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OKA'S PRINCIPLE FOR HOLOMORPHIC SECTIONS OF ELLIPTIC BUNDLES

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Dedicated to the memory of Volodia Eidlin

CONTENTS

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0. The h-principle for holomorphic maps

Let X and Y be complex analytic manifolds. One says that holomorphic maps $X \to Y$ satisfy the *h-principle* (h for homotopy, see [Gro]) if every continuous map $X \to Y$ is homotopic to a holomorphic map.

0.1. **Examples of the h-principle.** (a) If either X or Y is contractible (e.g., $X = \mathbf{C}^n$ or $Y = \mathbf{C}^n$), then the h-principle is trivially satisfied as every continuous map $X \to Y$ is homotopic to a constant map.

(b) Let $X \subset \mathbf{C}$ be a connected open subset with finitely generated fundamental group and $Y = \mathbf{C}^\times = \mathbf{C} - \{0\}$. Then every continuous map $X \to Y$ is homotopic to a holomorphic map of the form $X \to \prod_{i=1}^k (x - a_i)^{n_i}$ for some points $a_1, \dots, a_k$ in $\mathbf{C} - X$ and some integers $n_1, \dots, n_k$. Notice that we use here the multiplicative group structure in $\mathbf{C}^\times$.

(b′) If $\pi_1(X)$ is infinitely generated, the above construction does not directly apply. Yet holomorphic maps $X \to \mathbf{C}^\times$ do satisfy the h-principle for all open $X \subset \mathbf{C}$. In fact, this h-principle remains valid for all open Riemann surfaces X and, moreover, for all *Stein manifolds* X (see 0.2 for definitions) by a theorem of Arens and Royden [Ar, Roy] which illustrates a special case of the Grauert theorem (see 0.4).

(b″) Let Y be the punctured disk, $Y = D^* = \{y \in \mathbf{C} | \; 0 < |y| < 1\}$, and X be a *bounded* domain in $\mathbf{C}$ with *finitely generated* fundamental group $\pi_1(X)$. Then the scaled products $x \mapsto \varepsilon \prod_i (x - a_i)^{n_i}$ for small $\varepsilon > 0$ insure the h-principle of holomorphic maps $X \to D^*$.

On the other hand, the h-principle may fail if $\pi_1(X)$ is infinitely generated. For example, if $X = D - S$, where S is a closed *countable* subset in D, then every bounded holomorphic function on X extends to the disk $D \supset X$. It follows that there are at most *countably many* homotopy classes of holomorphic maps $X \to D^*$, while the homotopy classes of continuous maps are obviously uncountable.

(c) Let $Y \subset \mathbf{C}$ be a connected open subset which is not simply connected. Moreover, assume Y cannot be made simply connected by adding a single point to Y (e.g., we rule out $Y = D^*$, but every domain with noncyclic π_1 will do).

If holomorphic maps $X \to Y$ satisfy the h-principle then the cohomology $H^1(X; \mathbf{Z})$ vanishes. For example, if $X \subset \mathbf{C}$, then $\pi_1(X) = 0$.

Proof. Since the complement $\mathbf{C} - Y$ contains at least two points, Y is covered (uniformized) by the *hyperbolic plane*, and so Y admits a (unique) complete metric of curvature -1, which is called the *hyperbolic metric* on Y. Moreover, the point adding condition shows that Y is not biholomorphic to D or to D^*. It follows that there exists a closed geodesic, say γ, in Y for the hyperbolic metric in Y. On the other hand, the *Schwartz lemma* bounds the length of curves in Y coming from X by holomorphic maps $f: X \to Y$, length $f(C) \leq$ const, where C is a closed curve in X and where the constant depends on C but not on f. Therefore, $f(C)$ cannot be homotoped to a sufficiently high multiple of the above geodesic γ. In fact, no geodesic (in particular, no multiple of γ) can be shortened by a homotopy as the hyperbolic metric in Y has negative curvature. This clearly contradicts the h-principle unless every map of X to the circle is contractible, i.e., $H^1(X; \mathbf{Z}) = 0$. Q.E.D.

The first interesting case where the h-principle breaks down is that of X the annulus $A(a, b) = \{x \in \mathbf{C} \mid a < |x| < b\}$ and Y the complex line with two punctures, $Y = \mathbf{C} - \{0, 1\}$.

Remark. The failure of the h-principle for maps into Y is a sign of certain holomorphic rigidity, often called *hyperbolicity*, of Y. For example, if holomorphic maps $A(a, b) \to Y$ violate the h-principle, then the following (holomorphically invariant length) function $|\alpha|$ on the conjugacy classes α in $\pi_1(Y)$ is *nontrivial* (i.e., $|\alpha| = 0 \Rightarrow \alpha = [id]$), $|\alpha| = \inf(\log b/a)^{-1}$, where inf is taken over all (a, b) for which there exists a holomorphic map $A(a, b) \to Y$ representing α.

0.2. **Stein Manifolds.** The following properties of a complex manifold X say, in effect, that X has "sufficiently many" holomorphic maps $f: X \to \mathbf{C}$.

St_1. Every two distinct points x_1 and x_2 in X can be *separated* by some holomorphic function f on X, where the separation means $f(x_1) \neq f(x_2)$. Furthermore, for every infinite divergent sequence $x_i \in X$, there exists an f such that $\limsup_{i \to \infty} |f(x_i)| = \infty$.

St_2. For every finite or infinite countable discrete subset $\{x_i\} \subset X$, there exists a holomorphic function f on X such that $f(x_i) = c_i$ for prescribed values $c_i \in \mathbf{C}$.

St_3. X is biholomorphic to a complex analytic submanifold in $\mathbf{C}^N$ for some $N = N(X)$.

Proposition-Definition. *The conditions* St_1, St_2, *and* St_3 *are equivalent and a manifold satisfying these is called Stein.*

About the proof. The implications $\mathrm{St}_2 \Rightarrow \mathrm{St}_1$ and $\mathrm{St}_3 \Rightarrow \mathrm{St}_1$ are obvious. The implication $\mathrm{St}_3 \Rightarrow \mathrm{St}_2$ is an elementary exercise. The difficult step $\mathrm{St}_1 \Rightarrow \mathrm{St}_3$ is due (up to some technical refinements by Grauert [Gra_1]) to Narasimhan [Na] who bounds $N = \dim \mathbf{C}^N$ by $N \leq 2\dim X + 1$. (This bound was improved in [Fo_1, Scha, and G-E_2].)

Example. Let X be an open subset in $\mathbf{C}^n$. Then linear functions separate points in X. To get the second part of St_1, we assume that for each point z in the boundary of $X \subset \mathbf{C}$ there exists a complex hypersurface H in $\mathbf{C}^n$ lying in the complement $\mathbf{C}^n - X$ and containing z. (Notice that such an H always exists for $n = 1$; namely, $H = \{z\}$.) One knows that there is a holomorphic function F_z on $\mathbf{C}^n$ whose zero set is exactly H (see [G-R]), and then one easily satisfies the second part of St_1 with the function F_z^{-1} on $X \subset \mathbf{C}^n$. In particular, every domain $X \subset \mathbf{C}$ is Stein.

0.3. **Positive forms and convex functions.** A real differential 2-form ω on X is called *positive* if $\omega(\tau, \sqrt{-1}\tau) > 0$ for all nonzero tangent vectors τ on X. Next, a C^2-function $p : X \to \mathbf{R}$ is called *strictly* **C**-*convex* or just *convex* if the form $dJdp$ is positive, where d is the exterior differential and J is the (real) operator on the cotangent bundle corresponding to the multiplication by $\sqrt{-1}$. The ordinary convexity in $\mathbf{R}^n$ is called, whenever no confusion is possible, **R**-*convexity*, but we avoid "plurisubharmonicity" as much as "plurisublinearity."

Every Stein manifold admits a proper positive convex function. In fact, the function $\|z\|^2$ on $\mathbf{C}^N$ is obviously convex and proper on every submanifold. The converse is also true but not as easily seen.

0.3.A. **Theorem** [Gra_3]. *If a complex manifold X admits a proper positive convex function, then X is Stein.*

0.3.B. *An application.* Using this theorem, one can show that every real analytic manifold X_0 admits a complexification $X \supset X_0$ which is Stein and which is diffeomorphic to the tangent bundle $T(X_0)$. It easily follows that every countable, locally finite, finite dimensional polyhedron P is homotopy equivalent to a Stein manifold X. In fact, by a recent result of Eliashberg, one can find such an X satisfying $\dim_{\mathbf{C}} X = \dim P$.

0.4. **The h-principle of Grauert.** *If G is a complex Lie group and X is Stein then holomorphic maps $X \to G$ satisfy the h-principle, that is, every continuous map $X \to G$ can be made holomorphic by a homotopy.*

This result was proven in [Gra_2] and then improved and generalized in [Ca_2, Ra, F-R, Fo_2, H-L]. For example, the h-principle of Grauert extends to holomorphic sections of principle G-fibrations over X. Furthermore, this h-principle remains valid for every associated fibration whose fibers are G-homogeneous. In particular, one has the h-principle for holomorphic maps $X \to Y$, where Y is a G-homogeneous space.

0.4.A. *The h-principle and Oka's principle.* Oka's principle (as interpreted by the author) is an expression of an optimistic expectation with regard to the validity of the h-principle for holomorphic maps in the situation where the source manifold is Stein. The above theorem of Grauert as well as more general results proven in the present paper confirm Oka's principle.

0.4.B. **Example.** The space $Y = \mathbf{C}^n - \{0\}$ is GL_n-homogeneous, and so holomorphic maps $X \to Y$ satisfy the h-principle for all Stein manifolds X. If we remove several points from $\mathbf{C}^n$, then the resulting Y is no longer homogeneous, but the h-principle remains true for $n \geq 2$ (compare 0.1.(c)), as we shall see later on. In fact, we shall prove the h-principle for holomorphic maps $X \to Y = \mathbf{C}^n - Z$ for all *algebraic* subvarieties $Z \subset \mathbf{C}^n$ of codimension ≥ 2.

0.5. **Elliptic spaces and sprays.** Intuitively, a space Y is *elliptic* if it contains "sufficiently many" **C**-*lines* that are holomorphic maps $\mathbf{C} \to Y$. Here is an instance of an elliptic property that insures "many" maps $\mathbf{C}^N \to Y$.

0.5.A. *Spray spaces.* Loosely speaking, an (N-dimensional) spray s over Y is a holomorphic family of holomorphic maps $s_y : \mathbf{C}^N \to Y$ such that $s_y(0) = y$ for all $y \in Y$. More precisely, we have an (N-dimensional) vector bundle $p : E \to Y$ and a holomorphic map $s : E \to Y$ which is the identity on the zero section $Y \subset E$. Thus $s = \{s_y : E_y \to Y\}$ for the fibers $E_y = p^{-1}(y)$ $(= \mathbf{C}^N)$. Call s *dominating* at $y \in Y$ if the differential of $s_y : E_y \to Y$ at $0 \in E_y$ is a *surjection* $E_y \to T_y(Y)$. A spray is called *dominating* if it dominates at all points $y \in Y$.

0.5.B. **Examples.** (i) Suppose there are N one-parameter groups of biholomorphisms of Y denoted $(t, y) \to t_i \circ y$ for $(t, y) \in \mathbf{C} \times Y$ and $i = 1, \dots, N$. If the vector fields on Y corresponding to these groups span the tangent bundle $T(Y)$, then the *composed spray* $s : Y \times \mathbf{C}^N \to Y$ defined by $s(y, t_1, \dots, t_N) = t_1 \circ t_2 \circ \cdots \circ t_N \circ y$ dominates everywhere on Y.

(ii) The above construction provides dominating sprays for the complex Lie groups G (where one could also use $s = \exp : T(G) \to G$) and over the G-homogeneous spaces.

(iii) Here is a more interesting example (compare 0.4.B). Let $A \subset \mathbf{C}^n$ be an algebraic subset of codimension ≥ 2, and observe that for every linear map $l : \mathbf{C}^n \to \mathbf{C}^{n-1}$, there exists a *nonzero* polynomial a on $\mathbf{C}^{n-1}$ such that $l \circ a | A = 0$. Denote by ∂_l a nonzero constant vector field on $\mathbf{C}^n$ parallel to the line $\operatorname{Ker} l \subset \mathbf{C}^n$, and observe that the field $\partial_l' = a\partial_l$ preserves $Y = \mathbf{C}^n - A$ and integrates to a one-parameter group. It is clear that there exist finitely many linear maps l_i such that the fields ∂_{l_i}' span the tangent bundle of Y and thus provide a dominating spray over $Y = \mathbf{C}^n - A$.

(iv) Let Y be an *algebraic* manifold which is obtained from $\mathbf{C}P^n$ by a sequence of blow-ups along nonsingular subvarieties. Such a Y does not usually carry regular vector fields, and the group of biholomorphisms of Y is trivial. Yet there exist meromorphic fields which provide a dominating spray $s : E = \bigoplus_{i=1}^N H^{-m_i} \to Y$, where H is an *ample* line bundle over Y and m_i are (large) positive integers. As in the previous example, the construction of s is purely algebraic (see 3.5) and makes sense over an arbitrary field in place of $\mathbf{C}$.

0.6. **Main h-principle.** *Let X be Stein and Y admit a dominating spray. Then every continuous map $X \to Y$ is homotopic to a holomorphic one. Moreover,* *the inclusion between the spaces of maps* $\operatorname{Holo}(X, Y) \subset \operatorname{Cont}(X, Y)$ *is a weak homotopy equivalence, that is, the induced homomorphisms on the homotopy groups are bijective.*

We prove this in §§2 and 4 along with a more general h-principle for *elliptic fiber bundles* over X. The idea of the proof is roughly as follows. Cover X by small convex neighborhoods X_μ, and take a collection of holomorphic maps $f_\mu : X_\mu \to Y$. These maps do not have to agree on the intersections $X_\mu \cap X_\nu$ and need not define any global map $X \to Y$. In order to make the maps f_μ agree, we modify them using our spray $x : E \to Y$. An individual modification of f_μ, called an *s-deformation*, is determined by a section, say α_μ, of the induced bundle $f_\mu^*(E) \to X_\mu$, and is defined as the composition of the following three maps, $X_\mu \to^{\alpha_\mu} f_\mu^*(E) \to E \to^{s} Y$, where the middle arrow is the tautological map. By applying appropriate s-deformations to f_μ, we are able to obtain new local maps, say $f'_\mu : U_\mu \to Y$, which agree on the intersections $U_\mu \cap U_\nu$ and for which the resulting global map $f' : X \to Y$ belongs to a given homotopy class.

Remark. Our s-deformation process is similar to what was done in the original papers [Gra$_2$] and [Ca$_2$] while the arrangement of U_μ follows an idea in [H-L] and uses a positive proper convex function on X.

0.7. **Essential applications of the h-principle.** With the help of the h-principle, one reduces the study of the space $\operatorname{Holo}(X, Y)$ to that of $\operatorname{Cont}(X, Y)$, where many topological techniques are available. In particular, the following topological property of X becomes very useful.

0.7.A. **Lefschetz Theorem.** *Every Stein manifold* X *has homotopy type of an n-dimensional polyhedron for* $n = \dim_{\mathbf{C}} X = \frac{1}{2}\dim_{\mathbf{R}} X$.

Sketch of the proof. The index of a critical point of a *convex* function on X does not exceed n by an elementary linear algebraic argument. Then the result follows by the Morse theory applied to a generic proper positive convex function on X.

0.7.A$'$. **Corollary.** *If* Y *is k-connected, that is, the homotopy groups* $\pi_i(Y)$ *vanish for* $i \le k$, *then the space* $\operatorname{Holo}(X, Y)$ *is* $(k-n)$*-connected provided the h-principle applies.*

0.7.A$''$. **Example.** Take $Y = \mathbf{C}^N - A$ for an algebraic subset A of dimension m. Then, obviously, Y is k-connected for $k = 2N - 2m - 2$. If $m \le N-2$, the h-principle does apply and so the space $\operatorname{Holo}(X, Y)$ is $(2N-2m-2-n)$-connected.

0.7.B. One can take an opposite point of view and regard the h-principle as a tool to express *topological* information on X in holomorphic terms. For example, the h-principle for holomorphic maps $X \to \mathbf{C}^\times$ provides the following description of the first cohomology group $H^1(X; \mathbf{Z})$ in terms of the algebra A of holomorphic functions a on X. Denote by $A^\times \subset A$ the multiplicative subgroup of inevitable elements in A, and notice that $A^\times$ consists of holomorphic maps $a : X \to \mathbf{C}^\times$. The exponential map $\exp : A \to A^\times$ sends A onto the set of those maps $X \to \mathbf{C}^\times$ which lift to the universal covering $\mathbf{C}$ of $\mathbf{C}^\times$ mapped to $\mathbf{C}^\times$ by $z \to \exp z$. Hence

$$A^\times / \exp A = \pi_0(\mathrm{Holo}(X, \mathbf{C}^\times)).$$

If X is Stein, one can substitute " Cont " for " Holo " and then identify π_0 with $H^1(X, \mathbf{Z})$. Thus

$$(*) \qquad H^1(X, \mathbf{Z}) = A^\times / \exp A.$$

This observation is due to Arens and Royden [Ar, Roy] who proved the following version of $(*)$ for an arbitrary Banach algebra B of functions on a compact space S such that $S = \operatorname{Spec} B$. Here $\operatorname{Spec} B$ denotes the set of nonzero homomorphisms $B \to \mathbf{C}$, and the equality sign refers to bijectivity of the *evaluation* map $S \to \operatorname{Spec}$, where each s goes to $h : B \to \mathbf{C}$ defined by $h(b) = b(s)$.

0.7.B$'$. **Theorem.** *The Čech cohomology of* $S = \operatorname{Spec} B$ *satisfies*

$$(**) \qquad \check{H}^1(S; \mathbf{Z}) = B^\times / \exp B.$$

Sketch of the proof. If B is generated by finitely many elements, say by $b_1, \dots, b_n$, then these $b_i : S \to \mathbf{C}$ *embed* S into $\mathbf{C}^n$. The embedding property follows from the equality $S = \operatorname{Spec} B$ which also shows that the image of the embedding, say $\overline{S} \subset \mathbf{C}^n$, admits a basis of neighborhoods $U \supset \overline{S}$ in $\mathbf{C}^n$ which are Stein and whose function algebras approximate B in an appropriate sense.

Thus one deduces $(**)$ from $(*)$ applied to the algebra of holomorphic functions on U, for all Stein neighborhoods U of $\overline{S}$. Finally, one takes care of infinitely generated algebras by using a simple limit argument.

0.7.B$''$. The above theorem was generalized by Eidlin [Eid] (also see [No]) who gave a similar formula for $\check{H}^{\mathrm{odd}}(S)/\mathrm{Torsion}$ and $\check{H}^{\mathrm{even}}(S)/\mathrm{Torsion}$ using holomorphic maps into GL_∞ and $\mathrm{Gr}_k(\mathbf{C}^\infty)$ for $k \to \infty$. (The author first learned about Grauert's theorem from Eidlin in this context.) Yet no such formula is known for $\check{H}^i$ for $i \geq 3$. (For $i = 2$, one can use maps to CP^∞.) The difficulty stems from the following.

Problem. Does there exist an elliptic space Y which is homotopy equivalent to a given compact polyhedron? Here "elliptic" signifies the validity of the h-principle of holomorphic maps $X \to Y$ for all Stein manifolds X.

Example. If we want to take hold of $H^i(X; \mathbf{Z})$, we need an elliptic space of homotopy type of (a finite skeleton of) the Eilenberg-Mac Lane space $K(\mathbf{Z}, i)$. If i is odd, say $i = 2m - 1$, then a promising candidate is the symmetric product $Y_k = S^k Y_0$ for $Y_0 = \mathbf{C}^m - \{0\}$ and large $k \to \infty$.

Unfortunately, Y_k is singular for $m \geq 2$ and $k \geq 2$, and we cannot prove the desired ellipticity by our methods.

Remark. It seems one can give a "holomorphic formula" for H^i by developing a kind of analytic étale cohomology theory.

Let us look at the structure of the paper. Our proof follows the usual route of soft nonlinear analysis, where the major problem is to keep track of domains of definitions of various maps and sections and a uniform control on certain norms of these sections. An abstract language for this purpose is suggested in [Gro], and a systematic use of that kind of formalism would have significantly shortened the length of the present exposition. We take here a less formal approach and do not state our theorems immediately in the most general form but show how these slowly develop from special key cases. On the other hand, we sometimes go into the anatomy of technical lemmas when we feel this may clarify the matter and be useful for future applications (such as indicated in 5.4). For a similar reason, we discuss in §3 various generalizations and modifications of the idea of ellipticity without proving (or even stating) any theorem.

1. Basic properties of s-deformation

From now on, we adopt the fiber bundles point of view, and we fix our fibration, that is, a holomorphic map $h: Z \to X$. We deal exclusively with *vertical* sprays $s: E \to Z$, where each fiber E_z of E is mapped by s to a single fiber of $Z \to X$. In other words, $h(e) = h(p(e))$ for all $e \in E$, where $p: E \to Z$ denotes the implied projection of the vector bundle E over Z.

1.1. **s-Homotopies and fiber dominating sprays.** A point $z' \in Z$ is called an *s-deformation of* $z \in Z$ if it is contained in the image $s(E_z) \subset Z$. Two points z and z' are called *s-homotopic* if there exist intermediate points, say $z_0 = z, z_1, z_2, \dots, z_l = z'$, such that z_{i+1} is an s-deformation of z_i for $i = 0, 1, \dots, l-1$.

1.1.A. **Proposition** (Compare St_2 in 0.2). *Let $f_0: X \to Z$ be a holomorphic section, $\{x_j\} \subset X$, $j = 1, 2, \dots$, a discrete subset, and let $z'_j \in Z$ be s-homotopic to $z_j = f_0(x_j)$ for $j = 1, 2, \dots$. If X is Stein, then there exists a holomorphic section $f: X \to Z$ homotopic to f_0, such that $f(x_j) = z'_j$ for all j.*

Proof. The s-homotopy of $\{z_j\}$ to $\{z'_j\}$ can be achieved by a sequence of s-deformations moving only one point at a time. We denote the result of the kth move by $\{z_{j,k}\}$, let $j(k)$ denote the (single) index for which $z_{j,k+1} \neq z_{j,k}$, and assume that $j(k)$ is monotone in $k = 0, 1, \dots$. Suppose we have already constructed a section $f_k: X \to Z$ such that $f_k(x_j) = z_{j,k}$. Since $z_{j,k+1}$ is an s-deformation of $z_{j,k}$, there exists (by basic properties of vector bundles over Stein manifolds, see [G-R] and 1.1.A$'$ below) a section α_k of the bundle $p: E \to Z$ restricted to $f_k(X) \subset Z$, say $\alpha_k: X \to \widetilde{E}_k = E|f_k(X)$, such that

(i) $s \circ \alpha_k(x_j) = z_{j,k+1}$ for $j = j(k)$,

(ii) $\alpha_k(x_j) = 0$ for $j \neq j(k)$,

(iii) $\|\alpha_k\| \mid X_k \leq 2^{-k}$,

where $X_k \subset X$ are some compact subsets exhausting X and $\| \; \|$ is a fixed norm in the bundle $E \to Z$.

Now we take $f_{k+1} = s \circ \alpha_k$ and $f = \lim_{k\to\infty} f_k$. Q.E.D.

1.1.A$'$. *Remark.* If the subset $\{x_j\} \subset X$ is finite, then (iii) above is redundant. The only property of vector bundles over X we need in this case is the existence of a holomorphic section α with prescribed values at the points x_j.

1.1.B. *Submersions, vertical bundles, and sprays.* Let us assume that $h : Z \to X$ is a submersion, that is, the differential of h is everywhere surjective. In this case, the fibers $h^{-1}(x) \subset Z$ are smooth analytic submanifolds in Z for all $x \in X$. A vertical spray $s : E \to Z$ is called *fiber dominating* if the differential of s at $0 \in E_z$ sends E_z (or rather $T_0(E_z)$) onto the tangent space of the fiber of Z through z for all $z \in Z$. For fiber dominating sprays, the s-homotopy obviously reduces to the fiber homotopy. Namely, points z and z' are s-homotopic if and only if they lie in the same connected component of a fiber of $Z \to X$. For example, if the fibers of $Z \to Z$ are connected, then by 1.1.A every section can be homotoped to another which takes given values on the set $\{x_j\} \subset X$. In particular, if $Z = X \times Y \to X$ and sections $X \to Z$ correspond to maps $X \to Y$, one can start with a *constant* map f_0 and then get a holomorphic map f assuming given values on $\{x_j\}$.

1.1.B$'$. *Notations* $\mathrm{VT}(Z)$ *and* $D_s(0)$. The first is the *vertical* (or *fiberwise*) tangent bundle of $h : Z \to X$, that is, the kernel of the differential of h,

$$\mathrm{VT}(Z) = \operatorname{Ker} Dh \subset T(Z).$$

The fiberwise property of s implies that the *differential* of s on the zero section $Z \subset E$ sends $E \subset T(E) \mid Z$ to $\mathrm{VT}(Z) \subset T(Z)$. This is denoted by

$$D = D_s = D_s(0) : E \to \mathrm{VT}(Z).$$

Now the domination condition can be expressed by saying D is *surjective.*

1.2. **Homotopies and s-homotopies over compact subsets.** An s-deformation of a holomorphic section $f_0 : X_0 \to Z$, where $X_0 \subset X$ is a complex submanifold, is defined as a map $f : X_0 \to Z$ of the form $f = s \circ \alpha_0$ for some holomorphic section α_0 of the fibration $E|f_0(X_0)$ over $X_0 = f_0(X_0) \subset Z$. Then f and f' are called *s-homotopic* if there exist sections $f_0 = f, f_1, \dots, f_l = f'$ such that f_{i+1} is an s-deformation of f_i for $i = 0, \dots, l-1$.

1.2.A. Let $X_0 \subset X$ be an open relatively compact subset and f and f' holomorphic sections of $Z|X_0$. Let $X_1 \subset X$ be a larger open subset containing the closure of X_0, and suppose f and f' extend to holomorphic sections F and F' of $Z|X_1$ which are *homotopic* over X_1 by a homotopy of holomorphic section of $Z|X_1$. Using such homotopy, we want to produce an *s-homotopy* between f and f' in the case where $h : Z \to X$ is a submersion and the spray s is fiber dominating, that is, the differential $D = D_s(0) : E \to \mathrm{VT}(Z)$ is a *surjective* homomorphism (compare 1.1.B$'$). To clarify the ideas, we start with the following stronger assumption on D.

1.2.A$'$. *If D is bijective, then f and f' are s-homotopic.*

Proof. Since D is bijective, the map s restricted to $E_f = E|f(X_0)$ is a biholomorphism of some neighborhood of the zero section of E_f onto a neighborhood of $f(X_0) \subset Z$. It obviously follows that the map $\alpha \mapsto s \circ \alpha$ is a bijection of the space of small holomorphic sections α of $E|f(X_0)$ onto a neighborhood of f in the space of sections $X_0 \to Z$ with the uniform topology. In other words, every small deformation of f is an s-deformation. Now we prove our assertion by dividing the homotopy between F and F' over X_1 into sufficiently small steps which are reducible over X_0 to s-deformations.

1.2.A$''$. *Remark.* "Division" of a deformation F_t, $t \in [0, 1]$, means a division of $[0, 1]$ into smaller intervals, say into $[t_i, t_{i+1}]$, $i = 1, \dots, k$. Then our s-homotopy is uniquely determined by sections α_i of E restricted to $F_{t_i}(X_0) \subset Z$. If the intervals $[t_{i+1}, t_i]$ are sufficiently small, then α_i is obtained as the unique small solution of the equation $s \circ \alpha_i = F_{t_{i+1}}$ over X_0.

Notice that this $\alpha_i = \alpha_i(F_t)$ depends *continuously* on F_t.

1.2.B. Now we turn to the general case, where the homomorphism $D = D_s : E \to \mathrm{VT}(Z)$ is assumed surjective but not necessarily bijective.

If X_1 (containing the closure of $X_0 \subset X$) is Stein, then f and f' are s-homotopic over X_0.

Proof. The easiest case is where the manifold Z is Stein. Then every surjective homomorphism over Z admits a right inverse (see [G-R]). In particular, there exists a holomorphic homomorphism $\delta : \mathrm{VT}(Z) \to E$ (inverting D) such that $D \circ \delta = \mathrm{Id} \,|\, \mathrm{VT}(Z)$, and our claim follows from 1.2.A$'$ applied to s restricted to the image of δ.

In the general case where X_1 rather than Z is assumed Stein, we may still construct δ over $X_1 = F(X_1) \subset Z$. Moreover, if F_t is a homotopy of holomorphic sections over X_1 between F and a, we can find (according to the standard theory of Stein manifolds) a continuous family of holomorphic homomorphisms, say $\delta_t : V_t \to E_t$, where V_t denotes the restriction of the bundle $\mathrm{VT}(Z)$ to $X_1 = F_t(X_1) \subset Z$, i.e., $V_t = \mathrm{VT}(Z)|F_t(X_1)$ and $E_t = \mathrm{VT}(Z)|F_t(X_1)$ such that $D \circ \delta_t = \mathrm{Id}$ over $F_t(X_1)$ for all t. The existence of δ_t implies that every *small* deformation of $f_t = F_t|X_0$ is an s-deformation, and then by dividing the homotopy F_t into small steps as in 1.2.A$'$, we construct the desired s-homotopy between f and f'.

1.2.B$'$. *Remark.* The essential ingredient of the above discussion is an implicit function theorem which allows us to pass from surjectivity (or bijectivity) of D_s to that of the map $\alpha \mapsto s \circ \alpha$ between pertinent spaces of sections. Let us indicate a more general implicit function theorem of this type for holomorphic fiber preserving maps $\sigma : \tilde{Z} \to Z$ between submersions $\tilde{Z} \to X$ and $Z \to X$. We consider holomorphic sections $f_0 : X \to Z$ and $\tilde{f}_0 : X \to \tilde{Z}$, where $\tilde{f}_0 = \sigma \circ \tilde{f}_0$, and we ask ourselves whether a small perturbation f of f_0 can be covered by a section $\tilde{f}$ of $\tilde{Z}$ close to f_0. Here is an answer for Stein manifolds X.

1.2.B$''$. *Let X be Stein, the map σ a submersion, and take an open relatively compact subset $Y \subset X$. Then there exists a neighborhood Ω_0 of f_0 in the space of holomorphic sections $X \to Z$ such that every $f \in \Omega_0$ can be covered by a holomorphic section $\tilde{f}$ of $\tilde{Z}$ over Y. That is, $\tilde{f}$ is a section $Y \to \tilde{Z}$*

such that $\sigma \circ \tilde{f}_0 = f|Y$. *Moreover, one can choose* $\tilde{f}$ *continuously* (*and even holomorphically*) *depending on* $f \in \Omega_0$.

The proof follows from the standard theory of Stein manifolds which shows that the map between the pertinent function spaces induced by σ, that is, $\tilde{f} \to f = \sigma \circ \tilde{f}_0$, is a *submersion* in an appropriate sense. Then an implicit function argument delivers a section $\tilde{f}$ satisfying $\sigma \circ \tilde{f}_0 = f|Y$.

1.3. **Composed bundles and sprays.** Consider the holomorphic vector bundles $p_1 : E_1 \to Y$ and $p_2 : E_2 \to E_1$, and let $\bar{p}_2 = p_1 \circ p_2 : E_2 \to Y$. This *composed bundle* $E_2 \to Y$ does not have any canonical vector bundle structure (though it can often be given one; see 1.3.A$'$ below). This remark suggests the following.

1.3.A. **Definition.** A composed bundle over Y is a holomorphic fibration $\bar{p} : \overline{E} \to Y$ with a given decomposition $\bar{p} = p_1 \circ p_2 \circ \cdots \circ p_k$ for some vector bundles $p_1 : E_1 \to Y$, $p_2 : E_2 \to E_1, \ldots, p_k : E_k = \overline{E} \to E_{k-1}$.

1.3.A$'$. *If* Y *is Stein, then every composed bundle* $\overline{E}$ *over* Y *admits a holomorphic vector bundle structure.*

This follows from the standard Stein theory which, in fact, provides a vector bundle structure which agrees in a natural way with the partial linear structure in $\overline{E}$ (this aspect is not relevant for the moment).

1.3.B. *Composed sprays.* Consider two sprays over Z, say (E_1, p_1, s_1) and (E_2, p_2, s_2), and define *the composed spray* $(E = E_1 * E_2,\ p = p_1 * p_2,\ s = s_1 * s_2)$ as follows.

$$E = \{(e_1, e_2) \in E_1 \times E_2 |\ s_1(e_1) = p_2(e_2)\}.$$

In other words, E is the total space of the fibration $s_1^*(E_2) \to E_1$ induced by s_1 from $p_2 : E_2 \to Z$.

Next we define

$$p_1 * p_2(e_1, e_2) = p_1(e_1)$$

and

$$s_1 * s_2(e_1, e_2) = s_1(e_2).$$

Notice that the fiber E_z is the total space of a vector bundle over a vector space. But E_z is *not* a vector space although it is (noncanonically!) biholomorphic to one. Thus, in general, the composed bundle $p : E \to Z$ is not a vector bundle. However, according to 1.3.A$'$, the restriction of E to any Stein submanifold $Y \subset Z$ does carry a structure of a vector bundle. In fact, this bundle is isomorphic to the *Whitney sum* $E_1 \oplus E_2|Y$, where the implied isomorphism $E_1 * E_2 \leftrightarrow E_1 \oplus E_2$ agrees with the partial linear structure and the decomposition in $E_1 * E_2$.

1.3.C. *Composition of deformation.* Let $(E^{(k)}, p^{(k)}, e^{(k)})$ denote the composition of k copies of a given spray s over Z. Then an s-homotopy between sections divided into k s-deformations becomes an $s^{(k)}$-deformation in a canonical (and obvious) way.

1.3.D. Let us define the *infinite* composition $E^{(\infty)}$ by embedding each $E^{(k)}$ to $E^{(k+1)}$ by the zero section and then by taking the union $E^{\infty} = \bigcup_{k=1}^{\infty} E^{(k)}$.

This space is similar to the space of piecewise geodesic paths in a Riemannian manifold, and our lemma in 1.2.B furnishes a holomorphic version of the homotopy lifting (Serre fibration) property of $p^{(\infty)} : E^{(\infty)} \to Z$ over *relatively compact* subsets $X_0 \subset X$.

Question. Let X be Stein and s dominating (as in 1.2.B). Does $p^{(\infty)} : E^{(\infty)} \to Z$ satisfy the holomorphic homotopy lifting property over X? That is, let $f_t : X \to Z$, $t \in [0, 1]$, be a homotopy of holomorphic sections. Then are f_0 and f_1 s-homotopic?

We shall see in 3.5.G that this question has some points in common with the celebrated conjecture by Serre (solved by Suslin and Quillen) concerning projective moduli over the rings of polynomials.

1.4. **Runge approximation property.** A fibration $E \to X$ is called *Runge* over an open subset $U \subset X$, or, equivalently, U is called *Runge for* (*sections of*) E if every holomorphic section $U \to E$ admits an *approximate* holomorphic extension to all of X. That is, the image of the restriction map $\mathrm{Holo}(X, E) \to \mathrm{Holo}(U, E)$ has *dense image* in $\mathrm{Holo}(U, E)$ for the topology of uniform convergence on compact subsets in U. Here are some examples.

1.4.A. **Classical Runge Theorem.** *An open subset $U \subset \mathbf{C}$ is Runge for holomorphic functions if and only if $\mathbf{C} - U$ is connected* (*see* [G-R]).

This generalizes to higher dimensions with the notion of a (globally) $\mathbf{C}$-*convex* subset as follows.

1.4.A$'$. **Runge for convex subsets in Stein manifolds.** *Let $p : X \to \mathbf{R}$ be a convex* (*see* 0.3) *function. Then the* (*level*) *subsets $X_t = \{x \in X \mid p(x) < t\}$ are Runge for all vector bundles* (*and hence for composed bundles*) *over X* (*see* [G-R]).

Another standard fact reads

1.4.B. *If X and U are both Stein, then the Runge property for functions $X \to \mathbf{C}$ is equivalent to that for sections of every vector bundle* (*and hence every composed bundle*) *over X.* In fact, if we assume only U is Stein, then Runge for functions implies that for sections of a vector bundle $E \to X$ provided there exist holomorphic sections $h_i : X \to E$, $i = 1, \dots, k$ which span the fiber E_x for all $x \in U$. To see this, we view $h = (h_1, \dots, h_k)$ as a homomorphism of the trivial bundle of rank k to E. Since h is surjective over U, it admits a right inverse for U Stein (see [G-R]) and our claim follows. (Notice that if X is Stein, there exists such a *surjective* h over all X; see [G-R].)

1.4.C. *h-Runge property.* The h-Runge (h for homotopy) property claims that the approximate extension of sections from U to $X \supset U$ is invariant under homotopies of sections. Namely, if some holomorphic section f_0 over U holomorphically extends to a section $\tilde{f}_0$ on X and if f_1 can be joined with f_0 by a homotopy of holomorphic sections over U, then f_1 can be approximated by holomorphic sections f which extend to holomorphic sections $\tilde{f}$ on X and which can be joined with $\tilde{f}_0$ by homotopies of holomorphic sections. Notice that "h-Runge" is equivalent to "Runge" if the implied fibration is a vector bundle, and so any two sections are homotopic.

1.4.C$'$. *Runge for compact subsets.* It is often more convenient to work with *compact* rather than open subsets in X. Given such a subset $C \subset X$, a holomorphic section over C (or near C) refers to a section defined in a (small) neighborhood $U \supset C$ in X. Two sections are called *close* if their restrictions to C are close in the uniform topology. With these preliminaries, we can speak of the Runge property for $C \subset X$. We also speak about *h-Runge for* C with the following additional convention: a homotopy of section over (i.e., near) C refers to sections f_t defined on a *fixed* (independent of t) neighborhood U of C, and f_t is continuous in t for the usual topology in the space of sections on U (compare 1.5). Notice that the Runge and the h-Runge properties for an open subset $U \subset X$ follow from those for compact subsets $C_i \subset X$, $i = 1, \dots,$ such that $\bigcup_{k=1}^{\infty} C_i = U$.

1.4.D. **Theorem.** *Let $Z \to X$ be a submersion with a spray $s: E \to Z$, and let $C \subset X$ be a compact subset which is h-Runge for the composed bundles over X. Then C is h-Runge for Z provided one of the following two conditions is satisfied.*

(1) *The differential $D = D_s(0)$ of s (at the zero section) is an isomorphism $E \to \mathrm{VT}(Z)$ (compare* 1.2.A$'$.).
(2) *D is surjective (i.e., s is fiber dominating), and C admits arbitrarily small Stein neighborhood $U \supset C$ in X (i.e., C admits a basis of Stein neighborhoods in X).*

Proof. We know in both cases that homotopy of sections over C implies s-homotopy (see 1.2). Namely, if f_1 is homotopic to f_0 over C, then $f_1 = s^{(k)} \circ \alpha$, where $\alpha = (\alpha_1, \dots, \alpha_k)$ is a section of the composed bundle $E^{(k)}$ restricted to $f_0(U)$ for a small $U \supset C$. In the first case, α depends *continuously* on the implied homotopy f_t by 1.2.A$''$. Hence α is homotopic to the zero section (as f_t can be deformed to the constant homotopy $f_t = f_0$ by sending $[0, 1] \to [0, 1-\tau]$ and by moving τ from 0 to 1), and the h-Runge for $E^{(k)}$ over $f_0(X)$ gives us an approximate extension $\tilde{\alpha}$ of α to $X = f_0(X)$. Then the composition $s^{(k)} \circ \tilde{\alpha}$ provides the required approximate extension of f_1 to X.

In the second case, the section α is not unique (see 1.2.B), but as U is Stein, the composed bundle $E^{(k)}$ over $f_0(U)$ admits a vector bundle structure, and α is homotopic to the zero section anyway. Therefore, the h-Runge for $E^{(k)}$ over $f_0(X)$ implies that for Z. Q.E.D.

1.4.D$'$. **Example.** Let Y be a manifold with a dominating spray, let X be Stein, and $X_0 \subset X$ be an open *globally* **C**-*convex* (in the sense of 1.4.A$'$) subset which is also holomorphically *contractible* (see 2.1). For example, all these conditions are satisfied if $X = \mathbf{C}^n$ and X_0 is **R**-convex (i.e., convex in the usual sense). Then every holomorphic map $f_0: X_0 \to Y$ is holomorphically contractible, and therefore, f_0 can be approximated by holomorphic maps extendable to X. In fact, X_0 is Runge for vector bundles (see 1.4.A$'$), and so 1.4.D applies.

1.4.E. *Remark.* Let us indicate another version of the above argument which avoids composed bundles. First, we notice that the h-Runge property is an approximate version of the homotopy lifting property for the restriction map

$$\operatorname{Holo}(X, Z) \to \operatorname{Holo}(\mathscr{O}\!\mathscr{P} A, Z),$$

where $\mathscr{O}\!\mathscr{P}$ stands for an "arbitrarily small neighborhood of" (compare 1.5).

A well-known and obvious argument allows a localization of lifts as follows. Every path f_t in $\operatorname{Holo}(\mathscr{O}\!\mathscr{P} A, Z)$ is divided into (arbitrarily) small pieces by partitioning the interval $[0, 1]$ into $k+1$ equal pieces by $t_j = j|k$, for $j = 0, \dots, k$, and large k. Then an arbitrarily small perturbation of f_t lifts to a path $\tilde{f}_t$ in $\operatorname{Holo}(X, Z)$ by induction on j as follows. We assume the existence of an approximate lift $\tilde{f}_t$ for $t \in [0, t_j]$, where $\tilde{f}_t|\mathscr{O}\!\mathscr{P} A$ is arbitrarily close to f_t, and then we invoke our spray over $X = X_j = f_{t_j}(X)$. Since the interval $[t_j, t_{j+1}]$ is small and $\tilde{f}_{t_j}|\mathscr{O}\!\mathscr{P} A$ is close to f_{t_j}, one can lift the homotopy f_t for $t \in [t_j, t_{j+1}]$ to $E_j = E|X_j$. Namely, there exists a homotopy of sections $\alpha_t \colon X_j \to E_j$ for $t \in [t_j, t_{j+1}]$, such that $s \circ \alpha_t = f_t$ and where the sections α_t are close to the zero section $\mathscr{O}\!\mathscr{P} A \to E_j$. Then the Runge property of E_j allows an approximate extension of α_t which yields the inductive step $j \Rightarrow j+1$.

1.4.E$'$. *Partial sprays controlled on* A. The above argument uses the spray s only over $f(X) \subset Z$ for pertinent holomorphic sections $f \colon X \to Z$. These partial sprays must dominate Z over $\mathscr{O}\!\mathscr{P} A$, and "the quality of domination" must be uniform as $\tilde{f}$ approaches f on A. In other words, the spray over $\tilde{f}(\mathscr{O}\!\mathscr{P} A)$ must be "controlled" by the behavior of $\tilde{f}$ on A in order to insure the existence of $\varepsilon > 0$ depending on f_t but not on $\tilde{f}_{t_j}$ such that the above α_t exists over the interval $[t_j, t_j + \varepsilon] \in t$. With these provisions, one can make the induction work for $k > \varepsilon^{-1}$. (We invite the reader to furnish precise definitions and proofs.)

1.4.E$''$. *On removing the control condition.* It seems plausible that the control condition is unneeded if one allows *base change.* That is, for every Stein manifold X' holomorphically mapped into Z, one requires the existence of a vector bundle $E' \to X'$ and of a dominating spray $E' \to Z$ over $X' \to Z$. This property (for the trivial bundle $Z = X \times Y \to X$) is called Ell_1 in [Gro] where the Runge property is stated in Exercises (d), (e), and (e$'$) in [Gro, p. 72]. As we could not solve these, we call them now *conjectures* rather than exercises.

1.4.F. *Lifts of homotopies to sprays.* One can go around the control problem by using composed sprays as follows. Consider a holomorphic section $F_0 \colon X \to Z$ and a homotopy of F_0 over C that is a homotopy of holomorphic sections $f_t \colon \mathscr{O}\!\mathscr{P} C \to Z$ such that $f_0 = F_0|\mathscr{O}\!\mathscr{P} C$. We say that f_t *can be lifted to a* *spray over* X (compare 1.3.D) if there exist a vector bundle $\widetilde{E} \to X$ and a holomorphic map $\widetilde{F} : \widetilde{E} \to Z$ with the following three properties

1. $\widetilde{F}$ equals F_0 on the zero section $X_0 = X \subset \widetilde{E}$.
2. $\widetilde{F}$ sends fibers of $\widetilde{E}$ to those of Z.
3. There exists a homotopy of holomorphic sections $\tilde{f}_t : \mathcal{OP}C \to \widetilde{E}$ such that $\tilde{f}_0 = 0$ and $\widetilde{F} \circ \tilde{f}_t = f_t$.

One can slightly generalize this definition by allowing composed bundles $\widetilde{E}$. Or one can be more restrictive and insist on trivial bundles $\widetilde{E} = X \times \mathbf{C}^N \to X$. However, these modifications do not affect our applications where X is a *Stein* manifold, and one can easily pass from one class of bundles to another.

1.4.F$'$. Using the above definition, the proof of the h-Runge Theorem can be divided into two steps.

Step 1. Lifting f_t to some $\widetilde{E}$ over X. This is accomplished in §§1.2 and 1.3 using a self-composition of a dominating spray over Z.

Step 2. Descending the h-Runge property from $\widetilde{E}$ to Z. This step is completely trivial, but it is useful to have it separated from Step 1.

1.5. **Cartan pairs.** In what follows, we deal with holomorphic functions on a small nonspecified neighborhood U of a subset $A \subset X$, where U may become even smaller in the course of an argument. Such a small variable neighborhood is denoted by $\mathcal{OP}A \subset X$ and dealt with according to the obvious rules (see [Gro, p. 36]). Sometimes we speak of functions "near A" meaning functions on $\mathcal{OP}A$.

1.5.A. **Definition.** A pair of compact subsets A and B in X is called *Cartan*, or a *C-pair*, if for every holomorphic function c near $C = A \cap B$ there exist holomorphic functions a near A and b near B such that $(a - b) \mid \mathcal{OP}C = c$. We require, moreover, that a and b can be chosen in a sufficiently canonical manner. Namely, if c_t is a continuous (holomorphic) family of sections, then the corresponding a_t and b_t can be chosen continuous (holomorphic) in t. We also agree to take $a = b = 0$ whenever $c = 0$. Since every function c is a member of the family tc for $t \in [0, 1]$, we can find a and b such that their L_∞-norms are bounded by that of c. In fact, we require a and b to be such that $\|a\| \mid \mathcal{OP}A$ and $\|b\| \mid \mathcal{OP}A$ are bounded by const$\|c\|$, where the constant depends only on A and B.

1.5.A$'$. *Cartan for convex sets.* It is well known that convexity of A and B implies Cartan's property (see [G-R]). We shall need this for globally convex sets (see 1.4.A$'$) of the form $\{x \in X \mid p(x) \le \text{const}\}$ for proper positive convex functions p on X, but Cartan's property holds even for *locally convex subsets* (which may be non-Runge).

1.5.B. Consider a trivial fibration $E' = X' \times \mathbf{C}^N \to X'$ for $X' = \mathcal{OP}(A \cup B) \subset X$. Let $U \subset E'$ be a neighborhood of $C = C \times 0$, and let $\varphi : U \to E'$ be a *fiber preserving* holomorphic map, i.e., $\varphi(x', z) = (x', \psi(x', z))$ for all $(x', z) \in U \subset X' \times \mathbf{C}^N$.

1.5.C. **Lemma.** *If (A, B) is a C-pair and φ is sufficiently close to the identity map $U \to U \subset E'$, then there exist holomorphic sections $\alpha' : \mathcal{OP}A \to E'$ and $\beta' : \mathcal{OP}B \to E'$ such that $\alpha'(C) \subset U$ and $\varphi \circ \alpha' | \mathcal{OP}C = \beta' | \mathcal{OP}C$, where the required closeness of φ to* Id *depends on U as well as on A, B, and N.*

Proof. Start with the zero sections α'_0 near A and β'_0 near B and define c_1 near C by $c_1 = \varphi \circ \alpha'_0 - \beta'_0$. Now we invoke Definition 1.5.A, write $c_1 = a_1 - b_1$, and set

$$\alpha'_1 = \alpha'_0 - a_1 \quad \text{and} \quad \beta'_1 = \beta'_0 - b_1 .$$

Then we define $c_2 = \varphi \circ \alpha'_1 - \beta'_1$ and observe that the L_∞-norm of c_2 on $\mathcal{OP}C$ is bounded by

$$\begin{aligned} \|c_1\| &= \|\varphi \circ (\alpha'_0 - a_1) - \beta'_0 + b_1\| = \|\varphi \circ (\alpha'_0 - a_1) - \beta'_0 + a_1 - c_1\| \\ &= \|\varphi \circ (\alpha'_0 - a_1) + a_1 - \varphi \circ \alpha'_0\| \le \|D_\varphi - \mathrm{Id}\| \, \|a_1\| + o(\|a_1\|), \end{aligned}$$

where the implied constant in $o(\|a_1\|)$ depends on $\|D^2_\varphi\|$ over $\mathcal{OP}C$. When φ is close to Id, the same is true for the differentials D_φ and D^2_φ (on a smaller neighborhood), and since $\|a_1\| \le \mathrm{const}\, \|c_1\|$, we can get

$$\|c_2\| \le \delta \|c_1\|,$$

for an arbitrarily small $\delta > 0$ by choosing φ sufficiently close to the identity. The same estimate remains true if we continue the iteration process with $\alpha'_2, \alpha'_3, \ldots$, and $\beta'_2, \beta'_3, \ldots$. Namely,

$$\|c_{i+1} = \varphi \circ \alpha'_i - \beta'_i\| \le \delta \|c_i\|$$

for a small δ, say $\delta = \frac{1}{2}$. Therefore, α'_i and β'_i converge to the required α' and β' as $i \to \infty$. Q.E.D.

1.5.C$'$. *Control on $\|\alpha'\|$ and $\|\beta'\|$.* It is clear from the proof that α' and β' can be made arbitrarily small by choosing φ sufficiently close to the identity. In other words, if $\varphi \to \mathrm{Id}$, then α' and β' converge to zero.

1.5.C$''$. *Remark.* The implicit function argument used in the proof of 1.5.C is standard and similar to Cartan's proof of the multiplicative decomposition lemma stated below.

1.5.D. *Multiplicative decomposition.* The classical multiplicative Cartan lemma claims the C-property for holomorphic maps into a complex Lie group G. Namely, if $c : \mathcal{OP}C \to G$ is a map close to the identity, then there exist a and b defined on $\mathcal{OP}A$ and $\mathcal{OP}B$, respectively, such that $ab \mid \mathcal{OP}C = c$. A slight adjustment of the proof of Lemma 1.5.C provides a similar decomposition for φ. That is, $\varphi = \varphi_1 \circ \varphi_2$, where φ_1 is defined on $\mathcal{OP}(A \times 0) \subset E'$ and φ_2 on $\mathcal{OP}(B \times 0) \subset E'$. Notice that the role of G is played here by the (infinite dimensional) pseudogroup of biholomorphisms of E'.

One arrives at another instance of Cartan's lemma if one takes a complex subvariety $X' \subset E'$ and takes for G the group of germs of biholomorphisms of $\mathcal{OP}X' \subset E$ fixing X'.

1.6. **Gluing sections over Cartan pairs.** Let us return to our submersion $Z \to X$ with a dominating spray $s: E \to Z$, and let $(A, B, C = A \cap B)$ be a C-pair in X. Consider holomorphic sections $\alpha_0 : \mathscr{O}\mathscr{P} A \to Z$ and $\beta_0 : \mathscr{O}\mathscr{P} B \to Z$ whose restrictions to $\mathscr{O}\mathscr{P} C$ are *mutually close* (this is made more precise later on), and try to construct s-deformations α of α_0 and β of β_0 which are equal on $\mathscr{O}\mathscr{P} C$,

$$(*) \qquad \alpha | \mathscr{O}\mathscr{P} C = \beta | \mathscr{O}\mathscr{P} C .$$

1.6.A. To make the ideas clear, we start with the case where

(i) the bundle E over Z is trivial, say $E = Z \times \mathbf{C}^m \to Z$;

(ii) $\dim \mathrm{VT}(Z) = \dim E$.

Since s is dominating, condition (ii) implies

(ii)$'$ the differential of the spray $s: E \to Z$ at the zero section of E is an isomorphism of E onto the vertical tangent bundle $\mathrm{VT}(Z)$ of $Z \to X$.

Next we consider the restrictions of $E = Z \times \mathbf{C}^m$ to $\alpha_0(\mathscr{O}\mathscr{P} A) \subset Z$ and to $\beta_0(\mathscr{O}\mathscr{P} A)$ and thus get two trivial bundles, say

$$E_1' = (\mathscr{O}\mathscr{P} A) \times \mathbf{C}^m \to \mathscr{O}\mathscr{P} A = \alpha_0(\mathscr{O}\mathscr{P} A)$$

and

$$E_2' = (\mathscr{O}\mathscr{P} B) \times \mathbf{C}^m \to \mathscr{O}\mathscr{P} B = \beta_0(\mathscr{O}\mathscr{P} B) .$$

We denote by s_1 and s_2 the restriction of s to these bundles and observe with (ii)$'$ that these maps $s_1 : E_1' \to Z$ and $s_2 : E_2' \to Z$ are bijective near their respective zero sections $\mathscr{O}\mathscr{P} A \to E_1'$ and $\mathscr{O}\mathscr{P} B \to E_2'$. Now we invoke our condition of α_0 being close to β_0 on $\mathscr{O}\mathscr{P} C \subset \mathscr{O}\mathscr{P} A \cap \mathscr{O}\mathscr{P} B$, which makes s_1 and s_2 *close over* $\mathscr{O}\mathscr{P} C$. This means the maps s_1 and s_2 are close on some neighborhood $U_0 \subset \mathscr{O}\mathscr{P} C \times \mathbf{C}^m$ of $C = C \times 0 \subset \mathscr{O}\mathscr{P} C \times \mathbf{C}^m$. We may assume that s_1 and s_2 are biholomorphisms on U_0 and that there exists a smaller neighborhood $U \subset U_0$ of $C \subset U_0$ such that

$$s_2(C) \subset s_1(U) \subset s_2(U_0) .$$

(The existence of U just needs s_1 and s_2 to be close on U_0, which is achieved by assuming α_0 and β_0 are sufficiently close on $\mathscr{O}\mathscr{P} C$). Finally, we apply 1.5.C to the trivial bundle $E' = \mathscr{O}\mathscr{P}(A \times B) \times \mathbf{C}^m \supset \mathscr{O}\mathscr{P} C \times \mathbf{C}^m \supset U_0$ and $\varphi = s_2^{-1} \circ s_1 : U \to U_0 \subset E'$, thus obtaining sections $\alpha' : \mathscr{O}\mathscr{P} A \to E_1' \subset E'$ and $\beta' : \mathscr{O}\mathscr{P} B \to E_2' \subset E$ satisfying $\beta' = \varphi \circ \alpha'$ near C. Hence, $s_2 \circ \beta' = s_2 \circ \varphi \circ \alpha' = s_2 \circ s_2^{-1} \circ s_1 \circ \alpha' = s_1 \circ \alpha'$, which is exactly the required agreement of $\alpha = s_1 \circ \alpha'$ and $\beta = s_2 \circ \beta'$ near C. Q.E.D.

1.6.B. Here we still assume E is trivial, $E = Z \times \mathbf{C}^m \to Z$, but we drop the assumption $\dim \mathrm{VT}(Z) = \dim E$. Now we want to define φ as a solution to the equation

$$(**) \qquad s_2 \circ \varphi = s_1 \quad \text{on } U_0 .$$

Since s is dominating, the maps s_1 and s_2 are submersions $U_0 \to Z$ which are close to one another as we assume α_0 and β_0 to be close on $\mathscr{O}\mathscr{P} C$. Therefore,

$(**)$ is solvable on a slightly smaller neighborhood U of $C \subset U_0$ provided U_0 is a Stein manifold (see 1.2.B″). Then with such a φ, the proof goes along as in 1.6.A.

To complete the discussion, we notice that we have a great deal of freedom in choosing the neighborhood U_0 of $C = C \times 0 \subset \mathcal{OP}C \times \mathbf{C}^m$. In particular, if $\mathcal{OP}C \subset X$ is Stein (i.e., if $C \subset X$ admits an arbitrarily small Stein neighborhood $\mathcal{OP}C \subset X$), then one clearly can choose U_0 Stein also. Thus *the required s-deformations α and β do exist in the case where $E \to Z$ is trivial and $\mathcal{OP}C \subset X$ is Stein.*

1.6.C. Now we impose no triviality assumption on the bundle $E \to Z$, and we try to replace E by a trivial bundle over a pertinent part of the space Z. Namely, we look for trivial bundles $E_1' = \mathcal{OP}A \times \mathbf{C}^m \to \mathcal{OP}A$ and $E_2' = \mathcal{OP}B \times \mathbf{C}^m \to \mathcal{OP}B$ and holomorphic homomorphisms $h_1 : E_1' \to \alpha_0^*(E) = E|\alpha_0(\mathcal{OP}A)$ and $h_2 : E_2' \to \beta_0^*(E) = E|\beta_0(\mathcal{OP}B)$ (where we identify $\mathcal{OP}A = \alpha_0(\mathcal{OP}A)$ and $\mathcal{OP}B = \beta_0(\mathcal{OP}B)$) such that h_1 and h_2 are surjective over $\mathcal{OP}C$ and h_1 is close to h_2 over $\mathcal{OP}C$ insofar as α_0 is close to β_0 over $\mathcal{OP}C$. Let us give a list of conditions which is sufficient for the existence of h_1 and h_2.

(i) *$\mathcal{OP}C$ is Stein.* In this case, there exist surjective homomorphisms of the trivial bundle $\mathcal{OP}C \times \mathbf{C}^m$ to $E|\alpha_0(\mathcal{OP}C)$ and $E|\beta_0(\mathcal{OP}C)$ which are mutually close.

(ii) *C satisfies the Runge condition* (*see* 1.4.C) *in $\mathcal{OP}A \supset C$ for all vector bundles over $\mathcal{OP}A$, and the same Runge is satisfied by $C \subset \mathcal{OP}B$.* This allows us to perturb the homomorphisms in (i) in order to make them extendible to required homomorphisms h_1 over $\mathcal{OP}A$ and h_2 over $\mathcal{OP}B$.

1.6.C′. *Remark.* Notice that in order to make the homomorphisms h_1 and h_2 close over $\mathcal{OP}C$, it is enough to apply the Runge approximation only to one of them, say to h_2 on $\mathcal{OP}B$. Thus the Runge requirement for $\mathcal{OP}C \subset \mathcal{OP}A$ can be relaxed to the following condition on the bundle $E|\alpha_0(\mathcal{OP}A)$.

(ii)′ *For every $x \in C$ and every vector $e \in E$ there exists a section $g : \mathcal{OP}A \to E$ such that $g(x) = e$.*

It is clear, that this, (ii)′ together with Runge for $\mathcal{OP}A \supset \mathcal{OP}C$ serves as well as (ii).

As soon as we have the homomorphisms h_1 and h_2 at our disposal, we compose them with our spray and thus obtain maps $s_1 : \mathcal{OP}A \times \mathbf{C}^m \to Z$ and $s_2 : \mathcal{OP}B \times \mathbf{C}^m \to Z$ which are submersions near $\mathcal{OP}C \times 0$. Furthermore, these submersions are mutually close, and by the discussion in the previous section, they deliver the desired s-deformations of α_0 and β_0.

Let us summarize the discussion in 1.6.A–1.6.C in the following.

1.6.D. **Lemma.** *Let (A, B) be a Cartan pair in X satisfying* (i) *and* (ii) *above* (*or* (ii)′ *instead of* (ii) *if one wishes*) *and $Z \to X$ be a submersion with a fiberwise dominating spray s. Then any two sections $\alpha_0 : \mathcal{OP}A \to Z$ and $\beta_0 : \mathcal{OP}B \to Z$ which are close over $\mathcal{OP}C$ can be s-deformed to sections α and β which are equal over $\mathcal{OP}C$ where the required closeness is made precise in the following.*

1.6.D$'$. *Remark.* Let us fix some metric in Z. Then we can speak of *ε-close* sections over a given subset in X for $\varepsilon \geq 0$. Now "close" above should be understood as "ε-close," where $\varepsilon > 0$ depends on the following data:

(a) the fibrations $Z \to X$, $E \to Z$ and the spray $s : E \to Z$;
(b) the subsets A and B in X;
(c) the behavior of α_0 and β_0 near C. This means for every compact set $\mathscr{C}$ of holomorphic sections of Z over $\mathscr{OP}C$, there is an $\varepsilon > 0$ which serves all α_0 and β_0 whose restrictions to $\mathscr{OP}C$ are contained in $\mathscr{C}$, where we stick to the following convention concerning $\mathscr{C}$. All sections of $\mathscr{C}$ are defined on a *fixed* neighborhood of C, and then "compactness" refers to the usual topology in the space of sections on this neighborhood.

1.6.D$''$. *Control on* $\operatorname{dist}(\alpha, \alpha_0)$ *and* $\operatorname{dist}(\beta, \beta_0)$. These distances (over A and B) can be bounded from above in terms of $\varepsilon = \operatorname{dist}_C(\alpha_0, \beta_0)$ on compact families of sections over A and B, respectively. Namely, let $\mathscr{A}_0$ be a compact set of holomorphic sections over A. Then $\operatorname{dist}_A(\alpha, \alpha_0) \geq \delta = \delta(\varepsilon, A_0)$ for all $\alpha_0 \in A_0$, where $\delta \to 0$ for $\varepsilon \to 0$, and a similar property holds true for compact sets $\mathscr{B}_0$ of sections over B. Notice that the above δ for α depends on $\mathscr{A}_0$ but not on $\mathscr{B}_0$.

1.7. **Gluing homotopic sections.** Let us return to the starting point of 1.6 where we have two holomorphic sections $\alpha_0 : \mathscr{OP}A \to Z$ and $\beta_0 : \mathscr{OP}B \to Z$. Now instead of requiring α_0 and β_0 to be close on $\mathscr{OP}C$, we assume there exists a homotopy c_t of holomorphic section over $\mathscr{OP}C$ between $\alpha_0|\mathscr{OP}C$ and $\beta_0|\mathscr{OP}C$. Then we look for holomorphic homotopies α_t over $\mathscr{OP}A$ and β_t over $\mathscr{OP}B$ which bring α_0 to $\alpha = \alpha_1$ and β_0 to $\beta = \beta_1$ such that

$$(*) \qquad \alpha|\mathscr{OP}C = \beta|\mathscr{OP}C.$$

1.7.A. **Lemma.** *Let A, B, Z, and the spray s be as in* 1.6.D, *and let us additionally assume that $C = A \cap B$ is h-Runge* (*see* 1.4.C) *in $\mathscr{OP}B$ for the fibration Z over $\mathscr{OP}B$. Then there exist homotopies α_t of α_0 and β_t of β_0 for which $\alpha = \alpha_1$ and $\beta = \beta_1$ agree on C, which means the above* $(*)$ *on $\mathscr{OP}C$.*

Proof. By the h-Runge, one can bring β_0 by a homotopy to a section β_0' whose restriction to $\mathscr{OP}C \subset \mathscr{OP}B$ is as close to $\alpha_0|\mathscr{OP}C$ as one wishes. Then one can apply 1.6.D to α_0 and β_0', thus obtaining the required α and β.

1.7.B. *Homotopy remark.* Notice that the assumptions of the lemma contain homotopy c_τ between $\bar{\alpha}_0 = \alpha_0|\mathscr{OP}C$ and $\bar{\beta}_0 = \beta_0|\mathscr{OP}C$, while the conclusion provides homotopies α_t and β_t whose restrictions to $\mathscr{OP}C$ are denoted $\bar{\alpha}_t$ and $\bar{\beta}_t$. Since $\bar{\alpha}_1 = \bar{\beta}_1$, we obtain a triangle of homotopies in the space of holomorphic sections $\mathscr{OP}C \to Z$. See Figure 1. This triangle is not a priori contractible. Yet it can be made contractible if one is careful with α_t and β_t, and (a generalization of) this contractibility is important for our h-principle. On the other hand, the contractibility discussion can be avoided in many interesting cases as we shall see presently.

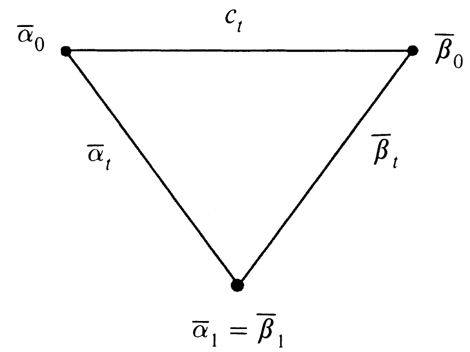

FIGURE 1

1.7.C. *Control on* $\|\alpha_t\|$ *and localization of the spray.* Since we do not move α_0 while bringing $\beta_0|C$ close to $\alpha_0|C$, the needed perturbation of α_0 can be assumed arbitrarily small. In other words, we may claim in the conclusion of the lemma that α_t is as close to α_0 over A as we wish. In fact, it is useful at this point to recall that $\alpha_t = s \circ \alpha'_t$, where $\alpha'_t = (1-t)\alpha'$ for some section α' of $E|\alpha_0(\mathscr{O}\mathscr{P}A)$, and that our bound applies to $\|\alpha'\|$. In other words, this α' can be chosen as small as we wish.

Now since α' is small, we do not need much of our spray $s : E \to Z$ in order to define $\alpha_t = s \circ \alpha_t$. In fact, we need s only *locally* in an arbitrarily small neighborhood of the zero section of $E|\alpha_0(\mathscr{O}\mathscr{P}A)$, and the existence of such a local s is automatic over *Stein* neighborhoods $\mathscr{O}\mathscr{P}A$ by the following standard.

1.7.C$'$. **Local Spray Lemma** (compare 1.2.B$''$). *Let $Z \to U$ be a submersion, where U is Stein, and let $\alpha : U \to Z$ be a holomorphic section. Then there exists a neighborhood E_0 of the zero section of the bundle $E^* = \alpha^*(\mathrm{VT}(Z))$ over U, that is, $\mathrm{VT}(Z)|\alpha(U)$, and a holomorphic map $s : E_0 \to Z$ such that the differential of s on the zero section, say $D : \mathrm{VT}(Z) \to \mathrm{VT}(Z)$ over $\alpha(U)$, is the identity homomorphism.*

Proof. Since U is Stein, the identity map $\mathrm{VT}(Z) \to \mathrm{VT}(Z)$ over $\alpha(U) \subset Z$ extends to a holomorphic jet of infinite order, that is, a formal map $E^* \to Z$. Instead of proving convergence, we first extend this formal map to some nonholomorphic C^∞-map $s_0 : E_0 \to Z$. Then again using the Stein property of U, we take some Hermitian norm $\|e\|$ in the bundle $E^* \to U$ *of negative curvature*, which means the convexity of the function $\|e\|^2$ on E^*. Notice that this function is *strictly* convex away from the zero section, and the decay of convexity is bounded by a power of $\|e\|$. Then it is clear that the function $z \mapsto \|s_0^{-1}(z)\|^2$ is convex on Z near $\alpha_0(U) \subset Z$ and strictly convex away from $\alpha_0(U)$.

Now using this convex function, one can easily show that every compact convex subset A in $U = \alpha_0(U) \subset Z$ is contained in some open convex subset in Z, say $Y \subset Z$. As soon as one knows A admits a convex and, hence, Stein (see [Gra$_3$]) neighborhood Y in Z, the construction of s is immediate over

A. For example, one can use the exponential spray of some affine connection in the (Z-fibers) $\bigcap Y$. (Affine connections are sections of affine bundles, and so their existence is no problem over Stein manifolds.) Finally, one can construct s over all of U by a simple "exhaustion by compact subsets" argument. We leave the details to the reader as we need s only over A anyway.

1.7.C$''$. **Corollary.** *If $\mathscr{OP}A$ is Stein, then the conclusion of Lemma* 1.7.A *holds true if we assume only the existence of a dominating spray for $Z|\mathscr{OP}B$ (without any assumptions on Z over $\mathscr{OP}A$ except submersivity).*

1.7.D. *Remark.* As we have seen in 1.4.F, the role of dominating sprays reduces to lifting homotopies (to some composed sprays). In the present case, we need a lift of a specific homotopy, namely, c_t over $\mathscr{OP}C$ (see the beginning of 1.7), to some spray over $\mathscr{OP}B \supset \mathscr{OP}C$. Granted such a lift, the proof of the above corollary (as well as of Lemma 1.7.A) goes through with no additional spray condition on Z.

2. The h-principle for locally trivial fibrations

Here the h-principle over $A \subset X$ means the validity of the following property:

(*) Every continuous section $\mathscr{OP}A \to Z$ can be homotoped to a holomorphic section.

Recall that $\mathscr{OP}A$ signifies a small nonspecified neighborhood of $A \subset X$. When we start the above homotopy, we can change $\mathscr{OP}$ and make it as small (but still open and containing A) as we wish. Yet, according to our convention, all sections in question (in particular, those constituting our homotopy) must be defined on a *fixed* $\mathscr{OP}A \subset X$.

In the following sections, we prove the h-principle over $X = \bigcup_{i=0}^{\infty} A_i$, where the h-principle is easy over A_0 and where the passage from A_i to $A_{i+1} = A_i \cup B_i$ is achieved with the gluing lemma of the previous section. Our argument is similar (but simpler) than that in [H-L], where the authors give a new proof of the original Grauert Theorem.

2.1. **The h-principle over small subsets.** Let $Z \to X$ be a *locally trivial* fibration. Then a subset $U \subset X$ is called *small* if Z is trivial over U, i.e., $Z|U$ is biholomorphic to the trivial fibration $Z|U = Z_u \times U \to U$.

If U is small, then the h-principle over U is obviously valid if either U or the fiber Z_u is **C**-*contractible*, which means a contracting homotopy of *holomorphic* self-mapping. For example, $\mathbf{C}^n$ is **C**-*contractible* as well as every star-shaped subset in $\mathbf{C}^n$.

2.2. **The h-principle over totally real submanifolds.** A smooth submanifold $Y \subset X$ is called *totally real* if the tangent subbundle $T(Y) \subset T(X)$ contains no complex line, i.e., $T(X) \cap JT(X) = 0$. If such a Y is real analytic and $\dim_{\mathbf{R}} Y = \dim_{\mathbf{C}} Y$, then the local geometry of Y in X depends only on Y. Namely, there exists a biholomorphism of a complexification $\mathbf{C}Y \supset Y$ into $\mathscr{OP}Y \subset X$ which is the identity on $Y \subset \mathbf{C}Y$. Therefore, holomorphic maps $\mathscr{OP}Y \to Z$ are in the one-to-one correspondence with real analytic maps $Y \to Z$.

One knows (see [Ca_1]) that the h-principle is valid for real analytic maps. In fact, for an arbitrary submersion $Z \to X$, every continuous section $Y \to Z$ can be approximated by real analytic ones. This yields the holomorphic h-principle over $\mathscr{OP}Y$, as real analytic sections uniquely extend to holomorphic sections over $\mathscr{OP}Y = \mathbf{C}Y$ by the very definition of $\mathbf{C}Y \supset Y$.

2.3. **Local extension of the h-principle.** A compact subset $A_1 \supset A$ in X is called a *local* extension of A if $A_1 \subset A \cup B$ for some *small* compact subset $B \subset X$, where "small" signifies that the fibration Z in question is trivial over $\mathscr{OP}B \subset X$. We say that such an extension is *Cartan* if (A, B) is a Cartan pair (see 1.5), and we recall that *convexity* of A and B is sufficient for the Cartan property.

2.3.A. **Local Extension Lemma.** *Let $A_1 \supset A$ be a local Cartan extension and $B \supset A_1 - A$. Let the fiber $Z_x \subset Z$, $x \in \mathscr{OP}B$, admit a dominating spray. Then in the following two cases the h-principle extends* (*as explained below*) *from A to $A_1 \subset A \cup B$.*

Case 1. The fiber Z_x is $\mathbf{C}$-contractible, e.g., $Z_x = \mathbf{C}^n$.

Case 2. $\mathscr{OP}A$ is a homotopy retract in $\mathscr{OP}A_1$, and $\mathscr{OP}C$ for $C = A \cap B$ is $\mathbf{C}$-contractible.

2.3.A$'$. *Explanation.* The extension of the h-principle from A to A_1 refers to the following property: *every continuous section of Z over $\mathscr{OP}A_1$ that is holomorphic over $\mathscr{OP}A$ can be homotoped to a holomorphic section over $\mathscr{OP}A_1$.*

In fact, we shall prove below a stronger property, called *Runge extension of the h-principle*, where the implied homotopy can be made almost constant over A. In particular, the holomorphic section we obtain over A_1 can be assumed as close as we wish to the starting section over A.

Proof. The dominating spray on Z_x obviously induces a fiber dominating spray on $Z|\mathscr{OP}B$. Then the homotopy gluing lemma (see 1.7.A and 1.7.C$''$) gives us a pair of sections α_1, β_1 which agree on $\mathscr{OP}C$ and where α_1 is close to the original section on A. Thus we obtain the required section on $A \cup B \supset A_1$. Q.E.D.

Notice that the Runge extension of the h-principle, which we have just proven, provides an approximate extension of holomorphic sections from $\mathscr{OP}A$ to $\mathscr{OP}A_1$, which is stronger than the h-Runge property for $A \subset A_1$, since we do not assume beforehand the existence of any holomorphic sections over A_1.

Warning. One cannot extend the h-principle from A to A_1 using a holomorphic homotopy moving $\mathscr{OP}A_1$ to $\mathscr{OP}A$ since no such homotopy, in general, exists. In fact, if $U \subset X$ is a relatively compact open subset, then "almost all" holomorphic self-maps of U are contractible.

2.3.B. *A homotopy remark.* To clarify the role of our assumptions, consider a more general situation where we want to extend the h-principle from A to $A_1 = A \cup B$ assuming the validity of the h-principle over B, as well as the applicability of the homotopy gluing lemma and corollary, 1.7.A and 1.7.C$''$. The h-principles over A and B provide us with holomorphic sections α_0 over $\mathscr{OP}A$

and β_0 over $\mathcal{OP}B$ in prescribed homotopy classes. Yet these h-principles do not insure the existence of a holomorphic homotopy between $\alpha_0|\mathcal{OP}C$ and $\beta_0|\mathcal{OP}C$ (needed for the homotopy gluing lemma) though the existence of a homotopy by continuous section is immediate with our assumptions. The additional conditions in Cases 1 and 2 ensure such a homotopy.

Even if we can find the above holomorphic homotopy over $\mathcal{OP}C$, we cannot be sure that the holomorphic section over $\mathcal{OP}A_1$ obtained with the gluing lemma is homotopic to the original continuous section. This problem does not arise, however, if the path triangle on Figure 1 is contractible in the space of continuous sections over $\mathcal{OP}C$. For example, this triangle is necessarily contractible if $\mathcal{OP}C$ is contractible and the fiber Z_x, $x \in \mathcal{OP}C$, is simply connected.

Summing up the above discussion, we arrive at an extra case where the h-principle does extend from A to $A_1 = A \cup B$.

Case 3. The fibration Z satisfies the h-principle over B, the fiber Z_x, $x \in \mathcal{OP}C$, is simply connected, and $\mathcal{OP}C$ is $\mathbf{C}$-contractible.

2.3.C. *Remark on the role of the spray.* According to the discussion in 1.4.F, we do not need a dominating spray over Z_x for the validity of Lemma 2.3.A but rather the following *lifting property.* For every holomorphic map $\beta_0 : \mathcal{OP}B \to Z_x$ and every homotopy of holomorphic sections $c_t : \mathcal{OP}C \to Z_x$, where $c_0 = \beta_0|\mathcal{OP}C$, there exists a lift of c_t to some *spray* $\widetilde{F} : E \to Z_x$, where $E \to \mathcal{OP}B$ is some holomorphic vector bundle.

2.4. **Localizable extension.** Consider a compact subset $A \subset X$, and say that X is a *localizable Cartan extension* of A if for an arbitrary covering A by open subsets U_j, $j \in J$, there exists an increasing sequence of compact subsets $A_0 = A$, A_1, A_2, $\dots$, with the following three properties.

(1) *Localization.* $A_{i+1} \subset A_i \cup B_i$ for some compact subset B_i which lies in a single open subset U_j for some $j = j(i) \in J$.

(2) *Cartan.* The pair (A_i, B_i), $i = 0, 1, \dots$, is Cartan for an appropriate choice of the above (small) B_i.

(3) *Exhaustion.* $\bigcup_{i=0}^{\infty} A_i = X$.

2.4.A. **Lemma** [H-L]. *Suppose there exists a proper smooth function $p : X \to \mathbf{R}_+$ whose zero set equals A and which is strictly convex outside A. Then X is a localizable Cartan extension of A. Moreover, if p has no critical point outside A, then one can choose the implied A_i and B_i with the following additional properties*:

(4) *$\mathcal{OP}A_i$ is a homotopy retract in $\mathcal{OP}A_{i+1} \supset \mathcal{OP}A_i$ for all i.*

(5) *$\mathcal{OP}C_i$ for $C_i = A_i \cap B_i$ is $\mathbf{C}$-contractible for all i.*

Proof. Start with the exhaustion of X by the level sets $A'_t = p^{-1}[0, t] \subset X$, $t \geq 0$, and observe that all A'_t are compact convex (see 1.5.A$'$) for $t > 0$ and the boundary $\partial A'_t$ is smooth for the noncritical levels. There obviously exists a perturbation of the sets A'_t by diffeomorphisms of X arbitrarily C^2-close to the identity which gives rise to a sequence A_i which satisfies (1) and (3) as well as (4) and (5) in the noncritical case. Since C^2-small perturbations do not disturb convexity, these A_i can be assumed convex, and then with (small) convex B_i, we have the Cartan property 1.5.A$'$.

2.4.A$'$. **Corollary.** *Let $Z \to X$ be a locally trivial fibration whose fiber admits a dominating spray. Then in the following two cases, the h-principle (Runge) extends from $A = p^{-1}(0) \subset X$ to all of X.*

(1) *The fibers of Z are* **C***-contractible.*
(2) *The function p has no critical points outside $A \subset X$.*

Proof. Lemma 2.3.A allows the Runge extension of the h-principle from A_i to A_{i+1} for all $i = 0, 1, \ldots$. This gives us a sequence of holomorphic sections $\alpha_i : \mathscr{O}\mathscr{P} A_i \to Z$ which converge (because of Runge) on each A_i as $i \to \infty$. Thus we obtain in the limit the desired holomorphic section over all X.

2.5. **The h-principle for $\mathbf{C}^m$-bundles.** Let $Z \to X$ be a locally trivial fibration whose fiber is biholomorphic to $\mathbf{C}^m$.

2.5.A. **Theorem.** *If X is Stein, then there exists a holomorphic section $X \to Z$.*

Proof. One applies (1) of 2.4.A$'$ to some convex proper function $p : X \to \mathbf{R}_+$ with a single minimum point $A = \{a\} = p^{-1}(0)$.

2.5.B. *Question.* Does every $\mathbf{C}^m$-bundle over a Stein manifold admit a vector bundle structure?

2.5.C. **Example.** The simplest nontrivial case of Theorem 2.5.A is that of a *flat* fibration $Z \to X$ coming from a holomorphic action of the fundamental group $\Gamma = \pi_1(X)$ on $\mathbf{C}^m$. For such a Z, holomorphic sections $X \to Z$ correspond to holomorphic Γ-equivariant maps of the universal covering $\widetilde{X}$ of X to $\mathbf{C}^m$. For example, one can take an open Riemann surface for X (e.g., $X = \mathbf{C}^\times = \mathbf{C} - \{0\}$), where Γ is a free group and there is an abundance of holomorphic actions of Γ on $\mathbf{C}^m$ for $m \geq 2$.

2.6. **Manifolds with totally real souls.** Call a compact subset $A \subset X$ a *soul* of X if there exists a proper function $p : X \to \mathbf{R}_+$ whose zero set $\{x \in X \mid p(x) = 0\}$ equals A and such that p is convex and has no critical point outside A.

2.6.A. **Proposition.** *Let $Z \to X$ be a locally trivial fibration whose fiber Z_x, $x \in X$, admits a dominating spray. If X admits a totally real analytic submanifold $A \subset X$ for a soul, then every continuous section $X \to Z$ can be homotoped to a holomorphic section.*

Proof. The h-principle is valid over A by 2.2, and the extension to X is achieved with Case 2 of Lemma 2.3.A.

2.6.B. **Examples.** (a) The circle $S^1 = \{x \in \mathbf{C} \mid |x| = 1\}$ is a totally real soul in $\mathbf{C}^\times = \mathbf{C} - \{0\}$.

(b) Every compact real analytic manifold A is a soul in same complexification $X = \mathbf{C}A \supset A$.

(c) Let X be an open Riemann surface with finitely generated fundamental group and $A \subset X$ a 1-dimensional subcomplex built of finitely many analytic arks such that the natural (inclusion) homomorphism $\pi_1(A) \to \pi_1(X)$ is an isomorphism. It is easy to see that A is a soul, and one can also show that every submersion $Z \to X$ satisfies the h-principle over $\mathscr{OP}A$ (compare 2.2). Hence, the *proof* (and the conclusion) of 2.6.A applies to X.

2.7. **Totally real extensions.** We say that $A_1 \supset A$ is a *totally real* extension if $A_1 = A \cup B$, where B is a real analytic totally real submanifold. In our applications below, B is a (topological) ball of dimension $k \leq \dim_{\mathbf{C}} X$ whose boundary sphere lies in A.

Since B is totally real, the study of holomorphic sections $\mathscr{OP}B \to Z$ reduces to that of real analytic sections $B \to Z$ (compare 2.2). In particular, we have the following obvious strengthening of the Runge property for an arbitrary submersion $Z \to X$.

2.7.A. **Lemma.** *Let $b_0 : \mathscr{OP}B \to Z$ be a continuous section whose restriction to $\mathscr{OP}C$ for $C = A \cap B$ is holomorphic. Then there exists a homotopy b_t of b over $\mathscr{OP}B$ which can be chosen arbitrarily close to the constant homotopy $b_t = b_0$ over B and such that b_t is holomorphic over $\mathscr{OP}C$ for all $t \in [0, 1]$ while b_1 is holomorphic over all $\mathscr{OP}B$.*

Now let us use Lemma 2.7.A instead of the h-Runge in the proof of Lemma 1.7.A. What we gain is the control over $\|\beta_t\|_B$ as well as $\|\alpha_t\|_A$. Namely, both homotopies α_t and β_t can now be assumed close to α_0 and β_0. In fact, we can even control the norm of the corresponding section β' of $E|\beta_0(\mathscr{OP}B)$ (compare 1.7.C), and so we do not need much of the spray over $\mathscr{OP}B$. In fact, we only need a *local* spray (see 1.7.C) over $\beta_0(\mathscr{OP}B) \subset Z$ as well as over $\alpha_0(\mathscr{OP}A) \subset Z$. Notice that one can always choose $\mathscr{OP}B$ Stein (since B is totally real), and so the existence of the local spray is automatic over $\beta_0(\mathscr{OP}B)$ for all submersions $Z \to X$. Thus we arrive at the following strengthening of the homotopy gluing lemma (see Lemma 1.7.A) for Cartan pairs (A, B), where B is totally real.

2.7.B. **Lemma.** *Let Z admit a local spray over $\alpha_0(\mathscr{OP}A) \subset Z$ (this is so, for example, if $\mathscr{OP}A$ is Stein (see Lemma 1.7.C′)). Then there exist homotopies of holomorphic sections α_t and β_t implied by Lemma 1.7.A (i.e., satisfying $\alpha_1|\mathscr{OP}C = \beta_1|\mathscr{OP}C$) which now can be chosen arbitrarily close to α_0 and β_0, respectively, for all $t \in [0, 1]$.*

2.7.B′. **Corollary.** *The h-principle extends from A to A_1. Moreover, every continuous section $a_0 : \mathscr{OP}A_1 \to Z$ which is holomorphic over $\mathscr{OP}A$ admits a homotopy a_t arbitrarily close to a_0 such that a_t is holomorphic over $\mathscr{OP}A$ for all $t \in [0, 1]$ and a_1 is holomorphic over $\mathscr{OP}A_1$.*

2.8. **Nicely localizable extensions.** A localizable extension X of $A \subset X$ (see 2.4) is called *nicely localizable* if for every $i = 0, 1, \ldots$ either conditions (4) and (5) in Lemma 2.4.A are satisfied by (A_{i+1}, A_i) or A_{i+1} is a totally real extension of A_i.

It is clear from the previous discussion that the h-principle (Runge) extends from A to X in the nice case provided the fiber Z_x, $x \in X$, admits a dominating spray.

2.8.A. **Lemma** [H-L]. *Suppose there is a proper smooth function* $p : X \to \mathbf{R}_+$ *as in Lemma* 2.4.A, *that is,* $p^{-1}(0) = A$ *and* p *is convex outside* A. *Then* X *is a nicely localizable extension of* A.

Proof. We assume (as we may) that the critical points of p are nondegenerate and there is at most one critical point x at every level $p^{-1}(t) \subset X$. Then we apply the usual Morse theory in order to see what happens when we go from a subcritical level $A_{t-\varepsilon} = p^{-1}[0, t-\varepsilon]$ to $A_{t+\varepsilon}$ across the critical point $x \in p^{-1}(t)$. Then by Morse theory, $A_{t+\varepsilon}$ is obtained first, by adding a k-dimensional disk B to $A_{t-\varepsilon}$ (where $k =$ index) and then by slightly thickening $A_{t-\varepsilon} \cup B$. Moreover, due to the convexity of p, the disk B can be assumed totally real and, if we wish, real analytic. The implied thickening process is of the same nature as going across a noncritical point, and so the localization is possible with properties (4) and (5). Summing up, we have $A_{i+1} \subset A_1 \cup B_i$ with properties (4) and (5) of Lemma 2.4.A if B_i contains no critical point of p, and we have the totally real extension otherwise.

2.8.B. **Corollary.** *If the fiber* Z_x *for* $x \in X$ *admits a dominating spray, then the h-principle extends from* A *to* X.

Remark. Notice that the local triviality of $Z \to X$ and the domination property are needed only over $X - A$.

2.9. **The basic h-principle for** $Z \to X$. *Let* X *be a Stein manifold and* $Z \to X$ *a locally trivial fibration such that the fiber* Z_x, $x \in X$, *admits a dominating spray* (*see* 0.5.A). *Then holomorphic sections* $X \to Z$ *satisfy the h-principle. That is, every continuous section is homotopic to a holomorphic section.*

The proof follows from Corollary 2.8.B and the existence of a proper convex function $X \to \mathbf{R}_+$ with a single minimum.

2.9.A. *Remark.* Instead of the existence of a dominating spray over Z_x, one may require the validity of the lifting property of 1.4.F for pairs (B, C), where B is a ball in $\mathbf{C}^n$ (for $n = \dim X$) and C is a convex subset in B. This is clear from our discussion in 1.4.F and Remark 1.7.D.

2.9.B. *The parametric h-principle.* The argument we used to prove the above h-principle can be applied to continuous families of holomorphic sections. Thus we can obtain the *parametric h-principle* (compare [Gro, p. 16]) which says that the inclusion between the spaces of sections

$$\mathrm{Holo}(X, Z) \to \mathrm{Cont}(X, Z)$$

is a weak homotopy equivalence. In particular, if two holomorphic sections can be joined by a homotopy of *continuous* sections, then there also exists a homotopy of *holomorphic* sections between the two. In other words, the inclusion Holo $\to$ Cont is injective (as well as surjective) on the sets of the connected components of our spaces.

Notice that there are more delicate situations (see §4 of this paper and [Gro, pp. 17, 18, and 76]), where parametric considerations enter in a crucial way even when we are interested in only the nonparametric h-principle.

2.9.C. *Extension from analytic subsets.* Let $Y \subset X$ be a complex subvariety and $\varphi : Y \to Z$ a holomorphic section. Then we look for a holomorphic extension of φ to X provided we are given a continuous extension. More generally, let $C_\varphi \subset \mathrm{cont}(X, Z)$ consist of sections which are equal to φ on Y and $H_\varphi = \mathrm{Holo} \cap C_\varphi$. It is easy to see that our basic constructions can be performed in the subspace $H_\varphi \subset \mathrm{Holo}$. This yields the parametric h-principle for H_φ: *the inclusion* $H_\varphi \to C_\varphi$ is a *weak homotopy equivalence.* The same conclusion remains valid for holomorphic sections $X \to Z$ with a prescribed *rth jet* along Y. Thus Y satisfies Ell_2 (see [Gro, p. 73]) as well as Ell_∞ in 3.1.

2.9.D. *Singular spaces.* The h-principle over a singular Stein variety can be proven by repeating our nonsingular arguments with minor adjustments. Alternatively, one may use induction by dimension and assume the validity of the h-principle for Z over the singular locus of X. Then the remaining extension problem (to all X) essentially concerns only the nonsingular part of X as in our original argument.

2.10. **Homomorphisms and holomorphic maps of rank** $> r$. Let us give an application of our h-principle.

2.10.A. **Theorem.** *Let A and B be vector bundles of ranks a and b over an n-dimensional (possibly singular) Stein variety of dimension n. If an integer $r \le \min(a, b)$ satisfies $2(a-r)(b-r) > \dim X$, then there exists a holomorphic homomorphism $h : A \to B$ whose rank everywhere is $> r$.*

Proof. The homomorphisms of rank $> r$ are sections of a fibration whose fiber Z_x is of the form $Z_x = \mathbf{C}^{ab} - \Sigma_r$, where $\Sigma_r \subset \mathbf{C}^{ab}$ is the variety of linear maps $\mathbf{C}^a \to \mathbf{C}^b$ of rank $\le r$, which has codim $\Sigma_r = (a-r)(b-r)$. If $a = b$ and $r = a - 1$, then $Z_x = \mathrm{GL}_a \mathbf{C}$, and apart from this case, codim $\Sigma_r \ge 2$. Hence Z_x always admits a dominating spray (see Examples 0.5.B). Next we observe that Z_x is k-connected for $k = 2\,\mathrm{codim}\,\Sigma_r - 2$ and recall that X is homotopy equivalent to an n-dimensional polyhedron for $n \le \dim_{\mathbf{C}} X$ by Lefschetz's Theorem (see Theorem 0.7.A). This insures the existence of a *continuous* homomorphism of rank $> r$ which can be made holomorphic according to the h-principle.

2.10.B. **Corollary.** *Every Stein manifold X (now nonsingular) of dimension n admits a holomorphic map $X \to \mathbf{C}^m$ of rank $> r$ provided $(m-r-1)(n-r) \ge n$.*

Proof. Combine the above with the *holomorphic h-principle* for maps of rank $> r$ (see [G-E$_2$] and [Gro, p. 70]).

3. Different notions of ellipticity

There are several a priori different notions of "ellipticity" of a complex space Y reflecting the idea of an abundance of holomorphic maps $\mathbf{C} \to Y$ and more generally of maps $\mathbf{C}^N \to Y$. Eventually, we want to construct "many" maps $X \to Y$, where X is a (possibly singular) Stein variety. The strongest possible property of this type is expressed by the following axioms (compare [Gro, p. 73]).

3.1. **Ell_∞-spaces.** The Ell_∞-property of Y refers to the h-principle for holomorphic maps $f: X \to Y$, where X is Stein and where the behavior of f is prescribed over certain subsets in X. Namely, we consider $X_0 \subset X$ such that $X_0 = X_0' \cup X_0''$, where X_0' is an analytic subset in X and X_0'' is a compact convex subset (which, as we know, satisfies the Runge approximation property for functions $X \to \mathbf{C}$). We want to formulate now a certain parametric h-principle for holomorphic maps $f: X \to Y$ which agree on X_0' (with certain order r) with a given holomorphic map $f_0': \mathscr{O}\mathscr{P} X_0' \to Y$ and which are arbitrarily close on X_0'' to some map $f_0'': X_0'' \to Y$. Since we want parameters in the picture, our starting object is a family rather than a single map, $f_0: X \times P \to Y$, where P is a finite polyhedron and where f_0 is *x-holomorphic* over $\mathscr{O}\mathscr{P}(X_0) \subset X$. That is, for every fixed $p \in P$, the map $f(x, p)$ is holomorphic in x for x running over $\mathscr{O}\mathscr{P} X_0 \subset X$. We also assume that for some subpolyhedron $P_0 \subset P$, the map $f|P_0$ is x-holomorphic on X. That is, for every fixed $p_0 \in P_0$ the map $f(x, p_0)$ is holomorphic over all of $X \ni x$.

Finally, we fix an integer $r = 0, 1, \ldots$, a real number $\varepsilon > 0$, and we insist that there exists a homotopy $f_t: X \times P \to Y$ for $t \in [0, 1]$ with the following six properties.

(0) The word "homotopy" means continuous for the implied map $X \times P \times [0, 1] \to Y$.

(1) The homotopy f_t is fixed over P_0, i.e., $f_t(x, p_0) = f_0(x, p_0)$ for $x \in X$, $p_0 \in P$, and $t \in [0, 1]$.

(2) The homotopy f_t is x-holomorphic over $\mathscr{O}\mathscr{P}(X_0) \subset X$, i.e., $f_t(x, p)$ is holomorphic in $x \in \mathscr{O}\mathscr{P}(X_0)$ for all t and p.

(3) The homotopy f_t is fixed over X_0' with order r. That is, for every pair (p, t) the two holomorphic maps $f_t(x, p)$ and $f_0(x, p)$ for $x \in \mathscr{O}\mathscr{P} X_0'$ are equal on X_0' with order r. (If X and Y are smooth, this is equivalent to the equality of jets $J^r f_t(x, p)|X_0' = J^r f_0(x, p)|X_0'$. Then the definition of the order for singular X and Y reduces to the nonsingular case with local embeddings of X and Y into $\mathbf{C}^n$.)

(4) The homotopy f_t is ε-fixed over X_0''. That is, $\operatorname{dist}(f_t(x, p), f_0(x, p)) \leq \varepsilon$ for all $x \in X_0''$, $p \in P$, $t \in [0, 1]$ and some metric on Y given beforehand.

(5) The result of the homotopy, the map $f_1: X \times P \to Y$, is x-holomorphic on X.

Let us sum up the above discussion in the following.

3.1.A. **Definition.** Y is called an Ell_∞-space if for arbitrary X, X_0, P, P_0, r, ε, and f_0 there exists a homotopy satisfying (0)–(5).

3.1.B. *Remark on singular spaces Y.* Our extendability condition (3) allows holomorphic extension of maps from X_0' to X provided such an extension is possible to $\mathscr{O}\mathscr{P} X_0' \subset X$. If we had required an unconditional extension, we would have ruled out singular spaces Y. In fact, the existence of holomorphic retraction $Y' \to Y$ of an ambient *nonsingular* space $Y' \supset Y$ ("retraction" means an extension of the identity map $Y \to Y$) is impossible (by an obvious argument) for the singular subsets $Y \subset Y'$. On the other hand, we have here no example of a *singular* Ell_∞-space. The first candidate to look at is $X = \mathbf{C}^2/\mathbf{Z}_2$ for the action $y \mapsto y$ of $\mathbf{Z}_2$ on $\mathbf{C}^2$.

3.2. **Ellipticity of the spray spaces.** One of the main results of the present paper claims (see 2.9.C) that *if Y is a nonsingular space with a dominating spray, then Y is* Ell_∞.

3.2.A. *Remark.* If Y is a *Stein manifold,* then the converse is true.

If Y is Ell_∞, *then there exists a dominating spray* $s : E \to Y$.

Proof. Since Y is Stein, it admits a holomorphic affine connection. In fact, connections correspond to sections of an affine bundle over Y, and the Stein property allows such sections. Then we recall the *exponential map* associated to a connection. This map, say s_0, is defined in a small neighborhood of the zero section $Y = Y_0 \subset E = T(Y)$, say $s_0 : \mathcal{OP}Y_0 \to Y$, and the differential of s_0 restricted to $E|Y_0$, say $D_0 : E \to T(Y)$, is the identity homomorphism.

Now if Y is Stein, then so is (the total space of the bundle) E, and by Ell_∞ (see (3) in 3.1), there exists a holomorphic map $s : E \to Y$ whose differential on $E|Y_0$ equals D_0. Q.E.D.

3.2.A$'$. *Holomorphic sprays on complex projective manifolds* (compare 3.5). Let $E \to Y$ be a "sufficiently negative" bundle of dimension N large compared to $\dim Y$. For example, if l is an *ample* line bundle over Y, then l^{-k} is "sufficiently negative" for large k, and one can take Nl^{-k} for E if N is large enough. Let us indicate the properties of negative bundles which we need for our purpose and which are known to be true for $E = Nl^{-k}$.

(i) There exists a surjective homomorphism $\delta_0 : E \to T(Y)$.

(ii) Denote by $(Y =)\ Y_0 \subset E$ the zero section of E, and let $\mathcal{OP}Y_0 \subset E$ be an "infinitely small" neighborhood of Y_0 as earlier (see 1.5). Then one can choose the above surjective δ_0 such that it "extends" to a holomorphic map (local spray) $s_0 : \mathcal{OP}Y_0 \to Y$. That is, the differential of s_0 on $E|Y_0$ equals δ_0.

(iii) There are "many" convex functions $p : E \to \mathbf{R}_+$ vanishing on Y_0. In particular, if Y is compact, there exists a proper convex function p whose zero set equals Y_0 (here, "convex" means strictly convex on $Y - Y_0$).

Using such a proper convex p and Ell_∞-property of Y, one can Runge extend s_0 to the desired spray $s : E \to X$. In fact, one may use essentially the same local extension argument as in §2, where one should replace the constructions based on sprays by existence theorems appealing to Ell_∞. The details are left to the reader.

3.2.A$''$. *Question.* The above discussion extends to manifolds $Y_1 \times Y_2$, where Y_1 is Stein and Y_2 is compact (and necessarily projective in the presence of a "negative" bundle). Moreover, one can generalize further to *holomorphically convex* manifolds Y. The next natural class of examples where the above may apply is constituted by *quasiprojective* manifolds and, more generally, by locally closed submanifolds in $\mathbf{C}P^N$. Yet, it is not at all clear if

$$\mathrm{Ell}_\infty \Rightarrow (\exists \text{ a dominating spray})$$

for *all* complex manifolds Y.

3.3. **Sprays over maps** $X \to Y$. Consider a holomorphic map $f : X \to Y$ and a vector bundle $E \to X$. A holomorphic map $s : E \to Y$ is called a *spray over* $X \to Y$ if the restriction of E to the zero section $X = X_0 \subset E$ equals f. We say as earlier that s is *dominating* if the differential of s restricted to $E|X_0$ is surjective. Notice that we recapture our earlier definition of a dominating spray $E \to Y$ if we take $f = \mathrm{Id} : Y = X \to X$ and observe that every dominating spray over Y induces that over all $X \to Y$. On the other hand, if Y is Ell_∞, then a slight modification of the proof of 3.2.A$'$ provides dominating sprays over all Stein spaces X (now singular ones are allowed) mapped to Y. Let us generalize this to *families* of holomorphic maps $X \to Y$.

3.3.A. *Sprays over maps* $X \times P \to Y$. Now we consider continuous x-holomorphic maps (see 3.1) and x-holomorphic (in the obvious sense) bundles $E \to P \times X$. Then we can speak of x-holomorphic sprays over (x-holomorphic maps) $P \times X \to Y$. Furthermore, these notions obviously generalize to open subsets $U \subset P \times X$. Namely, one may speak of x-holomorphic maps $U \to Y$, x-holomorphic bundles over U, etc. (A further natural generalization refers to *holomorphic foliations*, but we do not consider these in our paper.) In fact, almost all notions of complex analysis naturally extend to the x-holomorphic situation. In particular, one can define x-Stein spaces U and the Ell_∞-property of Y for x-holomorphic maps $U \to Y$. Then one can prove that the ordinary Ell_∞-property of Y implies the "x-holomorphic Ell_∞." (This is nearly obvious as Ell_∞ refers to families of maps $X \to Y$ in both cases.)

3.3.B. *Parametric lifting property* (compare 1.4.F). Consider an x-holomorphic map $F_0 : X \times P \to Y$ and a homotopy of F_0 over a compact subset $C \subset X$, say $f_t : \mathcal{OP}C \times P \to Y$ for $f_0 = F_0|(\mathcal{OP}C) \times P$. Next take an x-holomorphic spray over F_0, that is, a fibration $\widetilde{E} \to X \times P$ and an x-holomorphic map $\widetilde{F} : \widetilde{E} \to Y$ which extends F_0 on the zero section $X \times P \subset \widetilde{E}$. Then a homotopy of x-holomorphic sections, $\tilde{f}_t : \mathcal{OP}C \times P \to \widetilde{E}$, is called a *lift* of f_t if $\tilde{f}_0 = 0$ and $\widetilde{F} \circ \tilde{f}_t = f_t$.

3.3.B$'$. **Proposition.** *Let Y be* Ell_∞, *X be Stein, and $C \subset X$ be a compact convex* (*see* 1.5.A$'$) *subset. Then the above lift is possible for all F_0 and f_t where the implied $\widetilde{E}$ and $\widetilde{F}$ depend on F_0 and f_t.*

Since we do not use this proposition in this paper, we omit the proof (which is trivial, though tedious). What is more interesting for us is the converse to Proposition 3.3.B$'$.

3.3.C. **Theorem.** *The existence of the above $\widetilde{E}$, $\widetilde{F}$, and $\tilde{f}_t$ for all X, P, C, F_0, and f_t implies the* Ell_∞*-property for Y. In fact, to ensure* Ell_∞, *one needs the lifting property only for quite special X, C, and P, namely, for the unit open ball B in $\mathbf{C}^n$, for all $n = 1, 2, \dots$, for* $\mathbf{R}$*-convex subsets $C \subset B = X$, and for P* (*homeomorphic to*) *the unit closed ball in $\mathbf{R}^m$, for all $m = 0, 1, 2, \dots$.*

Proof. The nonparametric case has been discussed in Remark 2.9.A, and parameters bring no complication.

3.3.C$'$. **Corollary.** *Consider a locally trivial holomorphic fibration* $Y_1 \to Y_2$ *with* Ell_∞*-fibers. Then* (Ell_∞ *for* Y_1) $\Leftrightarrow$ (Ell_∞ *for* Y_2).

Proof. Since the fibers are Ell_∞, the fibration itself is Ell_∞ over every Stein manifold X mapped to Y_2. With this, one easily sees the implication "$\Leftarrow$" (compare B in [Gro, p. 73]). Now we prove "$\Rightarrow$" by observing that the lifting property required in Theorem 3.3.C descends from Y_1 to Y_2 as pertinent homotopies may be (first) lifted from Y_2 to Y_1.

3.3.C$''$. **Example.** If the universal covering $\widetilde{Y}$ of Y admits a dominating spray, then Y is Ell_∞.

Remarks. (a) Sprays can be obviously lifted from Y to $\widetilde{Y}$, but there is no *push forward* of sprays. Yet *the spray-lifting property* can be trivially pushed forward if the underlying Stein manifold is simply connected and holomorphic maps $X \to Y$ lift to $\widetilde{Y}$.

(b) One does not need Proposition 3.3.B$'$ to prove Example 3.3.C$''$, but Theorem 3.3.C is needed.

3.4. **Weaker notions of ellipticity.** We start with the weakest condition we can imagine.

(A) *Vanishing of the Kobayashi metric of* Y. This is equivalent to the following property.

For every *nonconstant* continuous function $d : Y \to \mathbf{R}$, there exists a holomorphic map f of the unit disk $D \subset \mathbf{C}$ to Y such that the composed map $d \circ f : D \to \mathbf{R}$ satisfies

$$|d \circ f(\tfrac{1}{2}) - d \circ f(0)| > \tfrac{1}{2}.$$

(B) $\mathbf{C}$*-connectedness.* This means that every two points in Y lie in the image of some holomorphic map $\mathbf{C} \to Y$.

Obviously (B) $\Rightarrow$ (A), but the implication (A) $\Rightarrow$ (B) seems to be unknown even for compact (algebraic) manifolds Y (where some holomorphic $\mathbf{C} \to Y$ can be obtained with Bloch-Brody limit argument).

(C) *Sub-Euclidean.* This means the existence of a surjective holomorphic map $\sigma : \mathbf{C}^N \to Y$ for some N. This is clearly stronger than (B).

(C$'$) *Densely sub-Euclidean.* This is a weakening of (C) above, where the map σ is required to contain only an open dense set in the image.

Further modifications of the definition are obtained by requiring (i) σ to be an immersion, (ii) σ to be a submersion, or (iii) the complement $Y - \sigma(\mathbf{C}^n)$ to be contained in an analytic subset in Y of positive codimension. Or one can require that Y is covered by the images of several maps $\sigma_i : \mathbf{C}^n \to Y$.

(D) *Runge spaces.* This refers to the Runge property for a class of domains $B \subset \mathbf{C}^N$. For example, one may insist that every holomorphic map of the unit ball $B \subset \mathbf{C}^N$ to Y extends to all $\mathbf{C}^N$ after an arbitrary small perturbation.

Again we have obvious implications $\mathrm{Ell}_\infty \Rightarrow$ (D) $\Rightarrow$ (C) $\Rightarrow$ (B), but we do not know when these arrows can be reversed.

(E) *Extension properties.* One may require, parallel to Runge, extendability of the holomorphic map $A \to Y$ from analytic subvarieties $A \subset \mathbf{C}^N$ to all of $\mathbf{C}^N$, and by varying the class of admissible subvarieties, one arrives at different (?) ellipticity conditions. The narrowest interesting class is formed by subvarieties $A = A_1 \cup A_2$, where A_1 and A_2 are linear subspaces in $\mathbf{C}^N$.

(F) *Examples. Smooth hypersurfaces* Y *of* (*small*) *degree* d *in* $\mathbf{C}P^{n+1}$. If $d = 1$ or 2, then Y is homogeneous and, hence, Ell_∞. The first interesting case is that of cubics. If $n = 1$, these are elliptic curves which are Ell_∞, since they are homogeneous. If $n = 2$, then Y is obtained from $\mathbf{C}P^2$ by blowing up six points and, hence (see 3.5.D), is Ell_∞. If $n = \dim Y = 3$, then Y is known to be unirational (but not rational) and, hence, sub-Euclidean. This Y is probably Ell_∞. On the other hand, cubic k-folds for $k \geq 4$ are often (always?) rational, and by local homogeneity (see 3.5.E$'''$), they are Ell_∞.

Curves Y of degree 4 in $\mathbf{C}P^2$ are far from being elliptic. In fact, they are hyperbolic (i.e., the Kobayashi metric on Y does not degenerate, and all holomorphic maps $\mathbf{C} \to Y$ are constant). Quartic surfaces in $\mathbf{C}P^3$ are special $K3$ surfaces. These cannot be covered by rational curves as they admit a nonzero holomorphic form. Yet some $K3$ are rationally dominated by complex tori $\mathbf{C}^2|\mathbf{Z}^4$, and these are densely sub-Euclidean by the argument in 3.5. We do not know if some (or all) $K3$ are Ell_∞.

One knows that the hypersurfaces Y of degree $d > \dim + 2$ are of general type, and every holomorphic map $f: \mathbf{C}_n \to Y$ has rank $f < \dim Y$ which makes Y non-Ell_∞.

If $d \leq \dim + 1$, then every two points in Y can be joined by a chain of rational curves which makes the Kobayashi metric zero. Furthermore, if $\dim X \geq 2^{d!}$, then Y is generally unirational (see [Ci] for a sharper result) and, hence, densely sub-Euclidean.

The case $d = \dim + 2$ is similar to that of $K3$.

The most optimistic conjecture concerning ellipticity of projective algebraic varieties over $\mathbf{C}$ is as follows. Y is Ell_∞ unless there exists a rational dominating map $Y \to Y'$, where Y' has general type (i.e., $\dim Y$ equals the Kodaira dimension where the case $\dim Y = 0$ is excluded).

3.5. **Algebraic ellipticity.** The ellipticity conditions from the previous section (except Runge) can be reformulated in the category of algebraic varieties and regular algebraic maps. One may also speak of algebraic sprays, and now we use these to define Ell_∞. This is motivated by Localization Lemma 3.5.B and the discussion in 3.2.

3.5.A. *Algebraic* Ell_∞. This refers to a dominating algebraic spray $E \to Y$ for a *vector bundle* $E \to Y$. It is clear that the algebraic Ell_∞ implies the analytic one, but the converse is by no means true. For example, $\mathbf{C}^\times$ and Abelian varieties are not algebraically Ell_∞. In fact, a linear algebraic group is Ell_∞ if and only if it is generated by unipotent elements (the proof is easy), though all groups are analytically Ell_∞ due to the exponential map. (This discussion suggests a generalized algebraic Ell_∞, where E is a group scheme over Y.)

3.5.B. **Localization Lemma.** *If each point $y \in Y$ admits a Zariski neighborhood $U \subset Y$ and a spray s over U (i.e., a bundle $E \to U$ with $s: E \to Y$) dominating at y, then Y is* Ell_∞.

Proof. One can assume that the implied bundle $E \to U$ extends to Y. In fact, by making U small enough, one may have E trivial. One may also assume $U = Y - D$ for an effective divisor (i.e., a hypersurface) $D \subset Y$. Then one considers the line bundle $L = L_D$ whose (local) sections represent a function on Y vanishing on D, and one observes that the natural homomorphism $h_i: E \otimes L^i \to E$, $i = 1, 2, \dots$, is isomorphic over U and vanishes with order i along D. It follows that the composition $s_i = s \circ h_i$ is defined over all of Y and regular for all sufficiently large i; thus s_i defines a spray over Y dominating at Y. Then a composition of finitely many such s_i corresponding to different points $y \in Y$ gives us the desired dominating spray over Y.

3.5.B$'$. **Corollary.** *If each point $y \in Y$ admits an* Ell_∞*-neighborhood, then Y is* Ell_∞.

3.5.B$''$. Ell_∞ *for fibrations.* Let $Y_1 \to Y$ be a Zariski locally trivial fibration. Then one immediately sees with Lemma 3.5.B that

$$(*) \qquad (\mathrm{Ell}_\infty \text{ for } Y_1) \Rightarrow (\mathrm{Ell}_\infty \text{ for } Y).$$

Conversely, let Y and the fibers be Ell_∞. *If the structure group of the fibration can be reduced to* GL_m *for some m, then Y_1 is* Ell_∞.

Proof. One may assume $Y = \mathbf{C}^n$ for some n and then invoke the Serre conjecture (solved by Suslin and Quillen) claiming that all GL_m-fibrations over $\mathbf{C}^n$ are trivial.

Questions. Is the above GL_m-condition essential? Does implication $(*)$ remain valid for *étale* locally trivial fibrations? For example, let $Y_1 \to Y$ be an unramified covering map. It is obvious that

$$(**) \qquad (\mathrm{Ell}_\infty \text{ for } Y) \Rightarrow (\mathrm{Ell}_\infty \text{ for } Y_1)$$

but the converse is unknown to the author. The most optimistic (and not very realistic) conjecture here reads: if $Y_1 \to Y$ is a dominant morphism (i.e., the image is Zariski dense in Y), then

$$(\mathrm{Ell}_\infty \text{ for } Y_1) \Rightarrow (\mathrm{Ell}_\infty \text{ for } Y).$$

From this and the discussion in the next section, one could easily derive that a *projective* manifold is Ell_∞ if and only if (the "only if" is trivial) it is *uni-rational.*

Notice that one does not even know if all rational projective manifolds are Ell_∞. On the other hand, there are uni-rational affine manifolds of the form C/Γ for a linear algebraic group G and a finite subgroup Γ in G (see [Bo]).

3.5.C. *Removal of subvarieties of codimension ≥ 2. If a Zariski closed subset $Y_0 \subset Y$ has* $\operatorname{codim} Y_0 \geq 2$, *then*

$$(\mathrm{Ell}_\infty \text{ for } Y) \Rightarrow (\mathrm{Ell}_\infty \text{ for } Y' = Y - Y_0).$$

Proof. Let $s : E \to Y$ be a dominating spray. Denote by s' the restriction of s to $E' = E|Y'$, and let us modify s' in order to eliminate the intersection between $s'(E')$ and Y_0. According to Localization Lemma 3.5.B, it suffices to make such a modification over each *affine* neighborhood $U \subset Y'$ such that E is trivial over U, i.e., $E \mid U = \mathbf{C}^N \times U \to U$. Since the manifold $X = \mathbf{C}^N \times U$ is *affine*, the spray $s : X \to Y$ can be brought into *general position* (see 3.5.C$'$ below) with respect to Y_0 by a "small" perturbation, say $\sigma : X \to Y$ of s, such that $\sigma^{-1}(Y_0) \subset \mathbf{C}^N \times U$ meets every fiber $\mathbf{C}^N \times U$ across a subset of codimension ≥ 2. Then by arguing as in (iii) of 0.5.B, we construct a fiber-preserving self-mapping $X \to X$ whose image misses $\sigma^{-1}(Y_0)$. This self-mapping composed with s gives us the desired modification of the spray if we take care not to change s on the zero section $0 \times U \subset X$ and keep the new spray dominating at a fixed point $Y \in U$.

3.5.C$'$. *Transversality discussion.* Let $f : X \to Y$ be a regular map and $E^f \to X$ the bundle induced from our E over Y. Then every section $X \to E^f$ defines a "deformation" of f which is just a composition of this section with s. If X is affine, then sections of vector bundles over X enjoy all forms of transversality (or general position) properties, and if Y is Ell_∞, these descend to similar properties of maps $Y \to X$.

3.5.D. *Blow-up.* Call a point $y \in Y$ *regular* if there exists a birational equivalence $\varphi : \mathbf{C}^n \to Y$ such that φ^{-1} is biregular on some Zariski neighborhood of Y. If, moreover, φ is regular on the complement of some subvariety of codimension ≥ 2 in $\mathbf{C}^n$, we say that Y is Ell-*regular.* According to 3.5.B and 3.5.C, the Ell-regularity of the points $y \in Y$ implies the Ell_∞-property of Y. Also notice that regularity implies Ell-regularity if Y is *projective.*

3.5.D$'$. **Example.** Let Y be obtained from $\mathbf{C}^n$ by blowing up the origin, and let $P^{n-1} \subset Y$ be the projective space which is the blow-up origin of $\mathbf{C}^n$. All points $y \in Y - P^{n-1}$ obviously are Ell-regular, as the blow-up map $\sigma^{-1} : \mathbf{C}^n \to Y$ is a birational equivalence of $\mathbf{C}^n - 0$ onto $Y - P^{n-1}$.

Now we recall that Y equals the total space of the canonical line bundle over P^{n-1} and let $p : Y \to P^{n-1}$ be the implied projection. This bundle is trivial on the complement of each hyperplane in P^{n-1} and, therefore, *each* point in Y has a neighborhood $Y' \subset Y$ of the form $Y' = p^{-1}(P^{n-1} - P^{n-2})$ which is biregular to $\mathbf{C}^n$. Notice that the blow-down map $\sigma : Y' = \mathbf{C}^n \to \mathbf{C}^n$ is given (in appropriate coordinates) by $(y_1, \dots, y_n) \mapsto (y_1, y_1y_2, \dots, y_1y_n)$.

3.5.D$''$. **Corollary.** Ell-*regularity is stable under blow-up of points in Y. Namely, if $\widehat{Y} \to Y$ is such blow-up and $\hat{y} \in Y$ projects to an* Ell-*regular point in Y, then $\hat{y}$ is* Ell-*regular in $\widehat{Y}$.*

3.5.E. *Blow-up of subvarieties.* Fedya Bogomolov explained to the author how to prove the following.

Proposition. *Regularity is stable under blow-ups with nonsingular centers.*

Proof. One immediately reduces the general case to that of $\mathbf{C}^n$ blown-up along a smooth connected subvariety $Z \subset \mathbf{C}^n$ of some dimension $m \leq n-2$. Then one observes that Z can be moved into $\mathbf{C}^{m+1} \subset \mathbf{C}^n = \mathbf{C}^{m+1} \times \mathbf{C}^k$, $k = m+1-n$, by a birational automorphism of $\mathbf{C}^n$ regular at a given point $z \in Z$. In fact, if the projection of Z to $\mathbf{C}^{m+1}$ has dimension m, then Z can be (obviously) moved to this projection. Now assuming $Z \subset \mathbf{C}^{m+1} \subset \mathbf{C}^n$, we consider the blow-up manifold Y and denote by $\sigma : Y \to \mathbf{C}^n$ the blow-down map. Let us construct a neighborhood $Y' \subset Y$ which is biregular to $\mathbf{C}^n$ and on which the map σ takes the form $(x, y) \mapsto (x, yf(x))$, where $x = (x_1, \dots, x_{m+1})$, $y = (y_1, \dots, y_k)$ for $k = n - m - 1$, and f is an irreducible polynomial on $\mathbf{C}^{m+1}$ whose zero set equals Z. First we observe that Z is defined in $\mathbf{C}^n \supset \mathbf{C}^{m+1}$ by the equations

$$\{f = 0, \ x_{m+2} = 0, \dots, x_n = 0\}$$

and that Y equals the Zariski closure of the graph of the map $\mathbf{C}^n \to P^k$ given by $(x_1, \dots, x_n) \mapsto (y_0, \dots, y_k)$, where $y_0 = f(x)$ and $y_i = x_{m+1+i}$ for $i = 1, \dots, k$. If we remove the hypersurface $y_0 = 0$ from $\mathbf{C}^n \times P^k$, we obtain $Y' = Y - \{y_0 = 0\} \subset \mathbf{C}^{n+k}$ defined as follows:

$$Y' = \{x_1, \dots, x_n, y_1, \dots, y_k \mid x_{m+1+i} = y_i f(x)\}, \qquad i = 1, \dots, k.$$

It is clear that $(x_1, \dots, x_{m+1}, y_1, \dots, y_k)$ give us the required biregular correspondence $\mathbf{C}^n \leftrightarrow Y'$.

Let us return to the general case of a smooth $Y \subset \mathbf{C}^n$. Then the above construction of Y' depends on a projection π of Y to a linear subspace $L = L^{m+1}$ in $\mathbf{C}^n$. If we fix a point $y \in Y$ and then take *generic* $(L, \pi : \mathbf{C}^n \to L)$, then our Y' will contain y as a regular point, which concludes the proof of 3.5.E.

3.5.E$'$. *Remark.* The above argument fails to prove Ell-regularity as birational transformations moving Y back from L to $\mathbf{C}^n$ may (and usually do) have poles. However, Ell-regularity follows from regularity in the projective case, and, therefore, we have the following.

3.5.E$''$. **Proposition.** Ell-*regularity is stable under blow-ups of projective manifolds along nonsingular subvarieties.*

3.5.E$'''$. *Remark.* It is unclear if *all* points in a smooth rational (projective) manifold are regular (and, thus, Ell-regular in the projective case). It is also unclear if the Ell_∞-property is a birational invariant of *projective* manifolds. On the other hand, the birational invariance of regularity is immediate for *locally homogeneous* manifolds where every two points have isomorphic Zariski neighborhoods. For example, every smooth cubic $Y \subset \mathbf{C}P^{n+1}$ is locally homogeneous. In fact, Y can be birationally reflected in every point $y_0 \in Y$ by mapping each $y \in Y$ to the third intersection point of Y with the line $(y_0 y)$. If y_0 is *generic* relative to given points y and y', then this involution is regular on some neighborhoods of y and y'. Notice that a similar argument proves local homogeneity of the intersection of two quadrics in $\mathbf{C}P^{n+2}$.

3.5.F. *Homogeneous manifolds.* If $Y = G/H$, where G is a linear algebraic group which is generated by unipotent elements, then, clearly, Y is Ell_∞. In some cases, one can even find a unipotent subgroup $U \subset G$ whose action on Y has an open orbit, say $Uy_0 \subset Y$. Such an orbit (clearly) is biregular to $\mathbf{C}^n$ and, consequently, Y is Ell-regular in these cases.

A large number of nonhomogeneous Ell_∞-manifolds can be obtained starting from homogeneous examples and then by blowing-up and removing subvarieties as well as by taking fiber bundles. Furthermore, if we divide such a manifold by an infinite discrete group of holomorphic (e.g., algebraic) automorphisms, we obtain an Ell_∞-manifold in the holomorphic category. Thus one obtains nearly all known examples of complex analytic Ell_∞-manifolds.

3.5.G. *Algebraic homotopies and s-homotopies.* One may expect that Ell_∞-manifolds Y have reasonable homotopy theory in the sense of Volodin (see [Vo]). Namely, one defines n-simplices in Y as regular maps $\sigma : L \to Y$, where $L = L^n$ is an n-dimensional affine space spanned by $n+1$ given points $l_0, \dots, l_n$ in L in general position. The set of all simplices in Y naturally forms a (semi)simplicial complex whose geometric realization, denoted $H(Y)$, represents *the algebraic homotopy type* of Y.

If Y comes with a spray $s : E \to Y$, then there is a natural subcomplex $H(Y, s) \subset H(Y)$ consisting of *s-contractible* simplices, where a map $\sigma : L \to Y$ is called *s-contractible* if it factors through a map into a fiber of the *iterated spray* $s^k : E^k \to Y$ (see 1.3) for some $k = 1, 2, \dots$. That is, there exists a map $\tilde{\sigma} : L \to E^k_y \subset E^k$ for some $y \in Y$ such that $s^k \circ \tilde{\sigma} = \sigma$.

Example. Let $Y = \mathrm{SL}_m$ and s be the spray corresponding to the unipotent subgroups. Then every simplex is s-contractible for $m \geq 3$ (see [Sus]), and there is a counterexample for $m = 2$. Namely, Cohn has shown (see [Co]) that the matrix

$$\begin{pmatrix} 1 - t_1 t_2 & t_1^2 \\ -t_1^2 & 1 + t_1 t_2 \end{pmatrix} \in \mathrm{SL}_2\, \mathbf{C}[t_1, t_2]$$

does not decompose into the product of unipotent matrices.

Remark. One can also ask the s-contractibility question for *holomorphic* maps $\sigma : \mathbf{C}^n \to Y$ (where the spray $s : E \to Y$ is assumed dominating), but one does not know the answer in most (if not in all) interesting cases. For example, one has the following.

Vaserstein Problem. Does every holomorphic map $\mathbf{C}^n \to \mathrm{SL}_m$ decompose into a finite product of holomorphic maps sending $\mathbf{C}^n$ into unipotent subgroups in SL_m?

3.5.G′. Using simplices $\sigma : X \times L \to Y$ in the space of maps $X \to Y$, one defines the homotopy space of the maps $H(X \to Y)$ as well as the s-part $H(X \to Y, s) \subset H(X \to Y)$ corresponding to s-deformations and s-homotopies. This definition looks reasonable if X is an *affine* variety while in the

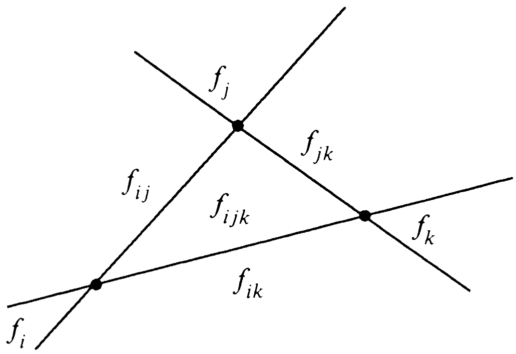

FIGURE 2

general case one should localize with respect to some Grothendieck topology (Zariski, étale, etc.). Namely, a "map" $f: X \to Y$ should be defined by the following data:

(1) a finite covering of X by U_i in the chosen topology,
(2) a regular map $f_i: U_i \to Y$ for all i,
(3) an m-simplex of maps for every intersection $U_I = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_m}$, that is, a regular map $f_I: U_I \times L^m \to Y$ such that $f_I|U_J = f_J$ for all multiindices I and J representing the faces of the simplices in question, where $I \subset J$ and the inclusion refers to the faces. For example, if $m = 1$, we have homotopies $f_{ij}: (U_i \cap U_j) \times L \to Y$, for $L = \mathbf{C}$ between $f|U_i \cap U_j$ and $f_j|U_i \cap U_j$. Then we have triangles $f_{ijk}: (U_i \cap U_j \cap U_k) \times L^2 \to Y$ filling in triples of homotopies (see Figure 2 and compare Figure 1), etc.

(The above notions of coverings and intersections can be taken literally only for the Zariski topology but not for general Grothendieck topologies.)

If X is an *affine* variety, this localized definition seems to give the same homotopy type as the nonlocalized $H(X \to Y)$ (or $H(X \to Y, s)$ if one is concerned with s-homotopies), at least if one uses the Zariski topology. This equivalence of the two homotopy types is somewhat similar to our holomorphic h-principle. A deeper generalization of this h-principle (as well as of the s-contractibility problem for simplices) should refer to the relationship between the homotopy types of spaces $H^{\text{et}}(X \to Y)$ (or $H^{\text{et}}(X \to Y, s)$) and maps $(X^{\text{et}} \to Y^{\text{et}})$, where H^{et} denotes the space of "maps" localized in the *étale* topology and X^{et} and Y^{et} denote the Cech-Grothendieck complexes (nerves) of X and Y for the étale coverings.

4. THE H-PRINCIPLE FOR SUBMERSIONS

We shall see in this section that the gluing lemmas over C-pairs (see 1.7) fit into the framework of continuous sheaves in [Gro]. This allows us to extend our holomorphic h-principle to nonlocally trivial submersions with fiber dominating sprays.

4.1. C-**fibrations.** Consider topological spaces $\mathscr{A}, \mathscr{B}, \mathscr{C}$ and continuous maps $\mathscr{A} \to \mathscr{C}$ and $\mathscr{B} \to \mathscr{C}$. Let $\mathscr{D} \subset \mathscr{A} \times \mathscr{B}$ denote the *product over* $\mathscr{C}$, that is, the set of the pairs (a, b) such that the images of a and b in $\mathscr{C}$, denoted $\bar{a}$ and $\bar{b}$, satisfy $\bar{a} = \bar{b}$. Then the *path product* of $\mathscr{A}$ and $\mathscr{B}$ over $\mathscr{C}$ is the space $\mathscr{D}^* = \{a, b; p\}$ for all $(a, b) \in \mathscr{A} \times \mathscr{B}$ and all *paths* p in $\mathscr{C}$ joining $\bar{a}$ and $\bar{b}$, and we observe that $\mathscr{D}$ is naturally embedded into $\mathscr{D}^*$.

4.1.A. **Definition.** The diagram $\mathscr{A} \to \mathscr{C} \leftarrow \mathscr{B}$ is called a C-fibration (C for Cartan) if the inclusion $\mathscr{D} \subset \mathscr{D}^*$ is a weak homotopy equivalence, that is, the relative homotopy groups $\pi_i(\mathscr{D}^*, \mathscr{D})$ vanish for $i = 0, 1, \dots$.

4.1.B. **Example.** If $\mathscr{A} \to \mathscr{C}$ is a Serre fibration, then, obviously, $\mathscr{A} \to \mathscr{C} \leftarrow \mathscr{B}$ is a C-fibration.

4.2. *Continuous sheaves.* Recall that a *topological presheaf* Φ over a topological space X is a contravariant functor from the category of open subsets and the inclusion maps in X to the category of topological spaces. A presheaf Φ is called a *sheaf* if it satisfies an axiom which mimics property $(*)$ in the following.

4.2.A. Basic Example. Let $Z \to X$ be a fibration. Assign to each open set $U \subset X$ the space of sections $U \to Z$, call it $\Phi(U)$, and assign to each inclusion $I: U_1 \subset U_2$ the *restriction map* on sections $\Phi(I): \Phi(U_2) \to \Phi(U_1)$ for $\Phi(I)(f) = f|U_1$ for all $f \in \Phi(U_2)$. This is clearly a presheaf which satisfies the following (sheaf) localization property for coverings of $U \subset X$ by smaller open subsets $U_\mu \subset U$.

$(*)$ If a collection of sections $f_\mu \in \Phi(U_\mu)$ satisfying $f_\mu|U_\mu \cap U_\nu = f_\mu|U_\mu \cap U_\nu$ for all pairs of the covering sets U_μ, then there exists a unique $f \in \Phi(U)$ such that $f|U_\mu = f_\mu$.

4.2.B. It is convenient to define $\Phi(A)$ for *closed* subsets $A \subset X$ by setting $\Phi(A) = \Phi(\mathscr{O}\mathscr{P} A)$ for an "arbitrarily small" neighborhood $\mathscr{O}\mathscr{P} A \subset X$ of A (compare 1.5). In other words, $\Phi(A)$ is the direct limit of the spaces $\Phi(U)$ over the neighborhoods $U \supset A$. Since the topological structure behaves rather badly in direct limits, one equips $\Phi(A)$ with a *quasitopology* which allows one to speak of continuous maps $P \to \Phi(A)$ that are continuous families of sections $f_p \in \Phi(\mathscr{O}\mathscr{P} A)$ (see [Gro, p. 36]).

4.2.C. *Cartan pairs.* A pair of compact subsets A and B is called *Cartan* relative to a given sheaf Φ over X if the diagram

$$\Phi(A) \to \Phi(A \cap B) \leftarrow \Phi(B)$$

is a C-fibration.

4.2.C$'$. **Example.** The old Cartan property (in the sense of 1.5) immediately implies the new one relative to (the sheaf of) holomorphic functions on X. Moreover, this and the parametric h-Runge for $A \cap B \subset B$ imply the Cartan property for holomorphic sections of a submersion Z over X, provided Z admits a dominating spray. This is a reformulation of Lemma 1.7.A.

4.2.D. Consider a string of subsets $\mathbf{A} = (A_0, \dots, A_m)$ in X, and let A_I for $I = (i_0, \dots, i_k)$ denote the intersection $A_{i_0} \cap \cdots \cap A_{i_k}$. Then for a given sheaf Φ over X, we denote by $\Phi^*(\mathbf{A})$ the space of the collections of maps $f_I : |I| \to \Phi(A_I)$, where $|I|$ denotes a geometric realization of the k-simplex spanned by $(i_0, \dots, i_k)$ such that for all $|J| \subset |I|$ the restriction $f_I | |J| : |J| \to \Phi_I$ equals $\Phi(\mathrm{Incl}(U_I \subset U_J)) \circ f_J$ (compare 3.5.G).

Example. If $m = 1$, this Φ^* is the same thing as $\mathscr{D}^*$ in 4.1.

4.2.D$'$. *Cartan property for* $(A_0, \dots, A_m)$. This is defined by induction on m starting from $m = 1$, where we use the definition of 4.2.C. Then $(A_0, \dots, A_m)$ is called Cartan for $m \geq 2$ if the sequences $A' = (A_0, \dots, A_{m-1})$ and $A^\cap = (A_m \cap A_0, A_m \cap A_1, \dots, A_m \cap A_{m-1})$ are Cartan and the pair $(A_0 \cup A_1 \cup \cdots \cup A_{m-1}, A_m)$ is Cartan. Notice that this definition depends on the ordering of A_i and that it extends to infinite sequences $A_0, A_1, \dots, A_m, \dots$, where the Cartan property by definition means Cartan for $(A_0, \dots, A_m)$ for all $m = 1, 2, \dots$.

4.2.D$''$. **Proposition** (Compare [Gro, p. 76]). *If* $\mathbf{A} = (A_0, \dots, A_m)$ *is Cartan, then the natural embedding of* $\Phi_m = \Phi(A_0 \cup A_1 \cup \cdots \cup A_m)$ *to* $\Phi^*(\mathbf{A})$ *is a weak homotopy equivalence.*

Proof. First we observe that $\Phi^*(\mathbf{A})$ equals the path product (see 4.1) of $\Phi^*(A')$ and $\Phi(A_m)$ over $\Phi^*(A^\cap)$. Then we look at the diagram

$$\begin{array}{ccccc} \Phi_{m-1} & \to & \Phi^\cap_{m-1} & & \\ & & & \nwarrow & \\ \cap & & \cap & & \Phi(A_m) \\ & & & \swarrow & \\ \Phi^*(A') & \to & \Phi^*(A^\cap) & & \end{array}$$

where

$$\Phi_{m-1} = \Phi(A_0 \cup \cdots \cup A_{m-1}),$$
$$\Phi^\cap_{m-1} = \Phi(A_m \cap (A_0 \cup \cdots \cup A_{m-1})),$$

and where the (vertical) inclusions are weak homotopy equivalences by induction on m. Since Φ_m equals the fiber product of Φ_{m-1} and $\Phi(A_m)$ over $\Phi^\cap_{m-1}$ and this is weakly homotopy equivalent to the path product, the desired weak homotopy equivalence property of the inclusion $\Phi_m \to \Phi^*(\mathbf{A})$ follows from the obvious homotopy invariance of path products.

4.3. **The deformation space** $\Phi^*(A)$. For every finite covering of a compact subset $A \subset X$, say $A = \bigcup_{i=1}^m A_i$, consider the above space $\Phi^* = \Phi^*(A_0, \dots, A_m)$ associated to this covering and denote by $\Phi^*(A)$ the direct limit of these spaces

Φ^* over all such coverings. Using 4.2.D$'$, we have

4.3.A. **Lemma.** *If A admits arbitrarily fine Cartan coverings, then the natural embedding $\Phi(A) \to \Phi^*(A)$ is a weak homotopy equivalence.*

4.3.B. **Example.** Let Φ be the sheaf of holomorphic sections of a submersion $Z \to X$. Then the space $\Phi^*(A)$ is weakly homotopy equivalent to the space of *continuous* sections $\mathcal{OP}A \to Z$. To see this, we denote by Φ_0 the sheaf of continuous sections of Z and observe that the embedding $\alpha : \Phi_0(A) \to \Phi_0^*(A)$ (obviously) is a weak homotopy equivalence (one way to see it is to use Lemma 4.3.A and observe that all pairs of compact subsets are Cartan for *continuous* sections). Next we look at the embedding $\beta : \Phi^*(A) \to \Phi_0^*(A)$ corresponding to the natural embedding of sheaves $\Phi \to \Phi_0$. Since $\Phi^*(A)$ is built of continuous families of *local* holomorphic section, the weak homotopy equivalence property of the map β follows from the fact that the *stalks* of the sheaves Φ and Φ_0 are (obviously) weakly homotopy equivalent at all points $a \in A$. (Compare the local h-principle in [Gro, p. 119].) Now we obtain the required weak homotopy equivalence by taking $\alpha \circ \beta^{-1}$.

4.4. **Convex coverings of Stein manifolds.** A covering $X = \bigcup_{i=0}^{\infty} A_i$ is called *convex* if all subsets A_i are convex and the finite union $\bigcup_{i=0}^{m} A_i$ is convex for all $m = 0, 1, \dots$.

4.4.A. **Lemma.** *A Stein manifold X admits an arbitrarily fine locally finite convex covering by compact subsets A_i.*

Proof. Use a convex function on X as in Lemma 2.4.A.

4.5. **Main Theorem.** *Let $Z \to X$ be a holomorphic submersion, where X is Stein and where Z admits a fiber dominating spray over a small neighborhood of each point in X. Then the inclusion between the spaces of holomorphic and continuous sections*

$$\mathrm{Holo}(X, Z) \subset \mathrm{Cont}(X, Z)$$

is a weak homotopy equivalence.

Proof. The above discussions and gluing of homotopies over Cartan pairs (see 1.7) show that the inclusion

$$\mathrm{Holo}(\mathcal{OP}A, Z) \subset \mathrm{Cont}(\mathcal{OP}A, Z)$$

is a weak homotopy equivalence for all *compact convex* subsets. Granted this, the proof can be concluded by the argument in §2, or by somewhat stretching the sheaf theory in order to include h-Runge.

4.5.A. *Remark.* One can sharpen the above theorem as in §2 and 3.1 by bringing forth the Runge approximation and the control of sections $X \to Z$ along given subvarieties in X. One can also somewhat relax the spray condition along the lines indicated in Remark 2.9.A.

4.6.B. **Examples.** (a) Let G be a complex Lie group and $\Gamma_x \subset G$ a discrete subgroup holomorphically depending on $x \in X$. Then the resulting fibration $Z \to X$ with fibers $Z_x = G/\Gamma_x$ is not, in general, holomorphically trivial and so the (locally trivial) theorem 2.9 does not apply here while 4.5 above does.

(a′) Let Y be an arbitrary Ell_∞-space and Γ_x a discrete group of biholomorphisms freely acting on Y. Then we have a similar $Z \to X$ with $Z_x = Y/\Gamma_x$, where 4.5 applies in view of Example 3.3.C″.

(b) Start with a vector bundle $V \to X$, and remove a complex submanifold $\Sigma \subset V$ whose intersections with the fibers V_x are algebraic subvarieties of codimension ≥ 2. Moreover, assume that for a generic homomorphism of V to a vector bundle V' over X with $\dim V' = \dim V - 1$, the image of S in V' is contained in a complex subvariety of positive codimension in V'. (In fact, we need such homomorphisms only *locally*, i.e., in a small neighborhood of each point $x \in X$.) Then we can construct a fiber dominating spray as in (iii) of Examples 0.5.B and apply the h-principle. Notice that for the validity of the h-principle, the resulting submersion $Z = V - \Sigma \to X$ need not be locally trivial even topologically, but topologically locally trivial fibrations behave better as far as continuous sections are concerned. For example, if $2\operatorname{codim}\Sigma > \dim X$, one knows there exists a continuous section $X \to Z$, assuming $Z \to X$ is topologically locally trivial and X is Stein (compare 2.10). Then the h-principle provides a holomorphic section $X \to Z$. Yet one can bypass the topological local triviality if one assumes that $\Sigma \subset V$ is *stratified* over X (which is the case in most interesting situations) and, consequently, X admits a stratification such that Z is topologically locally trivial over each stratum. If each stratum in X has dimension less than $2\operatorname{codim}(\Sigma \cap V_x \subset V_x)$ for x in this stratum, then one can construct a continuous section $\varphi : X \to Z = V - \Sigma$ using a simple induction by strata and Lefschetz Theorem 0.7.A. Furthermore, if $\operatorname{codim}(\Sigma \cap V_x \subset V_x) \geq 2$ for all $x \in X$, one can apply the h-principle and homotope φ to a holomorphic section.

(c) Go from Z to $\widehat{Z}$ by blowing up a subvariety in Z. By generalizing the discussion in 3.5.D, one obtains in this way many examples of nonlocally trivial fibrations (say, with projective fibers) where the h-principle applies.

5. The h-principle of elliptic sheaves and further generalizations

The notion of the h-principle makes sense for an arbitrary continuous sheaf Φ over X. In fact, this h-principle means that the embedding $\Phi(X) \to \Phi^*(X)$ is a weak homotopy equivalence for Φ^* defined in 4.3. (One can also use Φ^* from [Gro, Chapter 2.2] which leads to the same notion if the underlying space X can be locally triangulated at each point $x \in X$.) Then one can axiomatize the proof of our h-principle by introducing a notion of an *elliptic sheaf* which signifies the existence of "sufficiently many" homomorphisms $\Phi_N \to \Phi$, where Φ_N denotes the sheaf of holomorphic maps $X \to \mathbf{C}^N$. The resulting general theory is somewhat heavy (it contains, for example, basic results of [Gro, Chapters 2.1 and 2.2] as special cases), and we restrict ourselves in this paper to a brief survey of a few simple examples.

5.1. **Stratified submersions.** Let $X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_m$ be a descending sequence of analytic subvarieties, let $Z_i \to X_i$ be an analytic fibration for $i = 0, 1, \ldots, m$ (here fibration means a holomorphic map) and let $h_i : Z_i \to Z_{i-1}|X_i$ be holomorphic fiber preserving maps. We consider the sheaf

Φ of strings of holomorphic sections $f_i : X_i \to Z_i$, $i = 0, \dots, m$, such that $h_i \circ f_i = f_{i-1}|X_i$ for all i.

5.1.A. **Example.** Take a single fibration $Z \to X$ and a section $f_1 : X_1 \to Z$ for $X_1 \subset X$. Then the sheaf of sections $f : X \to Z$ satisfying $f|X_1 = f_1$ is an instance of the above Φ.

5.1.B. **Theorem.** *Suppose the fibrations Z_i over $\Sigma_i = X_i - X_{i+1} \subset X_i$, $i = 0, \dots, m$, are submersions admitting fiber dominating sprays. If X is Stein, then the embedding $\Phi(X) \to \Phi^*(X)$ is a weak homotopy equivalence.*

Proof. An obvious induction by strata Σ_i reduces the general problem to that of extensions of sections from X_1 to X as in Example 5.1.A. To prove the h-principle for this extension by our earlier argument, we need, besides a (global) spray over $X - X_1$, a "local spray" over all of X (or rather over compact subsets $A \subset X$ which, in general, meet X_1; compare 1.7.C). In other words, we need sufficiently many small holomorphic deformations of sections $f : X \to Z$, which come from holomorphic sections of a (trivial) vector bundle over X and which are fixed over X_1. To construct these deformations, we may integrate vertical holomorphic vector fields in Z vanishing over Z_1 which are defined near a given section $X \to Z$. Notice that Z may not be a submersion over X_1 (it may even be singular over X_1), but these fields make sense all the same became of the vanishing on X_1 condition. Moreover, these fields span all vertical tangent spaces of Z over $X - X_1$ and thus give us just as many local deformations as we need for the proof. At this point, we invite the reader to fill in the details.

5.1.B$'$. *Remarks.* (a) The above h-principle implies its more concrete counterpart referring to the embedding of $\Phi(X)$ to the corresponding sheaf of strings of *continuous* sections $X_i \to Z_i$ (compare Example 4.3.B).

(b) The above h-principle extends to certain subsheaves in Φ, where the sections are restricted by prescribing the behavior of the sets $J^{r_i} f_i$ on X_{i+1} in the directions "transversal" to $h_{i+1}(Z_i) \subset Z_{i+1}$ (compare [Gro, pp. 45 and 67] below).

5.2. **Open subsheaves of coherent sheaves.** We shall treat here only one particular example which is important for embeddings of Stein manifolds into $\mathbf{C}^N$. We start with a manifold Y and take the symmetric square $Y \times Y/\mathbf{Z}_2$ for X. Notice that this X is singular if $\dim Y \geq 2$. We denote by Φ_N the sheaf whose sections f correspond to holomorphic maps $\tilde{f} : Y \to \mathbf{C}^N$ satisfying $\tilde{f}(y_1, y_2) = -\tilde{f}(y_2, y_1)$. Notice that this Φ_N is a *coherent* sheaf over X which is *not* locally free for $\dim Y \geq 2$. Then we consider the *maximal* (open) subsheaf $\Phi \subset \Phi_N$ whose sections correspond to maps $\tilde{f} : Y \times Y \to \mathbf{C}^N$ such that

(a) $\tilde{f}(y_1, y_2) \neq 0$ unless $y_1 = y_2$ and small perturbations of $\tilde{f}$ have the same property.

To understand better the "small perturbation" condition, we restrict our sheaf Φ_N to the diagonally embedded $Y \subset X$ and observe a natural homomorphism from $\Phi_N|Y$ to the sheaf Φ' of holomorphic homomorphisms of the tangent bundle $T(Y)$ to the trivial bundle $L_N = \mathbf{C}^N \times Y \to Y$. Now we can define Φ by the conditions

(a$'$) $\tilde{f}(y_1, y_2) \neq 0$ for $y_1 \neq y_2$;
(a$''$) the homomorphism $\tilde{f}' : T(Y) \to L_N$ is injective over all $y \in Y$.

5.2.A. **Example.** Every map $\varphi : Y \to \mathbf{C}^N$ defines a section f of Φ for $\tilde{f}(y_1, y_2) = \varphi(y_1) - (y_2)$. The homomorphism f' equals here the differential of f. Thus condition (a$''$) says that φ is an immersion while (a$'$) makes φ one-to-one.

5.2.B. **Theorem.** *If Y is Stein, then the embedding $\Phi(X) \to \Phi^*(X)$ is a weak homotopy equivalence.*

Proof. One proceeds in two steps. First one proves the h-principle over $Y \subset X$, where the matter reduces to the (already known) h-principle for injective homomorphisms $T(Y) \to L_N$. Then one extends the h-principle to $X \supset Y$ as in 5.1. Now the "small deformations" one needs come from coherency of Φ_N, which ensures many homomorphisms of the sheaf of holomorphic functions to $\Phi_N \supset \Phi$, and the openness of Φ keeps the images of "small" functions in Φ_N inside Φ. These give one the required deformations. Here again, we leave the details to the reader.

5.2.B$'$. *Remark.* If we decipher the above h-principle and pass to continuous objects, we shall have pairs $(\tilde{f}, \tilde{f}')$, where $\tilde{f} : Y \times Y \to \mathbf{C}^N$ is a *continuous* antisymmetric map satisfying (a$'$) and $\tilde{f}' : T(Y) \to L_N$ is a continuous injective homomorphism such that $\tilde{f}$ and $\tilde{f}'$ agree near $Y \subset X$ in an obvious way.

5.2.B$''$. **Corollary.** *If $N > \max(n, \frac{3}{2}n - 1)$ for $n = \dim Y$, then Φ admits a section over X. Moreover, if the tangent bundle $T(Y)$ is trivial, then Φ admits a section for $N > n$.*

5.2.C. The h-principle for $\Phi \subset \Phi_N$ extends to the following more general situation. Suppose we are given some holomorphic section $f_0 \in \Phi_{N_0}$ for $N_0 < N$ and we look for $f_1 \in \Phi_{N_1}$, where $N_1 = N - N_0$ such that $f = f_0 \oplus f_1 \in \Phi_N$ is contained in our Φ (compare [G-E$_2$]). These sections f_1 form a subsheaf $\Phi_1 \subset \Phi_N$ (depending on f_0) which is "stratified" by "singularities" of f_0, and the proof of the h-principle for Φ_1 can be obtained by induction on the strata as earlier.

5.3. **Removal of singularities.** We want to explain here how the removal of singularities (see [G-E$_1$, G-E$_2$, Gro]) fits into the philosophy of elliptic sheaves. We consider as the simplest example the sheaf Φ of holomorphic immersions $f : X \to \mathbf{C}^N$. If $N > \dim X$, we have sufficiently many deformations of f coming from (the sheaf of) holomorphic functions $X \to \mathbf{C}$ as follows. Take a generic projection of an immersed manifold $f(X) \subset \mathbf{C}^N$ to $\mathbf{C}^{N-1}$, call the resulting map $f_0 : X \to \mathbf{C}^{N-1}$, and write $f = f_0 \oplus f_1$ for the remaining (coordinate) function $f_1 : X \to \mathbf{C}$. If $N - 1 \geq \dim X$, the map f_0 is an immersion away from a subvariety $\Sigma_0 \subset X$ of positive codimension, and so there exists a nonzero function ψ_0 on X whose first-order jet vanishes on Σ. Then *every* holomorphic function φ on X gives us a "deformation" of f, namely, $f_\varphi = f_0 \oplus (f_1 + \psi_0 \varphi)$ which is an *immersion* of X to $\mathbf{C}_N$. Using these deformations for various φ and an induction on strata, we can reduce the h-principle for immersions to Grauert's h-principle (see [G-E$_2$, Gro]). Moreover, we can incorporate the constructions in [G-E$_2$] and [Gro] into the framework of the present paper and thus obtain the h-principle for immersions $X \to Y$, where Y is an Ell_∞-manifold which admits "sufficiently many local splittings" of the form $Y = Y_0 \times \mathbf{C}$. We postpone a detailed discussion with definitions and proofs until another paper. Here we mention only that the (not defined) splitting condition is satisfied by

(a) the complex Lie groups Y,
(b) certain homogeneous spaces, such as the projective space and the Grassman manifold,
(c) $\mathbf{C}^N - \Sigma$ for algebraic Σ with $\operatorname{codim} \Sigma > 2$.

5.3.A. *Remark.* The removal of singularities applies to embeddings $X \to \mathbf{C}^N$ only in a limited way and yields no definite result for $N \leq \frac{2}{3} \dim X$ (see [G-E$_2$]). On the other hand, the removal of codimension two subvarieties dealt with several times in this paper suggests the h-principle for holomorphic embeddings of Stein manifolds X into $\mathbf{C}^N$ for $N \geq \dim X + 2$, where "embedding" may (or may not) include the (quasi)properness in the definition. For example, if the tangent bundle $T(X) \to X$ is trivial, we may expect an embedding $X \to \mathbf{C}^{n+2}$ for $n = \dim X$. On the other hand, one-to-one maps $X \to \mathbf{C}^{n+1}$ seem to display a great amount of "hyperbolic rigidity" which make the h-principle not very likely. (Notice that embeddings form a *presheaf* rather than a sheaf, but we can define the h-principle just the same.) The following is a test problem. Let f_1 be a holomorphic embedding of the unit ball B into $\mathbf{C}^N$. When does f_1 Runge extend to an embedding of the concentric ball of radius two? We expect the positive answer for $N \geq n + 2$ and, in general, negative for $N = n + 1$. (We may also look at $N = n$, but this appears to be an easy case.)

5.4. **Algebraic and holomorphic solutions of the undetermined partial differential equation.** Solutions f of certain partial differential equations over X can be deformed with functions φ on X in much the same way as immersions were deformed in the previous section.

5.4.A. **Example.** We denote by g_0 the standard quadratic differential form $\sum_{\nu=1}^N dz_i^2$ on $\mathbf{C}^N$, and study holomorphic maps $f \colon X \to \mathbf{C}^N$ which are *isometric* for a given form g on X. That is, $f^*(g_0) = g$, which can be equally expressed in local coordinates by partial differential equations

$$\langle \partial_i f, \partial_j f \rangle = g_{ij}, \qquad i, j = 1, \dots, n = \dim X$$

or by the relation

$$\sum_{\nu=1}^N df_\nu^2 = g$$

for the components $f_\nu : X \to \mathbf{C}$ of f. Given such an f, we consider the following system of *algebraic* equations for $\psi : X \to \mathbf{C}^N$,

$$\begin{aligned} \langle \psi , \partial_i f \rangle &= 0, \qquad i = 1, \dots, n, \\ \langle \psi , \partial_{ij} f \rangle &= 0, \qquad i, j = 1, \dots, n, \\ \langle \psi , \psi \rangle &= 0. \end{aligned}$$

Notice that the first two groups of equations are linear and indicate that ψ is g_0-orthogonal to the osculating bundle of $f(X) \subset \mathbf{C}^N$, while the last makes ψ g_0-isotropic. It is easy to see that the above system admits a nonzero solution ψ_0 if $N > n(n+3)$. Then $\varphi\psi_0$ is also a solution for all $\varphi : X \to \mathbf{C}$, and a straightforward computation (compare [Gro, pp. 116 and 147]) shows that the map $f_\varphi = f + \varphi\psi_0$ satisfies $f_\varphi^*(g_0) = f^*(g_0) = g$. These "deformations" of isometric maps by functions allow one to develop a holomorphic (and also algebraic) theory of isometric immersions $(X, g) \to \mathbf{C}^N$ for large N. Furthermore, similar ideas apply to *isometric symplectic immersions* (compare [Gro, Chapter 3.4]) and to connection inducing maps (compare [Gro, 2.2.6]). In fact, one can introduce a rather general class of Ell_∞-equations where the role of the spray is taken over by an appropriate differential operator $s(f, \varphi)$ which is defined on some space of jets associated to the equation. We shall pursue this in another paper.

5.5. **The h-principle for nonholomorphic maps.** The h-principle is known for many nonlinear partial differential equations (see [Gro]), but apart from the Cauchy-Riemann system (studied in this paper) the author knows of only one nonlinear *elliptic* (in the ordinary sense) system where the h-principle is valid. Namely, the argument we used for holomorphic maps (or rather a drastically simplified version of this argument) leads to the following.

5.5.A. **Proposition.** *Let X be a Riemannian manifold and Y an affine flat manifold. Then harmonic maps $X \to Y$ satisfy the h-principle provided X is open and Y geodesically complete.*

Remarks. (a) Harmonic maps into affine flat manifolds make sense since harmonic functions on X constitute a *linear space containing the constants.* With this in mind, one can extend Proposition 5.5.A to more general elliptic equations.

(b) Recall that X is *open* if each component of X is either noncompact or has nonempty boundary.

(c) The h-principle claimed by the proposition says, in effect, that every continuous map $X \to Y$ is homotopic to a harmonic map. (The "parametric" part of the h-principle is trivial in this case by the linearity of the harmonic equation for maps of X to the universal covering of Y which is an affine space.)

5.5.B. It seems there is no single "truly nonlinear" elliptic equation where one can prove the h-principle. Here are some candidates:

(i) harmonic maps of open Riemannian manifolds X to compact Riemannian manifolds Y of positive curvature, for example, maps to compact symmetric spaces Y,

(ii) holomorphic maps of open Riemann surfaces into almost complex manifolds (Y, J), where the structure J is a small perturbation of a complex structure J_0 such that (Y, J_0) is Ell_∞,
(iii) minimal immersions of smooth open manifolds X (with no additional structure) into, for example, Euclidean spaces and spheres of high dimension,
(iv) Einstein metrics on open manifolds, and
(v) Yang-Mills over $\mathbf{R}^4$ and over more general open Riemannian 4-manifolds.

Also notice that practically *all* "soft" properties related to the h-principle (Cartan, Runge, extension from submanifolds of codimension ≥ 2, etc.) remain unknown for these equations.

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