ON OKAMURA'S UNIQUENESS THEOREM

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Moyer [1] gives a uniqueness theorem for ordinary differential equations which includes many of the known criteria for uniqueness. It is demonstrated here that Moyer's results can be obtained as a special case of Okamura's theorem [2]. (See Yoshizawa, [3].)

If $f(t, y)$ is defined and continuous on D: $t_0 \leqq t \leqq t_0 + a$, $|y - y_0| \leqq b$, the problem of uniqueness for a solution $z(t)$ to the initial value problem,

(1) $\dot{y} = f(t, y)$, $y(t_0) = y_0$ is equivalent to the uniqueness of the solution $x(t) \equiv 0$ of

(2) $\dot{x} = F(t, x) = f(t, z(t)) - f(t, z(t) - x)$, where $F(t, 0) = 0$ on D.

THEOREM 1 (OKAMURA [2], SEE ALSO [3]). *The $x(t) \equiv 0$ solution of* (2) *is unique to the right iff there exists a function $V(t, x)$ defined on D such that* (i) $V(t, 0) = 0$, (ii) $V(t, x) > 0$, $x \neq 0$, (iii) $V(t, x)$ *satisfies a local Lipschitz condition with respect to x and*

$$V'(t, x) = \operatorname*{Lim\,inf}_{h \to 0} [V(t + h, x + hF(t, x)) - V(t, x)]$$
$$\leqq 0.$$

Or, equivalently, the solution $z(t)$ of (1) is unique if there exists a function $V(t, x)$ such that $V(t, z(t) - y)$ has the properties of Theorem 1. Moyer's results are obtained by defining

$$V(t, z(t) - y) = \exp[2W(t, z(t) - y)].$$

Moyer's Theorem 2.1 can be stated in the following manner.

THEOREM 2. *If there exists a $V(t, x)$ as in Theorem* 1 *and $z(t)$, $y(t)$ are two distinct solutions of* (1) *then $z(t) \equiv y(t)$ to the right of t_0 if and only if*

$$\operatorname*{Lim\,inf}_{t \to t_0} V(t, z(t) - y(t)) = 0.$$

REMARK. Okamura's theorem shows it may be possible to have uniqueness of solutions of (1) but not assert the existence of the $W(t, r)$ function of Moyer [1].

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