THE THREE-SPACE PROBLEM FOR L^1

MICHEL TALAGRAND

1. INTRODUCTION

The study of the structure of the classical Banach spaces C(K), $L^{p}(\mu)$, $1 \le p \le \infty$, their subspaces, their quotient spaces, and the operators between them is a central topic of Banach space theory. This paper focuses on the space L^{1} of (classes of) measurable functions on the unit interval.

In order to put our results in a proper perspective, let us recall a few of the most remarkable results on L^1 . A subspace of L^1 is either reflexive or it contains a subspace isomorphic to l^1 and complemented in L^1 (Kadec and Pelczynski [11]). A reflexive subspace of L^1 is isomorphic to a subspace of L^p for some p > 1 (Rosenthal [20]). Any subspace of L^1 contains a subspace isomorphic to l^p for some $1 \le p \le 2$ (Aldous [1]). The most challenging open question concerning the structure of L^1 might be to decide whether a complemented infinite-dimensional subspace of L^1 is isomorphic to l^1 or L^1 . (Interestingly, more is known in the case L^p , p > 1 [4].) An important step in that direction was made by Enflo and Starbird [6]. (See [7] or [12] for a different approach.) They showed that L^1 is primary; that is, if L^1 is isomorphic to a sum $E \oplus F$, then either E or F is isomorphic to L^1 . An easy consequence of that result is that if a direct sum $E \oplus F$ contains a complemented subspace isomorphic to L^1 , then either E or F has the same property.

Each separable Banach space is isomorphic to a quotient of L^1 . Thus, the study of quotients of L^1 naturally focuses on the study of quotients L^1/X , where X has some smallness property; e.g., X is reflexive [13, 18], separable dual [10], or "bien disposé" [8]. For simplicity, let us say that a Banach space contains a copy of L^1 if it contains a subspace isomorphic to L^1 . The threespace problem for L^1 raised in [14] belongs to the same circle of ideas. It is the following question: If X is a subspace of L^1 , does X or L^1/X contain a copy of L^1 ? The result of Enflo and Starbird shows that this is the case when X is complemented. The three-space problem for L^1 seems to have been the main motivation behind the papers [5, 2, 9, 10, 22] which contain many interesting results. In particular, Bourgain and Rosenthal [5] have shown that either X contains a copy of L^1 or L^1/X contains a copy of l^1 , and

Received by the editors October 15, 1988.

¹⁹⁸⁰ Mathematics Subject Classification (1985 Revision). Primary 46E30, 46B25.

This work was partially supported by an NSF grant.

Bourgain [2] (actually looking for a counterexample) proved the remarkable fact that L^{1}/H^{1} contains a copy of L^{1} .

The first main contribution of the present paper is the construction of (families of) subspaces X of L^1 that are small as Banach spaces (actually have the Radon-Nikodym property) but such that L^1/X does not contain a copy of L^1 . Also raised by the result of Enflo and Starbird and its consequences mentioned earlier is the following. If a direct sum $E \oplus F$ contains a copy of L^1 , does E or F contain a copy of L^1 ? The other main contribution of the present paper is a negative answer to that question.

Theorem 1.1. There exist two subspaces X_1 , X_2 of L^1 that are isomorphic to an l^1 -sum of spaces isomorphic (but not uniformly isomorphic) to l^1 such that L^1/X_1 and L^1/X_2 do not contain a copy of L^1 but that the canonical injection of L^1 in $L^1/X_1 \times L^1/X_2$ is an embedding.

In particular, X_1 , X_2 have the Radon-Nikodym property and, in particular, do not contain a copy of L^1 . If a subspace X of L^1 is isomorphic to a dual space, it does not seem to be known if L^1/X contains a copy of L^1 . We, however, have the following.

Proposition 1.2. If a subspace X of L^1 is isomorphic to l^1 , then L^1/X contains a copy of L^1 .

The following is related to [17, Problem 8.3] and remains open.

Problem 1.3. Consider a compact metric space S. Denote by C(S) the space of continuous functions on S and by $M(S) = C(S)^*$ the space of signed measures on S. Consider a weak^{*} (= $\sigma(M(S), C(S))$) closed subspace X of M(S). Suppose that X is a subspace of $L^1(\mu)$ for some probability measure μ on S. Does $L^1(\mu)/X$ contain a copy of L^1 ?

Following [15], we say that a (linear, bounded) operator T between two Banach spaces E and F is a *semiembedding* if it is one-to-one and if the image of the unit ball of E is closed. The following result, which is obtained by a variation of the construction of Theorem 1.1, answers questions of [5].

Theorem 1.4. There exist two Banach spaces E_1 , E_2 that do not contain a copy of L^1 but such that L^1 embeds in $E_1 \times E_2$ in such a way that the restrictions to L^1 of the projections on E_1 and E_2 are semiembeddings.

Once the proper approach has been found, it is a matter of standard (uninspiring) technique to reduce the proof of these theorems to a central statement concerning a remarkable family of functions on [0, 1]. This statement, which could be of independent interest, is the main new ingredient of this paper. Consider the family F_n of all the functions of the type

$$\frac{1}{2n} \sum_{0 \le i \le 2n-1} (-1)^i 2^{\sigma(i)} 1_{I_i}$$

for any sequence $\sigma(0) < \sigma(1) < \cdots < \sigma(2n-1)$ of integers, where the sets I_i are dyadic intervals of length $2^{-\sigma(i)}$, and where the sequence I_1, \ldots, I_{2n-1} decreases. Then (Theorem 3.3) given $\varepsilon > 0$, for *n* large enough and any function *f* in the convex hull of F_n , the set where |f| is greater than ε has Lebesgue measure $\leq \varepsilon$.

2. The approach

We say that an operator T between Banach spaces X and Y fixes a copy of L^1 if X contains a copy of L^1 on which the restriction of T is an isomorphism on its image.

We denote by λ Lebesgue's measure and by Σ the σ -algebra of measurable sets. To simplify notation, for $A \in \Sigma$, we write $|A| = \lambda(A)$.

Consider an atomless subalgebra Σ' of Σ and $A \in \Sigma'$, |A| > 0. We denote by $L_0^1(A, \Sigma')$ the subspace of L^1 that consists of Σ' -measurable functions that are zero outside A and such that $\int f d\lambda = 0$. Central to our approach will be the following theorem proved by Rosenthal [21], using the techniques developed by Kalton in [12].

Theorem 2.1 [21]. Consider a Banach space E and an operator T from L^1 to E that fixes a copy of L^1 . Then for some $\delta > 0$, some atomless subalgebra Σ' of Σ , and some $A \in \Sigma'$, |A| > 0, we have $||T(f)|| \ge \delta ||f||_1$ whenever $f \in L_0^1(A, \Sigma')$.

The application of this theorem to the three-space problem is as follows.

Corollary 2.2. Consider a subspace X of L^1 . The following are equivalent.

(2.1) The quotient map $L^1 \to L^1/X$ does not fix a copy of L^1 .

(2.2) For each $\delta > 0$, each atomless subalgebra Σ' of Σ , and each set $A \in \Sigma'$, $|A| \ge 0$, there exist $f \in L_0^1(A, \Sigma')$ and $g \in X$ such that $||f||_1 \ge \frac{1}{4}$ and $||f - g||_1 \le \delta$.

The choice of $\frac{1}{4}$ in the condition $||f||_1 \ge \frac{1}{4}$ is for convenience, any other positive number would be suitable.

Our approach will simply be to construct families of functions that satisfy (2.2) and then to take for X their closed linear span. While a priori this construction would only ensure that the quotient map $L^1 \to L^1/X$ is not an isomorphism on any space $L_0^1(A, \Sigma')$, Corollary 2.2 implies that it does fix any copy of L^1 , and the fact that L^1/X does not contain L^1 will then follow by a lifting argument (made possible by the special structure of X). The problem reduces then to finding an appropriate device that will prevent X from containing L^1 .

As a first step toward (2.2), we shall construct a countable family of functions C_n that satisfies the following condition.

 (P_n) For each atomless subalgebra Σ' of Σ and each set $A \in \Sigma'$, $|A| > 2^{-n}$, there exist $f \in L_0^1(A, \Sigma')$ and $g \in C_n$ such that $||f||_1 \ge \frac{1}{4}$ and $||f-g||_1 \le 2^{-n}$. The purpose of the restriction $|A| \ge 2^{-n}$ is to allow C_n to be uniformly bounded. How can we control the linear span of C_n ? Since we are allowed an error of 2^{-n} , there is nothing to lose by adding on another error of 2^{-n} . We add to each function in C_n a function of the type $2^{-n}|B|^{-1}1_B$, where |B| is very small. This can be done in such a way that C_n becomes 2^{n+1} -equivalent to the unit basis of l^1 while it still satisfies (P_{n-1}) . Thus, we can control the span of each C_n . However, X has to contain infinitely many families C_n in order to satisfy (2.2), and another idea is still needed there. This idea is to control the situation using convergence in measure. The functions of span C_{n+1} will be very "peaky" (that is, they will be essentially supported by very small sets), compared to those of span C_n , and the span X of $\bigcup_{n\geq 1} C_n$ will thus be isomorphic to the l^1 -sum of the spaces span C_n ; so X will be an l^1 -sum of spaces isomorphic to l^1 . The idea of using a convergence in measure for such a purpose is not new. It has been successfully used in the work of Roberts [19] and subsequent work, e.g. [3].

How are we to choose the class C_n ? The results of [9] indicate that we should not use functions f for which $||f||_{\infty}$ and $||f||_1$ are of the same order, so it is natural to use functions f for which ||f| resembles the function $(x \log(1/\varepsilon))^{-1} \mathbf{1}_{[\varepsilon, 1]}$ for ε small, or, more conveniently,

$$\frac{1}{m}\sum_{i=0}^{m-1}2^{i+1}\mathbf{1}_{[2^{-i-1},2^{-i}[}$$

for *m* large. How are we to choose the sign of f? Here there are several possibilities, but it turns out that one of the simplest choices, which is to take f resembling $\frac{1}{2n} \sum_{i=0}^{2n-1} (-1)^i 2^{i+1} \mathbf{1}_{[2^{-i-1}, 2^{-i}]}$, is suitable for our purposes.

We denote by \mathscr{I}_k the collection of the dyadic intervals $[l2^{-k}, (l+1)2^{-k}]$, $0 \le l < 2^k$ and by Σ_k the algebra they generate. For $f \in L^1$, we denote by $E^k(f)$ its conditional expectation with respect to Σ_k , so, for $I \in \mathscr{I}_k$, the constant value of $E^k(f)$ on I is $2^k \int_I f d\lambda$.

We now describe our basic class D_n of functions (we will have $C_n = D_{q(n)}$ for a fast growing sequence (q(n))). Consider a sequence B_0, \ldots, B_{2n-1} of sets and a sequence $\sigma(0) < \cdots < \sigma(2n-1)$ of numbers with the following properties:

(2.3) B_i is $\Sigma_{\sigma(i)}$ measurable.

(2.4) For $0 \le i < 2n - 1$, $E^{\sigma(i)}(1_{B_{i+1}}) = \frac{1}{2} 1_{B_i}$.

This latter condition means that $B_{i+1} \subset B_i$ and that for each $I \in \mathscr{I}_{\sigma(i)}$, $I \subset B_i$, we have $|I \cap B_{i+1}| = \frac{1}{2}|I|$.

We set

(2.5)
$$g = \frac{1}{2n|B_0|} \sum_{0 \le i \le 2n-1} (-1)^i 2^i 1_{B_i}.$$

If we set $B_{2n} = \emptyset$, we observe that

(2.6)
$$g = \frac{1}{2n|B_0|} \sum_{0 \le i \le 2n-1} a_i \mathbf{1}_{B_i \setminus B_{i+1}}$$

where $a_i = \sum_{0 \le l \le i} (-1)^l 2^l = ((-1)^i 2^{i+1} + 1)/3$. We denote by D_n the class of functions given by (2.5) for all possible choices

We denote by D_n the class of functions given by (2.5) for all possible choices of B_0, \ldots, B_{2n-1} that satisfy (2.3), (2.4), and $|B_0| \ge 2^{-n}$. We observe that

(2.7)
$$g = D_n \Rightarrow \|g\|_{\infty} \le 2^{3n} .$$

This might be the place to mention that throughout this paper we always use the simplest estimates sufficient for our purpose, even when they are rather crude as in (2.7).

We also have

(2.8)
$$g \in D_n \Rightarrow \|g\|_1 \ge \frac{1}{3} .$$

Indeed, since $|B_i| = 2^{-i} |B_0|$,

$$\begin{split} \|g\|_{1} &= \frac{1}{2n|B_{0}|} \sum_{0 \le i \le 2n-1} |a_{i}||B_{i} \setminus B_{i+1}| \\ &\ge \frac{1}{2n|B_{0}|} \sum_{0 \le i \le 2n-1} \frac{2^{i+1} + (-1)^{i}}{3} 2^{-i-1} |B_{0}| \\ &= \frac{1}{6n} \left(\sum_{0 \le i \le 2n-1} (1 + (-1)^{i} 2^{-i-1}) \right) \ge \frac{1}{3} \end{split}$$

since $\sum_{0 \le i \le 2n-1} (-1)^i 2^{-i-1} \ge 0$. We now prove that property (P_n) (and even more) holds.

Proposition 2.3. Consider an atomless subalgebra Σ' of Σ and a set $A \in \Sigma'$, $|A| > 2^{-n}$. Then for each $\eta > 0$, there exist $f \in L_0^1(A, \Sigma')$ and $g \in D_n$ such that $||f - g||_1 \le \eta$.

It is not stated here that we control $||f||_1$, but of course $||f||_1 \ge ||g||_1 - \eta$, and, as we can take $\eta < \frac{1}{12}$, we have from (2.8) that $||f||_1 \ge \frac{1}{4}$. The proof of Proposition 2.3 is a simple exercise. It will rely on the following

The proof of Proposition 2.3 is a simple exercise. It will rely on the following lemma.

Lemma 2.4. Consider $n \ge 0$, $A \in \Sigma$, $\varepsilon > 0$, $k \ge 0$, $B \in \Sigma_k$ such that $|A\Delta B| \le \varepsilon$. Consider $A' \subset A$ such that for all $I \in \mathcal{F}_k$ we have $|A' \cap I| = \frac{1}{2}|A \cap I|$. Then we can find m > k and $B' \in \Sigma_m$, $B' \subset B$, such that $|A'\Delta B'| \le 4\varepsilon$ and $E^k(1_{B'}) = \frac{1}{2}1_B$.

Proof. Consider first $A_1 = A' \cap B$. Since $A' \setminus A_1 = A' \setminus B \subset A \setminus B$, we have $|A' \setminus A_1| \leq \varepsilon$, and thus $|A' \Delta A_1| \leq \varepsilon$. We consider now a set $A_2 \in \Sigma$, $A_1 \subset A_2 \subset B$, such that whenever $I \in \mathcal{F}_k$, $I \subset B$, we have

$$|A_2 \cap I| = \frac{1}{2}|I|;$$
 $A \cap A_2 \cap I = A_1 \cap I \ (= A' \cap I).$

Since $A_2 \cap A = A_1$, we have $A_2 \setminus A_1 \subset B \setminus A$, and hence $|A_2 \setminus A_1| \leq \varepsilon$, so that $|A_2 \Delta A_1| \leq \varepsilon$. We can find m > k and $B_1 \in \Sigma_m$ such that $|B_1 \Delta A_2| \leq 2^{-k} \varepsilon$. For each $I \in \mathscr{I}_k$, $I \subset B$, we have

$$||B_1 \cap I| - 2^{-k-1}| = ||B_1 \cap I| - |A_2 \cap I|| \le |B_1 \Delta A_2| \le 2^{-k} \varepsilon.$$

Clearly, we can find $B' \in \Sigma_m$, $B' \subset B$, such that for all $I \in \mathscr{I}_k$, $I \subset B$,

$$|B' \cap I| = 2^{-k-1} = \frac{1}{2}|B' \cap I|; \qquad |(B' \cap I)\Delta(B_1 \cap I)| \le 2^{-k}\varepsilon.$$

It follows that $|B'\Delta B_1| \leq \varepsilon$. Thus,

$$|A'\Delta B'| \le |A'\Delta A_1| + |A_1\Delta A_2| + |A_2\Delta B_1| + |B_1\Delta B'| \le 4\varepsilon. \quad \Box$$

Proof of Proposition 2.3. Let $\varepsilon > 0$ such that $\varepsilon < |A| - 2^{-n}$. By induction over $i, 0 \le i \le 2n - 1$, we construct numbers $\sigma(i)$, sets A_i, B_i that satisfy conditions (2.3), (2.4), and the following:

$$A_0 = A; \quad A_{i+1} \subset A_i; \quad A_{i+1} \in \Sigma'; \quad |A_{i+1}| = \frac{1}{2}|A_i|; \quad |A_i \Delta B_i| \le 4^i \varepsilon.$$

To start the induction, we take $B_0 \in \Sigma_{\sigma(0)}$ such that $|A \Delta B_0| \le \varepsilon$.

Suppose that A_i , B_i , $\sigma(i)$ have been constructed. Since Σ' is atomless, its restriction to A_i is atomless. By Liapounov's convexity theorem, we can find $A_{i+1} \subset A_i$, $A_{i+1} \in \Sigma'$, such that

$$\int_{A_{i+1}} 1_I \, d\lambda = |A_{i+1} \cap I| = \frac{1}{2} |A_i \cap I|$$

for all $I \in \mathscr{I}_{\sigma(i)}$. In particular, $|A_{i+1}| = \frac{1}{2}|A_i|$. By Lemma 2.4, we can find $\sigma(i+1) > \sigma(i)$ and $B_{i+1} \in \Sigma_{\sigma(i+1)}$ such that $|A_{i+1}\Delta B_{i+1}| \le 4|A_i\Delta B_i|$ and $E^{\sigma(i+1)}(1_{B_{i+1}}) = \frac{1}{2}1_{B_i}$. This completes the construction.

The function

$$f = \frac{1}{2n|B_0|} \sum_{0 \le i \le 2n-1} (-1)^i 2^i 1_{A_i}$$

is Σ' -measurable and supported by $A_0 = A$. Moreover, since $|A_i| = 2^{-i}|A_0|$ for $i \leq 2n-1$, we have $\int f d\lambda = 0$. Thus, $f \in L^0(A, \Sigma')$. We consider the function

$$g = \frac{1}{2n|B_0|} \sum_{0 \le i \le 2n-1} (-1)^i 2^i 1_{B_i}.$$

By definition, $g \in D_n$, since $|B_0| \ge |A| - \varepsilon \ge 2^{-n}$. Moreover,

$$\|f - g\|_{1} = \frac{1}{2n|B_{0}|} \sum_{0 \le i \le 2n-1} 2^{i} \|\mathbf{1}_{B_{i}} - \mathbf{1}_{A_{i}}\|_{1}$$
$$\le \frac{1}{2n|B_{0}|} \sum_{0 \le i \le 2n-1} 2^{i} 4^{i} \varepsilon = c(n)\varepsilon$$

for some number c(n) depending on *n* only. Since *n* is fixed, $||f - g||_1$ can be made arbitrarily small. \Box

3. CONTROL IN MEASURE

We denote by $\operatorname{conv} D_n$ the absolute convex hull of D_n , i.e., the set of functions $\sum_{l \in L} c_l f_l$, where L is a finite set, $f_l \in D_n$, $\sum_{l \in L} |c_l| \le 1$. The cornerstone of our construction is the following.

Theorem 3.1. Given $\varepsilon > 0$, there exists n > 0 such that each function f in conv D_n satisfies $|\{|f| \ge \varepsilon\}| \le \varepsilon$.

The proof will show that one can actually take n of order $\varepsilon^{-6} \log(\frac{1}{\varepsilon})$. This estimate (that has no reason to be sharp) is irrelevant for our purposes.

The proof of Theorem 3.1 will be considerably clarified by the fact that D_n is contained in the convex hull of a family F_n (briefly considered in the introduction), which has the property of Theorem 3.1, but that consists of functions much simpler than those of D_n . We recall the definition of these functions now. Given $x \in [0, 1]$ and $k \ge 0$, we denote by I(x, k) the unique dyadic interval $I \in \mathcal{F}_k$ that contains x. Given an increasing sequence $\sigma = (\sigma(0), \ldots, \sigma(2n-1)), \sigma(0) < \sigma(1) < \cdots < \sigma(2n-1)$, we set

(3.1)
$$f(x, \sigma) = \frac{1}{2n} \sum_{0 \le i \le 2n-1} (-1)^i 2^{\sigma(i)} \mathbf{1}_{I(x, \sigma(i))} \, .$$

The reader might have observed that $(f_i(x, \sigma))_{0 \le i < n}$ is a martingale difference sequence. Unfortunately, the filtration $(\Sigma_{\sigma(2i)})_{0 \le i < n}$ with respect to which it has this property depends on f; this seems to prevent any simple use of martingale theory to prove Theorem 3.3. Since $|I(x, \sigma(i))| = 2^{-\sigma(i)}$, we have $||f(x, \sigma)||_1 \le 1$. It is worthwhile to note that if $C = I(x, \sigma(2n-1))$, we have $E^{\sigma(i)}(1_C) = 2^{\sigma(i)-\sigma(2n-1)} \mathbf{1}_{I(x, \sigma(i))}$, and thus,

(3.2)
$$f(x, \sigma) = \frac{1}{2n} \sum_{0 \le i \le 2n-1} \frac{(-1)^i}{|C|} E^{\sigma(i)}(1_C) \, .$$

We consider the class F_n of all functions of type (3.1) for all possible choices of x and σ .

Proposition 3.2. The convex hull of F_n contains D_n .

Proof. Consider g given by (2.5). From condition (2.4) we see that for $0 \le i \le 2n - 1$, we have

$$\frac{1}{|B_{2n-1}|}E^{\sigma(i)}(1_{B_{2n-1}}) = \frac{2^{2n-1}}{|B_0|}E^{\sigma(i)}(1_{B_{2n-1}}) = \frac{2^i}{|B_0|}1_{B_i}$$

so that

(3.3)
$$g = \frac{1}{2n} \sum_{0 \le i \le 2n-1} \frac{(-1)^i}{|B_{2n-1}|} E^{\sigma(i)}(1_{B_{2n-1}}).$$

We can write B_{2n-1} as a disjoint union $B_{2n-1} = \bigcup_{l \in L} C_l$, $C_l \in \mathscr{I}_{\sigma(2n-1)}$. Thus, (3.3) becomes

$$g = \frac{1}{\operatorname{card} L} \sum_{l \in L} \left(\frac{1}{2n} \sum_{0 \le i \le 2n-1} \frac{(-1)^{i}}{|C_{l}|} E^{\sigma(i)}(1_{C_{l}}) \right)$$

and, from (3.2), this shows that g belongs to the convex hull of F_n . \Box

Thus, in order to prove Theorem 3.1, it suffices to prove the following.

Theorem 3.3. Given $\varepsilon > 0$, there exists n > 0 such that each function f in conv F_n satisfies $|\{|f| \ge \varepsilon\}| \le \varepsilon$.

Before we can start the proof, we need some notation. In the entire sequel, n is fixed. We denote by S_n the collection of sequences $\sigma = (\sigma(i))_{0 \le i \le 2n-1}$; $\sigma(0) < \sigma(1) < \cdots < \sigma(2n-1)$. For $0 \le i < n$, $\sigma \in S_n$, we set

$$f_i(x, \sigma) = 2^{\sigma(2i)} \mathbf{1}_{I(x, \sigma(2i))} - 2^{\sigma(2i+1)} \mathbf{1}_{I(x, \sigma(2i+1))}.$$

For $p \leq n$, we set

$$f(x, \sigma, p) = \frac{1}{2n} \sum_{0 \le i \le p-1} f_i(x, \sigma)$$

= $\frac{1}{2n} \sum_{0 \le i \le 2p-1} (-1)^i 2^{\sigma(i)} \mathbf{1}_{I(x, \sigma(i))}$.

Thus, $f(x, \sigma, n) = f(x, \sigma)$ and $||f(x, \sigma, p)||_1 \le p/n$. The main ingredient of the proof of Theorem 3.3 is the following.

Proposition 3.4. Consider a finite index set L and numbers c_l with $\sum_{l \in L} |c_l| \leq 1$. 1. For $l \in L$, consider $x_l \in [0, 1]$, $\sigma_l \in S_n$, $p_l \leq n$. Consider $f = \sum_{l \in L} c_l f(x_l, \sigma_l, p_l)$ and r > 0. Assume $||f||_1 \geq 2^{-r}$ and $n \geq 3(r+1)2^{3r+7}$. Then we can find $A \in \Sigma$, $|A| \leq 2^{-2r}$, and numbers $q_l \leq p_l$ that satisfy the following conditions:

(3.4)
$$\frac{1}{n} \sum_{l \in L} c_l q_l \le \frac{1}{n} \sum_{l \in L} c_l p_l - 2^{-r-1}$$

(3.5) The function $f' = \sum_{l \in L} c_l f(x_l, \sigma_l, q_l)$ coincides with f outside A.

Observe that, since $||f(x, \sigma, p)||_1 \le p/n$, we have $||f||_1 \le \frac{1}{n} \sum_{l \in L} |c_l|p_l$. If we think of $\frac{1}{n} \sum_{l \in L} |c_l|p_l$ as measuring the size of f, this means that outside the very small set A, f coincides with the function f', which has a size significantly smaller than the size of f. Iterating the procedure will yield the proof of Theorem 3.3.

Proof. Let us first single out the main ingredient of the proof. For a function g, $||g||_{\infty} \leq 1$, $k \geq 0$, and $\alpha > \beta$, let us set

$$A(g, k, \alpha, \beta) = \{x; \exists m(0) < \dots < m(2k-1); \forall i, 0 \le i < k, \\ E^{m(2i)}(g)(x) \ge \alpha; E^{m(2i+1)}(g)(x) \le \beta\}.$$

Then

(3.6)
$$|A(g, k, \alpha, \beta)| \leq \left(\frac{1-\alpha}{1-\beta}\right)^k.$$

This follows, e.g., from an upcrossing inequality of Dubin as in [16, p. 27] applied to the positive martingale $E^n(1-g)$. Let us note also that (3.6) can be easily proved directly by induction over k and that we can even replace the bound of (3.6) by $((1-\alpha)/(1-\beta))^k((1+\beta)/(1+\alpha))^{k-1}$ (which makes no difference for our purposes).

Consider $g = \operatorname{sign} f$, so $||g||_{\infty} \leq 1$. We have

$$2^{-r} \le \|f\|_1 = \int fg \, d\lambda = \frac{1}{2n} \sum_{l \in L} c_l \sum_{0 \le i \le p_l - 1} \int gf_i(x_l, \sigma_l) \, d\lambda \, .$$

We set $k = 3(r+1)2^{r+3}$. For $l \in L$, denote by q_l the largest integer $\leq p_l$ such that

$$\operatorname{card}\left\{i: \ 0 \le i \le q_l; \ \int gf_i(x_l, \sigma_l) \, d\lambda \ge 2^{-r-1}\right\} \le 2^{r+3}k$$

Denote by L' the set of $l \in L$ such that $q_l < p_l$. For $l \in L'$, we have

(3.7)
$$\operatorname{card}\left\{i: 0 \le i \le q_l; \int gf_i(x_l, \sigma_l) d\lambda \ge 2^{-r-1}\right\} = 2^{r+3}k$$

We observe that

$$\int g 2^k \mathbf{1}_{I(x,k)} d\lambda = E^k(g)(x) \, .$$

Thus,

$$\int gf_i(x_l, \sigma_l) = E^{\sigma_l(2i)}(g)(x_l) - E^{\sigma_l(2i+1)}(g)(x_l) \,.$$

Hence, when $\int g f_i(x_l, \sigma_l) \ge 2^{-r-1}$, we can find an integer $-2^{r+2} \le s < 2^{r+2}$ such that

(3.8)
$$E^{\sigma_l(2i)}(g)(x_l) \ge (s+1)2^{-r-2}, \qquad E^{\sigma_l(2i+1)}(g)(x_l) \le s2^{-r-2}$$

Thus, (3.7) shows that for $l \in L'$, we can find s, $-2^{r+2} \le s < 2^{r+2}$, such that (3.8) holds for at least k indexes $i \le q_l - 1$. Hence, we have

$$I(x_l, \sigma_l(2q_l - 1)) \subset A_s = A(g, k, (s+1)2^{-r-2}, s2^{-r-2}).$$

From (3.6) we have

$$|A_s| \le \left(\frac{1 - (s+1)2^{-r-2}}{1 - s2^{-r-2}}\right)^k \le (1 - 2^{-r-3})^k \le e^{-k2^{-r-3}} \le 2^{-3r-3}.$$

We define $A = \bigcup_{2^{r+2} \le s \le 2^{r+2}} A_s$, so that $|A| \le 2^{-2r}$, and we have $I(x_l, \sigma_l(i)) \subset A$ for $i \ge 2q_l$, and hence f and $f' = \sum_{l \in L} |c_l| f(x_l, \sigma_l, q_l)$ coincide outside A. Since for each $l \in L$,

$$\left|\int gf_i(x_l,\,\sigma_l)\,d\lambda\right| \leq \|g\|_{\infty}\|f_i(x_l,\,\sigma_l)\|_1 \leq 2\,,$$

we have

$$\left| \int gf(x_l, \sigma_l, q_l) \, d\lambda \right| \le \frac{1}{2n} (2^{r+4}k + n2^{-r-1})$$
$$\le 2^{r+3} \frac{k}{n} + 2^{-r-2} \le 2^{-r-1}$$

This implies that $\sum_{l \in L} c_l \int gf(x_l, \sigma_l, q_l) d\lambda \le 2^{-r-1}$. On the other hand, since $\int gf d\lambda \ge 2^{-r}$, this implies that

(3.9)
$$2^{-r-1} \leq \sum_{l \in L} c_l \int g(f(x_l, \sigma_l, p_l) - f(x_l, \sigma_l, q_l)) d\lambda \\ \leq \sum_{l \in L} |c_l| \| f(x_l, \sigma_l, p_l) - f(x_l, \sigma_l, q_l) \|_1.$$

Now $||f(x_l, \sigma_l, p_l) - f(x_l, \sigma_l, q_l)||_1 \le \frac{1}{n}(p_l - q_l)$, and thus, (3.4) follows from (3.9). \Box

We now prove Theorem 3.3 and, actually, the following more precise statement. Consider r > 0, and suppose that $n \ge 3(r+1)2^{3r+7}$. Consider a finite index set L, and for $l \in L$, consider $x_l \in [0, 1]$, $\sigma_l \in S_n$, $c_l \in \mathbf{R}$. Assume $\sum_{l \in L} |c_l| \le 1$. Let $f = \sum_{l \in L} c_l f(x_l, \sigma_l)$. Then $|\{|f| \ge 2^{-r/2}\}| \le 2^{-r/2+2}$.

By induction over k, $k \ge 0$, we will construct numbers $(p_l^k)_{l \in L}$, $0 \le p_l^k \le n$, and sets $A_k \in \Sigma'$ that have the following properties:

(3.10)
$$\frac{1}{n} \sum_{l \in L} |c_l| p_l^k \le \frac{1}{n} \sum_{l \in L} |c_l| p_l^{k-1} - 2^{-r-1},$$

 $|A_k| \le 2^{-2r}$, f and $f^k = \sum_{l \in L} c_l f(x_l, \sigma_l, p_l^k)$ coincide outside $A'_k = \bigcup_{i \le k} A_i$. To start the induction, we set $p_i^0 = n$ for $l \in L$, $A_0 = \emptyset$.

To start the induction, we set $p_l^0 = n$ for $l \in L$, $A_0 = \emptyset$. The induction step is done by applying Proposition 3.2. The construction stops at the first value of k for which $||f^k||_1 \le 2^{-r}$. Thus, $|\{|f^k| \ge 2^{-r/2}\}| \le 2^{-r/2}$. From (3.10), we have

$$0 \le \frac{1}{n} \sum_{l \in L} |c_l| p_l^k \le 1 - k 2^{-r-1}$$

and thus, $k \le 2^{r+1}$. Hence, we have $|A'_k| \le k2^{-2r} \le 2^{-r+1}$. Since f and f^k coincide outside A'_k , we have $|\{|f| \ge 2^{-r/2}\}| \le 2^{-r/2} + 2^{-r+1} \le 2^{2-r/2}$. \Box

4. The three-space problem

Our task is, as described earlier, to construct small perturbations of the functions in D_n in such a way that their span will be isomorphic to l^1 . By induction over $l \ge 1$, we construct a sequence q(l) and a disjoint sequence (B_l) of measurable sets that satisfy the following conditions:

$$(4.1) 0 < |B_l| \le 2^{-3q(l-1)-2l-2}.$$

(4.2)
$$\forall f \in \operatorname{conv} D_{q(l)}, |\{|f| \ge 2^{-2l}\}| \le 2^{-l-1} \min_{i \le l} |B_i|.$$

18

We start the induction with $B_1 = [0, 2^{-4}]$. At each stage of the construction, we pick q(l) such that (4.2) holds, using Theorem 3.1, then B_{l+1} satisfying (4.1). We consider an enumeration $(f_{n,k})_{k\geq 1}$ of $C_n = D_{q(n)}$. We consider a one-to-one map $\varphi : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$, and we set $A(n, k) = B_{\varphi(n,k)}$. Finally, we set

$$g_{n,k} = f_{n,k} + 2^{-n} \frac{1}{|A(n,k)|} \mathbf{1}_{A(n,k)}.$$

Thus, $\|g_{n,k}\|_1 \le 1 + 2^{-n} \le 2$, and $\|g_{n,k} - f_{n,k}\|_1 \le 2^{-n}$.

Proposition 4.1. Consider a family $(a_{n,k})_{n,k\geq 1}$ of numbers. Assume that $\sum_{n\geq 1} 2^{-n} \sum_{k\geq 1} |a_{n,k}| < \infty$. Then the series $\sum_{n\geq 1} (\sum_{k\geq 1} a_{n,k}g_{n,k})$ converges in measure. Denoting, for simplicity, its sum by $\sum_{n,k\geq 1} a_{n,k}g_{n,k}$, we have

(4.3)
$$\left\|\sum_{n,k\geq 1} a_{n,k} g_{n,k}\right\|_{1} \le 1 \Rightarrow \sum_{n,k\geq 1} 2^{-n} |a_{n,k}| \le 4$$

Proof. For $n \ge 1$, let $b_n = \sum_{k\ge 1} |a_{n,k}|$, so that $\sum_{n\ge 1} 2^{-n} b_n < \infty$. Let $h_n = \sum_{k\ge 1} a_{n,k} f_{n,k}$, so that $h_n \in b_n N_n$, where N_n is the norm closure of conv C_n . It follows from (4.2) that the series $\sum_{n\ge 1} (\sum_{k\ge 1} a_{n,k} g_{n,k})$ converges in measure. From (2.7) we have that $||h_n||_{\infty} \le b_n 2^{3q(n)}$ and thus, for l > n, by (4.1),

(4.4)
$$\int_{B_l} |h_n| \, d\lambda \le b_n 2^{3q(n)} |B_l| \le b_n 2^{-2l-2} \le b_n 2^{-l-n-2}$$

Let $A_n = |\{|h_n| \ge b_n 2^{-2n}\}|$. From (4.2), we have

$$|A_n| \le 2^{-n-1} \min_{i \le n} |B_i|.$$

Let $A'_l = \bigcup_{n \ge l} A_n$. Since $|A_n| \le 2^{-n-1} |B_l|$ for $n \ge l$, we have $|A'_l| \le 2^{-l} |B_l| \le |B_l|/2$. Let $B'_l = B_l \setminus A'_l$. Thus, for $l \le n$,

(4.5)
$$\int_{B'_l} |h_n| \, d\lambda \le b_n 2^{-2n} |B'_l| \le b_n 2^{-l-n-2} \, .$$

From (4.4) and (4.5) we see that for all l, $n \ge 1$,

$$\int_{B_l'} |h_n| \, d\lambda \le b_n 2^{-l-n-2}$$

Thus, we have

(4.6)
$$\int_{B_{l}'} \left| \sum_{n, k \ge 1} a_{n, k} f_{n, k} \right| d\lambda = \int_{B_{l}'} \left| \sum_{n \ge 1} h_{n} \right| d\lambda \le 2^{-l-2} \sum_{n \ge 1} 2^{-n} b_{n}.$$

Consider now $r, s \ge 1$ and $l = \varphi(r, s)$. We have $B_l \cap A(n, k) = \emptyset$ unless n = r, k = s. So, since $|B'_l| \ge |B_l|/2$, we have

$$\int_{B_{l}'} \left| \sum_{n,k \ge 1} 2^{-n} a_{n,k} \frac{1}{|A(n,k)|} \mathbf{1}_{A(n,k)} \right| d\lambda \ge \int_{B_{l}'} \left| 2^{-r} a_{r,s} \frac{1}{|B_{l}|} \mathbf{1}_{B_{l}} \right| d\lambda \ge 2^{-r-1} |a_{r,s}|.$$

Thus, with (4.6) we get

$$\int_{B_l'} \left| \sum_{n,k\geq 1} a_{n,k} g_{n,k} \right| d\lambda \ge 2^{-r-1} |a_{r,s}| - 2^{-l-2} \sum_{n\geq 1} 2^{-n} b_n.$$

Summing this inequality over all the values of r, s gives, since the sets B'_l are disjoint,

$$\left\|\sum_{n,k} a_{n,k} g_{n,k}\right\|_{1} \ge \frac{1}{2} \sum_{r,s \ge 1} 2^{-r} |a_{r,s}| - \frac{1}{4} \sum_{n \ge 1} 2^{-n} b_{n} = \frac{1}{4} \sum_{n,k \ge 1} 2^{-n} |a_{n,k}| \quad \Box$$

A consequence of (4.3) is that for each n and each sequence (a_k) of numbers, all but finitely many zero, we have

$$2^{-n-2} \sum_{k \ge 1} |a_k| \le \left\| \sum_{k \ge 1} a_k g_{n,k} \right\|_1 \le 2 \sum_k |a_k|$$

Thus, the closed linear span H_n of the sequence $(g_{n,k})_{k\geq 1}$ is isomorphic to l^1 . We now show that the span of the spaces H_n is isomorphic to their l^1 -sum. **Proposition 4.2.** For each sequence $h_n \in H_n$ that is eventually zero, we have

$$\frac{1}{20}\sum_{n\geq 1}\|h_n\|_1 \le \left\|\sum_{n\geq 1}h_n\right\|_1 \le \sum_{n\geq 1}\|h_n\|_1.$$

Proof. It is enough to consider the case where each function h_n is a finite sum $\sum_{k\geq 1} a_{n,k} g_{n,k}$. We have to prove that if $\|\sum_{n\geq 1} h_n\|_1 \leq 1$, we have $\sum_{n\geq 1} \|h_n\|_1 \leq 20$. From Proposition 4.1, we know that $\sum_{n,k\geq 1} 2^{-n} |a_{n,k}| \leq 4$. Since $\|g_{n,k} - f_{n,k}\|_1 = 2^{-n}$, we have $\|\sum_{n\geq 1} u_n\|_1 \leq 5$, where $u_n = \sum_{k\geq 1} a_{n,k} f_{n,k}$. Since $\sum_{k\geq 1} |a_{n,k}| \leq 2^{n+2}$, u_n belongs to the closure of 2^{n+2} conv C_n . Thus, Proposition 4.2 is a consequence of the inequality

$$\sum_{n\geq 1} \|h_n\|_1 \le \sum_{n\geq 1} \|u_n\|_1 + \sum_{n,k\geq 1} 2^{-n} |a_{n,k}| \le \sum_{n\geq 1} \|u_n\|_1 + 4$$

and of the following.

Proposition 4.3. Consider a sequence $u_n \in 2^{n+2}N_n$, where N_n is the closure of conv C_n . Then the series $\sum_{n\geq 1} u_n$ converges in measure, and if its sum belongs to L^1 , we have

$$\sum_{n \ge 1} \|u_n\|_1 \le \left\|\sum_{n \ge 1} u_n\right\|_1 + 6$$

Proof. From (2.7), we have $||u_n||_{\infty} \le 2^{3q(n)+n+2}$. Set $A_n = \{|u_n| \ge 2^{-n+2}\}$. From (4.2), (4.1), we have

$$|A_n| \le 2^{-n-1} |B_n| \le 2^{-3q(n-1)-3n-3}$$

Hence, if we define $A'_n = \bigcup_{i>n} A_i$, we have $|A'_n| \le 2^{-3q(n)-3n-2}$, and thus, for $i \le n$,

(4.7)
$$\int_{A'_n} |u_i| \, d\lambda \le 2^{3q(i)+i+2} |A'_n| \le 2^{-2n}$$

We define
$$L_n = A_n \setminus A'_n$$
. For $i > n$, since $|u_i| \le 2^{-i+2}$ on L_i , we have
$$\int_{L_n} |u_i| \, d\lambda \le 2^{-i+2} |L_n| \le 2^{-i-2} \,.$$

Thus,

$$\int_{L_n} \left| \sum_{i \neq n} u_i \right| d\lambda \le n 2^{-2n} + 2^{-n-2} \le 2^{-n},$$

and hence,

$$\int_{L_n} |u_n| \, d\lambda \leq \int_{L_n} \left| \sum_{i \geq 1} u_i \right| \, d\lambda + 2^{-n} \, .$$

Summing over *n* gives, since the sets L_n are disjoint,

(4.8)
$$\sum_{n\geq 1} \int_{L_n} |u_n| \, d\lambda \leq \left\| \sum_{n\geq 1} u_n \right\|_1 + 1 \, .$$

Since $|u_n| \le 2^{-n+2}$ outside A_n , using (4.7), we have

(4.9)
$$||u_n||_1 \leq \int_{L_n} |u_n| d\lambda + 2^{-n+2} + 2^{-2n}.$$

From (4.8), this shows that $\sum_{n\geq 1} \|u_n\|_1 \le \|\sum_{n\geq 1} u_n\|_1 + 6$. \Box

We now prove Theorem 1.1. We denote by X_1 (resp. X_2) the closed linear span of $\bigcup_{n\geq 1} H_{2n}$ (resp. $\bigcup_{n\geq 1} H_{2n+1}$). From Propositions 4.1 and 4.2, both X_1 and X_2 are isomorphic to an l^1 -sum of spaces isomorphic to l^1 (and thus do not contain L^1).

We prove that the canonical map from L^1 to $(L^1/X_1) \times (L^1/X_2)$ is an embedding. Let $f \in L^1$, and suppose that for $f_1 \in X_1$, $f_2 \in X_2$ we have $\|f - f_1\|_1 \leq 1$, $\|f - f_2\|_1 \leq 1$. Then $\|f_1 - f_2\|_1 \leq 2$. However, Proposition 4.2 implies that $\|f_1\|_1 + \|f_2\|_1 \leq 40$. Thus, $\|f\|_1 \leq 1 + \|f_1\|_1 \leq 41$, which proves the claim.

It follows from our construction, Proposition 2.3, and Corollary 2.2 that for $X = X_1$ or X_2 , the quotient map T from L^1 onto L^1/X does not fix a copy of L^1 . It remains to show that L^1/X does not contain a copy of L^1 . The proof, which will be given at the end of §5, relies on a well-known compactness argument. This argument will be used twice. To avoid duplication, we present a somewhat abstract statement (in the spirit of [10]) that will cover all our needs.

Proposition 4.4. Consider an operator T from a Banach space Y onto a Banach space Z. For $z \in Z$, ||z|| < 1, let $U_z = \{y \in Y; ||y|| \le 1, T(y) = z\}$. Suppose that for some constant K, to each sequence $(y_n)_{n\geq 1} \in U_z$, we can associate a point $\varphi((y_n)_{n\geq 1}) \in Y$ with the following properties.

(4.10) If
$$x_n = y_n$$
 for n large enough, $\varphi((x_n)_{n\geq 1}) = \varphi((y_n)_{n\geq 1})$.

$$(4.11) \quad \|\varphi((y_n)_{n>1})\| \le K.$$

- (4.12) $T(\varphi((y_n)_{n>1})) = z$.
- (4.13) If $y_n \in U_z$, $y'_n \in U_z$, we have $\varphi\left(\left(\frac{y_n + y'_n}{2}\right)_{n \ge 1}\right) = \frac{1}{2}[\varphi((y_n)_{n \ge 1}) + \varphi((y'_n)_{n \ge 1})].$

Then if Z contains a copy of L^1 , T fixes a copy of L^1 .

Proof. Since Z contains a copy of L^1 , there exists an isomorphism V from L^1 into Z, and we can assume that ||V|| < 1. For $I \in \mathcal{I}_k$, $k \ge 1$, let $a_I = V(2^k 1_I)$. Thus, $||a_I|| < 1$. Take $b_I \in U_a$.

For $I \in \mathcal{I}_k$, $n \ge k$, set $b_{I,n} = 2^{k-n} \sum_{J \in \mathcal{I}_n, J \subset I} b_J$ and $b_{I,n} = b_I$ for $n \le k$. Obviously, $b_I \in U_{a_I}$. If I_1 , I_2 are the two elements of \mathcal{I}_{k+1} contained in I, we clearly have for $n \ge k+1$,

(4.14)
$$b_{I_1,n} = \frac{1}{2}(b_{I_1,n} + b_{I_2,n}).$$

We now set $c_I = \varphi((b_{I_1,n})_{n\geq 1})$. From (4.11), $||c_I|| \leq K$. From (4.10), (4.13), (4.14), $c_I = \frac{1}{2}(c_{I_1} + c_{I_2})$. From (4.12), $T(c_I) = a_I$. Thus, one can define an operator W from L^1 to Y such that $W(2^k 1_I) = c_I$ for $I \in \mathcal{I}_k$, and this operator satisfies $||W|| \leq K$, $V = T \circ W$. Since V is an isomorphism, T is an isomorphism from $W(L^1)$ to its image, and thus T fixes L^1 . \Box

5. SEMIEMBEDDINGS

In this section, we denote by $M_n = 2^n N_n$ the absolute closed convex hull of $2^n C_n = 2^n D_{q(n)}$. We denote by L^0 the space of all measurable functions on [0, 1]. We provide L^0 with the topology of convergence in measure. We denote by G the set of all functions in L^0 that can be represented as $f_1 + \sum_{n \ge 1} \alpha_n g_n$, where $f_1 \in L^1$, $||f_1||_1 \le 1$, $g_n \in M_n$, $\sum_{n \ge 1} |\alpha_n| \le 1$. We observe that the series $\sum_{n\ge 1} \alpha_n g_n$ converges in measure and that G is bounded in L^0 . We denote by N the gauge of G, given by $N(h) = \inf\{a > 0; h \in aG\}$ if this set is nonempty and $N(h) = \infty$ otherwise. We set $E = \{h \in L^0; N(h) < \infty\}$, and we provide E with the norm N. Since G is bounded in L^0 , E is a Banach space, and the canonical map j from L^1 to E is one-to-one. Since the unit ball of L^1 is closed in L^0 , j is a semiembedding. If we consider the space E_1 (resp. E_2) constructed like E but replacing the condition $g_n \in M_n$ by $g_n \in M_{2n}$ (resp. $l_n \in M_{2n+1}$), Proposition 4.3 shows that the canonical injection from L^1 into $E_1 \times E_2$ is an isomorphism. Indeed, if $f \in L^1$, and if we have f_1 , $f_2 \in L^1$, $\|f_1\|_1$, $\|f_2\|_1 \le 1$, $g_n \in M_n$ such that $f = f_1 + \sum_{n\ge 1} g_{2n} = f_2 + \sum_{n\ge 0} g_{2n+1}$, then $\sum_{n\ge 1} (-1)^n g_n = f_2 - f_1$ so that $\sum \|g_n\|_1 \le 8$ and $\|f\|_1 \le 9$. Thus, in order to prove Theorem 1.4, it suffices to show that E does not contain L^1 .

We know by our construction, Corollary 2.2 , and the fact that $\|j(f)\|_1 \leq 2^{-n}$ for $f \in D_{q(n)}$ that j does not fix a copy of L^1 . Suppose that there is an isomorphism V from L^1 to E. We can assume ||V|| < 1 and that we have $\|V(f)\| \ge \delta \|f\|_1$ for some $\delta > 0$. We will show that j must fix a copy of L^1 , thereby concluding the proof. The first part of the argument will be the application of Proposition 4.4 to an auxiliary space which we construct now.

We denote by X_n the Banach space $\bigcup_{\lambda \in \mathbf{R}} \lambda M_n$ provided with the norm given by the gauge $\|\cdot\|_n$ of M_n . We denote by $\bigoplus_1 X_n$ the l^1 -sum of the spaces X_n . We denote by Y the space $L^1 \times (\bigoplus_1 X_n)$ provided with the norm $\|(f, (g_n))\| = \max(\|f\|_1, \sum_{n \ge 1} \|g_n\|_n)$. Consider the operator $T: Y \to E$ given by $T((f, (g_n))) = f + \sum_{n \ge 1} g_n$. Clearly, $||T|| \le 1$, and T is onto. Let $z \in E$, ||z|| < 1, and consider a sequence $y_k = (f_k, (g_{n,k})_{n \ge 1})$ in

 $U_z = \{y \in Y; \|y\| \le 1, T(y) = z\}$. Hence, for each k,

(5.1)
$$z = f_k + \sum_{n \ge 1} g_{n,k} .$$

Let us fix an ultrafilter \mathscr{U} on N. Since M_n is convex, uniformly bounded, and closed, it is $\sigma(L^1, L^{\infty})$ -compact, and we can consider $g_n = \lim_{k \to \mathcal{U}} g_{n,k}$, where the limit is taken for this topology. Let $f = z - \sum_{n \ge 1} g_n$. We show that $\|f\|_1 \le 1$. Given $m \ge 0$, we can find coefficients α_k , all but finitely many zero, with $\sum_{k>1} \alpha_k = 1$ such that

$$\left\|g_n - \sum_{k \ge 1} \alpha_k g_{n,k}\right\|_1 \le 2^{-m}$$

for all $n \leq m$. We have from (5.1)

(5.2)
$$f - \sum_{k \ge 1} \alpha_k f_k = -\sum_{n \le m} \left(g_n - \sum_{k \ge 1} \alpha_k g_{n,k} \right) - \sum_{n > m} g_n + \sum_{n > m} \sum_{k \ge 1} \alpha_k g_{n,k} .$$

By definition of U_z and the definition of the norm of Y, we have $||f_k||_1 \le 1$ for all k; thus, (5.2) shows that f belongs to the closure in measure of the unit ball of L^1 ; thus, $||f||_1 \le 1$.

If we define $\varphi((y_k)_{k\geq 1}) = (f, (g_n)) \in Y$, it is obvious that (4.10)-(4.13) hold (with K = 1). Thus, from Proposition 4.4, we can find an operator W from L^1 to Y such that $V = T \circ W$.

The second part of the argument is crystalized in the following lemma. For $m \ge 1$, denote by Y_n the subspace of Y consisting of the $y = (f, (g_n))$ such that $g_n = 0$ whenever $n \ge m$.

Lemma 5.1. Consider an operator W from L^1 to Y. Then for each $\varepsilon > 0$, there exists an m, a set of positive measure A, and an operator W' from L^1 to Y_m such that $||W(f) - W'(f)|| \le \varepsilon ||f||$ whenever f is supported by A.

Before we prove this lemma, let us conclude the proof of Theorem 1.4. Recall that $||V(f)|| \ge \delta ||f||_1$ for $f \in L_1$. We apply Lemma 5.1 with $\varepsilon = \delta/2$. Thus, when $f \in L^{1}(A)$, the subspace of L^{1} consisting of functions supported by A, we have

$$||T \circ W(f) - T \circ W'(f)|| \le \frac{\delta}{2} ||f||_1$$

Since $||T \circ W(f)|| = ||V(f)|| \ge \delta ||f||_1$, we have $||T \circ W'(f)|| \ge \frac{\delta}{2} ||f||_1$ when $f \in L^1(A)$. We observe now that $T(Y_m) \subset L^1$ and that $T: Y_m \to L^1$ is bounded. Considering now $T \circ W'$ as valued in L^1 , we have

$$||j \circ T \circ W'(f)|| \ge \frac{\delta}{2} ||f||_1 \ge \frac{\delta}{2||T \circ W'||} ||T \circ W'(f)||_1$$

so that j fixes $T \circ W'(L^1(A))$. This completes the proof. \Box

Proof of Lemma 5.1. Denote by π_n the canonical projection of Y on X_n , and let $W_n = \pi_n \circ W$. By definition of the norm of Y, for $f \in L^1$, we have

$$\sum_{n \ge 1} \|W_n(f)\|_n \le \|W(f)\| \, .$$

For $n \ge 1$, $k \ge 1$ define a \sum_{k} -measurable function $h_{n,k}$ by $h_{n,k} = \|\frac{1}{|I|}W_n(1_I)\|_n$ for $x \in I \in \mathcal{I}_k$. Thus, $\sum_{n\ge 1}h_{n,k} \le \|W\|$. For each n, the sequence $(h_{n,k})_{k\ge 1}$ is a supermartingale; denote by h_n its limit a.e. Clearly, $\sum_{n\ge 1}h_n \le \|W\|$. For $I \in \sum_k$, $n \ge 1$, we have

$$\|W_n(1_I)\|_n = \int_I h_{n,k} \, d\lambda \leq \int 1_I h_n \, d\lambda \, .$$

Thus, by approximation, for any $f \in L^1$, we have

$$\left\|W_n(f)\right\|_n \le \int |f|h_n \, d\lambda$$

Thus, for any m,

$$\sum_{n\geq m} \left\| W_n(f) \right\|_n \leq \int |f| \left(\sum_{n\geq m} h_n \right) d\lambda.$$

Since $\sum_{n\geq 1} h_n \leq ||W||$, for any ε , we can find *m* large enough that |A| > 0, where $A = \{\sum_{n\geq m} h_n \leq \varepsilon\}$. If we denote by W' the composition of W and of the canonical projection of Y onto Y_m , for f supported by A, we have

$$\|W(f) - W'(f)\| \leq \sum_{n>m} \|W_n(f)\|_n \leq \int |f|\varepsilon \, d\lambda \leq \varepsilon \|f\|_1.$$

This completes the proof. \Box

We now turn back to the proof that L^1/X does not contain a copy of L^1 . We denote by Z the closure of j(X) in E. **Proposition 5.2.** Z is isomorphic to l^1 . *Proof.* Since

(5.3)
$$2^{n}g_{n,k} = \frac{1}{|A(n,k)|} \mathbf{1}_{A(n,k)} + 2^{n}f_{n,k},$$

the definition of $\|\cdot\|_E$ shows that $\|u_{n,k}\|_E \leq 2$, where $u_{n,k} = j(2^n g_{n,k})$. Consider now numbers $a_{n,k}$, all but finitely many zero, and suppose that $\|\sum_{n,k\geq 1} a_{n,k} u_{n,k}\|_E \leq 1$. By definition of the norm of E, we can find $f \in L^1$, $\|f\|_1 < 2$, $g_n \in M_n$, $(\alpha_n)_{n\geq 1}$, with $\sum |\alpha_n| \leq 2$ such that

$$\sum_{n,k\geq 1}a_{n,k}u_{n,k}=f+\sum_{n\geq 1}\alpha_ng_n.$$

There is no loss of generality to assume that $g_n \in 2^n \operatorname{conv} C_n$. Thus, we can write

$$\alpha_n g_n = \sum_{k\geq 1} 2^n b_{n,k} f_{n,k},$$

where $\sum_{k\geq 1} |b_{n,k}| \leq \alpha_n$, and thus, $\sum_{n,k\geq 1} |b_{n,k}| \leq 2$. Hence, using (5.3) again, for some $f' \in L^1$, $||f'||_1 \leq 4$, we have

$$\sum_{n,k\geq 1} 2^n a_{n,k} g_{n,k} = f' + \sum_{n\geq 1} \sum_{k\geq 1} 2^n b_{n,k} g_{n,k}$$

or, equivalently,

$$\sum_{n,k\geq 1} 2^{n} (b_{n,k} - a_{n,k}) g_{n,k} = -f'.$$

We observe that $\sum_{n,k} |b_{n,k} - a_{n,k}| < \infty$. Since $||f'||_1 \le 4$, Proposition 4.1 implies that $\sum_{n,k} |b_{n,k} - a_{n,k}| \le 16$, and thus, $\sum_{n,k} |a_{n,k}| \le 18$.

Thus, the span of $(u_{n,k})_{n,k\geq 1}$ is isomorphic to l^1 , but it is obviously the closure of j(X). \Box

Proposition 5.3. L^1/X is isomorphic to E/Z.

Proof. Denote by U (resp. V) the quotient map from L^1 onto L^1/X (resp. E onto E/Z). Since $V \circ j$ is zero on X, there exists an operator T from L^1/X into E/Z such that $T \circ U = V \circ j$. Since $j(L^1)$ is dense in E, the image of T is dense, so we are left to show that T is an isomorphism.

Consider $x \in L^1$ such that $||T \circ U(x)||_1 < 1$. Hence, $||V \circ j(x)||_1 < 1$ so that there exists $y \in X$ with $||j(x) - j(y)||_E < 1$. Thus,

$$x-y=f+\sum_{n\geq 1}\alpha_ng_n\,,$$

where $||f||_1 \le 1$, $\sum_{n\ge 1} |\alpha_n| \le 1$, $g_n \in M_n$. From Proposition 4.2, we see that $\sum_{n\ge 1} \alpha_n ||g_n||_1 < \infty$; hence, we can write

$$x - y = f' + \sum_{1 \le n \le N} \alpha_n g_n,$$

where $||f'||_1 \leq 2$; and we can assume that $g_n \in 2^n \operatorname{conv} C_n$. Denote by B the unit ball of L^1 . It follows from (5.3) that $2^n f_{n,k} \in B + X$, and thus, $2^n \operatorname{conv} C_n \subset B + X$. Since $\sum_{n\geq 1} |\alpha_n| \leq 1$, we have $\sum_{1\leq n\leq N} \alpha_n g_n \in B + X$. Thus, we can find $y' \in X$ such that $||x - y - y'||_1 \leq 3$. This shows that $||U(x)|| \leq 3$ and concludes the proof. \Box

In view of Proposition 5.3, to prove that L^1/X does not contain a copy of L^1 , it suffices to show that E/Z does not contain a copy of L^1 . We already know that E does not contain a copy of L^1 , and we will conclude by application of Proposition 4.2. From Proposition 5.2, Z is isomorphic to l^1 , which is a dual space. We denote by τ the weak * topology induced by such an isomorphism. Let us fix an ultrafilter \mathscr{U} on N. For a sequence $(y_n)_{n\geq 1}$ in $U_z = \{y \in E; \|y\| \leq 1, V(y) = z\}$, we set $\varphi((y_n)_{n\geq 1}) = y + \lim_{n\to \mathscr{U}} (y_n - y)$, where y is any point of $V^{-1}(z)$, and where the limit is taken in the weak * sense in Z. It is obvious that (4.10)–(4.13) are satisfied. This completes the proof of Theorem 1.1.

6. Quotients of L^1 by a subspace isomorphic to l^1

In this section, we prove Proposition 1.2. The proof relies on an elementary property of l^1 that must have been observed long ago. We provide a sketch of a proof for the convenience of the reader.

Lemma 6.1. Consider a sequence $(g_n)_{n\geq 1}$, $g_n \in l^1$, such that $||g_n|| = 1$. Let $\varepsilon > 0$. Then there exists a sequence n(k) such that each finite convex combination g of vectors $g_{n(k)}$ satisfies $||g|| \geq 1 - \varepsilon$.

Proof. Let $g_n = (g_{n,p})_{p \ge 1}$. We can assume that $a_p = \lim_{n \to \infty} g_{n,p}$ exists for all $p \ge 1$. We have $\sum_{p \ge 1} |a_p| \le 1$. We choose m(0) such that $\sum_{p > m(0)} |a_p| \le \frac{\varepsilon}{8}$. We then construct increasing sequences of integers m(k), n(k) such that

$$\sum_{p \leq m(k)} |a_p - g_{n(k),p}| \leq \frac{\varepsilon}{8}, \qquad \sum_{p > m(k+1)} |g_{n(k),p}| \leq \frac{\varepsilon}{8}.$$

Define $b_p = \operatorname{sign} a_p$ for $p \le m(0)$, $b_p = \operatorname{sign} g_{n(k),p}$ for $k \ge 1$, $m(k) . Simple computations show that <math>\sum_{p\ge 1} b_p g_{n(k),p} \ge 1-\varepsilon$ for all $k\ge 1$; this implies the result. \Box

For $k \ge 1$, we consider the set

$$B_{k} = \bigcup_{0 \le l < 2^{k-1}} [2l2^{-k}, (2l+1)2^{-k}].$$

Thus, $B_k \in \Sigma_k$ and $E^l(1_{B_k}) = \frac{1}{2}$ for l < k. We denote by S_p the collection of sequences $s = (\sigma(1), \ldots, \sigma(p))$ such that $1 \le \sigma(1) < \sigma(2) < \cdots < \sigma(p)$. For $s \in S_p$, we denote by $\Sigma(s)$ the algebra generated by $B_{\sigma(1)}, \ldots, B_{\sigma(p)}$. We observe that each of its atoms has measure $= 2^{-p}$. For a subalgebra Σ' of Σ , we denote by $L_0^1(\Sigma')$ the set of Σ' -measurable functions that are integrable and of integral zero. The core of the proof of Proposition 1.2 is the following.

Lemma 6.2. Consider an operator T from l^1 to L^1 and $s = (\sigma(1), \ldots, \sigma(p))$, $s \in S_p$. Let $\alpha > 0$. Suppose that for each $f \in L_0^1(\Sigma(s))$ and $g \in l^1$, ||g|| = 1, we have $||f - T(g)||_1 \ge \alpha$. Let $\beta < \alpha$. Then we can find $\sigma(p+1) > \sigma(p)$ such that if we set $s' = (\sigma(1), \ldots, \sigma(p), \sigma(p+1))$, then for each $f \in L_0^1(\Sigma(s'))$ and $g \in l^1$, $||g||_1 = 1$, we have $||f - T(g)||_1 \ge \beta$.

Proof. For $n > \sigma(p)$, we set $s_n = (\sigma(1), \ldots, \sigma(p), n) \in S_{p+1}$. Suppose that the conclusion of the lemma fails. Then for some $\beta < \alpha$ and each $n > \sigma(p)$, we can find $f_n \in L_0^1(\Sigma(s_n))$ and $g_n \in l^1$, $||g_n|| = 1$, such that $||f_n - T(g_n)||_1 \leq \beta$. Consider $\varepsilon > 0$, to be determined later. Lemma 6.1 gives a sequence n(k) such that each convex combination g of the sequence $(g_{n(k)})_{k\geq 1}$ satisfies $1 - \varepsilon \leq ||g|| \leq 1$. Since each atom of $\Sigma(s_n)$ has measure $= 2^{-p-1}$, the sequence $(f_n)_{n\geq 1}$ is uniformly bounded. There is no loss of generality to assume that the sequence $(f_{n(k)})_{k\geq 1}$ converges weakly to some function f. Since $\Sigma(s_n)$ is generated by $\Sigma(s)$ and B_n , where $E^l(1_{B_n}) = \frac{1}{2}$ for n > l, for $n > l > \sigma(p)$ we have $\int_U f_n d\lambda = \int_V f_n d\lambda$ for any two atoms U, V of Σ_l that are contained in the same atom of $\Sigma(s)$. It follows that $\int_U f d\lambda = \int_V f d\lambda$, and thus, f is constant on each atom of $\Sigma(s)$. Since $\int f d\lambda = 0$, we have $f \in L_0^1(\Sigma(s))$. There is a finite convex combination $h = \sum a_k f_{n(k)}$ such that $||f - h||_1 \leq \varepsilon$. Since $||g_n|| = 1 - ||g'|| \leq \varepsilon$. We have ||g|| = 1, and $||f - T(g)||_1 \leq \beta + \varepsilon(1 + ||T||)$. Thus, if $\varepsilon(1 + ||T||) < \alpha - \beta$, this is a contradiction. \Box

We now prove Proposition 1.2. Let T be an isomorphism from l^1 to X. Let $\alpha = \inf\{\|T(g)\|_1; \|g\| = 1\} > 0$. We set $\Sigma(\emptyset) = \{\emptyset, [0, 1]\}$, and thus, $L_0^1(\Sigma(\emptyset)) = \{0\}$. Thus, $\|f - T(g)\|_1 \ge \alpha$ for $g \in l^1$, $\|g\| = 1$, $f \in L_0^1(\Sigma(\emptyset))$. Using Lemma 6.2, we construct by induction a sequence $\sigma(p)$ such that if $s(0) = \emptyset$, $s(p) = (\sigma(1), \ldots, \sigma(p))$ for $p \ge 1$, for each $f \in L_0^1(\Sigma(s(p)))$ and each $g \in l^1$, $\|g\| = 1$, we have $\|f - T(g)\|_1 \ge \alpha(2^{-p} + \frac{1}{2})$.

Denote by Σ' the σ -algebra generated by the union of the algebras $\Sigma(s(p))$. Then $\bigcup_{p\geq 1} L_0^1(\Sigma(s(p)))$ is dense in norm in $L_0^1(\Sigma')$, and thus, $||f - T(g)||_1 \geq \frac{\alpha}{2}$ for $f \in L_0^1(\Sigma')$, $g \in l^1$, ||g|| = 1. It follows easily that for some $\beta > 0$, we have $||f - g||_1 > \beta$, whenever $f \in L_0^1(\Sigma')$, $||f||_1 = 1$, $g \in X$. Thus, the quotient map $L^1 \to L^1/X$ is an isomorphism on $L_0^1(\Sigma')$, which is isomorphic to L^1 . The proof is complete.

ACKNOWLEDGMENT

Thanks are due to N. Ghoussoub for useful comments.

MICHEL TALAGRAND

References

- 1. D. J. Aldous, Subspaces of L_1 via random measures, Trans. Amer. Math. Soc. 267 (1981), 445-463.
- 2. J. Bourgain, *Embedding* L_1 in L_1/H^1 , Trans. Amer. Math. Soc. 278 (1983), 689-702.
- J. Bourgain and H. P. Rosenthal, Martingales valued in certain subspaces of L₁, Israel J. Math 37 (1980), 54-75.
- J. Bourgain, H. P. Rosenthal and G. Schechtman, An ordinal L^p index for Banach spaces with application to complemented subspaces of L^p, Ann. of Math. (2) 114 (1981), 193-228.
- 5. J. Bourgain and H. P. Rosenthal, Applications of the theory of semi-embeddings to Banach space theory, J. Funct. Anal. 52 (1983), 149-188.
- P. Enflo and T. W. Starbird, Subspaces of L¹ containing L¹, Studia Math 65 (1979), 203-225.
- 7. H. Fakhoury, Représentation d'operateurs à valeurs dans $L^{1}(X, \Sigma, P)$, Math. Ann. 240 (1979), 203-212.
- G. Godefroy, Sous espaces bien disposés de L¹, Trans. Amer. Math. Soc. 286 (1984), 227-249.
- 9. N. Ghoussoub and S. Szarek, A note on sign preserving operators on L^1 , Publ. Math. Univ. Paris VII (1982), 133–138.
- 10. N. Ghoussoub and H. P. Rosenthal, Martingales, G_{δ} -embeddings and quotients of L_1 , Math. Ann. 264 (1983), 321-332.
- 11. M. I. Kadec and A. Pelczynski, Bases, lacunary sequences and complemented subspaces of L_p , Studia Math. 21 (1962), 161-176.
- 12. N. J. Kalton, The endomorphisms of L_p , 0 , Indiana Univ. Math. J. 27 (1978), 353-381.
- S. V. Kisliakov, On spaces with "small" annihilators, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Stekov. (LOMI) 65 (1976), 192-195. (Russian)
- J. Lindenstrauss and A. Pelczynski, Contributions to the theory of classical Banach spaces, J. Funct. Anal. 8 (1971), 225-249.
- H. P. Lotz, N. T. Peck, and H. Porta, Semi-embedding of Banach spaces, Proc. Edinburg Math. Soc. 22 (1979), 233-240.
- 16. J. Neveu, Discrete parameter martingales, North-Holland, Amsterdam, 1975.
- 17. A. Pelczynski, Banach spaces of analytic functions and absolutely summing operators, CBMS Regional Conf. Ser. in Math., no. 30, Amer. Math. Soc., Providence, R. I., 1978.
- G. Pisier, Une nouvelle classe d'espaces vérifiant le théorème de Grothendieck, Ann. Inst. Fourier (Grenoble) 28 (1978), 69-80.
- 19. J. W. Roberts, Pathological compact convex sets in the space $L^{p}([0, 1])$, $0 \le p < 1$, The Altgeld Book 1975/76, Univ. of Illinois, lecture X.
- 20. H. P. Rosenthal, On subspaces of L^{p} , Ann. of Math. (2) 97 (1973), 344-373.
- 21. ____, Embedding of L^1 in L^1 , Contemp. Math., Vol. 26, Amer. Math. Soc., Providence, R. I., 1984, pp. 335-349.
- Sign embedding of L¹, Banach Spaces, Harmonic Analysis, and Probability Theory, Lecture Notes in Math., vol. 995, Springer-Verlag, Berlin and New York, 1983, pp. 155–165.

ABSTRACT. We construct a subspace X of L^1 such that X is an l^1 -sum of spaces isomorphic to l^1 but such that L^1/X does not contain a copy of L^1 . We also construct two Banach spaces E_1 , E_2 that do not contain a copy of L^1 but such that $E_1 \times E_2$ contains a copy of L^1 . Moreover, the projections

of L^1 on each factor are one-to-one, and the images of the unit ball of L^1 are closed. These examples settle questions of J. Lindenstrauss, P. Pelczynski, J. Bourgain, and H. P. Rosenthal.

Equipe d'Analyse-Tour 46, U.A. au C.N.R.S. n $^\circ\,$ 754, Université Paris VI, 4 Place Jussieu, 75230 Paris Cedex 05

Department of Mathematics, Ohio State University, 231 W. 18th Avenue, Columbus, Ohio 43210