$L^2 - \bar{\partial}$ -COHOMOLOGY OF COMPLEX PROJECTIVE VARIETIES

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As part of a “Kähler package” for complex varieties, it was conjectured in [CGM] that one could generalize the Hodge (p, q) -decomposition of the cohomology of a smooth complex projective variety M

$$H^{p+q}(M) = \bigoplus_{p,q} H^{p,q}(M)$$

to nonsmooth varieties V by taking the L^2 -cohomology $H_{(2)}^{p+q}(V - \text{Sing } V)$ of the smooth part of V on the left and the $L^2 - \bar{\partial}$ -cohomology $H_{(2)}^{p,q}(V - \text{Sing } V)$ of the smooth part on the right. Here “ L^2 -cohomology” is in the sense of de Rham’s book [dR] and the Riemannian (or Hermitian) metric on $V - \text{Sing } V$ is that induced from the imbedding of V in projective space.

When M is smooth the alternating sum of the dimensions of the terms of type $(0, q)$ in the Hodge decomposition, the arithmetic genus of M ,

$$\chi(M) := \sum (-1)^q \dim H^{0,q}(M)$$

is a birational invariant of M . The conjectured extension of the Hodge decomposition to all varieties led MacPherson [M] to ask whether $\chi(M)$ extends to a birational invariant of all varieties.

Conjecture. *If V is a complex projective variety then*

$$\chi_{(2)}(V - \text{Sing } V) := \sum (-1)^q \dim H_{(2)}^{0,q}(V - \text{Sing } V) = \chi(X),$$

where $\pi: X \rightarrow V$ is any resolution of singularities of V .

The most prominent analytic difficulty with $L^2 - \bar{\partial}$ -cohomology is caused by the incompleteness of the metric on $V - \text{Sing } V$: there are (at least) two possible “boundary conditions,” which may be imposed in the definition of $H_{(2)}^{p,q}(V - \text{Sing } V)$. These are L^2 -versions of classical Dirichlet and Neumann conditions in the definitions of $H^{p,q}(\bar{M})$, where $\bar{M}$ is a complex manifold with boundary. It follows from our main results that they lead to different

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cohomology groups, which we denote $H_D^{p,q}(V - \text{Sing } V)$ and $H_N^{p,q}(V - \text{Sing } V)$, respectively. (The subscript “(2)” is dropped.)

Rather than identifying their alternating sums, we compute the individual terms $H_D^{0,q}(V - \text{Sing } V)$ and $H_N^{0,q}(V - \text{Sing } V)$ (but the latter only if $\dim V \leq 2$). Indeed, for smooth M , the terms, $H^{0,q}(M)$, of the Hodge decomposition are themselves birational invariants, and this property persists for arbitrary projective varieties V if one uses Dirichlet boundary conditions. This is a consequence of the first of our two main results.

Theorem A. *If V is a complex projective variety and $V - \text{Sing } V$ is given the Hermitian metric induced by any embedding of V in projective space, then the groups $H_D^{0,q}(V - \text{Sing } V)$ are birational invariants of V , and in fact, for $0 \leq q \leq n$,*

$$H_D^{0,q}(V - \text{Sing } V) \cong H^{0,q}(X),$$

where $X \rightarrow V$ is any resolution of singularities of X .

Theorem B. *For V in Theorem A, with isolated singularities, $\dim V \leq 2$, and $0 \leq q \leq 2$,*

$$H_N^{0,q}(V - \text{Sing } V) \cong H^q(X; \mathcal{O}(Z - |Z|)),$$

where $X \rightarrow V$ is a resolution of singularities of V , Z is the (unreduced) exceptional divisor, and Z is supported along a divisor with normal crossings.

Theorem B settles a conjecture made by the first author [P] in studying MacPherson's conjecture, and Theorem A settles MacPherson's conjecture itself. In case $V - \text{Sing } V$ is given a certain *complete* metric and V has only isolated singularities, L. Saper has proved Theorem A as part of his identification of intersection cohomology of V with L^2 -cohomology of $V - \text{Sing } V$ in the complete metric. (Here there is no difference between $H_N^{p,q}(V - \text{Sing } V)$ and $H_D^{p,q}(V - \text{Sing } V)$, due to the completeness of the metric.) Using arguments like those of [P], Haskell has proved Theorem A in case $\dim V \leq 2$ [H]. Using different techniques, Brüning, Peyerimhoff, and Schröder have proved the alternating sum versions of Theorems A and B, in case $\dim V = 1$ ([BPS]). Notice that Theorem B implies that the groups $H_N^{0,q}(V - \text{Sing } V)$ are not birational invariants of V .

The proof of Theorem A has two main points, together forming a bridge between cohomology of X and L^2 -cohomology of $V - \text{Sing } V$. The first is a theorem of Grauert and Riemenschneider [GR], which says that the direct image, $\pi_* \mathcal{A}_X^{n,*}$, is a fine resolution of $\pi_* \mathcal{K}_X$, where $\mathcal{A}_X^{n,*}$ is the $\bar{\partial}$ -complex of sheaves of differential forms on X and $\mathcal{K}_X$ is the sheaf of holomorphic n -forms on X . Notice that the cohomology of global sections of $\pi_* \mathcal{A}_X^{n,*}$ is $H^{n,*}(X)$. The second point is to show that the natural sheafification of the $L^2 - \bar{\partial}$ -complex on V is also a fine resolution of $\pi_* \mathcal{K}_X$; the cohomology of global sections of this complex is by definition $H_N^{n,*}(V - \text{Sing } V)$. In effect then,

$H^{n,*}(X)$ and $H_N^{n,*}(V - \text{Sing } V)$ are isomorphic because each is isomorphic to $H^*(V; \pi_* \mathcal{K}_X)$. Now Serre duality and its L^2 -version yield Theorem A.

The second point above is essentially a local vanishing theorem, first proved by Ohsawa [O1] in case V has isolated singularities and without a connection being made to the cohomology of X . The idea, which appears with essentially the same argument in an earlier paper of Demailly [D], is to approximate the incomplete metric by a sequence of complete ones, for which one already knows the local vanishing result. In this case the local vanishing is furnished by a theorem of Donnelly and Fefferman [DF], and our choice of the sequence of complete metrics uses the ideas of [O2].

Grauert and Riemenschneider's local vanishing result has a global counterpart, a generalization of the Kodaira vanishing theorem to singular varieties [GR, Satz 2.1]. An essentially immediate corollary of the proof of Theorem A and this vanishing theorem is

Corollary C. *If L is an ample line bundle on a complex projective variety V , then*

$$H_N^{n,q}(V - \text{Sing } V; \mathcal{O}(L)) = 0, \quad q > 0.$$

The analytic half of the proof of Theorem A (as distinct from the result of Grauert and Riemenschneider) makes no use of resolution of singularities. The analysis in Theorem B must do so and, in order to translate the proof to one about Dirichlet L^2 -cohomology, uses the same kind of duality as in the proof of Theorem A.

In the body of the paper Theorem A is reformulated in §1 as Theorem (1.10), which is then proved in §2, where Corollary C is also proved. Theorem B is proved in §3. Finally, an appendix by the first author corrects the proof of a crucial proposition in [P].

1. DEFINITIONS AND BASIC RESULTS

In this section we review definitions, sketch proofs showing the relation between various L^2 -cohomology groups, state the main technical result (Theorem (1.10)), and show how it implies Theorem A. We adhere to the following notational convention: If X is a complex manifold, γ will always denote a positive semidefinite hermitian metric on X , which is generically (i.e., almost everywhere) definite.

There is then a closed subset E of X on whose complement γ is positive definite and the smallest such subset is called the *degeneracy locus* of γ .

Let V be a complex projective variety and $\pi: X \rightarrow V$, a resolution of singularities such that $\pi|_{X - \pi^{-1}(\text{Sing } V)}$ is biholomorphic. Then the inverse image γ under π of the natural Kähler metric on $V - \text{Sing } V$ (induced by the imbedding of V in projective space) fulfills the conditions of the notational convention and will be the main example of it. Notice that $E := \pi^{-1}(\text{Sing } V)$ is the degeneracy locus of this γ . We will assume in this case that E is a divisor with normal crossings although this is not always necessary. We will not need

the stronger assumption made in [P] that γ be a pseudometric (of Hsiang-Pati type) until §3 (where we prove the conjecture made in [P]).

A positive semidefinite hermitian metric γ induces a pointwise inner product of (p, q) -forms α and β ,

$$\langle \alpha, \beta \rangle$$

and gives rise to an inner product on the vector space of (p, q) -forms,

$$(\alpha, \beta) := \int_X \langle \alpha, \beta \rangle d \text{Vol},$$

where $d \text{Vol}$ is the volume form of γ . (Remember that γ is assumed to be generically definite.) The resulting norm is the L^2 -norm used in the sequel.

We recall from [P] the following sheaves, where X is a complex manifold with positive semidefinite γ as above and closed set E containing the degeneracy locus of γ :

- $\mathcal{L}_\gamma^{p,q}$ = the sheaf of locally $L^2(p, q)$ -forms on X with measurable coefficients.
- $\mathcal{S}_{\gamma,E}^{p,q} := \mathcal{L}_\gamma^{p,q} \cap \mathcal{A}_{X-E}^{p,q}$, where $\mathcal{A}_{X-E}^{p,q}$ is a sheaf of smooth (p, q) -forms on $X - E$.

The corresponding $\bar{\partial}$ -complexes of sheaves on X are, for each p ,

- $(\mathcal{E}_{\gamma,E}^{p,*}, \bar{\partial}_w(X-E))$, where $\mathcal{E}_{\gamma,E}^{p,q} := \mathcal{L}_\gamma^{p,q} \cap \bar{\partial}_w(X-E)^{-1} \mathcal{L}_\gamma^{p,q+1}$ and $\bar{\partial}_w(X-E)$ is the weak $\bar{\partial}$ -operator, defined with respect to compact subsets of $X - E$. (In [P, (1.7)] the definition was with respect to compact subsets of X .)

- $(\mathcal{A}_{\gamma,E}^{p,*}, \bar{\partial})$, where $\mathcal{A}_{\gamma,E}^{p,q} := \mathcal{S}_{\gamma,E}^{p,q} \cap \bar{\partial}^{-1} \mathcal{S}_{\gamma,E}^{p,q+1}$.

The global sections of a sheaf will be denoted by the corresponding upper case latin character, e.g., $A_{\gamma,E}^{p,q}(X) = \Gamma(X, \mathcal{A}_{\gamma,E}^{p,q})$. Unless otherwise stated, E is henceforth a closed subset of X containing the degeneracy locus of γ and, when it is omitted from the notation, it is assumed minimal (i.e., equal to the degeneracy locus of γ). In case $X \rightarrow V$ is a resolution of singularities of V , E is the exceptional set and γ is induced from the resolution and the metric on $V - \text{Sing } V$ coming from its imbedding, the cohomology of $(A_{\gamma,E}^{p,*}, \bar{\partial})$ is the $L_2 - \bar{\partial}$ -cohomology as defined in [P], and denoted $H_N^{0,*}(V - \text{Sing } V)$ in the introduction.

The vector space $L_\gamma^{p,q}(X)$ is a Hilbert space with respect to the inner product $(\ , \)$. Both $A_{\gamma,E}^{p,q}(X)$ and $A_c^{p,q}(X-E)$ ($:=$ the smooth compactly supported forms on $X-E$) are dense, and the operators $\bar{\partial}: A_{\gamma,E}^{p,q}(X) \rightarrow A_{\gamma,E}^{p,q+1}(X)$ and $\bar{\partial}_c: A_c^{p,q}(X-E) \rightarrow A_c^{p,q+1}(X-E)$ are graph closable [RSI, p. 250]. Let $\bar{\partial}_N$ and $\bar{\partial}_D$ denote their closures and let $C_{\gamma,N}^{p,q} := \text{dom } \bar{\partial}_N$ and $C_{\gamma,D}^{p,q} := \text{dom } \bar{\partial}_D$ be the domains of their closures in degree (p, q) . The cohomology of the complexes $(C_{\gamma,N}^{p,q}, \bar{\partial}_N)$ and $(C_{\gamma,D}^{p,q}, \bar{\partial}_D)$ is denoted $H_{\gamma,N}^{p,q}(X-E)$ and $H_{\gamma,D}^{p,q}(X-E)$, respectively.

Remarks. (a) The subscripts N and D stand for Neumann and Dirichlet, respective. It can be shown that there is an exhaustion $\{K_n\}$ of $V - \text{Sing } V$ by complex manifolds with boundary such that the Neumann (resp. Dirichlet) conditions on forms in K_n , in the sense of [FK, 1.3.2 and p. 82], tend to the Neumann (resp. Dirichlet) conditions on L^2 -forms in $V - \text{Sing } V$.

(b) For reasonable E of real codimension ≥ 2 , $H_{\gamma, N}^{p, q}(X - E)$ and $H_{\gamma, D}^{p, q}(X - E)$ are actually independent of E , in the sense that every closed set E containing the degeneracy locus of γ produces the same cohomology (even the same underlying complex). This follows by the argument of (1.12) below. The distinction is useful, however, when comparing cohomology with respect to different metrics (see (1.4)–(1.8)).

(1.1) **Proposition.** *Let X be a complex manifold with positive semidefinite hermitian γ and closed set E containing the degeneracy locus of γ . Then for each p the complexes $(C_{\gamma, E}^{p, *}, \bar{\partial}_w(X - E))$ and $(C_{\gamma, N}^{p, *}, \bar{\partial}_N)$ are equal.*

Proof. This is the “equality of weak and strong derivatives,” which can be deduced from [H, 1.2.3].

(1.2) **Corollary.** *Let $\vartheta := -\bar{*}\bar{\partial}\bar{*}$ denote the formal adjoint of $\bar{\partial}$. Then, restricted to compactly supported forms in $X - E$, its closure is the Hilbert space adjoint of $\bar{\partial}_N$:*

$$\bar{\partial}_N^* = \vartheta_D.$$

Proof. This is a formal consequence of the preceding proposition.

(1.3) **Proposition.** *Let X and γ be as in (1.1), and let p , $0 \leq p \leq n$, be given. Suppose $H_{N, \gamma}^{p, q}(X - E)$ is finite dimensional for all q . Then*

(a) (Hodge theorem) *The (unbounded) operator $\bar{\partial}_N: L_{\gamma}^{p, q}(X) \rightarrow L_{\gamma}^{p, q+1}(X)$ and its Hilbert space adjoint $\bar{\partial}_N^*$ have closed ranges, and there is an orthogonal decomposition for each q ,*

$$L_{\gamma}^{p, q}(X) = \text{range } \bar{\partial}_N \oplus \text{range } \bar{\partial}_N^* \oplus \mathcal{H}_{\gamma, N}^{p, q},$$

where

$$\mathcal{H}_{\gamma, N}^{p, q} := \ker \bar{\partial}_N \cap \ker \bar{\partial}_N^*.$$

There is consequently an isomorphism

$$\mathcal{H}_{\gamma, N}^{p, q}(X - E) \xrightarrow{\cong} H_{\gamma, N}^{p, q}(X - E)$$

induced in the usual way.

(b) *The same holds with D and $n - p$ in place of N and p .*

(c) (duality) *There is a nonsingular pairing*

$$\{\cdot, \cdot\}: H_{\gamma, N}^{p, q}(X - E) \times H_{\gamma, D}^{n-p, n-q}(X - E) \rightarrow \mathbb{C}$$

give by

$$\{\alpha, \beta\} = \int \alpha \wedge \beta.$$

Proof. (a) See [KK, Appendix]. (b) Apply the isometry $\bar{*}$ (the star operator followed by conjugation) to the Hodge decomposition of (a) and use (1.2). (c) The star operator induces by (1.2)

$$\mathcal{H}_{\gamma, N}^{p, q} \cong \mathcal{H}_{\gamma, D}^{n-p, n-q}$$

so the vector spaces $H_{\gamma, N}^{p, q}(X - E)$ and $H_{\gamma, D}^{n-p, n-q}(X - E)$ have the same rank. An easy argument using Stokes' theorem shows the pairing $\{\cdot, \cdot\}$ exists and is well defined; so it suffices to show that if $\{\alpha, \beta\} = 0$ for all β , then $\alpha = 0$. This uses the Hodge decompositions of (a) and (b).

Here are two immediate consequences.

(1.4) **Corollary** (Smoothing of cohomology). *With the assumptions of (1.3), the inclusion of complexes*

$$(A_{\gamma, E}^{p, *}, \bar{\partial}) \hookrightarrow (C_{\gamma, N}^{p, *}, \bar{\partial}_N)$$

induces isomorphisms

$$H^q(A_{\gamma, E}^{p, *}) \rightarrow H_{\gamma, N}^{p, q}(X - E), \quad q \geq 0.$$

Proof. Each element of $\mathcal{H}_N^{p, q}$ is weakly closed and coclosed on $X - E$ (where γ is positive definite), hence is smooth on $X - E$ by elliptic regularity. Hence $H^q(A_{\gamma, E}^{p, *}) \rightarrow H_{\gamma, N}^{p, q}(X - E)$ is surjective. Similarly, if a smooth form is weakly exact, one may use the decomposition of (1.3)(a) and elliptic regularity to see that it is smoothly exact.

In case γ is induced by a resolution of singularities $X \rightarrow V$, it is easy to see that, given any hermitian metric σ on X (i.e., σ is positive definite), there is a constant $K > 0$ such that $\sigma \geq K_\gamma$. A simple computation using the definition of $\langle \cdot, \cdot \rangle$, the pointwise inner products on (p, q) -forms induced by σ and γ , shows that there are natural inclusions of complexes of sheaves

$$(1.5) \quad (\mathcal{E}_{\sigma, E}^{0, *}, \bar{\partial}_w(X - E)) \hookrightarrow (\mathcal{E}_{\gamma, E}^{0, *}, \bar{\partial}_w(X - E)),$$

$$(1.6) \quad (\mathcal{E}_{\gamma, E}^{n, *}, \bar{\partial}_w(X - E)) \hookrightarrow (\mathcal{E}_{\sigma, E}^{n, *}, \bar{\partial}_w(X - E))$$

and inclusion of complexes

$$(1.7) \quad (C_{\sigma, B}^{0, *}, \bar{\partial}_B) \hookrightarrow (C_{\gamma, B}^{0, *}, \bar{\partial}_B),$$

$$(1.8) \quad (C_{\gamma, B}^{n, *}, \bar{\partial}_B) \hookrightarrow (C_{\sigma, B}^{n, *}, \bar{\partial}_B),$$

where $B = D$ or N .

The following is immediate.

(1.9) **Proposition.** *The inclusions (1.7) for $B = D$ and (1.8) for $B = N$ induce maps in cohomology that are dual with respect to the pairings of (1.3)(c).*

The main technical result to be proved in §2 is

(1.10) **Theorem.** *The inclusion (1.8) for $B = N$ induces an isomorphism*

$$H_{\gamma, N}^{n, *}(X - E) \xrightarrow{\cong} H_{\sigma, N}^{n, *}(X - E).$$

Now this theorem is equivalent by (1.9) to

$$(1.11) \quad H_{\sigma, D}^{0, q}(X - E) \xrightarrow{\cong} H_{\gamma, D}^{0, q}(X - E), \quad q \geq 0.$$

The next proposition will say that the complex $(C_{\sigma, D}^{0, *}, \bar{\partial}_D)$ is the same as $(C_{\sigma}^{0, *}, \bar{\partial}_w)$, the standard complex of L_2 measurable forms on X with weak exterior derivative. As this is well known to have cohomology equal to $H^{0, *}(X)$, the proof of Theorem A will be complete using (1.10) and (1.4).

(1.12) **Proposition.** *Let σ denote an hermitian metric on the complex manifold X , and let $E \subseteq X$ be a divisor with normal crossings. Then there are equalities of complexes induced by natural inclusions*

$$(C_{\sigma, D}^{p, *}, \bar{\partial}_D) \xrightarrow{=} (C_{\sigma, N}^{p, *}, \bar{\partial}_N) \xrightarrow{=} (C_{\sigma, E}^{p, *}, \bar{\partial}_w(X - E)) \xleftarrow{=} (C_{\sigma}^{p, *}, \bar{\partial}_w(X)).$$

Proof. The second equality is (1.1). The third is a consequence of (1.1) above and the arguments of [P, Proposition (3.2)].

Tracing through the definitions we see that the first equality is equivalent to the following L_2 -Stokes theorem:

$$(1.13) \quad \textbf{Lemma.} \text{ For any } \alpha \in A_{\sigma, E}^{p, q}, \beta \in A_{\sigma, E}^{n-p, n-q-1}$$

$$\int_X \bar{\partial}(\alpha \wedge \beta) = 0.$$

Proof. It follows from the standard Stokes' theorem that we must show

$$(1.14) \quad \lim_{\varepsilon \rightarrow 0} \int_{\partial T(\varepsilon)} \alpha \wedge \beta = 0,$$

where $T(\varepsilon)$ is a tube of radius ε about E . According to (1.3)(a) we have

$$\alpha = \bar{\partial}_N \alpha_1 + \bar{\partial}_N^* \alpha_2 + h,$$

$$\beta = \bar{\partial}_N B_1 + \bar{\partial}_N^* \beta_2 + g,$$

where h and g are harmonic. The integral in (1.14) then breaks up into nine integrals, each of which will tend to zero (two equal zero). The idea is the same for each, so only one will be treated. Let us write $\bar{\partial}$ for $\bar{\partial}_N$.

First we have

$$(1.15) \quad \left| \int_{\partial T(\varepsilon)} \bar{\partial}^* \alpha_2 \wedge \bar{\partial} \beta_1 \right| \leq \|\bar{\partial}^* \alpha_2\|_{\partial T(\varepsilon)} \|\bar{\partial} \beta_1\|_{\partial T(\varepsilon)},$$

and since $\bar{\partial} \beta_1 \in A_{\sigma, E}^{n-q, n-p-1}$ by assumption, there is a sequence of positive real numbers $\{\varepsilon_n\}$, $\varepsilon_n \rightarrow 0$, such that

$$(1.16) \quad \|\bar{\partial} \beta_1\|_{\partial T(\varepsilon_n)} = O(\varepsilon_n^{-1/2} |\log \varepsilon_n|^{-1/2})$$

(see [C, Lemma 1.2]). Evidently, we will need a sharper bound on the other factor on the right side of (1.15).

Since $\alpha \in A_{\sigma, E}^{p, q}$, $\bar{\partial}\alpha \in A_{\sigma, E}^{p, q+1}$. Applying $\bar{\partial}$ to the Hodge decomposition of α above, we see that $\bar{\partial}^*\alpha_2 \in \text{dom } \bar{\partial}$. Hence $\bar{\partial}^*\alpha_2 \in \text{dom } \bar{\partial} \cap \text{dom } \bar{\partial}^*$. But $\bar{\partial}$ equals $\bar{\partial}_w$ from the first part of the proof. So the fact that $\bar{\partial}_w + \bar{\partial}_w^*$ is elliptic implies that the components of $\bar{\partial}^*\alpha_2$ (with respect to any local coordinates) belong to $W_2^1(X)$, the space of L_2 -functions with L_2 -derivatives. Hence a Sobolev trace estimate [F, 6.11] implies that

$$\|\bar{\partial}^*\alpha_2\|_{\partial T(\varepsilon)} = O(\text{Vol}(\partial T(\varepsilon))^{1/2}) = O(\varepsilon^{1/2}).$$

This, combined with (1.16), completes the proof.

We need to globalize this proposition to the case of complexes used in [P, 1.8]. The proof of the result stated below is essentially the same as that of (1.12) and is omitted.

Let F be a divisor on X , given locally by the meromorphic function f . For $U \subseteq X$, let

$$\mathcal{A}^{p, q}(U; \mathcal{O}(F)) = \{\omega \in A^{p, q}(U \cap (X - E)) \mid f\omega \in A^{p, q}(X)\}.$$

(The sheaves $\mathcal{A}^{p, q}(\mathcal{O}(F))$ are canonically isomorphic to $\mathcal{A}^{p, q} \otimes \mathcal{O}(F)$.) Similarly define the sheaves $\mathcal{L}^{p, q}(\mathcal{O}(F))$ by replacing $A^{p, q}(U)$ with the measurable locally $L^2(p, q)$ -forms on U . Here square-summability is with respect to any Hermitian metric σ on X . Next define the complexes $C_{\sigma, D}^{p, *}(\mathcal{O}(F))$ and $C_{\sigma, N}^{p, *}(\mathcal{O}(F))$ by analogy with $C_{\sigma, D}^{p, *}$ and $C_{\sigma, N}^{p, *}$; and finally define $C_{\sigma, E}^{p, *}(\mathcal{O}(F))$ and $C_{\sigma}^{p, *}(\mathcal{O}(F))$, using the weak $\bar{\partial}$ on $X - E$ and X , respectively. The Hodge decomposition, for Dolbeault cohomology with coefficients in $\mathcal{O}(F)$, implies that the inclusion $(A^{p, *}(\mathcal{O}(F)), \bar{\partial}) \hookrightarrow (C^{p, *}(\mathcal{O}(F)), \bar{\partial}_w)$ induces isomorphisms $H^{p, q}(X; \mathcal{O}(F)) \xrightarrow{\cong} H^q(C^{p, *}(\mathcal{O}(F)))$ so any of the complexes in the following proposition computes $H^{p, *}(X; \mathcal{O}(F))$.

(1.17) **Proposition.** *Given X , σ , and E as in (1.12), and a divisor F on X , there are equalities of complexes for each p , induced by inclusions*

$$\begin{aligned} (C_{\sigma, D}^{p, *}(\mathcal{O}(F)), \bar{\partial}_D) &\xrightarrow{\cong} (C_{\sigma, N}^{p, *}(\mathcal{O}(F)), \bar{\partial}_N) \xrightarrow{\cong} (C_{\sigma, E}^{p, *}(\mathcal{O}(F)), \bar{\partial}_w(X - E)) \\ &\xleftarrow{=} (C_{\sigma}^{p, *}(\mathcal{O}(F)), \bar{\partial}_w). \end{aligned}$$

2. PROOF OF THEOREM (1.10)

Fix a resolution of singularities $\pi: (X, E) \rightarrow (V, \text{Sing } V)$, where E is a divisor with normal crossings. Let γ denote the induced pseudometric on X , and let σ be any hermitian metric on X . The crux of Theorem A and the

main point in the proof of Theorem (1.10) is the following:

(2.1) **Proposition.** *The complexes $\{\pi_* \mathcal{E}_{\gamma,E}^{n,q}, q \geq 0\}$ and $\{\pi_* \mathcal{E}_{\sigma,E}^{n,q}, q \geq 0\}$ are fine resolutions of $\pi_* \mathcal{K}_X$, where $\mathcal{K}_X$ is the sheaf of holomorphic n -forms on X .*

Proof. Since the sheaves $\mathcal{E}_{\sigma,E}^{n,q}$ are fine on X , their direct images $\pi_* \mathcal{E}_{\sigma,E}^{n,q}$ are fine on V . To prove that $\{\pi_* \mathcal{E}_{\sigma,E}^{n,q}, q \geq 0\}$ is an exact sequence of sheaves, recall first from (1.1) that it equals the complex $\{\pi_* \mathcal{E}_{\sigma}^{n,q}, q \geq 0\}$ where the differential is $\bar{\partial}_w$. However, since $\{\mathcal{E}_{\sigma}^{n,q}, q \geq 0\}$ is a fine resolution of $\mathcal{K}_X$, the sheaf cohomology groups $\mathcal{H}^q(\pi_* \mathcal{E}_{\sigma}^{n,*})$ equal the higher direct images $R^q \pi_* \mathcal{K}_X$. But these vanish for $q \geq 1$ by a theorem of H. Grauert and O. Riemenschneider [GR, 2.3], so $\{\pi_* \mathcal{E}_{\sigma,E}^{n,q}, q \geq 0\}$ is a resolution of $\pi_* \mathcal{K}_X$ as claimed.

If $U \subseteq V$ is open, then the complex $\{(\pi_* C_{\gamma,E}^{n,q}(U), q \geq 0\}$ consists of locally L^2 -forms ω with measurable coefficients on U (L^2 on every compact subset of U , with respect to the metric induced by the imbedding of V in projective space), such that (the weak) $\bar{\partial}_w \omega$ is locally L^2 , where $\bar{\partial}_w$ is the weak $\bar{\partial}$ -operator with respect to compact subsets of $U - U \cap \text{Sing } V$.

To prove that $\pi_* \mathcal{E}_{\gamma,E}^{n,q}$ is fine it is enough to show that for any open $W \subseteq \bar{W} \subseteq U$, there is a smooth function ρ supported in U and identically $= 1$ in W , such that $\langle \bar{\partial} \rho, \bar{\partial} \rho \rangle$ is a bounded function on U , where $\langle \bar{\partial} \rho, \bar{\partial} \rho \rangle$ denotes the pointwise length of the 1-form $\bar{\partial} \rho$ in the induced metric.

To prove this, and for use below, we make the following simple remark. Given a Riemannian manifold M and a submanifold S endowed with the induced metric, there is an orthogonal decomposition of the tangent space to M at each $s \in S$

$$T_s M = T_s S \oplus (T_s S)^\perp.$$

Consequently, the restriction map $T_s^* M \rightarrow T_s^* S$ and the corresponding restriction maps of exterior powers are all distance-decreasing. Hence, to establish a pointwise upper bound for a form on S , we will establish one for a smooth extension of the form to M .

As a first application of this remark, note that a smooth function ρ exists with the required properties in the ambient projective space. Hence its restriction to U has the required pointwise bound.

To prove that $\pi_* \mathcal{E}_{\gamma,E}^{n,*}$ is a resolution of $\pi_* \mathcal{K}_X$, we need the following lemmas due to Demailly [D], Ohsawa [O1], and Donnelly and Fefferman [DF].

(2.2) **Lemma** [DF, 1.1, 2.1]. *Let N be a complete Kähler manifold of dimension n , whose Kähler metric ω is given by a potential function $F: N \rightarrow \mathbb{R}$ ($\omega = -i\partial\bar{\partial}F$) such that $\langle \partial F, \partial F \rangle$ is bounded. Then the $L_2 - \bar{\partial}$ -cohomology with respect to ω , $H_{(2)}^{p,q}(N, \omega) = 0$, $p + q \neq n$. In fact, if $\langle \partial F, \partial F \rangle^{1/2} \leq B$, and ϕ is a (p, q) -form on N with $\bar{\partial}\phi = 0$, $q > 0$ and $p + q \neq n$, then there is a $(p, q - 1)$ -form ν such that $\bar{\partial}\nu = \phi$ and $\|\nu\| \leq 4B\|\phi\|$.*

In [O1, 1.1] Ohsawa gives a simple and elegant proof of this lemma using Kähler identities. Let us say $H^{p,q}(N)$ *vanishes with an estimate* if the condition of the last sentence in (2.2) is satisfied for some $B > 0$.

(2.3) **Lemma** [D, Theorem 4.1; O1, Proposition 4.1]. *Let N be a complex manifold of dimension n with a decreasing sequence of complete hermitian metrics h_k , $k \geq 1$, which converges pointwise to a hermitian metric h . If $H_{(2)}^{n,q}(N, h_k)$ vanishes with an estimate that is independent of k , then $H_{(2)}^{n,q}(N, h)$ vanishes with an estimate. (Here $H_{(2)}^{n,q}(N, h_k)$ denotes $L_2 - \bar{\partial}$ -cohomology with respect to the metric h_k .)*

Proof. An easy calculation shows that for $l \geq k$, $\|\phi\|_k \leq \|\phi\|_l \leq \|\phi\|$ for any (n, q) -form ϕ , where $\|\cdot\|$ (resp. $\|\cdot\|_k$) denotes the norm with respect to h (resp. h_k). Hence if $\bar{\partial}\phi = 0$ and $\|\phi\| < \infty$, there exists for each $k \geq 1$ and $(n, q-1)$ -form ν_k with $\bar{\partial}\nu_k = \phi$ and $\|\nu_k\|_k \leq B\|\phi\|_k$ for some $B > 0$, independent of k .

If K is a compact subset of N and ψ is a form, then we denote the norm of ψ restricted to K , with respect to h_k , by $\|\psi\|_{k,K}$. For fixed k and K and $l \geq k$, we have

$$\|\nu_l\|_{k,K} \leq \|\nu_l\|_k \leq \|\nu_l\|_l \leq B\|\phi\|_l \leq B\|\phi\|.$$

For this same fixed k and K , $h_k \leq Ch$ on K , where $C > 0$ depends in general on k and K . Hence, there is a constant $D > 0$, depending on C , such that

$$\|\nu_l\|_K \leq D\|\nu_l\|_{k,K} \leq DB\|\phi\|$$

for all $l \geq k$. We can pick a (single) subsequence $\{\nu_{k(i)}\}$ of $\{\nu_k\}$ that is weakly convergent in each Hilbert space $L^{n,q-1}(K, h)$ of $L^2 - (n, q-1)$ -forms on K with respect to h . Hence we get ν , the weak limit of $\{\nu_{k(i)}\} \in L^{n,q-1}(N, h; \text{loc})$, the locally $L^2 - (n, q-1)$ -forms on N with respect to h . Moreover, the weak $\bar{\partial}$ of ν is ϕ because for every smooth compactly supported $\psi \in A_c^{n,q}(N)$,

$$\begin{aligned} (\nu, \bar{\partial}\psi) &= \lim(\nu_{k(i)}, \bar{\partial}\psi) \quad (\text{by weak convergence on compact sets}) \\ &= \lim(\bar{\partial}\nu_{k(i)}, \psi) \quad (\text{Stokes' theorem}) \\ &= (\phi, \psi). \end{aligned}$$

Finally, ν is in $L^{n,q-1}(N, h)$, since we saw above that $\|\nu_l\|_{k,K} \leq B\|\phi\|$, so letting $l \rightarrow \infty$ we get $\|\nu\|_{k,K} \leq B\|\phi\|$. Next letting $k \rightarrow \infty$, $\|\nu\|_K \leq B\|\phi\|$, and hence $\|\nu\| \leq B\|\phi\|$ the claimed vanishing of $H_{(2)}^{n,q}(N, h)$ with an estimate. (It is in this last step that we have used the fact that B is independent of k .)

Now let $U \subseteq V$ be the intersection with V of an open ball in the ambient space. We work locally here so we assume the ambient space is $\mathbb{C}^L$, $L \gg n = \dim V$. Assume the ball B_c has very small radius $c > 0$ and is centered at a singular point of V . (If U does not intersect $\text{Sing } V$, the conclusion (2.6) of

the following discussion is well known.) Let $\{f_1, f_2, \dots, f_m\}$ be polynomials in $\mathbb{C}^L$ whose common zero set in B_c is $U \cap \text{Sing } V$. Set

$$F = -\log(c^2 - |z|^2)$$

and

$$F_k = -\log(c^2 - |z|^2) - (1/k) \log(-\log \sum |f_i|^2),$$

for $z \in B_c$ and $k > 1$, where c is so small that $\sum |f_i|^2 \ll 1$ on B_c .

(2.4) **Lemma.** (a) *The metric $h_k := -i\partial\bar{\partial}F_k$ on $U - U \cap \text{Sing } V$ is complete and decreases monotonically to $h := -i\partial\bar{\partial}F$, pointwise on $U - U \cap \text{Sing } V$.*

(b) *$\langle \partial F_k, \partial F_k \rangle_k$ is bounded, independently of k , where $\langle \cdot, \cdot \rangle_k^{1/2}$ denotes the pointwise norm on 1-forms with respect to h_k .*

Proof. Part (a) is an easy calculation. To prove (b), begin by calculating

$$\begin{aligned} \partial\bar{\partial}F_k &= \frac{\partial\bar{\partial}|z|^2}{c^2 - |z|^2} \\ &+ \frac{\partial|z|^2 \wedge \bar{\partial}|z|^2}{(c^2 - |z|^2)^2} + \frac{1}{k} \left(\frac{\partial\bar{\partial} \sum |f_i|^2}{-\sum |f_i|^2 \log \sum |f_i|^2} - \frac{\partial \sum |f_i|^2 \wedge \bar{\partial} \sum |f_i|^2}{-(\sum |f_i|^2)^2 \log \sum |f_i|^2} \right. \\ &\quad \left. + \frac{\partial \sum |f_i|^2 \wedge \bar{\partial} \sum |f_i|^2}{(\sum |f_i|^2)^2 \log^2 \sum |f_i|^2} \right) \end{aligned}$$

and

$$\partial F_k = \frac{\partial|z|^2}{c^2 - |z|^2} + \frac{1}{k} \left(\frac{\partial \sum |f_i|^2}{-\sum |f_i|^2 \log \sum |f_i|^2} \right).$$

To proceed further we let $\langle \cdot, \cdot \rangle$ denote the squared pointwise norm on 1-forms with respect to h and $\langle \cdot, \cdot \rangle'_k$, the squared pointwise norm on 1-forms with respect to

$$\frac{\partial\bar{\partial}|z|^2}{c^2 - |z|^2} + \frac{1}{k} \frac{\partial \sum |f_i|^2 \wedge \bar{\partial} \sum |f_i|^2}{(\sum |f_i|^2)^2 \log^2 \sum |f_i|^2}.$$

Note that $\langle \cdot, \cdot \rangle_k \leq \langle \cdot, \cdot \rangle'_k$, since the sum of the first two terms, inside the parentheses in the calculation of $\partial\bar{\partial}F_k$, is positive semidefinite. This is used to establish the second inequality in

$$\begin{aligned} \langle \partial F_k, \partial F_k \rangle_k^{1/2} &\leq \left\langle \frac{\partial|z|^2}{c^2 - |z|^2}, \frac{\partial|z|^2}{c^2 - |z|^2} \right\rangle_k^{1/2} \\ &+ \frac{1}{k} \left\langle \frac{\partial \sum |f_i|^2}{-\sum |f_i|^2 \log \sum |f_i|^2}, \frac{\partial \sum |f_i|^2}{-\sum |f_i|^2 \log \sum |f_i|^2} \right\rangle_k^{1/2} \\ &\leq \left\langle \frac{\partial|z|^2}{c^2 - |z|^2}, \frac{\partial|z|^2}{c^2 - |z|^2} \right\rangle^{1/2} \\ &+ \frac{1}{k} \left\langle \frac{\partial \sum |f_i|^2}{-\sum |f_i|^2 \log \sum |f_i|^2}, \frac{\partial \sum |f_i|^2}{-\sum |f_i|^2 \log \sum |f_i|^2} \right\rangle_k^{1/2}. \end{aligned}$$

At this point we use the remark preceding the proof of fineness of $\pi_* C_{\gamma, E}^{n, q}$ and make the remaining calculations in the ambient space $\mathbb{C}^L$ where the desired bound will follow from the easy

(2.5) **Sublemma.** *Let $a = (a_1, \dots, a_L) \in \mathbb{C}^L$ and $\lambda \in \mathbb{R}_+$. Then viewing a as a 1-by- L matrix,*

$$a(\lambda I_L + (a_i \bar{a}_j))^{-1} \bar{a}^t \leq 1.$$

Now combining our three lemmas, (2.2), (2.3), and (2.4) we see that the $L^2 - \bar{\partial}$ -cohomology with respect to h , $H^{n, q}(U - U \cap \text{Sing } V, h) = 0$, for $q \geq 1$. But on any U' with $\bar{U}' \subseteq U$, h is quasi-isometric to the metric induced by the imbedding $U' - U' \cap \text{Sing } V \subseteq \mathbb{C}^N$. Thus, we reach the end of the first half of the proof of $\pi_* \mathcal{E}_{\gamma, E}^{n, q}$ is a resolution of $\pi_* \mathcal{K}_X$, having shown that

$$(2.6) \quad \mathcal{K}^q(\pi_* \mathcal{E}_{\gamma, E}^{n, *}) = 0, \quad q \geq 1.$$

It remains to show that

$$\mathcal{K}^0(\pi_* \mathcal{E}_{\gamma, E}^{n, *}) = \pi_* \mathcal{K}_X.$$

This follows from the fact that a holomorphic n -form ϕ on the intersection of an open set U of X with $X - E$, which is locally L^2 with respect to γ , is actually holomorphic on U . For the L^2 -condition is

$$\int_K \phi \wedge \bar{\phi} < \infty, \quad \text{for all compact } K \subseteq U,$$

which forces the Laurent expansion of ϕ , in a neighborhood of any point of E , to have no terms of negative degree.

We are now ready to prove Theorem (1.10). Because of the equality of complexes in (1.1) we have a commutative diagram

$$\begin{array}{ccc} H_{\gamma, N}^{n, q}(X - E) & \longrightarrow & H_{\sigma, N}^{n, q}(X - E) \\ \cong \downarrow & & \cong \downarrow \\ H^q(\Gamma(V, \pi_* \mathcal{E}_{\gamma, E}^{n, *})) & \longrightarrow & H^q(\Gamma(V, \pi_* \mathcal{E}_{\sigma, E}^{n, *})) \end{array}$$

with horizontals induced by (1.8) with $B = N$, and by (1.6).

But Proposition (2.1) says there is a commutative diagram

$$\begin{array}{ccc} H^q(\Gamma(V, \pi_* \mathcal{E}_{\gamma, E}^{n, *})) & \longrightarrow & H^q(\Gamma(V, \pi_* \mathcal{E}_{\sigma, E}^{n, *})) \\ \cong \downarrow & & \cong \downarrow \\ H^q(V; \pi_* \mathcal{K}_X) & \xrightarrow{=} & H^q(V; \pi_* \mathcal{K}_X). \end{array}$$

The proof of Theorem (1.10) is complete.

To prove Corollary C, the global vanishing theorem, we observe that an ample line bundle L on V is quasi-positive in the sense of [GR, p. 266]: there is a

smooth hermitian metric on L that has positive curvature on a smooth dense subset of V .

On the other hand, sections of the sheaf $\pi_* \mathcal{E}_{\gamma, E}^{n, q} \otimes \mathcal{O}(L)$ admit an inner product in the usual way, taking into account the metric on L (cf. [W, p. 109] and the discussion following the proof of (2.1)). Since L is locally trivial and its metric is smooth (hence locally quasi-isometric to a constant metric) the arguments of Lemma (2.3) carry over to show that

$$H_N^{n, q}(V - \text{Sing } V; \mathcal{O}(L)) \cong H^{0, q}(V; \pi_* \mathcal{K}_X \otimes \mathcal{O}(L)),$$

where the left side is by definition the global section cohomology of $\pi_* \mathcal{E}_{\gamma, E}^{n, q} \otimes \mathcal{O}(L)$. But the right side vanishes by [GR, Satz 2.1], so Corollary C is proved.

3. PROOF OF THEOREM B

Here and below we retain the notation of §1. In addition, $\pi: (X, E) \rightarrow (V, \text{Sing } V)$ is now a resolution of singularities of a normal complex projective surface V , E is a divisor with normal crossings, and γ is a pseudometric of Hsiang-Pati type as defined in [P, 1.5]. In [P] it was shown that there is an injection

$$(3.1) \quad H^1(X; \mathcal{O}(Z - |Z|)) \rightarrow H_{\gamma, N}^{0, 1}(X - E)$$

and a surjection

$$(3.2) \quad H^2(X; \mathcal{O}(Z - |Z|)) \rightarrow H_{\gamma, N}^{0, 2}(X - E),$$

where $Z = \pi^{-1}(\text{Sing } V)$ is the unreduced exceptional divisor. Instead of proving directly that the first of these is a surjection and the second is an injection, we dualize to the following statement, using (1.3)(c).

(3.3) **Proposition.** *The duals of (3.1) and (3.2) are isomorphisms*

$$\begin{aligned} \alpha: H_{\gamma, D}^{2, 1}(X - E) &\xrightarrow{\cong} H^{2, 1}(X; \mathcal{O}(|Z| - Z)), \\ \beta: H_{\gamma, D}^{2, 0}(X - E) &\xrightarrow{\cong} H^{2, 0}(X; \mathcal{O}(|Z| - Z)). \end{aligned}$$

Proof. Rather than taking α and β to be the duals of (3.1) and (3.2), we need to define them from a map of complexes; we then leave to the reader the easy verification that α and β so defined are dual to (3.1) and (3.2) in the sense of (1.3)(c).

First note that the elements of the complex $C_{\gamma, D}^{2, *}$ (the cohomology of which is $H_{\gamma, D}^{2, *}(X - E)$) are not in general smooth near E , so to define α or β we need a complex of differential forms whose cohomology is $H^{2, *}(X; \mathcal{O}(|Z| - Z))$ and whose elements are equally nonsmooth near E . It is for this purpose that the complex $C_{\sigma, D}^{p, *}(\mathcal{O}(F))$ and its relatives were introduced at the end of §1. Their use here is justified by (1.17). The inclusions in the following lemma induce α and β .

(3.4) **Lemma.** *There is an inclusion of complexes $C_{\gamma,D}^{2,*} \rightarrow C_{\sigma,D}^{2,*}(\mathcal{O}(|Z| - Z))$.*

Proof. The proof that $L_{\gamma}^{2,*}(X) \subseteq L_{\sigma}^{2,*}(X; \mathcal{O}(|Z| - Z))$ is trivial for $* = 2$, while for $* = 1$ it is similar to [P, (2.4)]; this gives what we want for $* \geq 1$. If it were true that $L_{\gamma}^{2,0}(X) \subseteq L_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z))$ the lemma would be proved (and would not even need a proof). However, the inclusion for $* = 0$ is in fact reversed and what must be shown is that the domain condition gives what we want,

$$C_{\gamma,D}^{2,0} \subseteq C_{\sigma,D}^{2,0}(\mathcal{O}(|Z| - Z)).$$

So let $\psi \in C_{\gamma,D}^{2,0}$. This means that there is a sequence $\{\psi_k\}$ of smooth $(2, 0)$ -forms, compactly supported in $X - E$, and convergent in $L_{\gamma}^{2,0}(X)$ to ψ , such that $\bar{\partial}\psi_k$ converges to some ϕ in $L_{\gamma}^{2,1}(X)$. We have the following version of the Hodge decomposition (1.3).

$$L_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z)) = \text{range } \bar{\partial}_D^* \oplus \mathcal{H}_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z)).$$

As each ψ_k has compact support in $X - E$, it lies in $L_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z))$. Let

$$\psi_k = \psi_k^* + h_k,$$

where $\psi_k^* \in \text{range } \bar{\partial}_D^*$ and $h_k \in \mathcal{H}_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z))$. The finite dimensionality of $H^{2,1}(X; \mathcal{O}(|Z| - Z))$ implies that the range of $\bar{\partial}_D$ is closed, so by [H1, Theorem 1.1], there is a constant $C > 0$ such that

$$\|\eta\| \leq C \|\bar{\partial}_D \eta\|$$

for all $\eta \in \text{dom } \bar{\partial}_D \cap \text{range } \bar{\partial}_D^*$. In particular, since $\{\bar{\partial}_D \psi_k^*\} = \{\bar{\partial}_D \psi_k\}$ is Cauchy, so is $\{\psi_k^*\}$. Thus, $\{\psi_k^*\}$ converges in $L_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z))$. Now $\{h_k\}$ converges in $L_{\sigma}^{2,0}(X)$ because ψ_k converges in $L_{\gamma}^{2,0}(X)$, ψ_k^* converges in $L_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z))$ and $L_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z)) \subseteq L_{\sigma}^{2,0}(X) = L_{\gamma}^{2,0}(X)$. However, the inclusion $\mathcal{H}_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z)) \hookrightarrow \mathcal{H}_{\sigma}^{2,0}(X)$ is bounded (all inner products on a finite-dimensional vector space are equivalent), so $\{h_k\}$ must converge in $\mathcal{H}_{\sigma}^{2,0}(X; \mathcal{O}(|Z| - Z))$ as well. This means that $\psi = \lim \psi_k = \lim \psi_k^* + \lim h_k^*$ is in $C_{\sigma,D}^{2,0}(X; \mathcal{O}(|Z| - Z))$, which was to be shown.

Returning to the proof of Proposition (3.3), we suppose $\alpha[\phi] = 0$, where $[\phi]$ is the class of an element $\phi \in C_{\gamma,D}^{2,1}$. To show $[\phi] = 0$, we need to produce a sequence of $(2, 0)$ forms $\{\psi_k\}$, compactly supported in $X - E$, such that $\{\psi_k\}$ converges in $L_{\gamma}^{2,0}$ and $\bar{\partial}\psi_k \rightarrow \phi$ in $L_{\gamma}^{2,1}$. This will be done in the most straightforward manner: letting $\bar{\partial}\psi = \phi$, where $\psi \in C_{\sigma,D}^{2,0}(\mathcal{O}(|Z| - Z))$, we will prove that $\bar{\partial}(\mu_k \psi) \rightarrow \phi$ in $L_{\gamma}^{2,1}$ for a suitable sequence of cut-off functions $\{\mu_k\}$; from our observation above that $L_{\sigma}^{2,0}(\mathcal{O}(|Z| - Z)) \subseteq L_{\gamma}^{2,0}(X)$, it will be obvious that $\mu_k \psi \rightarrow \psi$ in $L_{\gamma}^{2,0}$.

To construct the sequence $\{\mu_k\}$ we work in the variety V , rather than in its resolution X . Suppose for simplicity that E contracts to a point, assumed to be the origin of an affine coordinate system $(z_1, \dots, z_N)$ for a part of the ambient space containing V . Let $\rho_k: \mathbb{R}^1 \rightarrow [0, 1]$, $k \geq 1$, be a smooth function such that

$$\rho_k(x) = \begin{cases} 1, & x \leq k, \\ 0, & x \geq k+1, \end{cases} \quad \text{and} \quad |\rho'_k(x)| \leq 2$$

for all x and all k . Let $r: \mathbb{C}^N \rightarrow [0, 1/2]$ be a smooth increasing function of $|z|^2$ such that for $z = (z_1, \dots, z_N)$ and $|z|^2 = \sum |z_i|^2$,

$$r(z) = \begin{cases} |z|^2, & |z|^2 < \frac{1}{4}, \\ \frac{1}{2}, & |z|^2 \geq \frac{1}{4}. \end{cases}$$

Finally, $\mu_k: \mathbb{C}^N \rightarrow [0, 1]$ is defined to be

$$\mu_k(z) = \rho_k(\log(-\log r(z)))$$

and has the properties

$$(3.5) \quad \begin{aligned} (a) \quad \mu_k(z) &= \begin{cases} 1, & r \geq e^{-e^k}, \\ 0, & r \leq e^{-e^{k+1}}; \end{cases} \\ (b) \quad \langle \bar{\partial} \mu_k, \bar{\partial} \mu_k \rangle &\leq \frac{2\chi_k(|z|^2)}{|z|^2 \log^2(|z|^2)}, \end{aligned}$$

where χ_k is the characteristic function of the interval $[e^{-e^{k+1}}, e^{-e^k}]$, and $\langle \cdot \rangle$ is the Euclidean inner product of $(0, 1)$ -forms. Without any change in notation, we regard the μ_k as smooth functions on X , defined on a neighborhood of E in X using the resolution and constant outside the neighborhood.

To finish the proof of the first part of (3.3) it suffices, because of (3.5)(a), to prove the following lemma.

(3.6) **Lemma.** *If $\psi \in C_{\sigma, D}^{2,0}(\mathcal{O}(|Z| - Z))$ then $\bar{\partial} \mu_k \wedge \psi \rightarrow 0$ in $L_{\gamma}^{2,1}(X)$.*

Proof. Since E is compact and $\bar{\partial} \mu_k$ is supported in sets tending to E as $k \rightarrow \infty$, it suffices to show that $\|\bar{\partial} \mu_k \wedge \psi\|_U \rightarrow 0$, for a covering of E by open sets U of X . Let U be a small open set containing a norm crossing point of E ; the case where U is disjoint from the crossings is similar and easier.

We may assume there are integers $m_1, m_2 \geq 1$ (the multiplicities in Z of the components of E passing through U) such that

$$|z|^2 = |u|^{2m_1} |v|^{2m_2} B(u, v)$$

in U where $E \cap U = \{uv = 0\}$ and $0 < A^{-1} \leq B(u, v) \leq A$ for some constant A (depending on U). This is easy to see directly, or is immediate from the more detailed information in [HP (12)]. For the remainder of the proof of the lemma it will be necessary to distinguish notationally the pointwise norms

$\langle \cdot, \cdot \rangle_\sigma$ and $\langle \cdot, \cdot \rangle_\gamma$, as well as the volume forms dV_σ and dV_γ . We thus have in U ,

$$\langle \bar{\partial}\mu_k, \bar{\partial}\mu_k \rangle_\gamma \leq \frac{2AX_k(u, v)}{|u|^{2m_1}|v|^{2m_2} \log^2(A|u|^{2m_1}|v|^{2m_2})},$$

where $X_k(u, v)$ is the characteristic function of $\{(u, v) \in U \mid |u|^{2m_1}|v|^{2m_2} \in B(u, v) \in [e^{-e^{k+1}}, e^{-e^k}]\}$.

Let $t = |uv|$ in U and suppose there is a constant $C > 0$ such that the integrals over boundaries of tubes about E satisfy the “trace estimate”

$$(3.7) \quad \int_{\partial T_t} \langle u^{-(m_1-1)} v^{-(m_2-1)} \psi, u^{-(m_1-1)} v^{-(m_2-1)} \psi \rangle_\sigma dV_\sigma(t) \leq Ct,$$

where $\partial T(t) = \{(u, v) \in U \mid |uv| = t\}$ and $dV_\sigma(t)$ is the volume form on $\partial T(t)$ of the metric induced by restriction from our metric σ . Then we have, for some sequences of positive real numbers $s_k > t_k$, tending to zero as $k \rightarrow \infty$,

$$\begin{aligned} \|\bar{\partial}\mu_k \wedge \psi\|_U &:= \int_U \langle \bar{\partial}\mu_k \wedge \psi, \bar{\partial}\mu_k \wedge \psi \rangle_\gamma dV_\gamma \\ &\leq \int_U \langle \bar{\partial}\mu_k, \bar{\partial}\mu_k \rangle_\gamma \langle \psi, \psi \rangle_\gamma dV_\gamma \\ &\leq 2A \int_U \frac{X_k(u, v)}{|u|^2|v|^2 \log^2(A|u||v|)} \langle u^{-(m_1-1)} v^{-(m_2-1)} \psi, u^{-(m_1-1)} v^{-(m_2-1)} \psi \rangle_\gamma dV_\gamma \\ &= 2A \int_U \frac{X_k(u, v)}{|u|^2|v|^2 \log^2(A|u||v|)} \langle u^{-(m_1-1)} v^{-(m_2-1)} \psi, u^{-(m_1-1)} v^{-(m_2-1)} \psi \rangle_\sigma dV_\sigma \\ &\leq 2A \int_{t_k}^{s_k} \frac{dt}{t^2 \log^2(At)} \int_{\partial T(t)} \langle u^{-(m_1-1)} v^{-(m_2-1)} \psi, u^{-(m_1-1)} v^{-(m_2-1)} \psi \rangle_\sigma dV_\sigma(t) \\ &\leq 2AC \int_{t_k}^{s_k} \frac{dt}{t \log^2(At)} = 2AC(\log^{-1}(At_k) - \log^{-1}(As_k)), \end{aligned}$$

which tends to zero as $k \rightarrow \infty$. (The equality above holds because for any (2.0) form ϕ , the expression $\langle \phi, \phi \rangle dV$ is independent of metric.) Hence it remains to prove the trace estimate (3.7).

Recall that by definition of $C_\sigma^{2,0}(\mathcal{O}(|Z| - Z))$, $u^{-(m_1-1)} v^{-(m_2-1)} \psi \in L_\sigma^{2,0}(U)$. Moreover $\bar{\partial}(u^{-(m_1-1)} v^{-(m_2-1)} \psi) = u^{-(m_1-1)} v^{-(m_2-1)} \phi$, which lies in $u^{-(m_1-1)} v^{-(m_2-1)} L_\gamma^{2,1}(U) \subseteq u^{-(m_1-1)} v^{-(m_2-1)} L_\sigma^{2,1}(U; \mathcal{O}(|Z| - Z)) = L_\sigma^{2,1}(U)$. By [H2, Theorem 4.2.5], we find that $u^{-(m_1-1)} v^{-(m_2-1)} \psi \in W_\sigma^{2,0}(U)$, the Sobolev space of forms with at least one L_2 -derivative. The estimate (3.7) is now immediate from [F, 6.11, 6.17].

To complete the proof of Proposition (3.3) we show that β is surjective. If $\psi \in \Omega^2(X; \mathcal{O}(|Z| - Z))$, then for the sequence of cut-offs $\{\mu_k\}$ constructed above, we claim

$$\mu_k \psi \rightarrow \psi \quad \text{and} \quad \bar{\partial}(\mu_k \psi) \rightarrow 0$$

in $L_{\gamma}^{2,0}$ and $L_{\gamma}^{2,1}$, respectively. The first limit obviously holds, while for the second we observe that ψ satisfies the trace estimate of (3.7) (for the same reasons) and so the proof that $\bar{\partial}\mu_k \wedge \psi \rightarrow 0$ in $L_{\gamma}^{2,1}$ is the same as that given above.

APPENDIX: PROOF OF (4.5) IN [P]

The statement of Proposition (4.5) in [P] is correct, but not its proof. The error is in line (–4) on p. 181: the inequality (4.6) is correct, but in general gives the vanishing of $H^0(\mathcal{O}_{n_{k-1}E_{k-1}}(D - N_{k-1}))$ only when $n_{k-1} = 1$. (There is also a misprint: replace k by $k - 1$ in its first two appearances on line (–4).) An entirely different proof will be given below. All references (.) will be to [P].

Because of the first part of (4.2), the assertion of (4.5) amounts to the injectivity of

$$H^1(X; \mathcal{O}(Z - E)) \rightarrow H^1(X; \mathcal{O}(Z - E + N)),$$

where $N = D_{\gamma} - Z + E$. We will show that

- (a) $H^1(X; \mathcal{O}(Z - E)) \rightarrow H^1(X; \mathcal{O}(Z))$ is injective,
- (b) $H^1(X; \mathcal{O}(Z)) \rightarrow H^1(X; \mathcal{O}(D))$ is injective for any $D \geq Z$.

Proof of (a). There is a holomorphic function h on a neighborhood U of E such that $(h) = Z + R$, where R is an effective divisor with support transverse to E . (See [G].) We have, for each component E_i of E ,

$$O = E_i \cdot (h) = E_i \cdot Z + E_i \cdot R \geq E_i \cdot Z,$$

because R is effective and transverse to E_i . By the negative-definiteness of $(E_i \cdot E_j)$, $E_k \cdot Z < 0$, for some k ; say $E_1 \cdot Z < 0$. If $E_2 \cap E_1 \neq \emptyset$, then $E_2 \cdot E_1 > 0$ so $E_2 \cdot (Z - E_1) < 0$. Continue in this way to find E_k such that $E_k \cdot (Z - (E_1 + \cdots + E_{k-1})) < 0$ for all k . This is possible because E is connected. Now use the exactness for $k = 1, 2, \dots$ of

$$\begin{aligned} H^0(X; \mathcal{O}_{E_k}(Z - E_1 - \cdots - E_{k-1})) &\rightarrow H^1(X; \mathcal{O}(Z - E_1 - \cdots - E_k)) \\ &\rightarrow H^1(X; \mathcal{O}(Z - E_1 - \cdots - E_{k-1})) \end{aligned}$$

and the vanishing of the left term to get (a). (This is the “correct case” of [P, p. 181, line –4]: (4.6) does imply vanishing of the H^0 when $n_k = 1$ —although the D and n_i ’s in (4.6) differ in this application.)

Proof of (b). Let ϕ be a 1-form representing an element of the kernel of $H^1(X; \mathcal{O}(Z)) \rightarrow H^1(X; \mathcal{O}(D))$. We represent ϕ as in [P] (see (1.9)(b)): $\phi \in \mathcal{A}_{|E|}^{0,1}(X) \otimes \mathcal{O}(Z)$ and (by our assumption), $\phi = \bar{\partial}f$, where $f \in \mathcal{A}_{|E|}^{0,0}(X) \otimes \mathcal{O}(D)$. There is an obvious inclusion

$$\mathcal{A}_{|E|}^{0,0}(X) \otimes \mathcal{O}(Z) \hookrightarrow \mathcal{A}_{|E|}^{0,0}(X) \otimes \mathcal{O}(D)$$

whose image, we claim, contains f . This will prove (b). Look at the sequence (not exact!)

$$H^1(X; \mathcal{O}(Z)) \xrightarrow{\text{restr}} H^1(U; \mathcal{O}(Z)) \xrightarrow{h} H^1(U; \mathcal{O}) \rightarrow H^1(U; \mathcal{O}(D - Z))$$

$$\{\phi\} \mapsto \{\phi|U\},$$

where the last map is the one induced by $\mathcal{O} \hookrightarrow \mathcal{O}(D - Z)$. Since $f \in \mathcal{A}^{0,0}(X; \mathcal{O}(D))$, $hf \in \mathcal{A}^{0,0}(U; \mathcal{O}(D - Z))$ and $\bar{\partial}(hf) = h\bar{\partial}f = h\phi$ on $U - E$. Hence $\{h\phi|U\} = 0$ in $H^1(U; \mathcal{O}(D - Z))$. But the last map in the sequence above is injective (see the second part of Corollary (4.2) of [P], so $\{h\phi|U\} = 0$ in $H^1(U; \mathcal{O})$. Hence there exists $s \in \mathcal{A}^{0,0}(U)$ such that $\bar{\partial}_s = h\phi$ on U . But now on $U - E$ we have $\bar{\partial}(s - hf) = 0$ so $s - hf$ is holomorphic on $U - E$; and the Riemann extension theorem [GRe, p. 144] says $s - hf$ is actually holomorphic on U . Hence hf is smooth on U since s was. Thus $f \in \mathcal{A}^{0,0}(X; \mathcal{O}(Z))$, as was claimed.

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