

ISOSPECTRAL CONFORMAL METRICS ON 3-MANIFOLDS

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1. INTRODUCTION

Let M be a compact 3-manifold without boundary. A metric g_0 on M determines a class of conformally equivalent metrics of the form $\{g = u^4 g_0\}$. Our main result is a compactness criterion for metrics in a given conformal class.

Theorem. *Let $g_j = u_j^4 g_0$ be a sequence of conformal metrics satisfying the following conditions.*

- (i) $\text{Vol}(M, g_j) = \alpha_0$ for some positive constant α_0 .
- (ii) $\int R^2(g_j) + |\rho(g_j)|^2 dV_j \leq \alpha_2$ for some positive constant α_2 where $R(g_j)$ is the scalar curvature of g_j and ρ is the Ricci tensor of g_j and $dV_j = u_j^6 dV_0$.
- (iii) $\lambda_1(g_j)$, the lowest eigenvalue of the Laplacian of the metric g_j , has a positive lower bound: $\lambda_1(g_j) \geq \Lambda > 0$; i.e., for each ϕ defined on M , we have

$$\left(\int_M \phi^2 dV_j \right) \leq \left(\int_M \phi dV_j \right)^2 / \left(\int_M dV_j \right) + \frac{1}{\Lambda} \int_M |\nabla_j \phi|^2 dV_0.$$

Then there exist constants c_1, c_2 so that

- (a) $c_1 \leq u(x) \leq 1/c_1$,
- (b) $\|u_j\|_{2,2} \leq c_2$,

except in the case where (M, g_0) is the standard 3-sphere. Then we need to modify the conformal factor u_j by a suitably chosen conformal transformation T_j of S^3 : $\tilde{u}_j^4 g_0 = T_j^*(u_j^4 g_0)$ (the metrics u_j^4 and $\tilde{u}_j^4 g_0$ thus defined are isometric). Then (a) and (b) hold for \tilde{u}_j instead of u_j .

Although the result may be of independent interest, its original motivation is the application to isospectral conformal metrics. Recall that the heat invariants a_k defined by the asymptotics of the heat kernel ([MP, MS, and M])

$$\text{Tr}(e^{-\Delta t}) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \sim \sum_{k=0}^{\infty} a_k t^{k-n/2}$$

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are spectral invariants ($n = \text{dimension of manifold}$). Thus, for an isospectral set of metrics g_j , the heat invariants $a_k(g_j)$ are identical. In particular, the low order heat invariants give the following information on the metric when the dimension of the manifold is 3:

$$\begin{aligned} a_0 &= \text{volume} \quad \left(\text{volume} = \int dV \right) \\ a_1 &= \text{constant} \int R dV \\ a_2 &= A_2 \int R^2 dV + B_2 \int |\rho|^2 dV, \quad A_2, B_2 > 0. \end{aligned}$$

Hence, our theorem yields second-order information when the $g_j = u_j^4 g_0$ are conformally related, and, in fact, we have

Corollary. *An isospectral set of conformal metrics on a compact 3-manifold is compact in the \mathcal{E}^∞ -topology.*

For the Dirichlet problem on domains in the plane, [M] has shown that the curvature function of an isospectral set of domains forms a compact set in the \mathcal{E}^∞ -topology. For the case of compact surfaces, [OPS1, OPS2, also for related development OPS3] have shown that an isospectral set of metrics is compact in the \mathcal{E}^∞ -topology. In that case, the metrics are not restricted to a fix conformal class, and a global spectral invariant, i.e., the determinant of the Laplacian (not expressible as an integral of local invariants of the metric), was used to pin down the set of conformal classes to a compact region in the moduli space. In the appendix, we will give an alternative argument, eliminating the need for the use of the determinant when the metrics are restricted to lie in a fixed conformal class.

Previously, we ([CY, CY2]) gave a proof of the main theorem when (M, g_0) is the standard 3-sphere, and also [BPY] gave a proof for when R_0 is negative. In this paper, we will give a unified argument.

The underlying analysis of this problem is the optimal Sobolev inequality:

$$Q(M) \left(\int_M u^6 dV_0 \right)^{1/3} \leq 8 \int_M |\nabla u|^2 dV_0 + \int_M R_0 u^2 dV_0.$$

The optimal constant $Q(M)$ is an invariant of the conformal class of M . For a conformal metric $g = u^4 g_0$, its scalar curvature R is given by the equation

$$8\Delta u + Ru^5 = R_0 u \quad \text{on } M.$$

Thus, the Sobolev quotient

$$Q[u] = \frac{\int (8|\nabla u|^2 + R_0 u^2) dV_0}{(\int u^6 dV_0)^{1/3}}$$

is exactly given by $\int Ru^6 dV_0$ if the volume is held to be 1 (i.e., $\int u^6 dV_0 = 1$). The celebrated recent solution of Yamabe's problem [A, S] asserts that (a) $Q(M) < Q(S^3)$ unless M is conformally S^3 and (b) a minimizing sequence

for $Q[u]$ is compact if $Q(M) < Q(S^3)$. Thus, in our compactness assertion, we have substituted an L^2 bound for the curvature in place of the condition $Q[u_j] < Q(S^3)$ and substituted the condition $\lambda_1(g_j) \geq \Lambda > 0$ in place of the minimizing property for $Q[u]$.

We give in the remainder of the introduction an outline of the argument for the theorem. We first locate an additional measure theoretic condition.

There exist positive constants γ_0, l_0 so that

$$(*) \quad \int_{\{u_j(x) \geq \gamma_0\}} dV_0 \geq l_0 \int_M dV_0$$

for all u_j in the sequence and prove in Proposition A (§2) that if in addition to the conditions (i), (ii), and (iii), (*) holds for the sequence of conformal factors u_j , then there is a uniform bound for the integrals $\int u_j^{6+\varepsilon} dV_0$, where ε is a constant depending only on M, α_0, α_2 , and Λ . We then point out that the argument in our previous paper [CY] goes through without further difficulty.

The remainder of the paper then is devoted to verifying the condition (*) under the assumptions (i), (ii), and (iii). The idea is to show that if (*) fails, then the measures $u_j^6 dV_0$ would have to (after passing to a subsequence) concentrate at a single point, say x_0 , and that off the point of concentration, u_j converges to zero in a well-controlled way (in §3). More precisely, if we normalize with $v_j = c_j u_j$ for a suitable sequence of numbers $c_j \rightarrow \infty$, v_j converges uniformly on compact subsets of $M \setminus \{x_0\}$ to a nontrivial solution v_∞ of the conformal Laplacian

$$\Delta v_\infty - R_0 v_\infty = 0 \quad \text{on } M.$$

This is done using Harnack inequality in §4, where we need to compare the L^p integrals of u_j on adjacent annuli regions centered at x_0 . The key idea is that off the concentration point, Δv_j is small in $L^{6/5}$. The inhomogeneity under scaling of the integrand $R^2(g) + |\rho(g)|^2 dV(g)$ shows that the limit function v_∞ gives rise to a conformal metric on $M \setminus \{x_0\}$, which is Ricci flat, hence flat. There are two possibilities, according to whether v_∞ has a removable singularity at x_0 . If x_0 is a removable singularity, v_∞ must be constant, hence (M, g_0) is a flat manifold. If x_0 is an essential singularity, then v_∞ has a singularity of the order $1/\text{dist}(\cdot, x_0)$; hence, $v_\infty^4 g_0$ is a complete flat metric on $M \setminus \{x_0\}$. This implies that $M \setminus \{x_0\}$ is, in fact, conformally \mathbf{R}^3 , so that M is conformally S^3 . But in the latter case, our previous [CY] argument applies. The only remaining case is when (M, g_0) is a flat manifold. We handle the last case in §6 by extending the function $u_j|_{B(x_0)}$ on a fixed geodesic (Euclidean) ball to a function \tilde{u}_j on S^3 , thinking of the ball B as the northern hemisphere, and we show that the resulting sequence of positive functions \tilde{u}_j ,

after a suitably chosen conformal transformation, violates the optimal Sobolev inequality for S^3 .

2. PRELIMINARIES. NASH-MOSER ITERATION SCHEME

In this section, we will prove the following proposition.

Proposition A. *On (M, g_0) , if $g = u^4 g_0$ is a metric satisfying*

- (1) $a_0(g) = \alpha_0$,
- (2) $a_1(g) \leq \alpha_1$,
- (3) $\int_M R^2 u^6 dV_0 \leq \alpha_2$,
- (4) $0 < \Lambda \leq \lambda_1(g)$, where $\lambda_1(g)$ is the first positive eigenvalue of the Laplacian operator,

where $\alpha_0, \alpha_1, \alpha_2$, and Λ are positive constants and assuming in addition condition (*):

there exist some positive constants γ_0, l_0 so that

$$(*) \quad \int_{\{x: u(x) \geq \gamma_0\}} dV_0 \geq l_0 \int_M dV_0,$$

then there exist some $\varepsilon_0 > 0$ and a constant C_0 depending only on the data $\alpha_0, \alpha_1, \alpha_2, \Lambda, \gamma_0, l_0$ with

$$(5) \quad \int_M u^{6+\varepsilon_0} dV_0 \leq C_0.$$

Our proof of Proposition A below is a modification of the arguments used in our earlier paper [CY] (see also [BPY]). For completeness we will outline the arguments here. The main procedure used in the proof is an application of the well-known Nash-Moser iteration scheme. Since this same procedure will be applied repeatedly throughout this paper, we will now state it separately.

Recall that the equation relating the metric $g = u^4 g_0$ to its scalar curvature function $R = R(g)$, $R_0 = R(g_0)$ is

$$(6) \quad 8\Delta u + R u^5 = R_0 u \quad \text{for a 3-dimensional manifold } M.$$

We now fix a real number β and a suitably chosen positive cut-off function η (β and η chosen differently on each different occasion) and multiply the equation by $\eta^2 u^\beta$ and integrate to get

$$(7a) \quad \begin{aligned} 8\beta \int \eta^2 u^{\beta-1} |\nabla u|^2 dV_0 + 16 \int \nabla u \cdot \nabla \eta \eta u^\beta dV_0 + R_0 \int \eta^2 u^{\beta+1} dV_0 \\ = \int R u^4 \eta^2 u^{\beta+1} dV_0. \end{aligned}$$

Estimating the cross term $\int \nabla u \cdot \nabla \eta \eta u^\beta dV_0$ as (we will henceforth denote $\int dV_0$ as \int)

$$2 \int \nabla u \cdot \nabla \eta \eta u^\beta \leq \frac{1}{t} \int |\nabla \eta|^2 u^{\beta+1} + t \int \eta^2 |\nabla u|^2 u^{\beta-1}$$

with t small ($0 < t < |\beta|$ if $\beta \neq 0$), we obtain

$$(7b) \quad 8(|\beta| - t) \int \eta^2 u^{\beta-1} |\nabla u|^2 \leq \frac{8}{t} \int |\nabla \eta|^2 u^{\beta+1} + |R_0| \int u^{\beta+1} \eta^2 + \int |R| u^{5+\beta} \eta^2.$$

To start the iterating process, we apply the following Sobolov inequality for a 3-dimensional manifold. For all $v \in W_2^1(M)$ we have

$$(8) \quad Q \left(\int v^6 dV_0 \right)^{1/3} \leq 8 \int |\nabla v|^2 dV_0 + R_0 \int v^2 dV_0,$$

where

$$Q = Q(M, g_0) = \inf_{v \neq 0} \frac{8 \int |\nabla v|^2 dV_0 + R_0 \int v^2 dV_0}{\left(\int v^6 dV_0 \right)^{1/3}}.$$

Applying (8) to (7a) and (7b) with $v = \eta w$, $w = u^{(\beta+1)/2}$ we get

(9a) when $\eta \equiv 1$, $\beta \neq -1$,

$$\begin{aligned} 8 \frac{4\beta}{(1+\beta)^2} \int |\nabla w|^2 + R_0 \int w^2 &= \int R u^4 w^2, \\ Q \frac{4\beta}{(\beta+1)^2} \left(\int w^6 dV_0 \right)^{1/3} &\leq \int R u^4 w^2 + |R_0| \left(\frac{4\beta}{(\beta+1)^2} + 1 \right) \int w^2; \end{aligned}$$

(9b) when $t = |\beta|/2$, $\beta \neq 0$, $\beta \neq -1$ together with the estimate

$$\begin{aligned} \int |R| u^4 w^2 \eta^2 &\leq \left(\int R^2 u^6 \right)^{1/2} \left(\int_{\text{supp } \eta} u^6 \right)^{1/6} \left(\int w^6 \eta^6 \right)^{1/3} \\ &\leq \alpha_2^{1/2} \left(\int_{\text{supp } \eta} u^6 \right)^{1/6} \left(\int w^6 \eta^6 \right)^{1/3} \end{aligned}$$

we get

$$\begin{aligned} Q \left(\int (w\eta)^6 \right)^{1/3} &\leq C \left[\left(\frac{|1+\beta|^2}{|\beta|^2} + 1 \right) \int |\nabla \eta|^2 w^2 \right. \\ &\quad + \left(\frac{|1+\beta|^2}{|\beta|} + 1 \right) |R_0| \int \eta^2 w^2 \\ &\quad \left. + \frac{|1+\beta|^2}{|\beta|} \alpha_2^{1/2} \left(\int_{\text{supp } \eta} u^6 \right)^{1/6} \left(\int w^6 \eta^6 \right)^{1/3} \right], \end{aligned}$$

where C is a universal constant.

In §4, we will repeatedly apply (9b) to a sequence of β and η to obtain a Harnack inequality for functions $\{u_j\}$ that fails to satisfy condition (*).

We will now apply (9a) to finish the proof of Proposition A.

Proof of Proposition A. Starting with the inequality (9a), we will now apply our assumptions (1), (2), (3), (4) to estimate the term $I = \int R u^4 w^2$ ($w = u^{(1+\beta)/2}$ with $\beta > 1$ chosen later). Taking a suitably large number b (again to be chosen later) on the region $|R| \geq b$ we have

$$b^2 \int_{|R| \geq b} u^2 dV_0 \leq \int_{|R| \geq b} R^2 u^6 dV_0 \leq \alpha_2.$$

Thus,

$$(10) \quad \begin{aligned} \int_{|R| \geq b} Ru^4 w^2 &\leq \left(\int R^2 u^6 \right)^{1/2} \left(\int_{|R| \geq b} u^6 \right)^{1/6} \left(\int w^6 \right)^{1/3} \\ &\leq \alpha_2^{1/2} \left(\frac{\alpha_2}{b^2} \right)^{1/6} \left(\int w^6 \right)^{1/3}. \end{aligned}$$

For the remaining part of §1, we will apply condition (*) in the statement of Proposition A (which replaces the conformal pulling argument on S^3 as in our earlier paper [CY]) as follows.

For $dV = v^6 dV_0$, we have from the Raleigh-Ritz characterization for λ_1 ,

$$(11) \quad \int_M \psi^2 dV \leq \left(\int_M \psi dV \right)^2 / \left(\int dV \right) + \frac{1}{\lambda_1} \int_M |\nabla_u \psi|^2 dV,$$

where $|\nabla_u \psi|^2 dV = |\nabla \psi|^2 u^2 dV_0$. We will denote $E_\gamma = \{x \in M, u(x) \geq \gamma\}$ and $|E_\gamma| = \int_{E_\gamma} dV_0$. By assumption (*), there exist some $\gamma_0 > 0$, $l_0 > 0$ so that $|E_{\gamma_0}| \geq l_0 |\int_M dV_0|$. Applying (11) and (4) to $\psi = u^\varepsilon$ with $\beta = 1 + 2\varepsilon$ and ε small, we have

$$(12) \quad \int u^{6+2\varepsilon} dV_0 \leq \left(\int u^{6+\varepsilon} dV_0 \right)^2 / \left(\int u^6 dV_0 \right) + \frac{1}{\Lambda} \int |\nabla u^\varepsilon|^2 u^2 dV_0.$$

For simplicity, we will now normalize u and assume $\alpha_0 = \int u^6 dV_0 = 1$. We may then estimate the term $\int u^{6+\varepsilon} dV_0$ as

$$\begin{aligned} \int u^{6+\varepsilon} dV_0 &= \int_{E_{\gamma_0}} u^{6+\varepsilon} dV_0 + \int_{E_{\gamma_0}^c} u^{6+\varepsilon} dV_0 \\ &= \int_{E_{\gamma_0}} (u^6 - \gamma_0^6) u^\varepsilon dV_0 + \int_{E_{\gamma_0}} \gamma_0^6 u^\varepsilon dV_0 + \int_{E_{\gamma_0}^c} u^{6+\varepsilon} dV_0 \\ &\leq \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) u^{2\varepsilon} dV_0 \right)^{1/2} \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) dV_0 \right)^{1/2} + C(\gamma_0), \end{aligned}$$

where $C(\gamma_0)$ is a constant depending only on γ_0 and $\int dV_0$. Thus, for each $\eta > 0$ we have

$$(13) \quad \begin{aligned} \left(\int u^{6+\varepsilon} dV_0 \right)^2 &\leq (1 + \eta) \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) u^{2\varepsilon} dV_0 \right) \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) dV_0 \right) \\ &\quad + \left(1 + \frac{1}{\eta} \right) C^2(\gamma_0) \\ &\leq (1 + \eta) (1 - \gamma_0^6 |E_{\gamma_0}|) \left(\int u^{6+2\varepsilon} dV_0 \right) \\ &\quad + \left(1 + \frac{1}{\eta} \right) C^2(\gamma_0) \end{aligned}$$

(we may assume w.l.o.g. that γ_0 is small and $\gamma_0^6 |E_{\gamma_0}| \ll 1$).

Since by our assumption on $|E_{\gamma_0}|$ we have $\gamma_0^6 |E_{\gamma_0}| \geq \gamma_0^6 l_0 > 0$, we may choose η so that $(1 + \eta)(1 - \gamma_0^6 |E_{\gamma_0}|) \leq 1 - \delta$ for some positive δ , $\delta = \delta(\gamma_0, l_0)$ and obtain from (12), (13)

$$(14) \quad \delta \int u^{6+2\varepsilon} dV_0 \leq C(\gamma_0, l_0) + \frac{1}{\Lambda} \int |\nabla u^\varepsilon|^2 u^2 dV_0,$$

where again $C(\gamma_0, l_0)$ is a constant depending only on γ_0, l_0 .

From this point on, we may estimate the term $\int |\nabla u^\varepsilon|^2 u^2 dV_0$ as in [CY]; namely,

$$\int |\nabla u^\varepsilon|^2 u^2 dV_0 = \frac{\varepsilon^2}{(1 + \varepsilon)^2} \int |\nabla u^{1+\varepsilon}|^2 dV_0$$

and notice that for $\beta = 1 + 2\varepsilon$, $w = u^{(1+\beta)/2} = u^{1+\varepsilon}$. Thus, combining (9a) and (14) we have

$$(15) \quad \int u^{6+2\varepsilon} dV_0 \leq \frac{\varepsilon^2}{\delta \Lambda (1 + \varepsilon)^2} \frac{(1 + \varepsilon)^2}{8(1 + 2\varepsilon)} I + L,$$

where

$$I = \int R u^4 w^2 \quad \text{and} \quad L = 0 \left(\frac{R_0}{\delta} \int u^{2+2\varepsilon} dV_0 \right) + \frac{1}{\delta} C(\gamma_0, l_0).$$

Combining (15) with (10), we find

$$(16) \quad \begin{aligned} I &= \int R u^4 w^2 \leq \left(\frac{\alpha_2^2}{b} \right)^{1/3} \left(\int w^6 dV_0 \right)^{1/3} + b \int u^4 w^2 dV_0 \\ &\leq \left(\frac{\alpha_2^2}{b} \right)^{1/3} \left(\int w^6 dV_0 \right)^{1/3} + \frac{b\varepsilon^2}{8\Lambda} I + bL \end{aligned}$$

so that

$$(17) \quad \left(1 - \frac{b\varepsilon^2}{8\Lambda} \right) I \leq \left(\frac{\alpha_2^2}{b} \right)^{1/3} \left(\int w^6 dV_0 \right)^{1/3} + bL.$$

Now choosing b sufficiently large so that $(\alpha_2^2/b)^{1/3} < (1/2)Q$, and then choosing ε sufficiently small and proceeding with the same proof as in [CY], we get

$$\begin{aligned} \frac{3}{4} Q \left(\int w^6 dV_0 \right)^{1/3} &\leq I + |R_0| \int w^2 dV_0 \\ &\leq \frac{2}{3} Q \left(\int w^6 dV_0 \right)^{1/3} + \frac{4}{3} bL + |R_0| \int w^2 dV_0. \end{aligned}$$

Recall $w = u^{1+\varepsilon}$; hence,

$$\begin{aligned} \left(\int u^{6+6\varepsilon} dV_0 \right)^{1/3} &= \left(\int w^6 dV_0 \right)^{1/3} < 16 bL + 12 |R_0| \int u^{2+2\varepsilon} dV_0 \\ &\leq C(b, |R_0|) \left(\int u^6 dV_0 \right)^{(2+2\varepsilon)/6} \left(\int dV_0 \right)^{(4-2\varepsilon)/6} \\ &= C_0 < \infty. \end{aligned}$$

This proves Proposition A with $\varepsilon_0 = 6\varepsilon$.

Remarks. Assume u satisfies conditions (1)–(4) and the conclusion condition (5) in the statement of Proposition A. Then we may apply similar arguments as in Lemmas 3 and 4 in [CY] to the general 3-dimensional manifold M instead of S^3 to obtain constants c_1, c_2, c_3 (depending only on the data $\alpha_0, \alpha_1, \alpha_2, \Lambda$, and C_0) with $0 < c_1 \leq u(x) \leq c_2$ and $\|u\|_{2,2}^2 = \int u^2 + |\nabla u|^2 + |\nabla^2 u|^2 \leq c_3$. We can apply Gilkey's computation [G] for the coefficients a_k of the heat kernel for $g = u^4 g_0$ as in the arguments in [BPY] to conclude that u is bounded in the \mathcal{E}^∞ -topology. We summarize this conclusion as follows.

Corollary 1. Assume $\{g_j = u_j^4 g_0\}$ is an isospectral sequences of metrics on the 3-dimensional manifold (M, g_0) with $\{u_j\}$ satisfying condition (*) in Proposition A. Then $\{u_j\}$ forms a compact family in the \mathcal{E}^∞ -topology.

The only change required is a suitable version of the Faber-Krahn inequality for a general compact manifold in place of the standard 3-sphere. Since we cannot easily locate it in the literature, we provide a simple argument.

Lemma. For a compact manifold M , given $\delta > 0$, there exists $\Lambda > 0$ so that for any domain Ω in M with $\text{meas}(\Omega) \leq \text{Vol}(M) - \delta$, the first Dirichlet eigenvalue $\lambda_1(\Omega)$ is at least Λ .

Proof. Suppose on the contrary that no such Λ exists. Then we can find a sequence of functions $0 \leq \varphi_j \in W_2^1(M)$ with $\text{meas}\{\varphi_j = 0\} \geq \delta$ satisfying $\int |\nabla \varphi_j|^2 \leq \frac{1}{j}$ but $\int \varphi_j^2 dV = 1$. By taking the weak limit in $W_2^1(M)$, we find a function $0 \leq \varphi$ in $W_2^1(M)$ with $\int |\nabla \varphi|^2 = 0$ but $\int \varphi^2 = 1$. This implies φ is a positive constant, but weak convergence in W^1 implies strong convergence in L^2 , contradicting the assumption $\text{meas}\{\varphi_j = 0\} \geq \delta$.

Remark. Condition (*) is satisfied for sequences of functions satisfying conditions (1)–(4) in Proposition A when $R(g_0) = R_0 < 0$ and when (M, g_0) is conformally equivalent to (S^3, g_0) with g_0 the standard metric. (In the latter case, condition (*) is satisfied for functions isometric to the original sequences.) To see this, when $R_0 < 0$, $\int R^6 u^2 \leq \alpha_2$ implies $\int 1/u^2$ is finite. This coupled with the fact that $\int u^6 = \alpha_0$ implies condition (*). When $(M, g_0) = (S^3, g_0)$, for a given u satisfying conditions (1)–(4), we may find v pointwise isometric to u with $\int v^6 x_j = 0$ for $j = 1, 2, 3, 4$ (x_j being the ambient coordinates of S^3) with v satisfying the same conditions (1)–(4). For this function v we have

$$\left(\int v^6 \right) \left(\int v^6 x_j^2 \right) \leq \frac{1}{\lambda_1} \int |\nabla x_j|^2 v^2 \quad \text{for } j = 1, 2, 3, 4.$$

Summing over j we have $\int v^2 \geq C(\lambda, \alpha_0)$. Thus, v satisfies condition (*).

3. CONCENTRATION PHENOMENON

For the rest of the paper, we will study the isospectral sequence $\{g_j = u_j^4 g_0\}$ for which condition (*) in Proposition A fails (i.e., for each $\gamma_0 > 0$, the measure $|E_{\gamma_0}(u_j)| = \int_{\{u_j(x) \geq \gamma_0\}} dV_0(x)$ tends to zero as $j \rightarrow \infty$). We will show this can happen only when some subsequence of u_j has its mass “concentrate” at some point $x_0 \in M$.

First we state an easy consequence (and actually an equivalent statement) for the failure of condition (*).

Lemma 1. *Suppose condition (*) fails for some sequence of positive functions $\{u_j\}$ with $\int u_j^6 dV_0 = \alpha_0$. Then $u_j \rightarrow 0$ in L^p for all $p < 6$.*

Proof. Suppose not, i.e., there exist some $p < 6$ and $\delta_0 > 0$ with $\int u_j^p dV_0 \geq \delta_0 > 0$ for some subsequence of u_j . Then for each $\delta > 0$, we have

$$\begin{aligned} \int u_j^p dV_0 &= \int_{E_\gamma} u_j^p dV_0 + \int_{E_\gamma^c} u_j^p dV_0 \\ &\leq \left(\int_{E_\gamma} u_j^6 dV_0 \right)^{p/6} |E_\gamma|^{(6-p)/6} + \gamma^p |E_\gamma^c|, \end{aligned}$$

where $E_\gamma = E_\gamma(u_j)$. So for γ sufficiently small, say $\gamma^p |\int dV_0| < \frac{1}{2}\delta$, we have

$$\frac{1}{2}\delta_0 \leq \gamma_0^{p/6} |E_\gamma|^{(6-p)/6}.$$

Thus $|E_\gamma(u_j)| \geq \alpha_0(\delta_0/2\alpha_0)^{6/(6-p)} = l_0$ for each u_j , which contradicts our assumption that condition (*) fails for the sequence $\{u_j\}$.

Proposition B. *Suppose $\{u_j\}$ is a sequence of positive functions defined on (M, g_0) satisfying conditions (1)–(4) in Proposition A, while failing condition (*). Then there exists some subsequence of $\{u_j\}$, still denoted by $\{u_j\}$, whose mass concentrates at some point $x_0 \in M$; i.e., given $\varepsilon > 0$ and $r > 0$ sufficiently small, there exists some j_0 with $\int_{B(x_0, r)} u_j^6 > \alpha_0 - \varepsilon$ for all $j \geq j_0$, where $B(x_0, r)$ denotes the geodesic ball of radius r centered at x_0 .*

Proof. We will establish the proof in two steps.

Step I. The set of points where the mass of some subsequence of $\{u_j\}$ accumulates is nonempty; i.e., the set

$$\left\{ x \in M \mid \lim_{r \rightarrow 0} \overline{\lim}_{j \rightarrow \infty} \int_{B(x, r)} u_j^6 \neq 0 \right\}$$

is nonempty.

Step II. We then apply condition (4) to conclude that Step I consists of exactly one point x_0 .

It will then become clear that the proof also indicates that at the unique point x_0 , some subsequence of $\{u_j\}$ must satisfy the description in the statement of Proposition B.

Proof of Step I. Suppose the contrary; i.e., we assume that for each $x \in M$,

$$m_x = \lim_{r \rightarrow 0} \overline{\lim}_j \int_{B(x, r)} u_j^6 = 0.$$

For a fixed point $x \in M$ and $\varepsilon > 0$, we have $\int_{B(x, r)} u_j^6 < \varepsilon$ for some (sub)-sequence $\{u_j\}$ as $j \rightarrow \infty$ and r sufficiently small. Fix r small and choose η , a cut-off function, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B(x, \frac{r}{2})$, and $\eta \equiv 0$ off $B(x, r)$. Applying inequality (9b) to $\beta = 1$, $w = u_j$, and this choice of η , we obtain

$$\begin{aligned} Q \left(\int (\eta u_j)^6 dV_0 \right)^{1/3} &\leq C \left[\alpha_2^{1/2} \left(\int_{B(x, r)} u_j^6 \right)^{1/6} \left(\int (\eta u_j)^6 \right)^{1/3} + \frac{1}{r^2} \int_{B(x, r)} u_j^2 dV_0 \right] \\ &\leq C \left[\alpha_2^{1/2} \varepsilon^{1/6} \left(\int (\eta u_j)^6 \right)^{1/3} + \frac{1}{r^2} \int_{B(x, r)} u_j^2 dV_0 \right]. \end{aligned}$$

Thus, for ε sufficiently small and j large we have

$$(18) \quad \frac{Q}{2} \left(\int_{B(x, r/2)} u_j^6 dV_0 \right)^{1/3} \leq \frac{Q}{2} \left(\int (\eta u_j)^6 dV_0 \right)^{1/3} \leq \frac{c}{r^2} \int_{B(x_0, r)} u_j^2 dV_0.$$

Hence, if $m_x = 0$ for all x , we can cover the manifold M by finitely many balls $B(x_k, r_k/2)$ for $k = 1, 2, \dots, N$ so that (18) holds for each such ball. Thus,

$$\begin{aligned} \alpha_0 = \int u_j^6 dV_0 &\leq \sum_{k=1}^N \int_{B(x_k, r_k/2)} u_j^6 dV_0 \\ &\leq \left(\frac{2}{Q} \right)^3 \sum_{k=1}^N \left(\frac{1}{r_k^2} \int_{B(x_k, r_k)} u_j^2 dV_0 \right)^3 \\ &\rightarrow 0 \quad \text{as } j \rightarrow \infty \text{ (by Lemma 1).} \end{aligned}$$

This is a contradiction. Hence, $m_x \neq 0$ for some $x \in M$.

Proof of Step II. Assume again to the contrary that there exist at least two points where the mass of u_j^6 concentrates, say x_1 and x_2 .

By picking a subsequence of $\{u_j\}$, we may arrange that

$$\lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \int_{B(x_k, r)} u_j^6 = m_k, \quad k = 1, 2.$$

Let $\rho = \text{dist}(x_1, x_2)$. Writing $B(x_1, 1)$ as a disjoint union

$$\bigcup_n (B(x_1, 2^{-n}) \setminus (x_1, 2^{-n-1})),$$

we set

$$\mu_1 = \lim_{j \rightarrow \infty} \sup \int_{B(x_1, 1) \setminus B(x_1, 1/2)} u_j^6 dV_0.$$

We then restrict to a subsequence still denoted as u_j so that in the equation above, the $\lim \sup$ is, in fact, limit. We then inductively set

$$\mu_l = \lim_{j \rightarrow \infty} \int_{B(x_1, 2^{-l+1}) \setminus B(x_1, 2^{-l})} u_j^6 dV_0,$$

each time restricting to a subsequence. Since we must have $\sum_{j=1}^l \mu_j \leq \alpha_0$, there is an l_0 so that for $l \geq l_0$ we have

$$\mu_l \leq \frac{\alpha_0^{1/2}}{4} \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^{-1/2}.$$

We do this similarly for the rings around x_2 , choosing a common l_0 . Now pick ρ_0 small so that $\rho_0 \leq \min\{\rho, 2^{-l_0}\}$ and that $|\int_{B(x_k, r)} u_j^6 dv - m_k| \leq \varepsilon$ for all $r \leq 2\rho_0$ and j sufficiently large. We set φ to be a \mathcal{C}^∞ -function on M with

$$\varphi = \begin{cases} 1/m_1 & \text{on } B(x_1, \rho_0), \\ -1/m_2 & \text{on } B(x_2, \rho_0), \\ 0 & (B(x_1, 2\rho_0) \cup B(x_2, 2\rho_0))^c \end{cases}$$

and we extend φ to be a version of a linear function in the appropriate distance function in the rest of M . Then

$$\int u_j^6 \varphi^2 \geq \frac{1}{m_1^2} (m_1 - \varepsilon) + \frac{1}{m_2^2} (m_2 - \varepsilon) \geq \frac{1}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right)$$

while

$$\begin{aligned} \left| \int \varphi u_j^6 \right| &\leq \left| \int_{B(x_1, \rho_0) \cup B(x_2, \rho_0)} \varphi u_j^6 \right| + \left| \int_{\bigcup_{k=1}^2 B(x_k, 2\rho_0) \setminus B(x_k, \rho_0)} |\varphi| u_j^6 \right| \\ &\leq \left(\frac{m_1 + \varepsilon}{m_1} - \frac{1}{m_2} (m_2 - \varepsilon) \right) + \frac{\alpha_0^{1/2}}{4} \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^{1/2} \\ &\leq 3\varepsilon/m_2 + \frac{\alpha_0^{1/2}}{4} \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^{1/2} \end{aligned}$$

and

$$\int |\nabla \varphi|^2 u_j^2 \leq \frac{1}{m_2^2} \frac{\text{const}}{\rho_0^2} \int_M u_j^2.$$

Thus, from the Raleigh-Ritz characterization of λ_1 for the metric $u_j^6 dV_0$, we have

$$\begin{aligned} \lambda_1 &\leq \int |\nabla \varphi|^2 u_j^2 / \left[\alpha_0 \int u_j^6 \varphi^2 - \left(\int u_j^6 \varphi \right)^2 \right] \\ &\leq \text{constant} \left(\frac{1}{m_2^2} \frac{1}{\rho_0^2} \int_M u_j^2 \right) \left(\frac{\alpha_0}{4} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \right)^{-1} \\ &\rightarrow 0 \quad \text{as } j \rightarrow \infty \quad (\text{by Lemma 1}). \end{aligned}$$

This is a contradiction.

To see that $m_{x_0} = \alpha_0$ in Step II, we need to observe only that the same proof as given in Step I above also proves that for any compact set K , where $u_j^6 dV_0$ has some uniform positive mass, the set $\{x \in K \mid \lim_{r \rightarrow 0} \overline{\lim} \int u_j^6 dV_0 \neq 0\}$ must be nonempty. The conclusion of Proposition B follows easily from this fact (i.e., $m_{x_0} = \alpha_0$) together with a standard diagonal subsequence argument.

From now on, we will work on the (sub)sequence, which we denote again by $\{u_j\}$, that has the concentration property in Proposition B. In the following section, we will establish a Harnack-type inequality for the concentration sequence $\{u_j\}$ that holds uniformly outside the concentration point x_0 .

4. HARNACK INEQUALITY OFF THE CONCENTRATION POINT

We will first establish the fact that positive functions u with $\int u^6 = \alpha_0$ and $\int R^2 u^6 \leq \alpha_2$ have uniformly bounded B.M.O. norm $\|\log u\|_*$ depending only on α_2 . (B.M.O. denotes the class of functions of bounded mean oscillation, which was originally introduced by John and Nirenberg [JN]. Properties of B.M.O. had been applied earlier by Moser in connection with the Harnack inequality.) We refer readers to [JN] for the definition and basic property of functions in B.M.O. Also for our purpose here, a good reference is Theorem 7.2.1 in Gilbarg and Trudinger [GT].

Lemma 2. *Suppose u is a positive function satisfying conditions (1) and (3) in Proposition A. Then for each point $x \in M$, there exists some neighborhood $\Omega(x)$ such that for every point $y \in \Omega(x)$ and geodesic ball $B(y, \rho) \subset \Omega(x)$ we have*

$$(19) \quad \int_{B(y, \rho)} |\nabla \log u| dV_0 \leq k\rho^2$$

for some constant k depending only on α_0, α_2 . As a consequence, we have the existence of some constant $p_0 > 0$ (p_0 depends only on α_0, α_2 and is of order $\frac{1}{k}$) such that

$$(20) \quad \int_{B(y, \rho)} u^{p_0} dV_0 \int_{B(y, \rho)} u^{-p_0} dV_0 \leq C\rho^6$$

for some universal constant C .

Proof. We prove (19) by essentially the same argument as in the proof of Theorems 8.17 and 8.18 in [GT], with the minor change of replacing the L^q -conditions in [GT] with conditions (2) and (3) in our case. For the purpose of making the proof of Proposition C below clear, we will outline the proof here. Fix a point $x \in M$. Choose some small geodesic ball $\Omega(x)$ centered at x where the normal coordinate system holds. Thus, in $\Omega(x)$, we may assume w.l.o.g. that the distance function in the metric behaves like the Euclidean distance (with errors of higher order).

For $y \in \Omega(x)$, fix a ball $B(y, \rho) = B_\rho$ with $B_{2\rho}$ contained in $\Omega(x)$. Choose η to be a cut-off function $\eta = 1$ on B_ρ and $\eta = 0$ off $B_{2\rho}$. Then $|\nabla \eta| \leq 2/\rho$ on $B_{2\rho}$. Choosing $\beta = -1$ and $t = \frac{1}{2}$ in inequality (7b) in §2, we get

$$(21) \quad 4 \int_{B_\rho} |\nabla u|^2 u^{-2} \leq 16 \int_{B_{2\rho}} \frac{1}{\rho^2} + |R_0| \int_{B_\rho} + \int_{B_{2\rho}} |R| u^4$$

since

$$\begin{aligned} \int_{B_{2\rho}} |R|u^4 &\leq \left(\int R^2 u^6 \right)^{1/2} \left(\int u^6 \right)^{1/6} |B_{2\rho}|^{1/3} \\ &\lesssim \alpha_2^{1/2} \alpha_0^{1/6} \rho. \end{aligned}$$

Thus, for some $c = c(\alpha_0, \alpha_2)$, we have from (21)

$$\int_{B_\rho} \left| \frac{\nabla u}{u} \right|^2 \lesssim c\rho$$

and

$$\int_{B_\rho} |\nabla \log u| dV = \int_{B_\rho} \left| \frac{\nabla u}{u} \right| \leq \left(\int_{B_\rho} \left| \frac{\nabla u}{u} \right|^2 \right)^{1/2} |B_\rho|^{1/2} \lesssim c\rho^2,$$

which establishes (19).

That (20) follows from (19) is an immediate consequence of the celebrated John-Nirenberg [JN] inequality for B.M.O. functions. To see this, we denote $w = \log u$. Then w is a function in B.M.O. in $\Omega(x)$. For fixed $y \in \Omega(x)$, $B_\rho = B(y, \rho) \subset \Omega(x)$. Denote $\bar{w}_\rho = \int_{B_\rho} w dV_0 / \int_{B_\rho} dV_0$. We then have

$$\int_{B_\rho} e^{(c/k)|w - \bar{w}_\rho|} dV_0 \leq C\rho^3$$

for some suitable universal constants c, C . Let $p_0 = c/k$. We have

$$(20') \quad \int_{B_\rho} e^{p_0 w} dV_0 \int_{B_\rho} e^{-p_0 w} dx \leq C\rho^6 e^{p_0 w_\rho} e^{-p_0 w_\rho} = C,$$

which is equivalent to (20).

We will now apply the Nash-Moser iteration scheme to derive a Harnack estimate for the sequence $\{u_j\}$ outside its concentration point.

Proposition C. *Suppose $\{u_j\}$ is a sequence of functions as in Proposition B with x_0 its concentration point. Then for each fixed r (sufficiently small), there exists some integer $j(r)$ so that*

$$(22) \quad \int_{B(x_0, r) - B(x_0, r/2)} u_j^2 dV_0 \leq C \int_{B(x_0, 2r) - B(x_0, r)} u_j^2 dV_0$$

for all $j \geq j(r)$ and for some universal constant C . ($C = C(p_0)$, where p_0 is the constant as in Lemma 2.)

Remark. The same proof given here also indicates that for any (high) power of p , inequality (22) holds for u_j^p as $j \geq j(r, p) \rightarrow \infty$.

Proof. The argument here is similar to the proof of Theorems 8.17 and 8.18 in [GT]. Fix ρ small with $B(x_0, 4\rho)$ contained in a normal coordinate patch at x_0 . Denote $B_\rho = B(x_0, \rho)$. Choose η to be a \mathcal{C}^∞ cut-off function $\eta \equiv 1$ on $B_\rho \setminus B_\sigma$ supported in $B_{\rho+\delta, \sigma-\delta}$ (for $\delta < \frac{\rho}{2}$). Then $|\nabla \eta| \lesssim \frac{1}{\delta}$ on its support.

Applying inequality (9b) in §2 to $\beta \neq -1$, $\beta \neq 0$, η , and $w = u^{(1+\beta)/2}$, we get

$$(23) \quad A_\beta \left(\int_{B_\rho \setminus B_\sigma} w^6 \right)^{1/3} \leq B_\beta \frac{1}{\delta^2} \int_{B_{\rho+\delta} \setminus B_{\sigma-\delta}} w^2,$$

where

$$A_\beta = Q - C \frac{|1+\beta|^2}{\beta} \alpha_2^{1/2} \left(\int_{\text{supp } \eta} u^6 \right)^{1/6}$$

and

$$B_\beta = C \left(\frac{|1+\beta|^2}{|\beta|} + \frac{|1+\beta|^2}{|\beta|} \cdot |R_0| + 1 \right)$$

with C a universal constant. If we denote

$$\Phi(u, p, \Omega) = \left(\int_{\Omega} u^p \right)^{1/p},$$

then if $A_\beta > 0$, we have

$$(24) \quad \Phi(u, 3(1+\beta), B_\rho \setminus B_\sigma) \leq C_{\beta, \delta} \Phi(u, 1+\beta, B_{\rho+\delta} \setminus B_{\sigma-\delta}) \quad \text{if } 1+\beta > 0$$

and

$$(25) \quad \Phi(u, 1+\beta, B_{\rho+\delta, \sigma-\delta}) \leq C_{\beta, \delta} \Phi(u, 3(1+\beta), B_\rho \setminus B_\sigma) \quad \text{if } 1+\beta < 0$$

where

$$C_{\beta, \delta} = \left(\frac{B_\beta}{A_\beta \delta^2} \right)^{1/(|1+\beta|)}.$$

Now for fixed $r > 0$, we will begin to apply (24) iteratively to a sequence of ρ_k , β_k , and δ_k and corresponding functions η_k and $u = u_j$. We first fix β_0 so that $1+\beta_0 = p_0$, and choose $m \geq 0$ so that $3^{m+1}p_0 = 2$ (we may assume that p_0 is small, say $p_0 < \frac{1}{3}$). Define β_k , $k = 0, 1, 2, \dots, m$ as $1+\beta_k = 3^k p_0$. We then choose $\rho_m = r$, $\sigma_m = \frac{r}{2}$, $\rho_{k-1} = \rho_k + \delta_k$, $\sigma_{k-1} = \sigma_k - \delta_k$, and $\delta_k = r/2^{k+3}$, $k = 0, 1, 2, \dots, m$. We observe that for this choice of δ_k , support of the corresponding function η_k is contained in $B_{r/4}^c$ for each k and

$$\frac{|1+\beta_k|^2}{|\beta_k|} \leq \frac{4}{|\beta_m|} = 12.$$

Since $\{u_j\}$ concentrates at $\{x_0\}$, we have $\int_{\text{supp } \eta_k} u_j^6 \leq \int_{B_{r/4}^c} u_j^6 \rightarrow 0$ as $j \rightarrow \infty$. Thus, for j sufficiently large, we have $A_{\beta_k} \geq \frac{Q}{2}$ and

$$(26) \quad \Phi(u_j, 2, B_r \setminus B_{r/2}) \leq \left(\prod_{k=0}^m C_{\beta_k, \delta_k} \right) \Phi(u_j, p_0, B_{3r} \setminus B_{r/4}).$$

A similar argument applies to the sequence $1+\tilde{\beta}_k = -p_0 3^k$, and inequality (25) yields, for j sufficiently large,

$$(27) \quad \Phi(u_j, -p_0, B_{3r} \setminus B_{r/4}) \leq \left(\prod_{k=0}^m C_{\tilde{\beta}_k, \delta_k} \right) \Phi(u_j, -2, B_{2r} \setminus B_r),$$

where

$$\begin{aligned} \prod_{k=0}^m C_{\beta_k, \delta_k} &= \prod_{k=0}^m \left(\frac{B_{\beta_k}}{A_{\beta_k} \delta_k^2} \right)^{1/(|1+\beta_k|)} \\ &\leq C 2^{(\sum k \cdot 3^{-k})/p_0} r^{-2(\sum_{k=0}^m 3^{-k})/p_0} \\ &= C 2^{c/p_0} r^{-3(1/p_0-1/2)}. \end{aligned}$$

Similarly,

$$\prod_{k=0}^m C_{\tilde{\beta}_k, \delta_k} = C 2^{c/p_0} r^{-3(-1/p_0+1/2)}$$

(both C and c denote universal constants). Now observe that we may rewrite

$$(\Phi(u_j, p_0, B_{3r} \setminus B_{r/4}))^{p_0} = \left(\int_{B_{3r} \setminus B_{r/4}} u_j^{p_0} \right) \leq \int_{B_{3r}} u_j^{p_0}$$

and similarly,

$$(\Phi(u_j, p_0, B_{3r} \setminus B_{r/4}))^{-p_0} \leq \int_{B_{3r}} u_j^{-p_0}.$$

Applying Lemma 2 to $u = u_j$ and with $B_\rho = B_{3r}$ and $B_{r/4}$, we get

$$\begin{aligned} (\Phi(u_j, p_0, B_{3r} \setminus B_{r/4}))^{p_0} (\Phi(u_j, p_0, B_{3r} \setminus B_{r/4}))^{-p_0} &\leq \int_{B_{3r}} u_j^{p_0} \int_{B_{3r}} u_j^{-p_0} \\ &\lesssim C r^6. \end{aligned}$$

Thus, combining (26), (27), and (28), we obtain for $j \geq j(r)$,

$$\begin{aligned} \Phi(u_j, 2, B_r \setminus B_{r/2}) &\leq C 2^{2c/p_0} r^3 \Phi(u_j, -2, B_{2r} \setminus B_r) \\ &\leq C 2^{2c/p_0} \Phi(u_j, 2, B_{2r} \setminus B_r), \end{aligned}$$

which is equivalent to the desired estimates (22).

We have thus finished the proof of Proposition C with constant $C \sim 2^{c/p_0}$.

5. PROOF OF THE THEOREM FOR THE $R_0 > 0$ CASE

First we will prove a general statement for all manifolds with scalar curvature $R_0 \geq 0$.

Proposition D. *Assume a sequence of positive smooth functions $\{u_j\}$ satisfies assumptions of Proposition B, and, in addition,*

$$(3)' \quad \int \text{Ric}^2(u_j^4 g_0) u_j^6 dV_0 \leq a_2 < \infty.$$

Then there exist constants $c_j > 0$ with $c_j \rightarrow \infty$ so that the sequence $v_j = c_j u_j$ converges uniformly on compact subset of $M \setminus \{x_0\}$ either (i) to the Green's

function of the conformal Laplacian $L_G = \Delta G - R_0 G = -\delta_{x_0}$ when $R_0 > 0$ or (ii) for $G =$ positive constant when $R_0 = 0$.

Proof. Fix a small ball $B(x_0, r)$ and choose a constant c_j so that

$$c_j^6 \int_{M \setminus B(x_0, r)} u_j^6 = 1.$$

Because of Proposition B, it is clear that $c_j \rightarrow \infty$. Denote $v_j = c_j u_j$, $Lu = 8\Delta u - R_0 u$.

Observe that

$$(29) \quad \begin{aligned} \int \text{Ric}^2(v_j^4 g_0) v_j^6 dV_0 &= \frac{1}{c_j^2} \int \text{Ric}^2(u_j^4 g_0) u_j^6 dV_0 \\ &\leq \frac{a_2}{c_j^2} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Similarly,

$$(30) \quad \begin{aligned} \int \left(\frac{Lv_j}{v_j^2} \right)^2 dV_0 &= \int R^2(v_j^4 g_0) v_j^6 dV_0 \leq \frac{1}{c_j^2} \int R(u_j^4 g_0) u_j^6 dV_0 \\ &\leq \frac{\alpha_2}{c_j^2} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{M \setminus B(x_0, r)} (Lv_j)^{6/5} &= \int_{M \setminus B(x_0, r)} \left(\frac{Lv_j}{v_j^2} \right)^{6/5} v_j^{12/5} \\ &\leq \left(\int_{M \setminus B(x_0, r)} \left(\frac{Lv_j}{v_j^2} \right)^2 \right)^{3/5} \left(\int_{M \setminus B(x_0, r)} v_j^6 \right)^{2/5} \\ &\rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

It follows from this bound that v_j remains bounded on compacta in W_2^1 on $M \setminus B(x_0, r)$. Thus, a subsequence of $\{v_j\}$ converges weakly in W_2^1 to a (weak, hence strong) solution w of the equation $Lw = 0$ on $M \setminus B(x_0, r)$. We need to verify that w is strictly positive; this will follow from the minimum principle (since $R_0 \geq 0$) for elliptic operators once we have shown that $w \not\equiv 0$ on $M \setminus B(x_0, r)$.

To do this, we will again apply arguments similar to the derivation of (9b) to $\beta = 1$ and with the cut-off function η , $\eta \equiv 1$ on $M \setminus B(x_0, r)$ and $\eta \equiv 0$ on $B(x_0, \frac{r}{2})$ with $|\nabla \eta| \leq \frac{c}{r}$. Denoting $B_r = B(x_0, r)$ and $B_r^c = M \setminus B_r$, we obtain

$$(31) \quad \begin{aligned} 1 &= \left(\int_{B_r^c} v_j^6 \right)^{1/3} \leq \left(\int (\eta v_j)^6 \right)^{1/3} \\ &\leq C \left(\int_{B_{r/2}^c} \left(\frac{Lv_j}{v_j^2} \right)^2 \right)^{1/2} \left(\int_{B_{r/2}^c} v_j^6 \right)^{1/6} \left(\int (\eta v_j)^6 \right)^{1/3} \\ &\quad + \frac{c}{r^2} \left(\int_{B_{r/2}^c} v_j^2 \right). \end{aligned}$$

Now assume to the contrary that $w \equiv 0$. Then $\lim_{j \rightarrow \infty} \int_{B_r^c} v_j^2 = \int_{B_r^c} w^2 = 0$. Applying Proposition C, we then conclude that

$$\begin{aligned} \int_{B_{r/2}^c} v_j^2 &= c_j^2 \left[\int_{B_r \setminus B_{r/2}} u_j^2 + \int_{B_r^c} u_j^2 \right] \\ &\leq C(p_0) c_j^2 \int_{B_r^c} u_j^2 = c(p_0) \int_{B_r^c} v_j^2 \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Thus, if we apply and divide both sides of inequality (31) by $\int (\eta v_j)^6$, we obtain

$$1 \leq c \frac{\alpha_2^{1/2}}{c_j} c_j \left(\int_{B_{r/2}^c} u_j^6 \right)^{1/6} + \frac{c(p_0)}{r^2} \int_{B_r^c} v_j^2 \rightarrow 0$$

as $j \rightarrow \infty$ as u_j concentrates at x_0 .

This is a contradiction. Hence, $\lim_{j \rightarrow \infty} \int_{B_r^c} v_j^2 \geq \delta > 0$ and $w \not\equiv 0$ (hence, $w > 0$ on B_r^c).

For any $r' < r$, we now apply the same argument and perhaps find a different set of constants c'_j and functions $v'_j = c'_j u_j$ that tend to a positive function w' on $M \setminus B_{r'}$. But

$$\lim_{j \rightarrow \infty} \frac{c'_j}{c_j} = \lim_{j \rightarrow \infty} \frac{v'_j(x)}{v_j(x)} = \frac{w'(x)}{w(x)} > 0.$$

Thus, $\lim_{j \rightarrow \infty} c'_j = \infty$. Hence, $w'(x)$ is proportional to $w(x)$, and we may readjust constants c'_j to make $w' = w$ on $M \setminus B_{r'}$. If we repeat this process to a sequence $r_j \rightarrow 0$, a diagonal subsequence construction then gives a sequence of functions $v_j = c_j u_j \rightarrow w$, a positive solution of the equation $Lw = 0$ on $M \setminus \{x_0\}$. According to the isolated singularity theorem of Gilbarg and Serrin [GS], either w has a pole at x_0 and $w(x) \sim d(x, x_0)^{2-n}$ ($n = 3$ in one case) or (ii) w has a removable singularity at x_0 in which case $w(x) \equiv \text{constant}$ and $R_0 = 0$, which finishes the proof of Proposition C.

Proof of the theorem for the $R_0 > 0$ case. To finish off the proof of the main theorem, we first observe that in either the $R_0 > 0$ or $R_0 = 0$ case, $w^4 g_0$ defines a flat metric. The reason is that on any compact subset K of $M \setminus \{x_0\}$, we have $Lv_j \rightarrow Lw \equiv 0$ in $L^{6/5}$; hence, $v_j - w \rightarrow 0$ on $W_{6/5}^2$, and

$$\int_K \text{Ric}^2(w^4 g_0) w^6 dV_0 \leq \liminf_j \int_K \text{Ric}^2(v_j^4 g_0) v_j^6 dV_0 = 0.$$

Hence, $w^4 g_0$ is Ricci flat, but in dimension 3, this means $w^4 g_0$ is flat. In the case when $R_0 > 0$ and w is the Green function, we have $w(x) \sim d(x, x_0)^{2-n}$ ($n = 3$) yielding that $w^4 g_0$ is a complete flat metric on $M \setminus \{x_0\}$. According to the classification theorem for flat forms (cf. Wolf [W]), in three dimensions the only complete flat space that is simply connected at infinity is the Euclidean space. This implies via Liouville's Theorem that M is conformally equivalent to S^3 , which finishes the proof of the theorem for the $R_0 > 0$ case.

6. PROOF OF THE THEOREM FOR THE CASE $R_0 = 0$

To continue the proof of the theorem when $R_0 = 0$, we first observe that in this case (via Proposition D), (M, g_0) is a compact, flat manifold; hence, the metric g_0 is locally Euclidean. By rescaling the metric g_0 if necessary, we may assume that there is a Euclidean ball of radius 2 around the point of concentration x_0 in M . If we isometrically map the point x_0 to $0 = (0, 0, 0)$ in \mathbf{R}^3 and the ball of radius 2 into $B_2 = B_2(0)$ in \mathbf{R}^3 , and denote the Euclidean metric on \mathbf{R}^3 by $dx^2 = \sum_{i=1}^3 dx_i^2$ and Euclidean Laplacian by Δ_e , then locally we have a sequence of positive functions $\{u_j\}$ defined on $B_2(0)$ satisfying the following conditions.

$$(1)' \quad \int_{B_2(0)} u_j^6 dx \approx \alpha_0.$$

$$(3)'' \quad \int_{B_2(0)} (\Delta_e u_j / u_j^2)^2 dx \leq \alpha_2.$$

(4)' For each function ϕ of compact support in $B_2(0)$, we have

$$\int u_j^6 \phi^2 dx \leq \left(\int u_j^6 \phi dx \right)^2 / \left(\int_{B_2(0)} u_j^6 dx \right) + \frac{1}{\lambda_1} \int |\nabla_e \phi|^2 u_j^2 dx$$

for some constant $\lambda_1 \geq \Lambda > 0$, and furthermore, we have the following.

$$(32) \quad \begin{array}{l} \text{There exist sequences of numbers } c_j \rightarrow \infty \text{ and } s_j \rightarrow 0 \text{ with} \\ u_j c_j \rightarrow 1 \text{ uniformly on } B_2(0) \setminus B_{s_j}(0). \end{array}$$

We will now derive another property satisfied by our concentrating sequence u_j .

Lemma 6.1. *Suppose $\{u_j\}$ is defined on (M, g_0) with $R(g_0) = R_0 = 0$ and $\{u_j\}$ satisfies conditions (1)–(4) and (32) on M . Then*

$$\int_{M \setminus B_{2s_j}(0)} |\nabla u_j|^2 dV = o\left(\frac{1}{c_j^2}\right).$$

Proof. Choose a cut-off function η_1 defined on M with $\eta_1 = 0$ on $B_{s_j}(0)$, $\eta_1 = 1$ on $M \setminus B_{2s_j}(0)$. Then

$$\begin{aligned} \int_{M \setminus B_{2s_j}(0)} |\nabla u_j|^2 dV_0 &\leq \int_M |\nabla u_j|^2 \eta_1 dV_0 \\ &= - \int (\Delta u_j) u_j \eta_1 dV_0 + \int \nabla u_j \cdot u_j \nabla \eta_1 dV_0 \\ &\lesssim \int R u_j^6 \eta_1 dV_0 + \frac{1}{s_j} \int_{B_{2s_j}(0) \setminus B_{s_j}(0)} |\nabla u_j| |u_j| dV_0 \end{aligned}$$

$$\begin{aligned}
&\leq \left(\int_M R^2 u_j^6 dV_0 \right)^{1/2} \left(\int_{\text{supp } \eta_1} u_j^6 dV_0 \right)^{1/2} \\
&\quad + \frac{1}{s_j} \frac{1}{c_j^2} \left(\int_M \left| \frac{\nabla u_j}{u_j} \right|^2 dV_0 \right)^{1/2} s_j^{3/2} \\
&\lesssim \alpha_2^{1/2} \frac{1}{c_j^3} + (s_j)^{1/2} / c_j^2 = o\left(\frac{1}{c_j^2}\right),
\end{aligned}$$

where the last estimate follows from the observation that

$$\int_M \frac{|\nabla u_j|^2}{u_j^2} dV_0 = - \int_M R u_j^4 dV_0 \leq \left(\int_M R^2 u_j^6 dV_0 \right)^{1/2} \left(\int_M u_j^2 dV_0 \right)^{1/2}$$

is finite. This finishes the proof of Lemma 6.1.

Thus, we may assume that our sequence $\{u_j\}$ satisfies

$$(32)' \quad \int_{B_2(0) \setminus B_{2s_j}(0)} |\nabla_e u_j|^2 dx = o\left(\frac{1}{c_j^2}\right).$$

We will indeed show that a sequence $\{u_j\}$ satisfying (1), (3)'', (4)', (32), and (32)' cannot exist. Suppose it does. Then we will yield a contradiction by constructing from $\{u_j\}$ a sequence of functions $\{w_j\}$ defined on S^3 that violates the following sharp Sobolov inequality

$$(33) \quad Q(S^3) \left(\int_{S^3} w_j^{6+6\varepsilon} dV \right)^{1/3} \leq 8 \int_{S^3} |\nabla w_j^{(1+\varepsilon)}|^2 dV + 6 \int_{S^3} w_j^{2(1+\varepsilon)} dV,$$

where $Q(S^3) = 6(2\pi^2)^{2/3}$ when $j \rightarrow \infty$ and for some suitable $\varepsilon > 0$ ($\varepsilon = \varepsilon(\alpha_0, \alpha'_2, \lambda_1)$). We first extend u_j to a function v_j defined on \mathbf{R}^3 (in fact, on S^3) by adding a tail function ϕ_j on $B_2^c(0)$ to u_j , where ϕ_j is chosen so that the corresponding function $\tilde{\phi}_j$ defined on S^3 is an extremal function for the Sobolov inequality (33). We then isometrically move v_j to a "balanced" position and prove that the new function w_j , created this way under the assumptions (1)', (3)'', (4)', (32), and (32)', violates (33).

We will now describe and justify this construction procedure in detail. First we will set some notation.

We adopt coordinates on S^3 through its stereographic projection mapping north pole of S^3 to $0 = (0, 0, 0)$ in \mathbf{R}^3 . In this coordinate system, volume form dV on S^3 is defined by

$$dV = \left(\frac{2}{1 + |x|^2} \right)^3 dx.$$

For each function f defined in \mathbf{R}^3 , define the corresponding function \tilde{f} on S^3 by

$$\tilde{f}(x) = f(x) \left(\frac{1 + |x|^2}{2} \right)^{1/2}.$$

Thus, $\int_{S^3} (\tilde{f}(x))^6 dV(x) = \int_{\mathbb{R}^3} (f(x))^6 dx$. For each fixed $\tau > 0$, denote

$$\phi_\tau(x) = \left(\frac{2\tau}{\tau^2 + |x|^2} \right)^{1/2}$$

($\tilde{\phi}_\tau$'s are extremum functions for inequality (33), where $\varepsilon = 0$), and denote conformal isometries $T_\tau : L^6(S^3) \rightarrow L^6(S^3)$ as defined by

$$(T_\tau(v))(y/\tau) = v(y)(\tilde{\phi}_\tau(y))^{-1}.$$

We now fix a function $u = u_j$ satisfying conditions (1)', (3)'', (4)', (32), and (32)'. We will extend u to be a function v defined on S^3 as follows.

Choose a cut-off function on S^3 with $\eta = 1$ on $B_1(0)$ and $\eta = 0$ off $B_{1+\delta}(0)$ and $|\nabla_\eta \eta| \leq (1 + o(1))/\delta$ on $A_{1,1+\delta} = B_{1+\delta}(0) \setminus B_1(0)$ and $\frac{\partial \eta}{\partial n}|_{\partial B_1} = \frac{\partial \eta}{\partial n}|_{\partial B_{1+\delta}} = 0$, where $\delta = \delta_j$ is chosen so that $|u_j - 1/c_j| \leq \delta^2/c_j$ off $B_{\delta_j}(0)$ (which is possible via (32)). We then choose $t_j > 0$ with $\phi_{t_j}(1) = (2t_j/(1+t_j^2))^{1/2} = 1/c_j$ and $v = v_j$ as an extension of u from $B_2(0)$ to S^3 as

$$v(x) = \tilde{u}(x)\eta + \tilde{\phi}_t(x)(1 - \eta).$$

Fix $\varepsilon > 0$ (ε is independent of j and will be chosen later). We will now apply $T_\tau = T_{\tau_j}$ to v , where τ_j is chosen so that the mass of $(T_\tau v)^{6+\varepsilon}$ is in a balanced position; i.e.,

$$(34) \quad \int_{S^3} (T_\tau v)^{6+\varepsilon} x_\alpha dV(x) = 0 \quad \text{for } \alpha = 1, 2, 3, 4$$

where the x_α are the ambient coordinates of S^3 (i.e., $S^3 \hookrightarrow \mathbb{R}^4$ is defined by $\sum_{\alpha=1}^4 x_\alpha^2 = 1$). Denote $w = w_j = (T_\tau v)$. We now claim that with some suitable choice of ε , (32) will be violated for w . To see this, recall that the equation relating w to its curvature function $R = R(w^4 g_0)$ is as follows.

$$(6)' \quad -8\Delta w + 6w = R w^5 \quad \text{on } S^3.$$

Since R is the scalar curvature, an intrinsic invariant quantity, we have

$$R = R(\tilde{u}^4 g_0) \quad \text{on } B_{1/\tau}(0), \quad \text{and} \quad R = R(\tilde{\phi}_t^4 g_0) = 6 \quad \text{on } B_{(1+\delta)/\tau}^c(0).$$

Thus, on the complement of $A_{1/\tau, (1+\delta)/\tau} = B_{(1+\delta)/\tau}(0) \setminus B_{1/\tau}(0)$, we have

$$(35) \quad \int_{A_{1/\tau, (1+\delta)/\tau}^c} R^2 w^6 dV \leq \alpha'_2 < \infty.$$

Multiply equation (6)' by $w^{1+2\varepsilon}$. Then applying Sobolev inequality (33), we obtain

$$(36) \quad \begin{aligned} Q \left(\int w^{6+6\varepsilon} dV \right)^{1/3} &\leq 8 \int |\nabla w^{1+\varepsilon}|^2 dV + 6 \int w^{2(1+\varepsilon)} dV \\ &< \frac{(1+\varepsilon)^2}{1+2\varepsilon} \int R w^{6+2\varepsilon} dV. \end{aligned}$$

We will estimate the right-hand side in detail in order to verify that it is in fact strictly less than the left-hand side, and that is the contradiction.

To estimate $\int R w^{6+2\epsilon} dV$, we split the integral into three parts: $\int R w^{6+2\epsilon} dV = \text{I} + \text{II} + \text{III}$, where

$$\begin{aligned} \text{I} &= \int_{A_{1/\tau, (1+\delta)/\tau}} R w^{6+2\epsilon} dV, \\ \text{II} &= \int_{(A_{1/\tau, (1+\delta)/\tau})^c \cap \{|R| \geq b\}} R w^{6+2\epsilon} dV, \\ \text{III} &= \int_{(A_{1/\tau, (1+\delta)/\tau})^c \cap \{|R| \leq b\}} R w^{6+2\epsilon} dV \leq b \int_{S^3} w^{6+2\epsilon} dV, \end{aligned}$$

where b is a large constant depending on α_0 , α'_2 and is to be chosen later. We claim

$$(37) \quad \text{I} \leq \frac{16\pi + 0(1)}{c_j^{2+2\epsilon} \tau_j^\epsilon},$$

$$(38) \quad \frac{8\pi}{6} \frac{1}{c_j^{6+6\epsilon} \tau_j^{3\epsilon}} \leq \int_{S^3} w^{6+6\epsilon} dV.$$

Since we will soon see that I is the dominant term on the right-hand side, it will be convenient to set the constant

$$K_0 = \frac{16\pi}{Q(S^3)(\frac{8\pi}{6})^{1/3}} = \frac{16\pi}{6(2\pi^2)^{2/3}(\frac{8\pi}{6})^{1/3}} < 1.$$

Assuming (37) and (38) for the moment, we proceed to estimate II as in (10):

$$\begin{aligned} (39) \quad \text{II} &\leq \left(\int_{A^c} R^2 w^6 dV \right)^{1/2} \left(\int_{A^c \cap \{|R| \geq b\}} w^6 dV \right)^{1/6} \left(\int_{S^3} w^{6+6\epsilon} dV \right)^{1/3} \\ &\leq (\alpha'_2)^{1/2} \left(\frac{\alpha'_2}{b^2} \right)^{1/6} \left(\int_{S^3} w^{6+6\epsilon} dV \right)^{1/3} \\ &\leq \left(\frac{\alpha'_2}{b} \right)^{1/3} \cdot \left(\int_{S^3} w^{6+6\epsilon} dV \right)^{1/3}, \end{aligned}$$

where $A = A_{1/\tau, (1+\delta)/\tau}$.

To estimate III, we will use the crucial assumption that we have made on the choice of τ_j , with $w = w_j = T_{\tau_j}(v_j)$. That is (from (34)), τ_j is chosen so that

$$\int_{S^3} w^{6+\epsilon} x_\alpha dV = 0 \quad \text{for } \alpha = 1, 2, 3, 4.$$

Thus, if we choose a C^∞ cut-off function ρ supported on $B_{2s_j}(0)$ with $\rho = 1$ on $B_{s_j}(0)$ and recall that $s_j \rightarrow 0$ is chosen so that $|u_j - 1/c_j| \leq (\delta^2/c_j)$ off

$B_{s_j}(0)$, then

$$\begin{aligned}
 \int_{S^3} w^{6+2\varepsilon}(x) dV(x) &= \sum_{\alpha=1}^4 \int_{S^3} w^{6+2\varepsilon}(x) x_\alpha^2 dV(x) \\
 &\leq 2 \sum_{\alpha=1}^4 \left[\int_{S^3} w^6(w^\varepsilon(x) \rho(\tau x) x_\alpha)^2 dV(x) \right. \\
 &\quad \left. + \int w^6(w^\varepsilon(x)(1-\rho)(\tau x) x_\alpha)^2 dV(x) \right] \\
 &\leq 2 \sum_{\alpha=1}^4 (VI_\alpha + V_\alpha).
 \end{aligned}$$

On $(VI)_\alpha$ we have, $(w)(x) = (T_\tau v)(x) = \tilde{u}_j(\tau x) \tilde{\phi}_\tau^{-1}(\tau x)$ for $\tau x \in B_{2s_j}(0)$. Thus, we may apply our λ_1 -assumption (3)'' on u_j and obtain

$$\begin{aligned}
 (VI)_\alpha &\leq \left(\int_{S^3} w^{6+\varepsilon}(x) \rho(\tau x) x_\alpha dV \right)^2 / \int w^6(x) dV \\
 &\quad + \frac{1}{\Lambda} \int_{S^3} |\nabla(w^\varepsilon(\rho \circ \tau) x_\alpha)|^2 w^2(x) dV \\
 (\text{via (34)}) &\leq \left(\int_{S^3} w^{6+\varepsilon}(x)(1-\rho)(\tau x) x_\alpha dV \right)^2 / \int w^6(x) dV \\
 &\quad + \frac{\varepsilon^2}{\Lambda} \int |\nabla w|^2 w^{2\varepsilon}(x) + \frac{1}{\Lambda} \int |\nabla \rho \circ \tau|^2 w^{2+2\varepsilon} dV \\
 &\quad + \frac{1}{\Lambda} \int_{B_{2s_j/\tau}} |\nabla x_\alpha|^2 w^{2+2\varepsilon}(x) dx \\
 &\leq \frac{\varepsilon^2}{\Lambda} \int R w^{6+2\varepsilon}(x) dV + \frac{\text{constant} \cdot s_j}{c_j^{2+2\varepsilon} \tau_j^\varepsilon} + \frac{\text{constant} \cdot s_j}{c_j^{12+2\varepsilon} \tau_j^\varepsilon} \\
 &= \frac{\varepsilon^2}{\Lambda} \int R w^{6+2\varepsilon} dV + o\left(\int w^{6+6\varepsilon} dV\right)^{1/3},
 \end{aligned}$$

where the next to the last line in the computation above follows from the pointwise estimate of the value of w outside $B_{s_j}(0)$. A similar pointwise estimate yields

$$V_\alpha \leq \left(\frac{1}{c_j^{6+\varepsilon} \tau_j^{\varepsilon/2}} \right)^2 = o\left(\int w^{6+6\varepsilon}(x) dV(x)\right)^{1/3}.$$

Adding up $(VI)_\alpha$ and $(V)_\alpha$ for $\alpha = 1, 2, 3, 4$, we obtain

$$(40) \quad III \leq \frac{\varepsilon^2}{\Lambda} \int_{S^3} R w^{6+2\varepsilon} dV + o\left(\int_{S^3} w^{6+6\varepsilon} dV\right)^{1/3}.$$

Combining (37)–(40), we get

$$\begin{aligned}
 (41) \quad & \left(\frac{1+2\varepsilon}{(1+\varepsilon)^2} \right) Q(S^3) \left(\int_{S^3} w^{6+6\varepsilon} \right)^{1/3} \leq \int_{S^3} R w^{6+2\varepsilon} \\
 & \leq \left(\left(\frac{\alpha'_2}{b} \right)^{1/3} + o(1) \right) \left(\int_{S^3} w^{6+6\varepsilon} dV \right)^{1/3} \\
 & \quad + K_0 Q(S^3) \left(\int w^{6+6\varepsilon} \right)^{1/3} + \frac{b\varepsilon^2}{\Lambda} \int R w^{6+2\varepsilon}.
 \end{aligned}$$

Thus, if we choose b large enough with $(\alpha'_2/b)^{1/3} \leq \frac{1}{2}(1-K_0)Q(S^3)$, and then chosen ε sufficiently small, then the left-hand side of inequality (41) is strictly bigger than the right-hand side; i.e., $w^{6+6\varepsilon}$ violates Sobolev inequality on S^3 . With this contradiction, we finish the proof of the theorem for the $R_0 = 0$ case except for the verification of (37) and (38).

Verification of (37).

$$\begin{aligned}
 \mathbf{I} &= \int_{A_{1/\tau, (1+\delta)/\tau}} R(x) w^{6+2\varepsilon}(x) dV(x) \\
 &= \int_{A_{1, 1+\delta}} R(\tau^{-1}y) v^{6+2\varepsilon}(y) \tilde{\phi}_\tau^{-2\varepsilon}(y) dV(y) \\
 &= \int_{A_{1, 1+\delta}} (-8\Delta v + 6v)(y) v^{1+2\varepsilon}(y) (\tilde{\phi}_\tau)^{-2\varepsilon}(y) dV(y).
 \end{aligned}$$

For the moment, denote the function f by $v^{1+2\varepsilon}(\tilde{\phi}_\tau)^{2\varepsilon}$ and replace v by $\tilde{u}\eta + (1-\eta)\tilde{\phi}_t$. A straight-forward computation then yields

$$\begin{aligned}
 \mathbf{I} &= \int_{A_{1, 1+\delta}} [(\tilde{R}\tilde{u}^5)\eta + 6(\tilde{\phi}_t)^5(1-\eta) - 16(\nabla\tilde{u}\nabla\eta - \nabla\tilde{\phi}_t\nabla\eta) \\
 &\quad - 8(\tilde{u}\Delta\eta - \tilde{\phi}_t\Delta\eta)] f(y) dV(y).
 \end{aligned}$$

Applying integration by parts to the term $-8(\tilde{u} - \tilde{\phi}_t)(\Delta\eta)f$, we get

$$\begin{aligned}
 \mathbf{I} &= \int_{A_{1, 1+\delta}} [(\tilde{R}\tilde{u}^5)\eta + 6(\tilde{\phi}_t)^5(1-\eta) - 8(\nabla\tilde{u}\nabla\eta - \nabla\tilde{\phi}_t\nabla\eta)] f(y) dV(y) \\
 &\quad + 8 \int_{A_{1, 1+\delta}} (\tilde{u} - \tilde{\phi}_t) \nabla f \cdot \nabla \eta dV(y).
 \end{aligned}$$

We now observe that in the expression of \mathbf{I} , the first term $\tilde{R} = R(u^4 g_0)$ satisfies condition (3)'' with $|\tilde{u}| \leq (1+o(1))/c_j$ on $B_{s_j}^c(0)$. Thus, $\int_{A_{1, 1+\delta}} \tilde{R}\tilde{u}^5 dV \leq (\alpha'_2)^{1/2} \delta / c_j^2 = o(1/c_j^2)$. For the second term, we may apply the direct estimate $|\tilde{\phi}_t| = 1 + o(1)/c_j$ on $A_{1, 1+\delta}$. For the third term (i.e., the $\nabla\tilde{u} \cdot \nabla\eta$ term), we

apply condition (32)' in the proof of Lemma 6.1. Thus,

$$(42) \quad \begin{aligned} I \leq 8 \left[\int_{A_{1,1+\delta}} (\nabla \tilde{\phi}_t \cdot \nabla \eta) f \, dV + \int_{A_{1,1+\delta}} (\tilde{u} - \tilde{\phi}_t) \nabla f \cdot \nabla \eta \, dV \right] \\ + o\left(\frac{1}{c_j^{2+2\varepsilon} \tau_j^\varepsilon}\right). \end{aligned}$$

We may accurately compute $\int_{A_{1,1+\delta}} (\nabla \tilde{\phi}_t \cdot \nabla \eta) f(y) \, dV(y)$ as

$$\int_{A_{1,1+\delta}} (\nabla \tilde{\phi}_t \cdot \nabla \eta) f(y) \, dV(y) \leq \frac{4\pi + o(1)}{c_j^{1+2\varepsilon} \tau_j^\varepsilon} \int_1^{1+\delta} |\nabla_e \tilde{\phi}_t \cdot \nabla_e \eta| \frac{2}{1+r^2} r^2 \, dr.$$

Since by our choice of η , $|\nabla_e \eta| \leq (1 + o(1))/\delta$ on $A_{1,1+\delta}$, while applying a straightforward computation we have $|\nabla_e \tilde{\phi}_t| \leq (t_j/2)^{1/2} = 1/2c_j$ on $A_{1,1+\delta}$. Thus,

$$(43) \quad \left| \int_{A_{1,1+\delta}} |\nabla \tilde{\phi}_t \cdot \nabla \eta(y)| f(y) \, dV(y) \right| \leq \frac{2\pi + o(1)}{c_j^{2+2\varepsilon} \tau_j^\varepsilon}.$$

Next we observe that on $A_{1,1+\delta}$, $|\tilde{u} - \tilde{\phi}_t| = |\tilde{u} - 1/c_j| + |1/c_j - \tilde{\phi}_t| \leq \delta^2/c_j + \delta/2c_j \leq \delta/c_j$ (the first estimate follows from our choice of δ , the second from our choice of $t = t_j$, with $\tilde{\phi}_{t_j}(1) = 1/c_j$ and the estimate $|\nabla_e \tilde{\phi}_t| \leq 1/2c_j$ on $A_{1,1+\delta}$). Thus, we have

$$\left| \int_{A_{1,1+\delta}} (\tilde{u} - \tilde{\phi}_t) \nabla f \nabla \eta \, dV \right| \leq \frac{\delta}{c_j} \cdot \frac{1 + o(1)}{\delta} \int_{A_{1,1+\delta}} |\nabla f| \, dV.$$

We now recall

$$f = v^{1+2\varepsilon} (\tilde{\phi}_\tau)^{-2\varepsilon} = v^{1+2\varepsilon} \left(\frac{\tau(1+|y|^2)}{\tau^2 + |y|^2} \right)^{-\varepsilon}.$$

Thus, a pointwise computation of ∇f with estimates similarly to (43) above yields

$$(44) \quad \left| \int_{A_{1,1+\delta}} (\tilde{u} - \tilde{\phi}_t) \nabla f \nabla \eta \, dV \right| \leq \frac{1}{c_j^{2+2\varepsilon} \tau_j^\varepsilon} (\varepsilon \delta + \delta^{1/2}) = o\left(\frac{1}{c_j^{2+2\varepsilon} \tau_j^\varepsilon}\right).$$

Inserting (43) and (44) into (42) we get the desired estimate (37) of I , i.e.,

$$I \leq \frac{16\pi + o(1)}{c_j^{2+2\varepsilon} \tau_j^\varepsilon}.$$

Verification of (38). It suffices to prove $w = w_j$ satisfies

$$\int_{S^3} w^{6+6\varepsilon} \, dV \geq \frac{1}{c_j^{6+6\varepsilon} \tau_j^{3\varepsilon}} \left(\frac{8\pi}{3 \cdot 2^{3\varepsilon}} - o(1) \right).$$

To verify this, we notice that by our definition of $w = T_\tau v$ we have

$$\int_{S^3} w^{6+6\varepsilon} \, dV \geq \int_{A_{s/\tau, 1/\tau}} w^{6+6\varepsilon} \, dV + \int_{B_{(1+\delta)/\tau}^c(0)} w^{6+6\varepsilon} \, dV = L_1 + L_2,$$

where

$$\begin{aligned}
 L_1 &= \int_{A_{s/\tau, 1/\tau}} (\tilde{u}(\tau x))^{6+6\epsilon} \left(\frac{\tau^2 + |\tau x|^2}{\tau(1 + |\tau x|^2)} \right)^{3+3\epsilon} dV(x) \\
 &= \int_{A_{s, 1}} (\tilde{u}(y))^{6+6\epsilon} \left(\frac{(\tau(1 + |y|^2))}{1 + |y|^2} \right)^{3\epsilon} dV(y) \\
 (\text{from (32)}) \quad &\geq \frac{1}{c_j^{6+6\epsilon}} 32\pi \frac{1}{\tau^{3\epsilon}} \int_s^1 \left(\frac{\tau^2 + r^2}{1 + r^2} \right)^{3\epsilon} \frac{r^2 dr}{(1 + r^2)^3} \\
 &\geq \frac{4\pi}{c_j^{6+6\epsilon} \tau^{3\epsilon} 2^{3\epsilon}} \int_s^1 r^{6\epsilon} r^2 dr \\
 &= \left(\frac{4\pi}{3} - o(1) \right) \frac{1}{c_j^{6+6\epsilon} \tau^{3\epsilon} 2^{3\epsilon}}
 \end{aligned}$$

and

$$\begin{aligned}
 L_2 &= \int_{B_{(1+\delta)/\tau}^c(0)} (\tilde{\phi}_t(\tau x))^{6+6\epsilon} \left(\frac{\tau^2 + |\tau x|^2}{\tau(1 + |\tau x|^2)} \right)^{3+3\epsilon} dV(x) \\
 &= \frac{(32\pi)}{\tau_j^{3\epsilon}} \int_{1+\delta}^\infty \left(\frac{t_j}{t_j^2 + r^2} \right)^{3+3\epsilon} (\tau_j^2 + r^2)^{3\epsilon} r^2 dr \\
 &= (32\pi + o(1)) \frac{t_j^{3+3\epsilon}}{\tau_j^{3\epsilon}} \int_{1+\delta}^\infty r^{-4} dr \quad \left(\text{recall } \left(\frac{2t_j}{1 + t_j^2} \right)^{1/2} = \frac{1}{c_j} \rightarrow 0 \right) \\
 &= \left(\frac{4\pi}{3} - o(1) \right) \frac{1}{c_j^{6+6\epsilon} \tau_j^{3\epsilon} 2^{3\epsilon}}.
 \end{aligned}$$

Adding up the estimates for L_1 and L_2 , we have established (38).

We have thus finished the proof of the theorem.

APPENDIX

We present an alternative argument (using λ_1 to replace the log determinant of the Laplacian) to show that isospectral conformal metrics on compact surfaces form a compact set in the \mathcal{C}^∞ -topology.

Theorem. Suppose (M, g_0) is a compact surface, $\{e^{2u_j}\}$ is a sequence of conformal factors on M with

- (1) $\int e^{2u_j} dV_0 = a_0$;
- (2) $\int K_j^2 e^{2u_j} dV_0 = a_2 < \infty$, where K_j = Gaussian curvature of the metric $e^{2u_j} g_0$.

Assume in addition that the first eigenvalue λ_1 of the Laplacian w.r.t. the metrics $e^{2u_j} g_0$ is bounded from below by $\Lambda > 0$, i.e.,

- (3) for each function ϕ defined on M

$$\int_M \phi^2 e^{2u_j} dV_0 \leq \left(\int_M \phi e^{2u_j} dV_0 \right)^2 / \left(\int_M e^{2u_j} dV_0 \right) + \frac{1}{\Lambda} \int_M |\nabla_0 \phi|^2 dV_0.$$

Then either (a) and (b): when K_0 (the Gaussian curvature of the metric g_0) is < 0 or $= 0$, respectively, $\{u_j\}$ forms a bounded family in W_2^1 (i.e., $\sup_j \int_M |\nabla u_j|^2 dV_0$ is finite); or (c): when $K_0 = 1$ and $(M, g_0) = (S^2, g_0)$ with g_0 = surface measure on S^2 , then the isometry class of u_j forms a bounded family in W_2^1 .

Once the above theorem is established, when $\{e^{2u_j} g_0\}$ is a sequence of metrics that is an isospectral family, one may iteratively apply the common bounds (a_k for the k th ($k = 2, 3, \dots$)) coefficient in the trace of the heat kernel for the metrics and obtain a W_2^k bound for the family $\{e^{2u_j}\}$ as in [OPS2]. We then conclude that $\{e^{2u_j}\}$ (after modulo isometry class in the special case of (S^2, g_0)) is a compact family w.r.t. the \mathcal{E}^∞ -topology.

Proof of the theorem. Recall that Gaussian curvature K_j satisfies the equation

$$(4) \quad \Delta u_j + K_j e^{2u_j} = K_0.$$

Through the uniformization theorem, we may assume w.l.o.g. $K_0 = -1, 0$, or 1 . Integrating (4) over M we get

$$(5) \quad \int_M K_j e^{2u_j} dV_0 = \int_M K_0 dV_0.$$

Multiplying the equation (4) by u_j and integrating, we obtain

$$(6) \quad \int_M |\nabla u_j|^2 dV_0 + \int_M K_0 u_j dV_0 = \int_M K_j e^{2u_j} u_j dV_0.$$

(a) When $K_0 = 0$ (contrary to the 3-dimensional situation, this is the easy case). We notice that from (5) and (6) we have for any constant c ,

$$\begin{aligned} \int_M |\nabla u_j|^2 dV_0 &= \int_M K_j e^{2u_j} (u_j - c) dV_0 \\ &\leq \left(\int_M K_j^2 e^{2u_j} dV_0 \right)^{1/2} \left(\int_M e^{2u_j} (u_j - c)^2 dV_0 \right)^{1/2} \\ &\leq a_2^{1/2} \left[\left(\int_M e^{2u_j} (u_j - c) dV_0 \right)^2 / a_0 + \frac{1}{\Lambda} \int_M |\nabla u_j|^2 dV_0 \right]^{1/2} \quad (\text{via (2) and (3)}). \end{aligned}$$

Thus, if we choose $c = \int_M e^{2u_j} u_j dV_0$, we obtain

$$\int_M |\nabla u_j|^2 dV_0 \leq (a_2/\Lambda)^{1/2} \left(\int_M |\nabla u_j|^2 dV_0 \right)^{1/2}$$

i.e.,

$$\text{i.e.} \quad \int_M |\nabla u_j|^2 dV_0 \leq (a_2/\Lambda).$$

(b) When $K_0 = -1$, the proof we will present resembles the proof given in [BPY] for the 3-dimensional manifold with $R_0 < 0$.

We observe that in this case,

$$\begin{aligned} \int_M K_j^2 e^{2u_j} dV_0 &= \int_M (1 + \Delta u_j)^2 e^{-2u_j} dV_0 \\ &= \int_M (\Delta u_j)^2 e^{-2u_j} dV_0 + 4 \int_M |\nabla u_j|^2 e^{-2u_j} dV_0 + \int_M e^{-2u_j} dV_0. \end{aligned}$$

Thus by (2) we have $\int_M e^{-2u_j} dV_0 \leq a_2$. From this we may conclude that u_j satisfies

$$(*) \quad \int_{\{x \in M \mid e^{2u_j(x)} \geq l_0\}} dV_0 \geq \left(\int_M dV_0 - a_2 l_0 \right) \geq \gamma_0 > 0$$

for some $l_0 > 0$ sufficiently small (i.e., $\{u_j\}$ is not a “concentrating” sequence).

Denote $E = \{x \in M \mid e^{2u_j(x)} \geq l_0\}$. We have

$$\begin{aligned} (7) \quad \left(\int e^{2u_j} u_j dV_0 \right)^2 &\leq \left(\int_E (e^{2u_j} - l_0) u_j dV_0 + 2l_0 \int |u_j| dV_0 \right)^2 \\ &\leq (a_0 - l_0) \int_M e^{2u_j} u_j^2 dV_0 + c, \end{aligned}$$

where $c = c(a_1, a_2, l_0)$ is a constant depending on a_0, a_2, l_0 , and $2 \int u_j \leq \log \int e^{2u_j} \leq \log a_0$. Similarly, $-2 \int u_j \leq \log a_2$.

Applying (7) to (3) with $\phi = u_j$, we obtain

$$(8) \quad \int_M e^{2u_j} u_j^2 dM \leq c_1 + \frac{c_2}{\Lambda} \int_M |\nabla u_j|^2 dV_0$$

for some finite constants c_1, c_2 .

We may now apply the Schwartz inequality to the right-hand side of (6), similarly as in the case of (a) to obtain

$$\int_M |\nabla u_j|^2 dV_0 \leq \frac{1}{2} \log a_0 + a_2^{1/2} \left(c_1 + \frac{c_2}{\Lambda} \int_M |\nabla u_j|^2 dV_0 \right)^{1/2}.$$

Hence, $\int_M |\nabla u_j|^2 dV_0$ is finite.

(c) When $K_0 = 1$ (and through the uniformization theorem), $M = S^2$ with the standard metric g_0 .

In this case, equation (6) reads

$$\begin{aligned} (9) \quad \int_{S^2} |\nabla u_j|^2 dV_0 &= \int_{S^2} K_n e^{2u_j} (u_j - \bar{u}_j) dV_0 \\ &\leq a_2^{1/2} \left(\int_{S^2} e^{2u_j} (u_j - \bar{u}_j)^2 dV_0 \right)^{1/2}, \end{aligned}$$

where $\bar{u}_j = \int_{S^2} u_j dV_0 / \int_{S^2} dV_0$.

For each conformal transformation $\phi : S^2 \rightarrow S^2$, we denote the corresponding transformation T_ϕ as $T_\phi(u) = u \circ \phi + \frac{1}{2} \log |J_\phi|$ for all functions u defined on S^2 , where J_ϕ is the Jacobian of ϕ . We observe that for each ϕ the metrics $e^{2u} g_0$ and $e^{2T_\phi(u)} g_0$ are isometric and, in particular, isospectral.

For each fixed u_j , we now choose ϕ_j with $v_j = T_{\phi_j}(u_j)$ satisfying

$$\int_{S^2} e^{2v_j} (v_j - \bar{v}_j) x_\alpha dV_0 = 0, \quad \alpha = 1, 2, 3.$$

Here again the x_α are ambient coordinates on S^2 . We may argue similarly as in Lemma 1 in [CY] to prove that such ϕ_j exist. Applying (3) to v_j , we get

$$\begin{aligned} \int_{S^2} e^{2v_j} (v_j - \bar{v}_j)^2 x_\alpha^2 dV_0 &\leq \frac{1}{\Lambda} \int_{S^2} |\nabla(v_j - \bar{v}_j) x_\alpha|^2 dV_0 \\ &\leq \frac{2}{\Lambda} \left(\int_{S^2} |\nabla v_j|^2 x_\alpha^2 dV_0 + \int_{S^2} |\nabla x_\alpha|^2 |v_j - \bar{v}_j| dV_0 \right) \\ &\leq \frac{\text{constant}}{\Lambda} \int_{S^2} |\nabla v_j|^2 dV_0. \end{aligned}$$

Adding up $\alpha = 1, 2, 3$, we obtain

$$(10) \quad \int_{S^2} e^{2v_j} (v_j - \bar{v}_j)^2 dV_0 \leq \frac{\text{constant}}{\Lambda} \int_{S^2} |\nabla v_j|^2 dV_0.$$

We may now apply (10) to inequality (9) for v_j instead of u_j and conclude that $\int_{S^2} |\nabla v_j|^2 dV_0$ is uniformly bounded. This finishes the proof.

We remark that, in the proof above, once the bound for $\int |\nabla u_j|^2$ is attained, it is relatively easy to verify $|\int u_j dV_0|$ is bounded. Hence, the sequence $\{u_j\}$ is bounded in W_2^1 .

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