda_{-t}$ for X_t^*. Then we define an ϵ-Hermitian pairing $(\ ,\ )_t$ on V_t by

$$(\lambda_i, \lambda_{-j})_t = \delta_{ij},\ (\lambda_i, x_0)_t = (x_0, \lambda_{-j})_t = 0,\ (x_0, x_0')_t = (x_0, x_0')_0$$

for $i, j = 1, \ldots, t$ and $x_0, x_0' \in V_0$. Here $(\ ,\ )_0$ is the pairing associated with V_0.

First, we state the conservation relation of Sun and Zhu [SZ15]. Let $V_0^\dagger$ be a right anisotropic ϵ-Hermitian space such that $\chi_{V_0^\dagger} = \chi_{V_0}$ and $V_0^\dagger \not\cong V_0$. Such $V_0^\dagger$ is determined uniquely. Take $\{V_t^\dagger\}_{t\geq 0}$ as the Witt tower containing $V_0^\dagger$. Let π be an irreducible representation of $G(W)$. There is a non-negative integer $r(\pi)$ such that $\Theta_\psi(\pi, V_{r(\pi)}) \neq 0$ and $\Theta(\pi, V_t) = 0$ for $t < r(\pi)$. It is known that $\Theta_\psi(\pi, V_{r(\pi)})$ is irreducible and supercuspidal if π is supercuspidal [MVW87, p. 69]. We call $r(\pi)$ the first occurrence index for the theta correspondence from π to the Witt tower $\{V_t\}_{t\geq 0}$. Denote by $r^\dagger(\pi)$ the first occurrence index for the theta correspondence from π to $\{V_t^\dagger\}_{t\geq 0}$.

Proposition 9.4. *Let π be an irreducible representation of $G(W)$. Then we have*

$$m(\pi) + m^\dagger(\pi) = 2n + 2 - \epsilon,$$

where $m(\pi) = 2r(\pi) + \dim_D V_0$, and $m^\dagger(\pi) = 2r^\dagger(\pi) + \dim_D V_0^\dagger$.

Proof. [SZ15]. ☐

Then, we explain the behavior of theta correspondence when we change indexes of Witt towers. However, before doing that, we state here the analogue of the Gelfand-Kazhdan Theorem [BZ76, Theorem 7.3] for $\mathrm{GL}_r(D)$, which we use in the proof of Proposition 9.6:

Lemma 9.5. *Let τ be an irreducible representation of $\mathrm{GL}_r(D)$, and let τ^θ be the irreducible representation of $\mathrm{GL}_r(D)$ defined by $\tau^\theta(g) = \tau({}^t g^{*-1})$ for $g \in \mathrm{GL}_r(D)$. Then, τ^θ is equivalent to the contragredient representation $\tau^\vee$ of τ.*

Proof. [Rag02, Theorem 3.1]. ☐

Proposition 9.6. *Let $\{W_t\}_{t\geq 0}$ be a Witt tower of right $(-\epsilon)$-Hermitian spaces.*

(1) *Let π be an irreducible representation of $G(W_i)$, and let $\sigma = \theta_\psi(\pi, V_j)$. Suppose that $j \geq r(\pi)$, and we denote by $\sigma_{j'}$ the representation $\theta_\psi(\pi, V_{j'})$ for $r(\pi) \leq j' \leq j$. Then, σ is a subquotient of an induced representation*

$$\mathrm{Ind}_{Q_{j',j}}^{G(V_i)} \sigma_{j'} \boxtimes \chi_W |N_{X_{j',j}}|^{l_{i,j}+j-r(\pi)}.$$

Here, $l_{i,j} = 2\dim W_i - 2\dim V_j - \epsilon$, $X_{j',j}$ is a subspace of $X_{j'}$ spanned by $\lambda_{j'+1}, \ldots, \lambda_j$, $N_{X_{j',j}}$ is the reduced norm of $\mathrm{End}(X_{j',j})$, and $Q_{j',j}$ is the parabolic subgroup preserving $X_{j',j}$.

(2) *Let π be an irreducible representation of $G(W_{i'})$, let $\sigma = \theta_\psi(\pi, V_{j'})$, let τ be a non-trivial supercuspidal irreducible representation of $\mathrm{GL}_r(D)$, let s be a complex number, and let π' be an irreducible subquotient of $\mathrm{Ind}_{P_{i',i}}^{G(W_i)}(\pi \boxtimes \tau_s\chi_V)$ where $i = i' + r$ and $P_{i',i}$ is the parabolic subgroup preserving an r-dimensional totally isotropic subspace of $W_{i'}$. Suppose that $\sigma \neq 0$. Then, we have that $\theta_\psi(\pi', V_j)$ is a subquotient of $\mathrm{Ind}_{Q_{j',j}}^{G(V_j)} \sigma \boxtimes \tau_s\chi_W$. Here, $j = j' + r$, and $\tau_s\chi_W$ is the representation of $\mathrm{GL}_r(D)$ defined by $\tau_s\chi_W(g) = \tau(g)\chi_W(N(g))|N(g)|^s$ for $g \in \mathrm{GL}_r(D)$, where N denotes the reduced norm.*

Proof. These properties are proved by analyzing the Jacquet module of Weil representations: it goes a similar line with [Mui06], however, we explain for the readers (see also [Han11]). In the proof, we denote by $\omega_\psi[j,i]$ the Weil representation associated with the reductive dual pair $(G(V_j), G(W_i))$. Moreover, for a representation ρ of $G(V_j) \times G(W_i)$, for $0 \leq i' \leq i$, and for $0 \leq j' \leq j$, we denote by $J_{j',i'}\rho$ the Jacquet module of ρ with respect to the parabolic subgroup $Q_{j',j} \times P_{i',i}$. Then, by [MVW87], we have a $G(V_{j'}) \times \mathrm{GL}_{j-j'}(D) \times G(W_i)$ equivalent filtration:

$$J_{j',i}(\omega_\psi[j,i]) = R_0 \supset R_1 \supset \cdots \supset R_t \supset R_{t+1} = 0.$$

Here,

$$\begin{aligned}
&t = \min\{j - j', i\},\\
&R_0/R_1 = \chi_W |N_{X_{j',j}}|^{l_{i,j}+j-j'} \boxtimes \omega_\psi[j', i],\\
&R_k/R_{k+1} = \mathrm{Ind}_{P_{i-k,i}}^{G(W_i)} \rho_k \text{ for some representation } \rho_k \ (k = 1, \ldots, t-1),
\end{aligned}$$

and moreover if $j - j' \leq i$, we have

$$R_t = \mathrm{Ind}_{P_{i',i}}^{G(W_i)} \mathcal{S}(\mathrm{GL}_{j-j'}(D)) \boxtimes \omega_\psi[j', i'],$$

where $i' = i - (j - j')$, and the action of $\mathrm{GL}_{j-j'}(D) \times \mathrm{GL}_{i-i'}(D)$ on $\mathcal{S}(\mathrm{GL}_{j-j'}(D))$ is given by

$$[(g_1, g_2) \cdot \varphi](g) = \chi_W(N(g_1))\chi_V(N(g_2))\varphi(g_1^{-1} g g_2)$$

for $g_1 \in \mathrm{GL}_{j-j'}(D), g \in \mathrm{GL}_{j-j'}(D)$, and $g_2 \in \mathrm{GL}_{i-i'}(D)$, where N denotes the reduced norm. Now we prove (1). Composing $J_{j',i}(\omega_\psi[j,i]) \to R_0/R_1$ with the $G(V_{j'}) \times G(W_i)$-equivalent surjection

$$\omega_\psi[j', i] \to \sigma \boxtimes \pi,$$

we have a non-zero morphism

$$J_{j',i}(\omega_\psi[j,i]) \to \chi_W |N_{X_{j',j}}|^{l+j-j'} \boxtimes \sigma \boxtimes \pi.$$

Hence we have (1). Then we prove (2). Let π' be an irreducible component of $\mathrm{Ind}_{P_{i',i}}^{G(W_i)} \pi \boxtimes \tau_s\chi_V$. First, we have

$$\mathrm{Hom}(R_k/R_{k+1}, \pi') = \mathrm{Hom}(\rho_k, J_{i-k}\pi').$$

Here, we denote by $J_{i-k}\pi'$ the Jacquet module with respect to the parabolic subgroup $P_{i',i}$. However, since τ is supercuspidal, one can prove $J_{i-k}\,\mathrm{Ind}_{P_{i',i}}^{G(W_i)} \pi \boxtimes \tau_s\chi_V = 0$ for $k = 1, 2, \ldots, t-1$ by considering the filtration of Bernstein and Zelevinsky [BZ77, Theorem 5.2], and thus the right-hand side is 0. Hence, we have

$$R_1 \otimes \pi'^{\vee} \cong R_t \otimes \pi'^{\vee}.$$

Moreover, since $\tau_s\chi_W \not\cong \chi_W|N_{j',j}|^{l_{i,j}+j-j'}$, we have

$$R_0 \otimes (\tau_s\chi_W)^\vee \cong R_1 \otimes (\tau_s\chi_W)^\vee.$$

On the other hand, the non-zero $\mathrm{GL}_r(D) \times \mathrm{GL}_r(D)$-equivalent map

$$\mathcal{S}(\mathrm{GL}_{j-j'}(D)) \otimes ((\tau_s\chi_V)^\vee \boxtimes \tau_s\chi_W) \to \mathbb{C}\colon (\varphi, x, x')$$
$$\mapsto \int_{\mathrm{GL}_r(D)} \varphi(g)\langle \tau_s(g)x, x'\rangle \chi_W\chi_V^{-1}(N_{j-j'}(g))\, dg$$

yields a non-zero $\mathrm{GL}_r(D) \times \mathrm{GL}_r(D)$-equivalent map

$$\mathcal{S}(\mathrm{GL}_{j-j'}(D)) \otimes (\tau_s\chi_V)^\vee \to (\tau_s\chi_W)^\vee.$$

By combining the above arguments, and by Lemma 9.5, we have a non-zero $G(V_{j'})\times \mathrm{GL}_{j-j'}(D) \times G(W_i)$-equivalent map

$$\begin{aligned}
&J_{j',i}(\omega_\psi[j,i]) \otimes (\sigma \boxtimes \tau_s\chi_W)^\vee \otimes (\pi')^\vee\\
&= R_t \otimes (\sigma \boxtimes \tau_s\chi_W)^\vee \boxtimes (\pi')^\vee\\
&= (\mathrm{Ind}_{P_{i',i}}^{G(W_i)} \mathcal{S}(\mathrm{GL}_{j-j'}(D)) \boxtimes \omega_\psi[i',j']) \otimes (\sigma \boxtimes \tau_s\chi_W)^\vee \boxtimes (\pi')^\vee\\
&\to (\mathrm{Ind}_{P_{i',i}}^{G(W_i)} (\tau_s\chi_V)^\vee \boxtimes \pi) \otimes (\pi')^\vee\\
&\cong (\mathrm{Ind}_{P_{i',i}}^{G(W_i)} (\tau^\theta\chi_V)_{-s} \boxtimes \pi) \otimes (\pi')^\vee\\
&\cong (\mathrm{Ind}_{P_{i',i}}^{G(W_i)} (\tau_s\chi_V) \boxtimes \pi) \otimes (\pi')^\vee\\
&\to \mathbb{C}.
\end{aligned}$$

Hence we have (2). □

By the proof of Proposition 9.6, we also have a slightly different property:

Corollary 9.7. *Let $\{W_t\}_{t\geq 0}$ be a Witt tower of right $(-\epsilon)$-Hermitian spaces, let i, j, j' be non-negative integers so that $j - j' > 0$, let π be an irreducible representation of $G(W_i)$, let $\sigma = \theta_\psi(\pi, V_j)$. Suppose that $\sigma \neq 0$, and σ is a subrepresentation of an induced representation $\mathrm{Ind}_{Q_{j',j}}^{G(V_j)} \sigma' \boxtimes \tau_s\chi_W$ where σ' is an irreducible representation of $G(V_{j'})$, τ is an irreducible supercuspidal representation of $\mathrm{GL}_{j-j'}(D)$, and $s \in \mathbb{C}$. Moreover, suppose that $\theta_\psi(\pi, V_{j'}) = 0$. Then, we have $i \geq j - j'$, and there exists an irreducible representation π' of $G(W_{i'})$ such that $\theta_\psi(\pi', V_{j'}) \cong \sigma'$. Here we put $i' = i-(j-j')$. Moreover, π is an irreducible subquotient of $\mathrm{Ind}_{P_{i',i}}^{G(W_i)} \pi' \boxtimes \tau_s\chi_W$.*

Proof. We use the notation of the proof of Proposition 9.6. Since there is a non-zero $G(V_j) \times G(W_i)$-equivalent map

$$\omega_\psi[j,i] \to \sigma \boxtimes \pi,$$

by the Frobenius reciprocity, we have a non-zero $G(V_{j'}) \times \mathrm{GL}_{j-j'}(D) \times G(W_i)$-equivalent map

$$(\tau_s\chi_W)^\vee \boxtimes \pi^\vee \otimes J_{j',i}\omega_\psi[j,i] \to \sigma'. \tag{9.2}$$

Then, the assumption $\theta_\psi(\pi, V_{j'}) = 0$ implies that

$$\pi^\vee \otimes R_0/R_1 = 0.$$

Moreover, as in the proof of Proposition 9.6(2), we have

$$(\tau_s\chi_W)^\vee \boxtimes \pi^\vee \otimes J_{j',i}\omega_\psi[j,i] = (\tau_s\chi_W)^\vee \boxtimes \pi^\vee \otimes R_{j-j'}.$$

(Here, we put $R_k = 0$ for $k > t$.) Thus, $R_{i-i'}$ is forced not to be zero, and we have $i \geq j - j'$. By using the Frobenius reciprocity again, we have a non-zero $G(V_{j'}) \times \mathrm{GL}_{j-j'}(D) \times G(W_{i'}) \times \mathrm{GL}_{i-i'}(D)$-equivalent map

$$((\tau_s\chi_W)^\vee \boxtimes (J_{i',j}\pi)^\vee) \otimes (\mathcal{S}(\mathrm{GL}_{i-i'}(D)) \boxtimes \omega_\psi[j', i']) \to \sigma'.$$

Thus, ${\sigma'}^\vee \otimes \omega_\psi[j', i'] \neq 0$. Put $\pi' := \theta_\psi(\sigma', W_{i'})$. Then, $\theta_\psi(\sigma, W_i)$ is non-zero, and it is an irreducible subquotient π'' of $\mathrm{Ind}_{P_{i',i}}^{G(W_i)} \pi' \boxtimes \tau_s\chi_V$. However, by the Howe duality (Theorem 9.1), π'' coincides with π. Thus we have the corollary. $\square$

10. The local Siegel-Weil formula

In this section, we state the local Siegel-Weil formula, which is a local analogue of the (bounded and first term) Siegel-Weil formula. **We assume $l = 1$ and $n \geq 0$ in this section**.

10.1. **The map $\mathcal{I}$.** We define the $\Delta G(W^\square) \times G(V) \times G(V)$-invariant map

$$\mathcal{I}\colon \omega_\psi^\square \otimes \overline{\omega_\psi^\square} \to \mathbb{C}$$

by

$$\mathcal{I}(\phi, \phi') = \int_{G(V)} (\omega_\psi^\square(h)\phi, \phi')\, dh$$

for $\phi, \phi' \in \omega_\psi^\square$ where $(\ , \ )$ is the L^2-norm of $\mathcal{S}(V \otimes W^\triangledown)$ as in Proposition 8.2. The integral defining $\mathcal{I}(\ , \ )$ converges absolutely by [Li89, Theorem 3.2].

10.2. **The map $\mathcal{E}$.** Let $V^\flat$ be the unique ϵ-Hermitian space over D so that $\dim_D V^\flat = m + 1$ and $\chi_{V^\flat} = \chi_V$. Such space exists since we have assumed that $l = 1$ and $n \geq 1$. Consider the $G(W^\square)$-invariant map

$$\mathcal{S}(V \otimes W^\triangledown) \to I(-\frac{1}{2}, \chi_V)\colon \phi \mapsto F_\phi$$

defined by $F_\phi(g) = [\omega_\psi^\square(1, g)\phi](0)$ for $\phi \in$ and $g \in G(W^\square)$. Similarly, there is a $G(W^\square)$-invariant map $\mathcal{S}(V^\flat \otimes W^\triangledown) \to I(\frac{1}{2}, \chi_V)$. We denote by $R^W(V)$ and $R^W(V^\flat)$ the images of the above maps respectively. Then we have the following exact sequence:

$$0 \longrightarrow R^W(V^\flat) \longrightarrow I(\tfrac{1}{2}, \chi_V) \xrightarrow{M(\frac{1}{2}, \chi_V)} R^W(V) \longrightarrow 0$$

[Yam11, Proposition 7.6]. For $\phi \in \mathcal{S}(V \otimes W^\triangledown)$, we denote by $F_\phi^\dagger \in I(\frac{1}{2}, \chi_V)$ a section such that $M(\frac{1}{2}, \chi_V)F_\phi^\dagger = F_\phi$. Then, we define the map $\mathcal{E}$ by

$$\mathcal{E}(\phi, \phi') = \int_{G(W)} F_\phi^\dagger(\iota(g, 1)) \cdot \overline{F_{\phi'}(\iota(g, 1))}\, dg.$$

The integral defining $\mathcal{E}$ converges absolutely by Proposition 7.1. Moreover, Lemma 10.1 implies that the definition of $\mathcal{E}(\phi, \phi')$ does not depend on the choice of $F_\phi^\dagger$.

Lemma 10.1. *If $f \in R^W(V^\flat)$ and $h \in R^W(V)$, then we have*

$$\int_{G(W)} f(\iota(g, 1)) \cdot \overline{h(\iota(g, 1))}\, dg = 0.$$

Proof. By the proof of Lemma 7.7, we have

$$\mathrm{Hom}_{G(V)\times G(V)}(I(\rho, 1_{F^\times}), \mathbb{C}) = \mathrm{Hom}_{G(V^\square)}(I(\rho, 1_{F^\times}), \mathbb{C}) = Z \cdot \mathbb{C},$$

where

$$Z(F) = \int_{G(V)} F(\iota(g,1))\, dg$$

for $F \in I(\rho, 1)$. Thus, if there are $f \in R^W(V^\flat), h \in R^W(V)$ so that $Z(f \cdot \overline{h}) \neq 0$, we would have $R^W(V^\flat) \cong \overline{R^W(V)}^\vee$. Since $\overline{I(-\frac{1}{2}, \chi_V)} \cong I(-\frac{1}{2}, \chi_V)$, we have $\overline{R^W(V)} \cong R^W(V)$. Put $\sigma := R^W(V^\flat)$. Then, we have

$$\Theta(\sigma, V^\flat) = 1_{V^\flat},\ \Theta(\sigma, V) = 1_V.$$

However, according to the conservation relation (Proposition 9.4), one of them must vanish since $\dim V + \dim V^\flat = 2n - \epsilon$. This is a contradiction, and we have the lemma. □

10.3. **Local Siegel-Weil formula.** Lemma 10.2 gives the definition of $\alpha_2(V, W)$, which is the second constant we are interested in.

Lemma 10.2. *There is a non-zero constant $\alpha_2(V, W)$ such that $\mathcal{I} = \alpha_2(V, W) \cdot \mathcal{E}$.*

Proof. The two maps $\mathcal{I}$, $\mathcal{E}$ are $\Delta G(W^\square) \times G(V) \times G(V)$-invariant map. On the other hand, we have

$$\begin{aligned}\dim \mathrm{Hom}_{\Delta G(W^\square)\times G(V)\times G(V)}&(\omega_\psi^\square \otimes \overline{\omega_\psi^\square}, \mathbb{C}) \\ &= \dim \mathrm{Hom}_{\Delta G(W^\square)}(R^W(V) \otimes \overline{R^W(V^\flat)}, \mathbb{C}) = 1.\end{aligned}$$

Hence, it suffices to show that $\mathcal{I}$ and $\mathcal{E}$ are non-zero. Let $\phi \in \mathcal{S}(V \otimes W^\triangledown)$ be a positive function. Choose a neighbourhood U of 1 in $G(V)$ so that $\omega_\psi^\square(h)\phi = \phi$ for all $h \in U$. Then, we have

$$\begin{aligned}\mathcal{I}(\phi, \phi) &\geq \int_U (\omega_\psi^\square(h)\phi, \phi)\, dh \\ &= |U| \cdot (\phi, \phi) > 0.\end{aligned}$$

Thus we have $\mathcal{I} \neq 0$. The non-vanishing of $\mathcal{E}$ is obtained by Proposition 13.3. However, we also give a short proof. Consider the non-zero pairing

$$I(\frac{1}{2}, \chi_V) \times I(-\frac{1}{2}, \chi_V) \to \mathbb{C}\colon (f, h) \mapsto Z(f \cdot \overline{h}), \tag{10.1}$$

where Z is the map as in the proof of Lemma 10.1. We assume $\mathcal{E} = 0$ to derive a contradiction. Then the pairing (10.1) factors through the quotient

$$I(\frac{1}{2}, \chi_V) \times I(-\frac{1}{2}, \chi_V) \to I(\frac{1}{2}, \chi_V) \times R(V^\flat)$$

by [Yam11, Theorems 1.3, 1.4]. But this implies $R(V) \cong \overline{R(V^\flat)}^\vee$, which contradicts the conservation relation as in the proof of Lemma 10.1. Hence we have $\mathcal{E} \neq 0$, and we finish the proof of Lemma 10.2. □

We will determine the constant $\alpha_2(V, W)$ completely in §19. However we calculate $\alpha_2(V, W)$ directly when either V or W is anisotropic. The proof will be given in §§12–13:

Proposition 10.3.

(1) *Suppose that $-\epsilon = 1$ and V is anisotropic, then we have*

$$\alpha_2(V,W) = |N(R(\underline{e}))|^{n+\frac{1}{2}}\chi_V(-1)^n \times \begin{cases} -|2|_F^{-\frac{5}{2}}(1+q^{-1}) & (n=1, \chi_V \textit{ is unramified}), \\ -2|2|_F^{-\frac{5}{2}}q^{-\frac{\mathfrak{f}(\chi_W)}{2}} & (n=1, \chi_V \textit{ is ramified}), \\ 2|2|_F^{-7}q^{-2}(1+q^{-2}) & (n=2, \chi_V \textit{ is unramified}), \\ 2|2|_F^{-7}(1+q^{-1})q^{-\frac{3}{2}\mathfrak{f}(\chi_W)-1} & (n=2, \chi_V \textit{ is ramified}), \\ -2|2|_F^{-\frac{27}{2}}q^{-6}(1+q^{-1})(1+q^{-2}) & (n=3). \end{cases}$$

(2) *Suppose that $-\epsilon = -1$ and either V or W is anisotropic, then we have*

$$\alpha_2(V,W) = |N(R(\underline{e}))|^{n-\frac{1}{2}} \times \begin{cases} |2|_F^{-\frac{1}{2}} & (n=1), \\ |2|_F^{-3}\cdot q^{-1}\cdot(1+q^{-1}) & (n=2), \\ -|2|_F^{-\frac{15}{2}}q^{-4}\cdot\frac{(1+q^{-1})(1-q^{-4})}{1-q^{-3}} & (n=3). \end{cases}$$

11. Formal degrees and local theta correspondence

In this section, we state the behavior of the formal degree under the local theta correspondence, which extends the result of Gan and Ichino [GI14]. Let G be a connected reductive group over F, and let π be a square-integrable irreducible representation of G. Then, the formal degree is a number $\deg\pi$ satisfying

$$\int_{G/A_G}(\pi(g)v_1, v_2)\cdot\overline{(\pi(g)v_3, v_4)}\,dg = \frac{1}{\deg\pi}(v_1, v_3)\cdot\overline{(v_2, v_4)}$$

for $v_1, \ldots, v_4 \in \pi$, where A_G is the maximal F-split torus of the center of G.

Again, we consider a right m-dimensional ϵ-Hermitian space and a left n-dimensional $(-\epsilon)$-Hermitian space. In this section, **we assume that** $l = 1$. The purpose of this section is to describe the behavior of the formal degree under the theta correspondence for the quaternionic dual pair $(G(V), G(W))$. Let π be an irreducible square-integrable representation of $G(W)$, and let $\sigma = \theta_\psi(\pi, V)$. Assume that $\sigma \neq 0$. Then, we recall that σ is also square-integrable.

Lemma 11.1. *The number*

$$\frac{\deg\pi}{\deg\sigma}\cdot c_\sigma(-1)\cdot\gamma^V(0, \sigma\times\chi_W, \psi)^{-1} \tag{11.1}$$

does not depend on π.

We will prove Lemma 11.1 later (Proposition 15.1). We denote the constant (11.1) by $\alpha_3(V,W)$. Now we state our main theorem:

Theorem 11.2. *We have*

$$\alpha_3(V,W) = \begin{cases} \epsilon(\frac{1}{2}, \chi_V, \psi)^{-1} & (-\epsilon = 1), \\ \frac{1}{2}\chi_W(-1)^m\epsilon(\frac{1}{2}, \chi_W, \psi)^{-1} & (-\epsilon = -1). \end{cases}$$

We prove Theorem 11.2 in later sections. In this section, we see an example:

Example 11.3. Consider the case where $\epsilon = 1$, $m = 1$, $n = 2$, and $\chi_W = 1_{F^\times}$. We denote by St the Steinberg representation of $G(W)$. Then, it is known that $\theta_\psi(\mathrm{St}, V)$ is the trivial representation $1_{G(V)}$ of $G(V)$. The local Langlands correspondence for $G(W)$ has been established (see [Cho17, §5]) and the L-parameter of St is the principal parameter of $\widehat{G}$ (see e.g. [GR10, §3.3]). Then, as representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$, we have

$$\mathrm{ad} \circ \phi_0 = (1_{W_F} \otimes r_3) \oplus (1_{W_F} \otimes r_3),$$

where 1_{W_F} is the trivial representation of W_F, and r_3 is the unique three-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Thus, we have

$$\gamma(s + \frac{1}{2}, \mathrm{St}, \mathrm{ad}, \psi) = q^{-4s} \cdot \frac{\zeta_F(-s + \frac{3}{2})^2}{\zeta_F(s + \frac{3}{2})^2}.$$

Moreover, the centralizer $C_{\phi_0}(\widehat{G})$ of $\mathrm{Im}\,\phi_0$ in $\widehat{G}$ is $\{\pm 1\} \subset \widehat{G}$, and the component group $\widetilde{S}_{\phi_0}(\widehat{G})$ is abelian. Since the formal degree conjecture for $G(W)$ is available (see §21), we have

$$\deg \mathrm{St} = \frac{1}{2} \cdot \frac{q^2}{(1 + q^{-1})^2}.$$

On the other hand, we have

$$\deg 1_{G(V)} = |G(V)|^{-1} = \frac{q}{1 + q^{-1}}.$$

(Recall that the volume $|G(V)|$ of $G(V)$ is given by Corollary 6.2.) Therefore, by Lemma 7.5, we have

$$\frac{\deg \mathrm{St}}{\deg 1_{G(V)}} = \frac{1}{2} \cdot \gamma(0, 1_{G(V)} \boxtimes 1_{F^\times}, \psi)$$

which agrees with Theorem 11.2.

We explain the strategy of the proof of Theorem 11.2. First, we consider the case where either W or V is anisotropic (i.e. the minimal cases in the sense of the parabolic induction). In these cases, we can express $\alpha_2(V, W)$ with $\alpha_1(W)$ which is already determined in §7.3. And hence we obtain Proposition 10.3 (§§12–13). Second, we relate $\alpha_3(V, W)$ with $\alpha_2(W)$ (§§14–15). Then we have Theorem 11.2 in the minimal cases. And finally, we prove that the constant $\alpha_3(V, W)$ is compatible with parabolic inductions (§§16–18), which completes the proof of Theorem 11.2. Moreover, once $\alpha_3(V, W)$ is determined, the above processes can be reversed to obtain the general formula for $\alpha_1(W)$ and $\alpha_2(W)$ (§19).

Remark 11.4. As written in [Kak20, §5.3], the definition of the doubling γ-factor of Lapid and Rallis [LR05] should be modified by a constant multiple. Thus, it is natural to ask whether the statement of the main theorem of [GI14] might change. However, [GI14, Theorem 15.1] is still true. This is because their proof uses the doubling γ-factor not to determine the "constant $\mathcal{C}$" (see [GI14, §20.2]) but to show the existence of the constant $\mathcal{C}$. Hence, the difference of constant multiples is offset at the time of calculation of $\mathcal{C}$.

12. Minimal cases (I)

In this section, we prove Proposition 10.3(2) in the case $\dim V = 2$.

Suppose that $\epsilon = 1$, $V_0 = 0$ and $\dim_D V = 2$. Then, we can take a basis $\underline{e}^V = (e_1^V, e_2^V)$ of V so that

$$(e_1^V, e_1^V) = (e_2^V, e_2^V) = 0, \text{ and } (e_1^V, e_2^V) = 1.$$

We take bases $\underline{e}$ of W and $\underline{e}'^{\square}$ of $W^{\square}$ as in §7.1. Let $\mathcal{L}$ be a lattice

$$\left(\bigoplus_{i=1}^{n} e_1^V \varpi_D^{-1} \mathcal{O}_D \otimes e'_{n+i}\right) \oplus \left(\bigoplus_{i=1}^{n} e_2^V \mathcal{O}_D \otimes e'_{n+i}\right)$$

of $V \otimes W^{\triangledown}$, and we denote by $1_{\mathcal{L}}$ the characteristic function of $\mathcal{L}$. By the fact that $\mathcal{L}$ is self-dual with respect to the pairing (8.1), we have $|\mathcal{L}| = 1$.

Lemma 12.1. *We have*

$$\mathcal{I}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = q^{-2} \frac{(1-q^{-2})(1+q^{-2})(1+q^{-5})}{1-q^{-3}}.$$

Proof. Let $\mathcal{B}$ be the subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(V) \;\middle|\; a, b, d \in \mathcal{O}_D, c \in \varpi_D \mathcal{O}_D \right\}$$

of $G(V)$, which fixes the lattice $\mathcal{L}$. Then, $\mathcal{B}$ is an Iwahori subgroup by Lemma 5.2 and the volume $|\mathcal{B}|$ is given by $q^{-4}(1-q^{-2})$ by Proposition 6.1(2). By [BT72, Théorèm 5.1.3], we have $G(V) = \mathcal{B} \cdot \mathcal{N} \cdot \mathcal{B}$ where $\mathcal{N}$ is the normalizer of the maximal F-split torus consisting of the diagonal matrices in $G(V)$. Moreover, we can take a system of representatives

$$\{a(t) \mid t \in \mathbb{Z}\} \cup \{w(t) \mid t \in \mathbb{Z}\}$$

for $\mathcal{B} \backslash G(V) / \mathcal{B}$, where

$$a(t) = \begin{pmatrix} \varpi_D^t & 0 \\ 0 & (-\varpi_D)^{-t} \end{pmatrix} \text{ and } w(t) = \begin{pmatrix} 0 & \varpi_D^t \\ (-\varpi_D)^{-t} & 0 \end{pmatrix}.$$

Hence we have

$$\begin{aligned}
\mathcal{I}(1_{\mathcal{L}}, 1_{\mathcal{L}}) &= |\mathcal{B}| \cdot \sum_{t \in \mathbb{Z}} (|\mathcal{L} \cap a(t)\mathcal{L}| \cdot [\mathcal{B} a(t) \mathcal{B} : \mathcal{B}] + |\mathcal{L} \cap w(t)\mathcal{L}| \cdot [\mathcal{B} w(t) \mathcal{B} : \mathcal{B}]) \\
&= |\mathcal{B}| \cdot \sum_{t \in \mathbb{Z}} (q^{-3|t|} + q^{-6|t-1|+|1+3t|}) \\
&= q^{-4}(1-q^{-2}) \cdot (\frac{1+q^{-3}}{1-q^{-3}} + \frac{q^2+q^{-5}}{1-q^{-3}}) \\
&= q^{-2} \frac{(1-q^{-2})(1+q^{-2})(1+q^{-5})}{1-q^{-3}}.
\end{aligned}$$

Thus we have the lemma. $\square$

Lemma 12.2. *We have*

$$\mathcal{E}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = m^{\circ}(\frac{1}{2})^{-1} \cdot \alpha_1(W),$$

where $m^{\circ}(s)$ is a function as in Lemma 7.8.

Proof. One can show that $1_{\mathcal{L}}$ is a $K(\underline{e}'^{\square})$ fixed function with $1_{\mathcal{L}}(0)=1$. Thus, we have $\mathcal{F}_{1_{\mathcal{L}}} = f^{\circ}_{-\frac{1}{2}}$ where we mean by f°_s the unique $K(\underline{e}'^{\square})$ fixed section in $I(-\frac{1}{2},1)$ with $f^{\circ}_s(1)=1$. By the Gindikin-Karperevich formula (see e.g. [Cas89]), we can take $\mathcal{F}^{\dagger}_{1_{\mathcal{L}}} = m^{\circ}(\frac{1}{2})^{-1} f^{\circ}_{\frac{1}{2}}$. Hence, we have

$$\begin{aligned}\mathcal{E}(1_{\mathcal{L}},1_{\mathcal{L}}) &= m^{\circ}(\frac{1}{2})^{-1}\int_{G(W)} f^{\circ}_{\rho}(\iota(g,1))\,dg\\ &= m^{\circ}(\frac{1}{2})^{-1}\cdot\alpha_1(W).\end{aligned}$$

□

Hence, by Lemmas 12.1 and 12.2, we have:

Proposition 12.3. *If $\epsilon=1$, $V_0=0$ and $\dim_D V=2$, then we have*

$$\alpha_2(V,W) = -|2|_F^{-\frac{15}{2}}\cdot|N(R(\underline{e}))|^{\frac{5}{2}}\cdot q^{-4}\cdot\frac{(1+q^{-1})(1-q^{-4})}{1-q^{-3}}.$$

13. Minimal cases (II)

In this section, we prove Proposition 10.3(1) and the remaining cases of Proposition 10.3(2).

Assume that V is anisotropic. Recall that $\tau\in G(W^{\square})$ denotes the certain Weyl element (see §7.1), and $\mathfrak{u}_n$ is the certain F subspace of $\mathrm{M}_n(D)$ (see §6.2). For $\Phi\in\mathcal{S}(\mathfrak{u}_n)$, we define a section $f(s,\Phi,-)\in I(s,\chi_V)$ by

$$f(s,\Phi,g) := \begin{cases} 0 & (g\notin P(W^{\triangle})\tau U(W^{\triangle})),\\ \chi_{V,s+\rho}(\Delta(p))\cdot\Phi(X) & \left(g=p\tau\begin{pmatrix}1&0\\X&1\end{pmatrix}\in P(W^{\triangle})\tau U(W^{\triangle})\right).\end{cases}$$

Here $G(W^{\square})$ is embedded in $\mathrm{GL}_{2n}(D)$ by the basis $\underline{e}'^{\square}$. For $t\in\mathbb{Z}$, $\phi\in\mathcal{S}(V\otimes W^{\triangledown})$, and $\Phi\in\mathcal{S}(\mathfrak{u}_n)$, we define $\phi_t\in\mathcal{S}(V\otimes W^{\triangledown})$, and $\Phi_t\in\mathcal{S}(\mathfrak{u}_n)$ by

$$\phi_t(x) := q^{-4mnt}\phi(x\varpi_F^t),\ \text{and}\ \Phi_t(X) := q^{-4n\rho t}\Phi(X\varpi^{2t}).$$

Then we have Lemma 13.1:

Lemma 13.1.

(1) *For $\phi\in\mathcal{S}(V\otimes W^{\triangledown})$, we have $\widehat{\phi_t} = q^{-4mnt}(\widehat{\phi})_{-t}$.*
(2) *Let $\phi\in\mathcal{S}(V\otimes W^{\triangledown})$, and let $\Phi\in\mathcal{S}(\mathfrak{u}_n)$ such that $M(\frac{1}{2},\chi_V)f(\frac{1}{2},\Phi,-)=F_{\phi}$. Then we have*

$$M(\frac{1}{2},\chi_V)f(\frac{1}{2},\Phi_t,-) = q^{-4mnt}F_{\phi_{-t}}.$$

Proof. We have

$$\begin{aligned}\widehat{\phi_t}(x) &= \int_{V\otimes W^{\triangledown}} q^{-4mnt}\phi(y\varpi^t)\psi(\frac{\epsilon}{2}\langle\langle x,y\tau\rangle\rangle)\,dy\\ &= \int_{V\otimes W^{\triangledown}} \phi(y)\psi(\frac{\epsilon}{2}\langle\langle x\varpi^{-t},y\tau\rangle\rangle)\,dy\\ &= q^{-4mnt}(\widehat{\phi})_{-t}(x).\end{aligned}$$

Hence we have (1).

$$
\begin{aligned}
&M(\frac{1}{2},\chi_V)f(\frac{1}{2},\Phi_t,\tau\begin{pmatrix}1&0\\X&1\end{pmatrix})\\
&=\int_{\mathfrak{u}} f(\frac{1}{2},\Phi_t,\tau\begin{pmatrix}1&0\\Y&1\end{pmatrix}\tau\begin{pmatrix}1&0\\X&1\end{pmatrix})\,dY\\
&=\int_{\mathfrak{u}} f(\frac{1}{2},\Phi_t,\begin{pmatrix}Y&0\\-\epsilon&\epsilon\cdot Y^{-1}\end{pmatrix}\tau\begin{pmatrix}1&0\\-\epsilon Y^{-1}+X&1\end{pmatrix})\,dY\\
&=q^{-4n\rho t}\int_{\mathfrak{u}}\chi_{\frac{1}{2}+\rho}(N(Y))^{-1}f(\frac{1}{2},\Phi,\tau\begin{pmatrix}1&0\\(-\epsilon Y^{-1}+X)\varpi^{2t}&1\end{pmatrix}))\,dY\\
&=q^{-4mnt}\int_{\mathfrak{u}}\chi_{\frac{1}{2}+\rho}(N(Y))^{-1}f(\frac{1}{2},\Phi,\tau\begin{pmatrix}1&0\\-\epsilon Y^{-1}+X\varpi^{2t}&1\end{pmatrix}))\,dY\\
&=q^{-4mnt}M(\frac{1}{2},\chi_V)f(\frac{1}{2},\Phi,\tau\begin{pmatrix}1&0\\X\varpi^{2t}&1\end{pmatrix})\,dY\\
&=q^{-4mnt}F_\phi(\tau\begin{pmatrix}1&0\\X\varpi^{2t}&1\end{pmatrix})\\
&=q^{-4mnt}\beta_V(\tau)\int_{V\otimes W^\triangledown}\phi(x)\psi(\frac{1}{4}\langle\langle x,x\begin{pmatrix}1&0\\X\varpi^{2t}&1\end{pmatrix}\rangle\rangle)\,dx\\
&=\beta_V(\tau)\int_{V\otimes W^\triangledown}\phi(x\varpi^{-t})\psi(\frac{1}{4}\langle\langle x,x\begin{pmatrix}1&0\\X&1\end{pmatrix}\rangle\rangle)\,dx\\
&=q^{-4mnt}F_{\phi_{-t}}(\tau\begin{pmatrix}1&0\\X&1\end{pmatrix}).
\end{aligned}
$$

Hence we have (2). □

Proposition 13.2. *Let $\phi,\phi'\in\mathcal{S}(V\otimes W^\triangledown)$. Then, for sufficiently large $t\in\mathbb{Z}$, we have*

$$\mathcal{I}(\phi_t,\phi')=(-1)^{mn}\chi_V(-1)^nq^{-4mnt}|G(V)|F_\phi(1)\overline{F_{\phi'}(\tau)}.$$

Proof. The Fourier transform on the space $\mathcal{S}(V\otimes W^\triangledown)$ is given by the action of the Weyl element τ of $G(W^\square)$. Hence we have

$$
\begin{aligned}
\mathcal{I}(\phi_t,\phi')&=\mathcal{I}(\widehat{\phi_t},\widehat{\phi'})\\
&=q^{-4mnt}\mathcal{I}((\widehat{\phi})_{-t},\widehat{\phi'})\\
&=q^{-4mnt}\int_{G(V)}((\widehat{\phi})_{-t},\overline{\omega_\psi^\square(h)\widehat{\phi'}})\,dh.
\end{aligned}
$$

When t is sufficiently large, the support of $(\widehat{\phi})_{-t}$ is sufficiently small. Hence this integral is

$$
\begin{aligned}
&q^{-4mnt}|G(V)|\widehat{(\widehat{\phi})_{-t}}(0)\overline{\widehat{\phi'}(0)}\\
&=q^{-4mnt}|G(V)|q^{4mnt}(\widehat{\widehat{\phi}})_t(0)\overline{\widehat{\phi'}(0)}\\
&=q^{-4mnt}|G(V)|\phi_t(0)\overline{\widehat{\phi'}(0)}\\
&=q^{-4mnt}\beta_V(\tau)|G(V)|F_\phi(1)\overline{F_{\phi'}(\tau)}.
\end{aligned}
$$

Hence we have the proposition. □

Proposition 13.3. *Let $\phi, \phi' \in \mathcal{S}(V \otimes W^{\triangledown})$. Then, for sufficiently large $t \in \mathbb{Z}$, we have*

$$\mathcal{E}(\phi_t, \phi') = m^{\circ}(\rho)^{-1}\alpha_1(W)q^{-4mnt}F_{\phi}(1)\overline{F_{\phi'}(\tau)}.$$

Proof. When t is sufficiently large, the support of Φ_{-t} is sufficiently small. Then, by using Lemma 7.8, we have

$$\begin{aligned}\mathcal{E}(\phi_t, \phi') &= q^{-4mnt}\int_{G(W)} f(\frac{l}{2}, \Phi_{-t}, (g, 1))\overline{F_{\phi'}(g, 1)}\, dg \\ &= m^{\circ}(\rho)^{-1}\alpha_1(W)q^{-4mnt}\int_{U(W^{\triangle})} f(\frac{l}{2}, \Phi_{-t}, \tau u)\overline{F_{\phi'}(\tau u)}\, du \\ &= m^{\circ}(\rho)^{-1}\alpha_1(W)q^{-4mnt}\left(\int_{\mathfrak{u}_n} \Phi_{-t}(X)\, dX\right)\overline{F_{\phi'}(\tau)} \\ &= m^{\circ}(\rho)^{-1}\alpha_1(W)q^{-4mnt}F_{\phi}(1)\overline{F_{\phi'}(\tau)}.\end{aligned}$$

Hence we have the proposition. □

By Propositions 13.2 and 13.3, we have the following:

Proposition 13.4. *If V is anisotropic, then we have*

$$\alpha_2(V, W) = (-1)^{mn}\chi_V(-1)^n \cdot |G(V)| \cdot m^{\circ}(\rho) \cdot \alpha_1(W)^{-1}.$$

By substituting the values of $|G(V)|$ (Corollary 6.2) and $\alpha_1(W)$ (Proposition 7.6) for Proposition 13.4, we obtain Proposition 10.3(2) and Proposition 10.3(1) with V anisotropic. Thus, we finish the proof of Proposition 10.3.

14. The behavior of the γ-factor under the local theta correspondence

The purpose of this section is to explain the behavior of the γ-factor under the local theta correspondence, which extends [GI14, Theorem 11.5]. Let V be a right ϵ-Hermitian space of dimension m, and let W be a left $(-\epsilon)$-Hermitian space of dimension n. In this section, we allow D to be split and l not to be 1.

Theorem 14.1. *Let π be an irreducible representation of $G(W)$ and let ω be a character of $F^{\times}$. We denote $\sigma = \theta(\pi, V)$ and we assume $\sigma \neq 0$.*

(1) *If $l > 0$, then we have*

$$\frac{\gamma^V(s, \sigma \times \omega\chi_V, \psi)}{\gamma^W(s, \pi \times \omega\chi_W, \psi)} = \prod_{i=1}^{l}\gamma_F(s + \frac{l+1}{2} - i, \omega\chi_V\chi_W, \psi)^{-1}.$$

(2) *If $l < 0$, then we have*

$$\frac{\gamma^V(s, \sigma \times \omega\chi_V, \psi)}{\gamma^W(s, \pi \times \omega\chi_W, \psi)} = \prod_{i=1}^{-l}\gamma_F(s + \frac{-l+1}{2} - i, \omega\chi_V\chi_W, \psi).$$

The proof of Theorem 14.1 consists of four subsections (§§14.1–14.4). In the first three subsections, we reduce Theorem 14.1 to the unramified cases by using properties of the doubling γ-factor. In the last subsection, we discuss the unramified cases to finish the proof of Theorem 14.1.

14.1. **Multiplicative argument.** We put

$$f_D(s,V,W,\omega,\psi)=\begin{cases}\prod_{i=1}^{l}\gamma_F(s+\frac{l+1}{2}-i,\omega\chi_V\chi_W,\psi)^{-1} & (l>0),\\ \prod_{i=1}^{-l}\gamma_F(s+\frac{-l+1}{2}-i,\omega\chi_V\chi_W,\psi) & (l<0),\end{cases}$$

and we put

$$e_D(s,V,W,\pi,\omega,\psi)=\frac{\gamma^W(s,\sigma\times\omega\chi_W,\psi)}{\gamma^V(s,\pi\times\omega\chi_V,\psi)}\cdot f_D(s,V,W,\omega,\psi).$$

Then, Theorem 14.1 is equivalent to $e_D(s,V,W,\pi,\omega,\psi)=1$. When D is split, this equality has been already proved [GI14, Theorem 11.5].

Let $\{V_p\}_{p\geq 0}$ and $\{W_q\}_{q\geq 0}$ be Witt towers containing V and W respectively. Put $V=V_r, W=W_t, m_0=\dim_D V_0, m_0=\dim_D W_0$, and $l_p=l_{V_p,W}$. We denote by $\mathcal{J}_q(\pi)$ the set of the $G(W_0)$-part of the non-zero irreducible quotients of the Jacquet modules $J_P(\pi)$ for all parabolic subgroups P whose Levi subgroups contain $G(W_q)$ as a direct factor. We first state the multiplicativity:

Lemma 14.2. *We denote by $r(\pi)$ the first occurrence index of π (see §9.3). Suppose that $\mathcal{J}_q(\pi)\neq\varnothing$ and $p\geq r(\pi)$. Then, for an irreducible representation $\pi_q\in\mathcal{J}_q(\pi)$, we have*

$$e_D(V_p,W_q,\pi_q,\omega,\psi)=e_D(V,W,\pi,\omega,\psi).$$

Proof. First, we consider the case $q=t$ and $\pi_q=\pi$. We may assume that $r=r(\pi)$. Put $\sigma=\theta_\psi(\pi,V)$ and $\sigma'=\theta_\psi(\pi,V_p)$. Then, by Proposition 9.6(1), we have

$$\begin{aligned}&\gamma^{V_p}(s,\sigma'\times\omega,\psi)\cdot\gamma^V(s,\sigma\times\omega,\psi)^{-1}\\ &=\gamma^{GJ}_{\mathrm{GL}_{p-r}(D)}(s+(\frac{l_p}{2}+p-r),\omega,\psi)\cdot\gamma^{GJ}_{\mathrm{GL}_{p-r}(D)}(s-(\frac{l_p}{2}+p-r),\omega,\psi)\\ &=\prod_{i=1}^{p-r}\gamma^{GJ}_{D^\times}(s+\frac{l_p}{2}+(2i-1),\omega,\psi)\cdot\prod_{i=1}^{p-r}\gamma^{GJ}_{D^\times}(s-(\frac{l_p}{2}+(2i-1),\omega,\psi)\\ &=\prod_{i=1}^{2(p-r)}\gamma_F(s+\frac{l_p-1}{2}+i,\omega,\psi)\cdot\prod_{i=1}^{2(p-r)}\gamma_F(s+\frac{-l_p+1}{2}-i,\omega,\psi)\\ &=f_D(V_p,W,\pi,\omega,\psi)f_D(V,W,\pi,\omega,\psi)^{-1}.\end{aligned}$$

Here, $\gamma^{GJ}_{\mathrm{GL}_u(D)}(s,\omega,\psi)$ is the γ-factor defined by

$$\epsilon^{GJ}_{\mathrm{GL}_u(D)}(s,\omega,\psi)\frac{L^{GJ}_{\mathrm{GL}_u(D)}(1-s,\omega^{-1})}{L^{GJ}_{\mathrm{GL}_u(D)}(s,\omega)},$$

where $\epsilon^{GJ}_{\mathrm{GL}_u(D)}(s,-,\psi)$ and $L^{GJ}_{\mathrm{GL}_u(D)}(s,-)$ are ϵ-and L-factors defined in [GJ72], and ω denotes the composition $\omega\circ N$ of ω with the reduced norm N of $\mathrm{GL}_u(D)$. Thus we have

$$e_D(V_p,W,\pi,\omega,\psi)=e_D(V,W,\pi,\omega,\psi).$$

Second, we consider the general case. Put

$$t(\pi_q)=\min\{q'=0,\ldots,q\mid\mathcal{J}_{q'}(\pi_q)\neq\varnothing\}.$$

Then, any $\pi_{t(\pi)}\in\mathcal{J}_{t(\pi_q)}(\pi_q)$ is supercuspidal. Take a positive integer p' so that $p'\geq\max\{r+q-t,r(\pi)+q-t(\pi)\}$. Then, by the first part of this proof, we have

$$e_D(s,V_p,W_q,\pi_q,\omega,\psi)=e_D(s,V_{p'},W_q,\pi_q,\omega,\psi).$$

Moreover, by using Proposition 9.6(2) repeatedly, we can show that

$$e_D(s, V_{p'}, W_q, \pi_q, \omega, \psi) = e_D(s, V_{p'-(q-t(\pi))}, W_{t(\pi)}, \pi_{t(\pi)}, \omega, \psi).$$

By tracing the above discussions conversely, the right-hand side is equal to

$$e_D(s, V_{p'+(t-q)}, W, \pi, \omega, \psi) = e_D(s, V, W, \pi, \omega, \psi).$$

Thus we have the lemma. □

14.2. **Global argument.** In this subsection, we explain the global argument which we use in the proof of Theorem 14.1.

Lemma 14.3. *Let $\mathbb{F}$ be a number field, let $\mathbb{A}$ be the ring of its adeles, let $\mathbb{D}$ be a division quaternion algebra over $\mathbb{F}$, let $\underline{V}$ be a right ϵ-Hermitian space over $\mathbb{D}$, let $\underline{W}$ be a left $(-\epsilon)$-Hermitian space over $\mathbb{D}$, let Π be an irreducible cuspidal automorphic representation of $G(\underline{W})(\mathbb{A})$, let $\underline{\omega}$ be a Hecke character of $\mathbb{A}^\times/\mathbb{F}^\times$, and let $\underline{\psi}$ be a non-trivial additive character of $\mathbb{A}/\mathbb{F}$. Then, we have*

$$\prod_v e_{\mathbb{D}_v}(s, \underline{V}_v, \underline{W}_v, \Pi_v, \underline{\omega}_v, \underline{\psi}_v) = 1.$$

Proof. Consider the Witt tower $\{\underline{V}_p\}_{p=0}^\infty$ so that $\underline{V}_r = \underline{V}$. Denote by $r(\Pi)$ the first occurrence index of Π in $\{\underline{V}_p\}_{p=0}^\infty$, by Σ the theta correspondence $\theta(\Pi, \underline{W}_{r(\Pi)})$ of Π, and by S the set of the places where $\mathbb{D}_v$ is a division algebra. Then, we have $\theta_{\underline{\psi}}(\Pi, \underline{V}_{r(\Pi)})$ is cuspidal, and we have

$$\begin{aligned}
\prod_v e_{\mathbb{D}_v}(s, \underline{V}_v, \underline{W}_v, \Pi_v, \underline{\omega}_v, \underline{\psi}_v) &= \prod_{v\in S} e_{\mathbb{D}_v}(s, \underline{V}_{r(\Pi)_v}, \underline{W}_v, \Pi_v, \underline{\omega}_v, \underline{\psi}_v) \\
&= \prod_{v\in S} \frac{\gamma^V(s, \Sigma \boxtimes \underline{\omega}\chi_{\underline{W}}, \underline{\psi})}{\gamma^W(\pi, \Pi \boxtimes \underline{\omega}\chi_{\underline{V}}, \underline{\psi})} \cdot f_{\mathbb{D}_v}(s, \underline{V}, \underline{W}, \underline{\omega}, \underline{\psi}) \\
&\quad \times \frac{L^S(s, \Sigma\boxtimes\underline{\omega}\chi_{\underline{W}}) L_f^S(s)}{L^S(s, \Pi\boxtimes\underline{\omega}\chi_{\underline{V}})} \cdot \frac{L^S(1-s, \Pi\boxtimes\underline{\omega}\chi_{\underline{V}})}{L^S(1-s, \Sigma\boxtimes\underline{\omega}\chi_{\underline{W}}) L_f^S(1-s)} \\
&= 1,
\end{aligned}$$

where $L_f^S(s) = \prod_{v\notin S} L_{f,v}(s)$ with

$$L_{f,v}(s) = \begin{cases} \prod_{i=1}^{l} L_{\mathbb{F}_v}(s + \frac{l+1}{2} - i, \underline{\omega}_v \chi_{\underline{V}_v} \chi_{\underline{W}_v}) & (l > 0), \\ \prod_{i=1}^{-l} L_{\mathbb{F}_v}(s + \frac{-l+1}{2} - i, \underline{\omega}_v \chi_{\underline{V}_v} \chi_{\underline{W}_v})^{-1} & (l < 0). \end{cases}$$

Hence we have the lemma. □

14.3. **Globalization.**

Lemma 14.4. *Let $\mathbb{F}$ be a global field of characteristic zero, let $v_1, \ldots, v_d$ be places of $\mathbb{F}$, and let $\omega_1, \ldots, \omega_d$ be unitary characters of $\mathbb{F}_{v_1}^\times, \ldots, \mathbb{F}_{v_d}^\times$ respectively.*

(1) *Suppose that ω_i is trivial for $i = 2, \ldots, d$. Then, there exists a Hecke character $\underline{\omega}$ of $\mathbb{A}^\times$ so that $\underline{\omega}_j = \omega_{v_j}$ for $j = 1, \ldots, d$.*
(2) *Suppose that $\mathbb{F}_{v_1} = \cdots = \mathbb{F}_{v_d}$ and $\omega_1 = \cdots = \omega_d$. Then, there exists a Hecke character $\underline{\omega}$ of $\mathbb{A}^\times/\mathbb{F}^\times$ so that $\underline{\omega}_j = \omega_{v_j}$ for $j = 1, \ldots, d$.*

Proof. Let η_j be the character of $\mathrm{Gal}(\mathbb{F}_{v_j}^s/\mathbb{F}_{v_j})$ associated with ω_j via the local class field theory for $j = 1, \ldots, d$. First, suppose that ω_i is trivial for $i = 2, \ldots, d$. Take a finite Galois extension $\mathbb{L}$ of $\mathbb{F}$ so that $\ker \eta_1 = \mathrm{Gal}(\mathbb{F}_{v_1}^s/\mathbb{L}_{w_1})$ and $\mathbb{L}_{w_i} = \mathbb{F}_{v_i}$

for $i = 2, \dots, d$. Here, $w_1, \dots, w_d$ are some places of $\mathbb{L}$ lying above $v_1, \dots, v_d$ respectively. Take a character $\widetilde{\eta}$ of $\mathrm{Gal}(\mathbb{L}/\mathbb{F})$ so that $\widetilde{\eta}|_{\mathrm{Gal}(\mathbb{L}_{w_1}/\mathbb{F}_{v_1})} = \eta$, and define $\underline{\omega}$ as the Hecke character of $\mathbb{A}^\times$ associated with $\widetilde{\eta}$ via the global class field theory. Then we have $\underline{\omega}_{v_1} = \omega_1$ and $\underline{\omega}_{v_i} = 1_{F_{v_i}^\times}$ for $i = 2, \dots d$, and thus we have (1).

Then, suppose that $\mathbb{F}_{v_1} = \cdots = \mathbb{F}_{v_d}$ and $\omega_1 = \cdots = \omega_d$. By (1), there exists a Hecke character $\underline{\chi}$ of $\mathbb{A}^\times$ so that $\underline{\chi}_{v_1} = \omega_1$ and $\underline{\chi}_{v_i} = 1_{\mathbb{F}_{v_i}^\times}$ for $i = 2, \dots, d$. Let $\underline{\omega}$ be the continuous unitary character of $\mathbb{A}^\times$ given by

$$\underline{\omega}_v = \begin{cases} \omega_1 & (v = v_1, \dots, v_d), \\ \underline{\chi}_v^d & (\text{otherwise}). \end{cases}$$

Then we have $\underline{\omega}(\mathbb{F}^\times) = 1$ and it is a Hecke character of $\mathbb{A}^\times$ satisfying $\underline{\omega}_{v_j} = \omega_j$ for $j = 1, \dots, d$. Thus we have (2), and we finish the proof of Lemma 14.4. □

Let ψ be a unitary non-trivial additive character of F. For $a \in F^\times$ we denote by ψ_a the character of F defined by $\psi_a(x) = \psi(ax)$ for $x \in F$.

Lemma 14.5. *Assume that D is a division quaternion algebra. Let F' be a non-Archimedean local field of characteristic zero, let ψ' be an additive non-trivial character of F', let D' be a division quaternion algebra over F', let V' be another right ϵ-Hermitian space of dimension m, and let W' be another left $(-\epsilon)$-Hermitian space of dimension n. Then, there exist*

- *a global field $\mathbb{F}$ and its places v_1, v_2 such that $\mathbb{F}_{v_1} = F, \mathbb{F}_{v_2} = F'$,*
- *a division quaternion algebra $\mathbb{D}$ over $\mathbb{F}$ such that $\mathbb{D}_{v_1} = D$, $\mathbb{D}_{v_2} = D'$, and $\mathbb{D}_v$ is split for $v \neq v_1, v_2$,*
- *a left $(-\epsilon)$-Hermitian space $\underline{W}$ over $\mathbb{D}$ such that $\underline{W}_{v_1} = W, \underline{W}_{v_2} = W'$,*
- *a right ϵ-Hermitian space $\underline{V}$ over $\mathbb{D}$ such that $\underline{V}_{v_1} = V, \underline{V}_{v_2} = V'$,*
- *a Hecke character $\underline{\omega}$ of $\mathbb{A}^\times$ such that $\underline{\omega}_{v_1} = \omega, \underline{\omega}_{v_2} = 1_{F_{v_2}^\times}$,*
- *a non-trivial additive character $\underline{\psi}$ of $\mathbb{A}/\mathbb{F}$ such that $\underline{\psi}_{v_1} = \psi_{a_1^2}, \underline{\psi}_{v_2} = \psi'_{a_2^2}$ for some $a_1 \in F^\times, a_2 \in F'^\times$.*

Proof. The existences of such $\mathbb{F}$ and $\mathbb{D}$ are well-known. The existences of such $\underline{W}$, $\underline{V}$ and $\underline{\psi}$ are due to the weak approximation. It remains to show the existence of $\underline{\omega}$. Let η be the character of $\mathrm{Gal}(F^s/F)$ associated with ω via the local class field theory, and let L be the Galois extension field of F so that $\ker \eta = \mathrm{Gal}(F^s/L)$. Here, F^s denotes the separable closure of F. Then there exists a Galois extension field $\mathbb{L}$ of $\mathbb{F}$ such that $\mathbb{L}_{w_1} = L$ and $\mathbb{L}_{w_2} = F'$ where w_1 (resp. w_2) is some place of $\mathbb{L}$ lying above v_1 (resp. v_2). Take a character $\widetilde{\eta}$ of $\mathrm{Gal}(\mathbb{L}/\mathbb{F})$ so that $\widetilde{\eta}|_{\mathrm{Gal}(L/F)} = \eta$, and define $\underline{\omega}$ as the Hecke character of $\mathbb{F}$ associated with $\widetilde{\eta}$ via the global class field theory. Then, we have $\underline{\omega}_{v_1} = \omega$. Moreover, by $\mathbb{L}_{w_2} = F_{v_2}$, we have $\ker \widetilde{\eta} \supset \mathrm{Gal}(F'^s/F')$ which implies that $\underline{\omega}_{v_2} = 1_{F_{v_2}^\times}$. Hence we have the lemma. □

Moreover, by using the Poincare series, we obtain a globalization lemma of representations.

Lemma 14.6. *Let $\mathbb{F}$ be a global field, let G be a reductive group over $\mathbb{F}$, let Z be the center of G, let A be a maximal $\mathbb{F}$-split torus of Z, let $\underline{\chi}$ be a unitary character $A(\mathbb{A})/A(\mathbb{F}) \to \mathbb{C}^\times$, let S_0 be a non-empty set of places of $\mathbb{F}$ such that all Archimedean places are contained in S_0, let S be a finite set of places of $\mathbb{F}$ such that $S_0 \cap S = \varnothing$. Suppose that $Z(\mathbb{F}_v)/A(\mathbb{F}_v)$ is compact for $v \in S$, an irreducible* *supercuspidal representation π_v of $G(\mathbb{F}_v)$ so that $\pi_v|_{A(\mathbb{F}_v)}$ coincides with $\underline{\chi_v}$ is given for each $v \in S$, and a compact open subgroup K_v is given for each $v \notin S_0 \cup S$. Then there is an irreducible cuspidal automorphic representation Π of $G(\mathbb{A})$ such that*

- *$\Pi|_{A(\mathbb{A})}$ coincides with $\underline{\chi}$,*
- *$\Pi_v \cong \pi_v$ for $v \in S$,*
- *Π_v possesses a non-zero K_v-fixed vector for $v \notin S_0 \cup S$.*

Proof. Denote by ω_v the central character of π_v for each $v \in S$. Since $\prod_{v\in S} Z(\mathbb{F}_v)/A(\mathbb{F}_v)$ is compact, we have that $(\prod_{v\in S} Z(\mathbb{F}_v)) \cdot A(\mathbb{A})$ is a closed subgroup of $Z(\mathbb{A})$. Moreover, we have that $(\prod_{v\in S} Z(\mathbb{F}_v))A(\mathbb{A})/A(F)$ can be identified with a closed subgroup of $Z(\mathbb{A})/A(\mathbb{F})$. Consider a character $\widetilde{\underline{\chi}}$ on $(\prod_{v\in S} Z(\mathbb{F}_v))A(\mathbb{A})/A(F)$ given by

$$\widetilde{\underline{\chi}}((z_v)_{v\in S}, a) = (\prod_{v\in S} \omega_v(z_v)) \cdot \underline{\chi}(a)$$

for $z_v \in Z(\mathbb{F}_v)$ $(v \in S)$ and $a \in A(\mathbb{A})$. Then, this character can be extended to a character $\underline{\omega}$ on $Z(\mathbb{A})/A(F)$. Hence, by [Hen84, Appendice I], there exists an irreducible cuspidal automorphic representation Π of $G(\mathbb{A})$ so that $\Pi|_{Z(\mathbb{A})} = \underline{\omega}$, $\Pi_v \cong \pi_v$ for $v \in S$ and Π_v possesses a non-zero K_v-fixed vector for each $v \notin S_0 \cup S$. Thus we have the lemma. □

14.4. **Completion of the proof of Theorem 14.1.** Let π be an irreducible representation of $G(W)$, and let ω be a character of $F^\times$. By Lemma 14.2 and Corollary 9.7, we may assume that π and σ are irreducible supercuspidal representations. Take

- a global field $\mathbb{F}$ and its places v_1, v_2 such that $\mathbb{F}_{v_1} = F, \mathbb{F}_{v_2} = F$,
- a division quaternion algebra $\mathbb{D}$ over $\mathbb{F}$ such that $\mathbb{D}_{v_1} = D$, $\mathbb{D}_{v_2} = D$, and $\mathbb{D}_v$ is split for $v \neq v_1, v_2$,
- a left $(-\epsilon)$-Hermitian space $\underline{W}$ over $\mathbb{D}$ such that $\underline{W}_{v_1} = W$, the dimension of the anisotropic kernel of $\underline{W}_{v_2}$ is 0 or 1, and $\mathfrak{d}(\underline{W}_{v_2}) \in \mathcal{O}^\times_{\mathbb{F}_{v_2}}$,
- a right ϵ-Hermitian space $\underline{V}$ over $\mathbb{D}$ such that $\underline{V}_{v_1} = V$, the dimension of the anisotropic kernel of $\underline{V}_{v_2}$ is 0 or 1, and $\mathfrak{d}(\underline{V}_{v_2}) \in \mathcal{O}^\times_{\mathbb{F}_{v_2}}$,
- a non-trivial additive character $\underline{\psi}$ of $\mathbb{A}/\mathbb{F}$ such that $\underline{\psi}_{v_1} = \psi_a$ for some $a \in F^\times$.

Moreover, we can take a Hecke character $\underline{\omega}$ of $\mathbb{A}^\times$ such that $\underline{\omega}_{v_1} = \omega$ and $\underline{\omega}_{v_2} = 1_{F^\times_{v_2}}$ (Lemma 14.4(1)). Denote by $\{\underline{V}_i\}_{i=0}^\infty$ the Witt tower containing $\underline{V}$. Let K_{v_2} be the maximal compact subgroup fixing 0 of the apartment E of $G(\underline{W}_{v_2})$. Then, by Lemma 14.6, we can take an irreducible cuspidal automorphic representation Π of $G(\underline{W})(\mathbb{A})$ so that $\Pi_{v_1} = \pi$, and Π_{v_2} possesses a non-zero K_{v_2} fixed vector. Hence, by Lemma 14.3, we have

$$\begin{aligned} e_D(s, V, W, \pi, \omega, \psi) &= \prod_{v\neq v_1} e_{\mathbb{D}_v}(s, \underline{V}, \underline{W}, \Pi, \underline{\omega}, \underline{\psi})^{-1} \\ &= e_D(s, \underline{V}_{v_2}, \underline{W}_{v_2}, \Pi_{v_2}, 1_{F^\times_{v_2}}, \underline{\psi}_{v_2})^{-1}. \end{aligned} \tag{14.1}$$

Denote by ψ' the localization $\underline{\psi}_{v_2}$ and by W_0' the anisotropic kernel of $\underline{W}_{v_2}$. Then, we have $t(\Pi_{v_2}) = 0$ and $1_{G(W_0')} \in \mathcal{J}_0(\Pi_{v_2})$. Here, $1_{G(W_0')}$ is the trivial representation of $G(W_0')$. Hence, (14.1) is equal to

$$e_D(s, (\underline{V}_p)_{v_2}, W_0', 1_{G(W_0')}, 1_{F_{v_2}^\times}, \psi')^{-1},$$

where p is a sufficiently large integer so that $\Theta_{\psi'}(1_{G(W_0')}, (\underline{V}_p)_{v_2}) \neq 0$. By the above observation, it only suffices to consider the cases where $n = \dim W = 0, 1$ and $\pi = 1_{G(W)}$.

Lemma 14.7. *We denote by $1_{G(V)}$ (resp. $1_{G(W)}$) the trivial representation of $G(V)$ (resp. $G(W)$). Suppose that $n = 0$. Then we have $r(1_{G(W)}) = 0$ and $\theta_\psi(1_{G(W)}, V) = 1_{G(V)}$.*

For the rest of this subsection, we consider the case $n = 1$. In this case, we consider the accidental isomorphism:

$$G(V) \cong \mathrm{SU}_E(2), \text{ and } G(W) \cong \mathrm{U}'_E(1). \tag{14.2}$$

Here,

- E is the quadratic unramified extension field of F associated with the quadratic character χ_W of $F^\times$,
- $\mathrm{SU}_E(2)$ is the special unitary group preserving the Hermitian form
$$(\ ,\ )_E\colon E^2 \times E^2 \to E\colon (\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}) \mapsto \overline{x_1}y_1 - \varpi_F \cdot \overline{x_2}y_2,$$
where $\overline{x_i}$ denotes the conjugate of x_i with respect to E/F,
- $\mathrm{U}'_E(1)$ is the unitary group preserving the skew-Hermitian form
$$\langle\ ,\ \rangle_E\colon E \times E \to E\colon x, y \mapsto x\alpha\overline{y},$$
where $\alpha \in E$ is a non-zero trace zero element with $\mathrm{ord}_E(\alpha) = 0$.

In particular, for these groups, the L-parameters are defined.

Proposition 14.8. *Suppose that $n = m = 1$ and $\epsilon = 1$. Let π be a non-trivial irreducible representation of $G(W)$, and let ϕ be its L-parameter. Then, the representation $\Theta_\psi(\pi, V)$ is non-zero irreducible, and has L-parameter*

$$(\phi \otimes \chi_V\chi_W) \oplus \chi_W.$$

Proof. By [Ike19, §7], the accidental isomorphisms (14.2) are compatible with the local theta correspondences. We know the description of the local theta correspondence

$$\mathrm{Irr}(\mathrm{U}'_E(1)) \to \mathrm{Irr}(\mathrm{U}_E(2))$$

via L-parameters [GI16, Theorem 4.4]. Therefore, we have the claim. □

By tracing the converse of the global argument at the beginning of this subsection, we obtain:

Corollary 14.9. *Suppose that $n = 1$ and $\epsilon = 1$. Denote by $\{V_i\}_{i=0}^\infty$ the Witt tower containing V. Then we have $e_D(s, V_p, W, 1_{G(W)}, 1_{F^\times}, \psi) = 1$ for sufficiently large p.*

Similarly, by using the accidental isomorphism, we have:

Lemma 14.10. *Suppose that $n = 1$ and $\epsilon = -1$. Denote by $\{V_i\}_{i=0}^\infty$ the Witt tower containing V. Then we have $e_D(s, V_p, W, 1_{G(W)}, 1_{F^\times}, \psi) = 1$ for sufficiently large p.*

Hence, we complete the proof of Theorem 14.1.

15. The local analogue of the Rallis inner product formula

In this section, we discuss the local analogue of the Rallis inner product formula following [GI14], and describe the relation between $\alpha_2(V,W)$ and $\alpha_3(V,W)$.

We use the setting of §3, and we take a basis $\underline{e}$ of W as in §4. Suppose that $l=1$ and π is an irreducible square-integrable representation of $G(W)$. Consider the map

$$\mathcal{P}\colon \omega_\psi \otimes \overline{\omega_\psi} \otimes \overline{\omega_\psi} \otimes \omega_\psi \otimes \overline{\pi} \otimes \pi \otimes \pi \otimes \overline{\pi} \to \mathbb{C}$$

defined by

$$\begin{aligned}&\mathcal{P}(\phi_1,\phi_2,\phi_3,\phi_4;v_1,v_2,v_3,v_4)\\ &=\int_{G(V)} (\sigma(h)\theta(\phi_1,v_1),\theta(\phi_2,v_2))\cdot\overline{(\sigma(h)\theta(\phi_3,v_3),\theta(\phi_4,v_4))}\,dh.\end{aligned}$$

The integral defining $\mathcal{P}$ converges absolutely (Lemma 9.3). As in [GI14, §18], we compute $\mathcal{P}$ in two ways. First, we have

$$\begin{aligned}&\mathcal{P}(\phi_1\ldots,\phi_4,v_1,\ldots,v_4)\\ &=\frac{1}{\deg\sigma}\cdot(\theta(\phi_1,v_1),\theta(\phi_3,v_3))\cdot\overline{(\theta(\phi_2,v_2),\theta(\phi_4,v_4))}\\ &=\frac{1}{\deg\sigma}\cdot Z(-\frac{1}{2},F_{\phi_1\otimes\overline{\phi_3}},\overline{\xi}_{v_1,,v_3})\cdot\overline{Z(-\frac{1}{2},F_{\phi_2\otimes\overline{\phi_4}},\overline{\xi}_{v_2,v_4})}.\end{aligned}$$

Second, changing the order of integrals and using Proposition 8.2, we have

$$\begin{aligned}&\mathcal{P}(\phi_1\ldots,\phi_4,v_1,\ldots,v_4)\\ &=\int_{G(W)}\int_{G(W)}\left(\int_{G(V)}(\omega_\psi(gh)\phi_1,\phi_2)\overline{(\omega_\psi(g'h)\phi_3,\phi_4)}\,dh\right)\\ &\qquad(\pi(g)v_1,v_2)\overline{(\pi(g')v_3,v_4)}\,dgdg'\\ &=|2|_F^{2mn}\cdot|N(R(\underline{e}))|^{-m}\cdot\int_{G(W)}\int_{G(W)}\\ &\qquad\mathcal{I}(\omega_\psi(g)\phi_1\otimes\omega_\psi(g')\phi_3,\phi_2\otimes\phi_4)\cdot(\pi(g)v_1,v_2)\overline{(\pi(g')v_3,v_4)}\,dgdg'\\ &=\alpha_2(V,W)\cdot|2|_F^{2mn}\cdot|N(R(\underline{e}))|^{-m}\int_{G(W)}\int_{G(W)}\\ &\qquad\mathcal{E}(\omega_\psi(g)\phi_1\otimes\omega_\psi(g')\phi_3,\phi_2\otimes\phi_4)\cdot(\pi(g)v_1,v_2)\overline{(\pi(g')v_3,v_4)}\,dgdg'.\end{aligned}$$

Substituting the definition of $\mathcal{E}$, the expression is equal to

$$
\begin{aligned}
&\alpha_2(V,W)\cdot|2|_F^{2mn}\cdot|N(R(\underline{e}))|^{-m}\int_{G(W)}\int_{G(W)}\int_{G(W)}\\
&\quad F^\dagger_{\phi_1\otimes\phi_3}(\iota(g'^{-1}g''g,1))\cdot\overline{F_{\phi_2\otimes\phi_4}(\iota(g'',1))}\cdot(\pi(g)v_1,v_2)\overline{(\pi(g')v_3,v_4)}\,dg''dgdg'\\
&=\alpha_2(V,W)\cdot|2|_F^{2mn}\cdot|N(R(\underline{e}))|^{-m}\int_{G(W)}\int_{G(W)}\int_{G(W)}\\
&\quad F^\dagger_{\phi_1\otimes\phi_3}(\iota(g,1))\cdot\overline{F_{\phi_2\otimes\phi_4}(\iota(g'',1))}\cdot(\pi(g'g)v_1,\pi(g'')v_2)\overline{(\pi(g')v_3,v_4)}\,dg''dgdg'\\
&=\frac{\alpha_2(V,W)}{\deg\pi}\cdot|2|_F^{2mn}\cdot|N(R(\underline{e}))|^{-m}\int_{G(W)}\int_{G(W)}\\
&\quad F^\dagger_{\phi_1\otimes\phi_3}(\iota(g,1))\cdot\overline{F_{\phi_2\otimes\phi_4}(\iota(g'',1))}\cdot(\pi(g)v_1,v_3)\cdot\overline{(\pi(g'')v_2,v_4)}\,dg''dg\\
&=\frac{\alpha_2(V,W)}{\deg\pi}\cdot|2|_F^{2mn}\cdot|N(R(\underline{e}))|^{-m}\cdot Z(\frac{1}{2},F^\dagger_{\phi_1\otimes\overline{\phi}_3},\overline{\xi}_{v_1,v_3})\cdot\overline{Z(-\frac{1}{2},F_{\phi_2\otimes\overline{\phi}_4},\overline{\xi}_{v_2,v_4})}.
\end{aligned}
$$

The local functional equation of the doubling zeta integral says that

$$
\begin{aligned}
&Z(-\frac{1}{2},F_{\phi_1\otimes\overline{\phi}_3},\overline{\xi}_{v_1,v_3})\\
&=\left(c(s,\chi_V,A_0,\psi)R(s,\chi_V,A,\psi)^{-1}\gamma(s+\frac{1}{2},\pi\times\chi_V,\psi)\right)\Big|_{s=\frac{1}{2}}\\
&\quad\times c_\pi(-1)\cdot Z(\frac{1}{2},F^\dagger_{\phi_1\otimes\overline{\phi}_3},\overline{\xi}_{v_1,v_3}).
\end{aligned}
$$

By Theorem 14.1 and Proposition 7.2, we have

$$
\begin{aligned}
&c(s,\chi_V,A_0,\psi)R(s,\chi_V,A,\psi)^{-1}\gamma(s+\frac{1}{2},\pi\times\chi_V,\psi)\\
&=e(G(W))\cdot|N(R(\underline{e}))|^{-s}\cdot|2|^{-2ns+n(n-\frac{1}{2})}\cdot\chi_V^{-1}(4)\\
&\times\frac{\gamma(s+\frac{1}{2},1_{F^\times},\psi)}{\gamma(2s,1_{F^\times},\psi)}\cdot\prod_{i=1}^{n-1}\gamma(2s-2i,1_{F^\times},\psi)^{-1}\cdot\gamma^V(s+\frac{1}{2},\sigma\times\chi_W,\psi)\\
&\times\begin{cases}\gamma(s-n+\frac{1}{2},\chi_V,\psi)^{-1} & -\epsilon=1,\\ \epsilon(\frac{1}{2},\chi_W,\psi)^{-1} & -\epsilon=-1.\end{cases}
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\gamma^V(1,\sigma\times\chi_W,\psi)&=\gamma(1,\sigma^\vee\times\chi_W,\psi)\\
&=\gamma(0,\sigma\times\chi_W,\overline{\psi})^{-1}\\
&=\gamma(0,\sigma\times\chi_W,\psi)^{-1}\times\begin{cases}\chi_W(-1) & (\epsilon=1),\\ \chi_V(-1) & (\epsilon=-1).\end{cases}
\end{aligned}
$$

Summarizing the above discussions and substituting Proposition 9.2, we obtain:

Proposition 15.1. *Suppose that $l = 1$ and π is square-integrable. Then, we have*

$$\frac{\deg \pi}{\deg \sigma} \cdot c_\sigma(-1) \cdot \gamma^V(0, \sigma \times \chi_W, \psi)^{-1}$$
$$= \frac{1}{2} \cdot \alpha_2(V, W) \cdot e(G(W)) \cdot |2|_F^{2n\rho - n(n-\frac{1}{2})} \cdot |N(R(\underline{e}))|^{-\rho} \cdot \prod_{i=1}^{n-1} \frac{\zeta_F(2i)}{\zeta_F(1-2i)}$$
$$\times \begin{cases} \chi_V(-1)^{n+1}\gamma(1-n, \chi_V, \psi) & (-\epsilon = 1), \\ \chi_W(-1)^{m+1}\epsilon(\frac{1}{2}, \chi_W, \psi) & (-\epsilon = -1). \end{cases}$$

Hence we obtain Lemma 11.1. We write down the constant $\alpha_3(V, W)$ in the minimal cases, which proves a special case of Theorem 11.2.

Proposition 15.2.

(1) *In the case $\epsilon = -1$ and V is anisotropic, we have*

$$\alpha_3(V, W) = \epsilon(\frac{1}{2}, \chi_V, \psi)^{-1}.$$

(2) *In the case $\epsilon = 1$ and either V or W is anisotropic, we have*

$$\alpha_3(V, W) = \frac{1}{2} \cdot \chi_W(-1)^m \cdot \epsilon(\frac{1}{2}, \chi_W, \psi)^{-1}.$$

Proof. Recall that

$$\chi(-1) \cdot \epsilon(\frac{1}{2}, \chi, \psi) = \epsilon(\frac{1}{2}, \chi, \psi)^{-1}$$

for a quadratic character χ of $F^\times$. Then, for the case $m = 0$, one can verify this proposition directly. Otherwise, we obtain the claim by Proposition 10.3. $\square$

16. Plancherel measures

Our next goal is to prove Theorem 11.2 completely. This will be done in §18, and the Plancherel measure has important role in the proof. In this section, we recall some formulas of the Plancherel measure, and we discuss how the Plancherel measure behaves under the theta correspondence.

16.1. **Preliminaries.** Let G be a reductive group over F, let P be a parabolic subgroup of G, let M be a Levi subgroup of P, and let U be the unipotent radical of P. We denote by $X^*(M)$ the group of the algebraic characters of M, and by $E_{\mathbb{C}}^\vee$ the vector space $X^*(M) \otimes \mathbb{C}$. For a finite length representation π of M and for

$$\eta = \sum_{i=1}^t \chi_i \otimes s_i \in E_{\mathbb{C}}^\vee,$$

we denote by $\pi \otimes \eta$ the representation given by

$$[\pi \otimes \eta](g) := \pi(g) \prod_{i=1}^t |\chi_i(g)|^{s_i}$$

for $g \in M$. Take a maximal compact subgroup K of G so that $G = PK$. Then for $f \in \mathrm{Ind}_P^G(\pi)$, we define $f_\eta \in \mathrm{Ind}_P^G(\pi \otimes \eta)$ by

$$f_\eta(muk) = \prod_{i=1}^t |\chi_i(m)|^{s_i} f(muk)$$

for $m \in M, u \in U, k \in K$. Denote by $\overline{P}$ the opposite parabolic subgroup of P, and by $\overline{U}$ the unipotent radical of $\overline{P}$. It is known that for $f \in \mathrm{Ind}_P^G \pi$, the integral

$$[J_{\overline{P}|P}^G(\pi \otimes \eta) f_\eta](g) = \int_{\overline{U}} f_\eta(\overline{u} g)\, d\overline{u}$$

converges absolutely when η is contained in a certain open subset of $E_{\mathbb{C}}^\vee$, and it admits a meromorphic continuation to the whole space $E_{\mathbb{C}}^\vee$ with at most finitely many poles (see [Sha81]). Here, the measure $d\overline{u}$ is the normalized Haar measure as in §6.2. Therefore we have an intertwining operator

$$J_{\overline{P}|P}^G(\pi \otimes \eta)\colon\ \mathrm{Ind}_P^G(\pi \otimes \eta) \to \mathrm{Ind}_{\overline{P}}^G(\pi \otimes \eta)$$

for almost all $\eta \in E_{\mathbb{C}}^\vee$. This operator will be abbreviated to $J_{\overline{P}|P}(\pi \otimes \eta)$ unless it incurs confusion. The map $\eta \mapsto J_{\overline{P}|P}(\pi \otimes \eta)$ is rational (see [Wal03, §IV]). Since $\mathrm{Ind}_P^G(\pi \otimes \eta)$ is irreducible for all η in a certain Zariski open subset of $E_{\mathbb{C}}^\vee$ [Sau97, Theoréme 3.2], there exists a rational function $\mu(\eta, \pi)$ of η satisfying

$$J_{P|\overline{P}}(\pi \otimes \eta) \circ J_{\overline{P}|P}(\pi \otimes \eta) = \mu(\eta, \pi)^{-1}.$$

It is called the Plancherel measure.

Lemma 16.1. *If π is square-integrable, then we have $\mu(0, \pi) > 0$.*

Proof. Let (,) denote both the unitary inner product on $\mathrm{Ind}_P^G(\pi)$ and that of $\mathrm{Ind}_{\overline{P}}^G(\pi)$ as in the proof of [Wal03, IV, (6)]. Then, we have

$$\begin{aligned}\mu(0, \pi)^{-1}(f, f) &= (f, J_{P|\overline{P}}(\pi) \circ J_{\overline{P}|P}(\pi) f) \\ &= (J_{\overline{P}|P}(\pi) f, J_{\overline{P}|P}(\pi) f)\end{aligned}$$

for all $f \in \mathrm{Ind}_P^G(\pi)$. Choosing $f \in \mathrm{Ind}_P^G(\pi)$ so that $J_{\overline{P}|P}(\pi) f \neq 0$, we have $\mu(0, \pi) > 0$. Thus we have the lemma. $\square$

Let $W' \subset W$ be $(-\epsilon)$-Hermitian spaces, and let X, X^* be totally isotropic subspaces of W so that $W = X + W' + X^*$ and $X + X^*$ is the orthogonal complement of W'. Now we consider the case where $G = G(W)$, and $P = P(X)$. The restriction to $X + W'$ gives the identification $M \cong \mathrm{GL}(X) \times G(W')$. Then, for a finite length representation π' of $G(W')$ and a finite length representation τ of $\mathrm{GL}(X)$, we abbreviate $\mu(N \otimes s, \pi' \boxtimes \tau)$ to $\mu(s, \pi' \boxtimes \tau)$. Here, N denotes the reduced norm of $\mathrm{End}(X)$.

16.2. **Jacquet-Langlands correspondences.** Let F be a local field of characteristic zero, let d, t be positive integers, and let D be a central division algebra of F of $[D : F] = d^2$. We denote by $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{GL}_{dt}(F))$ (resp. $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{GL}_t(D))$) the set of the isomorphism classes of unitary irreducible representations of $\mathrm{GL}_{dt}(F)$ (resp. $\mathrm{GL}_t(D)$). Then, the local Jacquet-Langlands correspondence provides the map

$$|\,\mathrm{JL}_F\,|\colon\ \mathrm{Irr}_{\mathrm{unit}}(\mathrm{GL}_{dt}(F)) \to \mathrm{Irr}_{\mathrm{unit}}(\mathrm{GL}_t(D)) \cup \{0\}.$$

Let $\mathbb{F}$ be a global field of characteristic zero, let d, t be positive integers, let $\mathbb{D}$ be a central division algebra over $\mathbb{F}$ of $[\mathbb{D} : \mathbb{F}] = d^2$, and let Π be a discrete series of $\mathrm{GL}_{dt}(\mathbb{A})$ so that $|\,\mathrm{JL}_{\mathbb{F}_v}\,|(\Pi_v) \neq 0$. Then, the global Jacquet-Langlands correspondence provides a non-zero discrete series $|\,\mathrm{JL}_{\mathbb{F}}\,|(\Pi)$ of $\mathrm{GL}_t(\mathbb{D}_{\mathbb{A}})$. We do not explain the definition of the correspondences, but state some important properties, which are excerpts of the results of [DKV84], [Bad08].

Proposition 16.2.

(1) *If* $d = 1$, *then we have* $|\,\mathrm{JL}_F\,|$ *is the identity map* Id.
(2) *If* π *is an irreducible supercuspidal representation of* $\mathrm{GL}_{dt}(F)$, *then* $|\,\mathrm{JL}_F\,|(\pi)$ *is non-zero and supercuspidal.*
(3) *If* π *is an irreducible square-integrable representation of* $\mathrm{GL}_{dt}(F)$, *then* $|\,\mathrm{JL}_F\,|(\pi)$ *is non-zero and square-integrable.*
(4) *For all irreducible square-integrable representations* π' *of* $\mathrm{GL}_t(D)$, *there exists an irreducible square-integrable representation so that* $\pi' = |\,\mathrm{JL}_F\,|(\pi)$.

(5) *If* Π *is a discrete series of* $\mathrm{GL}_{dt}(\mathbb{D}_{\mathbb{A}})$ *such that* $|\,\mathrm{JL}_{\mathbb{F}_v}\,|(\Pi_v) \neq 0$ *for all* v, *then we have* $|\,\mathrm{JL}_{\mathbb{F}}\,|(\Pi)_v = |\,\mathrm{JL}_{\mathbb{F}_v}\,|(\Pi_v)$.
(6) *For all discrete series* Π' *of* $\mathrm{GL}_t(\mathbb{D}_{\mathbb{A}})$, *there exists a discrete series* Π *of* $\mathrm{GL}_{dt}(\mathbb{A})$ *so that* $|\,\mathrm{JL}_{\mathbb{F}_v}\,|(\Pi_v) \neq 0$ *and* $|\,\mathrm{JL}_{\mathbb{F}}\,|(\Pi) = \Pi'$.

Proof. By [Bad08, Theorem 5.1], we have the assertions (1), (5) and (6). The assertion (2) follows from [Bad08, §3.1]. Finally, we have the assertions (3) and (4) by [Bad08, Theorem 2.2]. □

16.3. **Plancherel measures for inner forms of general linear groups.** In this subsection, we denote by D' a central division algebra over F. Then there is a positive integer d so that $[D' : F] = d^2$. Let t_1 and t_2 be positive integers, and let $t = t_1 + t_2$. We consider the case where $M = \mathrm{GL}_{t_1}(D') \times \mathrm{GL}_{t_2}(D')$ and $G = \mathrm{GL}_t(D')$. Then, we have an identification $\mathbb{C}^2 \cong E_{\mathbb{C}}^{\vee}$ by

$$(\eta_1, \eta_2) \mapsto N_1 \otimes \eta_1 + N_2 \otimes \eta_2,$$

where N_i denotes the reduced norm of $\mathrm{GL}_{t_i}(D')$ for $i = 1, 2$ respectively.

Proposition 16.3. *Let* ρ_i *be a square-integrable irreducible representation of* $\mathrm{GL}_{dt_i}(F)$ *for* $i = 1, 2$, *and let* τ_i *be the representation* $|\,\mathrm{JL}_F\,|(\rho_i)$ *of* $\mathrm{GL}_{t_i}(D')$ *for* $i = 1, 2$. *Then we have*

$$\mu(\eta, \tau_1 \boxtimes \tau_2) = \gamma(s_1 - s_2, \rho_1 \boxtimes \rho_2^{\vee}, \psi)\gamma(s_2 - s_1, \rho_1^{\vee} \boxtimes \rho_2, \overline{\psi})$$

for $\eta = (s_1, s_2) \in \mathbb{C}^2$.

Proof. First, if we denote by P an F-rational parabolic subgroup of $\mathrm{GL}_t(D')$ having the Levi subgroup M, then we have

$$J_{\overline{P}|P}((\tau_1 \boxtimes \tau_2) \otimes \eta) \otimes |N|^{-\frac{1}{2}(s_1+s_2)} = J_{\overline{P}|P}((\tau_1 \boxtimes \tau_2) \otimes (\eta - (\frac{1}{2}(s_1 + s_2), \frac{1}{2}(s_1 + s_2)))).$$

Thus we have

$$\mu(\eta, \tau_1 \boxtimes \tau_2) = \mu((\frac{1}{2}(s_1 - s_2), \frac{1}{2}(s_2 - s_1)), \tau_1 \boxtimes \tau_2).$$

By [AP05, Theorem 7.2], this is equal to

$$\mu((\frac{1}{2}(s_1 - s_2), \frac{1}{2}(s_2 - s_1)), \rho_1 \boxtimes \rho_2).$$

We remark that our normalization of the Haar measure is different from that of [AP05]. Since ρ_1 and ρ_2 are generic [Zel80, Theorem 9.3], we have

$$\mu(\eta, \rho_1 \boxtimes \rho_2) = \gamma(s_1 - s_2, \rho_1 \boxtimes \rho_2^{\vee}, \psi)\gamma(s_2 - s_1, \rho_1^{\vee} \boxtimes \rho_2, \overline{\psi})$$

for $s \in \mathbb{C}$ [Sha90]. Hence we have the proposition. □

Proposition 16.4. *Let τ_i be an irreducible representation of $\mathrm{GL}_{t_i}(D')$ for $i = 1, 2$, let B_1 be an F-rational parabolic subgroup of $\mathrm{GL}_{t_1}(D')$, let M_1 be the Levi subgroup of B_1. The subgroup M_1 is of the form*

$$\mathrm{GL}_{t_{11}}(D) \times \cdots \times \mathrm{GL}_{t_{1\lambda}}(D)$$

with $t_{11} + \cdots + t_{1\lambda} = t_1$. Suppose that τ_1 is embedded into $\mathrm{Ind}_{B_1}^{\mathrm{GL}_{t_1}(D)} \sigma_1$ where

$$\sigma_1 = \sigma_{11}|N_{11}|^{a_1} \boxtimes \cdots \boxtimes \sigma_{1\lambda}|N_{1\lambda}|^{a_\lambda}$$

for some complex numbers $a_1, \ldots, a_{\lambda_1}$ and for some irreducible representations $\sigma_{11}, \ldots, \sigma_{1\lambda}$. Then we have

$$\mu(\eta, \tau_1 \boxtimes \tau_2) = \prod_{j=1}^{r} \mu(\eta + (a_j, 0)), \sigma_j \boxtimes \tau_2).$$

Proof. Let S be the center of $M_1 \times \mathrm{GL}_{t_2}(D')$, and let P be a parabolic subgroup of $\mathrm{GL}_t(D)$ whose Levi subgroup is $\mathrm{GL}_{t_1}(D) \times \mathrm{GL}_{t_2}(D)$. We denote by U (resp., U_1) the unipotent radical of P (resp. B_1), by $\Delta_S(U)$ (resp. $\Delta_S(U_1)$) the set of the roots of S whose root subspace is contained in U (resp. U_1). For $\alpha \in \Delta_S(U)$, we denote by S_α the kernel of α in S, and by M_α the centralizer of S_α in $\mathrm{GL}_t(D)$. We may suppose that the product $(B_1 \times \mathrm{GL}_{t_2}(D')) \cdot U$ is a parabolic subgroup of $\mathrm{GL}_t(D)$, and we denote it by B. Finally, we denote by P_α the parabolic subgroup $M_\alpha \cdot B$. Then we have

$$J_{\overline{P}|P}^{\mathrm{GL}_t(D)}(\tau_1 \boxtimes \tau_2, \eta) = \prod_{\alpha \in \Delta_S(U)} \mathrm{Ind}_{P_\alpha}^{\mathrm{GL}_t(D)}(J_{\overline{B}|B}^{M_\alpha}(\sigma_1 \boxtimes \sigma_2, \eta)).$$

Hence, we have the formula

$$\mu(\eta, \tau_1 \boxtimes \tau_2) = \prod_{j=1}^{\lambda} \mu(\eta, \sigma_{1j}|N_{1j}|^{a_j}| \boxtimes \tau_2),$$

which implies the proposition. $\square$

16.4. **Global property.** In this subsection, we recall a global property of the Plancherel measure for inner forms of general linear group and quaternionic unitary groups. Let $\mathbb{F}$ be a global field, and let $\mathbb{D}$ be a division quaternion algebra over $\mathbb{F}$.

Lemma 16.5. *Let $\underline{W}$ be a left $(-\epsilon)$-Hermitian space over $\mathbb{D}$, let $\underline{X}, \underline{X}^*$ be two left $\mathbb{D}$-vector spaces so that $\dim \underline{X} = \dim \underline{X}^* = r'$, and let $\underline{W'} = \underline{X} + \underline{W} + \underline{X}^*$ a $(-\epsilon)$-Hermitian space equipped with the $(-\epsilon)$-Hermitian form*

$$\langle\ ,\ \rangle' \colon (\underline{X} + \underline{W} + \underline{X}^*) \times (\underline{X} + \underline{W} + \underline{X}^*) \to \mathbb{D}$$

defined by

$$\langle (x_1, w_1, y_1), (x_2, w_2, y_2) \rangle' = x_1 \cdot J_{r'} \cdot {}^t y_2^* + \langle w_1, w_2 \rangle - \epsilon y_1 \cdot J_{r'} \cdot {}^t x_2^*.$$

Here, we recall that $J_{r'}$ is the anti-diagonal matrix defined in §4. Then, $M = \mathrm{GL}_{r'}(\mathbb{D}) \times G(\underline{W})$ is a Levi subgroup of a maximal parabolic subgroup of $G(\underline{W'})$. Then, for an irreducible cuspidal automorphic representation $\Pi \boxtimes \Xi$ of $M(\mathbb{A})$, we *have*

$$\prod_{v\in S}\mu_v(s,\Pi_v\boxtimes\Xi_v)=\frac{L^S(1-s,\Pi\boxtimes\Xi^\vee)}{L^S(s,\Pi^\vee\boxtimes\Xi)}\cdot\frac{L^S(1+s,\Pi^\vee\boxtimes\Xi)}{L^S(-s,\Pi\boxtimes\Xi^\vee)}\times\frac{L^S(1-2s,\Xi^\vee,\wedge^2)}{L^S(2s,\Xi,\wedge^2)}\cdot\frac{L^S(1+2s,\Xi,\wedge^2)}{L^S(-2s,\Xi^\vee,\wedge^2)}. \tag{16.1}$$

Here, S is a finite set of places of $\mathbb{F}$ such that

- *S contains all Archimedean places,*
- *$\mathbb{D}'_v$ is split for $v\notin S$, and*
- *$G(\underline{W}_v)$, Π_v, Ξ_v are unramified for $v\notin S$,*

and we denote

$$L^S(s,\Xi^\vee,\wedge^2)=\prod_{v\notin S}L(s,\Xi_v^\vee,\wedge^2),$$

where the right-hand side is an infinite product of the local exterior-square L-factor.

Proof. Let P be a parabolic subgroup so that M is the Levi subgroup of P, and let $f=\otimes_v f_v\in\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\Pi\boxtimes\Xi^\vee$. Consider the Eisenstein series

$$E_P[f](g)=\sum_{\gamma\in P(\mathbb{F})\backslash G(\underline{W'})(\mathbb{F})}f(\gamma g)$$

for $g\in G(\underline{W'})(\mathbb{A})$. If $v\notin S$, then by [Sha90] we have

$$\begin{aligned}&L(s,\Pi_v^\vee\boxtimes\Xi_v)L(-s,\Pi_v\boxtimes\Xi_v^\vee)L(2s,\Xi_v,\wedge^2)L(-2s,\Xi_v^\vee,\wedge^2)\cdot[J_{\overline{P}|P}\circ J_{P|\overline{P}}](f_v)\\&=L(s,\Pi_v^\vee\boxtimes\Xi_v)L(-s,\Pi_v\boxtimes\Xi_v^\vee)L(2s,\Xi_v,\wedge^2)L(-2s,\Xi_v^\vee,\wedge^2)\cdot f_v.\end{aligned}$$

Hence, by the functional equation of the Eisenstein series [Lan76, p.216, (i), (ii)], we have

$$\begin{aligned}&L^S(1-s,\Pi\boxtimes\Xi^\vee)L^S(1+s,\Pi^\vee\boxtimes\Xi)\\&\times L^S(1-2s,\Xi^\vee,\wedge^2)L^S(1+2s,\Xi,\wedge^2)\prod_{v\in S}\mu_v(\mu_v,\Pi_v\boxtimes\Xi_v)^{-1}E[f]\\&=L^S(s,\Pi^\vee\boxtimes\Xi)L^S(-s,\Pi\boxtimes\Xi^\vee)\\&\times L^S(2s,\Xi,\wedge^2)L^S(-2s,\Xi^\vee,\wedge^2)\cdot E[J_{\overline{P}|P}\circ J_{P|\overline{P}}(f)]\\&=L^S(s,\Pi^\vee\boxtimes\Xi)L^S(-s,\Pi\boxtimes\Xi^\vee)L^S(2s,\Xi,\wedge^2)L^S(-2s,\Xi^\vee,\wedge^2)\cdot E[f],\end{aligned}$$

which implies the lemma. □

16.5. **The behavior of the Plancherel measures under the theta correspondence.** Now we consider the Plancherel measures for quaternionic unitary groups. Let V be an m-dimensional right ϵ-Hermitian space, and let W be an n-dimensional left $(-\epsilon)$-Hermitian space. Note that, in this section, we allow the case where $l\neq 1$.

Proposition 16.6. *Let π be an irreducible representation of $G(W)$, let $\sigma:=\theta_\psi(\pi;V)$ and let τ be an irreducible representation of $\mathrm{GL}(X)$. Then we have*

$$\frac{\mu(s,\pi\boxtimes\tau\chi_V)}{\mu(s,\sigma\boxtimes\tau\chi_W)}=\gamma(s-\frac{l-1}{2},\tau,\psi)\cdot\gamma(-s-\frac{l-1}{2},\tau^\vee,\overline{\psi}).$$

The remaining part of this subsection is devoted to the proof of Proposition 16.6. Put

$$u_D(s;W,V,X,\pi,\tau) = \frac{\mu(s,\pi\boxtimes\tau\chi_V)}{\mu(s,\sigma\boxtimes\tau\chi_W)}\gamma(s-\frac{l-1}{2},\tau,\psi)^{-1}\gamma(-s-\frac{l-1}{2},\tau^\vee,\overline{\psi})^{-1}.$$

We will use global argument to prove Proposition 16.6, so that **we allow** D **to be split** in the rest of this section. We want to show $u_D(W,V,X,\pi,\tau)=1$ for all D,W,V,X,π,τ.

Lemma 16.7. *Let* $\{W_i\}_{i\geq 0}$ *be a Witt tower consisting of* $(-\epsilon)$*-Hermitian spaces and let* $\{V_j\}_{j\geq 0}$ *be a Witt tower consisting of* ϵ*-Hermitian spaces. Suppose that* $V=V_r$ *and* $W=W_t$.

(1) *If* D *is split, then we have*

$$u_D(s;W,V,X,\pi,\tau)=1.$$

(2) *Suppose that* π *is a subrepresentation of* $\mathrm{Ind}_{P_{t',t}}^{G(W)}\pi'\boxtimes\rho_{s_0}\chi_V$ *where* t' *is an integer so that* $\max\{t(\pi),r\}\leq t'\leq t$, $s_0\in\mathbb{C}$, π' *is an irreducible representation of* $G(W_{t'})$*, and* ρ *is an irreducible supercuspidal representation of* $\mathrm{GL}_{t-t'}(D)$*. Then, we have*

$$u_D(s;W,V,X,\pi,\tau)=u_D(s;W_{t'},V_{r'},X,\pi',\tau),$$

where $r'=r-(t-t')$.

(3) *Let* X',X'' *be two subspaces of* X *so that* $X=X'+X''$*. Suppose that* τ *is an irreducible subquotient of induced representation* $\mathrm{Ind}_{P(X')}^{\mathrm{GL}(X)}\tau'\boxtimes\tau''$ *where* τ' *(resp.* τ''*) is an irreducible representation of* $\mathrm{GL}(X')$ *(resp.* $\mathrm{GL}(X'')$*). Then, we have*

$$u_D(s;W,V,X,\pi,\tau)=u_D(s;W,V,X',\pi,\tau')u_D(s;W,V,X'',\pi,\tau'').$$

(4) *If* $r(\pi)$ *denotes the first occurrence index, then we have*

$$u_D(s;W,V_{r(\pi)},X,\pi,\tau)=u_D(s;W,V,X,\pi,\tau).$$

Proof. The claim (1) is proved in [GI14, Theorem 12.1]. Then, we prove (2). By [GI14, Proposition B.3], we have

$$\begin{aligned}\mu(s,\tau\chi_V\otimes\pi) &= \mu((s,s_0),\tau\chi_V\boxtimes\rho\chi_V)\mu((s,-s_0),\tau\chi_V\boxtimes\rho^\vee\chi_V)\mu(s,\tau\boxtimes\pi')\\ &= \mu((s,s_0),\tau\boxtimes\rho)\mu((s,-s_0),\tau\boxtimes\rho^\vee)\mu(s,\tau\boxtimes\pi').\end{aligned}$$

Hence, by Corollary 9.7 (with replacing V and W, σ and π), we have

$$\frac{\mu(s,\tau\chi_V\otimes\pi)}{\mu(s,\tau\chi_W\otimes\sigma)}=\frac{\mu(s,\tau\chi_V\otimes\pi')}{\mu(s,\tau\chi_W\otimes\sigma')}.$$

Thus, we have (2). We prove (3) similarly by using [GI14, Lemma B.2]. Finally, we prove (4). Put $t^\pi=r-r(\pi)$. Then, by using the local functional equation of the doubling γ-factor [Kak20, Theorem 5.7(4)], we have

$$\mu(s,\sigma\boxtimes\tau\chi_W)=\mu(s,|N|^{\frac{l}{2}+t^\pi}\boxtimes\tau\chi_W)\cdot\mu(s,|N|^{-\frac{l}{2}-t^\pi}\boxtimes\tau\chi_W)\cdot\mu(s,\sigma'\boxtimes\tau\chi_W).$$

By Proposition 16.4, this is equal to

$$\begin{aligned}
&\frac{\prod_{i=1}^{2t^\pi}\gamma(s+\frac{l}{2}+i-\frac{1}{2},\tau^\vee\chi_W,\psi)}{\prod_{i=1}^{2t^\pi}\gamma(s+\frac{l}{2}+i+\frac{1}{2},\tau^\vee\chi_W,\psi)}\cdot\frac{\prod_{i=1}^{2t^\pi}\gamma(s-\frac{l}{2}+\frac{1}{2}-i,\tau^\vee\chi_W,\psi)}{\prod_{i=1}^{2t^\pi}\gamma(s-\frac{l}{2}+\frac{3}{2}-i,\tau^\vee\chi_W,\psi)}\\
&\quad\times\mu(s,\sigma'\boxtimes\tau\chi_W)\\
&=\frac{\gamma(s+\frac{l+1}{2},\tau^\vee\chi_W,\psi)}{\gamma(s+\frac{l_0+1}{2},\tau^\vee\chi_W,\psi)}\cdot\frac{\gamma(s-\frac{l_0-1}{2},\tau^\vee\chi_W,\psi)}{\gamma(s-\frac{l-1}{2},\tau^\vee\chi_W,\psi)}\cdot\mu(s,\sigma'\boxtimes\tau\chi_W)\\
&=\frac{\gamma(-s-\frac{l_0-1}{2},\tau\chi_W,\overline{\psi})}{\gamma(-s-\frac{l-1}{2},\tau\chi_W,\overline{\psi})}\cdot\frac{\gamma(s-\frac{l_0-1}{2},\tau^\vee\chi_W,\psi)}{\gamma(s-\frac{l-1}{2},\tau^\vee\chi_W,\psi)}\cdot\mu(s,\sigma'\boxtimes\tau\chi_W)\\
&=\frac{\gamma(-s-\frac{l_0-1}{2},\tau\chi_W,\psi)}{\gamma(-s-\frac{l-1}{2},\tau\chi_W,\psi)}\cdot\frac{\gamma(s-\frac{l_0-1}{2},\tau^\vee\chi_W,\overline{\psi})}{\gamma(s-\frac{l-1}{2},\tau^\vee\chi_W,\overline{\psi})}\cdot\mu(s,\sigma'\boxtimes\tau\chi_W).
\end{aligned}$$

Hence we have

$$\begin{aligned}
u_D(s;W,V,X,\pi,\tau)&=\frac{\mu(s,\pi\boxtimes\tau\chi_V)}{\mu(s,\sigma\boxtimes\tau\chi_W)}\gamma(s-\frac{l-1}{2},\tau,\psi)^{-1}\gamma(-s-\frac{l-1}{2},\tau^\vee,\overline{\psi})^{-1}\\
&=\frac{\mu(s,\sigma'\boxtimes\tau\chi_W)}{\mu(s,\sigma\boxtimes\tau\chi_W)}\cdot\frac{\mu(s,\pi\boxtimes\tau\chi_V)}{\mu(s,\sigma'\boxtimes\tau\chi_W)}\\
&\quad\times\gamma(s-\frac{l-1}{2},\tau,\psi)^{-1}\gamma(-s-\frac{l-1}{2},\tau^\vee,\overline{\psi})^{-1}\\
&=\frac{\mu(s,\pi\boxtimes\tau\chi_V)}{\mu(s,\sigma'\boxtimes\tau\chi_W)}\gamma(s-\frac{l_0-1}{2},\tau,\psi)^{-1}\gamma(-s-\frac{l_0-1}{2},\tau^\vee,\overline{\psi})^{-1}\\
&=u_D(s;W,V_{r(\pi)},X,\pi,\tau).
\end{aligned}$$

Thus we have (4). □

Now we prove Proposition 16.6. By Corollary 9.7 and Lemma 16.7(2), (3), we may assume that π, σ, and τ are supercuspidal. Take

- a global field $\mathbb{F}$ and two distinct places v_1, v_2 of $\mathbb{F}$ so that $\mathbb{F}_{v_1}=\mathbb{F}_{v_2}=F$,
- a non-trivial additive character $\underline{\psi}$ of the ring of adeles $\mathbb{A}$ of $\mathbb{F}$,
- a division quaternion algebra $\mathbb{D}$ over $\mathbb{F}$ so that $\mathbb{D}_{v_1}=\mathbb{D}_{v_2}=D$ and $\mathbb{D}_v$ is split for all $v\neq v_1,v_2$,
- an ϵ-Hermitian space $\mathbb{V}$ over $\mathbb{D}$ so that $\mathbb{V}_{v_1}=\mathbb{V}_{v_2}=V$,
- a Witt tower $\{\mathbb{V}_i\}_{i=0}^\infty$ containing $\mathbb{V}$,
- a $-\epsilon$-Hermitian space $\mathbb{W}$ over $\mathbb{D}$ so that $\mathbb{W}_{v_1}=\mathbb{W}_{v_2}=W$,
- an irreducible cuspidal automorphic representation Π of $G(\mathbb{W})(\mathbb{A})$ so that $\Pi_{v_1}=\pi$,
- a vector space $\mathbb{X}$ over $\mathbb{D}$ so that $\dim_{\mathbb{D}}\mathbb{X}=\dim_D X$,
- an irreducible cuspidal automorphic representation Ξ of $\mathrm{GL}(\mathbb{X})(\mathbb{A})$ so that $\Xi_{v_1}=\tau$,
- a finite subset S of places so that $v_1,v_2\in S$, all Archimedean places are contained in S and Π_v, Ξ_v are unramified for all places $v\notin S$.

Let $r(\Pi)$ be the first occurrence index of the theta correspondence of Π to the Witt tower $\{\mathbb{V}_i\}_{i=0}^\infty$. Then, $\Theta_{\underline{\psi}}(\Pi,\mathbb{V}_{r(\Pi)})$ is a non-zero irreducible cuspidal automorphic representation. We denote by π' (resp. τ') the representation Π_{v_2} (resp. Ξ_{v_2}). Hence we have

(16.2)

$$\begin{aligned}
&u_D(s;V,W,X,\pi,\tau)\cdot u_D(s;V,W,X,\pi',\tau')\\
&= u_{\mathbb{D}_{v_1}}(s;\mathbb{V}_{v_1},\mathbb{W}_{v_1},\mathbb{X}_{v_1},\Pi_{v_1},\Xi_{v_1})\cdot u_{\mathbb{D}_{v_2}}(s;\mathbb{V}_{v_2},\mathbb{W}_{v_2},\mathbb{X}_{v_2},\Pi_{v_2},\Xi_{v_2})\\
&= u_{\mathbb{D}_{v_1}}(s;(\mathbb{V}_{r(\Pi)})_{v_1},\mathbb{W}_{v_1},\mathbb{X}_{v_1},\Pi_{v_1},\Xi_{v_1})\cdot u_{\mathbb{D}_{v_2}}(s;(\mathbb{V}_{r(\Pi)})_{v_2},\mathbb{W}_{v_2},\mathbb{X}_{v_2},\Pi_{v_2},\Xi_{v_2})\\
&\quad\times \prod_{v\neq v_1,v_2} u_{\mathbb{D}_v}(s;(\mathbb{V}_{r(\Pi)})_v,\mathbb{W}_v,\mathbb{X}_v,\Pi_v,\Xi_v)\\
&= 1.
\end{aligned}$$

Applying (16.2) when Π and Ξ are chosen so that $\pi'=\pi$ and $\tau'=\tau$, we have $u_D(s;V,W,X,\pi,\tau)^2=1$. Hence $u_D(s;V,W,X,\pi,\tau)=\pm 1$. It remains to determine the signature. By Lemma 16.7(4), we may assume that σ is also supercuspidal. Moreover, we may assume that τ is not of the form $\chi\circ N$ for any unramified character χ of $F^\times$, where N denotes the reduced norm. Then, the Godement-Jacquet L-factor of τ is 1 [GJ72, Propositions 4.4, 5.11]. By Lemma 16.1, we have

$$\mu(0,\pi\boxtimes\tau\chi_V)>0 \text{ and } \mu(0,\sigma\boxtimes\tau\chi_W)>0.$$

On the other hand, putting

$$\epsilon(s+\frac{1}{2},\tau,\psi)=a_\psi(\tau)\cdot q^{-\lambda s}$$

with $a_\psi(\tau)\in\mathbb{C}^\times$ and $\lambda\in\mathbb{Z}$, we have

$$\epsilon(-s+\frac{1}{2},\tau^\vee,\overline{\psi})=a_\psi(\tau)^{-1}\cdot q^{\lambda s},$$

and we have

$$\begin{aligned}
&\gamma(-\frac{l-1}{2},\tau,\psi)\gamma(-\frac{l-1}{2},\tau^\vee,\psi)\\
&= a_\psi(\tau)q^{\lambda l/2}\cdot\frac{L(\frac{l+1}{2},\tau^\vee)}{L(-\frac{l-1}{2},\tau)}\cdot a_\psi(\tau)^{-1}q^{\lambda l/2}\cdot\frac{L(\frac{l+1}{2},\tau)}{L(-\frac{l-1}{2},\tau^\vee)}\\
&= q^{\lambda l}>0.
\end{aligned}$$

Thus, the signature of $u_D(s;V,W,X,\pi,\tau)$ turns out to be 1. This completes the proof of Proposition 16.6.

17. Poles of Plancherel measures

In this section, we study the poles of the Plancherel measures, and we construct some irreducible supercuspidal representations whose Plancherel measures are not holomorphic on $\mathbb{R}_{>0}$. We will use this representation in the next section.

Let F be a local field, let D be a division quaternion algebra over F, and let V be an m-dimensional ϵ-Hermitian space. Denote by V_0 the anisotropic kernel of V, and write $V=X+V_0+X^*$ where X,X^* are totally isotropic subspaces so that $X+X^*$ is the orthogonal complement of V_0. Put $r=\dim_D X$.

Proposition 17.1.

(1) *There exists an irreducible supercuspidal representation ρ of* $\mathrm{GL}_{2r}(F)$ *such that the image of the L-parameter* $\phi_\tau\colon \mathrm{SL}_2(\mathbb{C})\times W_F\to\mathrm{GL}_{2r}(\mathbb{C})$ *is contained in* $\mathrm{Sp}_{2r}(\mathbb{C})$.

(2) *Take an irreducible supercuspidal representation* ρ *of* $\mathrm{GL}_{2r}(F)$ *as in* (1), *and denote by* τ *the representation* $|\mathrm{JL}_F|(\rho)$ *of* $\mathrm{GL}(X)$*. Then, for an irreducible representation* σ *of* $G(V_0)$*, the Plancherel measure* $\mu(s, \sigma \boxtimes \tau)$ *has at least one pole in* $\mathbb{R}_{>0}$*.*

Proposition 17.1 is proved at the end of this section. Before starting to prove it, we recall accidental isomorphisms to show the explicit formula of the Plancherel measure for some cases. To use the global argument, we write down them in the global setting. Let $\mathbb{F}$ be a global field, let $\mathbb{D}$ be a division quaternion algebra over $\mathbb{F}$, and $\underline{V_0}$ be an anisotropic ϵ-Hermitian space over $\mathbb{D}$. We denote by $\widetilde{G}(\underline{V_0})$ the similitude group of $\underline{V_0}$. Then it is known that $\widetilde{G}(\underline{V_0})$ is isomorphic to a certain more familiar group. By $\mathbb{E}$ we mean the etale $\mathbb{F}$ algebra associated with the quadratic (or trivial) character $\chi_{\underline{V_0}}$ of the idele group of $\mathbb{F}$.

- Suppose that $\epsilon = 1$ and $m = 1$. Then we have $\widetilde{G}(\underline{V_0}) = \mathbb{D}^\times$. If we denote by $\underline{V_0'}$ the two-dimensional symplectic space over $\mathbb{F}$, then $\widetilde{G}(\underline{V_0})$ is an inner form of $\mathrm{GSp}(\underline{V_0'}) = \mathrm{GL}_2(\mathbb{F})$.
- Suppose that $\epsilon = -1$ and $m = 1$. Let $\underline{V_0'}$ be a two-dimensional quadratic space such that $\chi_{\underline{V_0'}} = \chi_{\underline{V_0}}$. Then we have $\widetilde{G}(\underline{V_0}) = \mathrm{GSO}(\underline{V_0'})$ which is isomorphic to $\mathbb{E}^\times$.
- Suppose that $\epsilon = -1$ and $m = 2$. If we denote by $\underline{V_0'}$ the isotropic four-dimensional quadratic space such that $\chi_{\underline{V_0'}} = \chi_{\underline{V_0}}$, then $\widetilde{G}(\underline{V_0})$ is an inner form of $\mathrm{GSO}(\underline{V_0'})$ which is isomorphic to $\mathrm{GL}_2(\mathbb{E}) \times \mathbb{F}^\times/\{(z, N_{\mathbb{E}/\mathbb{F}}(z)^{-1}) \mid z \in \mathbb{E}^\times\}$ (cf. [GI11, A.2]). Thus, by using the non-commutative version of the Shapiro's lemma (cf. [PR94, Proposition 1.7]), we have $\widetilde{G}(\underline{V_0}) \cong \mathbb{B}^\times \times \mathbb{F}^\times/\{(z, N_{\mathbb{E}/\mathbb{F}}(z)^{-1}) \mid z \in \mathbb{E}^\times\}$ for some division quaternion algebra $\mathbb{B}$ over $\mathbb{E}$.
- Finally, suppose that $\epsilon = -1$ and $m = 3$. If we denote by $\underline{V_0'}$ the split six-dimensional quadratic space over $\mathbb{F}$, then $\widetilde{G}(\underline{V_0})$ is an inner form of $\mathrm{GSO}(\underline{V_0'})$ which is isomorphic to $\mathrm{GL}_4 \times \mathrm{GL}_1^\times/\{(z, z^{-2}) \mid z \in \mathrm{GL}_1\}$. Thus, we have $\widetilde{G}(\underline{V_0}) \cong \mathbb{D}_4^\times \times \mathbb{F}^\times/\{(z, z^{-2}) \mid z \in \mathbb{F}^\times\}$ for some central division algebra $\mathbb{D}_4$ with $[\mathbb{D}_4 : \mathbb{F}] = 16$.

Therefore, applying the Jacquet-Langlands correspondence, we have Lemma 17.2:

Lemma 17.2. *Let* $\widetilde{\Sigma}$ *be a discrete series of* $\widetilde{G}(\underline{V_0})(\mathbb{A})$*. Then, there is a discrete series* $\widetilde{\mathcal{R}}$ *of* $\mathrm{GSO}(\underline{V_0'})(\mathbb{A})$ *(or* $\mathrm{GSp}(\underline{V_0'})(\mathbb{A})$*) such that* $\widetilde{\Sigma}_v$ *and* $\widetilde{\mathcal{R}}_v$ *coincide for all place* v *of* $\mathbb{F}$*.*

Proof. We write $\widetilde{G}(\underline{V_0})$ as a quotient ${\mathbb{B}'}^\times \times \mathrm{GL}_1/C_1$ where $\mathbb{B}'$ is a central division algebra of a finite extension field $\mathbb{E}'$ of $\mathbb{F}$ and C_1 is a central subgroup of ${\mathbb{B}'}^\times \times \mathrm{GL}_1$. Then, $\mathrm{GSO}(\underline{V_0})$ (or $\mathrm{GSp}(\underline{V_0})$) is of the form $\mathrm{GL}_d(\mathbb{E}') \times \mathrm{GL}_1/C_2$ where d is the positive integer so that $d^2 = [\mathbb{B}' : \mathbb{E}']$, C_2 is a central subgroup of $\mathrm{GL}_d(\mathbb{E}') \times \mathrm{GL}_1$. Then, C_1 is isomorphic to C_2 via the inner twist isomorphism. By Proposition 16.2, we have that there exists a discrete series $\widetilde{\mathcal{R}}'$ of $\mathrm{GL}_d(\mathbb{E}') \times \mathrm{GL}_1$ so that $|\mathrm{JL}_{\mathbb{E}'}|(\widetilde{\mathcal{R}}') = \widetilde{\Sigma}$. Since the weak approximation holds for C_2 in each case, we have $\widetilde{\mathcal{R}}'|_{C_2} = 1_{C_2}$ by (1) and (5) of Proposition 16.2. Hence the representation $\widetilde{\mathcal{R}}$ of $\mathrm{GL}_d(\mathbb{E}') \times \mathrm{GL}_1/C_2$ yielded by $\widetilde{\mathcal{R}}'$ satisfies the conditions of the lemma. Thus we have the claim. □

Let F be a local field of characteristic 0, let D be the division quaternion algebra over F, and let V be an ϵ-Hermitian space over D. As in the global case, we denote V_0'

$$\begin{cases} \text{the } 2n_0\text{-dimensional equipped with the symplectic form} & (\epsilon = 1), \\ \text{the } 2n_0\text{-dimensional quadratic space of } \chi_{V_0'} = \chi_{V_0} & (\epsilon = -1). \end{cases}$$

Lemma 17.3. *Let σ be an irreducible representation of $G(V_0)$, and let ρ be an irreducible supercuspidal representation of $\mathrm{GL}_{2r}(F)$. Then, there exists a square-integrable representation σ' of $\mathrm{SO}(V_0')$ (or $\mathrm{Sp}(V_0')$) such that*

$$\mu(s, \sigma \boxtimes |\,\mathrm{JL}_F\,|(\rho)) = \frac{\gamma(s, \sigma'^{\vee} \boxtimes \rho, \psi)}{\gamma(1+s, \sigma'^{\vee} \boxtimes \rho, \psi)} \cdot \frac{\gamma(2s, \rho, \wedge^2, \psi)}{\gamma(1+2s, \rho, \wedge^2, \psi)}.$$

Here, $\gamma(s, \rho, \wedge^2, \psi)$ is the Langlands-Shahidi γ-factor [Sha90].

Proof. We prove this lemma only with $\epsilon = -1$ for simplicity. Take

- a unitary irreducible representation $\widetilde{\sigma}$ of the similitude group $\widetilde{G}(V_0)$ so that $\widetilde{\sigma}|_{G(V_0)}$ contains σ,
- a global field $\mathbb{F}$ and places v_1, v_2 of $\mathbb{F}$ such that $\mathbb{F}_{v_1} = \mathbb{F}_{v_2} = F$,
- a division quaternion algebra $\mathbb{D}$ over $\mathbb{F}$ such that $\mathbb{D}_{v_1} = \mathbb{D}_{v_2} = D$, and for all places $v \neq v_1, v_2$, $\mathbb{D}_v$ is split,
- an anisotropic ϵ-Hermitian space $\mathbb{V}_0$ such that $\mathbb{V}_{0v_1} = \mathbb{V}_{0v_2} = V_0$, and for all places $v \neq v_1, v_2$, $G(\mathbb{V}_0)$ is quasi-split,
- a $2n_0$-dimensional quadratic space $\mathbb{V}_0'$ such that $\chi_{\mathbb{V}_0'} = \chi_{\mathbb{V}_0}$,
- a vector space $\mathbb{X}$, $\mathbb{X}^*$ over $\mathbb{D}$ such that $\dim_{\mathbb{D}} \mathbb{X} = \dim_{\mathbb{D}} \mathbb{X}^* = \dim_D X$.

We denote by $\widetilde{Z}_1$ the center of $\widetilde{G}(\mathbb{V}_0)$ and by $\widetilde{Z}_2$ the center of $\mathrm{GSO}(\mathbb{V}_0')$. Then $\widehat{Z}_1$ and $\widehat{Z}_2$ are isomorphic to each other. Denote by $\omega_{\widetilde{\sigma}}$ the central character of σ and by ω_ρ the central character of ρ. Then, by Lemma 14.4(2), there exists a character $\underline{\omega}$ of $Z_1(\mathbb{A})/Z_1(\mathbb{F})$ so that $\underline{\omega}_{v_1} = \underline{\omega}_{v_2} = \omega_{\widetilde{\sigma}}$, and there exists a Hecke character $\underline{\omega}'$ of $\mathbb{A}^\times$ so that $\underline{\omega}'_{v_1} = \underline{\omega}'_{v_2} = \omega_\rho$. We take an auxiliary non-Archimedean place $v_3 \neq v_1, v_2$. Then, by Lemma 14.6, there exist

- a cuspidal automorphic irreducible representation $\widetilde{\Sigma}$ of $\widetilde{G}(\mathbb{V}_0)(\mathbb{A})$ so that $\widetilde{\Sigma}_{v_1} = \widetilde{\Sigma}_{v_2} = \widetilde{\sigma}$ and $\widetilde{\Sigma}_{v_3}$ is supercuspidal,
- a cuspidal automorphic irreducible representation $\widetilde{\Xi}$ of $\mathrm{GL}_{2r}(\mathbb{A})$ so that $\widetilde{\Xi}_{v_1} = \widetilde{\Xi}_{v_2} = \rho$ and $\widetilde{\Xi}_{v_3}$ is supercuspidal.

Take a discrete series $\widetilde{\mathcal{R}}$ as in Lemma 17.2. Then, $\widetilde{\mathcal{R}}$ is cuspidal since $\widetilde{\mathcal{R}}_{v_3} = \widetilde{\Sigma}_{v_3}$ is supercuspidal. By the product formula (Lemma 16.5), we have

$$\begin{aligned} \mu(s, \sigma \boxtimes |\,\mathrm{JL}_F\,|(\rho))^2 &= \mu(s, \widetilde{\sigma} \boxtimes |\,\mathrm{JL}_F\,|(\rho))^2 \\ &= \mu(s, \widetilde{\mathcal{R}}_{v_1} \boxtimes \Xi_{v_1}) \mu(s, \widetilde{\mathcal{R}}_{v_2} \boxtimes \Xi_{v_2}) \\ &= \mu(s, \widetilde{\mathcal{R}}_{v_1} \boxtimes \Xi_{v_1})^2. \end{aligned}$$

Thus, by the positivity (Lemma 16.1), we have

$$\mu(s, \sigma \boxtimes |\,\mathrm{JL}_F\,|(\rho)) = \mu(s, \widetilde{\mathcal{R}}_{v_1} \boxtimes \Xi_{v_1}).$$

Moreover, since $\widetilde{\mathcal{R}}_{v_1}$ and ρ are generic, this is equal to

$$\frac{\gamma(s, \widetilde{\mathcal{R}}_{v_1}^{\vee} \boxtimes \rho, \psi)}{\gamma(1+s, \widetilde{\mathcal{R}}_{v_1}^{\vee} \boxtimes \rho, \psi)} \cdot \frac{\gamma(2s, \rho, \wedge^2, \psi)}{\gamma(1+2s, \rho, \wedge^2, \psi)}$$

by [Sha90]. Thus, putting $\sigma' = \widetilde{\mathcal{R}}_{v_1}$, we have the lemma. □

Now we prove Proposition 17.1. (1) is a consequence of [Mie20, §4]. We prove (2). Let ρ be an irreducible supercuspidal representation so that the image of the L-parameter $\phi_\tau\colon \mathrm{SL}_2(\mathbb{C}) \times W_F \to \mathrm{GL}_{2r}(\mathbb{C})$ is contained in $\mathrm{Sp}_{2r}(\mathbb{C})$. Then, by a result of Jiang, Nien and Qin [JNQ10], we can conclude that $\gamma(s, \rho, \wedge^2, \psi)$ has a pole at $s = 1$. Let $\mathrm{Fr} \in W_F$ be a Frobenius element. Then, by [GR10, Lemma 3], $\phi_\rho(\mathrm{Fr})$ has finite order, hence, $[\wedge^2 \circ \phi_\rho](\mathrm{Fr})$ is a unitary operator. Thus all poles of $L(s, \rho, \wedge^2)$ lie in $\{\Re s = 0\}$, and we can conclude that $\gamma(s, \rho, \wedge^2, \psi)$ has a pole at $s = 1$, and all poles of $\gamma(s, \rho, \wedge^2, \psi)$ lie in $\{\Re s = 1\}$. Hence, the ratio

$$\frac{\gamma(2s, \rho, \wedge^2, \psi)}{\gamma(1 + 2s, \rho, \wedge^2, \psi)}$$

has a pole at $s = \frac{1}{2}$. Put

$$\mathcal{P} = \{s_0 \geq \frac{3}{2} \mid \ \gamma(s, {\sigma'}^\vee \boxtimes \rho, \psi) \text{ has a pole at } s = s_0\}.$$

If $\mathcal{P} = \varnothing$, then $\mu(s, \sigma \boxtimes |\ \mathrm{JL}_F\ |(\rho))$ has a pole at $s = \frac{1}{2}$ since all zeros of $\gamma(s, {\sigma'}^\vee \boxtimes \rho, \psi)$ lie in $\{\Re s \leq 0\}$. If $\mathcal{P} \neq \varnothing$, then the ratio $\gamma(s, \sigma' \boxtimes \rho, \psi)/\gamma(1 + s, {\sigma'}^\vee \boxtimes \rho, \psi)$ has a pole at $s = \sup \mathcal{P}$. Hence we finish the proof of Proposition 17.1.

18. Induction argument

In this section, we prove the compatibility of $\alpha_3(V, W)$ with the induction on the dimensions of V, W with $l = 1$, which completes the proof of Theorem 11.2. We emphasize that we allow V or W to be 0. Now, we explain more precisely. Let V be an m-dimensional right ϵ-Hermitian space, and let W be an n-dimensional left $(-\epsilon)$-Hermitian space. We assume that $l = 2n - 2m - \epsilon = 1$. Consider

- an ϵ-Hermitian space V' containing V and its totally isotropic subspaces X, X^* so that $\dim_D X = \dim_D X^* = t$, $X + V + X^* = V'$ and $X + X^*$ is the orthogonal complement of V,
- a $(-\epsilon)$-Hermitian space W' containing W and its totally isotropic subspaces Y, Y^* so that $\dim_D Y = \dim_D Y^* = t$, $Y + W + Y^* = W'$ and $Y + Y^*$ is the orthogonal complement of W.

We put $n' = n + 2t$ and $m' = m + 2t$. Then, we will prove

$$\alpha_3(V', W') = \alpha_3(V, W) \tag{18.1}$$

in this section. Let Q (resp. P) be the maximal parabolic subgroup of $G(V')$ (resp. $G(W')$) preserving X (resp. Y). Then, we can identify the Levi subgroup L_Q (resp. M_P) of Q (resp. P) with $\mathrm{GL}(X) \times G(V)$ (resp. $\mathrm{GL}(Y) \times G(W)$). Recall that $\alpha_3(V', W')$ and $\alpha_3(V, W)$ do not depend on the choices of the representations (see Proposition 15.1). Thus, it suffices to compare $\deg \pi' / \deg \theta_\psi(\pi', V)$ with $\deg \pi / \deg \theta_\psi(\pi, V)$ for at least one pair (π, π') of square-integrable representations π of $G(W)$ and π' of $G(W')$ so that both $\theta_\psi(\pi, V)$ and $\theta_\psi(\pi', V')$ are non-zero.

Proposition 18.1. *Suppose that there are $s_0 > 0$, an irreducible supercuspidal representation π of $G(W)$, an irreducible supercuspidal representation σ of $G(V)$, and a non-trivial irreducible supercuspidal representation τ of $\mathrm{GL}(X) \cong \mathrm{GL}(Y)$ so that*

- $\sigma \cong \theta_\psi(\pi, V)$,

- $\mathrm{Ind}_P^{G(W')} \pi \boxtimes \tau_{s_0}\chi_V$ *is reducible, and*
- $\mathrm{Ind}_Q^{G(V')} \sigma \boxtimes \tau_{s_0}\chi_W$ *is reducible.*

Then, $\mathrm{Ind}_P^{G(W')} \pi \boxtimes \tau_{s_0}\chi_V$ *and* $\mathrm{Ind}_Q^{G(V')} \sigma \boxtimes \tau_{s_0}\chi_W$ *have unique irreducible square-integrable representations* π' *and* σ' *respectively, and* σ' *coincides with* $\theta_\psi(\pi', V')$*. Moreover, we have* $\alpha_3(V', W') = \alpha_3(V, W)$*.*

We prove Proposition 18.1 in the former part of this section. Suppose that a quadruple (s_0, π, σ, τ) as in Proposition 18.1 is given. By Lemma 9.3 and Proposition 9.6, the representation $\theta_\psi(\pi', V')$ is the unique square-integrable irreducible subquotient of $\mathrm{Ind}_Q^{G(V')} \sigma \boxtimes \tau_{s_0}\chi_W$, which is nothing other than σ'. To prove the last assertion, we use Proposition 18.2, which is due to a result of Heiermann [Hei04].

Proposition 18.2. *Let* $s_0 > 0$*, let* π *be an irreducible supercuspidal representation of* $G(W)$ *and let* τ *be a supercuspidal representation of* $\mathrm{GL}_t(D)$*. Suppose that* $\mu(s, \pi \boxtimes \tau\chi_V)$ *has a pole at* $s = s_0$*. Then we have the following:*

(1) *The induced representation* $\mathrm{Ind}_Q^{G(W')} \pi \boxtimes \tau_{s_0}\chi_V$ *is reducible and it has a unique irreducible square-integrable constituent* π'*. Moreover we have*

$$\begin{aligned}\deg \pi' =& 2t \log q \cdot \deg \pi \deg \tau \cdot \mathrm{Res}_{s=s_0} \mu(s, \pi \boxtimes \tau\chi_V) \\ & \times \gamma(G(W')/P) \cdot \frac{|K_{M_P}|}{|K_{G(W')}|} \cdot |U_P \cap K_{W'}| \cdot |\overline{U_P} \cap K_{W'}|.\end{aligned}$$

Here, $\gamma(G(W')/P)$ *is the constant defined by*

$$\gamma(G(W')/P) = \int_{\overline{U}} \delta_P(\overline{u})\, d\overline{u},$$

where $\overline{U}$ *is the unipotent radical of the opposite parabolic subgroup* $\overline{P}$ *of* P*, and* f° *is the unique* $K_{W'}$*-invariant section of the representation* $\mathrm{Ind}_P^{G(W')} \delta_P^{\frac{1}{2}}$ *induced by the square root of the modular character* δ_P *so that* $f^\circ(1) = 1$*.*

(2) *The induced representation* $\mathrm{Ind}_Q^{G(V')} \sigma \boxtimes \tau_{s_0}\chi_V$ *is also reducible, and it has a unique irreducible square-integrable constituent* σ'*. Moreover we have*

$$\begin{aligned}\deg \sigma' =& 2t \log q \cdot \deg \sigma \deg \tau \cdot \mathrm{Res}_{s=s_0} \mu(s, \sigma \boxtimes \tau\chi_W) \\ & \times \gamma(G(V')/Q) \cdot \frac{|K_{L_Q}|}{|K_{G(V')}|} \cdot |U_Q \cap K_{V'}| \cdot |\overline{U_Q} \cap K_{V'}|.\end{aligned}$$

Here, $\gamma(G(V')/Q)$ *is the constant defined similarly as in* (1)*.*

Proof. This proposition is obtained by the proof of [GI14, Proposition 20.4]. $\square$

Now, take π as in Proposition 18.4, and put $\sigma = \theta(\pi, V)$. Then, by Proposition 18.2, we have

$$\begin{aligned}\frac{\deg \pi'}{\deg \sigma'} &= \frac{\deg \pi}{\deg \sigma} \cdot \frac{\mathrm{Res}_{s=s_0}\, \mu(s, \pi \boxtimes \chi_V)}{\mathrm{Res}_{s=s_0}\, \mu(s, \sigma \boxtimes \chi_W)} \cdot \frac{\gamma(G(W)/P)}{\gamma(G(V)/Q)} \cdot \frac{|K_{G(V')}||K_{M_P}|}{|K_{G(W')}||K_{L_Q}|} \\ &= \frac{\deg \pi}{\deg \sigma} \cdot \gamma(s_0 - \frac{l-1}{2}, |\ |^{s_0}, \psi)\gamma(-s_0 - \frac{l-1}{2}, |\ |^{s_0}, \overline{\psi}) \\ &\times \frac{|U_P \cap K_{W'}| \cdot |\overline{U_P} \cap K_{W'}|}{|U_Q \cap K_{V'}| \cdot |\overline{U_Q} \cap K_{V'}|} \cdot \frac{|\mathcal{B}^+_{V'}||\mathcal{B}^+_{M_P}|}{|\mathcal{B}^+_{W'}||\mathcal{B}^+_{L_Q}|} \\ &\times \frac{\prod_{\alpha \in \Sigma_{\mathrm{red}}(\overline{P})} [X_\alpha \cap K_{W'} : X_\alpha \cap \mathcal{B}^+_{W'}]^{-1}}{\prod_{\beta \in \Sigma_{\mathrm{red}}(\overline{Q})} [X_\beta \cap K_{V'} : X_\beta \cap \mathcal{B}^+_{V'}]^{-1}}.\end{aligned}$$

Here, we denote by $\mathcal{B}^+$ the pro-unipotent radical of $\mathcal{B}$, by $\Sigma_{\mathrm{red}}(\overline{P})$ (resp. $\Sigma_{\mathrm{red}}(\overline{Q})$) the set of positive reduced root with respect to the opposite parabolic subgroup $\overline{P}$ (resp. $\overline{Q}$) of P (resp. Q), and by X_α (resp. X_β) the root subgroup associated with $\alpha \in \Sigma_{\mathrm{red}}(\overline{P})$ (resp. $\beta \in \Sigma_{\mathrm{red}}(\overline{Q})$).

Lemma 18.3. *We have*

$$\frac{|\mathcal{B}^+_{V'}||\mathcal{B}^+_{M_P}|}{|\mathcal{B}^+_{W'}||\mathcal{B}^+_{L_Q}|} = q^{2t}.$$

Proof. Since $|\mathcal{B}^+_{M_P}| = |\mathcal{B}^+_W||\mathcal{B}^+_{\mathrm{GL}_r(D)}|$ and $|\mathcal{B}^+_{L_Q}| = |\mathcal{B}^+_V||\mathcal{B}^+_{\mathrm{GL}_r(D)}|$, we have

$$\begin{aligned}\frac{|\mathcal{B}^+_{V'}||\mathcal{B}^+_{M_P}|}{|\mathcal{B}^+_{W'}||\mathcal{B}^+_{L_Q}|} &= \frac{|\mathcal{B}^+_{V'}||\mathcal{B}^+_{W}|}{|\mathcal{B}^+_{W'}||\mathcal{B}^+_{V}|} \\ &= \begin{cases} q^{(n'^2-n^2)-(m'^2-m'-m^2+m)-\frac{1}{2}(a'_{V'}-a'_V)} & (-\epsilon = 1), \\ q^{(n'^2-n'-n^2+n)-(m'^2-m^2)+\frac{1}{2}(a'_{W'}-a'_W)} & (-\epsilon = -1), \end{cases}\end{aligned}$$

where

$$a'_W = \begin{cases} 0 & (\chi_W \text{ is unramified}), \\ -1 & (\chi_W \text{ is ramified}). \end{cases}$$

One can show that both coincide with q^{2t}. Hence we have the lemma. □

Moreover, we have

$$\begin{aligned}\frac{\prod_{\alpha \in \Sigma_{\mathrm{red}}(\overline{P})} [X_\alpha \cap K_{W'} : X_\alpha \cap \mathcal{B}^+_{W'}]^{-1}}{\prod_{\beta \in \Sigma_{\mathrm{red}}(\overline{Q})} [X_\beta \cap K_{V'} : X_\beta \cap \mathcal{B}^+_{V'}]^{-1}} &= q^{-2(n_0-m_0)t} \\ &= q^{-(1+\epsilon)t},\end{aligned}$$

and

$$\begin{aligned}\frac{|U_P \cap K_{W'}| \cdot |\overline{U_P} \cap K_{W'}|}{|U_Q \cap K_{V'}| \cdot |\overline{U_Q} \cap K_{V'}|} &= q^{-2(nt+\frac{1}{2}t(t-\epsilon))} \cdot q^{2(mt+\frac{1}{2}t(t+\epsilon))} \\ &= q^{-2(n-m)+2\epsilon t} \\ &= q^{-(1-\epsilon)t}.\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{\deg \pi'}{\deg \sigma'} &= \frac{\deg \pi}{\deg \sigma} \cdot \gamma(s_0 - \frac{l-1}{2}, \tau, \psi)\gamma(-s_0 - \frac{l-1}{2}, \tau^\vee, \overline{\psi}) \\ &= \frac{\deg \pi}{\deg \sigma} \cdot \gamma(s_0, \tau, \psi)\gamma(-s_0, \tau^\vee, \overline{\psi})\end{aligned}$$

since $l = 1$. Thus we have Proposition 18.1.

Now, we prove the existence of the quadruple (s_0, π, σ, τ) as in Proposition 18.1 when either V or W is anisotropic.

Proposition 18.4. *Suppose that V is anisotropic. Then, there exists an irreducible supercuspidal representation π of $G(W)$ such that $\Theta_\psi(\pi, V) \neq 0$.*

Proof. We use the following see-saw diagram to prove this:

$$\begin{array}{ccc} G(V^\square) & & G(W) \times G(W)\ . \\ \Big| & \times & \Big| \\ G(V) \times G(V) & & \Delta G(W) \end{array}$$

Let σ be an irreducible representation of $G(V)$. Since $G(V)$ is anisotropic, σ is supercuspidal, and then we have $\Theta_\psi(\sigma, V)$ is irreducible [MVW87, p. 69]. Then Theorem 9.1(3) implies that $\Theta_\psi(\sigma, V)^\vee \cong \Theta_{\overline{\psi}}(\sigma^\vee, V)$. Hence, by the see-saw property we have

$$\begin{aligned}\Theta_\psi(\sigma, W) \neq 0 &\Leftrightarrow \mathrm{Hom}_{\Delta G(W)}(\Theta_\psi(\sigma, W) \otimes \Theta_\psi(\sigma, W)^\vee, 1_{G(W)}) \neq 0 \\ &\Leftrightarrow \sigma \boxtimes \sigma^\vee \text{ appears as a quotient of } \Theta_\psi(1_{G(W)}, V^\square)|_{G(V)\times G(V)}.\end{aligned}$$

In the case where W is anisotropic, the proposition is clear by the above observation. Then suppose that W is isotropic. This only occurs in the case where $\epsilon = -1$. Thus we have $\chi_W = 1$. Hence, we have the isomorphism

$$R_s \colon I^V(s, 1_{F^\times}) \to \mathcal{S}(G(V))$$

by $[R_s f_s](g) = f_s(\iota(g, 1))$ for $f_s \in I^V(s, 1_{F^\times})$ and $g \in G(V)$.

Lemma 18.5. *For $u \in U(V^\triangle)$ there is a unique element $g_u \in G(V)$ such that $\iota(g_u, 1) \in P(V^\triangle)\tau u$ for some $p \in P(V^\triangle)$. Moreover, $u \mapsto g_u$ gives a homeomorphism*

$$U(V^\triangle) \to G(V) \setminus \{1\}.$$

By Lemma 18.5, if we take a non-zero function $\varphi \in \mathcal{S}(G(V))$ so that $\overline{\mathrm{supp}(\varphi)} \not\ni 1$ and $\varphi(g) \geq 0$ for all $g \in G(V)$, then the integral defining $M(s, 1_{F^\times})(R_s^{-1}\varphi)$ converges and $M(s, 1_{F^\times})(R_s^{-1}\varphi) \neq 0$ for all $s \in \mathbb{C}$. On the other hand, if we denote by W_i the i-dimensional $(-\epsilon)$-Hermitian space with $\chi_{W_i} = 1_{F^\times}$ and by l_i the integer $2i - 2m - \epsilon$, then we have

$$\Theta_\psi(1_{W_i}, V^\square) = \ker M(-\frac{l_i}{2}, 1_{F^\times})$$

for $i = 0, \ldots, n-1$ by [Yam11, Theorems 1.3, 1.4]. Thus, we have proved that

$$\sum_{i=0}^{n-1} R_{-l_i/2}(\Theta_\psi(1_{G(W_i0}, V^\square)) \subsetneq \mathcal{S}(G(V)).$$

Hence, there is an irreducible representation σ of $G(V)$ such that $n^+(\sigma) \geq n$ and $n^-(\sigma) \geq n+1$. Since we have assumed $l = 1$, the conservation relation (Proposition 9.4) says that $n^+(\sigma) + n^-(\sigma) = 2n+1$. Thus, we have $n^+(\sigma) = n$, and we have the lemma by putting $\pi = \Theta_\psi(\sigma, W)$. □

Proposition 18.6. *Suppose that W is anisotropic and V is isotropic. Then, there is an irreducible representation π of $G(W)$ such that $\theta_\psi(\pi, V)$ is non-zero supercuspidal.*

Proof. The situation in this proposition occurs only in the case where $\epsilon = 1$, $\dim W = 3$, $\dim V = 2$ and $\chi_W = \chi_V = 1_{F^\times}$. Then, as in §17, we have the accidental isomorphism

$$\widetilde{G}(W) \cong D_4^\times \times F^\times / \{(a, a^{-2}) \mid a \in F^\times\},$$

where D_4 is a central division algebra of F so that $[D_4 : F] = 16$. Now, we denote by π_0 an irreducible representation of $D_4^\times$ obtained as follows: let π_1 be an irreducible supercuspidal representation of $\mathrm{GL}_4(F)$ so that the image of its L-parameter is contained in $\mathrm{Sp}_4(\mathbb{C}) \times W_F$ (see [Mie20, §4]). Then we denote by π_0 the irreducible representation of $D_4^\times$ associated with π_1 by the Jacquet-Langlands correspondence. Since the central character of π_0 is trivial, we have the irreducible representation $\pi_0 \boxtimes 1_{F^\times}$ of $D_4^\times \times F^\times / \{(a, a^{-2}) \mid a \in F^\times\}$. We may regard it as a representation of $\widetilde{G}(W)$ by the accidental isomorphism. We denote by π an irreducible component of the restriction of $\pi_0 \boxtimes 1$ to $G(W)$. Then, the square exterior γ-factor $\gamma(s, \phi_{\pi_0}, \wedge^2, \psi)$ has a pole at $s = 1$ (see [JNQ10]). Hence we have $\Theta_\psi(\pi, V) \neq 0$ (see [GT14, Theorem 6.1], and [GT14, Proposition 3.3]). Moreover, since $\pi \neq 1_{G(W)}$, we have $m(\pi) > 0$. This forces that $m(\pi) = 2$, and $\theta_\psi(\pi, V)$ is supercuspidal. Hence we have the proposition. □

Corollary 18.7. *There exist (s_0, π, σ, τ) as in Proposition* 18.1 *when either V or W is anisotropic.*

Proof. Take an irreducible supercuspidal representation ρ of $\mathrm{GL}_2(F)$ so that the image of the L-parameter ϕ_τ is contained in Sp_{2r} (Proposition 17.1(1)). Moreover, we put $\tau = |\,\mathrm{JL}_F(\rho)|$.

Suppose first that V is anisotropic. Take π as Proposition 18.4, and put $\sigma = \theta_\psi(\pi, V)$. Then, there exists a positive real number s_0 so that $\mu(s, \pi \boxtimes \tau\chi_V)$ has a pole at $s = s_0$ (Proposition 17.1(2)). Since the Godement-Jacquet L-factor $L(s, \tau)$ is equal to 1, we have that $\mu(s, \sigma \boxtimes \tau\chi_W)$ also has a pole at $s = s_0$ by Proposition 16.6. This implies that the quadruple (s_0, π, σ, τ) satisfies the assumption of Proposition 18.1.

Then, suppose that W is anisotropic and V is isotropic. Take π as in Proposition 18.6, and put $\sigma = \theta_\psi(\pi, V)$. Then, by Proposition 17.1, $\mu(s, \pi \boxtimes \tau\chi_V)$ has a pole at a positive real number s_0. Then, we have that $\mu(s, \sigma \boxtimes \tau\chi_W)$ also has a pole at $s = s_0$ as in the above discussion. Hence, the quadruple (s_0, π, σ, τ) satisfies the assumption of Proposition 18.1. Hence we have the corollary. □

Corollary 18.7 completes the proof of (18.1), and we finish the proof of Theorem 11.2.

19. Determination of α_1 and α_2

In this section, we complete the formulas of $\alpha_1(W)$ and $\alpha_2(V,W)$ even when both V and W are isotropic. Let V be an m-dimensional right ϵ-Hermitian space, and let W be an n-dimensional left $(-\epsilon)$-Hermitian space. We assume that $2n-2m-\epsilon=1$. Take a basis $\underline{e}=(e_1,\ldots,e_n)$ for W. First, we have:

Theorem 19.1.

$$\alpha_2(V,W)=e(G(W))\cdot|2|^{-2n\rho+n(n-\frac{1}{2})}\cdot|N(R(\underline{e}))|^{\rho}\cdot\prod_{i=1}^{n-1}\frac{\zeta_F(1-2i)}{\zeta_F(2i)}$$
$$\times\begin{cases}2\cdot\chi_V(-1)^n\cdot\gamma(1-n,\chi_V,\psi)^{-1}\epsilon(\frac{1}{2},\chi_V,\psi) & (-\epsilon=1),\\ 1 & (-\epsilon=-1).\end{cases}$$

Proof. There is at least one irreducible square-integrable representation π of $G(W)$ such that $\Theta_\psi(\pi,V)\neq 0$ (this has been proved in §18 by replacing V with V'). Then, comparing the formula of $\alpha_3(V,W)$ of Proposition 15.1 with its definition in Theorem 11.2, we obtain

$$\frac{1}{2}\cdot\alpha_2(V,W)\cdot e(G(W))\cdot|2|_F^{2n\rho-n(n-\frac{1}{2})}\cdot|N(R(\underline{e}))|^{-\rho}\cdot\prod_{i=1}^{n-1}\frac{\zeta_F(2i)}{\zeta_F(1-2i)}$$
$$\times\begin{cases}\chi_V(-1)^{n+1}\gamma(1-n,\chi_V,\psi) & (-\epsilon=1),\\ \chi_W(-1)^{m+1}\epsilon(\frac{1}{2},\chi_W,\psi) & (-\epsilon=-1)\end{cases}$$
$$=\begin{cases}\chi_V(-1)\epsilon(\frac{1}{2},\chi_V,\psi) & (-\epsilon=1),\\ \frac{1}{2}\cdot\chi_W(-1)^{m+1}\epsilon(\frac{1}{2},\chi_W,\psi) & (-\epsilon=-1).\end{cases}$$

Hence, we have the claim. ☐

Suppose that $-\epsilon=-1$. We denote by W^u a $(-\epsilon)$-Hermitian space so that $\dim W^u=n$ and $W^{(u)}$ possesses a basis $\underline{e}^{(u)}$ with $R(\underline{e}^{(u)})\in\mathrm{GL}_n(\mathcal{O}_D)$. Then, by Theorem 19.1, we have:

Corollary 19.2.

$$\alpha_2(V,W)=|N(R(\underline{e}))|^{\rho}\cdot\alpha_2(V,W^u).$$

Proof. Since $|N(R(\underline{e}^{(u)})|=1$, the claim follows from Theorem 19.1. ☐

On the other hand, we may identify $W^{u\square}$ with $W^\square$ by identifying e_i' with $e_i^{(u)\prime}$ for $i=1,\ldots,2n$. Then we can compare $\mathcal{I}^{W^u}$ with $\mathcal{I}^W$:

Lemma 19.3. *For $\phi,\phi'\in\mathcal{S}(V\otimes W^\triangledown)=\mathcal{S}(V\otimes W^{u\triangledown})$, we have*

$$\mathcal{I}^W(\phi,\phi')=\mathcal{I}^{W^u}(\phi,\phi').$$

Proof. By writing down the definitions, we have the equation. ☐

Therefore, we have

$$\begin{aligned}\frac{\alpha_1(W)}{\alpha_1(W^u)}&=\frac{\mathcal{E}^{W^u}(\phi,\phi')}{\mathcal{E}^W(\phi,\phi')}\\&=\frac{\alpha_2(V,W^u)}{\alpha_2(V,W)}\\&=|N(R(\underline{e}))|^{-\rho}.\end{aligned}$$

Thus, we have a general formula of $\alpha_1(W)$:

Proposition 19.4. *In the case $-\epsilon = -1$, we have*

$$\alpha_1(W) = |2|_F^{2n\rho} \cdot |N(R(\underline{e}))|^{-\rho} \cdot q^{-(2\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor)} \cdot \prod_{i=1}^{n} (1 + q^{-(2i-1)}).$$

Proof. We already have a formula of $\alpha_1(W)$ either when $n_0 = 0$ or $n_0 = 1$ and χ_W is unramified (Proposition 7.6). Hence, we have the proposition by Lemma 19.3. □

20. The formal degree conjecture

In this section, we explain the refined version of formal degree conjecture [GR10]. Moreover, we give another proof of Theorem 11.2 assuming the local Langlands correspondence and the formal degree conjecture.

Let F be a non-Archimedean local field of characteristic 0, and let G be a connected reductive group over F. Let $\widehat{G}$ be the Langlands dual group of G, let ${}^L G$ be the L-group of G. By an L-parameter we mean the $\widehat{G}$-conjugacy class of L-homomorphisms

$$\phi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

whose images are not contained in any irrelevant parabolic subgroup of ${}^L G$ [Bor79]. In this section, we assume the Langlands correspondence, that is, we can associate an L-parameter with an irreducible representation of G. But we explain it more precisely for quaternionic unitary groups to clarify what we assume.

Hypothesis 20.1. *For an irreducible tempered representation π of $G(W)$, there is a tempered L-parameter ϕ of $G(W)$ satisfying the following two properties.*

- *For an irreducible supercuspidal representation τ of $\mathrm{GL}_{2r}(F)$ and $s \in \mathbb{C}$ we have*
$$\mu(s, \pi \boxtimes |\mathrm{JL}_F|(\tau)) = \frac{\gamma(s, \mathrm{std} \circ \phi^\vee \otimes \phi_\tau, \psi)}{\gamma(1+s, \mathrm{std} \circ \phi^\vee \otimes \phi_\tau, \psi)} \cdot \frac{\gamma(2s, \tau, \wedge^2, \psi)}{\gamma(1+2s, \tau, \wedge^2, \psi)},$$
where ϕ_τ is the L-parameter of τ.
- *For a character χ of $F^\times$ we have*
$$\gamma^W(s, \pi \boxtimes \chi, \psi) = \gamma(s, \mathrm{std} \circ \phi \otimes \chi, \psi),$$
where the left-hand side is the γ-factor defined in §7, and the right-hand side is the standard gamma factor.

It is known that the equality of the Plancherel measure characterizes ϕ uniquely ([GS12, Lemma 12.3], [GI14, p. 652]). Now we consider a general connected reductive group G again and we discuss the internal structure of the L-packet $\Pi_\phi(G(F))$. We denote by $Z(\widehat{G})$ the center of $\widehat{G}$, by $\widehat{G}_{\mathrm{ad}}$ the quotient $\widehat{G}/Z(\widehat{G})$, and by $\widehat{G}_{\mathrm{sc}}$ the simply connected covering group of $\widehat{G}_{\mathrm{ad}}$. Moreover, we denote by C_ϕ the centralizer of the image $\mathrm{Im}\,\phi$ of ϕ in $\widehat{G}$, by Γ the Galois group of F^{s}/F where F^{s} is the separable closure of F, by S_ϕ the quotient $C_\phi/Z(\widehat{G})^\Gamma \subset \widehat{G}_{\mathrm{ad}}$, and by $\widetilde{S}_\phi$ the preimage of S_ϕ by $\widehat{G}_{\mathrm{sc}} \to \widehat{G}_{\mathrm{ad}}$. We call the component group of $\widetilde{S}_\phi$ the S-group of ϕ, and we denote it by $\widetilde{\mathcal{S}}_\phi$. We choose a character ζ_G of $Z(\widehat{G}_{\mathrm{sc}})$ which is associated with G via the composition of the maps

$$\mathrm{Hom}(Z(\widehat{G}_{\mathrm{sc}}), \mathbb{C}^\times) \to \mathrm{Hom}(Z(\widehat{G}_{\mathrm{sc}})^\Gamma, \mathbb{C}^\times) \to \{\text{Inner forms of } G^{\mathrm{qs}}\},$$

where G^{qs} is the quasi-split inner form of G, the first map is the restriction, and the second map is the isomorphism of Kottwitz [Kot84]. Let A be the maximal split torus of the center of G. Then $\widehat{G/A}$ is a subset of $\widehat{G}$. We denote by C'_ϕ the intersection $C_\phi \cap \widehat{G/A}$ and by $\mathrm{Irr}(\widetilde{\mathcal{S}}_\phi, \zeta_G)$ the set of irreducible representations ρ of $\widetilde{\mathcal{S}}_\phi$ so that

$$\mathrm{Hom}_{Z(\widehat{G}_{\mathrm{sc}})}(\zeta_G, \rho) \neq 0.$$

Conjecture 20.2. *Let ϕ be a tempered L-parameter. Then there is a bijection*

$$\Pi_\phi(G) \to \mathrm{Irr}(\widetilde{\mathcal{S}}_\phi, \zeta_G) \tag{20.1}$$

such that for any square-integrable representation $\pi \in \Pi_\phi(G)$ we have

$$\deg(\pi) = \zeta_\pi \frac{\dim \eta_\pi}{\# C'_\phi} \gamma(0, \pi, \mathrm{ad}, \psi),$$

where $\mathrm{ad}\colon {}^L G \to \mathrm{GL}(\mathrm{Lie}(\widehat{G}_{\mathrm{ad}}))$ *is the adjoint representation, and*

$$\zeta_\pi = \frac{|\gamma(0, \mathrm{St}, \mathrm{ad}, \psi)|}{\gamma(0, \mathrm{St}, \mathrm{ad}, \psi)} \frac{\epsilon(\frac{1}{2}, \mathrm{St}, \mathrm{ad}, \psi)}{\epsilon(\frac{1}{2}, \pi, \mathrm{ad}, \psi)} \in \{\pm 1\},$$

where St *is the Steinberg representation of $G(W)$.*

We denote by η_π the image of π via the map (20.1).

Remark 20.3. It is expected that the characters of the irreducible representations belonging to $\Pi_\phi(G(F))$ satisfy linear equations called the "endoscopic character relations". (See for example [Kal16].) But we do not discuss them in this paper.

Now we deduce some properties of Langlands parameters and local theta correspondence. Assume that $l = 1$. For a unitary character χ of $F^\times$, we also denote by χ the corresponding character of the Weil group W_F via the local class field theory. For a non-negative integer k, let Q_k be the k-dimensional quadratic space over $\mathbb{C}$. Then, we have $\widehat{G(V)} = \mathrm{SO}(Q_M)$ and $\widehat{G(W)} = \mathrm{SO}(Q_{M+1})$ where $M = 2m + (1+\epsilon)/2$. Fix an isometric embedding $u\colon Q_M \to Q_{M+1}$. Then, u yields the embedding $\xi_0\colon \mathrm{O}(Q_M) \to \mathrm{SO}(Q_{M+1})$. Moreover, we fix an element $\varepsilon \in \mathrm{O}(Q_M)$ so that $\det(\varepsilon) = -1$. We extend the embedding $\xi_0|_{\mathrm{SO}(Q_M)}$ of the dual groups to an L-embedding from ${}^L G(V)$ into ${}^L G(W)$ by

$$\xi(w, g) = (w, g\xi_0(\varepsilon)^{a_V(w)}) \quad (w \in W_F, g \in \mathrm{SO}(M, \mathbb{C})),$$

where $a_V(w) = (1 - \chi_V(w))/2$ for $w \in W_F$.

Proposition 20.4. *Assume that Hypothesis* 20.1 *holds and that $l = 1$.*

(1) *For an irreducible tempered representation π of $G(W)$, $\theta_\psi(\pi, V)$ is non-zero if and only if* $\mathrm{std} \circ \phi_\pi$ *contains χ_V as representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$.*

(2) *For an irreducible tempered representation σ of $G(V)$, $\theta_\psi(\sigma, W)$ is non-zero if and only if* $\mathrm{std} \circ \phi_\sigma$ *does not contain χ_W as representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$.*

(3) *For an L-parameter ϕ of $G(W)$, by $\vartheta(\phi)$ we mean the set of L-parameters ϕ' of $G(V)$ so that $\xi \circ \phi' = \phi$ as L-parameters of $G(W)$. Then, the local theta correspondence defines a bijection*

$$\theta(-, V)\colon \Pi_\phi(G(W)) \to \bigcup_{\phi' \in \vartheta(\phi)} \Pi_{\phi'}(G(V))$$

if ϕ is tempered and $\vartheta(\phi) \neq \varnothing$.

Proof. Let σ be an irreducible tempered representation of $G(V)$ so that $\theta_\psi(\sigma, W) \neq 0$. Put $\pi = \theta_\psi(\sigma, W)$. Then as in [GI14, Theorem C.5], we have

$$\mathrm{std} \circ \phi_\pi \otimes \chi_V = (\mathrm{std} \circ \phi_\sigma \otimes \chi_W) \oplus 1_{W_F}$$

as representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$. This proves the "only if" part of (1). The "if" part of the property (1) is obtained by the similar argument to [HKS96, Theorem 6.2] of Harris, Kudla and Sweet. Let π be an irreducible tempered representation of $G(W)$ such that $\mathrm{std} \circ \phi_\pi$ contains χ_V. Then the standard L-function $L(s, \pi \boxtimes \chi_V)$ has a pole at $s = 0$. Since π is tempered, $L(s, \pi \boxtimes \chi_V)$ does not have a pole in $\Re s < 0$ [Yam11]. Hence we have

$$Z(M(s, \chi_V) f^{(s)}, \xi_\pi) \sim \frac{L(-s + \frac{1}{2}, \pi \boxtimes \chi_V)}{\zeta(2s - 1)} \cdot \frac{Z(f^{(s)}, \xi_\pi)}{L(s + \frac{1}{2}, \pi \boxtimes \chi_V)}.$$

Here, by $f_1 \sim f_2$ we mean f_1/f_2 is holomorphic at $s = \frac{1}{2}$. Hence, by [Yam11, Theorem 5.2] we can conclude that $Z(-, \xi_\pi)$ is non-zero on $\Theta_\psi(1_{G(V)}, W^\square) \subset I^W(-\frac{1}{2}, \chi_V)$. Then, by considering the see-saw diagram

$$\begin{tikzcd}
G(W^\square) \arrow[d, dash] \arrow[drr, dash] & & G(V) \times G(V) \arrow[d, dash] \arrow[dll, dash] \\
G(W) \times G(W) & & \Delta G(V)
\end{tikzcd},$$

we have

$$\mathrm{Hom}_{\Delta G(V)}(\Theta_\psi(\pi, V) \boxtimes \Theta_{\overline{\psi}}(\pi^\vee, V), 1_V) = \mathrm{Hom}_{G(V) \times G(V)}(\Theta_\psi(1_V, W^\square), \pi \boxtimes \pi^\vee) \neq 0.$$

Thus, we have $\theta_\psi(\pi, V) \neq 0$. By combining the property (1) with the conservation relation (Proposition 9.4), we have the property (2). The property (3) is a consequence of (1) and (2). $\square$

In the rest of this section, we assume Hypothesis 20.1 and that Conjecture 20.2 is true.

Proposition 20.5. *Let π be a square-integrable irreducible representation of $G(W)$, and let $\sigma = \theta_\psi(\pi, V)$. Suppose that $\sigma \neq 0$. Then we have*

$$\frac{\dim \eta_\pi}{\dim \eta_\sigma} = \begin{cases} 1 & (\epsilon = 1), \\ 1 & (\epsilon = -1, \phi_\sigma^\varepsilon \ncong \phi_\sigma), \\ 2 & (\epsilon = -1, \phi_\sigma^\varepsilon \cong \phi_\sigma). \end{cases}$$

Proof. Let ϕ_π be the L-parameter of π. Then, it is known that all representations in $\mathrm{Irr}(\widetilde{\mathcal{S}}_{\phi_\sigma}, \zeta_{G(W)})$ have the same dimension $[\widetilde{\mathcal{S}}_{\phi_\pi} : \widetilde{Z}_{\phi_\pi}]^{\frac{1}{2}}$ where $\widetilde{Z}_{\phi_\pi}$ denotes the center of $\widetilde{\mathcal{S}}_{\phi_\pi}$ [Art13, Lemma 9.2.2]. First, we claim that

$$C_{\phi_\pi} \subset C_{\phi_\pi} \cdot \xi(g_1) \text{ for some } g_1 \in \mathrm{O}(Q_M), \det(g_1) = -1. \tag{20.2}$$

Let $g \in C_{\phi_\pi}$. Since $\mathrm{std} \circ \phi_\pi = \mathrm{std} \circ \phi_\sigma \otimes \chi_W \chi_V + \chi_V \boxtimes 1_{\mathrm{SL}_2(\mathbb{C})}$, and $\mathrm{std} \circ \phi_\sigma \otimes \chi_W \chi_V$ does not contain $\chi_V \boxtimes 1_{\mathrm{SL}_2(\mathbb{C})}$ (Proposition 20.4), the action of g on Q_{M+1} preserves the subspace $u(Q_M)$. Hence we have $g \in \mathrm{O}(Q_M)$. Thus, if g' is also an element of C_{ϕ_π}, then we have $g' g^{-1} \in C_{\phi_\pi}$. This implies the claim (20.2).

Suppose that $\epsilon = 1$. Then, we have $C_{\phi_\pi} \supset \xi(C_{\pi_\sigma}) \times \{\pm 1\}$. By (20.2), we have $C_{\phi_\pi} = \xi(C_{\pi_\sigma}) \times \{\pm 1\}$, and we have $[\widetilde{\mathcal{S}}_{\phi_\pi} : \xi(\widetilde{\mathcal{S}}_{\phi_\sigma})] = [\widetilde{Z}_{\phi_\pi} : \widetilde{Z}_{\phi_\sigma}] = 2$. Thus we have $\dim \eta_\pi = \dim \eta_\sigma$.

Suppose that $\epsilon = -1$ and $\phi_\sigma^\varepsilon \not\cong \phi_\sigma$. Then, there is no element $g \in \mathrm{SO}(Q_M)$ so that $\xi(g\varepsilon) \in C_{\phi_\pi}$. Then, by (20.2), we have that ξ is a bijection between C_{ϕ_σ} and C_{ϕ_π}. Thus we have $\widetilde{\mathcal{S}}_{\phi_\sigma} \cong \widetilde{\mathcal{S}}_{\phi_\pi}$ and $\dim \eta_\pi = \dim \eta_\sigma$.

Finally, suppose that $\epsilon = -1$ and $\phi_\sigma^\varepsilon \cong \phi_\sigma$. Then, there exists an element $g \in \mathrm{SO}(Q_M)$ so that $\xi(g\varepsilon) \in C_{\phi_\pi}$. Hence we have $[\widetilde{\mathcal{S}}_{\phi_\pi} : \xi(\widetilde{\mathcal{S}}_{\phi_\sigma})] = 2$ by (20.2). Then, we have

$$\eta_\pi \subset \mathrm{Ind}_{\xi(\widetilde{\mathcal{S}}_{\phi_\sigma})}^{\widetilde{\mathcal{S}}_{\phi_\pi}} \eta$$

for some irreducible representation η of $\xi(\widetilde{\mathcal{S}}_{\phi_\sigma})$. Thus we have $\dim \eta_\pi \leq 2 \dim \eta = 2 \dim \eta_\sigma$. Besides, since the action of $g\varepsilon$ on $Z(\mathrm{Spin}(Q_M))$ is non-trivial, we have $[\widetilde{Z}_{\phi_\sigma} : \widetilde{Z}_{\phi_\pi}] > 1$, which implies $\dim \eta_\pi \geq 2 \dim \eta_\sigma$. Thus we have $\dim \eta_\pi = 2 \dim \eta_\sigma$. Therefore, we prove the proposition. ☐

Now we give the alternative proof of Theorem 11.2. By the proof of Proposition 20.5, we have

$$\frac{\dim \eta_\pi}{\dim \eta_\sigma} \cdot \frac{\# C_{\phi_\sigma}}{\# C_{\phi_\pi}} = \begin{cases} \frac{1}{2} & (\epsilon = 1), \\ 1 & (\epsilon = -1). \end{cases}$$

Moreover, as representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$, we have

$$\mathrm{ad} \circ \phi_\pi = \mathrm{ad} \circ \phi_\sigma \oplus (\mathrm{std} \circ \phi_\sigma) \otimes \chi_W. \tag{20.3}$$

By the substitution of the formulas of the formal degrees (Conjecture 20.2), we have

$$\begin{aligned} &\frac{\deg \pi}{\deg \sigma} \cdot c_\sigma(-1) \cdot \gamma(0, \sigma \times \chi_W, \psi)^{-1} \\ &= \zeta_\pi \zeta_\sigma^{-1} \cdot c_\sigma(-1) \cdot \frac{\dim \eta_\pi}{\dim \eta_\sigma} \cdot \frac{\# C_{\phi_\sigma}}{\# C_{\phi_\pi}} \cdot \frac{\gamma(0, \pi, \mathrm{ad}, \psi)}{\gamma(0, \sigma, \mathrm{ad}, \psi)} \cdot \gamma^V(0, \sigma \times \chi_W, \psi)^{-1} \\ &= \begin{cases} \frac{1}{2} c_\sigma(-1) \zeta_\pi \zeta_\sigma^{-1} & (\epsilon = 1), \\ c_\sigma(-1) \zeta_\pi \zeta_\sigma^{-1} & (\epsilon = -1). \end{cases} \end{aligned}$$

It remains to show that

$$\zeta_\pi \zeta_\sigma^{-1} = c_\sigma(-1) \chi_W(-1)^n \cdot \epsilon(\frac{1}{2}, \chi_V \chi_W, \psi)^{-1}.$$

It is known that

$$\mathrm{ad} \circ \phi_{\mathrm{St}} = \oplus_{d \geq 1} E'_d \otimes r_{2d-1}$$

as $W_F \times \mathrm{SL}_2(\mathbb{C})$-modules [GR10, §3.3]. Here E'_d is the W_F-modules obtained by the action by Γ on the submodule of homogeneous elements of degree d of E' (see §6.1), and r_{2d-1} is the unique $2d-1$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Then, by using the formula of the structure of the graded module E' (see the proof of Proposition 6.1), we have

$$\gamma(0, \mathrm{St}, \mathrm{ad}, \psi) = \epsilon(\frac{1}{2}, \mathrm{St}, \mathrm{ad}, \psi) \cdot |\gamma(0, \mathrm{St}, \mathrm{ad}, \psi)|,$$

and thus we have $\zeta_\pi = \epsilon(\frac{1}{2}, \pi, \mathrm{ad}, \psi)^{-1}$. Since σ is square-irreducible, the L-factor $L(s, \sigma \boxtimes \chi_W)$ does not have a pole at $s = 1/2$. Hence, by (20.3), we have

$$\zeta_\pi \zeta_\sigma^{-1} = \epsilon(\frac{1}{2}, \sigma \boxtimes \chi_W, \psi)^{-1}.$$

Moreover, by [Kak20, Proposition 8.2], this is equal to

$$c_\sigma(-1)\chi_W(-1)^m \epsilon(\frac{1}{2}, \chi_V \chi_W, \psi)^{-1}.$$

Thus, we complete the proof of Theorem 11.2 admitting that Hypothesis 20.1 and Conjecture 20.2 hold.

21. Formal degree conjecture for the non-split inner forms of Sp_4, GSp_4

The local Langlands correspondence for the non-split inner forms of GSp_4 and Sp_4 has been constructed by Gan and Tantono [GT14] and Choiy [Cho17]. Note that one can show the equation of the Plancherel measure by Proposition 16.6 and accidental isomorphisms. Hence Hypothesis 20.1 is true in these cases, and we have a bijection $\pi \mapsto \eta$. Thus, the refined formal degree conjecture for these groups can be stated unconditionally, and it suffices to show that the bijection $\pi \mapsto \eta$ satisfies the formula in Conjecture 20.2. In this section, we prove this as an application of Theorem 11.2. We denote by $G_{1,1}$, $H_{1,1}$, and $H_{3,0}$ the isometry groups of

- the two-dimensional Hermitian space W with $\chi_W = 1_{F^\times}$,
- the two-dimensional skew-Hermitian space W with $\chi_W = 1_{F^\times}$,
- the three-dimensional skew-Hermitian space W with $\chi_W = 1_{F^\times}$

respectively. We also denote by $\widetilde{G}_{1,1}$, $\widetilde{H}_{1,1}$, and $\widetilde{H}_{3,0}$ their similitude groups respectively. In this section, we assume that G is one of $G_{1,1}, H_{1,1}, H_{3,0}$, and we assume that $\widetilde{G}$ is the corresponding similitude group. We denote by $\mathfrak{p} \colon \widehat{\widetilde{G}} \to \widehat{G}$ the projection of [Lab85, Theorem 8.1]. Let $\widetilde{\phi}$ be an L-parameter for $\widetilde{G}$. We denote by $\phi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ the L-parameter given by the composition $\mathfrak{p} \circ \widetilde{\phi}$. According to [Cho17, §7.3], the L-parameter ϕ of $\widetilde{G}_{1,1}$ is classified into one of the following "Case I-(a), Case I-(b), Case II, Case III":

- Case I-(a): the parameter $\widetilde{\phi}$ comes from that of $\widetilde{H}_{1,1}$, and the cardinality of the L-packet $\Pi_{\widetilde{\phi}}$ is equal to 2, and the action of $\mathrm{Hom}(W_F, \mathbb{C}^1)$ is not transitive;
- Case I-(b): the parameter $\widetilde{\phi}$ comes from that of $\widetilde{H}_{1,1}$, and the cardinality of the L-packet $\Pi_{\widetilde{\phi}}$ is equal to 2, and the action of $\mathrm{Hom}(W_F, \mathbb{C}^1)$ is transitive;
- Case II: the parameter $\widetilde{\phi}$ comes from that of $\widetilde{H}_{1,1}$, and the cardinality of the L-packet $\Pi_{\widetilde{\phi}}$ is equal to 1;
- Case III: the parameter $\widetilde{\phi}$ comes from that of $\widetilde{H}_{3,0}$, and the cardinality of the L-packet $\Pi_{\widetilde{\phi}}$ is equal to 1.

Denote by $X(\widetilde{\phi})$ the stabilizer

$$\{a \in H^1(W_F, \widehat{\mathrm{GL}_1}) \mid a\widetilde{\phi} = \widetilde{\phi} \text{ as } L\text{-parameters}\}.$$

Then we have an exact sequence

$$\mathcal{S}_{\widetilde{\phi}} \to \mathcal{S}_\phi \to X(\widetilde{\phi}) \to 1.$$

In the case ϕ is a tempered parameter, the first map is injective [Cho19, Lemma 4.9].

21.1. **Restriction of representations from $\widetilde{G}$ to G.** It is known that such restriction problems have much information on Langlands parameters for G. We only use Lemma 21.1:

Lemma 21.1. *Let $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}$. Then, we have a decomposition*

$$\widetilde{\pi}|_G = (\bigoplus_{i=1}^{t} \pi_i)^{\oplus k},$$

where $\pi_1, \dots, \pi_t$ are irreducible representations of G and

$$k = \begin{cases} \frac{1}{2}\dim\eta & (G = G_{1,1} \textit{ and } \widetilde{\pi} \textit{ has the } L\textit{-parameter of Case } \mathrm{I}-(\mathrm{b})), \\ \dim\eta & (\textit{otherwise}). \end{cases}$$

Proof. It is obtained by [Cho17, Theorems 5.1, 6.1, 7.5]. □

In this paper, we need Lemma 21.1 to prove Lemmas 21.2 and 21.3.

Lemma 21.2. *Let π be a square-integrable irreducible representation of G, let (ϕ, η) be its Langlands parameter, let $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}$ so that its restriction $\widetilde{\pi}|_G$ to G contains π, and let $(\widetilde{\phi}, \widetilde{\eta})$ be the Langlands parameter of $\widetilde{\pi}$. Then, we have*

$$\deg\widetilde{\pi} = \frac{\dim\widetilde{\eta}}{\dim\eta}\cdot\frac{\#C_\phi}{\#C'_{\widetilde{\phi}}}\cdot\deg\pi, \ \textit{and}\ \mathrm{ad}\circ\widetilde{\phi} = \mathrm{ad}\circ\phi.$$

Proof. Put

$$X(\widetilde{\pi}) = \{\chi \in \mathrm{Hom}(F^\times, \mathbb{C}^\times) \mid (\chi\circ\lambda)\widetilde{\pi} \cong \widetilde{\pi}\}.$$

Then the reciprocity map of the local class field theory induces an embedding $X(\widetilde{\pi}) \to X(\widetilde{\phi})$. Moreover, we have

$$[X(\widetilde{\phi}) : X(\widetilde{\pi})] = \begin{cases} 2 & (G = G_{1,1} \text{ and } \widetilde{\pi} \text{ has the } L\text{-parameter of Case I-(b)}), \\ 1 & (\text{otherwise}). \end{cases}$$

Hence, by [GI14, Lemma 13.2] and by Lemma 21.1, we have

$$\begin{aligned} \deg\pi &= \frac{\#Z'(\widehat{\widetilde{G}})}{\#Z(\widehat{G})}\cdot\frac{k}{\#X(\widetilde{\pi})}\cdot\deg\widetilde{\pi} \\ &= \frac{\#Z'(\widehat{\widetilde{G}})}{\#Z(\widehat{G})}\cdot\frac{\dim\eta\cdot\#\mathcal{S}_{\widetilde{\phi}}}{\#\mathcal{S}_\phi}\cdot\deg\widetilde{\pi} \\ &= \frac{\dim\eta\cdot\#C'_{\widetilde{\phi}}}{\#C_\phi}\cdot\deg\widetilde{\pi}. \end{aligned}$$

Moreover, since the projection $\mathfrak{p}\colon \widehat{\widetilde{G}} \to \widehat{G}$ factors through the adjoint map ad, we have

$$\begin{aligned} \mathrm{ad}\circ\widetilde{\phi} &= \mathrm{ad}\circ\mathfrak{p}\circ\phi \\ &= \mathrm{ad}\circ\phi. \end{aligned}$$

Hence, we have the lemma. □

Lemma 21.3. *Let π be a square-integrable irreducible representation of $G_{1,1}$, and let σ be an irreducible representation of either $H_{1,1}$ or $H_{3,0}$ associated with π by the local theta correspondence. We assume that $\sigma \neq 0$. We denote by (ϕ_π, η_ϕ) (resp. $(\phi_\sigma, \eta_\sigma)$) the Langlands parameter associated with π (resp. σ). Then, we have*

$$\frac{\dim \eta_\sigma}{\dim \eta_\pi} = \begin{cases} \frac{1}{2} & (\pi \textit{ has the } L\textit{-parameter of Case } \mathrm{I}-(\mathrm{b})), \\ 1 & (\textit{otherwise}) \end{cases} \tag{21.1}$$

and we have

$$\frac{\#C'_{\phi_\sigma}}{\#C'_{\phi_\pi}} = \begin{cases} \frac{1}{2} & (\pi \textit{ has the } L\textit{-parameter of Cases } \mathrm{I}-(\mathrm{b}), \mathrm{III}), \\ 1 & (\textit{otherwise}). \end{cases}$$

Proof. Note that discrete series parameters are of Case I and Case III. By [GT14, Proposition 3.3] and Lemma 21.1, we have (21.1). The remaining equality is obtained by the case-by-case discussion in [Cho17, p. 1867–1874]. □

21.2. **Refined formal degree conjecture.** In this section, we discuss the refined formal degree conjecture [GR10, Conjecture 7.1]. We first prove it for inner forms of GL_N. Note that $\#C'_\phi = N$ if ϕ is a discrete parameter for GL_N.

Lemma 21.4. *Let G be an inner form of GL_N, and let π be a square-integrable irreducible representation of G. Then, we have*

$$\deg(\pi) = c_\pi (-1)^{N-1} \cdot \frac{1}{N} \cdot \gamma(0, \pi, \mathrm{ad}, \psi).$$

Here, ad *is the adjoint representation of ${}^LG$ on $\mathfrak{sl}_N(\mathbb{C})$.*

Proof. By [HII08, §3.1], we have

$$\deg(\pi) = \frac{1}{N} \cdot |\gamma(0, \rho, \mathrm{ad}, \psi)|.$$

Take a square-integrable representation ρ of $\mathrm{GL}_N(F)$ so that $\pi = |\,\mathrm{JL}_F\,|(\rho)$. Then, by [GI14, Proposition 14.1], we have

$$\begin{aligned} \frac{\gamma(0, \pi, \mathrm{ad}, \psi)}{|\gamma(0, \pi, \mathrm{ad}, \psi)|} &= \frac{\gamma(0, \rho, \mathrm{ad}, \psi)}{|\gamma(0, \rho, \mathrm{ad}, \psi)|} \\ &= \omega_\rho(-1)^{N-1} \\ &= \omega_\pi(-1)^{N-1}. \end{aligned}$$

Thus, by the positivity of $\deg \pi$, we have the lemma. □

Let $\widetilde{G}$ be one of $G_{1,1}$, $H_{1,1}$, $H_{3,0}$, $\widetilde{G}_{1,1}$, $\widetilde{H}_{1,1}$, and $\widetilde{H}_{3,0}$. Then the refined formal degree conjecture for $\widetilde{G}$ is true:

Theorem 21.5. *Let π be a square-integrable irreducible representation of $\widetilde{G}$, and let (ϕ, η) be its Langlands parameter. Then we have*

$$\deg \pi = c_\pi(-1) \cdot \frac{\dim \eta}{\#C'_\phi} \cdot \gamma(0, \mathrm{ad} \circ \phi, \psi).$$

Proof. When G' is either $\widetilde{H}_{1,1}$ or $\widetilde{H}_{3,0}$, we have the claim because of the accidental isomorphisms

$$\begin{aligned}\widetilde{H}_{1,1} &= D^\times \times \mathrm{GL}_2(F)/\{(t, t^{-1}\cdot I_2) \mid t \in F^\times\},\\ \widetilde{H}_{3,0} &= D_4^\times \times F^\times/\{(t, t^{-2}) \mid t \in F^\times\}\end{aligned}$$

as in §17. Here, D_4 is a central division algebra of F with $[D_4 : F] = 16$. Hence, we have the claim for $H_{1,1}$ and $H_{3,0}$ by Lemma 21.2. When $G' = G_{1,1}$, we have the claim by Theorem 11.2, equation (20.3) and Lemma 21.3. Hence, we also have the claim for $\widetilde{G}_{1,1}$. Thus we have the theorem. □

Appendix A. An explicit formula of zeta integrals

In [Kak20, Proposition 8.3], the author computed the doubling zeta integral of right $K(\underline{e}'^{\square})$-invariant sections. However, the formula does not tell us about the constant term and a certain multiplier polynomial factor $S(T)$. In this section, we complete the formula by applying the formula of $\alpha_1(W)$. Note that there are errors in [Kak20, Proposition 8.3]. We also point out and correct them. In this section, we assume **the residue characteristic of F is not** 2. We note that the results in this Appendix are not used in this paper but had been used in the previous version. Actually, we can prove Proposition 7.6 by them if we assume $q \nmid 2$.

Fix a basis $\underline{e}$ of W as in §4. We denote by $\underline{e}_0$ the basis $e_{r+1}, \ldots, e_{r+n_0}$ for W_0. Moreover, we may assume that

$$R_0 = R(\underline{e}_0) = \begin{cases} 1 & (-\epsilon = 1, n_0 = 1),\\ \alpha & (-\epsilon = -1, n_0 = 1),\\ \varpi_D^{-1} & (-\epsilon = -1, n_0 = 1),\\ \mathrm{diag}(\varpi_D^{-1}, \alpha\varpi_D^{-1}) & (-\epsilon = -1, n_0 = 2 \text{ with } \chi_W \text{ unramified}),\\ \mathrm{diag}(\alpha, \varpi_D^{-1}) & (-\epsilon = -1, n_0 = 2 \text{ with } \chi_W \text{ ramified}),\\ \mathrm{diag}(\alpha, \varpi_D^{-1}, \beta^{-1}) & (-\epsilon = -1, n_0 = 3).\end{cases}$$

Here, α is defined in §2, and β is an element of D so that $\mathrm{ord}_D(\beta) = -1$, $T_D(\beta) = 0$ and $\beta^2 = \alpha^2\varpi_D^2$. We recall that we have put $n_0 = \dim W_0$ and $r = \frac{n-n_0}{2}$. By this basis, we regard $G(W)$ as a subgroup of $\mathrm{GL}_n(D)$. Then, put

$$C_1 \coloneqq \{g \in G(W) \cap \mathrm{GL}_n(\mathcal{O}_D) \mid (g-1)R(\underline{e}) \in \mathrm{M}_n(\mathcal{O}_D)\}.$$

Note that C_1 is an open compact subgroup of $G(W)$. Let X_i be a subspace of X spanned by $e_1, \ldots, e_i$. We denote by $\mathfrak{f}$ the flag

$$\mathfrak{f} : 0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = X,$$

and by B the minimal parabolic subgroup preserving $\mathfrak{f}$.

Proposition A.1. *We have $G(W) = B \cdot C_1$.*

Proof. We use the setting and the notation of §5 in the proof of this proposition. By the result of Bruhat and Tits [BT72, Théorème (5.1.3)] and that of Heines and Rapoport [PR08, Appendix, Proposition 8], we have the decomposition

$$G(W) = B \cdot N_{G(W)}(S) \cdot \mathcal{B}.$$

Since $B \supset Z_{G(W)}(S)$, we can take a representative system $w_1, \ldots, w_t$ for $B \backslash (B \cdot N_{G(W)}(S))$ so that $w_i \in C_1$ for $i = 1, \ldots, t$. Moreover, $X_{a,0} \subset C_1$ for $a \in \Phi^+$ and $X_{a,\frac{1}{2}} \subset C_1$ for $a \in \Phi^-$. Hence, by Lemma 5.2, we have

$$\begin{aligned} B \cdot N_{G(W)}(S) \cdot \mathcal{B} &= \bigcup_{i=1}^{t} B \cdot w_i \cdot Z_{G(W)}(S)_1 \cdot \prod_{a \in \Phi^+} X_{a,0} \cdot \prod_{a \in \Phi^-} X_{a,\frac{1}{2}} \\ &= \bigcup_{i=1}^{t} B \cdot Z_{G(W)}(S)_1 \cdot w_i \cdot \prod_{a \in \Phi^+} X_{a,0} \cdot \prod_{a \in \Phi^-} X_{a,\frac{1}{2}} \\ &\subset B \cdot C_1. \end{aligned}$$

Thus we have the proposition. □

Let σ_0 be the trivial representation of $G(W_0)$, let s_i be a complex number for $i = 1, \ldots, r$, let σ_i be the character $| \ |^{s_i}$ of $\mathrm{GL}_1(D)$ for $i = 1, \ldots, r$. Then, $\sigma = \otimes_{i=0}^{r} \sigma_i$ is a character of the Levi subgroup of B. Let π be an irreducible subquotient representation of $\mathrm{Ind}_B^{G(W)}(\sigma)$ having a non-zero C_1-fixed vector. Then, we have the following formula of a zeta integral with a certain section and a matrix coefficient:

Proposition A.2. *Let $f_s^\circ \in I(s, 1_{F^\times})^{K(\underline{e}'^{\square})}$ be a non-zero $K(\underline{e}'^{\square})$-invariant section with $f_s^\circ(1) = 1$, let ξ° be the C_1-fixed matrix coefficient of π. Then, we have*

$$Z(f_s^\circ, \xi^\circ) = |C_1| \cdot \frac{S(q^{-s})}{d^W(s)} \prod_{i=0}^{r} L^{W_i}(s + \frac{1}{2}, \sigma_i)$$

for some self-reciprocal monic polynomial $S(T)$ of degree

$$f_W = \begin{cases} 1 & (-\epsilon = -1, n_0 = 2, \chi_W \text{ is unramified}), \\ 0 & (\text{otherwise}). \end{cases}$$

Here we set

$$d^W(s) = \begin{cases} \zeta_F(s + n + \frac{1}{2}) \prod_{i=1}^{\lfloor n/2 \rfloor} \zeta_F(2s + 2n + 1 - 4i) & (-\epsilon = 1), \\ \prod_{i=1}^{\lceil n/2 \rceil} \zeta_F(2s + 2n + 3 - 4i) & (-\epsilon = -1). \end{cases}$$

Note that if $n_0 = 0$, then $L^{W_0}(s, 1_{W_0} \times 1)$ denotes

$$\begin{cases} \zeta_F(s) & (-\epsilon = 1), \\ 1 & (-\epsilon = -1). \end{cases}$$

Note that we will determine $S(T)$ and $|C_1|$ later (Propositions A.4 and A.5).

Remark A.3. Proposition A.2 differs from [Kak20, Proposition 8.3] at the definition of f_W in the case $n_0 = 3$ and the definition of $L^{W_0}(s, 1_{G(W_0)} \times 1_{F^\times})$ in the case $n_0 = 0$, $-\epsilon = 1$. The former is caused by an error in the computation of the γ-factor, which is modified by (A.3). And the latter is caused by a typo.

Proof. We can deform the doubling zeta integral to the summation

$$Z^W(f_s^\circ, \xi^\circ) = \int_{C_1} \xi^\circ(g)\, dg + \int_{G(W) - C_1} f_s^\circ((g, 1)) \xi^\circ(g)\, dg.$$

If s_0 is a sufficiently large real number so that $Z^W(f_s^\circ, \xi^\circ)$ converges absolutely, then, by [Kak20, Lemma 8.4], we have

$$\left| \int_{G(W)-C_1} f_s^\circ((g,1))\xi^\circ(g) \right| \leq \int_{G(W)-C_1} |\Delta((g,1))|^{s-s_0} |f_{s_0}^\circ((g,1))\xi^\circ(g)|\, dg$$
$$\leq q^{-(\Re s - s_0)} \int_{G(W)} |f_{s_0}^\circ((g,1))\xi^\circ(g)|\, dg$$

for $\Re s > s_0$. Thus we have

$$\lim_{\Re s \to \infty} Z^W(f_s^\circ, \xi^\circ) = |C_1|. \tag{A.1}$$

Put

$$\Xi(q^{-s}) \coloneqq \frac{Z^W(f_s^\circ, \xi^\circ)}{\prod_{i=0}^r L^{W_i}(s+\frac{1}{2}, \sigma_i \times 1_{F^\times})}.$$

Then, the "g.c.d. property" ([Yam14, Theorem 5.2] and [Yam14, Lemma 6.1]) implies that $\Xi(q^{-s})$ is a polynomial in q^{-s} and q^s. Moreover, by (A.1), it is a polynomial of q^{-s} with the constant term $|C_1|$. Put $D(q^{-s}) \coloneqq d^W(s)$. Once we prove the equation

$$\Xi(q^{-s})D(q^s) = (q^{-s})^{f_W} \cdot \Xi(q^s)D(q^{-s}), \tag{A.2}$$

one can deduce that

$$\Xi(q^{-s}) = |C_1| \cdot S(q^{-s})D(q^{-s})$$

for some monic self-reciprocal monic polynomial of degree f_W since $q^{-ts}D(q^s)$ is a polynomial of q^{-s} which is coprime to $D(q^{-s})$ for sufficiently large t, which proves the proposition.

In the following, we prove equation (A.2). By the definition of the γ-factor, we have

$$c(s, 1_{F^\times}, A, \psi)^{-1} R(s, 1_{F^\times}, A, \psi) \cdot Z^W(M(s, 1_{F^\times})f_s^\circ, \xi^\circ)$$
$$= c_\pi(-1) \cdot \gamma^W(s+\frac{1}{2}, \pi \boxtimes 1_{F^\times}, \psi) Z^W(f_s^\circ, \xi^\circ).$$

By comparing this with the equation

$$c(s, 1_{F^\times}, A, \psi)^{-1} R(s, 1_{F^\times}, A, \psi) M^*(s, 1_{F^\times}, A, \psi) f_s^\circ$$
$$= q^{-n's} |N(R(\underline{e}))|^{-s} \epsilon(\frac{1}{2}, \chi_W, \psi) \cdot \frac{D(q^{-s})}{D(q^s)} f_{-s}^\circ,$$

where

$$n' = \begin{cases} 2\lceil \frac{n}{2} \rceil & (-\epsilon = 1), \\ 2\lfloor \frac{n}{2} \rfloor & (-\epsilon = -1), \end{cases}$$

we obtain

$$\Xi(q^{-s})D(q^s) = D(q^{-s})\Xi(q^s)$$
$$\times |N(R(\underline{e}))|^{-s} q^{-n's} \cdot \frac{\epsilon(\frac{1}{2}, \chi_W, \psi)}{\gamma(s+\frac{1}{2}, \pi \boxtimes 1_{F^\times}, \psi)}$$
$$\cdot \frac{\prod_{i=0}^r L^{W_i}(-s+\frac{1}{2}, \sigma_i^\vee \times 1_{F^\times})}{\prod_{i=0}^r L^{W_i}(s+\frac{1}{2}, \sigma_i \times 1_{F^\times})}.$$

Moreover, by Lemma 7.5, we have

$$\gamma^W(s+\frac{1}{2},\pi\times 1_{F^\times},\psi)=q^{-\lambda s}\cdot\epsilon^W(\frac{1}{2},\chi_W,\psi)\prod_{i=0}^{r}\frac{L^{W_i}(-s+\frac{1}{2},\sigma_i^\vee\times 1_{F^\times})}{L^{W_i}(s+\frac{1}{2},\sigma\times 1_{F^\times})}, \tag{A.3}$$

where

$$\lambda=\begin{cases}2\lceil\frac{n}{2}\rceil & (-\epsilon=1),\\ 2\lfloor\frac{n}{2}\rfloor & (-\epsilon=-1,\ n\not\equiv 3 \bmod 4,\ \chi_W \text{ is unramified}),\\ 2\lfloor\frac{n}{2}\rfloor+1 & (\text{otherwise}).\end{cases}$$

Therefore,

$$\begin{aligned}\Xi(q^{-s})D(q^s)&=D(q^{-s})\Xi(q^s)\cdot q^{-(n'-\lambda)s}\cdot|N(R(\underline{e}))|^{-s}\\&=D(q^{-s})\Xi(q^s)\cdot(q^{-s})^{f_W}.\end{aligned}$$

Hence we have equation (A.2), and we have the proposition. □

For the polynomial $S(T)$, we have the following:

Proposition A.4. *We have*

$$S(T)=\begin{cases}T^2+(q^{\frac{1}{2}}+q^{-\frac{1}{2}})T+1 & (-\epsilon=-1,\ n_0=2,\ \chi_W \textit{ is unramified}),\\ 1 & (\textit{ otherwise}).\end{cases}$$

Proof. We have $f_W=0$ in the cases other than $-\epsilon=-1$, $n_0=2$, and χ_W is unramified. Thus the proposition is clear for the second case. Consider the case $n=n_0=2$ and χ_W is unramified. Since $G(W)$ is compact, $Z(f_s^\circ,\xi^\circ)$ is a polynomial in q^{-s}. In other words,

$$S(q^{-s})\frac{\zeta_F(s+\frac{3}{2})L(s+\frac{1}{2},\chi_W)}{\zeta_F(2s+3)}$$

is a polynomial. Thus, we can conclude that $(1+q^{-\frac{1}{2}}T)$ divides $S(T)$. Such a self-reciprocal polynomial is only $(1+q^{-\frac{1}{2}}T)(1+q^{\frac{1}{2}}T)$. Hence we have

$$S(T)=T^2+(q^{\frac{1}{2}}+q^{-\frac{1}{2}})T+1.$$

Now, suppose that $-\epsilon=-1$, $n>n_0=2$, and χ_W is unramified. We recall a certain intertwining operator associated with the parabolic induction. Let $Q(X^\square)$ be the parabolic subgroup of $G(W^\square)$ preserving $X^\square$, let $U(X^\square)$ be the unipotent radical of $Q(X^\square)$, let M be the Levi-subgroup of $Q(X^\square)$, and let $I^X(s,1_{F^\times})$ be the space of smooth functions f on $\mathrm{GL}(X^\square)$ satisfying

$$f(pg)=|N(p|_{X^\triangle})|^{-(s+r)}|N(p|_{X^\triangledown})|^{s+r}f(g)$$

for $p\in P'(X^\triangle)$ and $g\in\mathrm{GL}(X^\square)$. Here, we denote by $P'(X^\triangle)$ the parabolic subgroup of $\mathrm{GL}(X^\square)$ preserving $X^\triangle$, by $p|_{X^\triangle}$ (resp. $p|_{X^\triangledown}$) the restriction of p to $X^\triangle$ (resp. $X^\triangledown$), and by N the reduced norm of $\mathrm{End}(X^\triangle)$ (resp. $\mathrm{End}(X^\triangledown)$). For a coefficient ξ of an irreducible representation of $\mathrm{GL}(X)$ and a section $f\in I(s,1_{F^\times})$, we define the doubling zeta integral by

$$Z^X(f,\xi)=\int_{\mathrm{GL}(X)}f(\iota_X(g,1))\xi(g)\,dg,$$

where $\iota_X \colon \mathrm{GL}(X) \times \mathrm{GL}(X) \to \mathrm{GL}(X^\square)$ is the embedding induced by the natural action of $\mathrm{GL}(X) \times \mathrm{GL}(X)$ on $X^\square$. Then, there is an intertwining map

$$\Psi(s) \colon I^W(s, 1_{F^\times}) \to \mathrm{Ind}_{Q(X^\square)}^{G(W^\square)}(I^X(s, 1_{F^\times}) \otimes I^{W_0}(s, 1_{F^\times}) \otimes |\Delta_{(X,W_0):W}|) \colon f_s \mapsto (g \mapsto [\Phi(s) f_s]_g)$$

(see [Yam14, Proposition 4.1]). Although we omit the definition, we note the relation

$$[\Phi(s) f_s^\circ]_e = J(s) f_s'^\circ \otimes f_s''^\circ,$$

where $f_s'^\circ$ (resp. $f_s''^\circ$) is the unique $\mathrm{GL}_r(\mathcal{O}_D)$-invariant section of $I^X(s, 1_{F^\times})$ (resp. the unique $K(\underline{e}_0'^\square)$-invariant section of $I^{W_0}(s, 1_{F^\times})$) so that $f_s'^\circ(1) = 1$ (resp. $f_s''^\circ(1) = 1$), and

$$J(s) = \int_{U(X^\square) \cap Q(W^\triangle) \backslash U(X^\square)} f_s^\circ(u) \, du.$$

Moreover, by Proposition A.1, we have

$$\begin{aligned} Z^W(f_s^\circ, \xi^\circ) &= |C_1| \int_Q f_s^\circ((g,1)) \, dg \\ &= |C_1| \int_M [\Psi(s) f_s^\circ]((m,1)) \, dm \\ &= |C_1| J(s) Z^{W_0}(f_s'^\circ, \xi'^\circ) Z^X(f_s''^\circ, \xi''^\circ) \\ &= |C_1| J(s) S(q^{-s}) \frac{L^{W_0}(s+\frac{1}{2}, 1_{G(W)} \times 1_{F^\times})}{d^{W_0}(s)} \cdot \frac{L^X(s+\frac{1}{2}, \sigma)}{d^X(s)} \\ &= |C_1| S^{W_0}(q^{-s}) \frac{J(s)}{d^{W_0}(s) d^X(s)} L^W(s + \frac{1}{2}, 1_{G(W)} \times 1_{F^\times}). \end{aligned}$$

Thus, we obtain

$$S^W(q^{-s}) = S^{W_0}(q^{-s}) \times J(s) \frac{d^W(s)}{d^{W_0}(s) d^X(s)}.$$

However, since $J(s)$ does not have a pole in $\Re s > -1$ [Yam14, Lemma 5.1] and $d^W(s), d^{W_0}(s), d^X(s)$ has neither a pole nor a zero at $s = \pi\sqrt{-1} \pm \frac{1}{2}$, we can conclude that $S^W(X)$ is divided by $(1 + q^{\pm\frac{1}{2}} T)$. Thus, we have $S^W(T) = S^{W_0}(T)$. Hence, we finish the proof of the proposition. □

Finally, by the formula of $\alpha_1(W)$ (Proposition 19.4), we can determine the volume $|C_1|$ of C_1:

Proposition A.5.

(1) *In the case $-\epsilon = 1$, we have*

$$|C_1| = q^{-2\lfloor n/2 \rfloor \lceil n/2 \rceil - \lceil n/2 \rceil} \prod_{i=1}^{\lfloor n/2 \rfloor} (1 + q^{-(2i-1)})(1 - q^{-2i}).$$

(2) *In the case $-\epsilon = -1$, we have*

$$|C_1| = |N(R(\underline{e}))|^{-\rho} q^{-(2\lfloor n/2\rfloor\lceil n/2\rceil - \lfloor n/2\rfloor)}$$
$$\times \begin{cases} \prod_{i=1}^{\lfloor n/2\rfloor}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor}(1-q^{-2i}) & (n_0=0), \\ \prod_{i=1}^{\lceil n/2\rceil}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor}(1-q^{-2i}) & (n_0=1, \chi_W : \textit{unramified}), \\ \prod_{i=1}^{\lfloor n/2\rfloor}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor}(1-q^{-2i}) & (n_0=1, \chi_W : \textit{ramified}), \\ \prod_{i=1}^{\lfloor n/2\rfloor-1}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor-1}(1-q^{-2i}) & (n_0=2, \chi_W : \textit{unramified}), \\ \prod_{i=1}^{\lfloor n/2\rfloor}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor-1}(1-q^{-2i}) & (n_0=2, \chi_W : \textit{ramified}), \\ \prod_{i=1}^{\lfloor n/2\rfloor}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor-1}(1-q^{-2i}) & (n_0=3). \end{cases}$$

Proposition A.4 and Proposition A.5 give a completion of the formula in Proposition A.2.

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