ENDS OF HYPERBOLIC 3-MANIFOLDS

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1. INTRODUCTION

In this paper we will investigate the geometric structure of hyperbolic 3-manifolds and their ends. The geometry of geometrically finite hyperbolic 3-manifolds is fairly well understood, due to the work of Ahlfors, Bers, Marden, Maskit, Sullivan, Thurston, and others. From a differential geometric viewpoint, the key fact is that ends of geometrically finite hyperbolic 3-manifolds are exponentially expanding. In the late 1970s Bill Thurston proposed another possible model for the geometry of an end of a hyperbolic 3-manifold. An end of a hyperbolic 3-manifold is said to be simply degenerate if it has a neighborhood U homeomorphic to $S \times [0, \infty)$ and there is a sequence of (not necessarily embedded) finite area “hyperbolic surfaces” which leave any compact subset of U . (By a “hyperbolic surface” we mean one whose intrinsic geometry has sectional curvature ≤ -1 .) A hyperbolic 3-manifold with finitely generated fundamental group is said to be geometrically tame if all its ends are either geometrically finite or simply degenerate. Thurston [46] proved that if $\pi_1(N)$ is freely indecomposable and N is the algebraic limit of a sequence of geometrically finite hyperbolic 3-manifolds, then N is geometrically tame. He used this fact in the proof of his geometrization theorem (see Morgan [34].) In 1984, Francis Bonahon proved that all hyperbolic structures on compact 3-manifolds with freely indecomposable fundamental group are geometrically tame. In this paper, we use Bonahon’s result to prove that all hyperbolic 3-manifolds which are topologically tame (i.e. homeomorphic to the interior of a compact 3-manifold) are also geometrically tame.

Let N be a complete (orientable) Riemannian 3-manifold with constant sectional curvature -1 and finitely generated fundamental group. N may be represented as $\mathbf{H}^3/\Gamma$ where Γ is a discrete, torsion-free subgroup of the isometry group of $\mathbf{H}^3$. If Γ is nonabelian the *convex core* $C(N)$ of N is the smallest convex submanifold of N such that the inclusion map is a homotopy equivalence. Explicitly, $C(N) = CH(L_\Gamma)/\Gamma$ where $CH(L_\Gamma)$ denotes the convex hull in $\mathbf{H}^3$ of the limit set L_Γ for Γ ’s action on the sphere at infinity S_∞^2 .

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for $\mathbf{H}^3$ (see Maskit [30] for basic definitions in the theory of Kleinian groups.) There is a retraction $R : N \rightarrow C(N)$ called the nearest point retraction where $R(x)$ is defined to be the (unique) point of $C(N)$ which is nearest to x (see [12, 17].) R gives $N - C(N)$ a natural product structure as $\partial C_1(N) \times [0, \infty)$ where $C_1(N)$ is the neighborhood of radius 1 of $C(N)$ and the real coordinate is the distance to $\partial C(N)$. (Notice that $\partial C_1(N)$ is homeomorphic to $\partial C(N)$.) In these coordinates the metric on $N - C_1(N)$ is quasi-isometric to the metric $\cosh^2 t ds_{\partial C_1(N)}^2 + dt^2$ where $ds_{\partial C_1(N)}^2$ is a hyperbolic metric on $\partial C_1(N)$ (see Perry [38] for example.) If $C(N)$ has finite volume or Γ is abelian, then N is said to be *geometrically finite*. We will see that the convex core is of fundamental importance in understanding the geometry of N .

We recall that the injectivity radius of N at a point x , denoted $\text{inj}(x)$, is defined to be half the length of the shortest (homotopically nontrivial) loop through x . There exists a constant $\mathcal{M}_3$, called the Margulis constant, such that if $\varepsilon < \mathcal{M}_3$ and

$$N_{\text{thin}(\varepsilon)} = \{x \in N | \text{inj}(x) < \varepsilon\}$$

then every component of $N_{\text{thin}(\varepsilon)}$ is either

- (a) a torus cusp, i.e. a horoball in $\mathbf{H}^3$ modulo a parabolic action of $\mathbf{Z} \oplus \mathbf{Z}$,
- (b) a rank one cusp, i.e. a horoball in $\mathbf{H}^3$ modulo a parabolic action of $\mathbf{Z}$,
- or
- (c) a solid torus neighborhood of a geodesic

(see Thurston [46] or Morgan [34].) We let $N_{\text{thick}(\varepsilon)} = N - N_{\text{thin}(\varepsilon)}$ and N_ε^0 denote N with all components of types (a) and (b) removed. It is the ends of N_ε^0 whose geometry we will attempt to understand. (We remark, at this point, that upon first reading it is probably best to ignore the presence of parabolics, in which case $N = N_\varepsilon^0$.)

Even when $C(N)$ does not have finite volume, there exists a compact submanifold M of N_ε^0 , called the *compact core*, such that the inclusion of M into N is a homotopy equivalence (see Scott [39]). We may furthermore choose M in such a way that M intersects each component of ∂N_ε^0 in either an annulus, if the corresponding component of $N_{\text{thin}(\varepsilon)}$ is of type (b), or in a torus, if the corresponding component is of type (a) (see Feighn-McCullough [18]). We will call such a submanifold a *relative compact core* for M and $P = \partial N_\varepsilon^0 \cap M$ will be called the *parabolic locus*. The ends of N_ε^0 are in one-to-one correspondence with components of $\partial M - P$ and there are finitely many such ends.

We will say that an end E of N_ε^0 is *geometrically finite* if it has a neighborhood U which does not intersect $C(N)$. The geometry of such ends is virtually the same as the geometry of an end of a geometrically finite hyperbolic 3-manifold. We will say that an end E is *simply degenerate* if it has a neighborhood U which is homeomorphic to $S \times [0, \infty)$ (where S is a compact surface) and there exists a sequence of (not necessarily embedded) “hyperbolic surfaces” $\{f_n : S \rightarrow U\}$ leaving every compact set such that for each n , $f_n(S)$

is homotopic to $S \times \{0\}$ within U . For “hyperbolic surfaces” one can use either the pleated surfaces of Thurston (see [46] or [12]), the simplicial hyperbolic surfaces of Bonahon (see [6]), or images of harmonic maps of surfaces, as in Minsky [33]. We will follow Bonahon and use simplicial hyperbolic surfaces; see §2 for a detailed definition.

A hyperbolic 3-manifold N with finitely generated fundamental group is said to be *geometrically tame* if all the ends of N_e^0 are either geometrically finite or simply degenerate. Francis Bonahon [6] proved that if each component of $\partial M - P$ is incompressible then N is geometrically tame. (A component S of $\partial M - P$ is said to be incompressible if $\pi_1(S)$ injects into $\pi_1(M)$.) We will say that a hyperbolic 3-manifold is *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold. Notice that a geometrically tame manifold is topologically tame. Our main result is

Main Theorem. *If a hyperbolic 3-manifold N is topologically tame, then it is also geometrically tame.*

We can combine this result with a minimum principle for positive superharmonic functions on the convex core to obtain several interesting corollaries.

Corollary 1. *If $N = \mathbf{H}^3/\Gamma$ is a topologically tame hyperbolic 3-manifold then either its limit set L_Γ is S_∞^2 or L_Γ has measure zero. Moreover, if $L_\Gamma = S_\infty^2$, then Γ acts ergodically on S_∞^2 .*

Corollary 2. *If N is a topologically tame hyperbolic 3-manifold, its geodesic flow is ergodic if and only if $L_\Gamma = S_\infty^2$.*

In 1966, Ahlfors [1] conjectured that if Γ is a finitely generated Kleinian group, then its limit set is either all of S_∞^2 or has measure zero. In that paper he proved his conjecture for geometrically finite groups; subsequently, Thurston [46] and Bonahon [6] proved it for groups whose associated 3-manifolds have compact cores with incompressible boundary. Corollary 1 is a generalization of their results to the setting of topologically tame Kleinian groups. The proof of Corollary 2 relies heavily on work of Sullivan [44] who proved that the geodesic flow of a hyperbolic 3-manifold is ergodic if and only if it has no (positive) Green’s function.

We will also establish that each simply degenerate end has a well-defined ending lamination and establish (in the absence of parabolics) that such a lamination lies in the Masur domain [31]. We note that Thurston [47] has conjectured that the isometry type of a geometrically tame hyperbolic 3-manifold is completely determined by its topology, its parabolic locus, the ending laminations of its simply degenerate ends and the conformal structure on $(S_\infty^2 - L_\Gamma)/\Gamma$.

We will finish the description of the results of this paper by stating the following ambitious conjecture which was originally posed as a question by Marden [29].

Conjecture. *If N is a hyperbolic 3-manifold with finitely generated fundamental group, then N is topologically tame and hence geometrically tame.*

In a sequel to this paper [11] we will use our main theorem to prove a generalization of a theorem of Thurston [46] which characterizes how simply degenerate ends cover.

Covering Theorem [11]. *Let $\tilde{N}$ be a topologically tame hyperbolic 3-manifold which covers $\pi : \tilde{N} \rightarrow N$ another hyperbolic 3-manifold N (where π is a local isometry). If $\tilde{E}$ is a simply degenerate end of $\tilde{N}_e^0$ then either*

- (a) *there exists a neighborhood $\tilde{U}$ of $\tilde{E}$ such that $\pi|_{\tilde{U}}$ is finite to one,*

or

- (b) *N has finite volume and has a finite cover $\hat{N}$ which fibers over the circle such that if N_S denotes the cover of $\hat{N}$ associated to the fiber subgroup then $\tilde{N}$ is finitely covered by N_S . Moreover, if $\tilde{N} \neq N_S$, then $\tilde{N}$ is homeomorphic to the interior of a twisted I-bundle which is doubly covered by N_S .*

If N is topologically tame and has infinite volume then all its covers with finitely generated fundamental group are also topologically tame (see §3). This provides us with a new class of hyperbolic 3-manifolds to which our main results apply. In particular, this observation along with the above covering theorem allows one to understand exactly which covers of an infinite volume topologically tame hyperbolic 3-manifold are geometrically finite. For example, if $N = \mathbf{H}^3/\Gamma$ and Γ is isomorphic to the fundamental group of a closed surface and contains no parabolics, then every infinite index, finitely generated subgroup of Γ is geometrically finite and hence a Schottky group. We remark that Scott and Swarup [40] previously obtained this result in the special case where Γ is the fibre subgroup of a hyperbolic 3-manifold which fibres over the circle. We will make this discussion more explicit in [11].

The basic idea of our proof is to realize the end of a topologically tame hyperbolic 3-manifold as the end of a negatively curved 3-manifold which is homeomorphic to the interior of a 3-manifold with incompressible boundary. We accomplish this by branching over a sufficiently complicated set of geodesics in the original hyperbolic 3-manifold. This branched cover may be given a metric of pinched negative curvature by using the techniques of Gromov and Thurston [20]. We then apply the techniques of Bonahon [6] and Thurston [46], many of which generalize to this setting, to derive the results described above. In §11 we will describe how to generalize these theorems to the setting of hyperbolic 3-orbifolds and nonorientable hyperbolic 3-manifolds.

Remark. J.P. Otal has previously proved that all algebraic limits of convex cocompact function groups of the form $\pi_1(S_1) * \pi_1(S_2)$ and $\pi_1(S_1) * \mathbf{Z}$ (where S_1 and S_2 are closed surfaces) are topologically and geometrically tame.

2. SIMPLICIAL HYPERBOLIC SURFACES

In this section we will recall from Bonahon [6] the definition of a simplicial hyperbolic surface and restate more precisely the definition of a simply degenerate end. The basic idea of a simplicial hyperbolic surface is very simple. One starts with a triangulation of a surface S and a topological map $g : S \rightarrow N$ of the surface into a hyperbolic 3-manifold. We then alter g so that each face of the triangulation is taken to a totally geodesic triangle in N , in such a way that the total angle about each vertex adds up to at least 2π . The intrinsic geometry of the resulting surface has constant negative curvature except at the vertices where there is “concentrated negative curvature.” One may think of simplicial hyperbolic surfaces as finite approximations to Thurston’s pleated surfaces. Again, the presence of parabolics will introduce some unfortunate technicalities into the discussion.

We begin by defining what we will mean by a triangulation of a punctured surface (following Hatcher [22]). Let $\bar{F}$ be a closed surface with a finite number of punctures $\{p_1, \dots, p_n\}$ and let F denote $\bar{F} - \{p_1, \dots, p_n\}$. (F is called a *punctured surface*.) Let V denote a finite collection of points in F . A *curve system* $\{\alpha_1, \dots, \alpha_m\}$ is a collection of arcs with endpoints in V no two of which are ambient isotopic (fixing V). A *triangulation* of (F, V) is simply a maximal curve system for (F, V) . The components of $F - \{\alpha_1, \dots, \alpha_m\}$ are either triangles or punctured monogons. If F has n punctures there will be n punctured monogons in $F - \{\alpha_1, \dots, \alpha_m\}$. In each of these punctured monogons we may append an arc β running from the vertex to the puncture to obtain a *completed triangulation* $\bar{T} = \{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n\}$ such that each component of $F - \bar{T}$ is actually a triangle. (These triangulations are not triangulations in the traditional sense as a single edge may be two edges of a face and a single vertex may be 1, 2, or all 3 of the vertices of a given face.)

Let S be a punctured surface. We say that a proper map $g : S \rightarrow N$ *weakly preserves parabolicity* if $g_* : \pi_1(S) \rightarrow \pi_1(N)$ has the property that $g_*(\gamma)$ is parabolic if γ is parabolic. Let $f : S \rightarrow N$ be a proper map weakly preserving parabolicity. f is said to be a *simplicial prehyperbolic surface* if there exists a triangulation T such that f maps each edge of the completed triangulation $\bar{T}$ to a geodesic arc and each face of T to a nondegenerate, totally geodesic triangle in N . The map f induces a piecewise Riemannian metric on S which has constant negative curvature except (perhaps) at the vertices. f is said to be a *simplicial hyperbolic surface* if the total angle about each interior vertex of T is at least 2π .

The condition that the total angle about each vertex is at least 2π ensures that the intrinsic geometry of $f(S)$ is like that of a surface with curvature ≤ -1 . For example, $f(S)$ has area $\leq 2\pi\chi(S)$. The key property, for our purposes, is summarized by Bonahon’s bounded diameter lemma (see [6] or [11] for the proof). We recall that a homotopically nontrivial closed curve γ on S will be called *compressible* if $f(\gamma)$ is homotopically trivial in N . (In the bounded diameter lemma we require that all compressible curves have length at least ε ,

to assure that the ε -thin portion of $f(S)$ is contained within $N_{\text{thin}(\varepsilon)}$.

Proposition 2.1 (*Bounded diameter lemma*). *Let $f : S \rightarrow N$ be a simplicial hyperbolic surface such that if γ is a compressible curve on S then $f(\gamma)$ has length at least ε . Then $f(S) \cap N_{\text{thick}(\varepsilon)}$ has diameter $\leq C$ where C depends only on ε and the topological type of S ; i.e., any two points $x, y \in S$ may be joined by a path P in S such that the portion of $f(P)$ contained in $N_{\text{thick}(\varepsilon)}$ has length $\leq C$.*

Let E be an end of N_ε^0 which has a neighborhood U homeomorphic to $\bar{S} \times [0, \infty)$. If the interior of $\bar{S}$, denoted S , is noncompact, a simplicial hyperbolic surface $f : S \rightarrow N$ (homotopic to $S \times \{0\}$) will not actually map into U since $f(S)$ will intersect a rank-one cusp of N . So we need to construct the *parabolic extension* U_p of U which will serve as the natural space a collection of simplicial hyperbolic surfaces exiting E will map into. If $\bar{S}$ is closed then $U_p = U$, if not we need to append to U a portion of each thin part which lies adjacent to U .

Let $(\partial\bar{S})_i$ be a component of the boundary of $\bar{S}$. $(\partial\bar{S})_i \times [0, \infty)$ lies on the boundary of a component P_i of $N_{\text{thin}(\varepsilon)}$ of type (b). This component P_i has a natural parameterization as $S^1 \times (-\infty, \infty) \times [0, \infty)$ which is induced by normalizing so that the horoball is centered at ∞ (in the (z, t) coordinates for the upper half-space model) and is determined by the equation $t \geq c$ for some constant c (depending only on ε) and the parabolic action of $\mathbf{Z}$ is generated by $z \mapsto z + 2\pi$. We give it coordinates (θ_i, r_i, s_i) in which the Riemannian metric is of the form

$$e^{-2s_i}(G^2 d\theta_i^2 + dr_i^2) + ds_i^2$$

where G is a constant depending only on ε . In these coordinates, $\theta_i = \text{Re } z \pmod{2\pi}$, $r_i = H(\text{Im } z)$ and $s_i = \log(t) + J$ for some constants H and J depending only on ε .

In this normalization, $(\partial\bar{S})_i \times [0, \infty)$ is a subset C_i of $S^1 \times (-\infty, \infty) \times \{0\}$. Let

$$B(C_i) = \{(\theta_i, r_i, s_i) | (\theta, r_i, 0) \in C_i \text{ and } s_i \geq 0\}$$

denote the portion of P_i lying “above” C_i in this parameterization. To form U_p we append $\bigcup_{P_i} B(\partial U \cap P_i)$ to U , where the union is taken over all rank-one cusps adjacent to U . U_p is homeomorphic to $S \times [0, \infty)$ where S is homeomorphic to the interior of $\bar{S}$.

We are now fully equipped to give a more precise definition of a simply degenerate end. We say that an end E of N_ε^0 is *simply degenerate* if there exists a neighborhood U of E homeomorphic to $\bar{S} \times [0, \infty)$ and there exists a sequence of simplicial hyperbolic surfaces $\{f_n : S \rightarrow U_p\}$ with image in the parabolic extension U_p of U , such that $f_n(S)$ is (properly) homotopic to $S \times \{0\}$ within U_p and $\{f_n(S)\}$ leaves every compact subset of N . Such a sequence of simplicial hyperbolic surfaces is said to *exit* E .

Remark. This definition is different than Bonahon and Thurston's original definition of a simply degenerate end of a hyperbolic 3-manifold whose compact core has incompressible boundary. However, in that restricted setting our definition is equivalent to theirs. We need to introduce our slightly more technical definition due to the added topological difficulties of the situation when the compact core has compressible boundary. In particular, the existence of a collection of surfaces homotopic to the boundary of the compact core and leaving every compact subset of the end does not imply, topologically, that the end has a product structure (see Example 3 of Tucker [49] or Example 5 in Scott-Tucker [41]).

We now explain one method to construct simplicial hyperbolic surfaces. (This method is due to Bonahon [6].) We start with a proper map $g : S \rightarrow N$ which weakly preserves parabolicity and a triangulation T such that if e is any edge of T with both endpoints at a single vertex v then $g(e \cup v)$ is homotopically nontrivial in N . We first homotope g to a map $g_1 : S \rightarrow N$ such that if e is an edge of T both of whose vertices v_1 and v_2 are interior vertices, then $g_1(e)$ is the unique geodesic arc joining $g(v_1)$ to $g(v_2)$ in the homotopy class of $g(e)$. We then (properly) homotope g_1 to a map g_2 such that each infinite edge e of $\bar{T}$ with interior vertex v is taken to the infinite geodesic with one endpoint $g_1(v)$ which is properly homotopic to $g_1(e)$. (This can be visualized best in the universal cover, where a lift of the infinite geodesic joins a lift of $g_1(v)$ to the fixed point of the appropriate parabolic element of Γ .) We then homotope g_2 to a simplicial prehyperbolic surface $f : S \rightarrow N$ such that each face of $\bar{T}$ is taken to the totally geodesic triangle spanned by the image of its edges. Bonahon [6] observes that if each vertex v_i of T has a distinguished edge e_i which has both endpoints at v_i and one chooses g such that $g(v_i)$ lies on the closed geodesic $g(e_i \cup v_i)^*$ homotopic to $g(e_i \cup v_i)$, then the resulting map is, in fact, a simplicial hyperbolic surface.

In our paper [11], it will be convenient to work with an especially simple type of simplicial hyperbolic surface which can be constructed in this manner. A simplicial hyperbolic surface $f : S \rightarrow N$ is said to be *useful* if its associated triangulation T has only one vertex v and T has a distinguished edge e such that $f(v)$ lies on the closed geodesic in the homotopy class of $f(e \cup v)$. We will later observe that simply degenerate ends of topologically tame hyperbolic 3-manifolds admit exiting sequences of useful simplicial hyperbolic surfaces.

3. TOPOLOGICAL PRELIMINARIES

In this section we will recall some topological constructions which will play a role in the rest of the paper.

3.1. Branched coverings. Let $\widehat{M}$ and M be orientable 3-manifolds. A map $p : \widehat{M} \rightarrow M$ is called a *branched covering* if there exists a finite collection of disjoint, simple closed curves $\gamma = \{\gamma_1, \dots, \gamma_n\}$ in $\widehat{M}$ with disjoint tubular neighborhoods $N(\gamma_i)$ such that $p|_{M - \bigcup_i N(\gamma_i)}$ is a covering map and, for each

$i, p : N(\gamma_i) \rightarrow p(N(\gamma_i))$ is topologically conjugate to a map $d_n : D^2 \times S^1 \rightarrow D^2 \times S^1$ where $d_n(z, \theta) = (z^n, \theta)$ for some n . In this case $\widehat{M}$ is called a *branched cover of M branched over $p(\gamma)$* .

Given a finite, disjoint collection of curves γ in the interior of a (orientable) 3-manifold M and an embedded (oriented) surface S which they bound, we may construct a *n -fold cyclic branched cover $\widehat{M}_n$ of M branched over γ* . If we cut M along S we obtain a manifold $\widetilde{M}$ whose boundary consists of ∂M and 2 copies of S , denoted S_+ and S_- , attached to each other along γ . M may be reconstructed from M' by reattaching S_+ to S_- , by a map $\phi : S_+ \rightarrow S_-$. The n -fold cyclic branched cover $\widehat{M}$ is constructed from n copies of $\widetilde{M}$, denoted $\widetilde{M}_1, \dots, \widetilde{M}_n$, by attaching $\widetilde{M}_i$ to $\widetilde{M}_{i+1}$ by gluing S_+^i to S_-^{i+1} by the gluing map ϕ (where $\widetilde{M}_{n+1} = \widetilde{M}_1$).

The following lemma is useful in establishing the existence of cyclic branched covers.

Lemma 3.1. *If γ is a null-homologous collection of curves in a compact (orientable) 3-manifold M , then γ bounds an embedded (oriented) surface S .*

Proof. Let $\gamma = \{\gamma_1, \dots, \gamma_m\}$ and $N(\gamma) = \{N(\gamma_1), \dots, N(\gamma_m)\}$ be a collection of disjoint tubular neighborhoods of γ_i . Since γ is null-homologous it bounds a, not necessarily embedded, surface T in M . $T \cap M - N(\gamma)$ determines a homology class in $H_2(M - N(\gamma), \partial N(\gamma))$. Every such homology class has an embedded representative S_0 ; see Thurston [48]. S_0 extends to an embedded surface S which bounds γ . $\square$

3.2. Topological tameness and covers. It will often be convenient to pass to various covers of our topologically tame hyperbolic 3-manifold N . The following proposition assures us that when we do this the cover is also topologically tame. This proposition, also provides us with a new class of examples to which our results apply. It will play a central role in [11].

Proposition 3.2. *If N is an infinite volume, topologically tame hyperbolic 3-manifold and $\widetilde{N}$ is a cover of N with finitely generated fundamental group, then $\widetilde{N}$ is also topologically tame.*

Proof. N is homeomorphic to the interior M_0 of a compact 3-manifold M . Since M is irreducible, atoroidal and has nonempty boundary, Thurston's geometrization theorem assures us that M_0 is homeomorphic to a geometrically finite, infinite volume hyperbolic 3-manifold N' (see Morgan [34]). We also recall that if $\widetilde{N}'$ is a cover of N' with finitely generated fundamental group that $\widetilde{N}'$ is itself geometrically finite (see Proposition 7.1 in [34]), and hence topologically tame. The result then follows, since any cover of N with finitely generated fundamental group is homeomorphic to a cover of N' with finitely generated fundamental group.

Remark. This result may also be proved by combining results of Hempel [21] and Simon [43].

3.3. Compression bodies. Bonahon [4] proved that one can decompose any compact, irreducible 3-manifold M into a finite union of compression bodies and 3-manifolds with incompressible boundary. This decomposition is unique up to isotopy. A (orientable) *compression body* is a boundary connected sum of a 3-ball, a handlebody and trivial I-bundles over closed (orientable) surfaces of positive genus, in which the other summands are attached to the 3-ball along disjoint disks. Notice that a compression body has only one compressible boundary component S_0 and the inclusion map $i : S_0 \rightarrow M$ induces a surjection $i_* : \pi_1(S_0) \rightarrow \pi_1(M)$.

Let N be a topologically tame hyperbolic 3-manifold homeomorphic to the interior of the compact 3-manifold M . An end E of N is associated to a component S of ∂M . If this component is compressible, it is the compressible boundary component of a compression body component C of the above decomposition. If S is incompressible the end E is associated to a collar neighborhood C of S in M . In either case, we will call C the *compression body associated to the end E* .

3.4. The Masur domain. Let S be a finite area hyperbolic surface. A *geodesic lamination* on S is a closed subset L of S which is a disjoint union of simple geodesics. A *measured lamination* is an invariant (with respect to projection along L) measure on arcs transverse to some compact geodesic lamination L with support on L . $\text{ML}(S)$ is the space of all measured laminations on S and projective lamination space $\text{PL}(S)$ is $(\text{ML}(S) - \{\emptyset\})/\mathbf{R}^+$. We may identify $\text{PL}(S)$ with the set of measured laminations which have length 1. (See Kerckhoff [26] for a discussion of measured laminations.) More generally, a *current* on S is a (positive) transverse invariant measure on the geodesic flow of S . $\text{ML}(S)$ may be identified with the set of currents with compact support and self-intersection number zero. (See Bonahon [6] for a discussion of currents.)

Let M be a compact 3-manifold and S be a compressible component of ∂M . Let $\text{Mod}(M, S)$ be the group of isotopy classes of homeomorphisms of S which extend to homeomorphisms of M . Moreover, let $\text{Mod}_0(M, S)$ denote the subgroup of $\text{Mod}(M, S)$ consisting of (isotopy classes of) homeomorphisms of S which extend to homeomorphisms of M which are homotopic to the identity. $\text{Mod}(M, S)$ acts naturally on $\text{PL}(S)$. We will call a finite-leaved measured lamination compressible if its support is a collection of closed geodesics on S which are homotopically trivial in M . Let $\mathcal{L}(S, M)$ be the closure in $\text{PL}(S)$ of the set of compressible finite-leaved measured laminations. $\mathcal{L}(S, M)$ is the limit set for $\text{Mod}(M, S)$'s action on $\text{PL}(S)$. Let

$$Z(\mathcal{L}(S, M)) = \{\lambda \in \text{PL}(S) \mid i(\mu, \lambda) = 0 \text{ for some } \mu \in \mathcal{L}(S, M)\}.$$

If S has more than one simple compressible curve, let

$$\mathcal{D}(S, M) = \text{PL}(S) - Z(\mathcal{L}(S, M)).$$

If S has just one simple compressible curve, let

$$\mathcal{D}(S, M) = \text{PL}(S) - Z(Z(\mathcal{L}(S, M))).$$

$\text{Mod}(S, M)$ acts properly discontinuously on the Masur domain $\mathcal{D}(S, M)$ (see Masur [31] and Otal [36]). Kerckhoff [27] has shown that $\mathcal{D}(S, M)$ has full measure in $\text{PL}(S)$.

Let γ be a disjoint collection of simple closed geodesics on the boundary of a compression body M whose associated projective measured lamination lies in the Masur domain. If there existed a compressible curve C which did not intersect any component of γ , then γ would be fixed by the Dehn twist about C , which is an element of $\text{Mod}(M, S)$, and hence could not lie in the Masur domain. In particular, every component of $S - \gamma$ is incompressible. Similarly, if A is any essential annulus, γ must intersect the boundary of A . (A properly embedded annulus $i : (A, \partial A) \rightarrow (M, \partial M)$ is said to be essential if $i_* : \pi_1(A) \rightarrow \pi_1(M)$ is an injection and i is not homotopic, as a map of pairs, to a map $i' : (A, \partial A) \rightarrow (M, \partial M)$ such that $i'(A) \subset \partial M$.) Otal [36] combined these observations to prove.

Proposition 3.3 (Otal [36]). *Let M be a compression body and S its compressible boundary component. Let γ and γ' be two collections of simple closed geodesics on S whose associated measured laminations lie in the Masur domain. If γ and γ' are freely homotopic in M , then there exists a homeomorphism $h \in \text{Mod}_0(M, S)$, such that $h(\gamma) = \gamma'$.*

We will sketch Otal's proof. Let $\mathcal{N}(\gamma)$ be a collection of disjoint annular neighborhoods of components of γ . $(M, \mathcal{N}(\gamma))$ is then an acylindrical pared manifold. The free homotopy of γ to γ' induces a homotopy equivalence $h' : (M, \mathcal{N}(\gamma)) \rightarrow (M, \mathcal{N}(\gamma'))$ of acylindrical pared manifolds. A theorem of Johannson [24] then assures us that there exists a homeomorphism $h : (M, \mathcal{N}(\gamma)) \rightarrow (M, \mathcal{N}(\gamma'))$ homotopic to h' . It should be pointed out that Proposition 3.3 is just the first step of Otal's proof that $\mathcal{D}(M, S)/\text{Mod}_0(S, M)$ maps homeomorphically into the space $\mathcal{E}(M)$ of currents on M .

A curve γ in M is said to be *diskbusting* if any curve freely homotopic to γ intersects every compressing disk in M . (A disk $i : (D, \partial D) \rightarrow (M, \partial M)$ is said to be a *compressing disk* if $i_* : \pi_1(\partial D) \rightarrow \pi_1(\partial M)$ is injective.) In the proof of Proposition 5.1 we will need a countable collection of diskbusting curves in M no two of which are freely homotopic.

Proposition 3.4. *Let γ be a simple closed curve on the boundary of a compression body M whose associated projective measured lamination lies in the Masur domain $\mathcal{D}(S, M)$. Then γ is a diskbusting curve.*

Proof. Let $M \cup D_\gamma$ be the manifold obtained by adding a 2-handle to M along γ . Jaco [23] proved that if every component of $\partial M - \gamma$ is incompressible then $M \cup D_\gamma$ has incompressible boundary. Casson and Gordon [13] further established that $M \cup D_\gamma$ is irreducible. In particular, $\pi_1(M \cup D_\gamma)$ has no non-trivial free decomposition.

Let w be a word in $\pi_1(M)$ which is associated to γ (this word is only well defined up to conjugacy.) Then $\pi_1(M \cup D_\gamma) \cong \langle \pi_1(M); w \rangle$ (where $\langle \pi_1(M); w \rangle$ denotes the group $\pi_1(M)$ with the relation $w = 1$ added).

Suppose γ' is homotopic to γ and misses a compressing disk D ; then $\pi_1(M)$ has a nontrivial free decomposition (induced by cutting along D) $\pi_1(M) \cong G_1 * G_2$ where γ' is represented by a word $w' \in G_1$. This implies that

$$\pi_1(M \cup D_\gamma) \cong \langle \pi_1(M); w \rangle \cong \langle \pi_1(M); w' \rangle \cong \langle G_1; w' \rangle * G_2$$

which is a contradiction unless $\langle G_1; w' \rangle \cong 1$ and G_2 is indecomposable. But since each G_i is isomorphic to the free product of a free group with a finite number of surface groups, this implies that $G_1 \cong \mathbf{Z}$ (since otherwise the abelianization of G_1 has rank at least two, and thus the abelianization of $\langle G_1; w' \rangle$ has rank at least one). Moreover, $\pi_1(G_2) = \pi_1(S)$ as the only other possible indecomposable group is $\mathbf{Z}$ and if $\pi_1(M \cup D_\gamma) \cong \mathbf{Z}$ then it could not have incompressible boundary.

If $\pi_1(M) \cong \pi_1(S) * \mathbf{Z}$ then M is homeomorphic to the boundary connected sum of a solid torus with a thickened surface. Moreover, by the above analysis, γ must be freely homotopic to a curve in the solid torus summand. Let A be any essential annulus contained in the thickened surface summand and let h_A denote the Dehn twist about the portion of ∂A which intersects S . Then $h_A \in \text{Mod}(M, S)$ but does not lie in $\text{Mod}_0(M, S)$. $h_A(\gamma)$ is freely homotopic to γ , so Proposition 3.3 guarantees that there exists $h \in \text{Mod}_0(M, S)$ such that $h(\gamma) = h_A(\gamma)$. Therefore, γ is fixed by $h \circ h_A^{-1}$ which contradicts its presence in the Masur domain. $\square$

We now conclude by stating the corollary which will be of use in the proof of Proposition 5.1.

Corollary 3.5. *Let M be a compression body with compressible boundary component S . There exists a countable collection of diskbusting curves in S no two of which are freely homotopic in M .*

Proof. Recall (see e.g. [6]) that the set of projective measured laminations whose support is a simple closed geodesic is dense in $\text{PL}(S)$. Proposition 3.4 guarantees that any curve whose associated lamination lies in the Masur domain is diskbusting. Thus since $\text{Mod}_0(M, S)$ acts properly discontinuously on the open set $\mathcal{D}(M, S)$ we may choose a countable collection of curves which arise as supports of measured laminations no two of which are related by elements of $\text{Mod}_0(M, S)$. Proposition 3.3 then guarantees that no two elements of the collection are freely homotopic in M . $\square$

Remark. It seems likely that one could use combinatorial group theoretic techniques to give a simpler proof that there exists a countable collection of diskbusting curves in M . One simply needs to find a countable collection of non-conjugate words in $\pi_1(M)$ none of which are conjugate to a word in any free

factor of $\pi_1(M)$. (One could not easily guarantee that these curves were homotopic to simple closed curves in S , but this property is not necessary in the proof of Proposition 5.1.) We have chosen to use the technology of the Masur domain as it seems to arise very naturally in the theory of hyperbolic 3-manifolds with freely decomposable fundamental group (see [36, 8, 10]).

4. MANIFOLDS OF PINCHED NEGATIVE CURVATURE

In this section we will state a version of Bonahon's theorem in the setting of Riemannian 3-manifolds with pinched negative curvature and hyperbolic cusps. Bonahon [6] observes that such a generalization is straightforward and most of the details are carried out in [8]. After stating the generalization we will sketch Bonahon's proof and describe the (minor) alterations which need to be made to obtain the stated generalization. We will assume throughout the section that N has finitely generated fundamental group.

We will say that a complete Riemannian 3-manifold has *pinched negative curvature* if there exist constants $-a^2$ and $-b^2$ such that the sectional curvature K of N lies between the two constants, i.e. $-b^2 \leq K \leq -a^2$. If N has pinched negative curvature, there exists a constant $\mathcal{M}$ (again called the Margulis constant) such that if $\varepsilon < \mathcal{M}$ every component of $N_{\text{thin}(\varepsilon)}$ is homeomorphic to either $S^1 \times \mathbf{R} \times [0, \infty)$, $T^2 \times [0, \infty)$, or a solid torus $S^1 \times D^2$ (see Ballman-Gromov-Schroeder [3]). N will be said to have *hyperbolic cusps* if there exists some $\varepsilon < \mu$ such that every noncompact component of $N_{\text{thin}(\varepsilon)}$ is isometric to a cusp of a hyperbolic 3-manifold. We then define N_ε^0 to be N with the noncompact components of $N_{\text{thin}(\varepsilon)}$ deleted.

To state precisely the notion of a simply degenerate end in this setting we first need to generalize the notion of a simplicial hyperbolic surface. The only obstacle is that if e_1 , e_2 and e_3 are 3 totally geodesic arcs which form a triangle in N which bounds a topological disk, they need not bound a totally geodesic surface. A *ruled triangle* is constructed by choosing the mutual vertex v_{12} of e_1 and e_2 and taking the collection of geodesics (in the appropriate homotopy class) with one endpoint v_{12} and the other endpoint on e_3 . Let $f: S \rightarrow N$ be a proper map which weakly preserves parabolicity. f is said to be a *simplicial ruled surface* if there exists triangulation T of S , such that each face of the completed triangulation $\bar{T}$ is taken to a nondegenerate ruled triangle and the total angle about each vertex is at least 2π . Each ruled triangle has a Riemannian metric with curvature $\leq -a^2$, and thus our surface inherits a piecewise Riemannian metric defined everywhere except the 1-skeleton of T . Again, this metric can be thought of as a metric with curvature $\leq -a^2$ with concentrated negative curvature at the vertices. Simplicial ruled surfaces may be constructed just as in §2. We will call a simplicial ruled hyperbolic surface *useful* if its associated triangulation has only one vertex v and there exists a distinguished edge e such that $f(v)$ lies on the closed geodesic in the homotopy class of $f(e \cup v)$. Again the key properties of a simplicial ruled

surface are summarized by the bounded diameter lemma, which continues to be true as long as we allow the constant C to depend on the topological type of S , ε , and $-a^2$.

We will say that an end E of N_ε^0 is *geometrically infinite* if there exists a sequence of closed geodesics $\{\alpha_j^*\}$ in N_ε^0 , such that if U is any neighborhood of E there exists J such that if $j \geq J$ then α_j^* lies in U . Such a collection of geodesics is said to *exit* the end E . We will further say that E is *simply degenerate* if it has a neighborhood U which is homeomorphic to $\bar{S} \times [0, \infty)$ and there exists a sequence of simplicial ruled surfaces $\{f_n : S \rightarrow U_P\}$ with image in the parabolic extension U_P of U , such that $\{f_n(S)\}$ leaves every compact subset of N and $f_n(S)$ is properly homotopic to $S \times \{0\}$ (within U_P) for all n . Such a sequence of simplicial ruled surfaces is said to *exit* the end.

We are now prepared to state the generalization of Bonahon's theorem.

Theorem 4.1. *Let N be a complete Riemannian 3-manifold with finitely generated fundamental group whose compact core has incompressible boundary. Further suppose that N has pinched negative curvature and hyperbolic cusps. If E is a geometrically infinite end of N_ε^0 , then E is simply degenerate. Moreover, E has an exiting sequence of useful simplicial hyperbolic surfaces.*

We will now sketch Bonahon's proof and comment on the alterations necessary to prove Theorem 4.1. We first scale so that N has curvature ≤ -1 . The first step in Bonahon's argument is the following intersection number lemma.

Proposition 4.2 (The Intersection Number Lemma). *Let N be a manifold with pinched negative curvature (≤ -1) and hyperbolic cusps and S an incompressible properly embedded surface whose inclusion map weakly preserves parabolicity. There exists a constant K such that if α_1^* and α_2^* are two geodesics in N of distance $\geq D$ from S , homotopic to 2 curves α_1 and α_2 in S by two homotopies which meet S only in α_i and arrive on the same side, and each geodesic is disjoint from $N_{\text{thin}(\varepsilon)}$, or is itself the core of a Margulis tube, then*

$$i(\alpha_1, \alpha_2) \leq Ke^{-D}l(\alpha_1)l(\alpha_2) + 4$$

where i is intersection number in S , and l is length measured on S .

To prove the intersection lemma, one first straightens the homotopies of α_1 to α_1^* and of α_2 to α_2^* to obtain simplicial ruled annuli A_1 and A_2 . One may observe that $i(\alpha_1, \alpha_2)$ is bounded from above by $i(\alpha_1^*, A_2) + i(\alpha_2^*, A_1)$. One may then use estimates on the area of these annuli to bound their intersection numbers with the geodesics α_i^* . For example, if neither geodesic is the core curve of a Margulis tube, then any two (inequivalent) intersections of α_1^* with A_2 are separated by at least $\frac{\varepsilon}{2}$ in the product metric on $\alpha_1^* \times A_2$. Thus about each intersection point which is at least a distance $\frac{\varepsilon}{8}$ from $\partial(\alpha_1^* \times A_2)$ there is an embedded ball of radius $\frac{\varepsilon}{8}$ which does not intersect the $\frac{\varepsilon}{8}$ -ball about any

other (inequivalent) intersection point. The number of (inequivalent) intersection points within $\frac{\varepsilon}{8}$ of $\partial(\alpha_1^* \times A_2)$ can be bounded by the area of $\partial(\alpha_1^* \times A_2)$. The estimate needed on the area of the simplicial ruled annuli is a simple exercise in differential geometry.

Lemma 4.3. *If A is a simplicial ruled annulus with boundary $C_1 \cup C_2$ in a manifold N with pinched negative curvature (≤ -1), then*

$$\text{area}(A) \leq l(C_1) + l(C_2).$$

Moreover,

$$\text{area}\{x \in A \mid d(x, C_1) \geq D\} \leq l(C_2) + e^{-D}l(C_1).$$

Now let E be a geometrically infinite end of N_ε^0 and $\{\alpha_j^*\}$ an exiting sequence of closed geodesics. Let M be a relative compact core for N_ε^0 and let U be the component of $N_\varepsilon^0 - M$ which is a neighborhood of E . We may modify the sequence so that each curve is either a core curve of a Margulis tube or never intersects a Margulis tube. (If α_j^* intersects a Margulis tube we simply replace it, in our sequence, by the core curve of that Margulis tube.) Each α_j^* is homotopic to a closed curve α_j on $S = \partial U_p$. We take α_∞ to be the limit of a subsequence of the sequence $\{\alpha_j/l_S(\alpha_j)\}$ in $\mathcal{E}(S)$. (If S is not closed, we must use an analogue of Proposition 5.13 in [6] to guarantee that a nontrivial limit exists.) The intersection number lemma assures us that $i(\alpha_\infty, \alpha_\infty) = 0$ and hence that α_∞ is a measured lamination.

Recall that every (unit length) measured lamination is the limit of a sequence of the form $\{\beta_j/l(\beta_j)\}$ where each curve β_j is a simple closed geodesic on S . Let $\{\beta_j\}$ be a sequence of simple closed geodesics in S such that $\{\beta_j/l(\beta_j)\}$ converges to α_∞ . The next step consists of showing that $\{\beta_j^*\}$ is an exiting sequence. The key observation is that if β_j^* stayed within any compact set they would converge to a “realization” of α_∞ . The precise statement is

Proposition 4.4. *Let S be a finite area hyperbolic surface and $\phi: S \rightarrow N$ a proper, incompressible map of S (which weakly preserves parabolicity) into a complete 3-manifold N with pinched negative curvature and hyperbolic cusps. Let α be a measured lamination on S . Then either*

- (1) ϕ may be homotoped so that $l_N(\phi(\alpha))$ is arbitrarily small,¹ or
- (2) for all $\varepsilon > 0$ and $t < 1$, there exists ϕ' homotopic to ϕ so that if γ is a closed geodesic such that $\gamma/l_S(\gamma)$ is close enough to $\alpha/l_S(\alpha)$ (in $\mathcal{E}(S)$), $\phi'(\gamma)$ is homotopic to a geodesic γ^* which lies within a ε -neighborhood of $\phi'(\gamma)$ for at least $tl_N(\phi'(\gamma))$.

The original proof of Proposition 4.4 depends very weakly on N 's being hyperbolic. One need only replace Bonahon's use of specific inequalities in

¹Perhaps we ought to explain this notation. We measure the length of the image of a measured lamination $l_N(\phi(\alpha))$ by integrating the measure on $\mathbf{P}(S)$ given by transverse measure cross Lebesgue measure of the image of α under ϕ . When α is a simple closed geodesic this agrees with the length of the image.

hyperbolic geometry by the appropriate inequalities resulting from the comparison theorems for triangles in manifolds of pinched negative curvature. (See Klingenberg [28] for an extensive discussion of these comparison theorems.)

Since α_∞ satisfies conclusion (i) of Proposition 4.4, $\{l_N(\beta_j^*)/l_S(\beta_j)\}$ converges to 0 which implies that $\{\beta_j^*\}$ must leave every compact subset of N . It remains to check that $\{\beta_j^*\}$ is an exiting sequence for the end E and not perhaps for some other end of the manifold. However, Bonahon's very delicate arguments in §6.4 generalize directly.

Now we may extend β_j to a triangulation of S and construct a useful simplicial ruled surface $f_j : S \rightarrow N$ whose image contains β_j^* . By the bounded diameter lemma we can see that $\{f_j : S \rightarrow N\}$ is an exiting sequence for the end E . It then remains to check that E has a neighborhood of the form $S \times [0, \infty)$. We may assume that the images $\{f_j(S)\}$ are disjoint and we may then use results of Freedman, Hass and Scott [19] to obtain a sequence of properly embedded surfaces exiting the end E . We then observe that since any two surfaces are homotopic and each surface is incompressible, the submanifold they bound is homeomorphic to $S \times [0, 1]$. We may piece these submanifolds together to obtain the desired product structure. (Notice that we have made key use of incompressibility here.)

Remark. In this formulation we avoid two delicate issues. The first is that of generalizing Bonahon's result to a general 3-manifold with pinched negative curvature; Bonahon's original proof makes use of various properties of horocycles in the cusps. The other issue is that of geometric finiteness for manifolds with pinched negative curvature. In [7], Bowditch defines a 3-manifold with pinched negative curvature to be geometrically finite if it contains a finite volume convex submanifold such that the inclusion map is a homotopy equivalence. However, work still remains to be done to check that there exists a decomposition of the ends of a 3-manifold with pinched negative curvature and finitely generated fundamental group into geometrically finite (in the obvious analogue of Bowditch's definition) and geometrically infinite (in our definition) ends.

5. BRANCHING TO INCOMPRESSIBILITY

In this section we will prove that an end E of a topologically tame hyperbolic 3-manifold may be realized as an end $\hat{E}$ of a complete Riemannian 3-manifold $\hat{N}$ with pinched negative curvature and hyperbolic cusps whose compact core has incompressible boundary. We will then, in §6, apply Theorem 4.1 to $\hat{N}$ to prove our main result. The entire section will be devoted to the proof of Proposition 5.1.

Proposition 5.1. *Let N be a topologically tame hyperbolic 3-manifold. If E is an end of N , then there exists an end $\hat{E}$ of a complete Riemannian 3-manifold $\hat{N}$ with pinched negative curvature and hyperbolic cusps whose compact core has incompressible boundary such that E has a neighborhood U which is isometric to some neighborhood $\hat{U}$ of $\hat{E}$.*

Proof. Let N be homeomorphic to the interior of a compact 3-manifold M , and let C be the compression body associated to the end E (see §3.3). We may lift to the (Riemannian) cover N_C of N associated to $\pi_1(C) \subset \pi_1(N)$. By Proposition 3.2, N_C is topologically tame and N_C has an end E' which occurs as the lift of E . If C has incompressible boundary we are done. In any case, we may assume that our original manifold N is homeomorphic to the interior of a compression body C .

Our manifold $\widehat{N}$ will be obtained by forming a 2-fold branched cover of N branched over a “sufficiently complicated” collection of geodesics in N . We first obtain topological conditions guaranteeing that a 2-fold branched cover has a compact core with incompressible boundary.

Lemma 5.2. *Let γ be a null-homologous, simple collection of closed curves in the interior of a compact 3-manifold M such that γ intersects every compressing disk in M . Then any 2-fold branched cover $\widehat{M}$ of M branched over γ has incompressible boundary.*

Proof. This lemma is an easy consequence of the equivariant Dehn’s lemma of Meeks-Yau which we state here for reference.

Theorem 5.3 (Equivariant Dehn’s lemma [32]). *If M is a smooth 3-manifold with compressible boundary and G is a (smooth) action of the finite group G on M , there exists a disjoint collection of compressing disks $\{D_i\}_{i=1}^k$ such that either $g(D_i) \cap D_j = \emptyset$ or $g(D_i) = D_j$ for all i, j and $g \in G$.*

There exists an action of $\mathbf{Z}_2$ on $\widehat{M}$ such that $\widehat{M}/\mathbf{Z}_2$ is M . So if $\widehat{M}$ has compressible boundary there exists a collection of compressing disks $\{D_i\}$ which are invariant by $\mathbf{Z}_2$. The projection of this collection of disks to M gives a compressing disk which either misses γ or intersects it at most once. (If it intersected it more than once the cover could not be a disk.) But since γ is null-homologous any compressing disk which intersects γ once intersects it at least twice. Thus the compressing disk would have to miss γ entirely, which is a contradiction. $\square$

Corollary 3.5 guarantees that there is a countable collection of diskbusting curves in ∂C no two of which are freely homotopic. Let $h : N \rightarrow C$ be a homeomorphism from N to the interior of C . As there are only finitely many conjugacy classes of parabolic elements in Γ , we may choose a curve β from this collection such that $h^{-1}(\beta)$ is homotopic to a closed geodesic in N . Let β_1 be obtained by appending a null-homologous, but not null-homotopic, loop to β . (Again we must be sure that $h^{-1}(\beta_1)$ is homotopic to a closed geodesic.) Then the collection $\gamma = \{h^{-1}(\beta), h^{-1}(\beta_1)\}$ is null-homologous and if γ' is any simple collection of curves homotopic to γ , γ' intersects every compressing disk. Thus by Lemma 5.2, any 2-fold cyclic branched cover $\widehat{C}$ of C branched over γ' has incompressible boundary.

Let γ^* be the collection of geodesics in N which are homotopic to γ . If γ^* is a simple collection of geodesics, we can let $\widehat{N}$ be a 2-fold cyclic branched

cover of N branched over γ^* . Then $\widehat{N}$ is homeomorphic to the interior of a 2-fold cyclic branched cover $\widehat{C}$ of C branched over $h(\gamma^*)$. Hence, $\widehat{N}$ has a compact core with incompressible boundary.

$\widehat{N}$ inherits a metric of constant negative curvature except along the branching locus where the metric is singular and has concentrated negative curvature. In Gromov and Thurston's paper [20], they show how to impart a metric of pinched negative curvature to the branched cover of a hyperbolic manifold, when the branching locus is totally geodesic. (We will not worry about the bounds on the sectional curvature obtained in this process.)

Lemma 5.4 (Gromov-Thurston [20]). *Let V be a complete hyperbolic 3-manifold, γ a null-homologous collection of geodesics in V , and $\tilde{V}$ a cyclic branched covering of V branched over γ . Then $\tilde{V}$ may be given a metric of pinched negative curvature, which is the metric induced by the branched covering outside of some compact neighborhood of the branching locus.*

So our manifold $\widehat{N}$, with the metric imparted by the lemma above, is a Riemannian manifold with pinched negative curvature and hyperbolic cusps and, by construction, it has an end $\widehat{E}$ with a neighborhood $\widehat{U}$ which is isometric to a neighborhood U of our original end E .

We are now done, except for one potential obstacle: γ^* may not be simple. Here is one way to get around this problem.

Lemma 5.5. *If γ is a collection of closed geodesics in a hyperbolic 3-manifold N which is not simple, we may alter the metric of N (in a compact neighborhood of γ) to obtain a metric N' with pinched negative curvature, such that the geodesic representative of γ in N' is simple and has a neighborhood of constant sectional curvature -1 .*

Proof. Choose a collection of small disjoint balls $\{B_i\}$ of radius 2μ (for some $\mu > 0$) about points x_i on γ , one ball on either side of each intersection point. We require that μ be chosen such that $\text{inj}_x(N) > 2\mu$ for all points on γ . We further require that no ball contains a self-intersection point of γ and that the intersection of γ with each ball is connected. Let $\widehat{B}_i$ be the balls of radius μ about x_i and $\widetilde{B}_i$ be the balls of radius $\frac{\mu}{2}$ about x_i . We may now choose a simple curve γ' which is homotopic to γ , geodesic on $N - \bigcup \widetilde{B}_i$ and as close to γ^* (in a C^∞ -sense) as we like. If γ' is close enough to γ^* , then the exponential map $p_i : \nu(\gamma', \widehat{B}_i) \rightarrow N$ is injective on the neighborhood of radius μ of the zero section where $\nu(\gamma', \widehat{B}_i)$ denotes the portion of the normal bundle $\nu(\gamma')$ of γ' which lies over $\gamma' \cap \widehat{B}_i$. Let C_i denote the image by the exponential map of the neighborhood of radius $\frac{\mu}{2}$ of the zero section of $\nu(\gamma', \widehat{B}_i)$. (Notice that $C_i \subset B_i$.) p_i defines cylindrical coordinates (x, r, θ) in C_i such that $\gamma' = \{r = 0\}$. Let g_i denote the metric on C_i in these coordinates. By choosing γ' close to γ^* we may assume that g_i is as close to h as we like, where

$$h = dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dx^2$$

is the hyperbolic metric in Fermi coordinates about a geodesic in $\mathbf{H}^3$. Now choose a bump function $\eta : \mathbf{R} \rightarrow [0, 1]$ which is 0 on the region $[0, \frac{\mu}{4}]$ and 1 on a neighborhood of $\frac{\mu}{2}$. Let g' be the metric which is obtained by taking the metric $(1 - \eta(r))h + \eta(r)g_i$ on C_i (for each i) and leaving the metric unchanged on $N - \bigcup C_i$. If γ' was chosen close enough to γ^* and hence g_i was close enough to h , then the resulting metric has negative curvature. Let N' denote N with the metric g' , where γ' was chosen close enough to γ^* to make g' a metric of negative curvature. (The key point in this proof is that we are averaging our arbitrarily small deformation with the hyperbolic metric on a neighborhood of fixed diameter and with a fixed bump function.) Notice that γ' is geodesic in the metric g' and that if x is a point on γ' then the ball of radius $\frac{\mu}{4}$ about x is isometric to a ball of radius $\frac{\mu}{4}$ in $\mathbf{H}^3$. If one chooses $\delta < \frac{\mu}{4}$ such that the exponential map $p : \nu(\gamma') \rightarrow N$ is injective on the neighborhood of radius δ (i.e. if the neighborhood $\mathcal{N}_\delta(\gamma')$ of γ' of radius δ is an embedded cylinder) then the metric on $\mathcal{N}_\delta(\gamma')$ is hyperbolic and has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dx^2$$

in Fermi coordinates. (Notice that, in general, δ will have to be chosen to be much less than $\frac{\mu}{4}$ to ensure that $\mathcal{N}_\delta(\gamma')$ is an embedded cylinder.) $\square$

This allows us to complete our original argument in the case where γ^* is self-intersecting. We will simply observe that Gromov and Thurston's argument was entirely local. Let N' and $\mathcal{N}_\delta(\gamma')$ be chosen as in the proof of Lemma 5.5. Let $\hat{N}'$ be a 2-fold cyclic branched cover of N' branched over γ' . Then the (singular) Riemannian metric on the neighborhood $\mathcal{N}'$ of radius δ of the branching locus in $\hat{N}'$ may be written

$$dr^2 + 4 \sinh^2 r d\theta^2 + \cosh^2 r dx^2.$$

In their paper [20], Gromov and Thurston construct a smooth function $\sigma : [0, \delta] \rightarrow \mathbf{R}_+$ such that $\sigma(r) = \sinh r$ for small $r \geq 0$ and $\sigma(r) = 2 \sinh r$ for $r > \frac{\delta}{2}$, such that the metric

$$dr^2 + \sigma^2(r) d\theta^2 + \cosh^2 r dx^2$$

has negative curvature. Therefore, we may obtain $\hat{N}$ from $\hat{N}'$ by replacing the singular metric on $\mathcal{N}'$ by the new metric constructed by Gromov and Thurston. (Notice that the new metric on $\mathcal{N}'$ agrees with the old metric on a neighborhood of the boundary.) $\square$

Remarks. (1) Topological tameness was used to assure that the branched cover has finitely generated fundamental group. If N is not topologically tame it is possible that the complement of γ^* has infinitely generated fundamental group, and hence that $\hat{N}$ does (see Tucker [49]).

(2) It is possible to construct a 3-fold branched cover of N_C whose compact core is acylindrical (see [8]). This has the advantage of simplifying some of

the steps in the proof of Bonahon's theorem. In particular, if the manifold is acylindrical then the difficult arguments in §6.4 of [6] are not necessary.

(3) Lemma 5.5 is not particularly aesthetically pleasing as a solution to our problem. A better solution would be to prove that any cyclic branched cover of a 3-manifold with pinched negative curvature branched over a collection of simple geodesics may be given a metric of pinched negative curvature.

6. PROOF OF MAIN THEOREM

In this section we will put together the results of §§4 and 5 to prove our main result.

Main theorem. *If N is a topologically tame hyperbolic 3-manifold, then N is also geometrically tame. Moreover, every simply degenerate end of N_ϵ^0 has an exiting sequence of useful simplicial hyperbolic surfaces.*

Proof of main theorem. We first need to recall from [6] the following characterization of ends which are not geometrically finite.

Proposition 6.1. *Let N be a hyperbolic 3-manifold with finitely generated fundamental group, and let E be an end of N_ϵ^0 . Then E is not geometrically finite if and only if there exists a sequence of closed geodesics exiting E .*

Let E be an end of N_ϵ^0 which is not geometrically finite. Then, by Proposition 6.1, there exists a sequence of closed geodesics $\{\alpha_j^*\}$ exiting E . There exists an end $\bar{E}$ of N such that any neighborhood of $\bar{E}$ contains a neighborhood of E . By Proposition 5.1 there exists a complete Riemannian 3-manifold $\hat{N}$ with pinched negative curvature and hyperbolic cusps whose compact core has incompressible boundary, such that $\hat{N}$ has an end $\hat{E}$ with a neighborhood $\hat{U}$ which is isometric to a neighborhood $\bar{U}$ of $\bar{E}$. Let this isometry be denoted $j : \hat{U} \rightarrow \bar{U}$. Let $U^+ \subset \bar{U}$ be a neighborhood of E ; then $j^{-1}(U^+)$ is a neighborhood of an end E' of $\hat{N}_\epsilon^0$. Then $\{j^{-1}(\alpha_j^*)\}$ is a sequence of closed geodesics exiting E' . So E' is geometrically infinite, and hence, by Theorem 4.1 it is simply degenerate. Therefore E' has a neighborhood $U' \subseteq j^{-1}(U^+)$ which is homeomorphic to $S \times [0, \infty)$ and there exists a sequence $\{f_n : S_0 \rightarrow U'_p\}$ of useful simplicial ruled surfaces exiting E' . Then $U = j(U')$ is a neighborhood of E which is homeomorphic to $S \times [0, \infty)$ and $\{j \circ f_n : S_0 \rightarrow U_p\}$ is a sequence of useful simplicial ruled surfaces exiting E . But any simplicial ruled surface in a hyperbolic 3-manifold is also a simplicial hyperbolic surface and we are done. $\square$

7. A TECHNICAL LEMMA

In this section we will prove that we may choose a neighborhood U of any simply degenerate end E such that if $\{f : S_0 \rightarrow U_p\}$ is a simplicial hyperbolic surface then if γ is any compressible curve or accidental parabolic on S_0 , then $f(\gamma)$ is long. (We will say that a curve γ in S_0 is an *accidental parabolic* if it is

not homotopic to a cusp of S_0 but $f(\gamma)$ is homotopic into the parabolic locus of N .) This property will play a role in our proof of the minimum principle for superharmonic functions on the convex core. It will also play a role in our proof of the covering theorem in [11].

Lemma 7.1. *Let N be a topologically tame hyperbolic 3-manifold and let E be a simply degenerate end of N_ε^0 . Let U be a neighborhood of E which is homeomorphic to $\bar{S} \times [0, \infty)$. Then given any $K > 0$ there exists a subneighborhood U^K of U which is homeomorphic to $\bar{S} \times [0, \infty)$ such that if $f : S \rightarrow U_P^K$ is a simplicial hyperbolic surface (properly) homotopic to ∂U_P (within U_P) and γ is any compressible curve or accidental parabolic on S , then $f(\gamma)$ has length at least K and $f(\gamma)$ is not contained entirely within any component of $N_{\text{thin}(\varepsilon)}$.*

Proof. Let E' , $\hat{N}$, N_C , j , and U' be as in the proofs of Proposition 5.1 and the main theorem. Let B be the neighborhood of the branching locus in $\hat{N}$ on which the branched covering of N_C by $\hat{N}$ is not a local isometry. Let γ be any compressible curve on S . If $j^{-1}(\gamma)$ is homotopic to a closed geodesic $\hat{\gamma}^*$, then $\hat{\gamma}^*$ must intersect B ; otherwise γ would be homotopic to the closed geodesic $j(\hat{\gamma}^*)$ which is impossible. Similarly, γ cannot be homotoped entirely within any cuspidal region of $\hat{N}_{\text{thin}(\varepsilon)}$. Therefore there exists a compact region B of $\hat{N}$, such that if γ is any compressible curve on S then $j^{-1}(\gamma)$ must be homotopic to a closed geodesic $\hat{\gamma}^*$ and this geodesic must intersect B . Let A be the length of the shortest closed geodesic which intersects B .

Now let $\hat{f} : S \rightarrow \hat{N}$ be a simplicial ruled surface (properly homotopic to $\partial U'_P$) and γ a compressible curve on S . If every point on $\hat{f}(\gamma)$ is at least $2D$ away from B then the length of $\hat{f}(\gamma)$ is at least $\min\{2D, e^D A/2\}$. If $\hat{f}(\gamma)$ is a distance $\geq D$ from its geodesic representative then $l(\hat{f}(\gamma)) \geq e^D l(\hat{\gamma}^*)/2 \geq e^D A/2$ and if $\hat{\gamma}^*$ has a point which lies a distance $\geq D$ from B then $l(f(\gamma)) \geq l(\hat{\gamma}^*) \geq 2D$.

Now suppose that γ is an accidental parabolic in S . If $j^{-1}(\gamma)$ is not homotopic into the parabolic locus of $\hat{N}$, then its geodesic representative must intersect B , as above. In this case, we see as above that if every point on $\hat{f}(\gamma)$ is at least $2D$ away from B then the length of $\hat{f}(\gamma)$ is at least $\min\{2D, e^D A/2\}$.

To handle accidental parabolics in S which are homotopic into cusps of $\hat{N}$ it is convenient to lift to the cover $\hat{N}_S$ of $\hat{N}$ associated to $\hat{f}_*(\pi_1(S))$. Then U' lifts homeomorphically to an end U'_S of $(\hat{N}_S)_\varepsilon^0$. This homeomorphism extends to a homeomorphism $h : U'_P \rightarrow (U'_S)_P$ between their parabolic extensions. If $\hat{f}(\gamma)$ is homotopic into the parabolic locus of $\hat{N}$ (but γ is not peripheral on S) then $h \circ \hat{f}(\gamma)$ is not homotopic into a cusp which intersects $(U'_S)_P$ since $(h \circ \hat{f})_*$ is an isomorphism. Therefore, if every point on $h \circ \hat{f}(\gamma)$ is at least $2D$ away from any noncompact component of $(\hat{N}_S)_{\text{thin}(\varepsilon)}$ which does not intersect $(U'_S)_P$ we see that $\hat{f}(\gamma)$ lies a distance at least $2D$ away from the component of the thin part P_γ which it is homotopic into and hence has length $\geq e^D \varepsilon$.

Now choose D such that $\min\{2D, e^D A/2, e^D \varepsilon\} \geq K$ and choose a subneighborhood $(U')^K$ of E' of the form $\bar{S} \times [R, \infty)$ (for some $R > 0$) such that every point in $(U')^K_p$ lies a distance at least $2D$ from B and any point in $h((U')^K_p)$ lies at least $2D$ away from any noncompact component of $(\hat{N}_S)_{\text{thin}(\varepsilon)}$ which does not intersect $(U'_S)_p$. We now simply take U_K to be $j((U')^K)$.

The fact that $\hat{f}(\gamma)$ is not contained entirely within a component of $N_{\text{thin}(\varepsilon)}$ follows from the fact that $\hat{f}$ is incompressible and $\hat{f}(\gamma)$ does not have a representative in $(U')^K$ of length $\leq \varepsilon$. (Notice that here we may assume that $K > \varepsilon$.) $\square$

8. ANALYTICAL TAMENESS

In this section we will attempt to isolate the properties of geometrically tame hyperbolic 3-manifolds which will allow us to analyze the behavior of the positive superharmonic functions they admit. This analysis has its first fruits in the minimum principle derived in the next section, which we will apply to give proofs of Corollaries 1 and 2. In another paper [9] we will use these properties to study the spectrum of the Laplacian and the Hausdorff dimension of the limit set.

Definition. A hyperbolic 3-manifold N with finitely generated fundamental group is *analytically tame* if $C(N)$ may be exhausted by a sequence of compact submanifolds $\{C_i\}$ such that

- (1) $C_i \subset C_j$ if $i < j$,
- (2) $\bigcup C_i^0 = C(N)$ (where C_i^0 denotes the interior of C_i considered as a subset of $C(N)$),
- (3) there exists K such that the boundary ∂C_i of C_i has area at most K for all i , and
- (4) there exists L such that the neighborhood of radius one of ∂C_i has volume at most L for all i .

We only require that our submanifolds C_i have Lipschitz boundary.

The condition on the volume of the neighborhood of the boundary will be used in our proof of the minimum principle, while the condition on the area of the boundary components will be more useful in studying the spectrum of the Laplacian. The regularity assumption is natural, as the boundary of the convex core itself is always a Lipschitz submanifold but is not in general a C^1 -submanifold (see Epstein-Marden [17]).

The main result in this section is a proof that geometrically tame hyperbolic 3-manifolds are also analytically tame. The proof formalizes ideas of Thurston contained in his proof [46] of the minimum principle for geometrically tame hyperbolic 3-manifolds which have compact cores with incompressible boundary.

Our definition of analytical tameness was motivated by the original version of Culler and Shalen's paper [15] and discussions with Peter Shalen. In their paper

they produce a dense set of analytically tame manifolds such that $C(N) = N$ on the boundary of Schottky space of genus 2. These manifolds are not known to be topologically tame.

Theorem 8.1. *If N is a topologically tame hyperbolic 3-manifold, then N is analytically tame.*

Proof. The compact submanifolds in the definition of analytical tameness will be constructed as the regions “enclosed” by a collection of simplicial hyperbolic surfaces (one in each simply degenerate end), horocyclic tori (one in each rank-two cusp), horocyclic annuli (one in each rank-one cusp) and the boundary of the convex core. We first observe that the volume of the neighborhood of radius one of a simplicial hyperbolic surface $f(S)$ is bounded above by a constant depending only on the Euler characteristic of S and the minimum length of a compressible curve on $f(S)$. Lemma 8.2 is the analogue of Proposition 8.12.1 in [46], which establishes the same property for pleated surfaces.

Lemma 8.2. *Let $f : S \rightarrow N$ be a simplicial hyperbolic surface such that every compressible curve in $f(S)$ has length $\geq a$. Then there exists a constant C (depending only on a) such that the volume of $\mathcal{N}_1(f(S))$ is less than $C|\chi(S)|$ (where $\mathcal{N}_1(f(S))$ denotes the set of points in N of distance less than 1 from $f(S)$).*

Proof. Given $x \in S$, let c_x denote the “characteristic function” of an immersed ball of radius $1 + \frac{a}{2}$; i.e. $c_x(y)$ is the number of distinct homotopy classes of paths from $f(x)$ to y of length $\leq 1 + \frac{a}{2}$. We further define

$$g(y) = \int_S c_x(y) dA.$$

Then

$$\int_N g dV \leq -2\pi\chi(S)\text{vol}(B_{1+\frac{a}{2}}^3),$$

where $\text{vol}(B_{1+\frac{a}{2}}^3)$ denotes the volume of a ball of radius $1 + \frac{a}{2}$ in $\mathbf{H}^3$, since

$$\int_N g dV = \int_N \int_S c_x(y) dA dV = \int_S \int_N c_x(y) dV dA = \int_S \text{vol}(B_{1+\frac{a}{2}}^3) dA.$$

(Recall that $f(S)$ has area $\leq 2\pi|\chi(S)|$.)

Now for any point $y \in \mathcal{N}_1(f(S))$, there is some point $x \in S$ such that $d(y, f(x)) \leq 1$. So there exists a contribution to $g(y)$ for every point z on S with $d(z, x) \leq \frac{a}{2}$ (and for each homotopy class of paths between z and x of length $\leq \frac{a}{2}$.) Thus, for all $y \in \mathcal{N}_1(f(S))$,

$$g(y) \geq \text{vol}(B_{\frac{a}{2}}^2)$$

where $\text{vol}(B_{\frac{a}{2}}^2)$ denotes the area of a ball of radius $\frac{a}{2}$ in $\mathbf{H}^2$. (Here we use our assumption that there are no compressible curves in S of length $\leq a$ to see that a homotopy class of paths in the surface corresponds to a homotopy class

of paths in the 3-manifold.) So,

$$\text{vol}(\mathcal{N}_1(f(S))) \leq \frac{1}{\text{vol}(B_{\frac{q}{2}}^2)} \int_N g \, dV \leq \frac{-2\pi \text{vol}(B_{1+\frac{q}{2}}^3)}{\text{vol}(B_{\frac{q}{2}}^2)} \chi(S). \quad \square$$

We first explain the proof of Theorem 8.1 in the simpler case when N has no cusps. A point x in a manifold M is said to be *enclosed* by a subset R of M , if every properly embedded ray originating at x intersects R . In particular, R always lies in the set enclosed by R . Let $\{E_i\}_{i=1}^m$ be the simply degenerate ends of N . For each i we choose a sequence of useful simplicial hyperbolic surfaces $\{f_j^i : S_i \rightarrow U^i\}$ exiting E_i where U^i is a neighborhood of E_i which is homeomorphic to $\bar{S}_i \times [0, \infty)$. By passing to a subsequence we can assume that $f_j^i(S_i) \cap f_k^i(S_i) = \emptyset$ and $f_j^i(S_i)$ is enclosed by $f_l^i(S) \cup \partial U^i$ whenever $j < l$. Furthermore we may choose $\{f_j^i\}$, by Lemma 7.1, such that if C is any compressible curve on S_i that $l_N(f_j^i(C)) \geq a$ for any prechosen a and all j . We now define $Z_j = \bigcup_{i=1}^m f_j^i(S_i)$. Then $\text{vol}(\mathcal{N}_1(Z_j)) \leq -C \sum \chi(S_i)$ (for some constant C given by Lemma 8.2) and $\text{area}(Z_j) \leq 2\pi \sum |\chi(S_i)|$. Let C_j be the set of points in N which are enclosed by Z_j and $\partial C(N)$. Then C_j is the desired exhausting sequence.

If N has cusps, some of our simplicial hyperbolic surfaces may be non-compact and we need to modify the construction of C_j given above. The essential step in this modification consists of cutting off a portion of the intersection of each simplicial hyperbolic surface with each cusp and appending a horocyclic cylinder joining the new boundary either to $\partial C(N)$ or to the boundary component of another truncated simplicial hyperbolic surface. This modification is essentially that suggested by the picture on p. 8.77 in [46].

We will construct a sequence $\{Z_j\}$ of compact surfaces in $C(N)$ such that

- (i) $\partial Z_j \subset \partial C(N)$,
- (ii) there exists a portion $\partial C(N)_j$ of $\partial C(N)$ such that the set of points enclosed by $Z_j \cup \partial C(N)_j$ is a compact submanifold C_j and $\bigcup_{j=1}^{\infty} C_{j_k}^0 = C(N)$ for any subsequence $\{C_{j_k}\}$,
- (iii) $\text{area}(Z_j) \leq K$ and $\text{vol}(\mathcal{N}_1(Z_j)) \leq L$ for some constants K and L and all j ,
- (iv) and, for any j there exists $k > j$ such that $C_j \subset C_k$.

Once we have constructed such a sequence we may pass to a subsequence to obtain an exhausting sequence of compact submanifolds satisfying conditions (1)–(4) in the definition of analytic tameness. Recall (see Epstein-Marden [17]) that the boundary of the convex hull is a pleated surface and thus has total area $2\pi|\chi(S)|$. Moreover, proposition 8.12.1 in [46] (whose proof is the same as that of Lemma 8.2) assures us that the neighborhood of radius one of the boundary of the convex core has finite volume.

We first construct a set of coordinates on each of the cusps of N . Let $\{P_1, \dots, P_m\}$ be the cuspidal components of $N_{\text{thin}(\varepsilon)}$. If P_k is a torus cusp it is homeomorphic to $S^1 \times S^1 \times [0, \infty)$ and we give it coordinates (θ_k, ϕ_k, s_k) such that $\gamma(s_k) = (\alpha, \beta, s_k)$ is a unit speed geodesic for all constants α and β , and the surface $\{s_k = d\}$ is a (projection of a) horosphere for all $d \geq 0$. In these coordinates the metric takes the form

$$e^{-2s_k}(f d\theta_k^2 + g d\theta_k d\phi_k + h d\phi_k^2) + ds_k^2$$

where $f d\theta_k^2 + g d\theta_k d\phi_k + h d\phi_k^2$ is a flat metric on the torus.

If P_k is a rank one cusp, then it is homeomorphic to $S^1 \times (-\infty, \infty) \times [0, \infty)$ and we give it coordinates (θ_k, r_k, s_k) as in §2. Notice that in these coordinates, $\gamma(s_k) = (\alpha, r, s_k)$ is a unit-speed geodesic for all constants α and r , the surface $\{s_k = d\}$ is a horosphere for all $d \geq 0$, and $\lambda(r_k) = (\alpha, r_k, 0)$ is a unit speed horocycle for all α . If P_k is a rank-one cusp we let $P_k(D)$ be the subset of P_k described by $\{(\theta_k, r_k, s_k) | s_k \geq D\}$ and if P_k is a rank-two cusp $P_k(D) = \{(\theta_k, \phi_k, s_k) | s_k \geq D\}$.

Let P_k be a rank-one cusp and F_n be a component of $\partial C(N)$. If we choose W_0^k large enough, F_n intersects $P_k(W_0^k)$ in a set of the form $\{r_k = b_k^n\}$ if at all (see Epstein-Marden [17] or [46]). If only one component of $\partial C(N)$ intersects P_k we normalize so that $b_k^n = 0$ and so that $\{r_k \geq 0\} \subset C(N)$; otherwise we normalize so that the curve $\{(\theta, 0, 0)\} \subset C(N)$.

Now let $\{E_i\}_{i=1}^m$ denote the collection of simply degenerate ends of N_ε^0 . For each i we choose a sequence of useful simplicial hyperbolic surfaces $\{f_j^i : S_i \rightarrow U_P^i\}$ exiting E_i where U^i is a neighborhood of E_i whose parabolic extension U_P^i is homeomorphic to $S_i \times [0, \infty)$. Again we may assume (by passing to a subsequence) that if C is any compressible curve on S_i then $l_N(f_j^i(C)) \geq a$ for some prechosen a and all j . Notice that since f_j^i is a useful simplicial hyperbolic surface, there exists D_j^i such that if $f_j^i(S_i)$ intersects $P_k(D_j^i)$, the intersection has the form $\{r_k = c_{jk}^i\}$ for some constant c_{jk}^i . (With a little more care, we could show that D_j^i can be chosen to be a uniform constant for all i and j .) Let $t_1 = \max\{|c_{1k}^1|, D_1^1, W_0^k, b_k^n, 1\}$. We inductively define $t_j = \max\{|c_{jk}^i|, D_j^i, t_{j-1} + 1\}$.

Let $N(t_j)$ denote $N - (\bigcup_{k=1}^m P_k(t_j))$. We define Y_j to be $(\bigcup_{i=1}^n f_j^i(S_i)) \cap N(t_j)$ and $\partial C(N)_j = \partial C(N) \cap N(t_j)$. Let Z_j consist of the union of Y_j with the set of points in $\partial N(t_j)$ which are enclosed (within $\partial N(t_j)$) by $Y_j \cup \partial C(N)_j$. We will now explicitly describe each piece of Z_j so that it will be clear that conditions (1)–(4) are verified.

There are 4 types of pieces of Z_j :

- (a) components of Y_j ;
- (b) if P_k intersects the parabolic extension of two simply degenerate ends,

then $Z_j \cap P_k - Y_j$ is a horocyclic annulus contained in

$$\{\theta_k \in [0, 2\pi), r_k \in [-t_j, t_j], s_k = t_j\}$$

(It has the precise form $\{\theta_k \in [0, 2\pi), r_k \in [-c_{jk}^i, c_{jk}^l], s_k = t_j\}$ for some i and l);

(c) if P_k intersects the parabolic extension of one simply degenerate end, then $Z_j \cap P_k - Y_j$ is a horocyclic annulus contained in

$$\{\theta_k \in [0, 2\pi), r_k \in [0, t_j], s_k = t_j\}$$

(it has the precise form $\{\theta_k \in [0, 2\pi), r_k \in [0, c_{jk}^i], s_k = t_j\}$ for some i);

(d) if P_k does not intersect the parabolic extension of any simply degenerate end, then $Z_j \cup P_k$ is the horocyclic surface

$$V_j^k = \{s_k = t_j\} \cap C(N) = \partial P_k(t_j) \cap C(N).$$

To complete the proof we now only need check that our Z_j satisfies conditions (i)–(iv) as claimed. Conditions (i) and (ii) are obviously satisfied, so we only need check conditions (iii) and (iv).

If R_j is the component of Y_j lying in E_i then it has area less than $-2\pi\chi(S_i)$ since it is a portion of a simplicial hyperbolic surface, and Lemma 8.2 guarantees that its neighborhood of radius one has volume less than $C|\chi(S_i)|$. Each component of type (b) or (c) has area less than $(4\pi G t_j)e^{-2t_j} < 4\pi G$ and its neighborhood of radius one has volume less than $8G\pi(t_j + e^{t_j})e^{-2t_j+2} < 32G\pi$. Suppose V_j^k is a component of type (d). Then the area of V_j^k is less than the area of V_1^k (in fact less than $e^{2(t_1-t_j)}\text{area}(V_1^k)$), while its neighborhood of radius one has volume less than the volume of the neighborhood of radius one of V_1^k (in fact, less than $e^{2(t_1-t_j)}\text{vol}(\mathcal{N}_1(V_1^k))$).

We will explain how one obtains the above estimates for a component of type (b), as the arguments for type (c) and (d) are virtually identical. Each component of type (b) is contained in the region parameterized by $S^1 \times [-t_j, t_j] \times \{t_j\}$ which has induced metric $e^{-2t_j}(G^2 d\theta_k^2 + dr_k^2)$ and thus has area $(4Gt_j\pi)e^{-2t_j}$. The neighborhood of radius one is contained in the submanifold of P_k parameterized by $S^1 \times [-t_j - e^{t_j}, t_j + e^{t_j}] \times [t_j - 1, t_j + 1]$ and thus has volume less than $2(2\pi G)(2(t_j + e^{t_j}))e^{-2t_j+2}$. Since the number of components is the same for all j and we have obtained uniform bounds on the area of each component and the volume of its neighborhood of radius one, this completes the verification of condition (iii).

Given any j we may choose $k > j$ such that $f_k(S_i) \cap f_j(S_i) = \emptyset$ for all i , and $f_j(S_i) \cap N_\epsilon^0$ is enclosed, within N_ϵ^0 , by $(f_k(S_i) \cap N_\epsilon^0) \cup \partial U^i$ for all i . Then we see, by construction, that $C_j \subset C_k$, which verifies condition (iv) and completes the proof. $\square$

9. ANALYTIC CONSEQUENCES

In this section we will derive a minimum principle for superharmonic functions on analytically tame hyperbolic 3-manifolds and apply it to prove Corollaries 1 and 2. Many of the arguments in this section are generalizations of ones given by Thurston [46] in proving the same results for geometrically tame hyperbolic 3-manifolds which have compact cores with incompressible boundary (see also Sullivan [45]).

Theorem 9.1. *If N is an analytically tame hyperbolic 3-manifold, then for every nonconstant positive superharmonic (i.e. $\Delta h \leq 0$) function h on N ,*

$$\inf_{C(N)} h = \inf_{\partial C(N)} h$$

where $C(N)$ denotes the convex core of N . In particular, if $C(N) = N$ (i.e. $L_\Gamma = S^2$) then there are no positive nonconstant superharmonic functions on N .

We will first give proofs of the main corollaries.

Corollary 1. *If $N = \mathbf{H}^3/\Gamma$ is an analytically tame, 3-manifold, then either L_Γ is all of S_∞^2 or it has measure zero. Moreover, if $L_\Gamma = S_\infty^2$ then Γ acts ergodically on S_∞^2 .*

Proof of Corollary 1. First recall that if ν is any measure on the sphere at infinity S_∞^2 , then

$$\phi_\nu(x) = \int_{S_\infty^2} |\alpha'_x|^2 d\nu$$

is a positive harmonic function on $\mathbf{H}^3$ where α_x is a hyperbolic isometry taking x to 0 (see Nicholls [35], Patterson [37] or Sullivan [44].) Explicitly,

$$|\alpha'_x(\xi)| = \frac{1 - |x|^2}{|x - \xi|^2}.$$

(All explicit calculations are done in the Poincaré ball model for $\mathbf{H}^3$.) Let A be any Γ -invariant subset of L_Γ and let μ_A be Lebesgue measure restricted to A (i.e. $\mu_A(E)$ is simply the Lebesgue measure m of $A \cap E$). Let $\mu'_A = \mu_A/4\pi$ and let $h_A(x)$ be the harmonic function $\phi_{\mu'_A}$. Since A is Γ -invariant, h_A is Γ -invariant and hence descends to a harmonic function on N . Notice that

$$h_A(x) = \int_{S_\infty^2} \frac{1}{4\pi} d\mu_{\alpha_x(A)} = \frac{m(\alpha_x(A))}{4\pi}.$$

(Topologically, $h_A(x)$ is the proportion of x 's visual sphere taken up by A , i.e. the proportion of the infinite rays beginning at x which have endpoints in A in the measure induced by the Riemannian metric on $T_1(\mathbf{H}^3)_x$.)

First suppose that $L_\Gamma \neq S_\infty^2$. Let x be any point in $\mathbf{H}^3$ which either lies in the complement of the convex hull $CH(L_\Gamma)$ or on the boundary of the convex hull. There exists a totally geodesic plane H bounding a half-space

which intersects $CH(L_\Gamma)$ only in H if at all (see Epstein-Marden [17]). Let α_x be an isometry taking x to 0. Then $\alpha_x(A)$ lies entirely in a hemisphere L bounded by $\alpha_x(H)$. Thus,

$$h_A(x) = m(\alpha_x(A))/4\pi \leq m(L)/4\pi = \frac{1}{2}.$$

Thus, $h_A(x) \leq \frac{1}{2}$ on $N - C(N)$. Since $(1 - h_A)$ is also positive and harmonic, Theorem 9.1 implies that

$$\inf_{C(N)}(1 - h_A) = \inf_{\partial C(N)}(1 - h_A) \geq \frac{1}{2}.$$

Therefore, $h_A \leq \frac{1}{2}$ on all of N .

Now suppose A has positive measure. Then there exists a point of density ξ such that

$$\lim_{\delta \rightarrow 0} \frac{\mu_A(B(\delta, \xi))}{m(B(\delta, \xi))} = 1$$

where $B(\delta, \xi)$ denotes the ball of radius δ about ξ . But this implies that if $x_i \in \mathbf{H}^3$ converges to ξ (along a geodesic), then $h_A(x_i)$ converges to 1, which contradicts the fact that $h_A \leq \frac{1}{2}$. In particular, L_Γ itself has measure zero.

If $L_\Gamma = S_\infty^2$, h_A must be constant, by Theorem 9.1. If A does not have measure zero, then there exists a point of density ξ for A . Thus, as above, if $\{x_i\}$ converges to ξ (along a geodesic), then $h_A(x_i)$ converges to 1. Therefore, $h_A(x) = 1$ for all $x \in \mathbf{H}^3$. Now we notice that $1 - h_A(x) = h_{A^C}(x)$ where $A^C = S_\infty^2 - A$. By the same argument, if A^C does not have measure zero, $h_{A^C}(x) = 1$ for all $x \in \mathbf{H}^3$. As these two possibilities are mutually exclusive, A must have either full or zero measure. Thus, Γ acts ergodically on S_∞^2 . $\square$

The following corollary of the minimum principle is a characterization of when an analytically tame hyperbolic 3-manifold is strongly parabolic. Recall that a Riemannian manifold is said to be *strongly parabolic* if and only if it has no nonconstant positive superharmonic functions.

Corollary 9.2. *An analytically tame hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$ is strongly parabolic if and only if $L_\Gamma = S_\infty^2$. Moreover, if N is not strongly parabolic it admits nonconstant bounded harmonic functions.*

Proof. If $L_\Gamma = S_\infty^2$ the minimum principle Theorem 9.1 assures us that N is strongly parabolic. If $L_\Gamma \neq S_\infty^2$, we may choose a Γ -invariant subset A of $S_\infty^2 - L_\Gamma$ which is neither of full or zero measure and let h_A be the bounded harmonic function obtained in the proof of Corollary 1. $\square$

Corollary 2. *Let $N = \mathbf{H}^3/\Gamma$ be an analytically tame hyperbolic 3-manifold. N 's geodesic flow is ergodic if and only if $L_\Gamma = S_\infty^2$.*

Proof of Corollary 2. In [44] (see also [35] and [46]) Sullivan proved that N 's geodesic flow is ergodic if and only if N has no (positive) Green's function. Recall that a (positive) Green's function G with pole at $x_0 \in N$ is a function

$G : N - \{x_0\} \rightarrow \mathbf{R}_+$ such that $\Delta(G) = 0$ and such that in a neighborhood of x_0 , $G(x) = g(x) + h(x)$ where $h(x)$ is bounded and $g(x) = (\frac{1}{r(x)} + r(x) - 2)$ where $r(x) = \tanh(d(x, x_0)/2)$.

Now suppose that N has a positive Green's function G . Let $f(x) = 1 - e^{-x}$. Then $f \circ G$ is a positive superharmonic function on N . Thus, Theorem 9.1 guarantees that $L_\Gamma \neq S^2$. (Alternatively, the proof of Theorem 9.1 may be modified to show directly that if $L_\Gamma = S^2$, then N does not admit a positive Green's function. Notice that the flow on $N - \{x_0\}$ determined by $-\nabla G$ is volume preserving and exists for all positive time.)

If $L_\Gamma \neq S_\infty^2$, then N admits nonconstant bounded harmonic functions which implies it admits a (positive) Green's function (see e.g. Ahlfors [2] or Nicholls [35]). If one wishes to be more explicit, the set of geodesics which intersect $\partial C(N)$ is an invariant subset of the geodesic flow which has neither full or zero measure, so the geodesic flow is not ergodic. (Observe that if D is a disk in $S_\infty^2 - L_\Gamma$, then no geodesic both of whose endpoints lie in D intersects the convex hull of the limit set. Conversely, if γ is any geodesic in H^3 which intersects $CH(L_\Gamma)$ transversely, one can find disks D_1 and D_2 about the endpoints of γ , such that any geodesic with one endpoint in D_1 and the other in D_2 intersects $CH(L_\Gamma)$.) $\square$

We now turn to the proof of Theorem 9.1.

Proof of Theorem 9.1. Let h be a nonconstant positive superharmonic function. The proof consists of examining the flow ϕ_t generated by $-\nabla h$. Notice that this flow is volume nondecreasing, since $\operatorname{div}(-\nabla h) = -\Delta h \geq 0$.

If $A \subset \{\phi_t(x) : t \geq 0\}$ is a portion of the forward trajectory of the point x , let $l(A)$ denote the length of A in N and $T(A)$ the amount of time the trajectory spends in A . Let ds denote the element of arc length on A . We apply the Schwartz inequality to obtain

$$l(A)^2 = \left(\int_A \frac{1}{\sqrt{|\nabla h|}} \sqrt{|\nabla h|} ds \right)^2 \leq \left(\int_A \frac{1}{|\nabla h|} ds \right) \left(\int_A |\nabla h| ds \right) \leq T(A)h(x)$$

since h is positive and decreases along flow lines. We may rewrite this as

$$T(A) \geq \frac{l(A)^2}{h(x)}.$$

Notice that this implies that our flow is defined for all positive time, as it takes an infinite amount of time to travel out an end.

Let $\{C_j\}$ be an exhausting sequence of submanifolds as in the definition of analytical tameness. Let Z_j denote $\partial C_j - \partial C(N)$. We may pass to a subsequence such that $\mathcal{N}_1(Z_j) \cap \mathcal{N}_1(Z_i) = \emptyset$ whenever $i \neq j$. Let $D = \{x \in C(N) | \nabla h = 0\}$. Notice that if $D = C(N)$ then h is constant on $C(N)$ and the theorem is trivially true.

We claim that almost all trajectories starting at points in $C(N) - D$ intersect the boundary $\partial C(N)$. Given $x \in C(N) - D$, we may choose a neighborhood U of x such that $\max_{x \in U} h(\phi_1(x)) < \min_{x \in U} h(x)$. This guarantees that $U \cap \phi_n(U) = \emptyset$ for all positive integers n . We may also assume, by scaling h if necessary, that $\max_{x \in U} h(x) < 1$. Suppose that there exists a subset B of U of positive measure such that the flow line beginning at any point in B does not intersect $\partial C(N)$. Since volume is nondecreasing under the flow and C_n is compact, given any n we may find a subset $B(n)$ of B and an integer $R(n)$ such that if $x \in B(n)$, then $\phi_t(x) \in N - C_n$ for some $t \leq R(n)$ and $2 \text{vol}(B(n)) > \text{vol}(B)$. (In fact, we may choose $R(n) \leq 2(\text{vol}(C_n)/\text{vol}(B)) + 1$.) Then given $\varepsilon > 0$, we may subdivide $B(n)$ into subsets $\{B^1(n, \varepsilon), \dots, B^m(n, \varepsilon)\}$ such that if $t \leq R(n)$, then $\phi_t(B^i(n, \varepsilon))$ has diameter less than ε for all i . Now choose $x_i \in B^i(n, \varepsilon)$ and let A_i be a portion of the flow line beginning at x_i which intersects each component of $\bigcup_{j=1}^n \mathcal{N}_{1-\varepsilon}(Z_j)$ in one arc whose end-points lie on opposite sides of the component. Since $l(A_i) > 2n - 2n\varepsilon$ and $h(x_i) < 1$, $T(A) \geq (2n - 2n\varepsilon)^2$. This assures us that $\phi_t(x_i) \in \bigcup_{j=1}^n \mathcal{N}_{1-\varepsilon}(Z_j)$ for at least $(2n - 2n\varepsilon)^2 - 2n$ integers t . But if $\phi_t(x_i) \in \bigcup_{j=1}^n \mathcal{N}_{1-\varepsilon}(Z_j)$, then $\phi_t(B^i(n, \varepsilon)) \subset \bigcup_{j=i}^n \mathcal{N}_1(Z_j)$. Therefore,

$$\text{vol}\left(\left\{\bigcup_{t=1}^{R(n)} \phi_t(B^i(n, \varepsilon))\right\} \cap \left\{\bigcup_{j=1}^n \mathcal{N}_1(Z_j)\right\}\right) \geq ((2n - 2n\varepsilon)^2 - 2n)\text{vol}(B^i(n, \varepsilon)).$$

Since this inequality is valid for all i we may sum to obtain

$$\text{vol}\left(\left\{\bigcup_{t=1}^{R(n)} \phi_t(B(n))\right\} \cap \left\{\bigcup_{j=1}^n \mathcal{N}_1(Z_j)\right\}\right) \geq ((2n - 2n\varepsilon)^2 - 2n)\text{vol}(B(n)).$$

If we combine this inequality with the observations that $2 \text{vol}(B(n)) > \text{vol}(B)$ and that $\text{vol}(\bigcup_{j=1}^n \mathcal{N}_1(Z_j)) < nL$ we obtain

$$2nL > ((2n - 2n\varepsilon)^2 - 2n)\text{vol}(B).$$

But the claim that this inequality is true for all n produces a contradiction.

So given $x \in C(N) - D$ we may choose a sequence of points $\{x_i\}$ approaching x such that each x_i is the starting point of a flow line which intersects $\partial C(N)$. For every x_i there exists $y_i \in \partial C(N)$ such that $h(x_i) \geq h(y_i)$, so $h(x) \geq \inf_{\partial C(N)} h$. Now suppose $x \in D$. Let $R_x = \max\{r | \bar{B}(r, x) \subset D\}$ where $\bar{B}(r, x) = \{y \in N | d(x, y) \leq r\}$. Since $\nabla h = 0$ on $\bar{B}(R_x, x)$, h is constant on $\bar{B}(R_x, x)$. But there exist a sequence of points $x_i \in C(N) - D$ converging to a point $x_\infty \in \partial \bar{B}(R_x, x)$. Thus, since $h(x_i) \geq \inf_{\partial C(N)} h$, we see that $h(x_\infty) \geq \inf_{\partial C(N)} h$. But since h is constant on $\bar{B}(R_x, x)$, $h(x) \geq \inf_{\partial C(N)} h$.

If $L_\Gamma = S^2$ then $\partial C(N) = \emptyset$ and for every point $x \in N - D$, $B = U$. Therefore, the conclusion that B has measure zero, implies that $D = C(N)$ and hence that h is constant. Therefore, N admits no nonconstant positive superharmonic functions. $\square$

Remarks. (1) Another consequence of Theorem 9.1 is that if $L_\Gamma = S_\infty^2$ the Patterson-Sullivan measure on L_Γ (see Sullivan [44], Patterson [37] or Nicholls [35] for definitions) is the same as Lebesgue measure. This observation is a key step in the proof of Culler and Shalen's [15] main theorem.

(2) In [16] Eberlein proved that if the limit set of a visibility manifold is the entire sphere at infinity, then the geodesic flow is topologically mixing.

(3) Cheng and Yau [14] proved that if N is a Riemannian manifold whose volume grows quadratically, then N is strongly parabolic (see also Karp [25]). The proof of Theorem 9.1 can be modified to give a (perhaps weaker) version of their results.

(4) Let N be a finite volume, noncompact hyperbolic 3-manifold which fibers over the circle. If $\hat{N}$ is the cover associated to the fiber subgroup, then $\hat{N}$ is strongly parabolic, but has exponential volume growth.

10. ENDING LAMINATIONS

In this section we will show that associated to each simply degenerate end of a hyperbolic 3-manifold there is a well-defined geodesic lamination, called the ending lamination. We will generalize the analysis of Thurston [46] (as corrected by Bonahon [5]). We will assume more familiarity with the techniques of [6] in this section.

Let E be a simply degenerate end of N_ε^0 which has a neighborhood U homeomorphic to $\bar{S} \times [0, \infty)$ and let S be a finite area hyperbolic surface homeomorphic to the interior of $\bar{S}$. Let $\{\alpha_j^*\}$ be a collection of closed geodesics exiting E such that each α_j^* is either the core geodesic of a compact component of $N_{\text{thin}(\varepsilon)}$ or is entirely disjoint from $N_{\text{thin}(\varepsilon)}$. Let α_j be a geodesic on S which is homotopic to α_j^* within U . Let $\{\alpha_j/l_{S_0}(\alpha_j)\}$ be the associated sequence of unit length currents in $\mathcal{C}(S)$. The intersection number lemma, Lemma 4.2, implies that $\{\alpha_j/l_{S_0}(\alpha_j)\}$ converges to a measured lamination α_∞ . Let $\lambda(E)$ be the geodesic lamination which is the support of α_∞ , we will call $\lambda(E)$ the *ending lamination* of E . Proposition 10.1 justifies this definition.

Proposition 10.1. *Let E be a simply degenerate end of N_ε^0 which has a neighborhood U homeomorphic to $\bar{S} \times [0, \infty)$. There exists a unique geodesic lamination $\lambda(E)$ such that if $\{\gamma_i^*\}$ is a collection of closed geodesics exiting E which are homotopic (within U) to curves γ_i on $S \times \{0\}$, such that $\{\gamma_i/l_S(\gamma_i)\}$ converges to a measured lamination γ_∞ , then γ_∞ has support $\lambda(E)$. Moreover, if μ is any measured lamination whose support is different than $\lambda(E)$, then $i(\mu, \gamma_\infty) \neq 0$.*

Proof. Let E' , U' and $\hat{N}$ be as in the proof of the main theorem. We first establish that if μ is any measured lamination whose support is different than α_∞ , then $i(\mu, \alpha_\infty) \neq 0$. Suppose that $i(\mu, \alpha_\infty) = 0$ but $\text{supp}(\mu) \neq \text{supp}(\alpha_\infty)$. Then either $\text{supp}(\mu) - \text{supp}(\alpha_\infty)$ or $\text{supp}(\alpha_\infty) - \text{supp}(\mu)$ is a non-trivial geodesic lamination ν . (Recall that the support of each measured lamination is made up of a finite number of disjoint minimal components.) We may

therefore choose $\varepsilon > 0$ such that the boundary of the neighborhood $\mathcal{N}_\varepsilon(\nu)$ of radius ε of ν does not intersect α_∞ . Since $S - \mathcal{N}_\varepsilon(\nu)$ contains the support of a nontrivial measured lamination, one of the boundary components β of $\mathcal{N}_\varepsilon(\nu)$ is homotopically nontrivial and $i(\beta, \gamma_\infty) = 0$. Further, choose a sequence of simple closed geodesics $\{\beta'_i\}$ on S , each of which does not intersect β such that $\{\beta'_i/l_{S_0}(\beta'_i)\}$ converges to α_∞ . We then see, as in the proof of Theorem 4.1, that the geodesic representatives $(\widehat{\beta}'_i)^*$ of $j^{-1}(\beta'_i \times \{0\})$ exit E' . We may now form a sequence of simplicial ruled surfaces $\{\widehat{f}_i: S \rightarrow U'_p\}$ (properly homotopic to $\partial U'_p$ within U'_p) such that both β'_i and β are mapped to their geodesic representatives. Then, by the bounded diameter lemma, $(\widehat{\beta}'_i)^*$ cannot exit E' , which is a contradiction.

Let C_k be a ruled annulus realizing the homotopy between $\widehat{\gamma}_k = j^{-1}(\gamma_k \times \{0\})$ and $\widehat{\gamma}_k^*$, and A_k a (ruled) homotopy between $\widehat{\alpha}_k = j^{-1}(\alpha_k \times \{0\})$ and $\widehat{\alpha}_k^*$. We recall, from [6], that if $\widehat{\gamma}_k^*$ does not intersect A_j , then

$$i(\alpha_j, \gamma_k) \leq i(\widehat{\alpha}_j^*, C_k).$$

We notice that this property holds for all k which are sufficiently large with respect to j . (By choosing k large enough we are avoiding the problems which may arise if γ_k^* intersects $N_{\text{thin}(\mu)}$ but is not the core of a Margulis tube.) Now, as in Bonahon's proof of the intersection number lemma,

$$i(\widehat{\alpha}_j^*, C_k) \leq Ke^{-D(j)} l_S(\alpha_j) l_S(\gamma_k) + 2,$$

where K is independent of j and k and $D(j)$ denotes the distance from $\widehat{\alpha}_j^*$ to $\widehat{S} = j^{-1}(S \times \{0\})$. So,

$$i\left(\frac{\alpha_j}{l_S(\alpha_j)}, \frac{\gamma_k}{l_S(\gamma_k)}\right) \leq Ke^{-D(j)} + \frac{2}{l_S(\alpha_j) l_S(\gamma_k)}.$$

Letting j and k go to infinity, we see that $i(\alpha_\infty, \gamma_\infty) = 0$. Therefore, α_∞ and γ_∞ have the same support. $\square$

The following corollary is of interest because it has potential application in the deformation theory of hyperbolic structures on 3-manifolds with compressible boundary (see [36, 8, 10]). We will restrict to the situation where our end is not adjacent to any cusps as the technology of the Masur domain has not been explicitly formulated in the relative situation. In the simplest situation, this corollary asserts that if E is a simply degenerate end of a topologically tame hyperbolic 3-manifold without cusps, then its ending lamination lies in the Masur domain. (See §3.4 for a discussion of the Masur domain.)

Corollary 10.2. *Let N be a topologically tame hyperbolic 3-manifold and let M be a compact core for N such that each component of $N - M$ is topologically a product. Let S be a boundary component of M and let U be the component of $N - M$ which has S on its boundary. Suppose that U intersects no cusps of N*

and is a neighborhood of a simply degenerate end E . Let α be any measured lamination with support $\lambda(E)$. Then $\alpha \in \mathcal{D}(S, M)$.

Proof. We first prove that α is not in $\mathcal{L}$. If it was, there would exist a sequence of simple compressible closed curves $\{\beta_j\}$ with $\{\beta_j/l_S(\beta_j)\}$ converging to α , in $\text{ML}(S)$. Let $\hat{N}$ and E' be as in the proof of the main theorem. As in Bonahon's proof, this implies that the geodesic representatives of $\{j^{-1}(\beta_j)\}$ exit E' . But this is a contradiction, as we have previously established (in the proof of Lemma 7.1) that the geodesic representative (in $\hat{N}$) of every compressible curve (in S) must intersect a fixed compact region B .

Now suppose that $i(\alpha, \lambda) = 0$. Then, by Proposition 10.1, α and λ have the same support, which implies, as above, that λ cannot be in $\mathcal{L}$. Therefore, α is in the Masur domain. $\square$

11. HYPERBOLIC 3-ORBIFOLDS

If Γ is a discrete subgroup of $\text{Isom}(\mathbf{H}^3)$, the group of isometries of $\mathbf{H}^3$, Γ is said to be a Kleinian group and the quotient $N = \mathbf{H}^3/\Gamma$ is a 3-dimensional hyperbolic orbifold (see Thurston [46] for a discussion of orbifolds.) It is a well-known result of Selberg's [42] that if Γ is finitely generated it has a finite index subgroup Γ' such that Γ' is torsion-free; i.e. $\mathbf{H}^3/\Gamma'$ has a finite cover $\mathbf{H}^3/\Gamma'$ which is a manifold. We may further assume, by passing to the orientable double cover if necessary, that $N' = \mathbf{H}^3/\Gamma'$ is an orientable hyperbolic 3-manifold. We will use this result to avoid giving an intrinsic definition of topological tameness for hyperbolic 3-orbifolds.

We will say that a Kleinian group Γ (equivalently a hyperbolic 3-orbifold $\mathbf{H}^3/\Gamma$) is *topologically tame* if it has a finite index subgroup Γ' such that $\mathbf{H}^3/\Gamma'$ is a topologically tame (orientable) 3-manifold. We will say that Γ is *geometrically tame* if it has a finite index subgroup Γ' such that $\mathbf{H}^3/\Gamma'$ is a geometrically tame (orientable) hyperbolic 3-manifold. Notice that these definitions, in particular, extend to nonorientable hyperbolic 3-manifolds. Also notice that this definition is reasonable, since if one finite cover of N which is a manifold is topologically tame then all finite covers which are manifolds are topologically tame.

Our main result then generalizes to

Theorem 11.1 (Main theorem for orbifolds). *A finitely generated Kleinian group is topologically tame if and only if it is geometrically tame.*

And since $L_\Gamma = L_{\Gamma'}$ if Γ' is a finite index subgroup of Γ , we also obtain

Corollary 11.2 (Corollary 1 for orbifolds). *If Γ is a topologically tame Kleinian group, its limit set L_Γ is either all of S_∞^2 or has measure zero. Moreover, if $L_\Gamma = S_\infty^2$, Γ acts ergodically on S_∞^2 .*

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ABSTRACT. Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold which is homeomorphic to the interior of a compact 3-manifold. We prove that N is geometrically tame. As a consequence, we prove that Γ 's limit set L_Γ is either the entire sphere at infinity or has measure zero. We also prove that N 's geodesic flow is ergodic if and only if L_Γ is the entire sphere at infinity.

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