# THE THREE-SPACE PROBLEM FOR $L^{1}$ 

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## 1. Introduction

The study of the structure of the classical Banach spaces $C(K), L^{p}(\mu)$, $1 \leq p \leq \infty$, their subspaces, their quotient spaces, and the operators between them is a central topic of Banach space theory. This paper focuses on the space $L^{\prime}$ of (classes of) measurable functions on the unit interval.

In order to put our results in a proper perspective, let us recall a few of the most remarkable results on $L^{1}$. A subspace of $L^{1}$ is either reflexive or it contains a subspace isomorphic to $l^{1}$ and complemented in $L^{1}$ (Kadec and Pelczynski [11]). A reflexive subspace of $L^{1}$ is isomorphic to a subspace of $L^{p}$ for some $p>1$ (Rosenthal [20]). Any subspace of $L^{1}$ contains a subspace isomorphic to $l^{p}$ for some $1 \leq p \leq 2$ (Aldous [1]). The most challenging open question concerning the structure of $L^{1}$ might be to decide whether a complemented infinite-dimensional subspace of $L^{1}$ is isomorphic to $l^{1}$ or $L^{1}$. (Interestingly, more is known in the case $L^{p}, p>1$ [4].) An important step in that direction was made by Enflo and Starbird [6]. (See [7] or [12] for a different approach.) They showed that $L^{1}$ is primary; that is, if $L^{1}$ is isomorphic to a sum $E \oplus F$, then either $E$ or $F$ is isomorphic to $L^{1}$. An easy consequence of that result is that if a direct sum $E \oplus F$ contains a complemented subspace isomorphic to $L^{1}$, then either $E$ or $F$ has the same property.

Each separable Banach space is isomorphic to a quotient of $L^{1}$. Thus, the study of quotients of $L^{1}$ naturally focuses on the study of quotients $L^{1} / X$, where $X$ has some smallness property; e.g., $X$ is reflexive [13, 18], separable dual [10], or "bien disposé" [8]. For simplicity, let us say that a Banach space contains a copy of $L^{1}$ if it contains a subspace isomorphic to $L^{1}$. The threespace problem for $L^{1}$ raised in [14] belongs to the same circle of ideas. It is the following question: If $X$ is a subspace of $L^{1}$, does $X$ or $L^{1} / X$ contain a copy of $L^{1}$ ? The result of Enflo and Starbird shows that this is the case when $X$ is complemented. The three-space problem for $L^{1}$ seems to have been the main motivation behind the papers $[5,2,9,10,22]$ which contain many interesting results. In particular, Bourgain and Rosenthal [5] have shown that either $X$ contains a copy of $L^{1}$ or $L^{1} / X$ contains a copy of $l^{1}$, and

[^0]Bourgain [2] (actually looking for a counterexample) proved the remarkable fact that $L^{1} / H^{1}$ contains a copy of $L^{1}$.

The first main contribution of the present paper is the construction of (families of) subspaces $X$ of $L^{1}$ that are small as Banach spaces (actually have the Radon-Nikodym property) but such that $L^{1} / X$ does not contain a copy of $L^{1}$. Also raised by the result of Enflo and Starbird and its consequences mentioned earlier is the following. If a direct sum $E \oplus F$ contains a copy of $L^{1}$, does $E$ or $F$ contain a copy of $L^{1}$ ? The other main contribution of the present paper is a negative answer to that question.
Theorem 1.1. There exist two subspaces $X_{1}, X_{2}$ of $L^{1}$ that are isomorphic to an $l^{1}$-sum of spaces isomorphic (but not uniformly isomorphic) to $l^{1}$ such that $L^{1} / X_{1}$ and $L^{1} / X_{2}$ do not contain a copy of $L^{1}$ but that the canonical injection of $L^{1}$ in $L^{1} / X_{1} \times L^{1} / X_{2}$ is an embedding.

In particular, $X_{1}, X_{2}$ have the Radon-Nikodym property and, in particular, do not contain a copy of $L^{1}$. If a subspace $X$ of $L^{1}$ is isomorphic to a dual space, it does not seem to be known if $L^{1} / X$ contains a copy of $L^{1}$. We, however, have the following.
Proposition 1.2. If a subspace $X$ of $L^{1}$ is isomorphic to $l^{1}$, then $L^{1} / X$ contains a copy of $L^{1}$.

The following is related to [17, Problem 8.3] and remains open.
Problem 1.3. Consider a compact metric space $S$. Denote by $C(S)$ the space of continuous functions on $S$ and by $M(S)=C(S)^{*}$ the space of signed measures on $S$. Consider a weak ${ }^{*}(=\sigma(M(S), C(S)))$ closed subspace $X$ of $M(S)$. Suppose that $X$ is a subspace of $L^{1}(\mu)$ for some probability measure $\mu$ on $S$. Does $L^{1}(\mu) / X$ contain a copy of $L^{1}$ ?

Following [15], we say that a (linear, bounded) operator $T$ between two Banach spaces $E$ and $F$ is a semiembedding if it is one-to-one and if the image of the unit ball of $E$ is closed. The following result, which is obtained by a variation of the construction of Theorem 1.1, answers questions of [5].

Theorem 1.4. There exist two Banach spaces $E_{1}, E_{2}$ that do not contain a copy of $L^{1}$ but such that $L^{1}$ embeds in $E_{1} \times E_{2}$ in such a way that the restrictions to $L^{1}$ of the projections on $E_{1}$ and $E_{2}$ are semiembeddings.

Once the proper approach has been found, it is a matter of standard (uninspiring) technique to reduce the proof of these theorems to a central statement concerning a remarkable family of functions on [ 0,1 ]. This statement, which could be of independent interest, is the main new ingredient of this paper. Consider the family $F_{n}$ of all the functions of the type

$$
\frac{1}{2 n} \sum_{0 \leq i \leq 2 n-1}(-1)^{i} 2^{\sigma(i)} 1_{I_{i}}
$$

for any sequence $\sigma(0)<\sigma(1)<\cdots<\sigma(2 n-1)$ of integers, where the sets $I_{i}$ are dyadic intervals of length $2^{-\sigma(i)}$, and where the sequence $I_{1}, \ldots, I_{2 n-1}$ decreases. Then (Theorem 3.3) given $\varepsilon>0$, for $n$ large enough and any function $f$ in the convex hull of $F_{n}$, the set where $|f|$ is greater than $\varepsilon$ has Lebesgue measure $\leq \varepsilon$.

## 2. The approach

We say that an operator $T$ between Banach spaces $X$ and $Y$ fuxes a copy of $L^{1}$ if $X$ contains a copy of $L^{1}$ on which the restriction of $T$ is an isomorphism on its image.

We denote by $\lambda$ Lebesgue's measure and by $\Sigma$ the $\sigma$-algebra of measurable sets. To simplify notation, for $A \in \Sigma$, we write $|A|=\lambda(A)$.

Consider an atomless subalgebra $\Sigma^{\prime}$ of $\Sigma$ and $A \in \Sigma^{\prime},|A|>0$. We denote by $L_{0}^{1}\left(A, \Sigma^{\prime}\right)$ the subspace of $L^{1}$ that consists of $\Sigma^{\prime}$-measurable functions that are zero outside $A$ and such that $\int f d \lambda=0$. Central to our approach will be the following theorem proved by Rosenthal [21], using the techniques developed by Kalton in [12].

Theorem 2.1 [21]. Consider a Banach space $E$ and an operator $T$ from $L^{1}$ to $E$ that fixes a copy of $L^{1}$. Then for some $\delta>0$, some atomless subalgebra $\Sigma^{\prime}$ of $\Sigma$, and some $A \in \Sigma^{\prime},|A|>0$, we have $\|T(f)\| \geq \delta\|f\|_{1}$ whenever $f \in L_{0}^{1}\left(A, \Sigma^{\prime}\right)$.

The application of this theorem to the three-space problem is as follows.
Corollary 2.2. Consider a subspace $X$ of $L^{1}$. The following are equivalent.
(2.1) The quotient map $L^{1} \rightarrow L^{1} / X$ does not fix a copy of $L^{1}$.
(2.2) For each $\delta>0$, each atomless subalgebra $\Sigma^{\prime}$ of $\Sigma$, and each set $A \in \Sigma^{\prime}$, $|A| \geq 0$, there exist $f \in L_{0}^{1}\left(A, \Sigma^{\prime}\right)$ and $g \in X$ such that $\|f\|_{1} \geq \frac{1}{4}$ and $\|f-g\|_{1} \leq \delta$.

The choice of $\frac{1}{4}$ in the condition $\|f\|_{1} \geq \frac{1}{4}$ is for convenience, any other positive number would be suitable.

Our approach will simply be to construct families of functions that satisfy (2.2) and then to take for $X$ their closed linear span. While a priori this construction would only ensure that the quotient map $L^{1} \rightarrow L^{1} / X$ is not an isomorphism on any space $L_{0}^{1}\left(A, \Sigma^{\prime}\right)$, Corollary 2.2 implies that it does fix any copy of $L^{1}$, and the fact that $L^{1} / X$ does not contain $L^{1}$ will then follow by a lifting argument (made possible by the special structure of $X$ ). The problem reduces then to finding an appropriate device that will prevent $X$ from containing $L^{1}$.

As a first step toward (2.2), we shall construct a countable family of functions $C_{n}$ that satisfies the following condition.
$\left(P_{n}\right)$ For each atomless subalgebra $\Sigma^{\prime}$ of $\Sigma$ and each set $A \in \Sigma^{\prime},|A|>2^{-n}$, there exist $f \in L_{0}^{1}\left(A, \Sigma^{\prime}\right)$ and $g \in C_{n}$ such that $\|f\|_{1} \geq \frac{1}{4}$ and $\|f-g\|_{1} \leq 2^{-n}$.

The purpose of the restriction $|A| \geq 2^{-n}$ is to allow $C_{n}$ to be uniformly bounded. How can we control the linear span of $C_{n}$ ? Since we are allowed an error of $2^{-n}$, there is nothing to lose by adding on another error of $2^{-n}$. We add to each function in $C_{n}$ a function of the type $2^{-n}|B|^{-1} 1_{B}$, where $|B|$ is very small. This can be done in such a way that $C_{n}$ becomes $2^{n+1}$-equivalent to the unit basis of $l^{1}$ while it still satisfies $\left(P_{n-1}\right)$. Thus, we can control the span of each $C_{n}$. However, $X$ has to contain infinitely many families $C_{n}$ in order to satisfy (2.2), and another idea is still needed there. This idea is to control the situation using convergence in measure. The functions of span $C_{n+1}$ will be very "peaky" (that is, they will be essentially supported by very small sets), compared to those of span $C_{n}$, and the span $X$ of $\bigcup_{n \geq 1} C_{n}$ will thus be isomorphic to the $l^{1}$-sum of the spaces span $C_{n}$; so $X$ will be an $l^{1}$-sum of spaces isomorphic to $l^{1}$. The idea of using a convergence in measure for such a purpose is not new. It has been successfully used in the work of Roberts [19] and subsequent work, e.g. [3].

How are we to choose the class $C_{n}$ ? The results of [9] indicate that we should not use functions $f$ for which $\|f\|_{\infty}$ and $\|f\|_{1}$ are of the same order, so it is natural to use functions $f$ for which $|f|$ resembles the function $(x \log (1 / \varepsilon))^{-1} 1_{[\varepsilon, 1]}$ for $\varepsilon$ small, or, more conveniently,

$$
\frac{1}{m} \sum_{i=0}^{m-1} 2^{i+1} 1_{\left[2^{-i-1}, 2^{-i}[ \right.}
$$

for $m$ large. How are we to choose the sign of $f$ ? Here there are several possibilities, but it turns out that one of the simplest choices, which is to take $f$ resembling $\frac{1}{2 n} \sum_{i=0}^{2 n-1}(-1)^{i} 2^{i+1} 1_{\left[2^{-i-1}, 2^{-i}[ \right.}$, is suitable for our purposes.

We denote by $\mathscr{F}_{k}$ the collection of the dyadic intervals $\left[l 2^{-k},(l+1) 2^{-k}[\right.$, $0 \leq l<2^{k}$ and by $\Sigma_{k}$ the algebra they generate. For $f \in L^{1}$, we denote by $E^{k}(f)$ its conditional expectation with respect to $\Sigma_{k}$, so, for $I \in \mathcal{F}_{k}$, the constant value of $E^{k}(f)$ on $I$ is $2^{k} \int_{I} f d \lambda$.

We now describe our basic class $D_{n}$ of functions (we will have $C_{n}=D_{q(n)}$ for a fast growing sequence $(q(n))$ ). Consider a sequence $B_{0}, \ldots, B_{2 n-1}$ of sets and a sequence $\sigma(0)<\cdots<\sigma(2 n-1)$ of numbers with the following properties:
(2.3) $\quad B_{i}$ is $\Sigma_{\sigma(i)}$ measurable.
(2.4) For $0 \leq i<2 n-1, E^{\sigma(i)}\left(1_{B_{i+1}}\right)=\frac{1}{2} 1_{B_{i}}$.

This latter condition means that $B_{i+1} \subset B_{i}$ and that for each $I \in \mathscr{I}_{\sigma(i)}$, $I \subset B_{i}$, we have $\left|I \cap B_{i+1}\right|=\frac{1}{2}|I|$.

We set

$$
\begin{equation*}
g=\frac{1}{2 n\left|B_{0}\right|} \sum_{0 \leq i \leq 2 n-1}(-1)^{i} 2^{i} 1_{B_{i}} . \tag{2.5}
\end{equation*}
$$

If we set $B_{2 n}=\varnothing$, we observe that

$$
\begin{equation*}
g=\frac{1}{2 n\left|B_{0}\right|} \sum_{0 \leq i \leq 2 n-1} a_{i} 1_{B_{i} \backslash B_{i+1}}, \tag{2.6}
\end{equation*}
$$

where $a_{i}=\sum_{0 \leq l \leq i}(-1)^{l} 2^{l}=\left((-1)^{i} 2^{i+1}+1\right) / 3$.
We denote by $\bar{D}_{n}$ the class of functions given by (2.5) for all possible choices of $B_{0}, \ldots, B_{2 n-1}$ that satisfy (2.3), (2.4), and $\left|B_{0}\right| \geq 2^{-n}$. We observe that

$$
\begin{equation*}
g=D_{n} \Rightarrow\|g\|_{\infty} \leq 2^{3 n} \tag{2.7}
\end{equation*}
$$

This might be the place to mention that throughout this paper we always use the simplest estimates sufficient for our purpose, even when they are rather crude as in (2.7).

We also have

$$
\begin{equation*}
g \in D_{n} \Rightarrow\|g\|_{1} \geq \frac{1}{3} \tag{2.8}
\end{equation*}
$$

Indeed, since $\left|B_{i}\right|=2^{-i}\left|B_{0}\right|$,

$$
\begin{aligned}
\|g\|_{1} & =\frac{1}{2 n\left|B_{0}\right|} \sum_{0 \leq i \leq 2 n-1}\left|a_{i}\right|\left|B_{i} \backslash B_{i+1}\right| \\
& \geq \frac{1}{2 n\left|B_{0}\right|} \sum_{0 \leq i \leq 2 n-1} \frac{2^{i+1}+(-1)^{i}}{3} 2^{-i-1}\left|B_{0}\right| \\
& =\frac{1}{6 n}\left(\sum_{0 \leq i \leq 2 n-1}\left(1+(-1)^{i} 2^{-i-1}\right)\right) \geq \frac{1}{3}
\end{aligned}
$$

since $\sum_{0 \leq i \leq 2 n-1}(-1)^{i} 2^{-i-1} \geq 0$. We now prove that property $\left(P_{n}\right)$ (and even more) holds.
Proposition 2.3. Consider an atomless subalgebra $\Sigma^{\prime}$ of $\Sigma$ and a set $A \in \Sigma^{\prime}$, $|A|>2^{-n}$. Then for each $\eta>0$, there exist $f \in L_{0}^{1}\left(A, \Sigma^{\prime}\right)$ and $g \in D_{n}$ such that $\|f-g\|_{1} \leq \eta$.

It is not stated here that we control $\|f\|_{1}$, but of course $\|f\|_{1} \geq\|g\|_{1}-\eta$, and, as we can take $\eta<\frac{1}{12}$, we have from (2.8) that $\|f\|_{1} \geq \frac{1}{4}$.

The proof of Proposition 2.3 is a simple exercise. It will rely on the following lemma.
Lemma 2.4. Consider $n \geq 0, A \in \Sigma, \varepsilon>0, k \geq 0, B \in \Sigma_{k}$ such that $|A \Delta B| \leq \varepsilon$. Consider $A^{\prime} \subset A$ such that for all $I \in \mathscr{J}_{k}$ we have $\left|A^{\prime} \cap I\right|=\frac{1}{2}|A \cap I|$. Then we can find $m>k$ and $B^{\prime} \in \Sigma_{m}, B^{\prime} \subset B$, such that $\left|A^{\prime} \Delta B^{\prime}\right| \leq 4 \varepsilon$ and $E^{k}\left(1_{B^{\prime}}\right)=\frac{1}{2} 1_{B}$.
Proof. Consider first $A_{1}=A^{\prime} \cap B$. Since $A^{\prime} \backslash A_{1}=A^{\prime} \backslash B \subset A \backslash B$, we have $\left|A^{\prime} \backslash A_{1}\right| \leq \varepsilon$, and thus $\left|A^{\prime} \Delta A_{1}\right| \leq \varepsilon$. We consider now a set $A_{2} \in \Sigma, A_{1} \subset A_{2} \subset$ $B$, such that whenever $I \in \mathscr{J}_{k}, I \subset B$, we have

$$
\left|A_{2} \cap I\right|=\frac{1}{2}|I| ; \quad A \cap A_{2} \cap I=A_{1} \cap I\left(=A^{\prime} \cap I\right)
$$

Since $A_{2} \cap A=A_{1}$, we have $A_{2} \backslash A_{1} \subset B \backslash A$, and hence $\left|A_{2} \backslash A_{1}\right| \leq \varepsilon$, so that $\left|A_{2} \Delta A_{1}\right| \leq \varepsilon$. We can find $m>k$ and $B_{1} \in \Sigma_{m}$ such that $\left|B_{1} \Delta A_{2}\right| \leq 2^{-k} \varepsilon$. For each $I \in \mathscr{J}_{k}, I \subset B$, we have

$$
\left|\left|B_{1} \cap I\right|-2^{-k-1}\right|=\left\|B _ { 1 } \cap I \left|-\left|A_{2} \cap I \| \leq\left|B_{1} \Delta A_{2}\right| \leq 2^{-k} \varepsilon\right.\right.\right.
$$

Clearly, we can find $B^{\prime} \in \Sigma_{m}, B^{\prime} \subset B$, such that for all $I \in \mathscr{I}_{k}, I \subset B$,

$$
\left|B^{\prime} \cap I\right|=2^{-k-1}=\frac{1}{2}\left|B^{\prime} \cap I\right| ; \quad\left|\left(B^{\prime} \cap I\right) \Delta\left(B_{1} \cap I\right)\right| \leq 2^{-k} \varepsilon
$$

It follows that $\left|B^{\prime} \Delta B_{1}\right| \leq \varepsilon$. Thus,

$$
\left|A^{\prime} \Delta B^{\prime}\right| \leq\left|A^{\prime} \Delta A_{1}\right|+\left|A_{1} \Delta A_{2}\right|+\left|A_{2} \Delta B_{1}\right|+\left|B_{1} \Delta B^{\prime}\right| \leq 4 \varepsilon
$$

Proof of Proposition 2.3. Let $\varepsilon>0$ such that $\varepsilon<|A|-2^{-n}$. By induction over $i, 0 \leq i \leq 2 n-1$, we construct numbers $\sigma(i)$, sets $A_{i}, B_{i}$ that satisfy conditions (2.3), (2.4), and the following:

$$
A_{0}=A ; \quad A_{i+1} \subset A_{i} ; \quad A_{i+1} \in \Sigma^{\prime} ; \quad\left|A_{i+1}\right|=\frac{1}{2}\left|A_{i}\right| ; \quad\left|A_{i} \Delta B_{i}\right| \leq 4^{i} \varepsilon
$$

To start the induction, we take $B_{0} \in \Sigma_{\sigma(0)}$ such that $\left|A \Delta B_{0}\right| \leq \varepsilon$.
Suppose that $A_{i}, B_{i}, \sigma(i)$ have been constructed. Since $\Sigma^{\prime}$ is atomless, its restriction to $A_{i}$ is atomless. By Liapounov's convexity theorem, we can find $A_{i+1} \subset A_{i}, A_{i+1} \in \Sigma^{\prime}$, such that

$$
\int_{A_{i+1}} 1_{I} d \lambda=\left|A_{i+1} \cap I\right|=\frac{1}{2}\left|A_{i} \cap I\right|
$$

for all $I \in \mathscr{J}_{\sigma(i)}$. In particular, $\left|A_{i+1}\right|=\frac{1}{2}\left|A_{i}\right|$. By Lemma 2.4, we can find $\sigma(i+1)>\sigma(i)$ and $B_{i+1} \in \Sigma_{\sigma(i+1)}$ such that $\left|A_{i+1} \Delta B_{i+1}\right| \leq 4\left|A_{i} \Delta B_{i}\right|$ and $E^{\sigma(i+1)}\left(1_{B_{i+1}}\right)=\frac{1}{2} 1_{B_{i}}$. This completes the construction.

The function

$$
f=\frac{1}{2 n\left|B_{0}\right|} \sum_{0 \leq i \leq 2 n-1}(-1)^{i} 2^{i} 1_{A_{i}}
$$

is $\Sigma^{\prime}$-measurable and supported by $A_{0}=A$. Moreover, since $\left|A_{i}\right|=2^{-i}\left|A_{0}\right|$ for $i \leq 2 n-1$, we have $\int f d \lambda=0$. Thus, $f \in L^{0}\left(A, \Sigma^{\prime}\right)$. We consider the function

$$
g=\frac{1}{2 n\left|B_{0}\right|} \sum_{0 \leq i \leq 2 n-1}(-1)^{i} 2^{i} 1_{B_{i}}
$$

By definition, $g \in D_{n}$, since $\left|B_{0}\right| \geq|A|-\varepsilon \geq 2^{-n}$. Moreover,

$$
\begin{aligned}
\|f-g\|_{1} & =\frac{1}{2 n\left|B_{0}\right|} \sum_{0 \leq i \leq 2 n-1} 2^{i}\left\|1_{B_{i}}-1_{A_{i}}\right\|_{1} \\
& \leq \frac{1}{2 n\left|B_{0}\right|} \sum_{0 \leq i \leq 2 n-1} 2^{i} 4^{i} \varepsilon=c(n) \varepsilon
\end{aligned}
$$

for some number $c(n)$ depending on $n$ only. Since $n$ is fixed, $\|f-g\|_{1}$ can be made arbitrarily small.

## 3. Control in measure

We denote by conv $D_{n}$ the absolute convex hull of $D_{n}$, i.e., the set of functions $\sum_{l \in L} c_{l} f_{l}$, where $L$ is a finite set, $f_{l} \in D_{n}, \sum_{l \in L}\left|c_{l}\right| \leq 1$.

The cornerstone of our construction is the following.
Theorem 3.1. Given $\varepsilon>0$, there exists $n>0$ such that each function $f$ in conv $D_{n}$ satisfies $|\{|f| \geq \varepsilon\}| \leq \varepsilon$.

The proof will show that one can actually take $n$ of order $\varepsilon^{-6} \log \left(\frac{1}{\varepsilon}\right)$. This estimate (that has no reason to be sharp) is irrelevant for our purposes.

The proof of Theorem 3.1 will be considerably clarified by the fact that $D_{n}$ is contained in the convex hull of a family $F_{n}$ (briefly considered in the introduction), which has the property of Theorem 3.1, but that consists of functions much simpler than those of $D_{n}$. We recall the definition of these functions now. Given $x \in[0,1]$ and $k \geq 0$, we denote by $I(x, k)$ the unique dyadic interval $I \in \mathscr{I}_{k}$ that contains $x$. Given an increasing sequence $\sigma=(\sigma(0), \ldots, \sigma(2 n-1)), \sigma(0)<\sigma(1)<\cdots<\sigma(2 n-1)$, we set

$$
\begin{equation*}
f(x, \sigma)=\frac{1}{2 n} \sum_{0 \leq i \leq 2 n-1}(-1)^{i} 2^{\sigma(i)} 1_{I(x, \sigma(i))} \tag{3.1}
\end{equation*}
$$

The reader might have observed that $\left(f_{i}(x, \sigma)\right)_{0 \leq i<n}$ is a martingale difference sequence. Unfortunately, the filtration $\left(\Sigma_{\sigma(2 i)}\right)_{0 \leq i<n}$ with respect to which it has this property depends on $f$; this seems to prevent any simple use of martingale theory to prove Theorem 3.3. Since $|I(x, \sigma(i))|=2^{-\sigma(i)}$, we have $\|f(x, \sigma)\|_{1} \leq 1$. It is worthwhile to note that if $C=I(x, \sigma(2 n-1))$, we have $E^{\sigma(i)}\left(1_{C}\right)=2^{\sigma(i)-\sigma(2 n-1)} 1_{I(x, \sigma(i))}$, and thus,

$$
\begin{equation*}
f(x, \sigma)=\frac{1}{2 n} \sum_{0 \leq i \leq 2 n-1} \frac{(-1)^{i}}{|C|} E^{\sigma(i)}\left(1_{C}\right) \tag{3.2}
\end{equation*}
$$

We consider the class $F_{n}$ of all functions of type (3.1) for all possible choices of $x$ and $\sigma$.

Proposition 3.2. The convex hull of $F_{n}$ contains $D_{n}$.
Proof. Consider $g$ given by (2.5). From condition (2.4) we see that for $0 \leq$ $i \leq 2 n-1$, we have

$$
\frac{1}{\left|B_{2 n-1}\right|} E^{\sigma(i)}\left(1_{B_{2 n-1}}\right)=\frac{2^{2 n-1}}{\left|B_{0}\right|} E^{\sigma(i)}\left(1_{B_{2 n-1}}\right)=\frac{2^{i}}{\left|B_{0}\right|} 1_{B_{i}}
$$

so that

$$
\begin{equation*}
g=\frac{1}{2 n} \sum_{0 \leq i \leq 2 n-1} \frac{(-1)^{i}}{\left|B_{2 n-1}\right|} E^{\sigma(i)}\left(1_{B_{2 n-1}}\right) \tag{3.3}
\end{equation*}
$$

We can write $B_{2 n-1}$ as a disjoint union $B_{2 n-1}=\bigcup_{l \in L} C_{l}, C_{l} \in \mathscr{J}_{\sigma(2 n-1)}$. Thus, (3.3) becomes

$$
g=\frac{1}{\operatorname{card} L} \sum_{l \in L}\left(\frac{1}{2 n} \sum_{0 \leq i \leq 2 n-1} \frac{(-1)^{i}}{\left|C_{l}\right|} E^{\sigma(i)}\left(1_{C_{l}}\right)\right)
$$

and, from (3.2), this shows that $g$ belongs to the convex hull of $F_{n}$.
Thus, in order to prove Theorem 3.1, it suffices to prove the following.
Theorem 3.3. Given $\varepsilon>0$, there exists $n>0$ such that each function $f$ in $\operatorname{conv} F_{n}$ satisfies $|\{|f| \geq \varepsilon\}| \leq \varepsilon$.

Before we can start the proof, we need some notation. In the entire sequel, $n$ is fixed. We denote by $S_{n}$ the collection of sequences $\sigma=(\sigma(i))_{0 \leq i \leq 2 n-1}$; $\sigma(0)<\sigma(1)<\cdots<\sigma(2 n-1)$. For $0 \leq i<n, \sigma \in S_{n}$, we set

$$
f_{i}(x, \sigma)=2^{\sigma(2 i)} 1_{I(x, \sigma(2 i))}-2^{\sigma(2 i+1)} 1_{I(x, \sigma(2 i+1))}
$$

For $p \leq n$, we set

$$
\begin{aligned}
f(x, \sigma, p) & =\frac{1}{2 n} \sum_{0 \leq i \leq p-1} f_{i}(x, \sigma) \\
& =\frac{1}{2 n} \sum_{0 \leq i \leq 2 p-1}(-1)^{i} 2^{\sigma(i)} 1_{I(x, \sigma(i))}
\end{aligned}
$$

Thus, $f(x, \sigma, n)=f(x, \sigma)$ and $\|f(x, \sigma, p)\|_{1} \leq p / n$. The main ingredient of the proof of Theorem 3.3 is the following.
Proposition 3.4. Consider a finite index set $L$ and numbers $c_{l}$ with $\sum_{l \in L}\left|c_{l}\right| \leq$ 1. For $l \in L$, consider $x_{l} \in[0,1], \sigma_{l} \in S_{n}, p_{l} \leq n$. Consider $f=$ $\sum_{l \in L} c_{l} f\left(x_{l}, \sigma_{l}, p_{l}\right)$ and $r>0$. Assume $\|f\|_{1} \geq 2^{-r}$ and $n \geq 3(r+1) 2^{3 r+7}$. Then we can find $A \in \Sigma,|A| \leq 2^{-2 r}$, and numbers $q_{l} \leq p_{l}$ that satisfy the following conditions:

$$
\begin{equation*}
\frac{1}{n} \sum_{l \in L} c_{l} q_{l} \leq \frac{1}{n} \sum_{l \in L} c_{l} p_{l}-2^{-r-1} \tag{3.4}
\end{equation*}
$$

(3.5) The function $f^{\prime}=\sum_{l \in L} c_{l} f\left(x_{l}, \sigma_{l}, q_{l}\right)$ coincides with $f$ outside $A$.

Observe that, since $\|f(x, \sigma, p)\|_{1} \leq p / n$, we have $\|f\|_{1} \leq \frac{1}{n} \sum_{l \in L}\left|c_{l}\right| p_{l}$. If we think of $\frac{1}{n} \sum_{l \in L}\left|c_{l}\right| p_{l}$ as measuring the size of $f$, this means that outside the very small set $A, f$ coincides with the function $f^{\prime}$, which has a size significantly smaller than the size of $f$. Iterating the procedure will yield the proof of Theorem 3.3.
Proof. Let us first single out the main ingredient of the proof. For a function $g,\|g\|_{\infty} \leq 1, k \geq 0$, and $\alpha>\beta$, let us set

$$
\begin{aligned}
A(g, k, \alpha, \beta)=\{x ; \exists m(0)<\cdots<m(2 k-1) & ; \forall i, 0 \leq i<k \\
& \left.E^{m(2 i)}(g)(x) \geq \alpha ; E^{m(2 i+1)}(g)(x) \leq \beta\right\}
\end{aligned}
$$

Then

$$
\begin{equation*}
|A(g, k, \alpha, \beta)| \leq\left(\frac{1-\alpha}{1-\beta}\right)^{k} \tag{3.6}
\end{equation*}
$$

This follows, e.g., from an upcrossing inequality of Dubin as in [16, p. 27] applied to the positive martingale $E^{n}(1-g)$. Let us note also that (3.6) can be easily proved directly by induction over $k$ and that we can even replace the bound of $(3.6)$ by $((1-\alpha) /(1-\beta))^{k}((1+\beta) /(1+\alpha))^{k-1}$ (which makes no difference for our purposes).

Consider $g=\operatorname{sign} f$, so $\|g\|_{\infty} \leq 1$. We have

$$
2^{-r} \leq\|f\|_{1}=\int f g d \lambda=\frac{1}{2 n} \sum_{l \in L} c_{l} \sum_{0 \leq i \leq p_{l}-1} \int g f_{i}\left(x_{l}, \sigma_{l}\right) d \lambda
$$

We set $k=3(r+1) 2^{r+3}$. For $l \in L$, denote by $q_{l}$ the largest integer $\leq p_{l}$ such that

$$
\operatorname{card}\left\{i: 0 \leq i \leq q_{l} ; \int g f_{i}\left(x_{l}, \sigma_{l}\right) d \lambda \geq 2^{-r-1}\right\} \leq 2^{r+3} k
$$

Denote by $L^{\prime}$ the set of $l \in L$ such that $q_{l}<p_{l}$. For $l \in L^{\prime}$, we have

$$
\begin{equation*}
\operatorname{card}\left\{i: 0 \leq i \leq q_{l} ; \int g f_{i}\left(x_{l}, \sigma_{l}\right) d \lambda \geq 2^{-r-1}\right\}=2^{r+3} k \tag{3.7}
\end{equation*}
$$

We observe that

$$
\int g 2^{k} 1_{I(x, k)} d \lambda=E^{k}(g)(x)
$$

Thus,

$$
\int g f_{i}\left(x_{l}, \sigma_{l}\right)=E^{\sigma_{l}(2 i)}(g)\left(x_{l}\right)-E^{\sigma_{l}(2 i+1)}(g)\left(x_{l}\right)
$$

Hence, when $\int g f_{i}\left(x_{l}, \sigma_{l}\right) \geq 2^{-r-1}$, we can find an integer $-2^{r+2} \leq s<2^{r+2}$ such that

$$
\begin{equation*}
E^{\sigma_{l}(2 i)}(g)\left(x_{l}\right) \geq(s+1) 2^{-r-2}, \quad E^{\sigma_{l}(2 i+1)}(g)\left(x_{l}\right) \leq s 2^{-r-2} \tag{3.8}
\end{equation*}
$$

Thus, (3.7) shows that for $l \in L^{\prime}$, we can find $s,-2^{r+2} \leq s<2^{r+2}$, such that (3.8) holds for at least $k$ indexes $i \leq q_{l}-1$. Hence, we have

$$
I\left(x_{l}, \sigma_{l}\left(2 q_{l}-1\right)\right) \subset A_{s}=A\left(g, k,(s+1) 2^{-r-2}, s 2^{-r-2}\right)
$$

From (3.6) we have

$$
\left|A_{s}\right| \leq\left(\frac{1-(s+1) 2^{-r-2}}{1-s 2^{-r-2}}\right)^{k} \leq\left(1-2^{-r-3}\right)^{k} \leq e^{-k 2^{-r-3}} \leq 2^{-3 r-3}
$$

We define $A=\bigcup_{-2^{r+2} \leq s<2^{r+2}} A_{s}$, so that $|A| \leq 2^{-2 r}$, and we have $I\left(x_{l}, \sigma_{l}(i)\right)$ $\subset A$ for $i \geq 2 q_{l}$, and hence $f$ and $f^{\prime}=\sum_{l \in L}\left|c_{l}\right| f\left(x_{l}, \sigma_{l}, q_{l}\right)$ coincide outside $A$. Since for each $l \in L$,

$$
\left|\int g f_{i}\left(x_{l}, \sigma_{l}\right) d \lambda\right| \leq\|g\|_{\infty}\left\|f_{i}\left(x_{l}, \sigma_{l}\right)\right\|_{1} \leq 2
$$

we have

$$
\begin{aligned}
\left|\int g f\left(x_{l}, \sigma_{l}, q_{l}\right) d \lambda\right| & \leq \frac{1}{2 n}\left(2^{r+4} k+n 2^{-r-1}\right) \\
& \leq 2^{r+3} \frac{k}{n}+2^{-r-2} \leq 2^{-r-1}
\end{aligned}
$$

This implies that $\sum_{l \in L} c_{l} \int g f\left(x_{l}, \sigma_{l}, q_{l}\right) d \lambda \leq 2^{-r-1}$. On the other hand, since $\int g f d \lambda \geq 2^{-r}$, this implies that

$$
\begin{align*}
2^{-r-1} & \leq \sum_{l \in L} c_{l} \int g\left(f\left(x_{l}, \sigma_{l}, p_{l}\right)-f\left(x_{l}, \sigma_{l}, q_{l}\right)\right) d \lambda  \tag{3.9}\\
& \leq \sum_{l \in L}\left|c_{l}\right|\left\|f\left(x_{l}, \sigma_{l}, p_{l}\right)-f\left(x_{l}, \sigma_{l}, q_{l}\right)\right\|_{1}
\end{align*}
$$

Now $\left\|f\left(x_{l}, \sigma_{l}, p_{l}\right)-f\left(x_{l}, \sigma_{l}, q_{l}\right)\right\|_{1} \leq \frac{1}{n}\left(p_{l}-q_{l}\right)$, and thus, (3.4) follows from (3.9).

We now prove Theorem 3.3 and, actually, the following more precise statement. Consider $r>0$, and suppose that $n \geq 3(r+1) 2^{3 r+7}$. Consider a finite index set $L$, and for $l \in L$, consider $x_{l} \in[0,1], \sigma_{l} \in S_{n}, c_{l} \in \mathbf{R}$. Assume $\Sigma_{l \in L}\left|c_{l}\right| \leq 1$. Let $f=\sum_{l \in L} c_{l} f\left(x_{l}, \sigma_{l}\right)$. Then $\left|\left\{|f| \geq 2^{-r / 2}\right\}\right| \leq 2^{-r / 2+2}$.

By induction over $k, k \geq 0$, we will construct numbers $\left(p_{l}^{k}\right)_{l \in L}, 0 \leq p_{l}^{k} \leq$ $n$, and sets $A_{k} \in \Sigma^{\prime}$ that have the following properties:

$$
\begin{equation*}
\frac{1}{n} \sum_{l \in L}\left|c_{l}\right| p_{l}^{k} \leq \frac{1}{n} \sum_{l \in L}\left|c_{l}\right| p_{l}^{k-1}-2^{-r-1} \tag{3.10}
\end{equation*}
$$

$\left|A_{k}\right| \leq 2^{-2 r}, f$ and $f^{k}=\sum_{l \in L} c_{l} f\left(x_{l}, \sigma_{l}, p_{l}^{k}\right)$ coincide outside $A_{k}^{\prime}=\bigcup_{i \leq k} A_{i}$.
To start the induction, we set $p_{l}^{0}=n$ for $l \in L, A_{0}=\varnothing$.
The induction step is done by applying Proposition 3.2. The construction stops at the first value of $k$ for which $\left\|f^{k}\right\|_{1} \leq 2^{-r}$. Thus, $\left|\left\{\left|f^{k}\right| \geq 2^{-r / 2}\right\}\right| \leq$ $2^{-r / 2}$. From (3.10), we have

$$
0 \leq \frac{1}{n} \sum_{l \in L}\left|c_{l}\right| p_{l}^{k} \leq 1-k 2^{-r-1}
$$

and thus, $k \leq 2^{r+1}$. Hence, we have $\left|A_{k}^{\prime}\right| \leq k 2^{-2 r} \leq 2^{-r+1}$. Since $f$ and $f^{k}$ coincide outside $A_{k}^{\prime}$, we have $\left|\left\{|f| \geq 2^{-r / 2}\right\}\right| \leq 2^{-r / 2}+2^{-r+1} \leq 2^{2-r / 2}$.

## 4. The three-space problem

Our task is, as described earlier, to construct small perturbations of the functions in $D_{n}$ in such a way that their span will be isomorphic to $l^{1}$. By induction over $l \geq 1$, we construct a sequence $q(l)$ and a disjoint sequence $\left(B_{l}\right)$ of measurable sets that satisfy the following conditions:

$$
\begin{gather*}
0<\left|B_{l}\right| \leq 2^{-3 q(l-1)-2 l-2},  \tag{4.1}\\
\forall f \in \operatorname{conv} D_{q(l)},\left|\left\{|f| \geq 2^{-2 l}\right\}\right| \leq 2^{-l-1} \min _{i \leq l}\left|B_{i}\right| \tag{4.2}
\end{gather*}
$$

We start the induction with $B_{1}=\left[0,2^{-4}\right]$. At each stage of the construction, we pick $q(l)$ such that (4.2) holds, using Theorem 3.1, then $B_{l+1}$ satisfying (4.1). We consider an enumeration $\left(f_{n, k}\right)_{k \geq 1}$ of $C_{n}=D_{q(n)}$. We consider a one-to-one map $\varphi: \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$, and we set $A(n, k)=B_{\varphi(n, k)}$. Finally, we set

$$
g_{n, k}=f_{n, k}+2^{-n} \frac{1}{|A(n, k)|} 1_{A(n, k)}
$$

Thus, $\left\|g_{n, k}\right\|_{1} \leq 1+2^{-n} \leq 2$, and $\left\|g_{n, k}-f_{n, k}\right\|_{1} \leq 2^{-n}$.
Proposition 4.1. Consider a family $\left(a_{n, k}\right)_{n, k \geq 1}$ of numbers. Assume that $\sum_{n \geq 1} 2^{-n} \sum_{k \geq 1}\left|a_{n, k}\right|<\infty$. Then the series $\sum_{n \geq 1}\left(\sum_{k \geq 1} a_{n, k} g_{n, k}\right)$ converges in measure. Denoting, for simplicity, its sum by $\sum_{n, k \geq 1} a_{n, k} g_{n, k}$, we have

$$
\begin{equation*}
\left\|\sum_{n, k \geq 1} a_{n, k} g_{n, k}\right\|_{1} \leq 1 \Rightarrow \sum_{n, k \geq 1} 2^{-n}\left|a_{n, k}\right| \leq 4 \tag{4.3}
\end{equation*}
$$

Proof. For $n \geq 1$, let $b_{n}=\sum_{k \geq 1}\left|a_{n, k}\right|$, so that $\sum_{n \geq 1} 2^{-n} b_{n}<\infty$. Let $h_{n}=\sum_{k \geq 1} a_{n, k} f_{n, k}$, so that $h_{n} \in b_{n} N_{n}$, where $N_{n}$ is the norm closure of conv $C_{n}$. It follows from (4.2) that the series $\sum_{n \geq 1}\left(\sum_{k \geq 1} a_{n, k} g_{n, k}\right)$ converges in measure. From (2.7) we have that $\left\|h_{n}\right\|_{\infty} \leq b_{n} 2^{3 q(n)}$ and thus, for $l>n$, by (4.1),

$$
\begin{equation*}
\int_{B_{l}}\left|h_{n}\right| d \lambda \leq b_{n} 2^{3 q(n)}\left|B_{l}\right| \leq b_{n} 2^{-2 l-2} \leq b_{n} 2^{-l-n-2} \tag{4.4}
\end{equation*}
$$

Let $A_{n}=\left|\left\{\left|h_{n}\right| \geq b_{n} 2^{-2 n}\right\}\right|$. From (4.2), we have

$$
\left|A_{n}\right| \leq 2^{-n-1} \min _{i \leq n}\left|B_{i}\right|
$$

Let $A_{l}^{\prime}=\bigcup_{n \geq l} A_{n}$. Since $\left|A_{n}\right| \leq 2^{-n-1}\left|B_{l}\right|$ for $n \geq l$, we have $\left|A_{l}^{\prime}\right| \leq 2^{-l}\left|B_{l}\right| \leq$ $\left|B_{l}\right| / 2$. Let $B_{l}^{\prime}=B_{l} \backslash A_{l}^{\prime}$. Thus, for $l \leq n$,

$$
\begin{equation*}
\int_{B_{l}^{\prime}}\left|h_{n}\right| d \lambda \leq b_{n} 2^{-2 n}\left|B_{l}^{\prime}\right| \leq b_{n} 2^{-l-n-2} \tag{4.5}
\end{equation*}
$$

From (4.4) and (4.5) we see that for all $l, n \geq 1$,

$$
\int_{B_{l}^{\prime}}\left|h_{n}\right| d \lambda \leq b_{n} 2^{-l-n-2}
$$

Thus, we have

$$
\begin{equation*}
\int_{B_{l}^{\prime}}\left|\sum_{n, k \geq 1} a_{n, k} f_{n, k}\right| d \lambda=\int_{B_{l}^{\prime}}\left|\sum_{n \geq 1} h_{n}\right| d \lambda \leq 2^{-l-2} \sum_{n \geq 1} 2^{-n} b_{n} \tag{4.6}
\end{equation*}
$$

Consider now $r, s \geq 1$ and $l=\varphi(r, s)$. We have $B_{l} \cap A(n, k)=\varnothing$ unless $n=r, k=s$. So, since $\left|B_{l}^{\prime}\right| \geq\left|B_{l}\right| / 2$, we have

$$
\int_{B_{l}^{\prime}}\left|\sum_{n, k \geq 1} 2^{-n} a_{n, k} \frac{1}{|A(n, k)|} 1_{A(n, k)}\right| d \lambda \geq \int_{B_{l}^{\prime}}\left|2^{-r} a_{r, s} \frac{1}{\left|B_{l}\right|} 1_{B_{l}}\right| d \lambda \geq 2^{-r-1}\left|a_{r, s}\right|
$$

Thus, with (4.6) we get

$$
\int_{B_{l}^{\prime}}\left|\sum_{n, k \geq 1} a_{n, k} g_{n, k}\right| d \lambda \geq 2^{-r-1}\left|a_{r, s}\right|-2^{-l-2} \sum_{n \geq 1} 2^{-n} b_{n} .
$$

Summing this inequality over all the values of $r, s$ gives, since the sets $B_{l}^{\prime}$ are disjoint,

$$
\left\|\sum_{n, k} a_{n, k} g_{n, k}\right\|_{1} \geq \frac{1}{2} \sum_{r, s \geq 1} 2^{-r}\left|a_{r, s}\right|-\frac{1}{4} \sum_{n \geq 1} 2^{-n} b_{n}=\frac{1}{4} \sum_{n, k \geq 1} 2^{-n}\left|a_{n, k}\right|
$$

A consequence of (4.3) is that for each $n$ and each sequence $\left(a_{k}\right)$ of numbers, all but finitely many zero, we have

$$
2^{-n-2} \sum_{k \geq 1}\left|a_{k}\right| \leq\left\|\sum_{k \geq 1} a_{k} g_{n, k}\right\|_{1} \leq 2 \sum_{k}\left|a_{k}\right| .
$$

Thus, the closed linear span $H_{n}$ of the sequence $\left(g_{n, k}\right)_{k \geq 1}$ is isomorphic to $l^{1}$. We now show that the span of the spaces $H_{n}$ is isomorphic to their $l^{1}$-sum.
Proposition 4.2. For each sequence $h_{n} \in H_{n}$ that is eventually zero, we have

$$
\frac{1}{20} \sum_{n \geq 1}\left\|h_{n}\right\|_{1} \leq\left\|\sum_{n \geq 1} h_{n}\right\|_{1} \leq \sum_{n \geq 1}\left\|h_{n}\right\|_{1}
$$

Proof. It is enough to consider the case where each function $h_{n}$ is a finite $\operatorname{sum} \sum_{k \geq 1} a_{n, k} g_{n, k}$. We have to prove that if $\left\|\sum_{n \geq 1} h_{n}\right\|_{1} \leq 1$, we have $\sum_{n \geq 1}\left\|h_{n}\right\|_{1} \leq 20$. From Proposition 4.1, we know that $\sum_{n, k \geq 1} 2^{-n}\left|a_{n, k}\right| \leq$ 4. Since $\left\|g_{n, k}-f_{n, k}\right\|_{1}=2^{-n}$, we have $\left\|\sum_{n \geq 1} u_{n}\right\|_{1} \leq 5$, where $u_{n}=$ $\sum_{k \geq 1} a_{n, k} f_{n, k}$. Since $\sum_{k \geq 1}\left|a_{n, k}\right| \leq 2^{n+2}, u_{n}$ belongs to the closure of $2^{n+2} \operatorname{conv} C_{n}$. Thus, Proposition 4.2 is a consequence of the inequality

$$
\sum_{n \geq 1}\left\|h_{n}\right\|_{1} \leq \sum_{n \geq 1}\left\|u_{n}\right\|_{1}+\sum_{n, k \geq 1} 2^{-n}\left|a_{n, k}\right| \leq \sum_{n \geq 1}\left\|u_{n}\right\|_{1}+4
$$

and of the following.
Proposition 4.3. Consider a sequence $u_{n} \in 2^{n+2} N_{n}$, where $N_{n}$ is the closure of conv $C_{n}$. Then the series $\sum_{n \geq 1} u_{n}$ converges in measure, and if its sum belongs to $L^{1}$, we have

$$
\sum_{n \geq 1}\left\|u_{n}\right\|_{1} \leq\left\|\sum_{n \geq 1} u_{n}\right\|_{1}+6
$$

Proof. From (2.7), we have $\left\|u_{n}\right\|_{\infty} \leq 2^{3 q(n)+n+2}$. Set $A_{n}=\left\{\left|u_{n}\right| \geq 2^{-n+2}\right\}$. From (4.2), (4.1), we have

$$
\left|A_{n}\right| \leq 2^{-n-1}\left|B_{n}\right| \leq 2^{-3 q(n-1)-3 n-3}
$$

Hence, if we define $A_{n}^{\prime}=\bigcup_{i>n} A_{i}$, we have $\left|A_{n}^{\prime}\right| \leq 2^{-3 q(n)-3 n-2}$, and thus, for $i \leq n$,

$$
\begin{equation*}
\int_{A_{n}^{\prime}}\left|u_{i}\right| d \lambda \leq 2^{3 q(i)+i+2}\left|A_{n}^{\prime}\right| \leq 2^{-2 n} \tag{4.7}
\end{equation*}
$$

We define $L_{n}=A_{n} \backslash A_{n}^{\prime}$. For $i>n$, since $\left|u_{i}\right| \leq 2^{-i+2}$ on $L_{i}$, we have

$$
\int_{L_{n}}\left|u_{i}\right| d \lambda \leq 2^{-i+2}\left|L_{n}\right| \leq 2^{-i-2}
$$

Thus,

$$
\int_{L_{n}}\left|\sum_{i \neq n} u_{i}\right| d \lambda \leq n 2^{-2 n}+2^{-n-2} \leq 2^{-n}
$$

and hence,

$$
\int_{L_{n}}\left|u_{n}\right| d \lambda \leq \int_{L_{n}}\left|\sum_{i \geq 1} u_{i}\right| d \lambda+2^{-n}
$$

Summing over $n$ gives, since the sets $L_{n}$ are disjoint,

$$
\begin{equation*}
\sum_{n \geq 1} \int_{L_{n}}\left|u_{n}\right| d \lambda \leq\left\|\sum_{n \geq 1} u_{n}\right\|_{1}+1 \tag{4.8}
\end{equation*}
$$

Since $\left|u_{n}\right| \leq 2^{-n+2}$ outside $A_{n}$, using (4.7), we have

$$
\begin{equation*}
\left\|u_{n}\right\|_{1} \leq \int_{L_{n}}\left|u_{n}\right| d \lambda+2^{-n+2}+2^{-2 n} \tag{4.9}
\end{equation*}
$$

From (4.8), this shows that $\sum_{n \geq 1}\left\|u_{n}\right\|_{1} \leq\left\|\sum_{n \geq 1} u_{n}\right\|_{1}+6$.
We now prove Theorem 1.1. We denote by $X_{1}$ (resp. $X_{2}$ ) the closed linear span of $\bigcup_{n \geq 1} H_{2 n}$ (resp. $\bigcup_{n \geq 1} H_{2 n+1}$ ). From Propositions 4.1 and 4.2, both $X_{1}$ and $X_{2}$ are isomorphic to an $l^{1}$-sum of spaces isomorphic to $l^{1}$ (and thus do not contain $L^{1}$ ).

We prove that the canonical map from $L^{1}$ to $\left(L^{1} / X_{1}\right) \times\left(L^{1} / X_{2}\right)$ is an embedding. Let $f \in L^{1}$, and suppose that for $f_{1} \in X_{1}, f_{2} \in X_{2}$ we have $\left\|f-f_{1}\right\|_{1} \leq 1,\left\|f-f_{2}\right\|_{1} \leq 1$. Then $\left\|f_{1}-f_{2}\right\|_{1} \leq 2$. However, Proposition 4.2 implies that $\left\|f_{1}\right\|_{1}+\left\|f_{2}\right\|_{1} \leq 40$. Thus, $\|f\|_{1} \leq 1+\left\|f_{1}\right\|_{1} \leq 41$, which proves the claim.

It follows from our construction, Proposition 2.3, and Corollary 2.2 that for $X=X_{1}$ or $X_{2}$, the quotient map $T$ from $L^{1}$ onto $L^{1} / X$ does not fix a copy of $L^{1}$. It remains to show that $L^{1} / X$ does not contain a copy of $L^{1}$. The proof, which will be given at the end of $\S 5$, relies on a well-known compactness argument. This argument will be used twice. To avoid duplication, we present a somewhat abstract statement (in the spirit of [10]) that will cover all our needs.
Proposition 4.4. Consider an operator $T$ from a Banach space $Y$ onto a Banach space $Z$. For $z \in Z,\|z\|<1$, let $U_{z}=\{y \in Y ;\|y\| \leq 1, T(y)=z\}$. Suppose that for some constant $K$, to each sequence $\left(y_{n}\right)_{n \geq 1} \in U_{z}$, we can associate a point $\varphi\left(\left(y_{n}\right)_{n \geq 1}\right) \in Y$ with the following properties.
(4.10) If $x_{n}=y_{n}$ for $n$ large enough, $\varphi\left(\left(x_{n}\right)_{n \geq 1}\right)=\varphi\left(\left(y_{n}\right)_{n \geq 1}\right)$.
(4.11) $\left\|\varphi\left(\left(y_{n}\right)_{n \geq 1}\right)\right\| \leq K$.
(4.12) $T\left(\varphi\left(\left(y_{n}\right)_{n \geq 1}\right)\right)=z$.

If $y_{n} \in U_{z}, y_{n}^{\prime} \in U_{z}$, we have

$$
\begin{equation*}
\varphi\left(\left(\frac{y_{n}+y_{n}^{\prime}}{2}\right)_{n \geq 1}\right)=\frac{1}{2}\left[\varphi\left(\left(y_{n}\right)_{n \geq 1}\right)+\varphi\left(\left(y_{n}^{\prime}\right)_{n \geq 1}\right)\right] . \tag{4.13}
\end{equation*}
$$

Then if $Z$ contains a copy of $L^{1}, T$ fixes a copy of $L^{1}$.
Proof. Since $Z$ contains a copy of $L^{1}$, there exists an isomorphism $V$ from $L^{1}$ into $Z$, and we can assume that $\|V\|<1$. For $I \in \mathscr{\mathcal { F }}_{k}, k \geq 1$, let $a_{I}=V\left(2^{k} 1_{I}\right)$. Thus, $\left\|a_{I}\right\|<1$. Take $b_{I} \in U_{a_{I}}$.

For $I \in \mathscr{J}_{k}, n \geq k$, set $b_{I, n}=2^{k-n} \sum_{J \in \mathcal{J}_{n}, J \subset I} b_{J}$ and $b_{I, n}=b_{I}$ for $n \leq k$. Obviously, $b_{I} \in U_{a_{I}}$. If $I_{1}, I_{2}$ are the two elements of $\mathscr{J}_{k+1}$ contained in $I$, we clearly have for $n \geq k+1$,

$$
\begin{equation*}
b_{I, n}=\frac{1}{2}\left(b_{I_{1}, n}+b_{I_{2}, n}\right) . \tag{4.14}
\end{equation*}
$$

We now set $c_{I}=\varphi\left(\left(b_{I, n}\right)_{n \geq 1}\right)$. From (4.11), $\left\|c_{I}\right\| \leq K$. From (4.10), (4.13), (4.14), $c_{I}=\frac{1}{2}\left(c_{I_{1}}+c_{I_{2}}\right)$. From (4.12), $T\left(c_{I}\right)=a_{I}$. Thus, one can define an operator $W$ from $L^{1}$ to $Y$ such that $W\left(2^{k} 1_{I}\right)=c_{I}$ for $I \in \mathscr{J}_{k}^{\sim}$, and this operator satisfies $\|W\| \leq K, V=T \circ W$. Since $V$ is an isomorphism, $T$ is an isomorphism from $W\left(L^{1}\right)$ to its image, and thus $T$ fixes $L^{1}$.

## 5. Semiembeddings

In this section, we denote by $M_{n}=2^{n} N_{n}$ the absolute closed convex hull of $2^{n} C_{n}=2^{n} D_{q(n)}$. We denote by $L^{0}$ the space of all measurable functions on $[0,1]$. We provide $L^{0}$ with the topology of convergence in measure. We denote by $G$ the set of all functions in $L^{0}$ that can be represented as $f_{1}+\sum_{n \geq 1} \alpha_{n} g_{n}$, where $f_{1} \in L^{1},\left\|f_{1}\right\|_{1} \leq 1, g_{n} \in M_{n}, \sum_{n \geq 1}\left|\alpha_{n}\right| \leq 1$. We observe that the series $\sum_{n \geq 1} \alpha_{n} g_{n}$ converges in measure and that $G$ is bounded in $L^{0}$. We denote by $N$ the gauge of $G$, given by $N(h)=\inf \{a>0 ; h \in a G\}$ if this set is nonempty and $N(h)=\infty$ otherwise. We set $E=\left\{h \in L^{0} ; N(h)<\infty\right\}$, and we provide $E$ with the norm $N$. Since $G$ is bounded in $L^{0}, E$ is a Banach space, and the canonical map $j$ from $L^{1}$ to $E$ is one-to-one. Since the unit ball of $L^{1}$ is closed in $L^{0}, j$ is a semiembedding. If we consider the space $E_{1}$ (resp. $E_{2}$ ) constructed like $E$ but replacing the condition $g_{n} \in M_{n}$ by $g_{n} \in M_{2 n}$ (resp. $g_{n} \in M_{2 n+1}$ ), Proposition 4.3 shows that the canonical injection from $L^{1}$ into $E_{1} \times E_{2}$ is an isomorphism. Indeed, if $f \in L^{1}$, and if we have $f_{1}, f_{2} \in L^{1}$, $\left\|f_{1}\right\|_{1},\left\|f_{2}\right\|_{1} \leq 1, g_{n} \in M_{n}$ such that $f=f_{1}+\sum_{n \geq 1} g_{2 n}=f_{2}+\sum_{n \geq 0} g_{2 n+1}$, then $\sum_{n \geq 1}(-1)^{n} g_{n}=f_{2}-f_{1}$ so that $\sum\left\|g_{n}\right\|_{1} \leq 8$ and $\|f\|_{1} \leq 9$. Thus, in order to prove Theorem 1.4, it suffices to show that $E$ does not contain $L^{1}$.

We know by our construction, Corollary 2.2 , and the fact that $\|j(f)\|_{1} \leq 2^{-n}$ for $f \in D_{q(n)}$ that $j$ does not fix a copy of $L^{1}$. Suppose that there is an isomorphism $V$ from $L^{1}$ to $E$. We can assume $\|V\|<1$ and that we have $\|V(f)\| \geq \delta\|f\|_{1}$ for some $\delta>0$. We will show that $j$ must fix a copy of $L^{1}$, thereby concluding the proof. The first part of the argument will be the application of Proposition 4.4 to an auxiliary space which we construct now.

We denote by $X_{n}$ the Banach space $\bigcup_{\lambda \in \mathbf{R}} \lambda M_{n}$ provided with the norm given by the gauge $\|\cdot\|_{n}$ of $M_{n}$. We denote by $\bigoplus_{1} X_{n}$ the $l^{1}$-sum of the spaces $X_{n}$. We denote by $Y$ the space $L^{1} \times\left(\bigoplus_{1} X_{n}\right)$ provided with the norm $\left\|\left(f,\left(g_{n}\right)\right)\right\|=\max \left(\|f\|_{1}, \sum_{n \geq 1}\left\|g_{n}\right\|_{n}\right)$. Consider the operator $T: Y \rightarrow E$ given by $T\left(\left(f,\left(g_{n}\right)\right)\right)=f+\sum_{n \geq 1} g_{n}$. Clearly, $\|T\| \leq 1$, and $T$ is onto.

Let $z \in E,\|z\|<1$, and consider a sequence $y_{k}=\left(f_{k},\left(g_{n, k}\right)_{n \geq 1}\right)$ in $U_{z}=\{y \in Y ;\|y\| \leq 1, T(y)=z\}$. Hence, for each $k$,

$$
\begin{equation*}
z=f_{k}+\sum_{n \geq 1} g_{n, k} \tag{5.1}
\end{equation*}
$$

Let us fix an ultrafilter $\mathscr{U}$ on $\mathbf{N}$. Since $M_{n}$ is convex, uniformly bounded, and closed, it is $\sigma\left(L^{1}, L^{\infty}\right)$-compact, and we can consider $g_{n}=\lim _{k \rightarrow \mathscr{U}} g_{n, k}$, where the limit is taken for this topology. Let $f=z-\sum_{n \geq 1} g_{n}$. We show that $\|f\|_{1} \leq 1$. Given $m \geq 0$, we can find coefficients $\alpha_{k}$, all but finitely many zero, with $\sum_{k \geq 1} \alpha_{k}=1$ such that

$$
\left\|g_{n}-\sum_{k \geq 1} \alpha_{k} g_{n, k}\right\|_{1} \leq 2^{-m}
$$

for all $n \leq m$. We have from (5.1)

$$
\begin{equation*}
f-\sum_{k \geq 1} \alpha_{k} f_{k}=-\sum_{n \leq m}\left(g_{n}-\sum_{k \geq 1} \alpha_{k} g_{n, k}\right)-\sum_{n>m} g_{n}+\sum_{n>m} \sum_{k \geq 1} \alpha_{k} g_{n, k} \tag{5.2}
\end{equation*}
$$

By definition of $U_{z}$ and the definition of the norm of $Y$, we have $\left\|f_{k}\right\|_{1} \leq 1$ for all $k$; thus, (5.2) shows that $f$ belongs to the closure in measure of the unit ball of $L^{1}$; thus, $\|f\|_{1} \leq 1$.

If we define $\varphi\left(\left(y_{k}\right)_{k \geq 1}\right)=\left(f,\left(g_{n}\right)\right) \in Y$, it is obvious that (4.10)-(4.13) hold (with $K=1$ ). Thus, from Proposition 4.4, we can find an operator $W$ from $L^{1}$ to $Y$ such that $V=T \circ W$.

The second part of the argument is crystalized in the following lemma. For $m \geq 1$, denote by $Y_{n}$ the subspace of $Y$ consisting of the $y=\left(f,\left(g_{n}\right)\right)$ such that $g_{n}=0$ whenever $n \geq m$.
Lemma 5.1. Consider an operator $W$ from $L^{1}$ to $Y$. Then for each $\varepsilon>0$, ihere exists an $m$, a set of positive measure $A$, and an operator $W^{\prime}$ from $L^{1}$ to $Y_{m}$ such that $\left\|W(f)-W^{\prime}(f)\right\| \leq \varepsilon\|f\|$ whenever $f$ is supported by $A$.

Before we prove this lemma, let us conclude the proof of Theorem 1.4. Recall that $\|V(f)\| \geq \delta\|f\|_{1}$ for $f \in L_{1}$. We apply Lemma 5.1 with $\varepsilon=\delta / 2$. Thus,
when $f \in L^{1}(A)$, the subspace of $L^{1}$ consisting of functions supported by $A$, we have

$$
\left\|T \circ W(f)-T \circ W^{\prime}(f)\right\| \leq \frac{\delta}{2}\|f\|_{1}
$$

Since $\|T \circ W(f)\|=\|V(f)\| \geq \delta\|f\|_{1}$, we have $\left\|T \circ W^{\prime}(f)\right\| \geq \frac{\delta}{2}\|f\|_{1}$ when $f \in L^{1}(A)$. We observe now that $T\left(Y_{m}\right) \subset L^{1}$ and that $T: Y_{m} \rightarrow L^{1}$ is bounded. Considering now $T \circ W^{\prime}$ as valued in $L^{1}$, we have

$$
\left\|j \circ T \circ W^{\prime}(f)\right\| \geq \frac{\delta}{2}\|f\|_{1} \geq \frac{\delta}{2\left\|T \circ W^{\prime}\right\|}\left\|T \circ W^{\prime}(f)\right\|_{1}
$$

so that $j$ fixes $T \circ W^{\prime}\left(L^{1}(A)\right)$. This completes the proof.
Proof of Lemma 5.1. Denote by $\pi_{n}$ the canonical projection of $Y$ on $X_{n}$, and let $W_{n}=\pi_{n} \circ W$. By definition of the norm of $Y$, for $f \in L^{1}$, we have

$$
\sum_{n \geq 1}\left\|W_{n}(f)\right\|_{n} \leq\|W(f)\|
$$

For $n \geq 1, k \geq 1$ define a $\Sigma_{k}$-measurable function $h_{n, k}$ by $h_{n, k}=$ $\left\|\frac{1}{|T|} W_{n}\left(1_{I}\right)\right\|_{n}$ for $x \in I \in \mathscr{F}_{k}$. Thus, $\sum_{n \geq 1} h_{n, k} \leq\|W\|$. For each $n$, the sequence $\left(h_{n, k}\right)_{k \geq 1}$ is a supermartingale; denote by $h_{n}$ its limit a.e. Clearly, $\sum_{n \geq 1} h_{n} \leq\|W\|$. For $I \in \Sigma_{k}, n \geq 1$, we have

$$
\left\|W_{n}\left(1_{I}\right)\right\|_{n}=\int_{I} h_{n, k} d \lambda \leq \int 1_{I} h_{n} d \lambda
$$

Thus, by approximation, for any $f \in L^{1}$, we have

$$
\left\|W_{n}(f)\right\|_{n} \leq \int|f| h_{n} d \lambda
$$

Thus, for any $m$,

$$
\sum_{n \geq m}\left\|W_{n}(f)\right\|_{n} \leq \int|f|\left(\sum_{n \geq m} h_{n}\right) d \lambda
$$

Since $\sum_{n \geq 1} h_{n} \leq\|W\|$, for any $\varepsilon$, we can find $m$ large enough that $|A|>0$, where $A=\left\{\sum_{n \geq m} h_{n} \leq \varepsilon\right\}$. If we denote by $W^{\prime}$ the composition of $W$ and of the canonical projection of $Y$ onto $Y_{m}$, for $f$ supported by $A$, we have

$$
\left\|W(f)-W^{\prime}(f)\right\| \leq \sum_{n>m}\left\|W_{n}(f)\right\|_{n} \leq \int|f| \varepsilon d \lambda \leq \varepsilon\|f\|_{1}
$$

This completes the proof.
We now turn back to the proof that $L^{1} / X$ does not contain a copy of $L^{1}$. We denote by $Z$ the closure of $j(X)$ in $E$.

Proposition 5.2. $Z$ is isomorphic to $l^{1}$.
Proof. Since

$$
\begin{equation*}
2^{n} g_{n, k}=\frac{1}{|A(n, k)|} 1_{A(n, k)}+2^{n} f_{n, k}, \tag{5.3}
\end{equation*}
$$

the definition of $\|\cdot\|_{E}$ shows that $\left\|u_{n, k}\right\|_{E} \leq 2$, where $u_{n, k}=j\left(2^{n} g_{n, k}\right)$. Consider now numbers $a_{n, k}$, all but finitely many zero, and suppose that $\left\|\sum_{n, k \geq 1} a_{n, k} u_{n, k}\right\|_{E} \leq 1$. By definition of the norm of $E$, we can find $f \in L^{1}$, $\|f\|_{1}<2, g_{n} \in M_{n},\left(\alpha_{n}\right)_{n \geq 1}$, with $\sum\left|\alpha_{n}\right| \leq 2$ such that

$$
\sum_{n, k \geq 1} a_{n, k} u_{n, k}=f+\sum_{n \geq 1} \alpha_{n} g_{n} .
$$

There is no loss of generality to assume that $g_{n} \in 2^{n} \operatorname{conv} C_{n}$. Thus, we can write

$$
\alpha_{n} g_{n}=\sum_{k \geq 1} 2^{n} b_{n, k} f_{n, k},
$$

where $\sum_{k \geq 1}\left|b_{n, k}\right| \leq \alpha_{n}$, and thus, $\sum_{n, k \geq 1}\left|b_{n, k}\right| \leq 2$. Hence, using (5.3) again, for some $f^{\prime} \in L^{1},\left\|f^{\prime}\right\|_{1} \leq 4$, we have

$$
\sum_{n, k \geq 1} 2^{n} a_{n, k} g_{n, k}=f^{\prime}+\sum_{n \geq 1} \sum_{k \geq 1} 2^{n} b_{n, k} g_{n, k}
$$

or, equivalently,

$$
\sum_{n, k \geq 1} 2^{n}\left(b_{n, k}-a_{n, k}\right) g_{n, k}=-f^{\prime} .
$$

We observe that $\sum_{n, k}\left|b_{n, k}-a_{n, k}\right|<\infty$. Since $\left\|f^{\prime}\right\|_{1} \leq 4$, Proposition 4.1 implies that $\sum_{n, k}\left|b_{n, k}-a_{n, k}\right| \leq 16$, and thus, $\sum_{n, k}\left|a_{n, k}\right| \leq 18$.

Thus, the span of $\left(u_{n, k}\right)_{n, k \geq 1}$ is isomorphic to $l^{1}$, but it is obviously the closure of $j(X)$.

Proposition 5.3. $L^{1} / X$ is isomorphic to $E / Z$.
Proof. Denote by $U$ (resp. $V$ ) the quotient map from $L^{1}$ onto $L^{1} / X$ (resp. $E$ onto $E / Z)$. Since $V \circ j$ is zero on $X$, there exists an operator $T$ from $L^{1} / X$ into $E / Z$ such that $T \circ U=V \circ j$. Since $j\left(L^{1}\right)$ is dense in $E$, the image of $T$ is dense, so we are left to show that $T$ is an isomorphism.

Consider $x \in L^{1}$ such that $\|T \circ U(x)\|_{1}<1$. Hence, $\|V \circ j(x)\|_{1}<1$ so that there exists $y \in X$ with $\|j(x)-j(y)\|_{E}<1$. Thus,

$$
x-y=f+\sum_{n \geq 1} \alpha_{n} g_{n},
$$

where $\|f\|_{1} \leq 1, \sum_{n \geq 1}\left|\alpha_{n}\right| \leq 1, g_{n} \in M_{n}$. From Proposition 4.2, we see that $\sum_{n \geq 1} \alpha_{n}\left\|g_{n}\right\|_{1}<\infty$; hence, we can write

$$
x-y=f^{\prime}+\sum_{1 \leq n \leq N} \alpha_{n} g_{n}
$$

where $\left\|f^{\prime}\right\|_{1} \leq 2$; and we can assume that $g_{n} \in 2^{n} \operatorname{conv} C_{n}$. Denote by $B$ the unit ball of $L^{1}$. It follows from (5.3) that $2^{n} f_{n, k} \in B+X$, and thus, $2^{n} \operatorname{conv} C_{n} \subset B+X$. Since $\sum_{n \geq 1}\left|\alpha_{n}\right| \leq 1$, we have $\sum_{1 \leq n \leq N} \alpha_{n} g_{n} \in B+X$. Thus, we can find $y^{\prime} \in X$ such that $\left\|x-y-y^{\prime}\right\|_{1} \leq 3$. This shows that $\|U(x)\| \leq 3$ and concludes the proof.

In view of Proposition 5.3, to prove that $L^{1} / X$ does not contain a copy of $L^{1}$, it suffices to show that $E / Z$ does not contain a copy of $L^{1}$. We already know that $E$ does not contain a copy of $L^{1}$, and we will conclude by application of Proposition 4.2. From Proposition 5.2, $Z$ is isomorphic to $l^{1}$, which is a dual space. We denote by $\tau$ the weak ${ }^{*}$ topology induced by such an isomorphism. Let us fix an ultrafilter $\mathscr{U}$ on $\mathbf{N}$. For a sequence $\left(y_{n}\right)_{n \geq 1}$ in $U_{z}=\{y \in E ;\|y\| \leq 1, V(y)=z\}$, we set $\varphi\left(\left(y_{n}\right)_{n \geq 1}\right)=y+\lim _{n \rightarrow \mathscr{U}}\left(y_{n}-y\right)$, where $y$ is any point of $V^{-1}(z)$, and where the limit is taken in the weak ${ }^{*}$ sense in $Z$. It is obvious that (4.10)-(4.13) are satisfied. This completes the proof of Theorem 1.1.

## 6. QUotients of $L^{1}$ by a subspace isomorphic to $l^{1}$

In this section, we prove Proposition 1.2. The proof relies on an elementary property of $l^{1}$ that must have been observed long ago. We provide a sketch of a proof for the convenience of the reader.
Lemma 6.1. Consider a sequence $\left(g_{n}\right)_{n \geq 1}, g_{n} \in l^{1}$, such that $\left\|g_{n}\right\|=1$. Let $\varepsilon>$ 0 . Then there exists a sequence $n(k)$ such that each finite convex combination $g$ of vectors $g_{n(k)}$ satisfies $\|g\| \geq 1-\varepsilon$.
Proof. Let $g_{n}=\left(g_{n, p}\right)_{p \geq 1}$. We can assume that $a_{p}=\lim _{n \rightarrow \infty} g_{n, p}$ exists for all $p \geq 1$. We have $\sum_{p \geq 1}\left|a_{p}\right| \leq 1$. We choose $m(0)$ such that $\sum_{p>m(0)}\left|a_{p}\right| \leq \frac{\varepsilon}{8}$. We then construct increasing sequences of integers $m(k), n(k)$ such that

$$
\sum_{p \leq m(k)}\left|a_{p}-g_{n(k), p}\right| \leq \frac{\varepsilon}{8}, \quad \sum_{p>m(k+1)}\left|g_{n(k), p}\right| \leq \frac{\varepsilon}{8}
$$

Define $b_{p}=\operatorname{sign} a_{p}$ for $p \leq m(0), b_{p}=\operatorname{sign} g_{n(k), p}$ for $k \geq 1, m(k)<p \leq$ $m(k+1)$. Simple computations show that $\sum_{p \geq 1} b_{p} g_{n(k), p} \geq 1-\varepsilon$ for all $k \geq 1$; this implies the result.

For $k \geq 1$, we consider the set

$$
B_{k}=\bigcup_{0 \leq l<2^{k-1}}\left[2 l 2^{-k},(2 l+1) 2^{-k}[\right.
$$

Thus, $B_{k} \in \Sigma_{k}$ and $E^{l}\left(1_{B_{k}}\right)=\frac{1}{2}$ for $l<k$. We denote by $S_{p}$ the collection of sequences $s=(\sigma(1), \ldots, \sigma(p))$ such that $1 \leq \sigma(1)<\sigma(2)<\cdots<\sigma(p)$. For $s \in S_{p}$, we denote by $\Sigma(s)$ the algebra generated by $B_{\sigma(1)}, \ldots, B_{\sigma(p)}$. We observe that each of its atoms has measure $=2^{-p}$. For a subalgebra $\Sigma^{\prime}$ of $\Sigma$, we denote by $L_{0}^{1}\left(\Sigma^{\prime}\right)$ the set of $\Sigma^{\prime}$-measurable functions that are integrable and of integral zero. The core of the proof of Proposition 1.2 is the following.

Lemma 6.2. Consider an operator $T$ from $l^{1}$ to $L^{1}$ and $s=(\sigma(1), \ldots, \sigma(p))$, $s \in S_{p}$. Let $\alpha>0$. Suppose that for each $f \in L_{0}^{1}(\Sigma(s))$ and $g \in l^{1},\|g\|=1$, we have $\|f-T(g)\|_{1} \geq \alpha$. Let $\beta<\alpha$. Then we can find $\sigma(p+1)>\sigma(p)$ such that if we set $s^{\prime}=(\sigma(1), \ldots, \sigma(p), \sigma(p+1))$, then for each $f \in L_{0}^{1}\left(\Sigma\left(s^{\prime}\right)\right)$ and $g \in l^{1},\|g\|_{1}=1$, we have $\|f-T(g)\|_{1} \geq \beta$.

Proof. For $n>\sigma(p)$, we set $s_{n}=(\sigma(1), \ldots, \sigma(p), n) \in S_{p+1}$. Suppose that the conclusion of the lemma fails. Then for some $\beta<\alpha$ and each $n>\sigma(p)$, we can find $f_{n} \in L_{0}^{1}\left(\Sigma\left(s_{n}\right)\right)$ and $g_{n} \in l^{1},\left\|g_{n}\right\|=1$, such that $\left\|f_{n}-T\left(g_{n}\right)\right\|_{1} \leq \beta$. Consider $\varepsilon>0$, to be determined later. Lemma 6.1 gives a sequence $n(k)$ such that each convex combination $g$ of the sequence $\left(g_{n(k)}\right)_{k \geq 1}$ satisfies $1-\varepsilon \leq$ $\|g\| \leq 1$. Since each atom of $\Sigma\left(s_{n}\right)$ has measure $=2^{-p-1}$, the sequence $\left(f_{n}\right)_{n \geq 1}$ is uniformly bounded. There is no loss of generality to assume that the sequence $\left(f_{n(k)}\right)_{k \geq 1}$ converges weakly to some function $f$. Since $\Sigma\left(s_{n}\right)$ is generated by $\Sigma(s)$ and $B_{n}$, where $E^{l}\left(1_{B_{n}}\right)=\frac{1}{2}$ for $n>l$, for $n>l>\sigma(p)$ we have $\int_{U} f_{n} d \lambda=\int_{V} f_{n} d \lambda$ for any two atoms $U, V$ of $\Sigma_{l}$ that are contained in the same atom of $\Sigma(s)$. It follows that $\int_{U} f d \lambda=\int_{V} f d \lambda$, and thus, $f$ is constant on each atom of $\Sigma(s)$. Since $\int f d \lambda=0$, we have $f \in L_{0}^{1}(\Sigma(s))$. There is a finite convex combination $h=\sum a_{k} f_{n(k)}$ such that $\|f-h\|_{1} \leq \varepsilon$. Since $\left\|f_{n(k)}-T\left(g_{n(k)}\right)\right\|_{1} \leq \beta$, we have $\left\|f-T\left(g^{\prime}\right)\right\|_{1} \leq \beta+\varepsilon$, where $g^{\prime}=\sum \alpha_{k} g_{n(k)}$. The choice of the sequence $n(k)$ implies that $\left\|g^{\prime}\right\| \geq 1-\varepsilon$. Let $g=g^{\prime} /\left\|g^{\prime}\right\|$, so $\left\|g-g^{\prime}\right\|=1-\left\|g^{\prime}\right\| \leq \varepsilon$. We have $\|g\|=1$, and $\|f-T(g)\|_{1} \leq \beta+\varepsilon(1+\|T\|)$. Thus, if $\varepsilon(1+\|T\|)<\alpha-\beta$, this is a contradiction.

We now prove Proposition 1.2. Let $T$ be an isomorphism from $l^{1}$ to $X$. Let $\alpha=\inf \left\{\|T(g)\|_{1} ;\|g\|=1\right\}>0$. We set $\Sigma(\varnothing)=\{\varnothing,[0,1]\}$, and thus, $L_{0}^{1}(\Sigma(\varnothing))=\{0\}$. Thus, $\|f-T(g)\|_{1} \geq \alpha$ for $g \in l^{1},\|g\|=1, f \in L_{0}^{1}(\Sigma(\varnothing))$. Using Lemma 6.2 , we construct by induction a sequence $\sigma(p)$ such that if $s(0)=\varnothing, s(p)=(\sigma(1), \ldots, \sigma(p))$ for $p \geq 1$, for each $f \in L_{0}^{1}(\Sigma(s(p)))$ and each $g \in l^{1},\|g\|=1$, we have $\|f-T(g)\|_{1} \geq \alpha\left(2^{-p}+\frac{1}{2}\right)$.

Denote by $\Sigma^{\prime}$ the $\sigma$-algebra generated by the union of the algebras $\Sigma(s(p))$. Then $\bigcup_{p \geq 1} L_{0}^{1}(\Sigma(s(p)))$ is dense in norm in $L_{0}^{1}\left(\Sigma^{\prime}\right)$, and thus, $\|f-T(g)\|_{1} \geq \frac{\alpha}{2}$ for $f \in L_{0}^{1}\left(\Sigma^{\prime}\right), g \in l^{1},\|g\|=1$. It follows easily that for some $\beta>0$, we have $\|f-g\|_{1}>\beta$, whenever $f \in L_{0}^{1}\left(\Sigma^{\prime}\right),\|f\|_{1}=1, g \in X$. Thus, the quotient map $L^{1} \rightarrow L^{1} / X$ is an isomorphism on $L_{0}^{1}\left(\Sigma^{\prime}\right)$, which is isomorphic to $L^{1}$. The proof is complete.

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Abstract. We construct a subspace $X$ of $L^{1}$ such that $X$ is an $l^{1}$-sum of spaces isomorphic to $l^{1}$ but such that $L^{1} / X$ does not contain a copy of $L^{1}$. We also construct two Banach spaces $E_{1}, E_{2}$ that do not contain a copy of $L^{1}$ but such that $E_{1} \times E_{2}$ contains a copy of $L^{1}$. Moreover, the projections
of $L^{1}$ on each factor are one-to-one, and the images of the unit ball of $L^{1}$ are closed. These examples settle questions of J. Lindenstrauss, P. Pelczynski, J. Bourgain, and H. P. Rosenthal.

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