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# STABILITY OF $\imath$CANONICAL BASES OF IRREDUCIBLE FINITE TYPE OF REAL RANK ONE

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ABSTRACT. It has been known since their birth in Bao and Wang's work that the $\imath$canonical bases of $\imath$quantum groups are not stable in general. In the author's previous work, the stability of $\imath$canonical bases of certain quasi-split types turned out to be closely related to the theory of $\imath$crystals. In this paper, we prove the stability of $\imath$canonical bases of irreducible finite type of real rank 1, for which the theory of $\imath$crystals has not been developed, by means of global and local crystal bases.

## 1. INTRODUCTION

Let $A = (a_{i,j})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix, $\mathfrak{g}$ the associated Kac-Moody algebra, and $\mathbf{U} = U_q(\mathfrak{g})$ the Drinfeld-Jimbo quantum group (quantized enveloping algebra) [10], [14], [15] with weight lattice $X$. Let $X^+$ denote the set of dominant weights. For each $\lambda \in X^+$, let $V(\lambda)$ (resp., $V(-\lambda)$) denote the irreducible integrable highest (resp., lowest) weight $\mathbf{U}$-module of highest weight $\lambda$ (resp., lowest weight $-\lambda$) with highest weight vector $v_\lambda$ (resp., lowest weight vector $v_{-\lambda}$). The canonical bases (also known as global crystal bases) of the negative part $\mathbf{U}^-$ and the positive part $\mathbf{U}^+$ of $\mathbf{U}$, and of $V(\pm\lambda)$ for all $\lambda \in X^+$, were constructed for type ADE in [22] and for general in [23] geometrically and in [16] algebraically.

In [24], Lusztig constructed the canonical basis of the tensor product $V(-\lambda) \otimes V(\mu)$ for arbitrary $\lambda, \mu \in X^+$, from the canonical bases of $V(-\lambda)$ and $V(\mu)$. A key ingredient of his construction is the quasi-$R$-matrix, which intertwines the bar-involutions on $\mathbf{U} \otimes \mathbf{U}$ and $\mathbf{U}$.

The canonical bases thus constructed are stable in the following sense. For each $\lambda, \mu, \nu \in X^+$, there exists a unique $\mathbf{U}$-module homomorphism

$$\pi_{\lambda,\mu,\nu} : V(-\lambda-\nu) \otimes V(\mu+\nu) \to V(-\lambda) \otimes V(\mu)$$

which sends $v_{-\lambda-\nu} \otimes v_{\mu+\nu}$ to $v_{-\lambda} \otimes v_\mu$. Then, each canonical basis element of $V(-\lambda-\nu) \otimes V(\mu+\nu)$ is sent to either a canonical basis element of $V(-\lambda) \otimes V(\mu)$ or 0, and the kernel of $\pi_{\lambda,\mu,\nu}$ is spanned by a subset of the canonical basis. In other words, the homomorphism $\pi_{\lambda,\mu,\nu}$ is a based $\mathbf{U}$-module homomorphism.

From this stability property, we see that for each $\zeta \in X$, the family

$$\{V(-\lambda) \otimes V(\mu) \mid \lambda, \mu \in X^+,\ \mu - \lambda = \zeta\} \tag{1.1}$$

of $\mathbf{U}$-modules with $\pi_{\lambda,\mu,\nu}$ for each $\lambda, \mu, \nu \in X^+$ with $\mu - \lambda = \zeta$ forms a projective system of based $\mathbf{U}$-modules. Then, the subspace $\dot{\mathbf{U}}1_\zeta$ of the modified quantum

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group $\dot{\mathbf{U}} = \bigoplus_{\zeta \in X} \dot{\mathbf{U}} 1_\zeta$ with canonical basis can be regarded as the projective limit of the projective system above in a certain category of based $\mathbf{U}$-modules. This construction led to an explicit description of the crystal basis of modified quantum group in [18].

Our main interest in the present paper is the $\imath$quantum group counterpart of the construction above. The $\imath$quantum group (also known as the quantum symmetric pair coideal subalgebra) $\mathbf{U}^\imath$ associated with an admissible pair $(I_\bullet, \tau)$ (in the sense of [19, Definition 2.3]) and parameters $\varsigma_i \in \mathbb{Q}(q)^\times$ and $\kappa_i \in \mathbb{Q}(q)$ for $i \in I \setminus I_\bullet$ is a certain right coideal subalgebra of $\mathbf{U}$ which forms a quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ [21], [19] (we more or less follow the notation in [6]). For each Kac-Moody algebra $\mathfrak{g}$, there exists a quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath) = (U_q(\mathfrak{g} \oplus \mathfrak{g}), U_q(\mathfrak{g}))$. Such a quantum symmetric pair is said to be of diagonal type. Therefore, the quantum group $\mathbf{U}$ itself is an instance of $\imath$quantum groups.

Let $w_\bullet$ denote the longest element of the Weyl group associated with $I_\bullet$ (by the definition of admissible pairs, $I_\bullet$ is of finite type). Bao and Wang constructed a $\mathbf{U}^\imath$-module $V(w_\bullet \lambda, \mu)$ for each $\lambda, \mu \in X^+$ with a distinguished basis, called the $\imath$canonical basis, and $\mathbf{U}^\imath$-module homomorphisms

$$\pi^\imath_{\lambda,\mu,\nu} : V(w_\bullet(\lambda + \tau\nu), \mu + \nu) \to V(w_\bullet \lambda, \mu)$$

for finite type in [6] and for the general case in [8]. To be more precise, the $\mathbf{U}^\imath$-module $V(w_\bullet \lambda, \mu)$ is obtained by restriction from the $\mathbf{U}$-submodule of $V(\lambda) \otimes V(\mu)$ generated by $v_{w_\bullet \lambda} \otimes v_\mu$, where $v_{w_\bullet \lambda} \in V(\lambda)$ denotes the unique canonical basis element of weight $w_\bullet \lambda$. And the $\mathbf{U}^\imath$-module homomorphism $\pi^\imath_{\lambda,\mu,\nu}$ is the unique one which sends $v_{w_\bullet(\lambda+\tau\nu)} \otimes v_{\mu+\nu}$ to $v_{w_\bullet \lambda} \otimes v_\mu$. A key ingredient of their construction of the $\imath$canonical bases is the quasi-$K$-matrix, which intertwines the bar-involutions on $\mathbf{U}$ and $\mathbf{U}^\imath$. The existence of quasi-$K$-matrix and bar-involution on $\mathbf{U}^\imath$ in general was formulated in [7] and the former was reformulated and proved in [1] while the latter in [20] after many partial results (see Section 1 of *loc. cit.*). The bar-involution was also found in [11], whose arXiv version appeared almost the same time as that of [7].

Set $X^\imath := X/\{\lambda + w_\bullet \tau \lambda \mid \lambda \in X\}$, and let $\bar{\cdot} : X \to X^\imath$ denote the quotient map. Then, for each $\zeta \in X^\imath$, we obtain a projective system

$$\{V(w_\bullet \lambda, \mu) \mid \lambda, \mu \in X^+,\ \overline{\mu + w_\bullet \lambda} = \zeta\} \tag{1.2}$$

of $\mathbf{U}^\imath$-modules. This projective system can be seen as a natural generalization of Lusztig's one (1.1). In fact, they coincide with each other when the quantum symmetric pair is of diagonal type since the quasi-$K$-matrix and the $\mathbf{U}^\imath$-module $V(w_\bullet \lambda, \mu)$ become the quasi-$R$-matrix and $V(-\lambda) \otimes V(\mu)$, respectively. However, in contrast to the projective system (1.1), the $\imath$canonical bases are not stable in the projective system (1.2). They are merely asymptotically stable; nevertheless this weaker stability property could still lead to the canonical basis (i.e., the $\imath$canonical basis) of the modified $\imath$quantum group in [6], [8].

On the other hand, in [28], the author proved that the $\imath$canonical bases are stable in the projective system (1.2) when $I_\bullet$ is empty, $a_{i,\tau(i)} \in \{2, 0, -1\}$ for all $i \in I$, and the parameters $\varsigma_i, \kappa_i$ are chosen appropriately. As a result, he interpreted the subspace $\dot{\mathbf{U}}^\imath 1_\zeta$ of the modified $\imath$quantum group $\dot{\mathbf{U}}^\imath = \bigoplus_{\zeta \in X^\imath} \dot{\mathbf{U}}^\imath 1_\zeta$ with $\imath$canonical basis as the projective limit of (1.2) in a certain category of based $\mathbf{U}^\imath$-modules. In the proof, the theory of $\imath$crystals developed in [28] plays a crucial role.

It is natural to expect that one can prove the stability of ıcanonical bases for general quantum symmetric pairs by developing the theory of ıcrystals. The theory of ıcrystals in [28] is based on many explicit calculations involving the quantum symmetric pairs of real rank 1, just like the theory of crystals is based on calculation involving the quantum group of rank 1. Here, the real rank of a quantum symmetric pair refers to the number of $\tau$-orbits in $I \setminus I_\bullet$. The quantum group in a quantum symmetric pair of real rank 1 considered in [28] is of type $A_1$, $A_1 \times A_1$, or $A_2$. Hence, its structure is relatively simple. In general, the quantum group in a quantum symmetric pair of real rank 1 is not of finite type. Even if we restrict our attention to a quantum symmetric pair of finite classical type, its rank can be arbitrarily high. Hence, the same strategy as [28] is not applicable for general quantum symmetric pairs.

In the present paper, after choosing the parameters $\varsigma_i, \kappa_i$ appropriately, we prove the stability of ıcanonical bases for the quantum symmetric pair of irreducible finite type of real rank 1 without developing the theory of ıcrystals. Here, "irreducible" means that the Dynkin diagram $I$, extended by adding edges between $i$ and $\tau(i)$ for all $i \in I$, is connected as a graph. This is the first step toward the generalization of the stability theorem of ıcanonical bases to general quantum symmetric pairs. Since the stability of ıcanonical bases is closely related to ıcrystals, the author expects that our new stability theorem, in turn, stimulates an attempt to extend the theory of ıcrystals to general quantum symmetric pairs.

After submitting the first version of the paper on arXiv, Weiqiang Wang told the author that in [6, Remark 6.18], Bao and he had conjectured that the ıcanonical bases are stable ("strongly compatible" in their terminology) if the parameters $\varsigma_i, \kappa_i$ are chosen properly. The stability theorem in this paper settles their conjecture affirmatively for irreducible finite type of real rank 1.

Let us summarize our proof of the stability of ıcanonical bases. First, we study the $\mathbf{U}$-module structure of $V(w_\bullet\lambda, \mu)$ by investigating the crystal structure of its crystal basis. This enables us to construct a $\mathbf{U}$-module homomorphism

$$V(w_\bullet(\lambda + \tau\nu), \mu + \nu) \to V(\nu + w_\bullet\tau\nu) \otimes V(w_\bullet\lambda, \mu). \tag{1.3}$$

Note that $\nu + w_\bullet\tau\nu$ is dominant. Next, verifying a sufficient condition for a $\mathbf{U}$-module homomorphism to be based, which is given in Proposition 2.4.6, we prove that the $\mathbf{U}$-module homomorphism above is based. Finally, we prove that there exists a based $\mathbf{U}^\imath$-module homomorphism $V(\nu + w_\bullet\tau\nu) \to \mathbb{Q}(q)$ (here, $\mathbb{Q}(q)$ is the trivial $\mathbf{U}^\imath$-module with ıcanonical basis $\{1\}$) which sends the highest weight vector to 1. Composing this based $\mathbf{U}^\imath$-module homomorphism with the based $\mathbf{U}$-module homomorphism in the second step, we obtain a based $\mathbf{U}^\imath$-module homomorphism, which is identical to $\pi^\imath_{\lambda,\mu,\nu}$. The first and second steps are applicable for general quantum symmetric pairs. The final step is achieved by studying the quantum symmetric pairs of irreducible finite type of real rank 1 one-by-one; there are only eight kinds of such quantum symmetric pairs. For the proof of one of them (type FII) we use computer program GAP [12] with package QuaGroup [13] because it is a discrete exceptional type in the sense that its underlying quantum group is of type $F_4$.

In [26], the notion of generalized Satake diagram, which generalizes admissible pair, was introduced. And it was observed that the theory of quantum symmetric pairs can be established for generalized Satake diagrams. There are generalized

Satake diagrams of irreducible finite type of real rank 1 which are not admissible pairs. The author expects that the results in the paper are valid for those generalized Satake diagrams, but we will not treat them here.

The paper is organized as follows. In Section 2, we review well-known results concerning canonical, or global crystal, and crystal bases, and based **U**-modules. Then, we give a sufficient condition for two based **U**-modules with a crystal morphism between their crystal bases having a based **U**-module homomorphism which lifts the crystal morphism in Proposition 2.4.6. In Section 3, after recalling the definition of $\imath$quantum groups and results of [8] about based $\mathbf{U}^\imath$-modules, we construct the based **U**-module homomorphism (1.3). In Section 4, we finish our proof of the stability of $\imath$canonical bases by studying certain $\mathbf{U}^\imath$-modules for each quantum symmetric pair of irreducible finite type of real rank 1 one-by-one.

## 2. Quantum group

The purpose of this section is to fix our notation about the quantum groups, and then prove Proposition 2.4.6, which provides a sufficient condition for a crystal morphism between the crystal bases of two based **U**-modules to be lifted to a based **U**-module homomorphism.

2.1. **Quantum group.** Let $A = (a_{i,j})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix with a symmetrizing matrix $D = \mathrm{diag}(d_i \mid i \in I)$ with $d_i \in \mathbb{Z}_{>0}$ being relatively prime. We often identify $I$ with the Dynkin diagram of $A$. Let $Y$ and $X$ be finitely generated free abelian groups with a perfect pairing $\langle , \rangle : Y \times X \to \mathbb{Z}$. Let $\Pi^\vee = \{h_i \mid i \in I\} \subset Y$ and $\Pi = \{\alpha_i \mid i \in I\} \subset X$ be linearly independent subsets satisfying

$$\langle h_i, \alpha_j \rangle = a_{i,j}$$

for all $i, j \in I$. Let $W$ denote the Weyl group associated with the generalized Cartan matrix $A$. For each $i \in I$, let $s_i \in W$ denote the simple reflection.

Let $q$ be an indeterminate. For each $i \in I$, $n \in \mathbb{Z}$, and $m \in \mathbb{Z}_{\geq 0}$, set

$$q_i := q^{d_i}, \ [n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \ [m]_i! := \prod_{n=1}^{m} [n]_i.$$

When $d_i = 1$, we sometimes omit the subscript "$i$" from notation above.

Let **U** denote the quantum group. Namely, **U** is the unital associative $\mathbb{Q}(q)$-algebra with generators $\{E_i, F_i, K_h \mid i \in I, \ h \in Y\}$ subject to the following relations: For each $i, j \in I$ and $h, h_1, h_2 \in Y$,

$$\begin{aligned}
&K_0 = 1, \ K_{h_1} K_{h_2} = K_{h_1 + h_2}, \\
&K_h E_i = q^{\langle h, \alpha_i \rangle} E_i K_h, \ K_h F_i = q^{\langle h, -\alpha_i \rangle} F_i K_h, \\
&E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\
&S_{i,j}(E_i, E_j) = S_{i,j}(F_i, F_j) = 0 \ \text{ if } i \neq j,
\end{aligned}$$

where

$$K_i := K_{d_i h_i}, \ S_{i,j}(x, y) := \sum_{r+s = 1 - a_{i,j}} (-1)^s \frac{1}{[r]_i! [s]_i!} x^r y x^s.$$

The $\mathbf{U}$ is a Hopf algebra with comultiplication map $\Delta$ given by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$
$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1},$$
$$\Delta(K_h) = K_h \otimes K_h$$

for all $i \in I$, $h \in Y$.

Let $\bar{\cdot}$ denote the bar-involution on $\mathbf{U}$, i.e., the $\mathbb{Q}$-algebra automorphism such that

$$\overline{E_i} = E_i,\ \overline{F_i} = F_i,\ \overline{K_h} = K_{-h},\ \bar{q} = q^{-1}$$

for all $i \in I$, $h \in Y$.

Let $\rho$ denote the anti-algebra automorphism on $\mathbf{U}$ such that

$$\rho(E_i) = q_i^{-1} F_i K_i,\ \rho(F_i) = q_i K_i^{-1} E_i,\ \rho(K_h) = K_h \tag{2.1}$$

for all $i \in I$, $h \in Y$.

For each $i \in I$, let $T_i$ denote both the algebra automorphism $T''_{i,1}$ on $\mathbf{U}$ in [25, Proposition 37.1.2] and the automorphism $T''_{i,1}$ on integrable $\mathbf{U}$-modules in [25, 5.2.1]. For each $w \in W$ with a reduced expression $w = s_{i_1} \cdots s_{i_r}$, set $T_w := T_{i_1} \cdots T_{i_r}$.

Let $\mathbf{U}^+$ (resp., $\mathbf{U}^-$) denote the subalgebra of $\mathbf{U}$ generated by $\{E_i \mid i \in I\}$ (resp., $\{F_i \mid i \in I\}$). Let $(\mathcal{L}(\pm\infty), \mathcal{B}(\pm\infty))$ and $\mathbf{B}(\pm\infty)$ denote the crystal base and global crystal basis of $\mathbf{U}^{\mp}$ in [16, Theorems 4 and 6]. In this paper, crystal lattices are considered over the subring $\mathbf{A}_\infty$ of $\mathbb{Q}(q)$ consisting of all rational functions regular at $q = \infty$. Let $G_{\pm\infty} : \mathcal{B}(\pm\infty) \to \mathbf{B}(\pm\infty)$ denote the bijection such that $G_{\pm\infty}(b) + q^{-1}\mathcal{L}(\pm\infty) = b$ for all $b \in \mathcal{B}(\pm\infty)$. Set $b_{\pm\infty} := 1 + q^{-1}\mathcal{L}(\pm\infty) \in \mathcal{B}(\pm\infty)$.

2.2. **Crystal.** Let $\mathcal{B}$ be a crystal in the sense of [17, Section 1.2]. For each $\lambda \in X$, set

$$\mathcal{B}_\lambda := \{b \in \mathcal{B} \mid \mathrm{wt}(b) = \lambda\}.$$

We say that $b \in \mathcal{B}_\lambda$ is a highest weight element of weight $\lambda$ if $\tilde{E}_i b = 0$ for all $i \in I$. Let $\mathcal{B}^{\mathrm{hi}} \subset \mathcal{B}$ denote the set of highest weight elements.

For each $b \in \mathcal{B}$, let $C(b)$ denote the connected component of $\mathcal{B}$ containing $b$.

Given two crystals $\mathcal{B}_1, \mathcal{B}_2$, their tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ is the crystal with the following structure:

$$\begin{aligned}
\mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2),\\
\varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1) - \langle h_i, \mathrm{wt}(b_2)\rangle, \varepsilon_i(b_2)),\\
\varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \mathrm{wt}(b_1)\rangle),\\
\tilde{E}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{E}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2),\\ b_1 \otimes \tilde{E}_i b_2 & \text{if } \varepsilon_i(b_1) \leq \varphi_i(b_2),\end{cases}\\
\tilde{F}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{F}_i b_2 & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2),\\ \tilde{F}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2).\end{cases}
\end{aligned} \tag{2.2}$$

Note that this structure is different from that in [17] due to difference of convention.

Lemma 2.2.1 is easily deduced from the above.

**Lemma 2.2.1.** *Let $\mathcal{B}_1, \mathcal{B}_2$ be crystals such that*

$$\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{E}_i^k b \neq 0\},\ \varphi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{F}_i^k b \neq 0\}$$

*for all $i \in I$, $b \in \mathcal{B}_1, \mathcal{B}_2$. Let $(b_1, b_2) \in \mathcal{B}_1 \times \mathcal{B}_2$. Then, we have $b_1 \otimes b_2 \in (\mathcal{B}_1 \otimes \mathcal{B}_2)^{\mathrm{hi}}$ if and only if $b_2 \in \mathcal{B}_2^{\mathrm{hi}}$ and $\varepsilon_i(b_1) \leq \langle h_i, \mathrm{wt}(b_2) \rangle$ for all $i \in I$.*

2.3. **Irreducible module $V(\pm\lambda)$.** Let $X^+$ denote the set of dominant weights:

$$X^+ = \{\lambda \in X \mid \langle h_i, \lambda \rangle \geq 0 \ \text{ for all } i \in I\}.$$

For each $\lambda \in X^+$, let $V(\lambda)$ (resp., $V(-\lambda)$) denote the irreducible integrable highest (resp., lowest) weight $\mathbf{U}$-module of highest weight $\lambda$ (resp., lowest weight $-\lambda$). Let $(\mathcal{L}(\pm\lambda), \mathcal{B}(\pm\lambda))$ and $\mathbf{B}(\pm\lambda)$ denote the crystal base and global crystal basis of $V(\pm\lambda)$ [16, Theorems 2 and 6]. For $w \in W$, let $b_{\pm w\lambda} \in \mathcal{B}(\pm\lambda)$ and $v_{\pm w\lambda} \in \mathbf{B}(\pm\lambda)$ denote the unique elements of weight $\pm w\lambda$.

*Remark* 2.3.1. Suppose that the Dynkin diagram $I$ is of finite type. Let $w_0 \in W$ denote the longest element. Then, for each $\lambda \in X^+$ and $w \in W$, the symbol $v_{-w\lambda}$ represents both the vectors $v_{-w\lambda} \in V(-\lambda)$ and $v_{ww_0(-w_0\lambda)} \in V(-w_0\lambda)$. In order to make our notation consistent, we identify $V(-\lambda)$ with $V(-w_0\lambda)$ under the $\mathbf{U}$-module isomorphism $V(-\lambda) \to V(-w_0\lambda)$ which sends $v_{-\lambda}$ to $v_{w_0(-w_0\lambda)}$.

Let $(,)$ denote the inner product on $V(\lambda)$ such that $(v_\lambda, v_\lambda) = 1$ and

$$(xu, v) = (u, \rho(x)v)$$

for all $x \in \mathbf{U}$, $u, v \in V(\lambda)$, where $\rho$ is the anti-algebra automorphism on $\mathbf{U}$ given in (2.1). By [25, Lemma 19.1.4], we have

$$(G(b_1), G(b_2)) \in \delta_{b_1,b_2} + q^{-1}\mathbf{A}_\infty$$

for all $b_1, b_2 \in \mathcal{B}(\lambda)$.

For each $\lambda \in X^+$, let

$$\pi_{\pm\lambda} : \mathbf{U}^\mp \to V(\pm\lambda)$$

denote the $\mathbf{U}^\mp$-module homomorphism given by $\pi_{\pm\lambda}(x) = xv_{\pm\lambda}$. By [16, Theorem 5], it induces a map $\pi_{\pm\lambda} : \mathcal{B}(\pm\infty) \to \mathcal{B}(\pm\lambda) \sqcup \{0\}$, and the induced map restricts to a bijection

$$\mathcal{B}(\pm\infty; \pm\lambda) := \{b \in \mathcal{B}(\pm\infty) \mid \pi_{\pm\lambda}(b) \neq 0\} \to \mathcal{B}(\pm\lambda).$$

Let

$$\iota_{\pm\lambda} : \mathcal{B}(\pm\lambda) \to \mathcal{B}(\pm\infty; \pm\lambda) \tag{2.3}$$

denote its inverse. For each $b \in \mathcal{B}(\pm\lambda)$, we have the following:

$$\begin{aligned}
&G(b) = G_{\pm\infty}(\iota_{\pm\lambda}(b))v_{\pm\lambda}, \\
&\iota_{\pm\lambda}(b_{\pm\lambda}) = b_{\pm\infty}, \\
&\mathrm{wt}(\iota_{\pm\lambda}(b)) = \mathrm{wt}(b) \mp \lambda, \\
&\varepsilon_i(\iota_\lambda(b)) = \varepsilon_i(b), \ \varphi_i(\iota_{-\lambda}(b)) = \varphi_i(b), \\
&\tilde{F}_i(\iota_\lambda(b)) = \iota_\lambda(\tilde{F}_i b) \text{ if } \tilde{F}_i b \neq 0, \ \tilde{E}_i \iota_{-\lambda}(b) = \iota_{-\lambda}(\tilde{E}_i b) \text{ if } \tilde{E}_i b \neq 0
\end{aligned}$$

for all $i \in I$, $b \in \mathcal{B}(\pm\lambda)$.

2.4. **Based module.** Set $\mathcal{A} := \mathbb{Q}[q,q^{-1}]$. Let $\dot{\mathbf{U}} = \bigoplus_{\lambda,\mu \in X} 1_\lambda \dot{\mathbf{U}} 1_\mu$ denote the modified quantum group, and ${}_\mathcal{A}\dot{\mathbf{U}}$ its $\mathcal{A}$-form.

Following [5, Section 2.1], we define a based $\mathbf{U}$-module to be an integrable $\mathbf{U}$-module $M$ equipped with a linear basis $\mathbf{B}_M$ satisfying the following:

- $\mathbf{B}_M \cap M_\lambda$ is a basis of $M_\lambda$ for all $\lambda \in X$, where
$$M_\lambda := \{m \in M \mid K_h m = q^{\langle h,\lambda \rangle} m \text{ for all } h \in Y\}.$$
- ${}_\mathcal{A}M := \mathcal{A}\mathbf{B}_M$ is a ${}_\mathcal{A}\dot{\mathbf{U}}$-submodule. We call it the $\mathcal{A}$-form of $M$.
- The $\mathbb{Q}$-linear map $\bar{\cdot} : M \to M$ sending $q^n v$ to $q^{-n} v$ for all $n \in \mathbb{Z}$ and $v \in \mathbf{B}_M$ satisfies $\overline{xm} = \bar{x}\bar{m}$ for all $x \in \mathbf{U}$, $m \in M$. We call it the bar-involution on $M$.
- Setting $\mathcal{L}_M := \mathbf{A}_\infty \mathbf{B}_M$ and $\mathcal{B}_M := \{v + q^{-1}\mathcal{L}_M \mid v \in \mathbf{B}_M\}$, the pair $(\mathcal{L}_M, \mathcal{B}_M)$ forms a crystal base of $M$.

Let $(M, \mathbf{B}_M)$ be a based $\mathbf{U}$-module. Then, the quotient map $\mathrm{ev}_\infty : \mathcal{L}_M \to \mathcal{L}_M / q^{-1}\mathcal{L}_M$ restricts to a $\mathbb{Q}$-linear isomorphism
$$\mathcal{L}_M \cap {}_\mathcal{A}M \cap \overline{\mathcal{L}_M} \to \mathcal{L}_M / q^{-1}\mathcal{L}_M.$$
Let $G_M$ denote its inverse. We sometimes omit the subscript $M$ of $G_M$ for simplicity.

A based submodule of a based module $(M, \mathbf{B}_M)$ is a $\mathbf{U}$-submodule $N \subset M$ spanned by a subset $\mathbf{B}_N \subset \mathbf{B}_M$. Note that $(N, \mathbf{B}_N)$ is a based $\mathbf{U}$-module in its own right.

Let $(N, \mathbf{B}_N)$ be a based submodule of a based $\mathbf{U}$-module $(M, \mathbf{B}_M)$. Then,
$$(M/N, \{v + N \mid v \in \mathbf{B}_M \setminus \mathbf{B}_N\})$$
is a based $\mathbf{U}$-module.

Let $(M, \mathbf{B}_M), (N, \mathbf{B}_N)$ be based $\mathbf{U}$-modules, and $f : M \to N$ a $\mathbf{U}$-module homomorphism. We say that $f$ is a based $\mathbf{U}$-module homomorphism if $f(\mathbf{B}_M) \subset \mathbf{B}_N \sqcup \{0\}$ and $\mathrm{Ker}\, f$ is a based submodule. This definition is reformulated as follows.

**Lemma 2.4.1.** *Let $(M, \mathbf{B}_M), (N, \mathbf{B}_N)$ be based $\mathbf{U}$-modules, and $f : M \to N$ a $\mathbf{U}$-module homomorphism. Then, $f$ is a based $\mathbf{U}$-module homomorphism if and only if it satisfies the following:*

- *$f(\mathcal{L}_M) \subset \mathcal{L}_N$; it induces a map*
$$\phi : \mathcal{B}_M \to \mathcal{B}_N;\ b \mapsto \mathrm{ev}_\infty(f(G_M(b))).$$
- *$f({}_\mathcal{A}M) \subset {}_\mathcal{A}N$.*
- *$f \circ \psi_M = \psi_N \circ f$, where $\psi_M, \psi_N$ denote the bar-involution on $M, N$, respectively.*
- *$\phi$ is injective on $\{b \in \mathcal{B}_M \mid \phi(b) \neq 0\}$.*

*Proof.* The "only if" part is obvious. Let us prove the opposite direction. By the assumption on $f$, we see that
$$f(G_M(b)) \in \mathcal{L}_N \cap {}_\mathcal{A}N \cap \overline{\mathcal{L}_N}$$
for all $b \in \mathcal{B}_M$. Hence, we obtain
$$f(G_M(b)) = (G_N \circ \mathrm{ev}_\infty)(f(G_M(b))) = G_N(\phi(b)) \tag{2.4}$$

for all $b \in \mathcal{B}_M$. This implies that $f(\mathbf{B}_M) \subset \mathbf{B}_N \sqcup \{0\}$. In order to complete the proof, let us investigate $\operatorname{Ker} f$. Let $v \in \operatorname{Ker} f$, and write

$$v = \sum_{b \in \mathcal{B}_M} c_b G_M(b)$$

for some $c_b \in \mathbb{Q}(q)$. Then, by equation (2.4), we have

$$0 = f(v) = \sum_{\substack{b \in \mathcal{B}_M \\ \phi(b) \neq 0}} c_b G_N(b).$$

Since $\phi$ is injective on $\{b \in \mathcal{B}_M \mid \phi(b) \neq 0\}$, we must have $c_b = 0$ for all $b \in \mathcal{B}_M$ with $\phi(b) \neq 0$. Therefore, we obtain

$$v = \sum_{\substack{b \in \mathcal{B}_M \\ \phi(b) = 0}} c_b G_M(b).$$

Consequently,

$$\operatorname{Ker} f = \mathbb{Q}(q)\{G_M(b) \mid b \in \mathcal{B}_M \text{ and } \phi(b) = 0\}.$$

This implies that $f$ is based. Thus, the proof completes. □

**Example 2.4.2.** Let $\lambda$, $\mu$, $\lambda_1, \dots, \lambda_r \in X^+$.

(1) $(V(\pm\lambda), \mathbf{B}(\pm\lambda))$ is a based $\mathbf{U}$-module.

(2) Let $\mathbf{B}(\lambda_1, \dots, \lambda_r)$ denote the canonical basis of

$$V(\lambda_1, \dots, \lambda_r) := V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$$

constructed in [5, Theorem 2.9]. Then, $(V(\lambda_1, \dots, \lambda_r), \mathbf{B}(\lambda_1, \dots, \lambda_r))$ is a based $\mathbf{U}$-module. Its crystal basis $\mathcal{B}(\lambda_1, \dots, \lambda_r)$ is the tensor product $\mathcal{B}(\lambda_1) \otimes \cdots \otimes \mathcal{B}(\lambda_r)$.

(3) By [25, Proposition 25.1.2], for each $\lambda, \mu \in X^+$, there exists a unique based $\mathbf{U}$-module homomorphism

$$\chi_{\lambda,\mu} : V(\lambda + \mu) \to V(\lambda, \mu)$$

such that $\chi_{\lambda,\mu}(v_{\lambda+\mu}) = v_\lambda \otimes v_\mu$.

(4) By [25, Proposition 25.1.4], for each $\lambda \in X^+$, there exists a unique based $\mathbf{U}$-module homomorphism

$$\delta_\lambda : V(-\lambda) \otimes V(\lambda) \to \mathbb{Q}(q)$$

such that $\delta_\lambda(v_{-\lambda} \otimes v_\lambda) = 1$. Here, we identify $(\mathbb{Q}(q), \{1\})$ with the based $\mathbf{U}$-module $(V(0), \mathbf{B}(0))$.

Let $\mathcal{C}^+$ denote the category of based $\mathbf{U}$-modules and based homomorphisms consisting of $(M, \mathbf{B}_M)$ with finite-dimensional weight spaces satisfying the following: There exist finitely many $\lambda_1, \dots, \lambda_r \in X$ such that for each $\lambda \in X$, we have $M_\lambda \neq 0$ only if $\lambda \in \bigcup_{k=1}^r \{\lambda_k - \alpha \mid \alpha \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i\}$. In particular, $M$ is semisimple with simple components of the form $V(\lambda)$, $\lambda \in X^+$.

Let $(M, \mathbf{B}_M) \in \mathcal{C}^+$. For each $\lambda \in X^+$, let $I_\lambda(M)$ denote the sum of all submodules of $M$ isomorphic to $V(\lambda)$. Set

$$M[> \lambda] := \bigoplus_{\substack{\mu \in X^+ \\ \mu > \lambda}} I_\mu(M), \quad M[\geq \lambda] := M[> \lambda] \oplus I_\lambda(M),$$

$$\mathcal{B}_M[> \lambda] := \bigsqcup_{\substack{b' \in \mathcal{B}_M^{\mathrm{hi}} \\ \mathrm{wt}(b') > \lambda}} C(b'), \quad \mathcal{B}_M[\geq \lambda] := \bigsqcup_{\substack{b' \in \mathcal{B}_M^{\mathrm{hi}} \\ \mathrm{wt}(b') \geq \lambda}} C(b'),$$

where $<$ denotes the partial order on $X^+$ defined by saying $\lambda \leq \mu$ if and only if $\mu - \lambda \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. Recall that $C(b')$ denotes the connected component of $\mathcal{B}_M$ containing $b'$ (cf. Section 2.2). By the same way as the proofs of [25, Propositions 27.1.7 and 27.1.8], we see that the submodules $M[> \lambda]$ and $M[\geq \lambda]$ are based submodules with crystal basis $\mathcal{B}_M[> \lambda]$ and $\mathcal{B}_M[\geq \lambda]$, respectively, and that there exists a based $\mathbf{U}$-module isomorphism $M[\geq \lambda]/M[> \lambda] \to V(\lambda)^{|\mathcal{B}_{M,\lambda}^{\mathrm{hi}}|}$ sending $G_M(b)$ to $v_\lambda^b$ for all $b \in \mathcal{B}_{M,\lambda}^{\mathrm{hi}}$, where $v_\lambda^b$ denotes the highest weight vector of the $b$-th component.

For each $\lambda \in X^+$, let

$$p_\lambda : M = \bigoplus_{\mu \in X^+} I_\mu(M) \to I_\lambda(M)$$

denote the projection. For each $b \in \mathcal{B}_M^{\mathrm{hi}}$, set

$$v_b := p_{\mathrm{wt}(b)}(G_M(b)) \in I_{\mathrm{wt}(b)}(M).$$

By the above, we see that $v_b$ is a highest weight vector of weight $\mathrm{wt}(b)$ and that

$$v_b \in \mathcal{L}_M, \ \mathrm{ev}_\infty(v_b) = b.$$

For each $b \in \mathcal{B}_M^{\mathrm{hi}}$, set

$$M[\geq b] := M[> \mathrm{wt}(b)] \oplus \mathbf{U} v_b, \quad \mathcal{B}_M[\geq b] := \mathcal{B}_M[> \mathrm{wt}(b)] \sqcup C(b).$$

The connected component $C(b)$ is isomorphic to $\mathcal{B}(\mathrm{wt}(b))$. Let

$$\iota_b : C(b) \to \mathcal{B}(\mathrm{wt}(b)) \xrightarrow{\iota_{\mathrm{wt}(b)}} \mathcal{B}(\infty; \mathrm{wt}(b))$$

denote the composition of the isomorphism $C(b) \to \mathcal{B}(\mathrm{wt}(b))$ and $\iota_{\mathrm{wt}(b)}$ (cf. equation (2.3)).

**Lemma 2.4.3.** *Let $(M, \mathbf{B}_M) \in \mathcal{C}^+$, and $b \in \mathcal{B}_M^{\mathrm{hi}}$. Then, $M[\geq b]$ is a based submodule of $M$ with crystal basis $\mathcal{B}_M[\geq b]$. Moreover, there exists a based $\mathbf{U}$-module isomorphism*

$$M[\geq b]/M[> \mathrm{wt}(b)] \to V(\mathrm{wt}(b))$$

*which sends $G_M(b) + M[> \mathrm{wt}(b)]$ to $v_{\mathrm{wt}(b)}$.*

*Proof.* The assertion is obvious from the above. ☐

**Lemma 2.4.4.** *Let $(M, \mathbf{B}_M) \in \mathcal{C}^+$ and $b \in \mathcal{B}_M^{\mathrm{hi}}$. Then, we have $\overline{v_b} = v_b$.*

*Proof.* Set $\lambda := \mathrm{wt}(b)$ and $u_b := G_M(b) - v_b \in M[> \lambda]$. Since the bar-involution on $M$ preserves $I_\mu(M)$ for all $\mu \in X^+$, we see that $\overline{v_b} \in I_\lambda(M)$ and $\overline{u_b} \in M[> \lambda]$. Hence, we obtain

$$\overline{v_b} - v_b = u_b - \overline{u_b} \in I_\lambda(M) \cap M[> \lambda] = 0.$$

This proves the assertion. ☐

**Lemma 2.4.5.** *Let* $(M, \mathbf{B}_M), (N, \mathbf{B}_N) \in \mathcal{C}^+$*, and* $\phi : \mathcal{B}_M \to \mathcal{B}_N$ *be a strict crystal morphism. Then, there exists a unique* $\mathbf{U}$*-module homomorphism* $f : M \to N$ *such that* $f(v_b) = v_{\phi(b)}$ *for all* $b \in \mathcal{B}_M^{\mathrm{hi}}$*. Here, we set* $v_{\phi(b)} = 0$ *if* $\phi(b) = 0$*. Moreover, the following hold:*

- $f \circ \psi_M = \psi_N \circ f$*, where* $\psi_M, \psi_N$ *denote the bar-involutions on* $M, N$*, respectively.*
- $f(\mathcal{L}_M) \subset \mathcal{L}_N$*.*
- $\phi \circ \mathrm{ev}_\infty = \mathrm{ev}_\infty \circ f$ *on* $\mathcal{L}_M$*.*

*Proof.* The existence of $f$ follows from easy observation that $v_{\phi(b)}$ is either 0 or a highest weight vector of weight $\mathrm{wt}(b)$. The commutativity of $f$ and the bar-involutions follows from Lemma 2.4.4.

In order to show the remaining assertions, let $b \in \mathcal{B}_M^{\mathrm{hi}}$ and $b' \in C(b)$. Then, we can write $b' = \tilde{F}_{i_1} \cdots \tilde{F}_{i_r} b$ for some $i_1, \dots, i_r \in I$. Set $v := \tilde{F}_{i_1} \cdots \tilde{F}_{i_r} v_b \in \mathcal{L}_M$. Then, we have

$$\begin{aligned} \mathrm{ev}_\infty(v) &= \tilde{F}_{i_1} \cdots \tilde{F}_{i_r} b = b', \\ f(v) &= \tilde{F}_{i_1} \cdots \tilde{F}_{i_r} v_{\phi(b)} \in \mathcal{L}_N, \\ \mathrm{ev}_\infty(f(v)) &= \tilde{F}_{i_1} \cdots \tilde{F}_{i_r} \phi(b) = \phi(b'). \end{aligned}$$

These imply the remaining assertions. Thus, the proof completes. □

**Proposition 2.4.6.** *Let* $(M, \mathbf{B}_M), (N, \mathbf{B}_N) \in \mathcal{C}^+$*,* $\phi : \mathcal{B}_M \to \mathcal{B}_N$ *be a strict crystal morphism, and* $f : M \to N$ *the* $\mathbf{U}$*-module homomorphism in* Lemma 2.4.5*. Suppose that* $f(E_i G_M(b)) = E_i G_N(\phi(b))$ *for all* $i \in I$ *and* $b \in \mathcal{B}_M^{\mathrm{hi}}$*. Then, we have* $f(G_M(b)) = G_N(\phi(b))$ *for all* $b \in \mathcal{B}_M$*. Furthermore, if* $\phi(b_1) = \phi(b_2) \neq 0$ *implies* $b_1 = b_2$ *for all* $b_1, b_2 \in \mathcal{B}_M^{\mathrm{hi}}$*, then* $f$ *is based.*

*Proof.* Let $b \in \mathcal{B}_M^{\mathrm{hi}}$ and set $\lambda := \mathrm{wt}(b)$. Then, the number $D(b)$ of elements $b_1 \in \mathcal{B}_M^{\mathrm{hi}}$ such that $\mathrm{wt}(b_1) > \lambda$ is finite. We prove that $f(G_M(b')) = G_N(\phi(b'))$ for all $b' \in C(b)$ by induction on $D(b)$.

Assume that our claim is true for all $b_1 \in \mathcal{B}_M^{\mathrm{hi}}$ with $D(b_1) < D(b)$; note that we assume nothing when $D(b) = 0$. Then, we have

$$f(G_M(b'')) = G_N(\phi(b'')) \tag{2.5}$$

for all $b'' \in \mathcal{B}[> \lambda]$.

Set

$$u_b := G_M(b) - v_b \in M[> \lambda], \quad u_{\phi(b)} := G_N(\phi(b)) - v_{\phi(b)} \in N[> \lambda].$$

We shall show that $f(u_b) = u_{\phi(b)}$. By the assumption on $f$, we have

$$E_i(f(u_b) - u_{\phi(b)}) = f(E_i G_M(b)) - E_i G_N(\phi(b)) = 0$$

for all $i \in I$. This implies that $f(u_b) - u_{\phi(b)}$ is either 0 or a highest weight vector of weight $\lambda$. However, the submodule $N[> \lambda]$, to which $f(u_b) - u_{\phi(b)}$ belongs, has no highest weight vector of weight $\lambda$. Hence, the equality $f(u_b) = u_{\phi(b)}$ follows.

Now, we have

$$f(G_M(b)) = f(v_b + u_b) = v_{\phi(b)} + u_{\phi(b)} = G_N(\phi(b)). \tag{2.6}$$

Let $b' \in C(b)$. We shall show that $f(G_M(b')) = G_N(\phi(b'))$. By Lemma 2.4.3 (see also Subsection 2.3), we have

$$G_M(b') = G_\infty(\iota_b(b'))G_M(b) + \sum_{b'' \in \mathcal{B}_M[>\lambda]} c_{b''} G_M(b'')$$

for some $b'' \in \mathcal{A}$. By equations (2.5) and (2.6), we obtain

$$f(G_M(b')) = G_\infty(\iota_b(b'))G_N(\phi(b)) + \sum_{b'' \in \mathcal{B}_M[>\lambda]} c_{b''} G_N(\phi(b'')).$$

This implies that $f(G_M(b')) \in {}_{\mathcal{A}}N$. By Lemma 2.4.5, we see that

$$f(G_M(b')) \in \mathcal{L}_N,\ \mathrm{ev}_\infty(f(G_M(b'))) = \phi(b'),\ \overline{f(G_M(b'))} = f(G_M(b')).$$

These imply that $f(G_M(b')) \in \mathcal{L}_N \cap {}_{\mathcal{A}}N \cap \overline{\mathcal{L}_N}$, and hence,

$$f(G_M(b')) = (G_N \circ \mathrm{ev}_\infty)(f(G_M(b'))) = G_N(\phi(b')),$$

as desired.

The remaining assertion follows from Lemma 2.4.1. Thus, the proof completes. □

2.5. **Based submodule $V(w\lambda, \mu)$.** For each $\lambda, \mu \in X^+$ and $w \in W$, let $V(w\lambda, \mu) \subset V(\lambda) \otimes V(\mu)$ denote the $\mathbf{U}$-submodule generated by $v_{w\lambda} \otimes v_\mu$. By [8, Theorem 2.2], the $V(w\lambda, \mu)$ is a based submodule of $V(\lambda, \mu)$. Set

$$\mathcal{L}(w\lambda, \mu) := \mathcal{L}(\lambda, \mu) \cap V(w\lambda, \mu),$$
$$\mathcal{B}(w\lambda, \mu) := \{v + q^{-1}\mathcal{L}(w\lambda, \mu) \mid v \in \mathbf{B}(\lambda, \mu) \cap V(w\lambda, \mu)\}.$$

Then, $(\mathcal{L}(w\lambda, \mu), \mathcal{B}(w\lambda, \mu))$ forms a crystal base of $V(w\lambda, \mu)$.

**Lemma 2.5.1.** *Let $\lambda, \mu, \nu \in X^+$ and $w \in W$. Then, $V(\nu) \otimes V(w\lambda, \mu)$ is a based submodule of $V(\nu, \lambda, \mu)$ (see* Example 2.4.2(2) *for the based module structure of $V(\nu, \lambda, \mu)$).*

*Proof.* It suffices to show that $G(b_1 \otimes b_2 \otimes b_3) \in V(\nu) \otimes V(w\lambda, \mu)$ for all $b_1 \otimes b_2 \otimes b_3 \in \mathcal{B}(\nu) \otimes \mathcal{B}(w\lambda, \mu) \subset \mathcal{B}(\nu, \lambda, \mu)$. By the construction of $G(b_1 \otimes b_2 \otimes b_3)$ in [5, Theorem 2.9], we have

$$G(b_1 \otimes b_2 \otimes b_3) = \sum_{b_1' \otimes b_2' \otimes b_3' \in \mathcal{B}(\nu, \lambda, \mu)} c_{b_1', b_2', b_3'} G(b_1') \otimes G(b_2' \otimes b_3')$$

for some $c_{b_1', b_2', b_3'} \in \mathcal{A}$ such that $c_{b_1', b_2', b_3'} = 0$ unless $G(b_2' \otimes b_3')$ belongs to the smallest based $\mathbf{U}$-submodule of $V(\lambda, \mu)$ containing $G(b_2 \otimes b_3)$. Since $V(w\lambda, \mu)$ is based submodule of $V(\lambda, \mu)$ containing $G(b_2 \otimes b_3)$, we see that $c_{b_1', b_2', b_3'} = 0$ unless $G(b_2' \otimes b_3') \in V(w\lambda, \mu)$. This proves the assertion. □

## 3. ıQuantum group

In this section, after recalling the notion of ıquantum groups and based $\mathbf{U}^\imath$-modules, we prove the existence of certain based $\mathbf{U}$-module homomorphisms in Proposition 3.4.7. This reduces the problem of stability of ıcanonical bases to that of existence of based $\mathbf{U}^\imath$-module homomorphism $\delta_\nu^\imath : V(\nu + w_\bullet \tau \nu) \to \mathbb{Q}(q)$ sending the highest weight vector to 1 for each $\nu \in X^+$.

3.1. **Admissible pair.** An admissible pair [19, Definition 2.3] is a pair $(I_\bullet, \tau)$ consisting of a subset $I_\bullet \subset I$ of finite type and a Dynkin diagram automorphism $\tau$ on $I$ satisfying the following:

- $\tau^2 = \mathrm{id}$.
- $w_\bullet(\alpha_j) = -\alpha_{\tau(j)}$ for all $j \in I_\bullet$, where $w_\bullet$ denotes the longest element of the Weyl group $W_{I_\bullet}$ for $I_\bullet$.
- $\langle \rho_\bullet^\vee, \alpha_i \rangle \in \mathbb{Z}$ for all $i \in I_\circ := I \setminus I_\bullet$ with $\tau(i) = i$, where $\rho_\bullet^\vee$ denotes half the sum of positive coroots for $I_\bullet$.

In order to clarify which Dynkin diagram $I$ is considered, we sometimes denote an admissible pair by the triple $(I, I_\bullet, \tau)$.

**Definition 3.1.1.**

(1) An admissible pair $(I_\bullet, \tau)$ is said to be *irreducible* if for each $i, j \in I$, there exists a sequence $i = i_1, \ldots, i_r = j \in I$ such that for each $k = 1, \ldots, r-1$, we have either $a_{i_k, i_{k+1}} \neq 0$ or $i_{k+1} = \tau(i_k)$.
(2) An admissible pair $(I, I_\bullet, \tau)$ is said to be of *finite type* if the Dynkin diagram $I$ is of finite type.
(3) The number of $\tau$-orbits in $I_\circ$ is called the *real rank* of $(I_\bullet, \tau)$.

*Remark* 3.1.2. In the literature, the real rank is called "relative rank" or "restricted rank". Also, note that the real rank of an admissible pair $(I, I_\bullet, \tau)$ equals the dimension of

$$\{h \in \bigoplus_{i \in I} \mathbb{Q} h_i \mid w_\bullet \tau h = h\},$$

where $\tau$ acts as $\tau(h_i) = h_{\tau(i)}$.

*Remark* 3.1.3. The admissible pairs of finite type with $I_\bullet \neq I$ are identical to the Satake diagrams in [2] (cf. [19, after Definition 2.3]).

Later, we will restrict our attention to the irreducible admissible pairs of finite type of real rank 1. Here is the list of such admissible pairs; for the labels of Dynkin diagrams (except type $A_1 \times A_1$), we follow [9, Figs. 2.5 and 2.6]:

- Type AI.
  - $I = \{1\}$: type $A_1$.
  - $I_\bullet = \emptyset$.
  - $\tau = \mathrm{id}$.
- Type AII.
  - $I = \{1, 2, 3\}$: type $A_3$.
  - $I_\bullet = \{1, 3\}$.
  - $\tau = \mathrm{id}$.
- Type AIII.
  - $I = \{1, 2\}$: type $A_1 \times A_1$.
  - $I_\bullet = \emptyset$.
  - $\tau(1) = 2,\ \tau(2) = 1$.
- Type AIV.
  - $I = \{1, \ldots, n\}$: type $A_n$ with $n \geq 2$.
  - $I_\bullet = \{2, \ldots, n-1\}$.
  - $\tau(i) = n - i + 1$.
- Type BII.
  - $I = \{1, \ldots, n\}$: type $B_n$ with $n \geq 2$.

  - $I_\bullet = \{2, \dots, n\}$.
  - $\tau = \mathrm{id}$.
- Type CII.
  - $I = \{1, \dots, n\}$: type $C_n$ with $n \geq 3$.
  - $I_\bullet = \{1, 3, \dots, n\}$.
  - $\tau = \mathrm{id}$.
- Type DII.
  - $I = \{1, \dots, n\}$: type $D_n$ with $n \geq 4$.
  - $I_\bullet = \{2, \dots, n\}$.
  - $\tau(i) = \begin{cases} n & \text{if } i = n-1 \text{ and } n \in 2\mathbb{Z}, \\ n-1 & \text{if } i = n \text{ and } n \in 2\mathbb{Z}, \\ i & \text{otherwise.} \end{cases}$
- Type FII.
  - $I = \{1, 2, 3, 4\}$: type $F_4$.
  - $I_\bullet = \{1, 2, 3\}$.
  - $\tau = \mathrm{id}$.

3.2. **ıQuantum group.** Assume that $\tau$ induces involutive automorphisms on $Y$ and $X$ such that

$$\tau h_i = h_{\tau(i)},\ \tau\alpha_i = \alpha_{\tau(i)},\ \langle \tau h, \tau\lambda \rangle = \langle h, \lambda \rangle$$

for all $i \in I$, $h \in Y$, $\lambda \in X$. Note that this assumption is always satisfied when $I$ is of finite type and $Y = \sum_{i \in I} \mathbb{Z} h_i$.

Set

$$Y^\imath := \{h \in Y \mid h + w_\bullet \tau h = 0\}, \quad X^\imath := X / \{\lambda + w_\bullet \tau\lambda \mid \lambda \in X\}.$$

For each $\lambda \in X$, let $\bar{\lambda} \in X^\imath$ denote the image of $\lambda$. The perfect pairing $\langle , \rangle : Y \times X \to \mathbb{Z}$ induces a bilinear map $\langle , \rangle : Y^\imath \times X^\imath \to \mathbb{Z}$ such that

$$\langle h, \bar{\lambda} \rangle = \langle h, \lambda \rangle$$

for all $h \in Y^\imath$, $\lambda \in X$.

For each $i \in I_\circ$, chose $\varsigma_i \in \mathbb{Z}[q, q^{-1}]^\times$ and $\kappa_i \in \mathbb{Z}[q, q^{-1}]$ in a way such that

- $\varsigma_i = \varsigma_{\tau(i)}$ if $\langle h_i, w_\bullet \alpha_{\tau(i)} \rangle = 0$.
- $\kappa_i = 0$ unless $\tau(i) = i$, $\langle h_i, \alpha_j \rangle = 0$ for all $j \in I_\bullet$, and $\langle h_k, \alpha_i \rangle \in 2\mathbb{Z}$ for all $k \in I_\circ$ with $\tau(k) = k$ and $\langle h_k, \alpha_j \rangle = 0$ for all $j \in I_\bullet$.
- $\varsigma_{\tau(i)} = (-1)^{\langle 2\rho_\bullet^\vee, \alpha_i \rangle} q_i^{-\langle h_i, 2\rho_\bullet + w_\bullet \alpha_{\tau(i)} \rangle} \overline{\varsigma_i}$, where $\rho_\bullet$ denotes half the sum of positive roots for $I_\bullet$.
- $\overline{\kappa_i} = \kappa_i$.

The first two conditions are needed for the ıquantum group below to be of a reasonable size (cf. [19, Theorem 10.8], see also [3, Remark 3.3]). The third one guarantees the existence of the bar-involution on $\mathbf{U}^\imath$ (cf. [20, Corollary 4.2]). The fourth one was used to construct the quasi-$K$-matrix in [4, Theorem 6.10]. The constraint $\varsigma_i, \kappa_i \in \mathbb{Z}[q, q^{-1}]$ is necessary for the theory of ıcanonical bases in [8].

**Definition 3.2.1.** The ıquantum group is the subalgebra $\mathbf{U}^\imath \subset \mathbf{U}$ generated by

$$\{E_j, B_i, K_h \mid j \in I_\bullet,\ i \in I,\ h \in Y^\imath\},$$

where

$$B_i := \begin{cases} F_i & \text{if } i \in I_\bullet, \\ F_i + \varsigma_i T_{w_\bullet}(E_{\tau(i)})K_i^{-1} + \kappa_i K_i^{-1} & \text{if } i \in I_\circ. \end{cases}$$

By [20, Corollary 4.2], there exists a unique $\mathbb{Q}$-algebra automorphism $\bar{\cdot}$ on $\mathbf{U}^\imath$ such that

$$\overline{E_j} = E_j,\ \overline{B_i} = B_i,\ \overline{K_h} = K_{-h},\ \bar{q} = q^{-1}$$

for all $j \in I_\bullet$, $i \in I$, $h \in Y^\imath$. We call it the $\imath$bar-involution on $\mathbf{U}^\imath$.

Let $\mathbf{U}_{I_\bullet} \subset \mathbf{U}^\imath$ denote the subalgebra generated by $\{E_j, F_j, K_j^{\pm 1} \mid j \in I_\bullet\}$. It is the quantum group associated with $I_\bullet$. Similarly, let $\mathbf{U}^+_{I_\bullet}$, $\mathcal{B}_{I_\bullet}(-\infty)$, $\mathcal{B}_{I_\bullet}(-\infty; -\lambda)$, $\mathcal{B}_{I_\bullet}(-\lambda)$, and so on denote the same things without subscripts, but associated with $I_\bullet$.

3.3. **Based module.** Let $\dot{\mathbf{U}}^\imath = \bigoplus_{\zeta,\eta \in X^\imath} 1_\zeta \dot{\mathbf{U}}^\imath 1_\eta$ denote the modified $\imath$quantum group (cf. [8, Section 3.5], [27, Section 3.3]), and ${}_\mathcal{A}\dot{\mathbf{U}}^\imath$ its $\mathcal{A}$-form.

Following [8, Definition 6.11], we define a based $\mathbf{U}^\imath$-module to be a pair $(M, \mathbf{B}^\imath_M)$ consisting of a weight $\mathbf{U}^\imath$-module $M$ and its linear basis $\mathbf{B}^\imath_M$ satisfying the following:

- $\mathbf{B}^\imath_M \cap M_\zeta$ forms a basis of $M_\zeta$, where
$$M_\zeta := \{m \in M \mid K_h m = q^{\langle h, \zeta\rangle} \text{ for all } h \in Y^\imath\}.$$
- ${}_\mathcal{A}M := \mathcal{A}\mathbf{B}^\imath_M$ is a ${}_\mathcal{A}\dot{\mathbf{U}}^\imath$-submodule. We call it the $\mathcal{A}$-form of $M$.
- The $\mathbb{Q}$-linear map $\bar{\cdot} : M \to M$ sending $q^n v$ to $q^{-n} v$ for all $n \in \mathbb{Z}$ and $v \in \mathbf{B}^\imath_M$ satisfies $\overline{xm} = \bar{x}\bar{m}$ for all $x \in \mathbf{U}^\imath$ and $m \in M$. We call it the $\imath$bar-involution on $M$.
- Setting $\mathcal{L}_M := \mathbf{A}_\infty \mathbf{B}^\imath_M$, the quotient map $\mathrm{ev}_\infty : \mathcal{L}_M \to \mathcal{L}_M / q^{-1}\mathcal{L}_M$ restricts to a $\mathbb{Q}$-linear isomorphism $\mathcal{L}_M \cap {}_\mathcal{A}M \cap \overline{\mathcal{L}_M} \to \mathcal{L}_M/q^{-1}\mathcal{L}_M$; let $G^\imath_M$ denote its inverse.

The notions of based $\mathbf{U}^\imath$-submodules and based $\mathbf{U}^\imath$-module homomorphisms are defined in the same way as based $\mathbf{U}$-modules. The following can be proved by the same way as Lemma 2.4.1.

**Lemma 3.3.1.** *Let $(M, \mathbf{B}_M), (N, \mathbf{B}_N)$ be based $\mathbf{U}^\imath$-modules, and $f : M \to N$ a $\mathbf{U}^\imath$-module homomorphism. Then, $f$ is a based $\mathbf{U}^\imath$-module homomorphism if and only if it satisfies the following:*

- *$f(\mathcal{L}_M) \subset \mathcal{L}_N$; it induces a map*
$$\phi : \mathcal{B}_M \to \mathcal{B}_N;\ b \mapsto \mathrm{ev}_\infty(f(G_M(b))).$$
- *$f({}_\mathcal{A}M) \subset {}_\mathcal{A}N$.*
- *$f \circ \psi^\imath_M = \psi^\imath_N \circ f$, where $\psi^\imath_M, \psi^\imath_N$ denote the $\imath$bar-involution on $M, N$, respectively.*
- *$\phi$ is injective on $\{b \in \mathcal{B}_M \mid \phi(b) \neq 0\}$.*

**Example 3.3.2.** Let $\lambda$, $\mu$, $\lambda_1, \dots, \lambda_r \in X^+$ and $w \in W$.

(1) Let $\mathbf{B}^\imath(\lambda_1, \dots, \lambda_r)$ denote the $\imath$canonical basis of the based $\mathbf{U}$-module $V(\lambda_1, \dots, \lambda_r)$ in the sense of [8, Theorem 6.12]. Then, $(V(\lambda_1, \dots, \lambda_r), \mathbf{B}^\imath(\lambda_1, \dots, \lambda_r))$ forms a based $\mathbf{U}^\imath$-module.

(2) By [8, Theorem 6.13 (2)], $V(w\lambda, \mu)$ is a based $\mathbf{U}^\imath$-submodule of $V(\lambda, \mu)$; set $\mathbf{B}^\imath(w\lambda, \mu) := \mathbf{B}^\imath(\lambda, \mu) \cap V(w\lambda, \mu)$.

(3) Let $\dot{\mathbf{B}}^\imath$ denote the ıcanonical basis of $\dot{\mathbf{U}}^\imath$ in the sense of [8, Theorem 7.2]. Then, $(\dot{\mathbf{U}}^\imath, \dot{\mathbf{B}}^\imath)$ is a based $\mathbf{U}^\imath$-module.

**Lemma 3.3.3.** *Let $\lambda, \mu, \nu \in X^+$ and $w \in W$. Then, $V(\nu) \otimes V(w\lambda, \mu)$ is a based $\mathbf{U}^\imath$-submodule of $V(\nu, \lambda, \mu)$.*

*Proof.* It suffices to show that $G^\imath(b_1 \otimes b_2 \otimes b_3) \in V(\nu) \otimes V(w\lambda, \mu)$ for all $b_1 \otimes b_2 \otimes b_3 \in \mathcal{B}(\nu) \otimes \mathcal{B}(w\lambda, \mu) \subset \mathcal{B}(\nu, \lambda, \mu)$. By the construction of $G^\imath(b_1 \otimes b_2 \otimes b_3)$ in [8, Theorem 6.12], we have

$$G^\imath(b_1 \otimes b_2 \otimes b_3) = \sum_{b_1' \otimes b_2' \otimes b_3' \in \mathcal{B}(\nu, \lambda, \mu)} c_{b_1', b_2', b_3'} G(b_1' \otimes b_2' \otimes b_3')$$

for some $c_{b_1', b_2', b_3'} \in \mathcal{A}$ such that $c_{b_1', b_2', b_3'} = 0$ unless $G(b_1' \otimes b_2' \otimes b_3')$ belongs to the smallest based $\mathbf{U}$-submodule of $V(\nu, \lambda, \mu)$ containing $G(b_1 \otimes b_2 \otimes b_3)$. Since $V(\nu) \otimes V(w\lambda, \mu)$ is a based $\mathbf{U}$-submodule containing $G(b_1 \otimes b_2 \otimes b_3)$, we see that $c_{b_1', b_2', b_3'} = 0$ unless $G(b_1' \otimes b_2' \otimes b_3') \in V(\nu) \otimes V(w\lambda, \mu)$. This proves the assertion. □

3.4. **Based submodule $V(w_\bullet\lambda, \mu)$.** Let $\lambda, \mu \in X^+$. Let $C_{I_\bullet}(b_\lambda) \subset \mathcal{B}(\lambda)$ denote the connected component of $\mathcal{B}(\lambda)$ containing $b_\lambda$ as the crystal of type $I_\bullet$. Namely,

$$C_{I_\bullet}(b_\lambda) = \{\tilde{F}_{j_1} \cdots \tilde{F}_{j_r} b_\lambda \mid j_1, \dots, j_r \in I_\bullet\} \setminus \{0\}.$$

Note that it is isomorphic to $\mathcal{B}_{I_\bullet}(w_\bullet\lambda)$ as a crystal of type $I_\bullet$; we regard $-w_\bullet\lambda$ as a dominant weight for $I_\bullet$.

**Lemma 3.4.1.** *We have*

$$C_{I_\bullet}(b_\lambda) = \{b \in \mathcal{B}(\lambda) \mid \mathrm{wt}(b) \geq w_\bullet\lambda\}.$$

*Proof.* Since $C_{I_\bullet}(b_\lambda) \simeq \mathcal{B}_{I_\bullet}(w_\bullet\lambda)$, we see that $\mathrm{wt}(b) \geq w_\bullet\lambda$ for all $b \in C_{I_\bullet}(b_\lambda)$. This proves the containment "$\subset$".

We shall prove the opposite direction. Let $b \in \mathcal{B}(\lambda)$ be such that $\mathrm{wt}(b) \geq w_\bullet\lambda$. Since $\mathrm{wt}(b) \leq \lambda$, we can write

$$\lambda - \mathrm{wt}(b) = \sum_{j \in I_\bullet} m_j \alpha_j \tag{3.1}$$

for some $m_j \geq 0$. On the other hand, we have

$$b = \tilde{F}_{i_1} \cdots \tilde{F}_{i_r} b_\lambda$$

for some $i_1, \dots, i_r \in I$. Taking equation (3.1) into account, we see that $i_1, \dots, i_r \in I_\bullet$. This implies that $b \in C_{I_\bullet}(b_\lambda)$. Thus, the proof completes. □

By Subsection 2.3, there exist maps

$$\pi = \pi_{I_\bullet; w_\bullet\lambda} : \mathcal{B}_{I_\bullet}(-\infty) \to C_{I_\bullet}(b_\lambda) \sqcup \{0\}, \ \iota = \iota_{I_\bullet; w_\bullet\lambda} : C_{I_\bullet}(b_\lambda) \hookrightarrow \mathcal{B}_{I_\bullet}(-\infty; w_\bullet\lambda)$$

such that

$$\begin{aligned} &\pi \circ \iota = \mathrm{id}, \\ &\iota(b_{w_\bullet\lambda}) = b_{-\infty}, \\ &\mathrm{wt}(\iota(b)) = \mathrm{wt}(b) - w_\bullet\lambda, \\ &\varphi_j(\iota(b)) = \varphi_j(b), \\ &\tilde{E}_j \iota(b) = \iota(\tilde{E}_j b) \text{ if } \tilde{E}_j b \neq 0 \end{aligned} \tag{3.2}$$

for all $j \in I_\bullet$ and $b \in C_{I_\bullet}(b_\lambda)$.

**Lemma 3.4.2.** *Let $b \in \mathcal{B}_{I_\bullet}(-\infty)$ and set $\pi := \pi_{I_\bullet;w_\bullet\lambda}$. Then, we have*
$$G(b)(v_{w_\bullet\lambda} \otimes v_\mu) = G(\pi(b)) \otimes v_\mu = G(\pi(b) \otimes b_\mu).$$
*In particular, $G(b' \otimes b_\mu) = G(b') \otimes v_\mu \in V(w_\bullet\lambda, \mu)$ for all $b' \in C_{I_\bullet}(b_\lambda)$.*

*Proof.* We have
$$G(b)(v_{w_\bullet\lambda} \otimes v_\mu) = G(b)v_{w_\bullet\lambda} \otimes v_\mu = G(\pi(b)) \otimes v_\mu.$$
The left-hand side belongs to ${}_{\mathcal{A}}V(w_\bullet\lambda, \mu)$, and is bar-invariant. On the other hand, the right-hand side belongs to $\mathcal{L}(\lambda, \mu)$, and its image under $\mathrm{ev}_\infty$ is $\pi(b) \otimes b_\mu$. Therefore, we have $G(\pi(b)) \otimes v_\mu \in \mathcal{L}(w_\bullet\lambda, \mu) \cap {}_{\mathcal{A}}V(w_\bullet\lambda, \mu) \cap \overline{\mathcal{L}(w_\bullet\lambda, \mu)}$, and consequently,
$$G(\pi(b)) \otimes v_\mu = (G \circ \mathrm{ev}_\infty)(G(\pi(b)) \otimes v_\mu) = G(\pi(b) \otimes b_\mu),$$
as desired. □

**Proposition 3.4.3.** *Let $\lambda, \mu \in X^+$ and $b \in \mathcal{B}(w_\bullet\lambda, \mu)$. Then, $b$ is a highest weight element if and only if $b = b_1 \otimes b_\mu$ for some $b_1 \in C_{I_\bullet}(b_\lambda)$ such that $\varepsilon_i(b_1) \leq \langle h_i, \mu\rangle$ for all $i \in I$.*

*Proof.* Let $b \in \mathcal{B}(w_\bullet\lambda, \mu)^{\mathrm{hi}}$. Since the $\mathbf{U}$-module $V(w_\bullet\lambda, \mu)$ is generated by a global crystal basis element $v_{w_\bullet\lambda} \otimes v_\mu$ of weight $w_\bullet\lambda + \mu$, we have $I_{\lambda'}(V(w_\bullet\lambda, \mu)) = 0$ unless $\lambda' \geq w_\bullet\lambda + \mu$. Hence, we obtain $\mathrm{wt}(b) \geq w_\bullet\lambda + \mu$. Now, the "only if" part follows from Lemmas 2.2.1 and 3.4.1.

Let us prove the opposite direction. Let $b_1 \in C_{I_\bullet}(b_\lambda)$ be such that $\varepsilon_i(b) \leq \langle h_i, \mu\rangle$ for all $i \in I$. By Lemma 2.2.1, we have $b := b_1 \otimes b_\mu \in \mathcal{B}(\lambda, \mu)^{\mathrm{hi}}$. By Lemma 3.4.2, we see that $G(b) \in V(w_\bullet\lambda, \mu)$. Hence, we obtain
$$b = \mathrm{ev}_\infty(G(b)) \in \mathcal{B}(w_\bullet\lambda, \mu).$$
This completes the proof. □

For each $\lambda, \mu \in X^+$, set
$$C_{I_\bullet;\mu}(b_\lambda) := \{b \in C_{I_\bullet}(b_\lambda) \mid \varepsilon_i(b) \leq \langle h_i, \mu\rangle \ \text{ for all } i \in I\}.$$
Then, by Proposition 3.4.3, we obtain
$$\mathcal{B}(w_\bullet\lambda, \mu)^{\mathrm{hi}} = \{b \otimes b_\mu \mid b \in C_{I_\bullet;\mu}(b_\lambda)\}. \tag{3.3}$$

For each $\nu \in X^+$, consider the compositions
$$\begin{aligned}\pi = \pi_{I_\bullet;w_\bullet\lambda,\nu} : C_{I_\bullet}(b_{\lambda+\tau\nu}) &\xrightarrow{\iota_{I_\bullet;w_\bullet(\lambda+\tau\nu)}} \mathcal{B}_{I_\bullet}(-\infty; w_\bullet(\lambda+\tau\nu)) \\ &\xrightarrow{\pi_{I_\bullet;w_\bullet\lambda}} C_{I_\bullet}(b_\lambda) \sqcup \{0\}, \\ \iota = \iota_{I_\bullet;w_\bullet\lambda,\nu} : C_{I_\bullet}(b_\lambda) &\xrightarrow{\iota_{I_\bullet;w_\bullet\lambda}} \mathcal{B}_{I_\bullet}(-\infty; w_\bullet\lambda) \xrightarrow{\pi_{I_\bullet;w_\bullet(\lambda+\tau\nu)}} C_{I_\bullet}(b_{\lambda+\tau\nu}).\end{aligned}$$
They satisfy the following (cf. equation (3.2)):
$$\begin{aligned} &\pi \circ \iota = \mathrm{id}, \\ &\iota(b_{w_\bullet\lambda}) = b_{w_\bullet(\lambda+\tau\nu)}, \\ &\varphi_j(\iota(b)) = \varphi_j(b), \\ &\mathrm{wt}(\iota(b)) = \mathrm{wt}(b) + w_\bullet\tau\nu, \\ &\tilde{E}_j\iota(b) = \iota(\tilde{E}_j b) \text{ if } \tilde{E}_j b \neq 0 \end{aligned} \tag{3.4}$$
for all $j \in I_\bullet$ and $b \in C_{I_\bullet}(b_\lambda)$.

**Lemma 3.4.4.** *Let $\lambda, \mu, \nu \in X^+$ and $b \in C_{I_\bullet}(b_\lambda)$. Set $\iota := \iota_{I_\bullet;w_\bullet\lambda,\nu}$. Then, we have $\iota(b) \in C_{I_\bullet;\mu+\nu}(b_{\lambda+\tau\nu})$ if and only if $b \in C_{I_\bullet;\mu}(b_\lambda)$.*

*Proof.* By equation (3.4), we have

$$\varepsilon_j(\iota(b)) = \varphi_j(\iota(b)) - \langle h_j, \mathrm{wt}(\iota(b))\rangle = \varphi_j(b) - \langle h_j, \mathrm{wt}(b) + w_\bullet\tau\nu\rangle = \varepsilon_j(b) + \langle h_j, \nu\rangle$$

for all $j \in I_\bullet$, and

$$\varepsilon_i(\iota(b)) = 0 = \varepsilon_i(b)$$

for all $i \in I_\circ$. Now, the assertion follows from the definition of $C_{I_\bullet;\mu}(b_\lambda)$. □

**Lemma 3.4.5.** *Let $\nu \in X^+$. Then, we have $\nu + w_\bullet\tau\nu \in X^+$.*

*Proof.* For each $i \in I$, we have

$$\langle h_i, \nu + w_\bullet\tau\nu\rangle = \langle h_i, \nu\rangle + \langle w_\bullet h_{\tau(i)}, \nu\rangle.$$

Now, the assertion follows from easy observation that $w_\bullet h_{\tau(i)} = -h_i$ if $i \in I_\bullet$ and that $w_\bullet h_{\tau(i)}$ is a positive coroot if $i \in I_\circ$. □

By Lemma 3.4.5, we can apply Lemma 2.5.1 to see that $V(\nu+w_\bullet\tau\nu)\otimes V(w_\bullet\lambda,\mu)$ is a based **U**-submodule of $V(\nu + w_\bullet\tau\nu, \lambda, \mu)$.

**Lemma 3.4.6.** *Let $b \in \mathcal{B}_{I_\bullet}(-\infty)$, and set $\pi := \pi_{I_\bullet;w_\bullet\lambda}$. Then, we have*

$$G(b)(v_{\nu+w_\bullet\tau\nu} \otimes v_{w_\bullet\lambda} \otimes v_\mu) = v_{\nu+w_\bullet\tau\nu} \otimes G(\pi(b)) \otimes v_\mu = G(b_{\nu+w_\bullet\tau\nu} \otimes \pi(b) \otimes b_\mu).$$

*In particular, $G(b_{\nu+w_\bullet\tau\nu} \otimes b' \otimes b_\mu) = v_{\nu+w_\bullet\tau\nu} \otimes G(b') \otimes v_\nu$ for all $b' \in C_{I_\bullet}(b_\lambda)$.*

*Proof.* Noting that

$$E_j v_{\nu+w_\bullet\tau\nu} = 0, \quad \langle h_j, \nu + w_\bullet\tau\nu\rangle = 0$$

for all $j \in I_\bullet$, we see that the same argument as the proof of Lemma 3.4.2 proves the assertion. □

**Proposition 3.4.7.** *Let $\lambda, \mu, \nu \in X^+$ and set $\iota := \iota_{I_\bullet;w_\bullet\lambda,\nu}$ and $\pi := \pi_{I_\bullet;w_\bullet\lambda,\nu}$. Then, there exists a unique based* **U***-module homomorphism*

$$f : V(w_\bullet(\lambda + \tau\nu), \mu + \nu) \to V(\nu + w_\bullet\tau\nu) \otimes V(w_\bullet\lambda, \mu)$$

*such that*

$$f(G(b \otimes b_{\mu+\nu})) = G(b_{\nu+w_\bullet\tau\nu} \otimes \pi(b) \otimes b_\mu)$$

*for all $b \in C_{I_\bullet;\mu+\nu}(b_{\lambda+\tau\nu})$.*

*Proof.* First, we observe that there exists a strict crystal morphism

$$\phi : \mathcal{B}(w_\bullet(\lambda + \tau\nu), \mu + \nu) \to \mathcal{B}(\nu + w_\bullet\tau\nu) \otimes \mathcal{B}(w_\bullet\lambda, \mu)$$

such that

$$\phi(b \otimes b_{\mu+\nu}) = b_{\nu+w_\bullet\tau\nu} \otimes \pi(b) \otimes b_\mu$$

for all $b \in C_{I_\bullet;\mu+\nu}(b_{\lambda+\tau\nu})$. This follows from equation (3.3), and that $b \otimes b_{\mu+\nu}$ is a highest weight element of weight $\mathrm{wt}(b) + \mu + \nu$ and $b_{\nu+w_\bullet\tau\nu} \otimes \pi(b) \otimes b_\mu$ is either 0 or a highest weight element of weight $\mathrm{wt}(b) + \mu + \nu$ (see Lemma 3.4.4) for all $b \in C_{I_\bullet;\mu+\nu}(b_\lambda)$.

Let $f : V(w_\bullet(\lambda+\tau\nu), \mu+\nu) \to V(\nu+w_\bullet\tau\nu)\otimes V(w_\bullet\lambda, \mu)$ denote the **U**-module homomorphism in Lemma 2.4.5. By Proposition 2.4.6, in order to prove the assertion, it suffices to show that

$$f(E_i G(b)) = E_i G(\phi(b))$$

for all $i \in I$ and $b \in \mathcal{B}(w_\bullet(\lambda+\tau\nu), \mu+\nu)^{\mathrm{hi}}$ by induction on $D(b)$ (see the proof of Proposition 2.4.6 for the definition of $D(b)$). By weight consideration, we see that $E_i G(b) = 0 = E_i G(\phi(b))$ if $i \in I_\circ$. Hence, we only need to consider the case when $i \in I_\bullet$.

Assume that our claim is true for all $b'$ with $D(b') < D(b)$; note that we assume nothing when $D(b) = 0$. Set $\zeta := \mathrm{wt}(b)$. Then, by Proposition 2.4.6, $f$ restricts to a based $\mathbf{U}$-module homomorphism

$$V(w_\bullet(\lambda+\tau\nu), \mu+\nu)[> \zeta] \to V(\nu + w_\bullet\tau\nu) \otimes V(w_\bullet\lambda, \mu).$$

Let $b_1 \in C_{I_\bullet;\mu+\nu}(b_{\lambda+\tau\nu})$ be such that $b = b_1 \otimes b_{\mu+\nu}$. Set $\tilde{b}_1 := \iota_{I_\bullet;w_\bullet(\lambda+\tau\nu)}(b_1)$. Let us write

$$E_i G(\tilde{b}_1) = \sum_{b' \in \mathcal{B}_{I_\bullet}(-\infty)} c_{b'} G(b')$$

for some $c_{b'} \in \mathcal{A}$. Then, using Lemma 3.4.2, we compute as

$$\begin{aligned} E_i G(b) &= E_i G(b_1 \otimes b_{\mu+\nu}) \\ &= E_i G(\tilde{b}_1)(v_{w_\bullet(\lambda+\tau\nu)} \otimes v_{\mu+\nu}) \\ &= \sum_{b'} c_{b'} G(b')(v_{w_\bullet(\lambda+\tau\nu)} \otimes v_{\mu+\nu}) \\ &= \sum_{b'} c_{b'} G(\pi'(b') \otimes b_{\mu+\nu}), \end{aligned}$$

where $\pi' := \pi_{I_\bullet;w_\bullet(\lambda+\tau\nu)}$. Since $E_i G(b) \in V(w_\bullet(\lambda+\tau\nu), \mu+\nu)[> \zeta]$, we have $c_{b'} = 0$ unless $\pi'(b') \otimes b_{\mu+\nu} \in \mathcal{B}(w_\bullet(\lambda+\tau\nu), \mu+\nu)[> \zeta]$. By our induction hypothesis, this implies that

$$f(E_i G(b)) = \sum_{b'} c_{b'} G(\phi(\pi'(b') \otimes b_{\mu+\nu})).$$

On the other hand, using Lemma 3.4.6, we compute as

$$\begin{aligned} E_i G(\phi(b)) &= E_i G(b_{\nu+w_\bullet\tau\nu} \otimes \pi(b_1) \otimes b_\mu) \\ &= E_i G(\tilde{b}_1)(v_{\nu+w_\bullet\tau\nu} \otimes v_{w_\bullet\lambda} \otimes v_\mu) \\ &= \sum_{b'} c_{b'} G(b')(v_{\nu+w_\bullet\tau\nu} \otimes v_{w_\bullet\lambda} \otimes v_\mu) \\ &= \sum_{b'} c_{b'} G(b_{\nu+w_\bullet\tau\nu} \otimes \pi''(b') \otimes b_\mu), \end{aligned}$$

where $\pi'' := \pi_{I_\bullet;w_\bullet\lambda}$. Hence, in order to complete the proof, it suffices to show that $\phi(\pi'(b') \otimes b_{\mu+\nu}) = b_{\nu+w_\bullet\tau\nu} \otimes \pi''(b') \otimes b_\mu$ for all $b'$ with $c_{b'} \neq 0$.

Let $b' \in \mathcal{B}_{I_\bullet}(-\infty)$ be such that $c_{b'} \neq 0$. Suppose first that $\pi'(b') = 0$. Then, we have $\pi''(b') = \pi(\pi'(b')) = 0$, and hence,

$$\phi(\pi'(b') \otimes b_{\mu+\nu}) = 0 = b_{\nu+w_\bullet\tau\nu} \otimes \pi''(b') \otimes b_\mu,$$

as desired. Next, suppose that $\pi'(b') \neq 0$. Then, there exists $b_2 \in C_{I_\bullet;\mu+\nu}(b_{\lambda+\tau\nu})$ such that $\pi'(b') \otimes b_\mu \in C(b_2 \otimes b_{\mu+\nu})$. This implies that there exist $j_1, \dots, j_r \in I_\bullet$ such that $\tilde{F}_{j_r} \cdots \tilde{F}_{j_1}(b_2 \otimes b_{\mu+\nu}) = \pi'(b') \otimes b_{\mu+\nu}$. Moreover, we must have

$$\varepsilon_{j_{k+1}}(b_2^k) \geq \langle h_{j_{k+1}}, \mu+\nu \rangle, \ \varphi_{j_{k+1}}(b_2^k) > 0 \tag{3.5}$$

for all $k = 0, \dots, r-1$, where

$$b_2^0 := b_2, \ b_2^{k+1} := \tilde{F}_{j_{k+1}} b_2^k.$$

Then, we compute as

$$\begin{aligned}\phi(\pi'(b') \otimes b_{\mu+\nu}) &= \tilde{F}_{j_r} \cdots \tilde{F}_{j_1} \phi(b_2 \otimes b_{\mu+\nu}) \\ &= \tilde{F}_{j_r} \cdots \tilde{F}_{j_1} (b_{\nu+w_\bullet \tau\nu} \otimes \pi(b_2) \otimes b_\mu) \\ &= b_{\nu+w_\bullet \tau\nu} \otimes \pi(b_2^r) \otimes b_\mu.\end{aligned}$$

The last equality follows from the tensor product rule (2.2) and equations (3.4) and (3.5). Since

$$\pi(b_2^r) = \pi(\pi'(b')) = \pi''(b'),$$

our claim follows. Thus, the proof completes. ☐

**Corollary 3.4.8.** *Let $\lambda, \mu, \nu \in X^+$. Assume that there exists a based $\mathbf{U}^\imath$-module homomorphism $\delta_\nu^\imath : V(\nu + w_\bullet \tau\nu) \to \mathbb{Q}(q)$ such that $\delta_\nu^\imath(v_{\nu+w_\bullet\tau\nu}) = 1$. Then, there exists a based $\mathbf{U}^\imath$-module homomorphism $\pi_{\lambda,\mu,\nu}^\imath : V(w_\bullet(\lambda+\tau\nu), \mu+\nu) \to V(w_\bullet\lambda, \mu)$ such that $\pi_{\lambda,\mu,\nu}^\imath(v_{w_\bullet(\lambda+\tau\nu)} \otimes v_{\mu+\nu}) = v_{w_\bullet\lambda} \otimes v_\mu$.*

*Proof.* Let $f : V(w_\bullet(\lambda + \tau\nu), \mu + \nu) \to V(\nu + w_\bullet\tau\nu) \otimes V(w_\bullet\lambda, \mu)$ denote the based $\mathbf{U}$-module homomorphism in Proposition 3.4.7. Then, we have

$$\begin{aligned}f(v_{w_\bullet(\lambda+\tau\nu)} \otimes v_{\mu+\nu}) &= f(G(b_{w_\bullet(\lambda+\tau\nu)} \otimes b_{\mu+\nu})) = G(b_{\nu+w_\bullet\tau\nu} \otimes b_{w_\bullet\lambda} \otimes b_\mu) \\ &= v_{\nu+w_\bullet\tau\nu} \otimes v_{w_\bullet\lambda} \otimes v_\mu.\end{aligned}$$

Set $\pi^\imath = \pi_{\lambda,\mu,\nu}^\imath := (\delta_\nu^\imath \otimes \mathrm{id}) \circ f$. Noting that $G^\imath(b_{\nu+w_\bullet\tau\nu}) = v_{\nu+w_\bullet\tau\nu}$, we obtain

$$\pi^\imath(v_{w_\bullet(\lambda+\tau\nu)} \otimes v_{\mu+\nu}) = (\delta_\nu^\imath \otimes \mathrm{id})(v_{\nu+w_\bullet\tau\nu} \otimes v_{w_\bullet\lambda} \otimes v_\mu) = v_{w_\bullet\lambda} \otimes v_\mu.$$

This implies that $\pi^\imath$ commutes with the $\imath$bar-involutions. On the other hand, both $f$ and $\delta_\nu^\imath \otimes \mathrm{id}$ preserve the crystal lattices and $\mathcal{A}$-forms. Let $\phi : \mathcal{B}(w_\bullet(\lambda+\tau\nu), \mu+\nu) \to \mathcal{B}(\nu+w_\bullet\tau\nu) \otimes \mathcal{B}(w_\bullet\lambda, \mu) \sqcup \{0\}$ and $\delta_\nu^\imath : \mathcal{B}(\nu+w_\bullet\tau\nu) \to \{1\} \sqcup \{0\}$ denote the induced maps. Then, $\pi^\imath$ preserves the crystal lattices, and the induced map

$$\pi^\imath = \pi_{\lambda,\mu,\nu}^\imath := (\delta_\nu^\imath \otimes \mathrm{id}) \circ \phi : \mathcal{B}(w_\bullet(\lambda + \tau\nu), \mu + \nu) \to \mathcal{B}(w_\bullet\lambda, \mu) \sqcup \{0\}$$

is injective on $\{b \in \mathcal{B}(w_\bullet(\lambda+\tau\nu), \mu+\nu) \mid \pi^\imath(b) \neq 0\}$. By Lemma 3.3.1, we see that $\pi^\imath$ is based. Thus, the proof completes. ☐

## 4. Irreducible finite type of real rank 1

In this section, we assume the following:

- The admissible pair $(I, I_\bullet, \tau)$ is of irreducible finite type of real rank 1 (see Definition 3.1.1 and the list after Remark 3.1.3).
- $Y = \sum_{i \in I} \mathbb{Z} h_i$.
- $X = \mathrm{Hom}_\mathbb{Z}(Y, \mathbb{Z})$.
- $\kappa_i = 0$ for all $i \in I_\circ$.

For each $i \in I$, let $\varpi_i \in X$ denote the $i$-th fundamental weight;

$$\langle h_j, \varpi_i \rangle = \delta_{i,j}$$

for all $j \in I$.

In Subsections 4.1–4.5, we study certain $\mathbf{U}^\imath$-modules in order to prove Proposition 4.6.1. Then, we prove the stability of $\imath$canonical bases in Theorem 4.6.2.

4.1. **Type A.** Let $n \geq 1$ and consider the Dynkin diagram $I = \{1, \dots, n\}$ of type $A_n$. For each $r \in I$ and $n+1 \geq i_1 > \cdots i_r \geq 1$, let $b_{i_1,\dots,i_r} \in \mathcal{B}(\varpi_r)$ denote the unique element of weight $\sum_{k=1}^{r}(\varpi_{i_k} - \varpi_{i_k-1})$; here, we set $\varpi_0 := 0$. Set $v_{i_1,\dots,i_r} := G(b_{i_1,\dots,i_r}) \in V(\varpi_r)$.

*Type* AI. Set

$$\varsigma_1 := q^{-1}, \quad \varpi := 2\varpi_1. \tag{4.1}$$

Since

$$\varpi_1 + w_\bullet \tau \varpi_1 = 2\varpi_1,$$

we have

$$\nu + w_\bullet \tau \nu \in \mathbb{Z}_{\geq 0}\varpi$$

for all $\nu \in X^+$.

**Lemma 4.1.1.** *Let $K : V(\varpi_1) \to V(\varpi_1)$ denote the linear isomorphism given by*

$$K(v_1) = v_2, \quad K(v_2) = v_1.$$

*Then, $K$ is a based $\mathbf{U}^\imath$-module isomorphism.*

*Proof.* By [6, Theorem 4.18], we see that there exists a $\mathbf{U}^\imath$-module isomorphism $K'$ such that $K'(v_1) = v_2$. Then, we have

$$K'(v_2) = K'(B_1 v_1) = B_1 v_2 = v_1.$$

Hence, we obtain $K' = K$.

It remains to show that $K$ is based. We have

$$v_2 = B_1 v_1 \in \mathcal{L}(\varpi_1) \cap {}_{\mathcal{A}}V(\varpi_1) \cap \psi^\imath(\mathcal{L}(\varpi_1)),$$

where $\psi^\imath$ denote the $\imath$bar-involution on $V(\varpi_1)$. This implies that $v_2 = G^\imath(b_2)$. Therefore, the $K$ is a based $\mathbf{U}^\imath$-module isomorphism. Thus, the proof completes. □

**Lemma 4.1.2.** *There exists $w_0 \in \mathcal{L}(\varpi)$ such that $\mathbf{U}^\imath w_0 \simeq \mathbb{Q}(q)$ and $\mathrm{ev}_\infty(w_0) = b_\varpi$.*

*Proof.* Noting that $\varpi = 2\varpi_1$ and $V(\varpi_1) = V(-\varpi_1)$ (see Remark 2.3.1), consider the composition

$$g : V(\varpi) \xrightarrow{\chi_{\varpi_1,\varpi_1}} V(\varpi_1) \otimes V(\varpi_1) \xrightarrow{K \otimes \mathrm{id}} V(\varpi_1) \otimes V(\varpi_1) \xrightarrow{\delta_{\varpi_1}} \mathbb{Q}(q).$$

For the definitions of based $\mathbf{U}$-module homomorphisms $\chi_{\varpi_1,\varpi_1}$ and $\delta_{\varpi_1}$, see Example 2.4.2(3) and (4), respectively.

By the definition, $g$ is a $\mathbf{U}^\imath$-module homomorphism preserving the crystal lattices. For each $b \in \mathcal{B}(\varpi)$, considering at $q = \infty$, we obtain

$$\mathrm{ev}_\infty(g(G(b))) = \delta_{b,b_\varpi}.$$

Hence, $\mathrm{Ker}\, g$ has a basis $\{w_b \mid b \in \mathcal{B}(\varpi) \setminus \{b_\varpi\}\}$ such that $w_b \in \mathcal{L}(\varpi)$ and $\mathrm{ev}_\infty(w_b) = b$. Therefore, the complement $W_0 \subset V(\varpi)$ of $\mathrm{Ker}\, g$ with respect to the inner product on $V(\varpi)$ (cf. Subsection 2.3) is spanned by a vector $w_0 \in \mathcal{L}(\varpi)$ satisfying $\mathrm{ev}_\infty(w_0) = b_\varpi$. Now, the assertion follows from the following:

$$W_0 \simeq V(\varpi)/\mathrm{Ker}\, g \simeq \mathrm{Im}\, g = \mathbb{Q}(q).$$

Thus, the proof completes. □

*Type* AII. Set

$$\begin{aligned} \varsigma_2 &:= q, \\ \varpi &:= \varpi_2. \end{aligned} \tag{4.2}$$

Since

$$\varpi_i + w_\bullet \tau \varpi_i = \begin{cases} \varpi_2 & \text{if } i = 1, 3, \\ 2\varpi_2 & \text{if } i = 2 \end{cases}$$

for all $i \in I$, we have

$$\nu + w_\bullet \tau \nu \in \mathbb{Z}_{\geq 0} \varpi$$

for all $\nu \in X^+$.

**Lemma 4.1.3.** *We have* $B_2 v_{4,3} = q^2 v_{3,1}$.

*Proof.* Using [25, Proposition 5.2.2], we compute as

$$\begin{aligned} B_2 v_{4,3} &= (F_2 + q T_{w_\bullet}(E_2) K_2^{-1})(v_{4,3}) \\ &= q^2 T_{w_\bullet}(E_2 T_{w_\bullet}^{-1}(v_{4,3})) \\ &= q^2 T_{w_\bullet}(E_2 v_{4,3}) \\ &= q^2 T_{w_\bullet}(v_{4,2}) \\ &= q^2 v_{3,1}. \end{aligned}$$

This proves the assertion. □

**Lemma 4.1.4.** *There exists* $w_0 \in \mathcal{L}(\varpi)$ *such that* $\mathbf{U}^\imath w_0 \simeq \mathbb{Q}(q)$ *and* $\mathrm{ev}_\infty(w_0) = b_\varpi$.

*Proof.* By direct calculation and Lemma 4.1.3, we see that the vector

$$w_0 := v_{2,1} - q^{-2} v_{4,3} \in V(\varpi)$$

spans the $\mathbf{U}^\imath$-submodule isomorphic to $\mathbb{Q}(q)$. Clearly, we have $w_0 \in \mathcal{L}(\varpi)$ and $\mathrm{ev}_\infty(w_0) = b_{2,1} = b_\varpi$. Thus, the proof completes. □

*Type* AIII. Set

$$\begin{aligned} \varsigma_1 = \varsigma_2 &:= 1, \\ \varpi &:= \varpi_1 + \varpi_2. \end{aligned} \tag{4.3}$$

Since

$$\varpi_i + w_\bullet \tau \varpi_i = \varpi_1 + \varpi_2$$

for all $i \in I$, we have

$$\nu + w_\bullet \tau \nu \in \mathbb{Z}_{\geq 0} \varpi$$

for all $\nu \in X^+$.

For each $i = 1, 2$, the $V(\varpi_i)$ is two-dimensional. Let $v_1^i$ (resp., $v_2^i$) denote the highest (resp., lowest) weight vector.

**Lemma 4.1.5.** *Let* $K : V(\varpi_1) \to V(\varpi_2)$ *denote the linear isomorphism given by*

$$K(v_1^1) = v_2^2, \ K(v_2^1) = v_1^2.$$

*Then,* $K$ *is a based* $\mathbf{U}^\imath$*-module isomorphism.*

*Proof.* Noting that

$$B_1 v_1^1 = v_2^1, \quad B_2 v_1^2 = v_2^2,$$

the assertion can be proved by the same way as Lemma 4.1.1 □

**Lemma 4.1.6.** *There exists* $w_0 \in \mathcal{L}(\varpi)$ *such that* $\mathbf{U}^\imath w_0 \simeq \mathbb{Q}(q)$ *and* $\mathrm{ev}_\infty(w_0) = b_\varpi$*.*

*Proof.* Using Lemma 4.1.5, the same argument as in the proof of Lemma 4.1.2 proves the assertion. □

*Type* AIV. Set

$$\begin{aligned} &\varsigma_1 := 1, \ \varsigma_n := (-1)^n q^{n-1}, \\ &\varpi := \varpi_1 + \varpi_n. \end{aligned} \tag{4.4}$$

Since

$$\varpi_i + w_\bullet \tau \varpi_i = \varpi_1 + \varpi_n$$

for all $i \in I$, we have

$$\nu + w_\bullet \tau \nu \in \mathbb{Z}_{\geq 0} \varpi$$

for all $\nu \in X^+$.

**Lemma 4.1.7.** *We have* $B_n v_n = v_{n+1} + v_1$*.*

*Proof.* Using [25, Proposition 5.2.2], we compute as

$$\begin{aligned} B_n v_n &= (F_n + (-1)^n q^{n-1} T_{w_\bullet}(E_1) K_n^{-1}) v_n \\ &= v_{n+1} + (-1)^n q^{n-2} T_{w_\bullet}(E_1 T_{w_\bullet}^{-1}(v_n)) \\ &= v_{n+1} + (-1)^2 T_{w_\bullet}(E_1 v_2) \\ &= v_{n+1} + v_1. \end{aligned}$$

This proves the assertion. □

**Lemma 4.1.8.** *Let* $K : V(\varpi_1) \to V(\varpi_1)$ *denote the linear isomorphism given by*

$$K(v_i) = \begin{cases} v_{n+1} & \text{if } i = 1, \\ v_i & \text{if } i = 2, \dots, n, \\ v_1 & \text{if } i = n+1. \end{cases}$$

*Then, it is a based* $\mathbf{U}^\imath$*-module isomorphism.*

*Proof.* Noting that

$$\begin{aligned} v_2 &= B_1 v_1, \\ v_3 &= B_2 v_2, \\ &\cdots, \\ v_n &= B_{n-1} v_{n-1} \end{aligned}$$

and that

$$v_{n+1} = B_n v_n - v_1$$

by Lemma 4.1.7, the assertion can be proved by the same way as Lemma 4.1.1. □

**Lemma 4.1.9.** *There exists* $w_0 \in \mathcal{L}(\varpi)$ *such that* $\mathbf{U}^\imath w_0 \simeq \mathbb{Q}(q)$ *and* $\mathrm{ev}_\infty(w_0) = b_\varpi$*.*

*Proof.* Using Lemma 4.1.8, the same argument as in the proof of Lemma 4.1.2 proves the assertion. □

4.2. **Type** BII. Let $n \geq 2$ and consider the Dynkin diagram $I = \{1, \dots, n\}$ of type $B_n$. Set $L := \{1, \dots, n, 0, \bar{n}, \dots, \bar{1}\}$, and equip the set $\mathcal{B} := \{b_i \mid i \in L\}$ with the crystal structure as in [9, Example 2.22]. Then, $\mathcal{B}$ is identical to $\mathcal{B}(\varpi_1)$. For each $i \in L$, set $v_i := G(b_i) \in V(\varpi_1)$.

Set

$$\varsigma_1 := q^{2n-3}, \qquad \varpi := \varpi_1. \tag{4.5}$$

Since

$$\varpi_i + w_\bullet \tau \varpi_i = \begin{cases} 2\varpi_1 & \text{if } i \neq n, \\ \varpi_1 & \text{if } i = n \end{cases}$$

for all $i \in I$, we have

$$\nu + w_\bullet \tau \nu \in \mathbb{Z}_{\geq 0} \varpi$$

for all $\nu \in X^+$.

**Lemma 4.2.1.** *We have* $B_1 v_{\bar{1}} = q^{2n-1} v_2$.

*Proof.* Using [25, Proposition 5.2.2], we compute as

$$\begin{aligned} B_1 v_{\bar{1}} &= (F_1 + q^{2n-3} T_{w_\bullet}(E_1) K_1^{-1}) v_{\bar{1}} \\ &= q^{2n-1} T_{w_\bullet}(E_1 T_{w_\bullet}^{-1}(v_{\bar{1}})) \\ &= q^{2n-1} T_{w_\bullet}(E_1 v_{\bar{1}}) \\ &= q^{2n-1} T_{w_\bullet}(v_{\bar{2}}) \\ &= q^{2n-1} v_2. \end{aligned}$$

This proves the assertion. ☐

**Lemma 4.2.2.** *There exists* $w_0 \in \mathcal{L}(\varpi)$ *such that* $\mathbf{U}^\imath w_0 \simeq \mathbb{Q}(q)$ *and* $\mathrm{ev}_\infty(w_0) = b_\varpi$.

*Proof.* By direct calculation and Lemma 4.2.1, we see that the vector

$$w_0 := v_1 - q^{-2n+1} v_{\bar{1}} \in V(\varpi)$$

spans the $\mathbf{U}^\imath$-submodule isomorphic to $\mathbb{Q}(q)$. Clearly, we have $w_0 \in \mathcal{L}(\varpi)$ and $\mathrm{ev}_\infty(w_0) = b_1 = b_\varpi$. Thus, the proof completes. ☐

4.3. **Type** CII. Let $n \geq 3$ and consider the Dynkin diagram $I = \{1, \dots, n\}$ of type $C_n$. Set $L := \{1, \dots, n, \bar{n}, \dots, \bar{1}\}$, and equip the set $\mathcal{B} := \{b_i \mid i \in L\}$ with the crystal structure as in [9, Example 2.23]. Then, $\mathcal{B}$ is identical to $\mathcal{B}(\varpi_1)$.

The $\mathbf{U}$-module $V(\varpi_2)$ can be embedded into $V(\varpi_1) \otimes V(\varpi_1)$ in a way such that

$$v_{\varpi_2} \mapsto v_2 \otimes v_1 - q^{-1} v_1 \otimes v_2.$$

Then, the crystal basis $\mathcal{B}(\varpi_2)$ is identified with the connected component $C(b_2 \otimes b_1)$ of $\mathcal{B} \otimes \mathcal{B}$. For each $(i,j) \in L^2$ such that $b_i \otimes b_j \in \mathcal{B}(\varpi_2)$, set

$$v_{i,j} = \begin{cases} v_{\bar{k}} \otimes v_k + q^{-1} v_{\overline{k-1}} \otimes v_{k-1} \\ -q^{-1} v_{k-1} \otimes v_{\overline{k-1}} - q^{-2} v_k \otimes v_{\bar{k}} & \text{if } (i,j) = (\bar{k}, k) \text{ for some } k \in \{2, \dots, n\}, \\ v_i \otimes v_j - q^{-1} v_j \otimes v_i & \text{otherwise.} \end{cases}$$

Then, $\{v_{i,j} \mid b_i \otimes b_j \in \mathcal{B}(\varpi_2)\}$ forms a free basis of $\mathcal{L}(\varpi_2)$.

For each $i \in I$ and $k \in \{2, \dots, n\}$, one can straightforwardly verify that

$$F_i v_{\bar{k},k} = \begin{cases} [2] v_{\bar{\imath},i+1} & \text{if } k = i+1, \\ v_{\bar{\imath},i+1} & \text{if } k = i+2 \text{ or } k = i < n, \\ 0 & \text{otherwise}, \end{cases} \tag{4.6}$$

and

$$E_i v_{\bar{k},k} = \begin{cases} [2] v_{\overline{i+1},i} & \text{if } k = i+1, \\ v_{\overline{i+1},i} & \text{if } k = i+2 \text{ or } k = i < n, \\ 0 & \text{otherwise}. \end{cases} \tag{4.7}$$

**Lemma 4.3.1.** *Set*

$$w_0' := -\frac{1}{[2]} v_{\bar{2},2} + \sum_{k=3}^{n} (-1)^{k-3} \frac{[n-k+1]}{[n-2]} v_{\bar{k},k}.$$

*Then, we have $E_j w_0' = F_j w_0' = 0$ for all $j \in I_\bullet$.*

*Proof.* Since $w_0'$ is a weight vector of weight 0, the identity $E_j w_0' = 0$ follows from $F_j w_0' = 0$.

By equation (4.6), we have

$$F_i w_0' = \begin{cases} [2] \cdot (-\frac{1}{[2]}) + 1 & \text{if } i = 1, \\ \frac{(-1)^{i-3}}{[n-2]}([n-i+1] - [2][n-i] + [n-i-1]) & \text{if } i = 3, \dots, n-1, \\ 0 & \text{if } i = n. \end{cases}$$

Now, the assertion follows from a well-known identity

$$[2][m] = [m+1] + [m-1]$$

valid for all $m \in \mathbb{Z}$. Thus, the proof completes. □

Set

$$\varsigma_2 := q^{n-1}, \qquad \varpi := \varpi_2. \tag{4.8}$$

Since

$$\varpi_i + w_\bullet \tau \varpi_i = \begin{cases} \varpi_2 & \text{if } i = 1, \\ 2\varpi_2 & \text{if } i \neq 1 \end{cases}$$

for all $i \in I$, we have

$$\nu + w_\bullet \tau \nu \in \mathbb{Z}_{\geq 0} \varpi$$

for all $\nu \in X^+$.

**Lemma 4.3.2.** *There exists $w_0 \in \mathcal{L}(\varpi)$ such that $\mathbf{U}^\imath w_0 \simeq \mathbb{Q}(q)$ and $\mathrm{ev}_\infty(w_0) = b_\varpi$.*

*Proof.* By direct calculation, equations (4.6) and (4.7), and Lemma 4.3.1, we see that the vector

$$w_0 := v_{2,1} - \frac{q^{-n+1}[2][n-2]}{[n]} w_0' + q^{-2n+1} v_{\bar{1},\bar{2}}$$

spans the $\mathbf{U}^\imath$-submodule isomorphic to $\mathbb{Q}(q)$. Clearly, we have $w_0 \in \mathcal{L}(\varpi)$ and $\mathrm{ev}_\infty(w_0) = b_{2,1} = b_\varpi$. Thus, the proof completes. □

4.4. **Type** DII. Let $n \geq 4$ and consider the Dynkin diagram $I = \{1, \dots, n\}$ of type $D_n$. Set $L := \{1, \dots, n, \bar{n}, \dots, \bar{1}\}$, and equip the set $\mathcal{B} := \{b_i \mid i \in L\}$ with the crystal structure as in [9, Example 2.24]. Then, $\mathcal{B}$ is identical to $\mathcal{B}(\varpi_1)$. For each $i \in L$, set $v_i := G(b_i) \in V(\varpi_1)$.

Set

$$\varsigma_1 := q^{n-2}, \quad \varpi := \varpi_1. \tag{4.9}$$

Since

$$\varpi_i + w_\bullet \tau \varpi_i = \begin{cases} 2\varpi_1 & \text{if } i \neq n-1, n, \\ \varpi_1 & \text{if } i = n-1, n \end{cases}$$

for all $i \in I$, we have

$$\nu + w_\bullet \tau \nu \in \mathbb{Z}_{\geq 0} \varpi$$

for all $\nu \in X^+$.

**Lemma 4.4.1.** *We have* $B_1 v_{\bar{1}} = q^{n-1} v_2$.

*Proof.* Using [25, Proposition 5.2.2], we compute as

$$\begin{aligned} B_1 v_{\bar{1}} &= (F_1 + q^{n-2} T_{w_\bullet}(E_1) K_1^{-1}) v_{\bar{1}} \\ &= q^{n-1} T_{w_\bullet}(E_1 T_{w_\bullet}^{-1}(v_{\bar{1}})) \\ &= q^{n-1} T_{w_\bullet}(E_1 v_{\bar{1}}) \\ &= q^{n-1} T_{w_\bullet}(v_{\bar{2}}) \\ &= q^{n-1} v_2. \end{aligned}$$

This proves the assertion. □

**Lemma 4.4.2.** *There exists* $w_0 \in \mathcal{L}(\varpi)$ *such that* $\mathbf{U}^\imath w_0 \simeq \mathbb{Q}(q)$ *and* $\mathrm{ev}_\infty(w_0) = b_\varpi$.

*Proof.* By direct calculation and Lemma 4.4.1, we see that the vector

$$w_0 := v_1 - q^{-n+1} v_{\bar{1}} \in V(\varpi)$$

spans the $\mathbf{U}^\imath$-submodule isomorphic to $\mathbb{Q}(q)$. Clearly, we have $w_0 \in \mathcal{L}(\varpi)$ and $\mathrm{ev}_\infty(w_0) = b_1 = b_\varpi$. Thus, the proof completes. □

4.5. **Type** FII. Consider the Dynkin diagram $I = \{1, 2, 3, 4\}$ of type $F_4$, and the crystal $\mathcal{B}(\varpi_4)$ (see [9, Fig. 5.8]). For each $\lambda \in X \setminus \{0\}$ with $\mathcal{B}(\varpi_4)_\lambda \neq \emptyset$, let $b_\lambda \in \mathcal{B}(\varpi_4)$ denote the unique element of weight $\lambda$. Set

$$b_0^1 := \tilde{F}_4 b_{-\varpi_3 + 2\varpi_4}, \quad b_0^2 := \tilde{F}_3 b_{-\varpi_2 + 2\varpi_3 - \varpi_4}.$$

Then, we have $\mathcal{B}(\varpi_4)_0 = \{b_0^1, b_0^2\}$.

Set

$$\varsigma_1 := q^5, \quad \varpi := \varpi_4. \tag{4.10}$$

Since

$$\varpi_i + w_\bullet \tau \varpi_i = \begin{cases} 2\varpi_4 & \text{if } i = 1, 4, \\ 4\varpi_4 & \text{if } i = 2, \\ 3\varpi_4 & \text{if } i = 3 \end{cases}$$

for all $i \in I$, we have

$$\nu + w_\bullet \tau \nu \in \mathbb{Z}_{\geq 0} \varpi$$

for all $\nu \in X^+$.

**Lemma 4.5.1.** *There exists $w_0 \in \mathcal{L}(\varpi)$ such that $\mathbf{U}^\imath w_0 \simeq \mathbb{Q}(q)$ and $\mathrm{ev}_\infty(w_0) = b_\varpi$.*

*Proof.* Using a computer program GAP [12] with a package QuaGroup [13], we see that

$$w_0 := G(b_{\varpi_4}) - \frac{q^{-5}[2]}{[3]}\left(G(b_0^2) - \frac{1}{[2]}G(b_0^1)\right) + q^{-11}G(b_{-\varpi_4})$$

spans a $\mathbf{U}^\imath$-submodule of $V(\varpi)$ isomorphic to $\mathbb{Q}(q)$. Clearly, we have $w_0 \in \mathcal{L}(\varpi)$ and $\mathrm{ev}_\infty(w_0) = b_{\varpi_4} = b_\varpi$. Thus, the proof completes. □

4.6. **Stability of $\imath$canonical bases.** In the sequel, we set $\varsigma_i$ for $i \in I_\circ$ and $\varpi \in X^+$ as in (4.1)–(4.5), (4.8)–(4.10).

**Proposition 4.6.1.** *For each $m \in \mathbb{Z}_{\geq 0}$, there exists a unique based $\mathbf{U}^\imath$-module homomorphism $g_m : V(m\varpi) \to \mathbb{Q}(q)$ such that $g_m(v_{m\varpi}) = 1$.*

*Proof.* We prove the assertion by induction on $m$. The case when $m = 0$ is trivial. Let us prove for $m = 1$.

Let $w_0 \in V(\varpi)$ be as in Lemmas 4.1.2, 4.1.4, 4.1.6, 4.1.9,, 4.2.2, 4.3.2, 4.4.2, and 4.5.1. Set $W_0 := \mathbb{Q}(q)w_0$ and $W_1 \subset V(\varpi)$ the complement of $W_0$ with respect to the inner product in Subsection 2.3. Then, we can write

$$v_\varpi = cw_0 + w_1$$

for some $c \in 1 + \mathbf{A}_\infty$ and $w_1 \in W_1$. Define a $\mathbf{U}^\imath$-module homomorphism

$$g_1 : V(\varpi) = W_0 \oplus W_1 \to \mathbb{Q}(q)$$

by

$$g_1(w_0) = c^{-1},\ g_1(W_1) = 0.$$

Then, it preserves the crystal lattices, and we have

$$g_1(v_\varpi) = 1.$$

This identity implies that $g_1$ preserves the $\mathcal{A}$-forms and commutes with the $\imath$bar-involutions. Moreover, the induced map $\gamma_1 : \mathcal{B}(\varpi) \to \{1\} \sqcup \{0\}$ satisfies

$$\gamma_1(b) = \delta_{b,b_\varpi}$$

for all $b \in \mathcal{B}(\varpi)$. Therefore, by Lemma 3.3.1, we see that $g_1$ is a based $\mathbf{U}^\imath$-module homomorphism.

Now, assume that $m > 1$ and the assertion is true for $m - 1$. Consider the composition

$$g : V(m\varpi) \xrightarrow{\chi_{(m-1)\varpi,\varpi}} V((m-1)\varpi) \otimes V(\varpi) \xrightarrow{g_{m-1}\otimes \mathrm{id}} V(\varpi) \xrightarrow{g_1} \mathbb{Q}(q).$$

Then, it preserves the crystal lattices and $\mathcal{A}$-forms, and we have

$$g(v_{m\varpi}) = g_1(g_{m-1}(v_{(m-1)\varpi}) \otimes v_\varpi) = g_1(v_\varpi) = 1.$$

This identity implies that $g = g_m$ and that $g_m$ commutes with the $\imath$bar-involutions. Moreover, the induced map $\gamma_m : \mathcal{B}(m\varpi) \to \{1\} \sqcup \{0\}$ satisfies

$$\gamma(b) = \delta_{b,b_{m\varpi}}$$

for all $b \in \mathcal{B}(m\varpi)$. Therefore, by Lemma 3.3.1, we see that $g_m$ is a based $\mathbf{U}^\imath$-module homomorphism. Thus, the proof completes. □

**Theorem 4.6.2.** *Let $\lambda, \mu, \nu \in X^+$. Then, there exists a unique based $\mathbf{U}^\imath$-module homomorphism*

$$\pi^\imath_{\lambda,\mu,\nu} : V(w_\bullet(\lambda + \tau\nu), \mu + \nu) \to V(w_\bullet\lambda, \mu)$$

*such that*

$$\pi^\imath_{\lambda,\mu,\nu}(v_{w_\bullet(\lambda+\tau\nu)} \otimes v_{\mu+\nu}) = v_{w_\bullet\lambda} \otimes v_\mu.$$

*Proof.* The assertion follows from Corollary 3.4.8 and Proposition 4.6.1 since we have $\nu + w_\bullet\tau\nu \in \mathbb{Z}_{\geq 0}\varpi$. ☐

**Corollary 4.6.3.** *For each $\lambda, \mu \in X^+$, there exists a unique based $\mathbf{U}^\imath$-module homomorphism*

$$\pi^\imath_{\lambda,\mu} : \dot{\mathbf{U}}^\imath 1_{\overline{\mu + w_\bullet\lambda}} \to V(w_\bullet\lambda, \mu)$$

*such that*

$$\pi^\imath_{\lambda,\mu}(1_{\overline{\mu+w_\bullet\lambda}}) = v_{w_\bullet\lambda} \otimes v_\mu.$$

*Proof.* Define a $\mathbf{U}^\imath$-module homomorphism $\pi^\imath_{\lambda,\mu}$ by

$$\pi^\imath_{\lambda,\mu}(x) := x \cdot (v_{w_\bullet\lambda} \otimes v_\mu).$$

We shall show that it is based.

Let $x \in \dot{\mathbf{B}}^\imath$ (see Example 3.3.2(3)). By [8, Theorem 7.2], there exists $\nu \in X^+$ such that

$$\pi^\imath_{\lambda+\tau\nu,\mu+\nu}(x) \in \mathbf{B}^\imath(w_\bullet(\lambda + \tau\nu), \mu + \nu).$$

Then, by Theorem 4.6.2, we see that

$$\pi^\imath_{\lambda,\mu}(x) = \pi^\imath_{\lambda,\mu,\nu}(\pi^\imath_{\lambda+\tau\nu,\mu+\nu}(x)) \in \mathbf{B}^\imath(w_\bullet\lambda, \mu) \sqcup \{0\}.$$

It remains to show that $\operatorname{Ker} \pi^\imath_{\lambda,\mu}$ is spanned by a subset of $\dot{\mathbf{B}}^\imath$. Let $v \in \operatorname{Ker} \pi^\imath_{\lambda,\mu}$, and write

$$v = \sum_{k=1}^{r} c_k x_k$$

for some $c_1, \ldots, c_r \in \mathbb{Q}(q)^\times$ and $x_1, \ldots, x_r \in \dot{\mathbf{B}}^\imath$. Again by [8, Theorem 7.2], there exists $\nu \in X^+$ such that the vectors $\pi^\imath_{\lambda+\tau\nu,\mu+\nu}(x_k)$ are distinct ıcanonical basis elements. Since the homomorphism $\pi^\imath_{\lambda,\mu,\nu}$ is based, the nonzero vectors of the form $\pi^\imath_{\lambda,\mu}(x_k) = \pi^\imath_{\lambda,\mu,\nu}(\pi^\imath_{\lambda+\tau\nu,\mu+\nu}(x_k))$ are distinct. Hence, we must have $\pi^\imath_{\lambda,\mu}(x_k) = 0$ for all $k = 1, \ldots, r$. This implies that

$$\operatorname{Ker} \pi^\imath_{\lambda,\mu} = \mathbb{Q}(q)\{x \in \dot{\mathbf{B}}^\imath \mid \pi^\imath_{\lambda,\mu}(x) = 0\},$$

as desired. Thus, the proof completes. ☐

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