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NAVIER-STOKES EQUATIONS ON THIN 3D DOMAINS. I: GLOBAL ATTRACTORS AND GLOBAL REGULARITY OF SOLUTIONS

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CONTENTS

1. INTRODUCTION

The modern mathematical theory of fluid dynamics began over 50 years ago when Leray (1933, 1934a, 1934b) published his pioneering works on the Navier-Stokes equations. These equations describe the time evolution of solutions of mathematical models of viscid incompressible fluid flows. Because of this basic role in the modelling of fluid flows, there is considerable interest in developing a good mathematical theory of the behavior of the solutions of the Navier-Stokes equations. Since the solutions of these equations depend on both space and time, one is especially interested in the phenomenon of the time evolution of the spatial variations of the solutions. This phenomenon, which is described with more precision later, is referred to as the *regularity* of solutions, and it is the primary focus of the theory we present in this paper.

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The Navier-Stokes equations on a bounded region $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, are given by

$$
\begin{aligned}
U_t - \nu\Delta U + (U \cdot \nabla)U + \nabla P = F, \\
\nabla \cdot U = 0,
\end{aligned}
\tag{1.1}
$$

where ∇ is the gradient operator and Δ is the Laplacian. In this paper we treat the case where $\Omega = \Omega_\varepsilon$ is a thin 3-dimensional domain, i.e., $\Omega_\varepsilon = Q_2 \times (0, \varepsilon)$, where Q_2 is a suitable bounded region in R^2 and ε is a small positive parameter. In particular we will study (1.1) with periodic boundary conditions where $Q_2 = (0, l_1) \times (0, l_2)$, and l_1 and l_2 are positive.

Recall that the Navier-Stokes equations (1.1) on Ω can be written in the abstract form

$$U' + \nu AU + B(U, U) = \mathbb{P}_n F, \tag{1.2}$$

where $\mathbb{P}_n$ is the orthogonal projection of $L^2(\Omega, \mathbb{R}^n)$ onto the space of divergence-free vector fields, $AU = -\mathbb{P}_n\Delta U$, and $B(U, V) = \mathbb{P}_n(U \cdot \nabla)V$. We will be interested in solutions of (1.2) under the assumption that the initial data U_0 satisfy

$$U_0 \in D(A^{1/2}), \tag{1.3}$$

where $D(A^{1/2})$ is the domain of $A^{1/2}$; see Temam (1977, 1983) and Constantin and Foiaş (1988). We also assume that the forcing function $F = F(t)$ satisfies

$$F(\cdot) \in W^{1,\infty}([0, \infty), L^2(\Omega)). \tag{1.4}$$

In the case of periodic boundary conditions one has $D(A^{1/2}) \subset H^1_{\text{per}}(\Omega)$. We will also assume, in this case, that

$$\int_\Omega U_0\, dy = \int_\Omega F\, dy = 0.$$

The phrase *global regularity* of solutions, or existence of *strong solutions*, refers to the property that when U_0 and F satisfy (1.3) and (1.4), then (1.2) has a solution $U(t)$ that satisfies $U(0) = U_0$ and $U \in C^0([0, \infty), H^1(\Omega))$. The *principal outstanding problem* for the 3-dimensional Navier-Stokes equations (3DNS) is to determine whether or not (1.2) has a global regular solution for every U_0 and F satisfying (1.3) and (1.4).

The study of the regularity of solutions, both in 2-dimensions and 3-dimensions, has attracted widespread interest beginning with Leray (1933, 1934a, 1934b). We are unable to give a complete history of this study here, but special mention should be made of the important contributions of Hopf (1951), Kiselev and Ladyzhenskaya (1957), Serrin (1962), Fujita and Kato (1964), Masuda (1967), Komatsu (1980), and Caffarelli, Kohn, and Nirenberg (1982). Additional references can be found in Giga (1988). Before describing our results on the global regularity of solutions of the 3DNS, let us review some aspects of the classical theory of regularity of these solutions.

For the 3DNS it is known that for every U_0 and F satisfying (1.3) and (1.4) there is a T, which depends on U_0 and F, $0 < T \leq \infty$, such that (1.2) has a unique strong, or regular, solution $U(t)$ that satisfies $U \in C^0([0, T), H^1(\Omega)) \cap L^2_{\mathrm{loc}}((0, T), D(A))$ and $U' \in L^2_{\mathrm{loc}}((0, T), L^2(\Omega))$. Furthermore, if the data U_0 and F are small, then (as is known and as we show in §2.11) one has $T = \infty$, i.e., (1.2) has a globally regular solution for small data. Other than several theorems which establish the global regularity of solutions for small data, it is essentially unknown whether there are any other initial conditions U_0 and F for which (1.2) has a globally regular solution, see Constantin and Foiaş (1988), Ladyzhenskaya (1969), Lions (1969), Temam (1977, 1983, 1988), and von Wahl (1985).

The theory of global regularity of solutions of the 2-dimensional Navier-Stokes equations (2DNS) is quite different. In this case there exists a globally regular solution of (1.2) for all U_0 and F satisfying (1.3) and (1.4). Furthermore, one has $U(t) \in H^2(\Omega)$ for all $t > 0$, and there exist positive constants K and L_1, L_2 where L_1 and L_2 do not depend on U_0, such that

$$\|U(t)\|_{H^1(\Omega)} \le K, \qquad 0 \le t < \infty, \tag{1.5}$$

and

$$\limsup_{t\to\infty} \|U(t)\|_{H^i(\Omega)} \le L_i, \qquad i = 1, 2. \tag{1.6}$$

These classical results can be found in Ladyzhenskaya (1969), as well as in Constantin and Foiaş (1988) and Temam (1977, 1983, 1988). Because of the relevance of (1.5) and (1.6) for the 3-dimensional theory presented here, proofs of these relations are included in §5.

As a result of (1.5) and (1.6), it follows that in 2-dimensions, when F is time independent, (1.2) has a global attractor $\mathfrak{A}$, and $\mathfrak{A}$ is a compact set in $H^1(\Omega)$ and compact in $H^2(\Omega)$; see Ladyzhenskaya (1972), Hale (1988), and Temam (1988). This means that $\mathfrak{A}$ is a Lyapunov stable attracting invariant set[1] in $H^1(\Omega)$. If F is time-varying, but has some compactness property (e.g., $F(t)$ is Bohr almost periodic in t), then by using the theory of skew-product flows (see Sacker and Sell (1977, 1990) and §2.11) one can show that (1.2) generates a global attractor in $H^1(\Omega) \times \mathscr{F}$, where $\mathscr{F}$ is a compact positively invariant subset of $W^{1,\infty}([0, \infty), L^2(\Omega))$.

It is the existence of the global attractor for the 2DNS which is the *raison d'être* of our study of the 3DNS on thin domains. Because a thin 3-dimensional domain is somehow *close to* a 2-dimensional domain, it is natural to ask whether one can use the good properties of the 2DNS to study the global regularity of the 3DNS. As we shall see, the theory presented here gives an affirmative answer to this question.

The idea of exploiting the existence of a global attractor of an evolutionary equation on an n-dimensional domain to obtain better information for a corresponding equation on a thin $(n+1)$-dimensional domain has already been used in Hale and Raugel (1990, 1992a, 1992b). Also see Raugel (1989).

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[1]It is also the case that the global attractor $\mathfrak{A}$ for the 2DNS has finite dimension; see Mallet-Paret (1976). The dimensionality of $\mathfrak{A}$ has been widely studied; see Temam (1988) for references to this literature.

The process of exploiting the fact that Ω_ε is close to Q_2, when ε is small, is far from being trivial. The main reason for the complication is due to the fact that the 3DNS on Ω_ε is a singular perturbation of the 2DNS on Q_2. The regularization of this singular perturbation is done in two steps, and it follows the methods introduced in Hale and Raugel (1992a, 1992b) for reaction diffusion equations and damped wave equations on thin domains. First one maps Ω_ε onto $Q_3 = Q_2 \times (0, 1)$ by means of dilation. The Navier-Stokes equations (1.1) on Ω_ε are then transformed to the dilated Navier-Stokes equations on Q_3; see (2.4). This dilation alone does not remove the singular perturbation because some of the differential operators in (2.4) contain coefficients with ε^{-1}, or ε^{-2}, and ε is small. Nevertheless, since the domain is now fixed to be Q_3 for all $\varepsilon > 0$, this opens up the possibility of using other techniques from the theory of partial differential equations to regularize the singular perturbation in (2.4). The second step is accomplished by introducing the orthogonal projection $v = Mu$ where

$$v(x_1, x_2) = \int_0^1 u(x_1, x_2, s)\, ds.$$

By applying M and $(I-M)$ to the dilated Navier-Stokes evolutionary equation (2.5) one finds an equivalent system (2.23). What we effectively show in this paper is that the system (2.23) is a regular perturbation of the 2DNS when ε is small.

In Raugel and Sell (1989), we described the above method and announced some of our existence results. Since that time, we noticed that a more accurate Sobolev inequality given in Hale and Raugel (1992a) (also see the appendix, §8) can be used to improve the existence theorem in $H^1(\Omega_\varepsilon)$ in a significant way.

The following theorem is the principal result in this paper:

Theorem A. *Consider the* 3DNS (1.1) *on* Ω_ε *with periodic boundary conditions. There is an* $\varepsilon_0 = \varepsilon_0(\nu, \lambda_1) > 0$ *such that for every* ε, $0 < \varepsilon \le \varepsilon_0$, *there are* large *sets* $\mathscr{R}(\varepsilon)$ *and* $\mathscr{S}(\varepsilon)$ *where*

$$\mathscr{R}(\varepsilon) \subset \left\{ U \in H^1(\Omega_\varepsilon) : \nabla \cdot U = 0, \int_{\Omega_\varepsilon} U\, dy = 0 \right\},$$

$$\mathscr{S}(\varepsilon) \subset \left\{ F \in W^{1,\infty}([0, \infty), L^2(\Omega_\varepsilon)) : \int_{\Omega_\varepsilon} F\, dy = 0 \right\},$$

such that if $U_0 \in \mathscr{R}(\varepsilon)$ *and* $F \in \mathscr{S}(\varepsilon)$, *then* (1.2) *has a strong solution* $U(t)$ *with* $U(0) = U_0$, *defined for all* $t \ge 0$, *and*

$$\|U(t)\|^2_{H^1(\Omega_\varepsilon)} \le \widehat{K}_1 < \infty,$$

where $\widehat{K}_1$ *depends on* U_0 *and* F. *Furthermore, there exist constants* $\widehat{L}_1$ *and* $\widehat{L}_2$, *which do not depend on* U_0 *and which satisfy*

$$\limsup_{t \to \infty} \|U(t)\|_{H^1(\Omega_\varepsilon)} \le \widehat{L}_1, \qquad \limsup_{t \to \infty} \|U(t)\|_{H^2(\Omega_\varepsilon)} \le \widehat{L}_2.$$

The proof of Theorem A, including a clarification of the significance of the assertion that $\mathscr{R}(\varepsilon)$ and $\mathscr{S}(\varepsilon)$ are *large* sets, will be incorporated into the H^1

and H^2-Regularity Theorems, which are discussed in the next three sections. It is a consequence of these Regularity Theorems that the set of strong solutions of the Navier-Stokes evolutionary equation has a local attractor $\mathfrak{A}_\varepsilon = \mathfrak{A}_\varepsilon(F)$ whenever F satisfies some compactness property and $F \in \mathscr{S}(\varepsilon)$. The basin of attraction of $\mathfrak{A}_\varepsilon$ contains the set $\mathscr{R}(\varepsilon) \times H^+(F)$; see §2.11. Moreover, we show that for $F \in \mathscr{S}(\varepsilon)$ the set $\mathfrak{A}_\varepsilon$ is the global attractor for the *Leray* solutions of the 3DNS on Ω_ε, i.e., those weak solutions that satisfy the energy inequality (3.35). Furthermore, when F is time-independent, $\mathfrak{A}_\varepsilon$ is a compact set in $H^2(\Omega_\varepsilon)$. We also show that, under reasonable assumptions, $\mathfrak{A}_\varepsilon$ is upper semicontinuous at $\varepsilon = 0$.

In the next section we introduce the notation used in this paper, and we state the main theorems proved herein. The proofs of the regularity theorems will be given in §3 and §4, and the theory of the reduced 3DNS is presented in §5. The reduced 3DNS describe solutions of the 3DNS which depend only on two spatial coordinates. In §6 we study the attractor $\mathfrak{A}_\varepsilon$ for the 3DNS and we show that, under reasonable hypotheses, $\mathfrak{A}_\varepsilon$ is *close to* the global attractor $\mathfrak{A}_0$ of the reduced 3DNS.

The theory of the 3DNS on thin domains, which we present in §§2–6, will be formulated in the context of spatially periodic boundary conditions; however, the methods we use are valid in other settings. In §7 we will show how the theory presented here can be extended to cover the Navier-Stokes equations with other homogeneous boundary conditions. In a forthcoming paper we will present the theory of global regularity for the Navier-Stokes equations with inhomogeneous boundary conditions on thin 3-dimensional domains, and we will consider other types of thin domains.

The results described in this paper were presented at the Workshop on Dynamical Systems Approaches to Turbulence held at the IMA at the University of Minnesota in May 1990. Related contributions appear in Raugel and Sell (1989, 1992a, 1992b). It should be noted that by imposing various symmetry conditions on the solutions of the 3DNS, one can show the global regularity of these solutions; see Ladyzhenskaya (1970) and Mahalov, Titi, and Leibovich (1990).

2. Notation: Statement of theorems

The Navier-Stokes equations on a bounded region Ω in $\mathbb{R}^n$, $n = 2, 3$, are given by

$$U_t - \nu\Delta U + (U \cdot \nabla)U + \nabla P = F, \qquad \nabla \cdot U = 0, \tag{2.1}$$

where ∇ is the gradient operator and Δ is the Laplacian. In this paper we will be especially interested in the case where $n = 3$ and Ω is a thin domain of the form $\Omega_\varepsilon = Q_2 \times (0, \varepsilon)$, where Q_2 is a suitable bounded domain in $\mathbb{R}^2$ and ε is a small positive number. In particular, we will assume that $Q_2 = (0, l_1) \times (0, l_2)$, where l_1 and l_2 are positive. We will assume that $\varepsilon \le l_2 \le l_1$ and $0 < \varepsilon \le 1$ and that the solutions U of (2.1) satisfy the periodic boundary

conditions:
$$\begin{cases} U(y+l_i e_i, t) = U(y, t), & i = 1, 2, \\ U(y+\varepsilon e_3, t) = U(y, t), \end{cases}$$

where $\{e_1, e_2, e_3\}$ is the natural basis in $\mathbb{R}^3$. In addition, we will require that F and the initial data U_0 satisfy

$$\int_{\Omega_\varepsilon} F\, dy = \int_{\Omega_\varepsilon} U_0\, dy = 0.$$

It then follows that any solution U of (2.1) with $U(0) = U_0$ will also satisfy $\int_{\Omega_\varepsilon} U\, dy = 0$ for $t > 0$. Set $Q_3 = Q_2 \times (0, 1)$, and define $a = (a_1, a_2, a_3)$, where $a_i = l_i^{-1}$, $i = 1, 2, 3$, and $l_3 = 1$. The change of variables $(y_1, y_2, y_3) \mapsto (x_1, x_2, x_3)$ where $x_i = y_i$, $i = 1, 2$, and $x_3 = \varepsilon^{-1} y_3$ maps Ω_ε onto Q_3.

We will present some aspects of the theory of the Navier-Stokes equations in this section. For further information, consult Constantin and Foiaş (1988), Ladyzhenskaya (1969), Lions (1969), Temam (1977, 1983), or von Wahl (1985).

2.1. **Dilated Navier-Stokes equations.** The linear operator J_ε given by $U = J_\varepsilon u$, where

$$U(y_1, y_2, y_3) = u(y_1, y_2, \varepsilon^{-1} y_3), \tag{2.2}$$

sets up a one-to-one correspondence between measurable functions on Ω_ε and measurable functions on Q_3. Furthermore, one has $J_\varepsilon(W^{k,p}(Q_3)) = W^{k,p}(\Omega_\varepsilon)$ for any Sobolev space $W^{k,p}$. We will need the identity

$$\|U\|^p_{L^p(\Omega_\varepsilon)} = \varepsilon \|u\|^p_{L^p(Q_3)}, \qquad 1 \le p < \infty, \tag{2.3}$$

where $U = J_\varepsilon u$. We shall use capital Roman letters to denote functions on Ω_ε and lower case Roman letters for functions on Q_3.

We want to let ε vary in our study of the solutions of (2.1). Rather than studying a *fixed* equation on a variable domain, it is more convenient to fix the domain and permit the equation to vary. Therefore, we shall follow the construction in Hale and Raugel (1992a). In particular, by using the operator J_ε, the Navier-Stokes equations (2.1) are transformed to the following system on Q_3:

$$\begin{gathered} u_t - \nu \Delta_\varepsilon u + (u \cdot \nabla_\varepsilon) u + \nabla_\varepsilon p = f, \\ \nabla_\varepsilon \cdot u = 0, \end{gathered} \tag{2.4}$$

where $\nabla_\varepsilon = (D_1, D_2, \varepsilon^{-1} D_3)$, $\Delta_\varepsilon = D_1^2 + D_2^2 + \varepsilon^{-2} D_3^2$, $D_i = \partial/\partial x_i$, $i = 1, 2, 3$, $u = J_\varepsilon^{-1} U$, $f = J_\varepsilon^{-1} F$, and $p = J_\varepsilon^{-1} P$. We will refer to (2.4) as the *dilated Navier-Stokes equations on* Q_3. Because of the terms $\varepsilon^{-1} D_3$ and $\varepsilon^{-2} D_3^2$ where ε is small, the system (2.4) is a singular perturbation of the two-dimensional Navier-Stokes equations.

2.2. **Abstract formulation.** The next step is to reformulate the initial value problem for (2.4) as an abstract nonlinear evolutionary equation on a suitable

Hilbert space H_ε. The approach we use is an adaptation of the theory presented in Temam (1983).

Let $L^2(Q_3) = L^2(Q_3, \mathbb{R}^3)$ denote the collection of all functions $u\colon Q_3 \to \mathbb{R}^3$ with the property that

$$\int_{\Omega_3} |u|^2\, dx = \int_{Q_3} u \cdot u\, dx < \infty,$$

and let

$$\|u\| \stackrel{\text{def}}{=} \|u\|_{L^2(Q_3)} = \left(\int_{Q_3} |u|^2\, dx\right)^{1/2}$$

denote the usual norm. For $m = 0, 1, 2, \ldots$ let the Sobolev spaces $H_p^m(Q_3) = H_p^m(Q_3, \mathbb{R}^3)$ be the closure in $H^m(Q_3, \mathbb{R}^3)$ of those smooth functions that are periodic in space, i.e.,

$$u(x + l_i e_i) = u(x), \qquad i = 1, 2, 3.$$

One then has $H_p^0(Q_3) = L^2(Q_3)$. Also the norm on $H_p^m(Q_3)$ is generated by the inner product

$$(u, v)_m = \sum_{|\alpha| \le m} \int_{Q_3} D^\alpha u \cdot D^\alpha v\, dx.$$

Let $H_\varepsilon = H_\varepsilon(Q_3)$ denote the closure in $L^2(Q_3)$ of those smooth functions u that are periodic on Q_3 and satisfy

$$\int_{Q_3} u\, dx = 0 \quad \text{and} \quad \nabla_\varepsilon \cdot u \stackrel{\text{def}}{=} D_1 u_1 + D_2 u_2 + \varepsilon^{-1} D_3 u_3 = 0.$$

Let $\mathbb{P}_\varepsilon$ denote the orthogonal projection of $L^2(Q_3)$ onto H_ε. By applying $\mathbb{P}_\varepsilon$ to (2.4) and using the fact that $\mathbb{P}_\varepsilon(\nabla_\varepsilon p) = 0$, we obtain the following abstract nonlinear evolutionary equation on H_ε:

$$u' + \nu A_\varepsilon u + B_\varepsilon(u, u) = \mathbb{P}_\varepsilon f, \tag{2.5}$$

where $\partial u/\partial t = u'$, $u = \mathbb{P}_\varepsilon u \in H_\varepsilon$, $A_\varepsilon u = -\mathbb{P}_\varepsilon \Delta_\varepsilon u$ (with the periodic boundary conditions), and the bilinear form B_ε satisfies $B_\varepsilon(u, v) = \mathbb{P}_\varepsilon(u \cdot \nabla_\varepsilon) v$. We shall refer to (2.5) as the *dilated Navier-Stokes evolutionary equation.* We define V_ε^m for $m = 0, 1, 2, \ldots$ by

$$V_\varepsilon^m = H_\varepsilon \cap H_p^m(Q_3).$$

Thus $V_\varepsilon^0 = H_\varepsilon$. Also A_ε is a selfadjoint operator with compact resolvent, and one has $D(A_\varepsilon) = V_\varepsilon^2$ and $D(A_\varepsilon^{1/2}) = V_\varepsilon^1$. Furthermore, A_ε satisfies

$$\lambda_1 \|u\|^2 \le \|A_\varepsilon^{1/2} u\|^2, \qquad u \in D(A_\varepsilon^{1/2}), \tag{2.6}$$

where $\lambda_1 > 0$ is the smallest eigenvalue of A_ε. Since $0 < \varepsilon \le l_2 \le l_1$, one has $\lambda_1 = 4\pi^2 a_1^2 = 4\pi^2 l_1^{-2}$.

The evolutionary equation (2.5) does not contain the pressure term $\nabla_\varepsilon p$. In order to recover the pressure term we apply $(I - \mathbb{P}_\varepsilon)$ to (2.4) to obtain

$$(I - \mathbb{P}_\varepsilon)(u_t - \nu \Delta_\varepsilon u + (u \cdot \nabla_\varepsilon) u) + \nabla_\varepsilon p = (I - \mathbb{P}_\varepsilon) f, \tag{2.7}$$

or in the case of periodic boundary conditions

$$(I - \mathbb{P}_\varepsilon)(u \cdot \nabla_\varepsilon)u + \nabla_\varepsilon p = (I - \mathbb{P}_\varepsilon)f. \tag{2.8}$$

If $u \in V_\varepsilon^1$ and $f \in L^2(Q_3)$ are known, one can solve (2.7) or (2.8) for p by classical techniques; see Constantin and Foiaş (1988) and Temam (1977, 1983).

We will assume the forcing term f in (2.4) to be a time-varying function in the space $L^\infty((0, \infty), L^2(Q_3))$, and we define the norm $\|f\|_\infty$ by

$$\|f\|_\infty \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{0<t<\infty} \|f(t)\|_{L^2(Q_3)}.$$

For some applications, we will assume that $f \in W^{1,\infty}([0, \infty), L^2(Q_3))$, in which case the function f is absolutely continuous and the mapping $t \to f(t)$ is uniformly continuous.

We shall say that $u(t)$ is a *strong solution* of (2.5) on an interval $[0, T)$, where $0 < T \leq \infty$, if for every τ, $0 < \tau < T$, one has

$$u(\cdot) \in C^0([0, \tau], V_\varepsilon^1) \cap L^2((0, \tau), V_\varepsilon^2). \tag{2.9}$$

Recall that if $u(t)$ is a strong solution of (2.5) on $[0, T)$, then it is uniquely determined; see Temam (1977, 1983) or Constantin and Foiaş (1988). Furthermore, if $u(t)$ is a solution of (2.5) on an interval $[0, T)$, where $0 < T \leq \infty$, and satisfies $u(\cdot) \in C^0([0, \tau], V_\varepsilon^1)$ for every τ, $0 < \tau < T$, then $u(\cdot) \in C^0((0, \tau], V_\varepsilon^2)$. (See §4.)

A strong solution $u(t)$ on an interval $[0, T)$ is said to be *maximally defined* if $u(t)$ does not have a proper extension to a strong solution of (2.5) on a larger interval. Recall that if $u(t)$ is a maximally defined strong solution of (2.5) on an interval $[0, T)$ and if $T < \infty$, then one has

$$\|A_\varepsilon^{1/2} u(t)\| \to \infty \quad \text{as } t \to T^-; \tag{2.10}$$

see Temam (1977, 1983), or Constantin and Foiaş (1988).

2.3. **Fourier series.** The spaces V_ε^m can also be described in terms of the Fourier series expansion for functions $u \in L^2(Q_3)$. For k in the integer lattice $\mathbb{Z}^3$, we define

$$ka \stackrel{\text{def}}{=} (k_1 a_1, k_2 a_2, k_3 a_3).$$

Then the Fourier series expansion for $u \in L^2(Q_3)$ is given by

$$u(x) = \sum_{k \in \mathbb{Z}^3} c^k e^{2\pi i ka \cdot x}, \tag{2.11}$$

where $c^k \in \mathbb{C}^3$, $\overline{c^k} = c^{-k}$, and

$$c^k = a_1 a_2 a_3 \int_{Q_3} u(x) e^{-2\pi i ka \cdot x}\, dx, \qquad k \in \mathbb{Z}^3.$$

Consequently, one has $u \in V_\varepsilon^0 = H_\varepsilon$ if and only if $c^0 = 0$ and

$$k_1 a_1 c_1^k + k_2 a_2 c_2^k + \varepsilon^{-1} k_3 a_3 c_3^k = 0 \quad \text{for all } k \in \mathbb{Z}^3. \tag{2.12}$$

Similarly one has $u \in V_\varepsilon^m$ for $m \geq 0$ if and only if $c^0 = 0$, condition (2.12) holds, and

$$\sum_{k\in\mathbb{Z}^3} |k|^{2m}|c^k|^2 < \infty,$$

where $|k|^2 = k_1^2 + k_2^2 + k_3^2$. Furthermore, it follows from the Parseval equality that

$$\|u\|^2_{H_p^m(Q_3)} = (a_1a_2a_3)^{-1} \sum_{|\alpha|\leq m} \sum_{k\in\mathbb{Z}^3} |(2\pi)^{|\alpha|}(ka)^\alpha \cdot c^k|^2, \qquad u \in H_p^m(Q_3), \tag{2.13}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, $\mathbb{N} = \{0, 1, 2, \dots\}$, and

$$(ka)^\alpha \cdot c^k = (k_1a_1)^{\alpha_1}c_1^k + (k_2a_2)^{\alpha_2}c_2^k + (k_3a_3)^{\alpha_3}c_3^k.$$

The eigenvalues of A_ε are given by

$$\lambda = 4\pi^2[(k_1a_1)^2 + (k_2a_2)^2 + \varepsilon^{-2}(k_3a_3)^2],$$

where $k \in \mathbb{Z}^3 - \{(0, 0, 0)\}$. If $u \in V_\varepsilon^2 = D(A_\varepsilon)$, then one has

$$A_\varepsilon u = 4\pi^2 \sum_{k\in\mathbb{Z}^3} [(k_1a_1)^2 + (k_2a_2)^2 + \varepsilon^{-2}(k_3a_3)^2]c^k e^{2\pi i ka\cdot x}.$$

By using the Fourier series representation, it is easily verified that if $u \in V_\varepsilon^2$, then $\nabla_\varepsilon \cdot \Delta_\varepsilon u = 0$. This implies that $A_\varepsilon u = -\Delta_\varepsilon u$ for all $u \in D(A_\varepsilon)$.

The Navier-Stokes equations (2.1) on Ω_ε can be written in the abstract form

$$U' + \nu AU + B(U, U) = \mathbb{P}_3 F, \tag{2.14}$$

where $\mathbb{P}_3$ is the orthogonal projection of $L^2(\Omega_\varepsilon)$ onto the space of divergence-free vector fields, $AU = -\mathbb{P}_3\Delta U$, and $B(U, V) = \mathbb{P}_3(U \cdot \nabla)V$; cf. Temam (1983). One can use the operator J_ε given by (2.2) to compare the solutions of (2.14) with those of (2.5). For example, if $U = J_\varepsilon u$, where u is given by (2.11), then U has the Fourier expansion

$$U(y) = \sum_{k\in\mathbb{Z}^3} c^k e^{2\pi i(k_1a_1, k_2a_2, \varepsilon^{-1}k_3a_3)\cdot(y_1, y_2, y_3)}.$$

The following identities are easily verified:

$$\frac{\partial}{\partial y_i} J_\varepsilon u = J_\varepsilon \varepsilon^{-\{i\}} \frac{\partial}{\partial x_i} u, \qquad i = 1, 2, 3, \text{ for } u \in W^{1,p}(Q_3),$$

where $\{1\} = \{2\} = 0$, $\{3\} = 1$. Also one has

$$\Delta J_\varepsilon u = J_\varepsilon \Delta_\varepsilon u, \quad AJ_\varepsilon u = J_\varepsilon A_\varepsilon u \qquad \text{for } u \in D(A_\varepsilon).$$

As a consequence of (2.3) one then obtains

$$\begin{cases} \|\partial U/\partial y_i\|^2_{L^2(\Omega_\varepsilon)} = \varepsilon\|\varepsilon^{-\{i\}}\partial u/\partial x_i\|^2_{L^2(Q_3)}, & u \in H^1(Q_3); \\ \|A^r U\|^2_{L^2(\Omega_\varepsilon)} = \varepsilon\|A_\varepsilon^r u\|^2_{L^2(Q_3)}, & u \in D(A_\varepsilon^r), \ r \geq 0. \end{cases} \tag{2.15}$$

If u is given by (2.11) and belongs to $D(A_\varepsilon^{1/2})$, we have

$$(2.16)\quad \|A_\varepsilon^{1/2}u\|^2 = (A_\varepsilon u, u)_0 = 4\pi^2(a_1a_2a_3)^{-1}\sum_{k\in\mathbb{Z}^3}(k_1^2a_1^2+k_2^2a_2^2+\varepsilon^{-2}k_3^2a_3^2)|c^k|^2.$$

Moreover, if u belongs to $D(A_\varepsilon)$, one has

$$\|A_\varepsilon u\|^2 = 16\pi^4(a_1a_2a_3)^{-1}\sum_{k\in\mathbb{Z}^3}(k_1^2a_1^2+k_2^2a_2^2+\varepsilon^{-2}k_3^2a_3^2)^2|c^k|^2.$$

From the Parseval equality (2.13), we conclude that there exist two positive constants C_6 and C_7, which are independent of ε, such that

$$(2.17)\quad \begin{aligned} C_6(\|u\|_{H^1(Q_3)} + \varepsilon^{-1}\|D_3u\|_{L^2(Q_3)}) &\le \|A_\varepsilon^{1/2}u\|_{L^2(Q_3)} \\ &\le C_7(\|u\|_{H^1(Q_3)} + \varepsilon^{-1}\|D_3u\|_{L^2(Q_3)}) \end{aligned}$$

and

$$(2.18)\quad \begin{aligned} &C_6(\|u\|_{H^2(Q_3)} + \varepsilon^{-1}\|D_3u\|_{H^1(Q_3)} + \varepsilon^{-2}\|D_3^2u\|_{L^2(Q_3)}) \\ &\le \|A_\varepsilon u\|_{L^2(Q_3)} \\ &\le C_7(\|u\|_{H^2(Q_3)} + \varepsilon^{-1}\|D_3u\|_{H^1(Q_3)} + \varepsilon^{-2}\|D_3^2u\|_{L^2(Q_3)}). \end{aligned}$$

From (2.15) and (2.18), we deduce that $U = J_\varepsilon u$ satisfies

$$(2.19)\quad C_6\|U\|_{H^2(\Omega_\varepsilon)} \le \|AU\|_{L^2(\Omega_\varepsilon)} \le 3C_7\|U\|_{H^2(\Omega_\varepsilon)}.$$

2.4. **The projection** M. For any $u\in L^2(Q_3)$ we define $v = Mu$ by

$$v(x_1, x_2) = \int_0^1 u(x_1, x_2, s)\,ds$$

and set $w = (I-M)u$. Since $w = (I-M)u$, one has $Mw = 0$, and M is an orthogonal projection on $L^2(Q_3)$ which satisfies

$$MD_iu = D_iMu, \qquad i = 1, 2, \ \text{for all } u\in W^{1,1}(Q_3),$$
$$MD_3u = D_3Mu = 0 \quad \text{for all } u\in H_p^1(Q_3),$$

and, therefore,

$$\nabla_\varepsilon\cdot Mu = M\nabla_\varepsilon\cdot u \quad \text{for all } u\in H_p^1(Q_3),$$

as well as

$$(2.20)\quad A_\varepsilon Mu = MA_\varepsilon u \quad \text{for all } u\in D(A_\varepsilon).$$

As a consequence of all these properties, we conclude that $M(V_\varepsilon^m)\subset V_\varepsilon^m$ and that M is an orthogonal projection in V_ε^m for all integers $m\ge 0$. In particular, we have

$$(2.21)\quad \|A_\varepsilon^r u\|^2 = \|A_\varepsilon^r v\|^2 + \|A_\varepsilon^r w\|^2, \qquad r = 0, \tfrac12, 1.$$

In terms of the Fourier series

$$w(x) = \sum_{k \in \mathbb{Z}^3} c^k e^{2\pi i k a \cdot x},$$

one has $Mw = 0$ if and only if

$$c^{(k_1, k_2, 0)} = 0 \quad \text{for all } (k_1, k_2) \in \mathbb{Z}^2.$$

To put it another way, if c^k is any nonzero Fourier coefficient for w, where $Mw = 0$, then $k = (k_1, k_2, k_3)$ satisfies $k_3 \neq 0$. Consequently, one has the Poincaré inequality

$$\|w\|^2 \leq C_5^2 \varepsilon^2 \|A_\varepsilon^{1/2} w\|^2, \qquad w \in V_\varepsilon^1, \ Mw = 0, \tag{2.22}$$

where C_5 does not depend on ε. Indeed, from the Parseval equality (2.13) with $m = 0$ and from (2.16) one obtains

$$\begin{aligned}
\|A_\varepsilon^{1/2} w\|^2_{L^2(Q_3)} &= 4\pi^2 (a_1 a_2 a_3)^{-1} \sum_{k \in \mathbb{Z}^3} (k_1^2 a_1^2 + k_2^2 a_2^2 + \varepsilon^{-2} k_3^2 a_3^2) |c^k|^2 \\
&= 4\pi^2 (a_1 a_2 a_3)^{-1} \sum_{k_3 \neq 0} (k_1^2 a_1^2 + k_2^2 a_2^2 + \varepsilon^{-2} k_3^2 a_3^2) |c^k|^2 \\
&\geq 4\pi^2 \varepsilon^{-2} (a_1 a_2 a_3)^{-1} \sum_{k \in \mathbb{Z}^3} |c^k|^2 = 4\pi^2 \varepsilon^{-2} \|w\|^2_{L^2(Q_3)},
\end{aligned}$$

which completes the proof of (2.22). As we shall see, (2.22) plays a critical role in the theory presented here.

2.5. **The v and w equations.** We now apply the projections M and $(I - M)$ to the equation (2.5) where $v = Mu$ and $w = (I - M)u$. Since one has $MB_\varepsilon(v, v) = B_\varepsilon(v, v)$, it follows from (2.20) that one obtains the system

$$\left\{ \begin{aligned} & v' + \nu A_\varepsilon v + B_\varepsilon(v, v) = M\mathbb{P}_\varepsilon f - M(B_\varepsilon(v, w) + B_\varepsilon(w, v) + B_\varepsilon(w, w)), \\ & w' + \nu A_\varepsilon w = (I - M)\mathbb{P}_\varepsilon f - (I - M)(B_\varepsilon(v, w) + B_\varepsilon(w, v) + B_\varepsilon(w, w)). \end{aligned} \right. \tag{2.23}$$

Since v does not depend on x_3, one has $A_\varepsilon v = D_1^2 v + D_2^2 v$, i.e., $A_\varepsilon v$ is independent of ε. Similarly $B_\varepsilon(v, v)$ is independent of ε. The initial condition $u(0) = u_0 = v_0 + w_0$ also splits into a v and w component. We will be studying solutions $(v, w) = (v(t), w(t))$, where $v(0) = v_0 = Mu_0$ and $w(0) = w_0 = (I - M)u_0$.

2.6. **Reduced 3D Navier-Stokes evolutionary equation.** The system (2.23) has an invariant set which occurs when

$$(I - M)\mathbb{P}_\varepsilon f = 0, \qquad w_0 = 0,$$

i.e., both the forcing term $\mathbb{P}_\varepsilon f$ and the initial condition u_0 depend only on x_1 and x_2. In this case $w(t) \equiv 0$ for all $t \geq 0$ and $\overline{v} = v(t)$ satisfies

$$\overline{v}' + \nu A_\varepsilon \overline{v} + B_\varepsilon(\overline{v}, \overline{v}) = M\mathbb{P}_\varepsilon f \tag{2.24}$$

with $\overline{v}(0) = v_0$. We refer to (2.24) as the *reduced* 3D *Navier-Stokes evolutionary equation.* Note that $\overline{v} = (\overline{v}_1, \overline{v}_2, \overline{v}_3)$ is a three dimensional vector field on Q_3, and $\overline{v}$ does not depend on x_3.

The reduced 3D Navier-Stokes evolutionary equation incorporates the 2DNS on Q_2. In order to see this, we let $L^2(Q_2, \mathbb{R}^2)$ denote the L^2 space of 2-dimensional vector fields $m = (m_1, m_2)$ which depend on $(x_1, x_2) \in Q_2$. Let $\mathbb{P}_2$ denote the orthogonal projection of $L^2(Q_2, \mathbb{R}^2)$ onto $H(Q_2)$, where $H(Q_2)$ is the closure in $L^2(Q_2, \mathbb{R}^2)$ of those smooth functions m that are periodic on Q_2 and satisfy $\int_{Q_2} m\, dx = 0$ and $(D_1 m_1 + D_2 m_2) = 0$. One then has

$$\mathbb{P}_\varepsilon \begin{pmatrix} (\overline{v}_1 D_1 + \overline{v}_2 D_2)\overline{v}_1 \\ (\overline{v}_1 D_1 + \overline{v}_2 D_2)\overline{v}_2 \\ (\overline{v}_1 D_1 + \overline{v}_2 D_2)\overline{v}_3 \end{pmatrix} = \begin{pmatrix} \mathbb{P}_2 \begin{pmatrix} (\overline{v}_1 D_1 + \overline{v}_2 D_2)\overline{v}_1 \\ (\overline{v}_1 D_1 + \overline{v}_2 D_2)\overline{v}_2 \end{pmatrix} \\ (\overline{v}_1 D_1 + \overline{v}_2 D_2)\overline{v}_3 \end{pmatrix}$$

and

$$\mathbb{P}_\varepsilon \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} \mathbb{P}_2 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \\ g_3 \end{pmatrix},$$

where $g = (g_1, g_2, g_3) \in ML^2(Q_2, \mathbb{R}^3)$. Furthermore, $\overline{v}$ is a solution of the reduced 3D Navier-Stokes evolutionary equation (2.24) if and only if $m = (\overline{v}_1, \overline{v}_2)$ is a solution of the 2D Navier-Stokes evolutionary equation

$$\frac{d}{dt} m - \nu(D_1^2 + D_2^2)m + \mathbb{P}_2(m \cdot \nabla)m = (g_1, g_2),$$

and $\overline{v}_3$ is a solution of the linear equation

$$\frac{d}{dt}\overline{v}_3 - \nu(D_1^2 + D_2^2)\overline{v}_3 + (\overline{v}_1 D_1 + \overline{v}_2 D_2)\overline{v}_3 = g_3,$$

where $g = (g_1, g_2, g_3) = M\mathbb{P}_\varepsilon f$.

If we want to emphasize that the terms in (2.24) do not depend on ε or x_3, we introduce the following notation. For $i = 1, 2, 3$ we set $V_0^i = MV_\varepsilon^i$, $H_0 = V_0^0 = MH_\varepsilon(Q_3)$. We denote by A_0 the restriction of A_ε to V_0^2. If $\overline{v}$ is in V_0^2, then $A_0\overline{v} = -(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)\overline{v}$. We also set $B_0(\overline{v}, \overline{v}) = \mathbb{P}_\varepsilon(\overline{v} \cdot \nabla)\overline{v}$ if $\overline{v}$ is in V_0^1. Note that $B_0(\overline{v}, \overline{v}) = B_\varepsilon(\overline{v}, \overline{v})$. The reduced 3D Navier-Stokes evolutionary equation (2.24) now becomes

$$\overline{v}' + \nu A_0 \overline{v} + B_0(\overline{v}, \overline{v}) = g$$

with $\overline{v}(0) = v_0$ in V_0^1 and $g = M\mathbb{P}_\varepsilon f$.

2.7. **The trilinear form** b_ε. The trilinear form $b_\varepsilon(u, v, w)$ is defined by

$$(2.25) \qquad b_\varepsilon(u, v, w) = \sum_{i,j=1}^{3} \int_{Q_3} \varepsilon^{-\{i\}} u_i (D_i v_j) w_j \, dx,$$

provided the integrals are all defined, where $\{1\} = \{2\} = 0$ and $\{3\} = 1$. If $u, v, w \in V_\varepsilon^1$, then (2.25) is well defined, and since $\mathbb{P}_\varepsilon$ is an orthogonal projection with $\mathbb{P}_\varepsilon w = w$, one has

$$\begin{aligned}\langle B_\varepsilon(u, v), w\rangle &= \langle \mathbb{P}_\varepsilon(u\cdot\nabla_\varepsilon)v, w\rangle = \langle (u\cdot\nabla_\varepsilon)v, \mathbb{P}_\varepsilon w\rangle \\ &= \langle (u\cdot\nabla_\varepsilon)v, w\rangle = b_\varepsilon(u, v, w).\end{aligned}$$

Note that

$$b_\varepsilon(v^1, w, v^2) = -b_\varepsilon(v^1, v^2, w) = 0, \qquad b_\varepsilon(w, v^1, v^2) = 0, \tag{2.26}$$

whenever $Mv^i = v^i$, $i = 1, 2$, and $Mw = 0$.

2.8. **Statement of regularity theorems.** Theorem A, which is stated in §1, gives a sufficient condition for the nonlinear evolutionary equation (2.5) to have a strong solution $u(t)$ that remains in V_ε^1 for all $t \geq 0$ and in V_ε^2 for all $t > 0$. In a moment we shall define the sets $\mathscr{R}(\varepsilon)$ and $\mathscr{S}(\varepsilon)$, and in §2.10 we explain why these are *large* sets. The key to this is the following Hypothesis H(a, b):

We shall say that the bounded monotone functions $\eta_i(\varepsilon)$ defined for $0 < \varepsilon \leq 1$, $i = 1, 2, 3, 4$, and constants r and p satisfy **Hypothesis H(a, b)**, where a and b are positive, provided:

(1) $p \geq -1$, $r > -2$.

(2) $\varepsilon^{1/4}\eta_i^{-1} \to 0$ as $\varepsilon \to 0$, $i = 1, 2$.

(3) $\varepsilon^{1/8}\eta_i^{-1} \to 0$ as $\varepsilon \to 0$, $i = 3, 4$.

(4) $\varepsilon^{1/4}Q(\varepsilon)$ is bounded for $0 < \varepsilon \leq 1$, where

$$Q(\varepsilon) = |\log(2C_5^2\nu^{-2}\varepsilon^{2+r-p}\eta_4^{-2}\eta_3^2)|.$$

(5) Let $a > 0$ be fixed. Then one has

$$\begin{cases} \varepsilon^{5/8}\eta^{-2}\exp(a\eta^{-4}) \to 0, \\ \eta^{-2} \to \infty \end{cases} \tag{2.27}$$

as $\varepsilon \to 0^+$, where

$$\eta^{-2} \stackrel{\text{def}}{=} \max(4\eta_1^{-2} + k_1^2\eta_3^{-4} + k_2^2\varepsilon^{2+r}\eta_4^{-2}, 1) \tag{2.28}$$

and $\varepsilon^{5/8}\exp(2a\eta_2^{-4})$ is bounded for $0 < \varepsilon \leq 1$. (The constants k_1 and k_2 are defined in Lemma 3.1.)

(6) Let $b > 0$ be fixed. Then for any λ, $0 < \lambda < 1$, there is an $\varepsilon_4 = \varepsilon_4(b, \lambda) > 0$ such that one has

$$\eta_2^{-2}\exp(b\eta_2^{-4}) \leq \lambda(4\eta_1^{-2} + k_1^2\eta_3^{-4}), \qquad 0 < \varepsilon < \varepsilon_4.$$

(7) The function $\varepsilon^{4+2r}\eta_4^{-4}(\log\eta^{-4} + 1)$ is bounded as $\varepsilon \to 0^+$.

Remarks. 1. The choice of the function η_i may depend on the parameters p, r, a, and b.

2. Here is an example where Hypothesis H(a, b) is satisfied for any given choice of $a > 0$ and $b > 0$. We begin with statement (5). If η is given by (2.28), then

$$1 \leq \eta^{-4} \leq (48\eta_1^{-4} + 3k_1^4\eta_3^{-8} + 3k_2^4\varepsilon^{2(2+r)}\eta_4^{-4}).$$

Next fix $p \geq -1$ and set

$$\begin{cases} r = -2 + \delta, & \delta > 0; \\ \eta_4^{-1} = -\log \varepsilon; \\ \eta_3^{-2} = \eta_1^{-1}. \end{cases}$$

If $\eta_1^{-2} = (-\log \varepsilon^\alpha)^{1/2}$, where $\alpha > 0$, then conditions (1), (2), and (3) hold. Furthermore (4) and (7) are valid. In addition, one has

$$\eta^{-4} \leq (48 + 3k_1^4)(-\log \varepsilon^\alpha) + 3k_2^4 \varepsilon^{2(2+r)}(-\log \varepsilon)^4.$$

Consequently (2.27) is valid provided

$$\varepsilon^{5/8}(-\log \varepsilon^\alpha)^{1/2} \varepsilon^{-a(48+3k_1^4)\alpha} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

In other words, if α is chosen such that $\frac{5}{8} > (48 + 3k_1^4)a\alpha$, then (2.27) is satisfied. Likewise, by choosing η_2^{-2} so that

$$\eta_2^{-2} = (\log(-\log \varepsilon^\alpha)^\beta)^{1/2}, \quad \text{where } \beta > 0 \text{ and } 2b\beta < 1,$$

we see that statement (6) is valid. ($\eta_2^{-2} = \log(\log(-\log \varepsilon))$ is another possible choice of η_2^{-2}.) Also note that this example satisfies condition (2.55) below, provided $-1 \leq p < 0$.

3. Another example of interest occurs when η_2 and η_4 are positive constants and η_1 and η_3 are as in Remark 2. This situation arises in the study of the semicontinuity of the attractors, see §6.

In order to prove Theorem A we shall analyze the dilated Navier-Stokes evolutionary equation on Q_3. This analysis consists of two major steps. The first step, which we call the H^1-Regularity Theorem, is to show that there is a constant K_1 such that $\|A_\varepsilon^{1/2}u(t)\|^2 \leq K_1^2$ for all $t \geq 0$ and that there is a constant L_5, which does not depend on the initial data, such that $\limsup_{t\to\infty} \|A_\varepsilon^{1/2}u(t)\|^2 \leq L_5^2$. The second step, which we call the H^2-Regularity Theorem, is to show that $u(t) \in D(A_\varepsilon)$ for all $t > 0$ and that there is a constant L_6, which does not depend on the initial data, such that $\limsup_{t\to\infty} \|A_\varepsilon u(t)\|^2 \leq L_6^2$.

Theorem 1. **H^1-Regularity.** *Let η_i, $i = 1, 2, 3, 4$, r, and p satisfying Hypothesis H(a, b), where a and b are sufficiently large. Then there exist $\varepsilon_0 > 0$, $k_2 > 0$, a continuous function $\Gamma \in C([0, \infty), \mathbb{R})$, and for all ε, $0 < \varepsilon \leq \varepsilon_0$, a time $\widehat{T}_1 = \widehat{T}_1(\varepsilon) > 0$ such that for all ε, $0 < \varepsilon \leq \varepsilon_0$, whenever $u_0 \in D(A_\varepsilon^{1/2})$ and $f \in L^\infty((0, \infty), L^2(Q_3))$ satisfy*

$$\begin{cases} \|A_\varepsilon^{1/2}v_0\|^2 \leq \eta_1^{-2}, & \|M\mathbb{P}_\varepsilon f\|_\infty^2 \leq \eta_2^{-2}, \\ \|A_\varepsilon^{1/2}w_0\|^2 \leq \varepsilon^p \eta_3^{-2}, & \|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 \leq \varepsilon^r \eta_4^{-2}, \end{cases} \tag{2.29}$$

then (2.5) *has a solution u that belongs to $C^0([0, \infty), V_\varepsilon^1) \cap L^\infty((0, \infty), V_\varepsilon^1)$, i.e., one has*

$$\|A_\varepsilon^{1/2}u(t)\|^2 \leq K_1^2, \qquad t \geq 0, \tag{2.30}$$

where K_1 *depends on* ν, λ_1, *and* η_i, $i = 1, 2, 3, 4$, *but not on* $t \geq 0$. *Moreover, the components of* $u = v + w$ *satisfy*

$$\|A_\varepsilon^{1/2} v(t)\|^2 \leq \Gamma(\eta_2^{-2}), \qquad t \geq \widehat{T}_1, \tag{2.31}$$

where Γ *is given by* (3.84), *and*

$$\|A_\varepsilon^{1/2} w(t)\|^2 \leq \max(\varepsilon^2, k_2^2 \varepsilon^{2+r} \eta_4^{-2}), \qquad t \geq \widehat{T}_1. \tag{2.32}$$

Remarks. 1. The principal objective in any study of the global regularity of solutions of the 3DNS is to show that the conclusions of Theorem 1 apply for all $u_0 \in V_\varepsilon^1$ and $\mathbf{P}_\varepsilon f \in L^\infty((0, \infty), L^2(Q_3))$. Since the techniques developed in this paper seem to fall short of achieving this goal, we seek, instead, to find the largest possible u_0 and f (see (2.29)) for which we can prove global regularity.

2. It follows from (2.21), (2.31), and (2.32) that

$$\|A_\varepsilon^{1/2} u(t)\|^2 \leq L_5^2, \qquad t \geq \widehat{T}_1, \tag{2.33}$$

where $L_5^2 = \Gamma(\eta_2^{-2}) + \max(\varepsilon^2, k_2^2 \varepsilon^{2+r} \eta_4^{-2})$. Since L_5^2 does not depend on η_1, η_3, or p, it is independent of the initial condition u_0. Furthermore, if η_2^{-2} and $\varepsilon^{2+r}\eta_4^{-2}$ are bounded for $0 < \varepsilon \leq 1$, it follows from Theorem 1 that L_5^2 can be chosen independent of ε.

Theorem 2. **H^2-Regularity.** *Let* r, p, *and* η_i, $i = 1, 2, 3, 4$, *satisfy Hypothesis H(a, b), where* a *and* b *are sufficiently large. If* $u_0 \in V_\varepsilon^1$ *and*

$$\mathbb{P}_\varepsilon f \in C^0([0, \infty), H_\varepsilon) \cap L^\infty((0, \infty), H_\varepsilon) \cap W^{1,\infty}((0, \infty), D(A_\varepsilon^{-1/2})) \tag{2.34}$$

satisfy (2.29), *then for* $0 < \varepsilon \leq \varepsilon_0$, *where* ε_0 *is given by Theorem* 1, *the solution* $u(t)$ *of* (2.5) *belongs to* $C^0((0, \infty), V_\varepsilon^2)$. *Furthermore there exist three positive constants* K_2, K_3 *and* K_4, *which depend on* ν, λ_1, η_i, $i = 1, 2, 3, 4$, *and* K_1, *where* K_1 *is given by* (2.30), *such that*

$$\begin{cases} \|A_\varepsilon u(t)\|^2 \leq K_2^2 + K_3^2 \|A_\varepsilon^{-1/2} \mathbb{P}_\varepsilon f'\|_\infty^2 + K_4^2 t^{-1} & \text{for } 0 < t \leq 1, \\ \|A_\varepsilon u(t)\|^2 \leq K_2^2 + K_3^2 \|A_\varepsilon^{-1/2} \mathbb{P}_\varepsilon f'\|_\infty^2 & \text{for } t \geq 1. \end{cases} \tag{2.35}$$

Moreover, there is a positive continuous function Γ_2 *on* $[0, \infty)$ *given by* (4.22), *such that*

$$\|A_\varepsilon u(t)\|^2 \leq L_6^2, \qquad t \geq \widehat{T}_1 + 1, \tag{2.36}$$

where $\widehat{T}_1$ *is given by Theorem* 1, $L_6^2 = \Gamma_2(L_5^2)$, *and* L_5^2 *is given in* (2.33).

If, in addition, u_0 *belongs to* $D(A_\varepsilon)$, *then the solution* u *of* (2.5) *belongs to the space* $C^0([0, \infty)), V_\varepsilon^2)$, *and one has*

$$\|A_\varepsilon u(t)\|^2 \leq K_5^2 + K_6^2 \|A_\varepsilon u_0\|^2 + K_7^2 \|A_\varepsilon^{-1/2} \mathbb{P}_\varepsilon f'\|_\infty^2, \qquad 0 \leq t \leq 1, \tag{2.37}$$

where K_5, K_6, *and* K_7 *are positive constants depending on* ν, λ_1, η_i, $i = 1, 2, 3, 4$, *and* K_1.

Let $\mathscr{B}_\varepsilon^0$, $\mathscr{B}_\varepsilon^1$, and $\mathscr{B}_\varepsilon^2$ denote the following bounded sets in $V_\varepsilon^1 = D(A_\varepsilon^{1/2})$:

$$\mathscr{B}_\varepsilon^0 \stackrel{\text{def}}{=} \{u = v + w \colon \|A_\varepsilon^{1/2} v\|^2 \le \eta_1^{-2},\ \|A_\varepsilon^{1/2} w\|^2 \le \varepsilon^p \eta_3^{-2}\}, \tag{2.38}$$

(2.39)

$$\mathscr{B}_\varepsilon^1 \stackrel{\text{def}}{=} \{u = v + w \colon \|A_\varepsilon^{1/2} v\|^2 \le 4\eta_1^{-2} + k_1^2 \eta_3^{-4},\ \|A_\varepsilon^{1/2} w\|^2 \le k_2^2 \varepsilon^{2+r} \eta_4^{-2}\},$$

$$\mathscr{B}_\varepsilon^2 \stackrel{\text{def}}{=} \bigcup_{t \ge 0} S_\varepsilon(\mathbb{P}_\varepsilon f, t)(\mathscr{B}_\varepsilon^0 \cup \mathscr{B}_\varepsilon^1), \tag{2.40}$$

where $u(t) = S_\varepsilon(\mathbb{P}_\varepsilon f, t)u_0$ is the strong solution of the equation (2.5) with initial data u_0. Due to the Lemmas 3.1 and 3.2, $\mathscr{B}_\varepsilon^2$ is well defined and is a bounded set in V_ε^1.

Remarks. 1. Since L_5^2 does not depend on the initial condition u_0, it follows from Theorem 2 that the bound L_6^2 does not depend on u_0. Furthermore, if f is chosen so that $\|\mathbb{P}_\varepsilon f\|_\infty$ and $\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty$ are bounded for $0 < \varepsilon \le 1$, then L_6^2 can be chosen to be independent of ε.

2. One can obtain other H^2-regularity results if one assumes that f has more spatial regularity, e.g., $\mathbb{P}_\varepsilon f \in L^\infty((0, \infty); V_\varepsilon^1)$ instead of satisfying (2.34). (See Foiaş, Guillopé, and Temam (1981).)

3. If in addition to the hypotheses of Theorem 2, the function f belongs to $W^{1,\infty}((0, \infty), H_\varepsilon)$, then from (2.6) and (2.22) one finds that

$$\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2 \le \lambda_1^{-1}\|M\mathbb{P}_\varepsilon f'\|_\infty^2 + C_5^2\varepsilon^2\|(I - M)\mathbb{P}_\varepsilon f'\|_\infty^2, \tag{2.41}$$

which can be used in (2.35) (2.36), and (2.37).

2.9. **Small data regularity.** As mentioned in the introduction, it is known that the 3DNS has a globally regular solution whenever the data of the problem are small. The global regularity with small data, which is valid for any reasonable bounded 3-dimensional region, is a simple consequence of the Stable Manifold Theorem. We emphasize that our theorems, which are valid for *thin* 3-dimensional regions, are not consequences of the small data arguments. Before showing this though, it will be useful to recall one of these small data arguments at this time.[2]

The argument we give here will be for the Navier-Stokes evolutionary equation (2.14) on an arbitrary bounded region Ω in $\mathbb{R}^3$. We will not exploit, at this time, the assumption that $\Omega = \Omega_\varepsilon$ is a thin domain, an assumption which is of special interest elsewhere in this paper. We assume here, for simplicity, that $F \in L^2(\Omega)$ does not depend on time.

We will use the standard 3D estimate for $B(U^1, U^2)$:

$$|\langle B(U^1, U^2), U^3\rangle| \le C_8\|A^{1/2}U^1\|_{L^2(\Omega)}\|A^{1/2}U^2\|_{L^2(\Omega)}^{1/2}\|AU^2\|_{L^2(\Omega)}^{1/2}\|U^3\|_{L^2(\Omega)};$$

see Temam (1977, 1983) or Constantin and Foiaş (1988). The constant $C_8 = C_8(\Omega)$ depends on Ω.

[2]There are several approaches to proving the global regularity of solutions with small data. For all practical purposes, these arguments all lead to the same conclusions described here.

By taking the scalar product of (2.14) with AU we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^{1/2}U\|^2_{L^2(\Omega)} &+ \nu\|AU\|^2_{L^2(\Omega)} \\ &\le \|\mathbb{P}_3 F\|_{L^2(\Omega)}\|AU\|_{L^2(\Omega)} + C_8\|A^{1/2}U\|^{3/2}_{L^2(\Omega)}\|AU\|^{3/2}_{L^2(\Omega)} \\ &\le \frac{\nu}{2}\|AU\|^2_{L^2(\Omega)} + \frac{1}{\nu}\|\mathbb{P}_3 F\|^2_{L^2(\Omega)} + \frac{27}{4\nu^3}C_8^4\|A^{1/2}U\|^6_{L^2(\Omega)}.\end{aligned}$$

We then get

$$\frac{d}{dt}\|A^{1/2}U\|^2_{L^2(\Omega)} + \nu\|AU\|^2_{L^2(\Omega)} \le \frac{2}{\nu}\|\mathbb{P}_3 F\|^2_{L^2(\Omega)} + \frac{27}{2\nu^3}C_8^4\|A^{1/2}U\|^6_{L^2(\Omega)},$$

which in turn implies that

(2.42)
$$\frac{d}{dt}\|A^{1/2}U\|^2_{L^2(\Omega)} + \lambda_1\nu\|A^{1/2}U\|^2_{L^2(\Omega)} \le \frac{2}{\nu}\|\mathbb{P}_3 F\|^2_{L^2(\Omega)} + \frac{27}{2\nu^3}C_8^4\|A^{1/2}U\|^6_{L^2(\Omega)}.$$

Now set $R_0^2 = \|A^{1/2}U_0\|^2_{L^2(\Omega)} + \|\mathbb{P}_3 F\|^2_{L^2(\Omega)}$ and $N > \max(1, 4\lambda_1^{-1}\nu^{-2})$. Since $R_0^2 \ge \|A^{1/2}U_0\|^2_{L^2(\Omega)}$ and $N > 1$, it follows from Lemma 3.0 below that there is a T^N, $0 < T^N \le \infty$, such that

$$\|A^{1/2}U(t)\|^2_{L^2(\Omega)} \le NR_0^2, \qquad 0 \le t < T^N. \tag{2.43}$$

We assume, without loss of generality, that $[0, T^N)$ is the maximal time interval for which (2.43) is valid.

Next assume that

$$\frac{27C_8^4}{2\nu^3}N^2R_0^4 \le \frac{\lambda_1\nu}{2}. \tag{2.44}$$

Inequality (2.44) is the precise assumption that the data for (2.14) are small. Because of (2.43) and (2.44), it follows from (2.42) that

$$\frac{d}{dt}\|A^{1/2}U\|^2_{L^2(\Omega)} + \frac{\lambda_1\nu}{2}\|A^{1/2}U\|^2_{L^2(\Omega)} \le \frac{2}{\nu}\|\mathbb{P}_3 F\|^2_{L^2(\Omega)}. \tag{2.45}$$

By applying the Gronwall inequality to (2.45) we get

$$\|A^{1/2}U(t)\|^2_{L^2(\Omega)} \le \exp\left(-\frac{\lambda_1\nu}{2}t\right)\|A^{1/2}U_0\|^2_{L^2(\Omega)} + \frac{4}{\lambda_1\nu^2}\|\mathbb{P}_3 F\|^2_{L^2(\Omega)} < NR_0^2$$

for $0 < t \le T^N$. Consequently (2.10) implies that $T^N = \infty$, which completes the proof of global regularity of solutions for small data.

Remark. In the case of a thin domain Ω_ε, one has $C_8 = C\varepsilon^{-1/2}$, where C is independent of ε. As a result, inequality (2.44) can be rewritten as

$$\|A^{1/2}U_0\|^2_{L^2(\Omega_\varepsilon)} + \|\mathbb{P}_3 F\|^2_{L^2(\Omega_\varepsilon)} \le C^*\varepsilon, \tag{2.46}$$

where C^* depends on ν and λ_1 but not on ε.

2.10. **Large data regularity.** We now show that the inequalities (2.29) describe large data conditions on both Q_3 and the thin domain Ω_ε. The inequalities (2.29) describe the norms of the data for (2.5) in the space $L^2(Q_3)$. We now set $p = r = -1$ and assume that $\eta_i(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ for $1 \le i \le 4$. By using the mapping J_ε together with (2.3), (2.15), and (2.21) one finds that

$$\begin{aligned}\|A^{1/2}U_0\|^2_{L^2(\Omega_\varepsilon)} &= \|A^{1/2}V_0\|^2_{L^2(\Omega_\varepsilon)} + \|A^{1/2}W_0\|^2_{L^2(\Omega_\varepsilon)} \\ &= \varepsilon\|A_\varepsilon^{1/2}v_0\|^2_{L^2(Q_3)} + \varepsilon\|A_\varepsilon^{1/2}w_0\|^2_{L^2(Q_3)} \\ &\le \varepsilon\eta_1^{-2} + \eta_3^{-2}.\end{aligned} \tag{2.47}$$

Similarly one has

$$\|\mathbb{P}_3F\|^2_{L^2(\Omega_\varepsilon)} = \varepsilon\|\mathbb{P}_\varepsilon f\|^2_{L^2(Q_3)} \le \varepsilon\eta_2^{-2} + \eta_4^{-2}. \tag{2.48}$$

Inequalities (2.47) and (2.48) imply that

$$\|A^{1/2}U_0\|^2_{L^2(\Omega_\varepsilon)} + \|\mathbb{P}_3F\|^2_{L^2(\Omega_\varepsilon)} \le \varepsilon(\eta_1^{-2} + \eta_2^{-2}) + \eta_3^{-2} + \eta_4^{-2}. \tag{2.49}$$

Assume that we choose $\eta_3(\varepsilon) = \eta_4(\varepsilon) = 1$. Then, even in this case, condition (2.49) is much weaker than condition (2.46) since we allow $\eta_1^{-2}(\varepsilon)$ and $\eta_2^{-2}(\varepsilon)$ to go to ∞ as $\varepsilon \to 0^+$. To put it another way, assume that F satisfies (2.46) and really depends on the three variables (y_1, y_2, y_3), and let $U_1 = U_1(y_1, y_2)$ satisfy $U_1 \in H^2(Q_2, \mathbb{R}^3)$, with periodic boundary conditions, divergence free, and $\|U_1\|_{L^2(Q_2)} = 1$. Then

$$U_0(y_1, y_2, y_3) = \eta_1^{-1}(\varepsilon)U_1(y_1, y_2)$$

will satisfy (2.49), but not (2.46), for small ε, whenever $\eta_1^{-1}(\varepsilon) \to \infty$ as $\varepsilon \to 0^+$.

For $0 < \varepsilon \le \varepsilon_0$ we define $R_1(\varepsilon)$ to be the collection of $v_0 \in MV_\varepsilon^1$ such that

$$\|A_\varepsilon^{1/2}v_0\|^2_{L^2(Q_3)} = \|v_0\|^2_{V_\varepsilon^1} \le \eta_1^{-2}$$

and $R_2(\varepsilon)$ to be the collection of $w_0 \in (I-M)V_\varepsilon^1$ such that

$$\|A_\varepsilon^{1/2}w_0\|^2_{L^2(Q_3)} = \|w_0\|^2_{V_\varepsilon^1} \le \varepsilon^{-1}\eta_3^{-2}.$$

Set $R(\varepsilon) = R_1(\varepsilon) + R_2(\varepsilon)$, and let $\mathscr{R}_1(\varepsilon) = J_\varepsilon R_1(\varepsilon)$, $\mathscr{R}_2(\varepsilon) = J_\varepsilon R_2(\varepsilon)$, and $\mathscr{R}(\varepsilon) = J_\varepsilon R(\varepsilon)$ denote the corresponding sets in $H^1(\Omega_\varepsilon)$. The sets $R_1(\varepsilon)$ and $R_2(\varepsilon)$ are bounded sets in MV_ε^1 and $(I-M)V_\varepsilon^1$ with V_ε^1-radius being η_1^{-1} and $\varepsilon^{-1/2}\eta_3^{-1}$, respectively. From (2.3) we see that $V_0 = J_\varepsilon v_0 \in \mathscr{R}_1(\varepsilon)$ and $W_0 = J_\varepsilon w_0 \in \mathscr{R}_2(\varepsilon)$ if and only if

$$\|A^{1/2}V_0\|^2_{L^2(\Omega_\varepsilon)} \le \varepsilon\eta_1^{-2} \quad \text{and} \quad \|A^{1/2}W_0\|^2_{L^2(\Omega_\varepsilon)} \le \eta_3^{-2}.$$

Thus $\mathscr{R}_1(\varepsilon)$ and $\mathscr{R}_2(\varepsilon)$ contain bounded sets in $MH^1(\Omega_\varepsilon)$ and $(I-M)H^1(\Omega_\varepsilon)$ with $H^1(\Omega_\varepsilon)$-radius being $C\varepsilon^{1/2}\eta_1^{-1}$ and $C\eta_3^{-1}$, respectively. The example constructed in §2.8 gives information on the size of these radii as $\varepsilon \to 0^+$. The point to note in this example is that $\eta_3^{-1} = (-\log \varepsilon^\alpha)^{1/8}$. The assertion in Theorem A that $\mathscr{R}(\varepsilon)$ is *large* is a heuristic formulation of the fact that $\eta_i^{-1} \to \infty$ as $\varepsilon \to 0^+$, $i = 1, 3$.

Similarly we define $S(\varepsilon)$ to be the collection of $f \in L^\infty((0, \infty), L^2(Q_3))$ that satisfy

$$\|M\mathbb{P}_\varepsilon f(t)\|^2_{L^2(Q_3)} \le \eta_2^{-2} \quad \text{and} \quad \|(I - M)\mathbb{P}_\varepsilon f(t)\|^2_{L^2(Q_3)} \le \varepsilon^r \eta_4^{-2}$$

for $0 < t < \infty$ and set $\mathscr{S}(\varepsilon) = J_\varepsilon S(\varepsilon)$, where $r = -2 + \delta$, say $0 < \delta \le \frac{1}{2}$. One then has $F = J_\varepsilon f \in \mathscr{S}(\varepsilon)$ if and only if

$$\|M\mathbb{P}_3 F(t)\|^2_{L^2(\Omega_\varepsilon)} \le \varepsilon\eta_2^{-2} \quad \text{and} \quad \|(I - M)\mathbb{P}_3 F(t)\|^2_{L^2(\Omega_\varepsilon)} \le \varepsilon^{r+1}\eta_4^{-2}.$$

Once again, the assertion that $\mathscr{S}(\varepsilon)$ is *large* is a heuristic formulation of the fact that $\eta_i^{-1} \to \infty$ as $\varepsilon \to 0^+$, $i = 2, 4$. The example in §2.8 shows that one can choose $\eta_4^{-1} = (-\log \varepsilon)$.

If the initial condition u_0 belongs to $D(A_\varepsilon)$, as in the case of Theorem 2, then one has $v_0, w_0 \in D(A_\varepsilon)$, and (2.6) and (2.22) imply that

$$\begin{cases} \|A_\varepsilon^{1/2} v_0\|^2_{L^2(Q_3)} \le \lambda_1^{-1} \|A_\varepsilon v_0\|^2_{L^2(Q_3)}, \\ \|A_\varepsilon^{1/2} w_0\|^2_{L^2(Q_3)} \le C_5^2 \varepsilon^2 \|A_\varepsilon w_0\|^2_{L^2(Q_3)}. \end{cases}$$

This means that $u_0 = v_0 + w_0$ will satisfy (2.29) provided one has

$$\begin{cases} \|A_\varepsilon v_0\|^2_{L^2(Q_3)} \le \lambda_1 \eta_1^{-2}, \\ \|A_\varepsilon w_0\|^2_{L^2(Q_3)} \le C_5^{-2} \varepsilon^{p-2} \eta_3^{-2}. \end{cases} \tag{2.50}$$

By using the mapping J_ε and (2.3), we see that (2.50) can be written in the equivalent form

$$\begin{cases} \|AV_0\|^2_{L^2(\Omega_\varepsilon)} \le \lambda_1 \varepsilon \eta_1^{-2}, \\ \|AW_0\|^2_{L^2(\Omega_\varepsilon)} \le C_5^{-2} \varepsilon^{p-1} \eta_3^{-2}. \end{cases}$$

Thus $\mathscr{R}_2(\varepsilon)$ contains a bounded set in $(I - M)H^2(\Omega_\varepsilon)$ of $H^2(\Omega_\varepsilon)$-radius $\ge C\varepsilon^{p-1}$, where $p \ge 1$.

2.11. **Skew-Product dynamics.** We will review here some aspects of the theory of skew-product flows in order to describe the (local) attractors for the Navier-Stokes equations with a time-varying forcing function f. We will formulate this general theory for the Navier-Stokes equations on an arbitrary bounded domain Ω in $\mathbb{R}^n$, where $n = 2, 3$. For simplicity[3] we will consider forcing functions f in the space

$$f \in W(\Omega) \stackrel{\text{def}}{=} C^0(\mathbb{R}, L^2(\Omega)) \cap L^\infty(\mathbb{R}, L^2(\Omega)), \tag{2.51}$$

[3]By using other topologies one can describe attractors when the forcing function f is discontinuous. See Miller and Sell (1970) or Sacker and Sell (1977) for details.

where $L^2(\Omega) = L^2(\Omega, \mathbb{R}^n)$. For any linear operator T on $L^2(\Omega)$ we let

$$TW(\Omega) = C^0(\mathbb{R}, TL^2(\Omega)) \cap L^\infty(\mathbb{R}, TL^2(\Omega)).$$

A metrizable topology is introduced in the space $W(\Omega)$ by defining sequential convergence $f_n \to f$ to mean that for any compact set $K \subset \mathbb{R}$ one has

$$\sup_{t\in K} \|f_n(t) - f(t)\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

We will denote the associated metric by $\operatorname{dist}_{W(\Omega)}$.

For any f satisfying (2.51) and any $\tau \in \mathbb{R}$ we define the translate $f_\tau(t) \overset{\text{def}}{=} f(\tau + t)$. Note that $f_\tau \in W(\Omega)$, and the mapping $(f, \tau) \to f_\tau$ is a continuous mapping of $W(\Omega) \times \mathbb{R}$ onto itself. This means that f_τ defines a (two-sided) flow on $W(\Omega)$. For each f satisfying (2.51) we define the *positive hull* $H^+(f)$ as

$$H^+(f) = \operatorname{Closure}_{W(\Omega)}\{f_\tau : \tau \geq 0\}$$

and the *hull* $H(f)$ as

$$H(f) = \operatorname{Closure}_{W(\Omega)}\{f_\tau : \tau \in \mathbb{R}\}.$$

If $f \in W(\Omega)$, then $H^+(f)$ and $H(f)$ lie in $W(\Omega)$. The *omega-limit set* $\omega(f)$ is defined by

$$\omega(f) = \bigcap_{\tau\geq 0} H^+(f_\tau).$$

Note that $\omega(f)$ is an invariant set in the translational flow on $W(\Omega)$.

Without further assumptions on the forcing function f, the omega-limit set $\omega(f)$ can be empty. However, if $H^+(f)$ is a compact set, then so is $H^+(f_\tau)$ for every $\tau \geq 0$. Since $H^+(f_\tau) \subset H^+(f)$ for $\tau \geq 0$, we see that if $H^+(f)$ is compact, then the omega-limit set $\omega(f)$ is nonempty and compact.

The question of the compactness of $H^+(f)$ can be resolved by using the Ascoli-Arzelá Theorem; see Sell (1967a, 1967b). In particular, if there is a compact set $\mathscr{K} \subseteq L^2(\Omega)$ such that $f(t) \in \mathscr{K}$ for all $t \geq 0$ and the mapping $t \to f(t)$ is a uniformly continuous mapping of $[0, \infty)$ into $L^2(\Omega)$, then $H^+(f)$ is a nonempty compact set; furthermore, the omega-limit set $\omega(f)$ is nonempty, compact, and invariant under the translational flow. We list here five examples of functions f for which $H^+(f)$ is compact:

(1) $f \in W^{1,\infty}((0, \infty), L^2(\Omega))$, and there is a compact set $\mathscr{K} \subseteq L^2(\Omega)$ such that $f(t) \in \mathscr{K}$ for all $t \geq 0$.
(2) $f \in W^{1,\infty}((0, \infty), H^1(\Omega))$.
(3) $f(t)$ is continuous and Bohr almost-periodic, or periodic, in t.
(4) $f = g + h$, where g and h satisfy (2.51), $\|h(t)\|_{L^2(\Omega)} \to 0$ as $t \to \infty$, and $g(t)$ is Bohr almost-periodic, or periodic, in t.
(5) $f \in L^2(\Omega)$ is independent of t.

The evaluation mapping $\operatorname{Ev}: W(\Omega) \to L^2(\Omega)$ given by $\operatorname{Ev}(f) = f(0)$ is a continuous mapping of $W(\Omega)$ in $L^2(\Omega)$. Therefore if $H^+(f)$ is a compact set in $W(\Omega)$, then

$$\mathrm{Ev}(H^+(f)) = \{g(0): g \in H^+(f)\}$$

is a compact set in $L^2(\Omega)$, and one has $g(t) \in \mathrm{Ev}(H^+(f))$ for all $t \geq 0$ and $g \in H^+(f)$.

Let H be a Banach space with a norm $\|\cdot\|_H$, and let $\mathscr{F}$ be any positively invariant subset of $W(\Omega)$. Let $\mathscr{O}$ be an open set in $H \times \mathscr{F} \times [0, \infty)$ with

$$\{(u, f, 0): (u, f) \in H \times \mathscr{F}\} \subset \mathscr{O},$$

and let $\pi: \mathscr{O} \to H \times \mathscr{F}$ be a mapping of the form

$$\pi(u, f; \tau) = (S(f, \tau)u, f_\tau), \qquad (u, f, \tau) \in \mathscr{O}. \tag{2.52}$$

For each $(u, f) \in H \times \mathscr{F}$, let $I_{(u,f)} = [0, \tau)$, where $\tau = \tau(u, f)$, denote the maximal time interval for which $(u, f, t) \in \mathscr{O}$ for $0 \leq t < \tau$. We say that π is a *skew-product semiflow* on $H \times \mathscr{F}$ if the following properties are satisfied:

(1) $S(f, 0)u = u$, for all $(u, f) \in H \times \mathscr{F}$.

(2) Whenever $t \in I_{(u,f)}$ and $s \in I_{(S(f,t)u, f_t)}$, then $(t+s) \in I_{(u,f)}$ and one has

$$S(f_t, s)S(f, t)u = S(f, t+s)u.$$

(3) The mapping $(u, f, t) \to \pi(u, f; t)$ is continuous in $(u, f) \in H \times \mathscr{F}$ for t fixed and continuous in t for (u, f) fixed.

(4) If $(u, f) \in H \times \mathscr{F}$ and $\tau(u, f) < \infty$, then one has

$$\limsup_{t \to \tau^-} \|S(f, t)u\|_H = \infty.$$

It is a consequence of (2) that one has

$$\pi(u, f; t+s) = \pi(\pi(u, f; t); s).$$

Let $\mathscr{K}$ be a subset of $H \times \mathscr{F}$ and assume that t satisfies $0 \leq t < \tau(u, f)$ for all $(u, f) \in \mathscr{K}$. For this t we define $\pi(\mathscr{K}; t)$ to be the collection of all $\pi(u, f; t)$ with $(u, f) \in \mathscr{K}$. A subset $\mathscr{K}$ in $H \times \mathscr{F}$ is said to be *invariant* for π if one has $\tau(u, f) = \infty$ for all $(u, f) \in \mathscr{K}$ and $\pi(\mathscr{K}; t) = \mathscr{K}$ for all $t \geq 0$. If $\mathscr{K}$ is any subset of $H \times \mathscr{F}$ with $\tau(u, f) = \infty$ for all $(u, f) \in \mathscr{K}$, we define the *omega-limit set* of $\mathscr{K}$ as $\omega(\mathscr{K})$ where

$$\omega(\mathscr{K}) = \bigcap_{\tau \geq 0} \mathrm{Closure}_{H \times \mathscr{F}} \left(\bigcup_{t \geq \tau} \pi(\mathscr{K}; t) \right).$$

In the case of the dilated Navier-Stokes evolutionary equation on Q_3 with periodic boundary conditions, for $U_0 \in V_\varepsilon^1$ and $F \in W(Q_3)$, we set

$$S(F, t)U_0 = U(t), \qquad 0 \leq t < \tau(U_0, F), \tag{2.53}$$

where $U(t)$ is the maximally defined strong solution on $[0, \tau(U_0, F))$ that satisfies $U(0) = U_0$. One can show that

$$\mathscr{O} \stackrel{\mathrm{def}}{=} \{(U, F, t) \in V_\varepsilon^1 \times W(Q_3) \times [0, \infty): 0 \leq t < \tau(U, F)\}$$

is an open set, and the mapping π defined by (2.52) and (2.53) generates a skew-product semiflow on $V_\varepsilon^1 \times W(Q_3)$.

The same construction generates a skew-product semiflow for the Navier-Stokes equations on any reasonable bounded domain Ω in $\mathbb{R}^n$, for $n = 2, 3$, and under other homogeneous boundary conditions; see Constantin and Foiaş (1988). In this case, the semiflow is on the space $V^1 \times W(\Omega)$, where $V^1 = D(A^{1/2})$. For the 2DNS, one has $\tau(u, f) = \infty$, for all $(u, f) \in V^1 \times W(\Omega)$, i.e., π is a *global* semiflow in this case. Furthermore, by using the Leray solutions of the 2DNS instead of the strong solutions, this global semiflow extends to a global semiflow on $H \times H(\Omega)$, where $H = \mathbb{P}_2 L^2(\Omega)$; see Constantin and Foiaş (1988).

For the Navier-Stokes equations we will be studying the semiflow generated by (2.53) on $V_\varepsilon^1 \times \mathscr{F}$, where $\mathscr{F}$ is a compact invariant set in $W(Q_3)$. For example, with equation (2.5) one might assume either $\mathscr{F} = H^+(f)$ to be compact, or $\mathscr{F} = H^+(\mathbb{P}_\varepsilon f)$ to be compact. Either assumption leads to a reasonable dynamical theory for (2.5). The stronger condition that $H^+(f)$ be compact is important primarily in the study of the original system (2.4) where $(I - \mathbb{P}_\varepsilon)f$ is used. Similarly for the reduced 3D Navier-Stokes evolutionary system (2.24), one gets a good dynamical theory by assuming any one of the following three sets to be compact: $H^+(f)$, $H^+(\mathbb{P}_\varepsilon f)$, or $H^+(M\mathbb{P}_\varepsilon f)$.

If the forcing term f is time-independent and in $L^2(\Omega)$, then the hull $H(f)$ consists of a single point $\{f\}$, and the Navier-Stokes equations generate (local) semiflows on appropriate Hilbert spaces. For the 3DNS equations the strong solutions $S(t)u_0$ generate a semiflow on the Hilbert space $\mathbb{P}_3(H^1(\Omega))$. The weak solutions of the 2DNS generate a semiflow on $\mathbb{P}_2(L^2(\Omega))$.

2.12. **Local and global attractors.** We will continue to use the notation introduced in §2.11. Let π be the skew-product semiflow on $H \times \mathscr{F}$ given by (2.52), where $\mathscr{F}$ is a compact, positively invariant subset of $W(Q_3)$. A subset $\mathfrak{A}$ in $H \times \mathscr{F}$ is said to be a (*local*) *attractor* for π if $\mathfrak{A}$ is a compact, invariant set for π, and $\mathfrak{A} = \omega(U)$ is the omega-limit set of some bounded neighborhood U of $\mathfrak{A}$ in $H \times \mathscr{F}$. The *basin of attraction* $B(\mathfrak{A})$ defined to be the collection of all $(u, f) \in H \times \mathscr{F}$ with the property that

$$\operatorname{dist}_{H\times\mathscr{F}}(\pi(u, f; t), \mathfrak{A}) \to 0 \quad \text{as } t \to \infty. \tag{2.54}$$

If it happens that $\mathfrak{A}$ is an attractor with $B(\mathfrak{A}) = H \times \mathscr{F}$, then $\mathfrak{A}$ is said to be a *global attractor* for π. Note that any attractor $\mathfrak{A}$ attracts the bounded set U, as well as every compact set K in the basin $B(\mathfrak{A})$. Recall that $\mathfrak{A}$ attracts a set $B \subset H \times \mathscr{F}$ provided that for every $\delta > 0$ there is a time $T = T(B, \delta) \geq 0$ such that

$$\pi(u, f, t) \in N_{(H\times\mathscr{F}, \delta)}(\mathfrak{A}), \quad \text{for all } (u, f) \in B \text{ and all } t \geq T,$$

where $N_{(H\times\mathscr{F}, \delta)}(\mathfrak{A})$ denotes the δ-neighborhood of $\mathfrak{A}$ in $H \times \mathscr{F}$. Recall that the skew-product semiflow π is said to be *compact*, if for every $\tau > 0$, the mapping

$$\pi_\tau : \mathscr{O} \to H \times \mathscr{F}$$

given by $\pi_\tau(u, f) = \pi(u, f, \tau)$, maps bounded sets into compact sets. For a compact semiflow, the attractor $\mathfrak{A}$ attracts all bounded sets in the basin $B(\mathfrak{A})$, see Hale (1988) and Sell and You (1993).

We note that the flow generated by the strong solutions of the 2DNS is compact and always has a global attractor, provided the positive hull $H^+(f)$ of the forcing function f is compact. As we shall see, the reduced 3DNS also has a global attractor, when $H^+(f)$ is compact. The theorems, which we describe in the next section, effectively state that when ε is small the full 3DNS has a local attractor $\mathfrak{A}_\varepsilon$ and that $\mathfrak{A}_\varepsilon$ has a *large* basin of attraction.

2.13. **Statement of theorems about attractors.** In this section we assume that $f \in W(Q_3)$ is chosen so that $H^+(f)$ is compact; see §2.11. This includes the case where $f \in \mathrm{L}^2(Q_3)$ is time-independent. We assume that Hypothesis H(a, b) is satisfied, where a and b are sufficiently large. Let $\mathscr{B}_\varepsilon^0$, $\mathscr{B}_\varepsilon^1$, and $\mathscr{B}_\varepsilon^2$ be given by (2.38), (2.39), and (2.40). It is an immediate consequence of Theorems 1 and 2 that for $u_0 \in \mathscr{B}_\varepsilon^0 \cup \mathscr{B}_\varepsilon^1$ the solution $S_\varepsilon(f, t)u_0 = S_\varepsilon(\mathbb{P}_\varepsilon f, t)u_0$ lies in a bounded set in V_ε^2 for $t \geq \widehat{T}_1$. Therefore, $S_\varepsilon(f, t)u_0$ lies in a compact set in V_ε^1 for $t \geq \widehat{T}_1$. As a matter of fact, we are able to prove the following compactness result:

Theorem 3. *Let η_i, $i = 1, 2, 3, 4$, r, and p satisfy Hypothesis H(a, b), where a and b are sufficiently large. Assume that $f \in W(Q_3)$ is chosen so that $\mathbb{P}_\varepsilon f \in W^{1,\infty}((0, \infty), H_\varepsilon)$, $H^+(f)$ is compact, and* (2.29) *is satisfied. Let $S_\varepsilon(f, t)u_0$ denote the strong solution of* (2.5) *with initial data $u_0 \in V_\varepsilon^1$. Then for any $\tau > 0$ there is a compact subset $\mathscr{K}(\tau)$ of V_ε^2 such that*

$$S_\varepsilon(f, t)(\mathscr{B}_\varepsilon^0 \cup \mathscr{B}_\varepsilon^1) \subset \mathscr{K}(\tau), \qquad t \geq \tau.$$

The proof of Theorem 3 is given in §4. If we do not assume $H^+(f)$ to be compact in Theorem 3, then we can only prove, for $t > 0$, $S_\varepsilon(t)(\mathscr{B}_\varepsilon^0 \cup \mathscr{B}_\varepsilon^1)$ belongs to a compact set $\widehat{\mathscr{K}}(t)$ which may depend on t.

The following theorems are proved in §6. Let $u(t) = S_\varepsilon(\mathbb{P}_\varepsilon f, t)u_0$ denote the strong solution of the equation (2.5) with initial data u_0 in V_ε^1, and let $\pi_\varepsilon(u_0, \mathbb{P}_\varepsilon f; \tau) = (S_\varepsilon(\mathbb{P}_\varepsilon f, \tau)u_0, (\mathbb{P}_\varepsilon f)_\tau)$ denote the skew-product semiflow generated by the strong solutions of the dilated 3D Navier-Stokes evolutionary equation (2.5).

Theorem 4. *Let η_i, $i = 1, 2, 3, 4$, r, and p satisfy Hypothesis H(a, b), where a and b are sufficiently large. Assume that $f \in W(Q_3)$ is chosen so that $\mathbb{P}_\varepsilon f \in W^{1,\infty}((0, \infty), H_\varepsilon)$, $H^+(f)$ is compact, and one has*

$$\|M\mathbb{P}_\varepsilon f\|_\infty^2 \leq \eta_2^{-2}, \qquad \|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 \leq \varepsilon^r \eta_4^{-2}.$$

Let $\varepsilon_0 > 0$ be given by Theorem 1. *Then, for $0 < \varepsilon \leq \varepsilon_0$, the skew-product flow π_ε generated by the strong solutions of the dilated* 3D *Navier-Stokes evolutionary equation* (2.5) *has a unique, maximal, compact* (*local*) *attractor $\mathfrak{A}_\varepsilon$ included in*

$\mathscr{B}_\varepsilon^2 \times \omega(\mathbb{P}_\varepsilon f)$, which attracts $\mathscr{B}_\varepsilon^2 \times H^+(\mathbb{P}_\varepsilon f)$ in the space $V_\varepsilon^1 \times \mathbb{P}_\varepsilon W(Q_3)$, where $\mathscr{B}_\varepsilon^2$ is given by (2.40). *Furthermore, one has*

$$\mathfrak{A}_\varepsilon \subset \{u = v + w : \|A_\varepsilon^{1/2} v\|^2 \leq \Gamma(\eta_2^{-2}),\ \|A_\varepsilon^{1/2} w\|^2 \leq k_2^2 \varepsilon^{2+r} \eta_4^{-2}\} \times \omega(\mathbb{P}_\varepsilon f).$$

Moreover, $\mathfrak{A}_\varepsilon$ is bounded and compact in $V_\varepsilon^2 \times \omega(\mathbb{P}_\varepsilon f)$ and attracts the bounded set $(\mathscr{B}_\varepsilon^2 \cap V_\varepsilon^2) \times H^+(\mathbb{P}_\varepsilon f)$ in the space $V_\varepsilon^2 \times \mathbb{P}_\varepsilon W(Q_3)$.

In the next result we show that, under an added condition on η_i, $i = 1, 2, 3, 4$, see (2.55) below, the attractor $\mathfrak{A}_\varepsilon$ is the global attractor for the Leray solutions of the dilated 3D Navier-Stokes evolutionary equation (2.5), i.e., the weak solutions that satisfy the energy inequality (3.35); see Foiaş and Temam (1987).[4] Note that the example given in Remark 2 prior to the statement of Theorem 1 satisfies (2.55).

Corollary 4.1. *Let the hypotheses of Theorem 4 be satisfied. Assume in addition that for every $\lambda > 0$ there is an $\varepsilon_{10} = \varepsilon_{10}(\lambda) > 0$ such that*

$$\eta_2^{-2} + \varepsilon^{2+r} \eta_4^{-2} \leq \lambda \min(\eta_1^{-2}, \varepsilon^p \eta_3^{-2}), \qquad 0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_{10}). \tag{2.55}$$

Then for any ε satisfying $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_{10})$ and any $\rho > 0$, there is a time $T = T(\rho, \varepsilon) > 0$, and, for every weak (Leray) solution $u(t)$ of (2.5) *with $\|u(0)\| \leq \rho$, there is a time t_0 satisfying $0 < t_0 \leq T(\rho, \varepsilon)$ and $u(t_0) \in \mathscr{B}_\varepsilon^0$. In particular, $u(t)$ is a regular solution of* (2.5) *for $t \geq t_0$, and the attractor $\mathfrak{A}_\varepsilon$ given in Theorem* 4 *is the global attractor for the Leray solutions of* (2.5), *provided $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_{10})$.*

Let us now consider the reduced 3D Navier-Stokes evolutionary equation (2.24), and let us denote by $S_0(g, t)\overline{v}_0$ the strong solution of (2.24) with initial data $\overline{v}_0$ in MV_ε^1, where $g = M\mathbb{P}_\varepsilon f$. We denote by $\pi_0(\overline{v}_0, g; \tau) = (S_0(g, \tau)\overline{v}_0, g_\tau)$ the skew-product semiflow generated by the strong solutions of (2.24). As noted in §2.6, the terms in (2.24) do not depend on x_3 and ε. We have the following result:

Corollary 4.2. *Assume that $f \in W(Q_3)$ is chosen so that $\mathbb{P}_\varepsilon f \in W^{1,\infty}((0, \infty); H_\varepsilon)$ and $H^+(f)$ is compact. Then π_0 admits a global attractor $\mathfrak{A}_0 = \mathfrak{A}_0(g)$ in $MV_\varepsilon^1 \times H^+(g)$. Furthermore, if $\|M\mathbb{P}_\varepsilon f\|_\infty^2 \leq \eta_2^{-2}$, where η_2 is given by Hypothesis H(a, b), then*

$$\mathfrak{A}_0(g) \subset \{u = v + w : v \in MV_\varepsilon^1,\ \|A_\varepsilon^{1/2} v\|^2 \leq \Gamma(\eta_2^{-2}),\ w = 0\} \times \omega(g). \tag{2.56}$$

If, in addition, one has

$$(I - M)\mathbb{P}_\varepsilon f = 0,$$

then the attractors $\mathfrak{A}_\varepsilon$ and $\mathfrak{A}_0(g)$ coincide for $0 < \varepsilon \leq \varepsilon_0$, where ε_0 is given by Theorem 1.

If $(I - M)\mathbb{P}_\varepsilon f \neq 0$, a comparison of the two attractors $\mathfrak{A}_\varepsilon$ and $\mathfrak{A}_0(g)$ is more difficult. Nevertheless, we are able to derive some results establishing the upper semicontinuity of $\mathfrak{A}_\varepsilon$ at $\varepsilon = 0$.

[4] Recall that the 3DNS can have *weak* solutions that do not satisfy the energy inequality (3.35); see Temam (1983).

Let us consider a sequence of positive numbers $\varepsilon_n \to 0$ when $n \to \infty$. We introduce a sequence of functions f_n in $W(Q_3) \cap W^{1,\infty}((0,\infty), L^2(Q_3))$ such that $f_n \to f_0$ in $W(Q_3)$, where $f_0 \in MW(Q_3) \cap W^{1,\infty}((0,\infty); ML^2(Q_3))$. We set $g_n = \mathbb{P}_{\varepsilon_n} f_n$ and $g_0 = M\mathbb{P}_{\varepsilon_n} f_0$. According to the comments made in §2.6, $\mathbb{P}_{\varepsilon_n} f_0(t)$ belongs to MH_{ε_n}, for every t, and consequently

$$g_0 = \mathbb{P}_{\varepsilon_n} f_0 = \begin{pmatrix} \mathbb{P}_2 \binom{f_{01}}{f_{02}} \\ f_{03} \end{pmatrix}, \tag{2.57}$$

where $f_0 = (f_{01}, f_{02}, f_{03})$. It follows from the above convergence hypothesis and from (2.57) that

$$\lim_{n\to\infty} \|g_n - g_0\|_\infty = 0.$$

We consider next the reduced 3D Navier-Stokes evolutionary equation

$$\overline{v}' + \nu A_0 \overline{v} + B_0(\overline{v}, \overline{v}) = g_0 \tag{2.58}$$

with initial data $\overline{v}(0) = \overline{v}_0$ in V_0^1, and we let $S_0(g_0, t)\overline{v}_0$ denote the strong solutions of (2.58) with initial data $\overline{v}_0$ in V_0^1. As a consequence of Corollary 4.2, the skew-product semiflow $\pi_0(\overline{v}_0, g_0; \tau) = (S_0(g_0, \tau)\overline{v}_0, g_{0\tau})$ admits a global compact attractor $\mathfrak{A}_0 = \mathfrak{A}_0(g_0)$ in $V_0^1 \times \omega(g_0)$, which is also the global compact attractor in $V_0^2 \times \omega(g_0)$.

Let E be a subset of $V_{\varepsilon_n}^1 \times W(Q_3)$. For any $\delta > 0$, we denote by $\mathcal{N}_{V_{\varepsilon_n}^1 \times W(Q_3)}(E, \delta)$ the δ-neighborhood of E in $V_{\varepsilon_n}^1 \times W(Q_3)$. We will prove the following result:

Theorem 5. *Let η_i, $i = 1, 2, 3, 4$, r, and p satisfy Hypothesis H(a, b), where a and b are sufficiently large, and assume that*

$$\varepsilon^{4+2r}\eta_4^{-4}(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{2.59}$$

Let ε_n be a sequence of positive numbers with $\varepsilon_n \to 0$ as $n \to \infty$. Let $\mathcal{F}$ be any positively invariant compact subset of $W(Q_3) \cap W^{1,\infty}((0,\infty), L^2(Q_3))$, and let f_n be a sequence of functions in $\mathcal{F}$ that satisfies

$$\lim_{n\to\infty} \|f_n - f_0\|_\infty = 0, \tag{2.60}$$

where $f_0 \in M\mathcal{F}$. Assume further that

$$\|M\mathbb{P}_{\varepsilon_n} f_n\|_\infty^2 < \eta_2^{-2}, \qquad \|(I - M)\mathbb{P}_{\varepsilon_n} f_n\|_\infty^2 \le \varepsilon_n^r \eta_4^{-2}.$$

Then the attractors $\mathfrak{A}_{\varepsilon_n}$, given in Theorem 4, are upper semicontinuous in $V_{\varepsilon_n}^1 \times \mathcal{F}$ at $\varepsilon = 0$, i.e., for any $\delta > 0$, there is an $n_0 > 0$ such that

$$\mathfrak{A}_{\varepsilon_n} \subset \mathcal{N}_{V_{\varepsilon_n}^1 \times W(Q_3)}(\mathfrak{A}_0(g_0), \delta)$$

for $n \ge n_0$, where $g_0 = M\mathbb{P}_{\varepsilon_n} f_0 = \mathbb{P}_{\varepsilon_n} f_0$.

Theorem 5 has some interesting extensions. The following result, which we formulate in the case where the forcing terms f_n are independent of time t, allows for the possibility that f_n can be chosen so that

$$\|(I-M)\mathbb{P}_{\varepsilon_n} f_n\|^2 \to \infty \quad \text{as } n \to \infty.$$

If f is independent of t, the mapping $S_\varepsilon(t) \stackrel{\text{def}}{=} S_\varepsilon(\mathbb{P}_\varepsilon f, t)$ is a (local) C^0-semigroup on V_ε^1. We then deduce from Theorem 4 that $S_\varepsilon(t)$ admits a unique maximal compact (local) attractor $\widetilde{\mathfrak{A}}_\varepsilon$ included in $\mathscr{B}_\varepsilon^2$ which attracts $\mathscr{B}_\varepsilon^2$ in the space V_ε^1. Actually, we have

$$\mathfrak{A}_\varepsilon = \widetilde{\mathfrak{A}}_\varepsilon \times \{\mathbb{P}_\varepsilon f\}.$$

Likewise, $S_0(t) \stackrel{\text{def}}{=} S_0(g, t)$ is a C^0-semigroup on $V_0^1 = MV_\varepsilon^1$, where $g = M\mathbb{P}_\varepsilon f$. We deduce from Corollary 4.2 that $S_0(t)$ admits a global, compact attractor $\widetilde{\mathfrak{A}}_0 = \widetilde{\mathfrak{A}}_0(g)$ in V_0^1; and we have

$$\mathfrak{A}_0 = \widetilde{\mathfrak{A}}_0 \times \{g\}.$$

The following result is proved in §6.

Corollary 5.1. *Let η_i, $i = 1, 2, 3, 4$, r, and p satisfy Hypothesis H(a, b), where a and b are sufficiently large, and assume that* (2.59) *holds. Let ε_n be a sequence of positive numbers with $\varepsilon_n \to 0$ as $n \to \infty$. Let f_n be a sequence in $L^2(Q_3)$ that satisfies*

$$\lim_{n\to\infty} \|M\mathbb{P}_{\varepsilon_n} f_n - g_0\| = 0$$

for some $g_0 \in H_0$. Assume further that

$$\|M\mathbb{P}_{\varepsilon_n} f_n\|^2 \le \eta_2^{-2}, \qquad \|(I-M)\mathbb{P}_{\varepsilon_n} f_n\|^2 \le \varepsilon_n^r \eta_4^{-2}.$$

Then the attractors $\widetilde{\mathfrak{A}}_{\varepsilon_n}$ of (2.5) *with forcing term $\mathbb{P}_{\varepsilon_n} f_n$ are upper semicontinuous at $\varepsilon = 0$ in $V_{\varepsilon_n}^1$, i.e.,*

$$\sup_{u_n \in \widetilde{\mathfrak{A}}_{\varepsilon_n}} \inf_{v \in \widetilde{\mathfrak{A}}_0} \|A_{\varepsilon_n}^{1/2}(u_n - v)\| \to 0 \quad \text{as } \varepsilon = \varepsilon_n \to 0, \tag{2.61}$$

where $\widetilde{\mathfrak{A}}_0 = \widetilde{\mathfrak{A}}_0(g_0)$ is the global attractor of (2.58).

Using the fact that $\mathfrak{A}_0 = \mathfrak{A}_0(g_0)$ is also the global compact attractor of the skew-product semiflow $\pi_0(\cdot, g_0; \tau)$ in $V_0^2 \times H^+(g_0)$, one also obtains the result:

Corollary 5.2. *Assume that the hypotheses of Theorem 5 hold and that*

$$\lim_{n\to\infty} \|f'_n - f'_0\|_\infty = 0. \tag{2.62}$$

Then the attractors $\mathfrak{A}_{\varepsilon_n}$ are also upper semicontinuous at $\varepsilon = 0$ in $V_\varepsilon^2 \times W(Q_3)$, i.e., for any $\delta > 0$, there exists an integer $n_1 > 0$ such that, for $n \ge n_1$,

$$\mathfrak{A}_{\varepsilon_n} \subset \mathscr{N}_{V_{\varepsilon_n}^2 \times W(Q_3)}(\mathfrak{A}_0(g_0), \delta). \tag{2.63}$$

Remark. We now give an example where the condition (2.60) is satisfied. Let $F = F(t, y)$ be given, where $F \in C^0(\mathbb{R}, W^{1,\infty}(Q_3)) \cap L^\infty(\mathbb{R}, W^{1,\infty}(Q_3))$. As in §2.1, we introduce the mapping f_ε by setting $F = J_\varepsilon f_\varepsilon$

$$f_\varepsilon(t, x_1, x_2, x_3) = F(t, x_1, x_2, \varepsilon x_3), \qquad (x_1, x_2, x_3) \in Q_3.$$

Next set $f_0(t, x_1, x_2, x_3) = F(t, x_1, x_2, 0)$. By applying the integral Taylor formula, we obtain

$$\begin{aligned} f_\varepsilon(t, x) - f_0(t, x) &= F(t, x_1, x_2, \varepsilon x_3) - F(t, x_1, x_2, 0) \\ &= \varepsilon \int_0^1 \frac{\partial F}{\partial x_3}(t, x_1, x_2, s x_3) x_3 \, ds, \end{aligned}$$

and therefore

$$\|f_\varepsilon - f_0\|_{C^0([0,\infty); L^\infty(Q_3))} \leq \varepsilon \|D_3 F\|_{C^0([0,\infty); L^\infty(Q_3))}.$$

In particular, we have

$$\|f_\varepsilon - f_0\|_\infty \leq C\varepsilon.$$

If, in addition, F belongs to $C^1(\mathbb{R}, W^{1,\infty}(Q_3)) \cap L^\infty(\mathbb{R}, W^{1,\infty}(Q_3))$, then the condition (2.62) is also satisfied; more precisely, we have

$$\|f'_\varepsilon - f'_0\|_\infty \leq \widetilde{C}\varepsilon.$$

3. H^1-Regularity: Theorem 1

In this section we prove Theorem 1, the H^1-Regularity Theorem, which gives the global regularity of the Navier-Stokes equations on Ω_ε in the Sobolev space H^1. We assume that the forcing function f is in $L^\infty((0, \infty), L^2(Q_3))$ and that the initial condition u_0 satisfies $u_0 \in D(A_\varepsilon^{1/2})$. Also we assume that one has

$$(3.1) \qquad \begin{cases} \|A_\varepsilon^{1/2} v_0\|^2 \leq \eta_1^{-2}, & \|M\mathbb{P}_\varepsilon f\|_\infty^2 \leq \eta_2^{-2}, \\ \|A_\varepsilon^{1/2} w_0\|^2 \leq \varepsilon^p \eta_3^{-2}, & \|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 \leq \varepsilon^r \eta_4^{-2}, \end{cases}$$

where $\eta_i(\varepsilon)$ is bounded and monotone for $0 < \varepsilon \leq 1$, $i = 1, 2, 3, 4$. (We are primarily interested in the case where $\eta_i(\varepsilon) \to 0$ as $\varepsilon \to 0$, $i = 1, 2, 3, 4$.) Throughout this section we let $D_1, D_2, \ldots$ denote positive functions of the viscosity ν and λ_1, the first eigenvalue of A_ε. These functions will not depend on ε for $0 < \varepsilon \leq 1$.

The proof of the H^1-Regularity Theorem is done in two steps. In the first lemma, which is the Short Time Argument, we show that the $w(t)$-term becomes small very rapidly. The second lemma is referred to as the Long Time Argument. This is the induction step needed for the proof of the H^1-Regularity Theorem. (See §§3.2 and 3.3.)

We will use the following auxiliary estimates concerning the trilinear form b_ε: If $v^1, v^2, v^3 \in R(M)$, then these functions depend only on x_1 and x_2, and one has

$$(3.2) \qquad |b_\varepsilon(v^1, v^2, v^3)| \leq C_1 \|v^1\|^{1/2} \|A_\varepsilon^{1/2} v^1\|^{1/2} \|A_\varepsilon^{1/2} v^2\|^{1/2} \|A_\varepsilon v^2\|^{1/2} \|v^3\|.$$

The proof of (3.2) is accomplished by using 2D Sobolev embeddings; see Temam (1983) or Constantin and Foiaş (1988) for details. If one has $v \in R(M)$ and $Mw^1 = Mw^2 = Mw = 0$, then the following inequalities hold:

$$\text{(3.3)} \quad \begin{cases} |b_\varepsilon(w^1, w^2, u)| \le C_2 \varepsilon^{1/2} \|A_\varepsilon^{1/2} w^1\| \, \|A_\varepsilon^{1/2} w^2\|^{1/2} \|A_\varepsilon w^2\|^{1/2} \|u\|, \\ |b_\varepsilon(w, u^2, u^3)| \le C_3 \varepsilon^{5/32} \|A_\varepsilon^{1/2} w\|^{15/32} \|A_\varepsilon w\|^{17/32} \|A_\varepsilon^{1/2} u^2\| \, \|u^3\|, \\ |b_\varepsilon(v, w, u)| \le C_4 \varepsilon^{1/4} \|A_\varepsilon^{1/2} v\| \, \|A_\varepsilon^{1/2} w\|^{1/2} \|A_\varepsilon w\|^{1/2} \|u\|. \end{cases}$$

The proof of (3.3) is given in the appendix, §8. It is important to note that the constants C_1, C_2, C_3, and C_4 above do not depend on ε, for $0 < \varepsilon \le 1$.

We shall also use the Young inequality

$$\text{(3.4)} \quad ab \le \frac{c^p a^p}{p} + \frac{b^q}{q c^q} = \delta a^p + c_\delta b^q,$$

where a, b, c, δ, and c_δ are positive, $1 \le p, q$, $p^{-1} + q^{-1} = 1$, as well as

$$\text{(3.5)} \quad (a+b)^3 \le 4(a^3 + b^3), \qquad a, b \ge 0.$$

The proof of the following result can easily be derived from the theory presented in Constantin and Foiaş (1988), Temam (1977, Chapter III, Lemma 1.2 and Theorem 3.11), Temam (1983, §3, Theorem 3.2), as well as the other references cited above.

Lemma 3.0. *Let $u_0 \in D(A_\varepsilon^{1/2})$ and $f \in L^\infty(0, T; H_\varepsilon)$. Then there exists a time T_*, $0 < T_* \le \infty$, such that there exists a unique solution u of (2.5) on $(0, T_*)$. Moreover, u satisfies: $u \in C^0([0, T_*]; V_\varepsilon^1) \cap L^2(0, T_*; V_\varepsilon^2)$, and $u_t \in L^2(0, T_*; H_\varepsilon)$. Assume furthermore that $R_0^2 \ge \|A_\varepsilon^{1/2} u_0\|^2$ and $N > 1$. Then there exists a positive time T^N, $0 < T^N \le T_*$, such that $\|A_\varepsilon^{1/2} u(t)\|^2 \le N R_0^2$ for $0 \le t < T^N$.*

3.1. **The short time argument.** We shall say that **Hypothesis H1** is satisfied if one has:

(1) $p \ge -1$, $r > -2$.
(2) $\varepsilon^{1/4} \eta_i^{-1} \to 0$ as $\varepsilon \to 0$, $i = 1, 2$.
(3) $\varepsilon^{1/8} \eta_i^{-1} \to 0$ as $\varepsilon \to 0$, $i = 3, 4$.
(4) $\varepsilon^{1/4} Q(\varepsilon)$ is bounded for $0 < \varepsilon \le 1$, where

$$\text{(3.6)} \quad Q(\varepsilon) = |\log(2 C_5^2 \nu^{-2} \varepsilon^{2+r-p} \eta_4^{-2} \eta_3^2)|.$$

Here $\eta_i(\varepsilon)$ denote bounded monotone functions defined for $0 < \varepsilon \le 1$, $i = 1, 2, 3, 4$, and r and p are negative constants.

We now prove the following result.

Lemma 3.1. *Assume that Hypothesis H1 is satisfied and that (3.1) is valid. Then there are positive constants k_1 and k_2 and an $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \le \varepsilon_1$ there exists a time $T_1 = T_1(\varepsilon) > 0$ such that $u(t) \in D(A_\varepsilon^{1/2})$ for $0 \le t \le T_1$ and*

$$\text{(3.7)} \quad \begin{cases} \|A_\varepsilon^{1/2} v(T_1)\|^2 \le 4\eta_1^{-2} + k_1^2 \eta_3^{-4}, \\ \|A_\varepsilon^{1/2} w(T_1)\|^2 \le k_2^2 \varepsilon^{2+r} \eta_4^{-2}. \end{cases}$$

Proof. Define R_0^2 by

$$R_0^2 \stackrel{\text{def}}{=} \eta_1^{-2} + \varepsilon^p \eta_3^{-2} + \eta_3^{-4} + \eta_2^{-2} + \varepsilon^{r/3}\eta_4^{-2}. \tag{3.8}$$

Since $R_0^2 \geq \|A_\varepsilon^{1/2} u_0\|^2$, it follows from Lemma 3.0 that for any $N > 1$ there is a time $T^N > 0$ such that

$$\|A_\varepsilon^{1/2} u(t)\|^2 \leq N R_0^2, \qquad 0 \leq t < T^N. \tag{3.9}$$

Without loss of generality we let $[0, T^N)$ denote the maximal time interval for which (3.9) is valid. If $T^N < \infty$, then one must have

$$\|A_\varepsilon^{1/2} u(T^N)\|^2 = N R_0^2. \tag{3.10}$$

For the remainder of the proof of this lemma we restrict our attention to $t \in (0, T^N)$.

The equation satisfied by $w = (I - M)u$ in (2.23) is

$$\frac{dw}{dt} + \nu A_\varepsilon w = (I - M)\mathbb{P}_\varepsilon f - (I - M)(B_\varepsilon(w, v) + B_\varepsilon(v, w) + B_\varepsilon(w, w)) \tag{3.11}$$

since $(I - M)B_\varepsilon(v, v) = 0$. By taking the scalar product of (3.11) with $A_\varepsilon w$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{1/2} w\|^2 + \nu\|A_\varepsilon w\|^2 \leq{}& |\langle (I - M)\mathbb{P}_\varepsilon f, A_\varepsilon w\rangle| + |b_\varepsilon(w, v, A_\varepsilon w)| \\ &+ |b_\varepsilon(v, w, A_\varepsilon w)| + |b_\varepsilon(w, w, A_\varepsilon w)|.\end{aligned}$$

By using (3.3) and the Young inequality (3.4) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{1/2} w\|^2 + \nu\|A_\varepsilon w\|^2 \leq{}& \frac{\nu}{2}\|A_\varepsilon w\|^2 + \frac{1}{2\nu}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 \\ &+ C_3 \varepsilon^{5/32}\|A_\varepsilon^{1/2} v\|\,\|A_\varepsilon^{1/2} w\|^{15/32}\|A_\varepsilon w\|^{49/32} \\ &+ C_4 \varepsilon^{1/4}\|A_\varepsilon^{1/2} v\|\,\|A_\varepsilon^{1/2} w\|^{1/2}\|A_\varepsilon w\|^{3/2} \\ &+ C_2 \varepsilon^{1/2}\|A_\varepsilon^{1/2} w\|^{3/2}\|A_\varepsilon w\|^{3/2}.\end{aligned} \tag{3.12}$$

Since $Mw = 0$, one can use (2.22) together with (3.12) to find

$$\begin{aligned}\frac{d}{dt}\|A_\varepsilon^{1/2} w\|^2 + \nu\|A_\varepsilon w\|^2 \leq{}& \frac{1}{\nu}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 + 2C_5^{15/32} C_3 \varepsilon^{5/8}\|A_\varepsilon^{1/2} v\|\,\|A_\varepsilon w\|^2 \\ &+ 2C_5^{1/2} C_4 \varepsilon^{3/4}\|A_\varepsilon^{1/2} v\|\,\|A_\varepsilon w\|^2 \\ &+ 2C_5^{1/2} C_2 \varepsilon\|A_\varepsilon^{1/2} w\|\,\|A_\varepsilon w\|^2.\end{aligned}$$

From the Pythagorean relation (2.21) one obtains

$$\frac{d}{dt}\|A_\varepsilon^{1/2} w\|^2 + (\nu - D_1 \varepsilon^{5/8}\|A_\varepsilon^{1/2} u\|)\|A_\varepsilon w\|^2 \leq \frac{1}{\nu}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2, \tag{3.13}$$

where $D_1 = 2(C_5^{15/32} C_3 + C_5^{1/2} C_4 + C_5^{1/2} C_2)$. From Hypothesis H1 we see that for $0 \leq t < T^N$ one has

$$D_1 \varepsilon^{5/8}\|A_\varepsilon^{1/2} u\| \leq D_1 \varepsilon^{5/8} N^{1/2} R_0 \leq D_1 N^{1/2} \varepsilon^{5/8}(\eta_1^{-1} + \varepsilon^{p/2}\eta_3^{-1} + \eta_3^{-2} + \eta_2^{-1} + \varepsilon^{r/6}\eta_4^{-1}), \tag{3.14}$$

which goes to 0, as $\varepsilon \to 0^+$. Consequently there is a positive number $\varepsilon_2 = \varepsilon_2(N)$ that satisfies

$$D_1 N^{1/2} \varepsilon^{5/8} R_0 \leq \frac{\nu}{2}, \qquad 0 < \varepsilon \leq \varepsilon_2. \tag{3.15}$$

(Later N will be fixed, and it will depend only on ν and λ_1.) For $0 < \varepsilon \leq \varepsilon_2$ and $0 \leq t < T^N$ it follows from (3.13), (3.14), and (3.15) that

$$\frac{d}{dt}\|A_\varepsilon^{1/2} w\|^2 + \frac{\nu}{2}\|A_\varepsilon w\|^2 \leq \frac{1}{\nu}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 \tag{3.16}$$

and from (2.22) that

$$\frac{d}{dt}\|A_\varepsilon^{1/2} w\|^2 + \frac{\nu C_5^{-2}\varepsilon^{-2}}{2}\|A_\varepsilon^{1/2} w\|^2 \leq \frac{1}{\nu}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2. \tag{3.17}$$

We then apply the Gronwall inequality to (3.17) to obtain

$$\|A_\varepsilon^{1/2} w(t)\|^2 \leq \exp\left(\frac{-\nu C_5^{-2}\varepsilon^{-2}}{2} t\right)\|A_\varepsilon^{1/2} w_0\|^2 + \frac{2C_5^2\varepsilon^2}{\nu^2}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 \tag{3.18}$$

for $0 \leq t < T^N$ and $0 < \varepsilon \leq \varepsilon_2$. By integrating (3.16) we also obtain

$$\int_0^t \|A_\varepsilon w(s)\|^2\, ds \leq \frac{2t}{\nu^2}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 + \frac{2}{\nu}\|A_\varepsilon^{1/2} w_0\|^2 \tag{3.19}$$

for $0 \leq t < T^N$ and $0 < \varepsilon \leq \varepsilon_2$.

For the remainder of the proof we shall restrict our attention to $0 < \varepsilon \leq \varepsilon_2$. We will need an estimate of $\int_0^t \|A_\varepsilon^{1/2} w\|^3 \|A_\varepsilon w\|\, ds$. From (3.5) and (3.18) we obtain

$$\begin{aligned}
&\int_0^t \|A_\varepsilon^{1/2} w\|^6\, ds \\
&\quad \leq 4\int_0^t \left[\exp\left(\frac{-3\nu C_5^{-2}\varepsilon^{-2}}{2} s\right)\|A_\varepsilon^{1/2} w_0\|^6 + \frac{8C_5^6\varepsilon^6}{\nu^6}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^6\right] ds \\
&\quad \leq 4\left(\frac{2C_5^2\varepsilon^2}{3\nu}\|A_\varepsilon^{1/2} w_0\|^6 + \frac{8C_5^6\varepsilon^6 t}{\nu^6}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^6\right).
\end{aligned} \tag{3.20}$$

By using the Schwarz inequality with (3.19) and (3.20) we next obtain

$$\begin{aligned}
\int_0^t \|A_\varepsilon^{1/2} w\|^3 \|A_\varepsilon w\|\, ds &\leq \left(\int_0^t \|A_\varepsilon^{1/2} w\|^6\, ds\right)^{1/2} \left(\int_0^t \|A_\varepsilon w\|^2\, ds\right)^{1/2} \\
&\leq 4C_5\nu^{-1}\varepsilon\left(\frac{1}{\sqrt{3}}\|A_\varepsilon^{1/2} w_0\|^3 + \frac{2C_5^2\varepsilon^2 t^{1/2}}{\nu^{5/2}}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty^3\right) \\
&\quad \times \left(\|A_\varepsilon^{1/2} w_0\| + \frac{t^{1/2}}{\nu^{1/2}}\|(I - M)\mathbb{P}_\varepsilon f\|_\infty\right).
\end{aligned} \tag{3.21}$$

Let us return to inequality (3.18). Note that there is a time $T_1 = T_1(\varepsilon) > 0$ such that

$$\varepsilon^p \eta_3^{-2} \exp\left(\frac{-\nu C_5^{-2}\varepsilon^{-2}}{2} T_1\right) = \frac{2C_5^2\varepsilon^2}{\nu^2}\varepsilon^r \eta_4^{-2}.$$

Indeed, this time T_1 is given by

$$T_1 \overset{\text{def}}{=} 2C_5^2\varepsilon^2\nu^{-1}Q(\varepsilon), \tag{3.22}$$

where $Q(\varepsilon)$ is given by (3.6). It follows from (3.18) that if $T_1 < T^N$, then one has

$$\|A_\varepsilon^{1/2}w(t)\|^2 \le k_2^2\varepsilon^{2+r}\eta_4^{-2}, \qquad T_1 \le t < T^N, \tag{3.23}$$

where $k_2^2 = 4C_5^2\nu^{-2}$.

The next step is to return to (2.23) and the equation satisfied by $v = Mu$:

$$\frac{dv}{dt} + \nu A_\varepsilon v = M\mathbb{P}_\varepsilon f - MB_\varepsilon(v, v) - MB_\varepsilon(v, w) - MB_\varepsilon(w, v) - MB_\varepsilon(w, w). \tag{3.24}$$

By taking the scalar product of (3.24) with $A_\varepsilon v$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{1/2}v\|^2 + \nu\|A_\varepsilon v\|^2 \le |\langle M\mathbb{P}_\varepsilon f, A_\varepsilon v\rangle - b_\varepsilon(v, v, A_\varepsilon v) - b_\varepsilon(w, w, A_\varepsilon v)| \tag{3.25}$$

since $b_\varepsilon(v, w, A_\varepsilon v) = b_\varepsilon(w, v, A_\varepsilon v) = 0$ from (2.26). By using the Young inequality (3.4) with (3.2) and (3.3) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{1/2}v\|^2 + \nu\|A_\varepsilon v\|^2 \le{}& \frac{\nu}{2}\|A_\varepsilon v\|^2 + \frac{1}{2\nu}\|M\mathbb{P}_\varepsilon f\|_\infty^2 \\ &+ C_1\|v\|^{1/2}\|A_\varepsilon^{1/2}v\|\,\|A_\varepsilon v\|^{3/2} \\ &+ C_2\varepsilon^{1/2}\|A_\varepsilon^{1/2}w\|^{3/2}\|A_\varepsilon w\|^{1/2}\|A_\varepsilon v\|,\end{aligned} \tag{3.26}$$

which implies that

$$\begin{aligned}\frac{d}{dt}\|A_\varepsilon^{1/2}v\|^2 + \nu\|A_\varepsilon v\|^2 \le{}& \frac{1}{\nu}\|M\mathbb{P}_\varepsilon f\|_\infty^2 + \frac{\nu}{2}\|A_\varepsilon v\|^2 + \frac{27}{2\nu^3}C_1^4\|v\|^2\|A_\varepsilon^{1/2}v\|^4 \\ &+ \frac{\nu}{2}\|A_\varepsilon v\|^2 + \frac{2}{\nu}C_2^2\varepsilon\|A_\varepsilon^{1/2}w\|^3\|A_\varepsilon w\|.\end{aligned} \tag{3.27}$$

Consequently

$$\begin{aligned}\frac{d}{dt}\|A_\varepsilon^{1/2}v\|^2 \le{}& \left(\frac{27}{2\nu^3}C_1^4\|v\|^2\|A_\varepsilon^{1/2}v\|^2\right)\|A_\varepsilon^{1/2}v\|^2 \\ &+ \frac{1}{\nu}\|M\mathbb{P}_\varepsilon f\|_\infty^2 + \frac{2C_2^2\varepsilon}{\nu}\|A_\varepsilon^{1/2}w\|^3\|A_\varepsilon w\|.\end{aligned} \tag{3.28}$$

By using the Gronwall inequality with (3.28) one finds that

$$\|A_\varepsilon^{1/2}v(t)\|^2 \le e^{G(t)}(\|A_\varepsilon^{1/2}v_0\|^2 + H(t)), \qquad t \ge 0, \tag{3.29}$$

where

$$H(t) = \int_0^t h(s)\,ds, \qquad G(t) = \int_0^t g(s)\,ds, \tag{3.30}$$

$$h(t) = \frac{1}{\nu}\|M\mathbb{P}_\varepsilon f\|_\infty^2 + \frac{2C_2^2\varepsilon}{\nu}\|A_\varepsilon^{1/2}w\|^3\|A_\varepsilon w\|, \tag{3.31}$$

$$g(t) = \frac{27}{2\nu^3}C_1^4\|v\|^2\|A_\varepsilon^{1/2}v\|^2. \tag{3.32}$$

Restricting to $t \le \min(T_1, T^N)$ and using (3.21) we see that

(3.33)
$$\frac{2C_2^2\varepsilon}{\nu}\int_0^t \|A_\varepsilon^{1/2}w\|^3\|A_\varepsilon w\|\,ds \le D_2\varepsilon^2(\|A_\varepsilon^{1/2}w_0\| + t^{1/2}\|(I-M)\mathbb{P}_\varepsilon f\|_\infty) \times (\|A_\varepsilon^{1/2}w_0\|^3 + \varepsilon^2 t^{1/2}\|(I-M)\mathbb{P}_\varepsilon f\|_\infty^3),$$

where $D_2 = 8C_2^2C_5\nu^{-2}\max(3^{-1/2}, 2C_5^2\nu^{-5/2})\max(1, \nu^{-1/2})$. By using (3.1) and (3.33) one obtains

$$H(t) \le T_1\nu^{-1}\eta_2^{-2} + D_2\varepsilon^2(\varepsilon^{3p/2}\eta_3^{-3} + T_1^{1/2}\varepsilon^{2+3r/2}\eta_4^{-3})(\varepsilon^{p/2}\eta_3^{-1} + T_1^{1/2}\varepsilon^{r/2}\eta_4^{-1})$$

for $0 \le t \le \min(T_1, T^N)$. Consequently from (3.22), the fact that $p \ge -1$, $r > -2$, and the Young inequality (3.4) one deduces that

$$H(t) \le E_1(\varepsilon) + \tfrac{7}{4}D_2\eta_3^{-4}, \qquad 0 \le t \le \min(T_1, T^N),$$

where

$$E_1(\varepsilon) \stackrel{\text{def}}{=} D_3(\varepsilon^2Q\eta_2^{-2} + \varepsilon^2Q\eta_4^{-4} + \varepsilon^2Q^2\eta_4^{-4} + \varepsilon^{3/2}Q^{1/2}\eta_3^{-1}\eta_4^{-3})$$

and

$$D_3 \stackrel{\text{def}}{=} \max(2C_5^2\nu^{-2}, 2D_2C_5^2\nu^{-1}, \sqrt{2}D_2C_5\nu^{-1/2}, D_2C_5^4\nu^{-2}).$$

By using Hypothesis H1 we see that

$$E_1(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0^+.$$

It follows from (3.1) and (3.29) that

$$\|A_\varepsilon^{1/2}v(t)\|^2 \le e^{G(t)}(\eta_1^{-2} + E_1(\varepsilon) + \tfrac{7}{4}D_2\eta_3^{-4}), \qquad 0 \le t \le \min(T_1, T^N). \tag{3.34}$$

The next objective is to show that $G(t)$ is small. By taking the scalar product of (2.5) with u and using the fact that $b_\varepsilon(u, u, u) = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|A_\varepsilon^{1/2}u\|^2 &\le |(\mathbb{P}_\varepsilon f, u)| = |\langle A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f, A_\varepsilon^{1/2}u\rangle| \\ &\le \frac{\nu}{2}\|A_\varepsilon^{1/2}u\|^2 + \frac{1}{2\nu}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f\|_\infty^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|u(t)\|^2 - \|u(t_0)\|^2 + \nu\int_{t_0}^t \|A_\varepsilon^{1/2}u(s)\|^2\,ds &\le \frac{t-t_0}{\nu}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f\|_\infty^2 \\ &\le \frac{2(t-t_0)}{\nu}(\|A_\varepsilon^{-1/2}M\mathbb{P}_\varepsilon f\|_\infty^2 + \|A_\varepsilon^{-1/2}(I-M)\mathbb{P}_\varepsilon f\|_\infty^2).\end{aligned} \tag{3.35}$$

By using (2.6), (2.22), and the Gronwall inequality one finds

(3.36)
$$\|v(t)\|^2 \le \|u(t)\|^2 \le \|u_0\|^2 + 2\lambda_1^{-2}\nu^{-2}\|M\mathbb{P}_\varepsilon f\|_\infty^2 + 2C_5^2\lambda_1^{-1}\nu^{-2}\varepsilon^2\|(I-M)\mathbb{P}_\varepsilon f\|_\infty^2.$$

Since $v_0, w_0 \in D(A_\varepsilon^{1/2})$, one has from (2.6) and (2.22) that

$$\|u_0\|^2 = \|v_0\|^2 + \|w_0\|^2 \le \lambda_1^{-1}\|A_\varepsilon^{1/2}v_0\|^2 + C_5^2\varepsilon^2\|A_\varepsilon^{1/2}w_0\|^2. \tag{3.37}$$

Now putting (3.36) and (3.37) together we obtain

$$\|v(t)\|^2 \leq D_4(\|A_\varepsilon^{1/2}v_0\|^2+\varepsilon^2\|A_\varepsilon^{1/2}w_0\|^2+\|M\mathbb{P}_\varepsilon f\|_\infty^2+\varepsilon^2\|(I-M)\mathbb{P}_\varepsilon f\|_\infty^2), \tag{3.38}$$

where $D_4 = \max(\lambda_1^{-1}, C_5^2, 2\lambda_1^{-2}\nu^{-2}, 2C_5^2\lambda_1^{-1}\nu^{-2})$. From (3.9) one has

$$\|A_\varepsilon^{1/2}v(t)\|^2 \leq \|A_\varepsilon^{1/2}u(t)\|^2 \leq NR_0^2, \qquad 0 \leq t < T^N. \tag{3.39}$$

Next we use (3.1), (3.22), (3.38), and (3.39) to observe that

$$G(t) \leq E_2(\varepsilon), \qquad 0 \leq t \leq \min(T_1, T^N), \tag{3.40}$$

where

$$E_2(\varepsilon) \stackrel{\text{def}}{=} \varepsilon^2 Q D_5(\eta_1^{-2} + \varepsilon^{2+p}\eta_3^{-2} + \eta_2^{-2} + \varepsilon^{2+r}\eta_4^{-2})NR_0^2 \tag{3.41}$$

and $D_5 = 27C_1^4C_5^2\nu^{-4}D_4$. After using (3.8) and expanding, one observes that the right-hand side of (3.41) contains 13 terms of the form

$$c\varepsilon^{b_0}\eta_i^{-2}\eta_j^{-2}\varepsilon^{1/4}Q(\varepsilon),$$

3 terms of the form

$$c\varepsilon^{3/4}\eta_i^{-2}\eta_3^{-2}\varepsilon^{1/4}Q(\varepsilon),$$

and 4 terms of the form

$$c\varepsilon^{b_1}\eta_i^{-2}\eta_3^{-4}\varepsilon^{1/4}Q(\varepsilon),$$

where $b_0 \geq 1$, $b_1 \geq \frac{7}{4}$, $i, j = 1, 2, 3, 4$, and where c denotes positive constants which are bounded as $\varepsilon \to 0^+$. By using Hypothesis H1, it is a straight-forward verification to see that each of these terms goes to 0 as $\varepsilon \to 0$. In other words, $E_2(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. By combining (3.34) and (3.40) one obtains

$$\|A_\varepsilon^{1/2}v(t)\|^2 \leq e^{E_2(\varepsilon)}(\eta_1^{-2}+E_1(\varepsilon)+\tfrac{7}{4}D_2\eta_3^{-4}), \qquad 0 \leq t \leq \min(T_1, T^N), \tag{3.42}$$

provided $0 < \varepsilon \leq \varepsilon_2$.

Now set $N \stackrel{\text{def}}{=} 1+\max(4, \frac{7}{2}D_2)$, where D_2 is given above, and choose ε_3 so that $0 < \varepsilon_3 \leq \varepsilon_2(N)$ and

$$e^{E_2(\varepsilon)} \leq 2, \qquad E_1(\varepsilon) \leq \eta_1^{-2}, \qquad 2C_5^2\varepsilon^{2/3} \leq \nu^2 \tag{3.43}$$

for $0 < \varepsilon \leq \varepsilon_3$.

We claim that for $0 < \varepsilon \leq \varepsilon_3$ one has $T_1 < T^N$. To prove this we assume on the contrary that $T^N \leq T_1 < \infty$. Then (3.1) and (3.18) imply that

$$\|A_\varepsilon^{1/2}w(T^N)\|^2 \leq \|A_\varepsilon^{1/2}w_0\|^2 + \tfrac{1}{2}k_2^2\varepsilon^2\|(I-M)\mathbb{P}_\varepsilon f\|_\infty^2 \leq \varepsilon^p\eta_3^{-2} + \tfrac{1}{2}k_2^2\varepsilon^{2+r}\eta_4^{-2},$$

where $k_2^2 \stackrel{\text{def}}{=} 4C_5^2\nu^{-2}$. Since $r > -2$, we have $2+r \geq \frac{2}{3}+\frac{r}{3}$, and consequently (3.43) implies that

$$\|A_\varepsilon^{1/2}w(T^N)\|^2 \leq \varepsilon^p\eta_3^{-2} + \varepsilon^{r/3}\eta_4^{-2}, \qquad 0 < \varepsilon \leq \varepsilon_3. \tag{3.44}$$

On the other hand, (3.42) and (3.43) imply that for $0 < \varepsilon \leq \varepsilon_3$ one has

$$\|A_\varepsilon^{1/2}v(t)\|^2 \leq 2(2\eta_1^{-2} + \tfrac{7}{4}D_2\eta_3^{-4}), \qquad 0 \leq t \leq \min(T_1, T^N). \tag{3.45}$$

By adding (3.44) and (3.45) we obtain

$$\|A_\varepsilon^{1/2}u(T^N)\|^2 \le (4\eta_1^{-2} + \tfrac{7}{2}D_2\eta_3^{-4} + \varepsilon^p\eta_3^{-2} + \varepsilon^{r/3}\eta_4^{-2}) < NR_0^2, \qquad 0 < \varepsilon \le \varepsilon_3,$$

which contradicts (3.10). Hence one has $T_1 < T^N$ for $0 < \varepsilon \le \varepsilon_3$.

Finally we set $k_1^2 \stackrel{\text{def}}{=} \frac{7}{2}D_2$ and $\varepsilon_1 \stackrel{\text{def}}{=} \varepsilon_3$. It then follows from (3.45) that

$$\|A_\varepsilon^{1/2}v(T_1)\|^2 \le 4\eta_1^{-2} + k_1^2\eta_3^{-4}$$

and from (3.23) that

$$\|A_\varepsilon^{1/2}w(T_1)\|^2 \le k_2^2\varepsilon^{2+r}\eta_4^{-2},$$

which completes the proof of Lemma 3.1. □

Remark. The proof of Lemma 3.1 still works if we take $r = -2$. However, the result of Lemma 3.1 is interesting only in the case where $\varepsilon^{2+r}\eta_4^{-2}$ is bounded as $\varepsilon \to 0^+$; see also Hypothesis H2(a, b) given below. If $\eta_4^{-2} \to \infty$ as $\varepsilon \to 0^+$, this implies that r must satisfy $r > -2$. In Theorems 7, 8, and 9 we will impose a stronger requirement, viz. that $\varepsilon^{2+r}\eta_4^{-2} \to 0$ as $\varepsilon \to 0^+$. This is the reason why we impose the requirement that $r > -2$ in Hypothesis H1.

3.2. **Strategy of proof.** The argument of Lemma 3.1 can of course be repeated with the new initial conditions satisfying (3.7) instead of (3.1). By making the realistic assumption that

$$k_2^2\varepsilon^{2+r}\eta_4^{-2} \le \varepsilon^p\eta_3^{-2} \tag{3.46}$$

one needs only to replace η_1^{-2} with $(4\eta_1^{-2} + k_1^2\eta_3^{-4})$, and the entire argument carries through. Unfortunately, this is not a good strategy, because one is forced to choose a smaller value for ε_2, and thereby a smaller value for ε_1. It is important to take advantage of the fact that $k_2^2\varepsilon^{2+r}\eta_4^{-2}$ can be made small instead of using the crude bound (3.46). As a result of Lemma 3.1 we can now assume the initial condition $\|A_\varepsilon^{1/2}w_0\|^2$ to be small for $0 < \varepsilon \le \varepsilon_1$.

For the Long Time Argument we begin by assuming $\|A_\varepsilon^{1/2}w_0\|^2$ is small, i.e., $\|A_\varepsilon^{1/2}w_0\|^2 \le k_2^2\varepsilon^{2+r}\eta_4^{-2}$ and $\|A_\varepsilon^{1/2}v_0\|^2 \le 4\eta_1^{-2} + k_1^2\eta_3^{-4}$. In the course of the argument we show that if ε is sufficiently small, then the dilated 3D Navier-Stokes evolutionary equation (2.5) has a strong solution on a suitable interval $[0, 2T_0]$ where $T_0 = T_0(\varepsilon)$ is finite but large. We also show that $\|A_\varepsilon^{1/2}w(t)\|^2 \le k_2^2\varepsilon^{2+r}\eta_4^{-2}$ and $\|A_\varepsilon^{1/2}v(t)\|^2 \le \frac{1}{2}(4\eta_1^{-2} + k_1^2\eta_3^{-4})$ on the half-interval $t \in [T_0, 2T_0]$. This, of course, permits one to prove the H^1-Regularity Theorem by using the Long Time Argument with induction.

3.3. **Long-Time argument.** In this section we continue the analysis of the Short-Time Argument. The terms $\eta_i(\varepsilon)$, $1 \le i \le 4$, r, and p will be assumed to satisfy Hypothesis H1. In addition, we will assume that the following **Hypothesis H2(a, b)** is satisfied, where a and b are sufficiently large:

(1) Let $a > 0$ be fixed. Then one has

$$\begin{cases} \varepsilon^{5/8}\eta^{-2}\exp(a\eta^{-4}) \to 0, \\ \eta^{-2} \to \infty \end{cases}$$

as $\varepsilon \to 0^+$, where

$$\eta^{-2} \stackrel{\text{def}}{=} \max(4\eta_1^{-2} + k_1^2\eta_3^{-4} + k_2^2\varepsilon^{2+r}\eta_4^{-2}, 1),$$

are $\varepsilon^{5/8}\exp(2a\eta_2^{-4})$ is bounded for $0 < \varepsilon \le 1$. (The constants k_1 and k_2 are given in Lemma 3.1.)

(2) Let $b > 0$ be fixed. Then for any λ, $0 < \lambda < 1$, there is an $\varepsilon_4 = \varepsilon_4(b, \lambda) > 0$ such that one has

$$\eta_2^{-2}\exp(b\eta_2^{-4}) \le \lambda(4\eta_1^{-2} + k_1^2\eta_3^{-4}), \qquad 0 < \varepsilon < \varepsilon_4.$$

(3) The function $\varepsilon^{4+2r}\eta_4^{-4}(\log\eta^{-4} + 1)$ is bounded as $\varepsilon \to 0^+$. Our objective now is to prove the following result:

Lemma 3.2. *Assume that both Hypotheses $H1$ and $H2(a, b)$ are satisfied, where a and b are sufficiently large. Then there is an $\varepsilon_0 > 0$ such that for every ε, $0 < \varepsilon \le \varepsilon_0$, there is a time $T_0 = T_0(\varepsilon) > 0$ with the property that, for $0 < \varepsilon \le \varepsilon_0$, whenever the initial conditions*

$$\left\{\begin{array}{ll} \|A_\varepsilon^{1/2}v_0\|^2 \le 4\eta_1^{-2} + k_1^2\eta_3^{-4}, & \|M\mathbb{P}_\varepsilon f\|_\infty^2 \le \eta_2^{-2}, \\ \|A_\varepsilon^{1/2}w_0\|^2 \le k_2^2\varepsilon^{2+r}\eta_4^{-2}, & \|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 \le \varepsilon^r\eta_4^{-2} \end{array}\right. \tag{3.47}$$

are satisfied, then the solution $u(t)$ of (2.5) satisfies $u(t) \in D(A_\varepsilon^{1/2})$ for $0 \le t \le 2T_0$ and

$$\left\{\begin{array}{l} \|A_\varepsilon^{1/2}v(t)\|^2 \le \frac{1}{2}(4\eta_1^{-2} + k_1^2\eta_3^{-4}), \\ \|A_\varepsilon^{1/2}w(t)\|^2 \le k_2^2\varepsilon^{2+r}\eta_4^{-2} \end{array}\right. \tag{3.48}$$

for $T_0 \le t \le 2T_0$.

Proof. The proof begins as in Lemma 3.1. For any positive numbers d_1 and d_2 we define $R_0^2 = R_0^2(\varepsilon, d_1, d_2)$ by

$$R_0^2 = 1 + (\eta^{-2} + \eta_2^{-2} + d_1)[1 + \exp(d_2\eta^{-4})\exp(2d_2\eta_2^{-4})]. \tag{3.49}$$

The values of d_1 and d_2 will be fixed later. Note that $R_0^2 \ge \eta^{-2} \ge \|A_\varepsilon^{1/2}u_0\|^2$. Therefore, it follows from Lemma 3.0 that for any $N > 1$ there is a time T^N, $0 < T^N \le \infty$, such that

$$\|A_\varepsilon^{1/2}u(t)\|^2 \le NR_0^2, \qquad 0 \le t < T^N. \tag{3.50}$$

Without loss of generality we let $[0, T^N)$ denote the maximal time interval for which (3.50) is valid. Therefore if $T^N < \infty$ one must have

$$\|A_\varepsilon^{1/2}u(T^N)\|^2 = NR_0^2. \tag{3.51}$$

By taking the scalar product of the w-equation (3.11) with $A_\varepsilon w$, we then obtain (3.13) with the same value for D_1. For $0 \le t < T^N$ one has

$$D_1^2\varepsilon^{5/4}\|A_\varepsilon^{1/2}u\|^2 \le D_1^2N\varepsilon^{5/4}R_0^2, \tag{3.52}$$

where R_0^2 is given by (3.49). From Hypotheses H1 and H2(a, b), the right-hand side of (3.52) goes to 0 as $\varepsilon \to 0^+$, provided $a \geq d_2$ and $b \geq 2d_2$. Consequently there is an $\varepsilon_5 = \varepsilon_5(N, d_1, d_2)$, $0 < \varepsilon_5 \leq \varepsilon_1$, where ε_1 is given by Lemma 3.1, such that

$$D_1 N^{1/2} \varepsilon^{5/8} R_0 \leq \nu/2, \qquad 0 < \varepsilon \leq \varepsilon_5.$$

As a result, (3.16), (3.17), (3.18), and (3.19) are valid for $0 \leq t < T^N$ and $0 < \varepsilon \leq \varepsilon_5$. We now restrict to $0 < \varepsilon \leq \varepsilon_5$ for the remainder of the argument. By using (3.47) we see that for $0 < \varepsilon \leq \varepsilon_5$ inequality (3.18) now assumes the form

$$\|A_\varepsilon^{1/2} w(t)\|^2 \leq \left[k_2^2 \exp\left(\frac{-\nu C_5^{-2} \varepsilon^{-2}}{2} t\right) + \frac{1}{2} k_2^2\right] \varepsilon^{2+r} \eta_4^{-2} \leq \frac{3}{2} k_2^2 \varepsilon^{2+r} \eta_4^{-2} \tag{3.53}$$

for $0 \leq t < T^N$. By using (3.47) once again, (3.19) becomes

$$\begin{aligned} \int_0^t \|A_\varepsilon w(s)\|^2\, ds &\leq \frac{2}{\nu} \|A_\varepsilon^{1/2} w_0\|^2 + \frac{2t}{\nu^2} \|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 \\ &\leq D_9^2 (\varepsilon^2 + t) \varepsilon^r \eta_4^{-2} \end{aligned}$$

for $0 \leq t < T^N$, where $D_9^2 = \max(2k_2^2 \nu^{-1}, 2\nu^{-2})$. In addition, by integrating (3.16) and using (3.53) we have for $0 < \varepsilon \leq \varepsilon_5 \leq 1$

$$\int_{t-1}^t \|A_\varepsilon w\|^2\, ds \leq \frac{2}{\nu} \|A_\varepsilon^{1/2} w(t-1)\|^2 + \frac{2}{\nu^2} \|(I - M)\mathbb{P}_\varepsilon f\|_\infty^2 \leq D_{10}^2 \varepsilon^r \eta_4^{-2},$$

for $1 \leq t < T^N$, where $D_{10}^2 = 3k_2^2 \nu^{-1} + 2\nu^{-2}$. It follows from (3.5) and (3.53) that

$$\begin{aligned} \int_0^t \|A_\varepsilon^{1/2} w\|^6\, ds &\leq 4 \int_0^t \left[k_2^6 \exp\left(\frac{-3\nu C_5^{-2} \varepsilon^{-2}}{2} s\right) + \frac{1}{8} k_2^6\right] \varepsilon^{6+3r} \eta_4^{-6}\, ds \\ &\leq D_{11}^2 (\varepsilon^2 + t) \varepsilon^{6+3r} \eta_4^{-6} \end{aligned}$$

for $0 \leq t < T^N$, where $D_{11}^2 = 4k_2^6 \max(2C_5^2 (3\nu)^{-1}, \frac{1}{8})$. Using $0 < \varepsilon \leq 1$ one has

$$\int_{t-1}^t \|A_\varepsilon^{1/2} w\|^6\, ds \leq 2D_{11}^2 \varepsilon^{6+3r} \eta_4^{-6}$$

for $1 \leq t < T^N$. Using the argument in (3.21) one then obtains

$$\int_0^t \|A_\varepsilon^{1/2} w\|^3 \|A_\varepsilon w\|\, ds \leq D_9 D_{11} (\varepsilon^2 + t) \varepsilon^{3+2r} \eta_4^{-4}, \qquad 0 \leq t < T^N, \tag{3.54}$$

and, since $0 < \varepsilon \leq 1$,

$$\int_{t-1}^t \|A_\varepsilon^{1/2} w\|^3 \|A_\varepsilon w\|\, ds \leq 2D_{10} D_{11} \varepsilon^{3+2r} \eta_4^{-4}, \qquad 1 \leq t < T^N. \tag{3.55}$$

Next we return to the v-equation (3.24). By taking the scalar product of (3.24) with v and using $b_\varepsilon(v, v, v) = 0$ together with (2.26) and (3.3) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2+\lambda_1\nu\|v\|^2 &\le \frac{1}{2}\frac{d}{dt}\|v\|^2+\nu\|A_\varepsilon^{1/2}v\|^2\\ &\le (\|M\mathbb{P}_\varepsilon f\|_\infty + C_2\varepsilon^{1/2}\|A_\varepsilon^{1/2}w\|^{3/2}\|A_\varepsilon w\|^{1/2})\|v\|\\ &\le \lambda_1^{-1/2}(\|M\mathbb{P}_\varepsilon f\|_\infty + C_2\varepsilon^{1/2}\|A_\varepsilon^{1/2}w\|^{3/2}\|A_\varepsilon w\|^{1/2})\|A_\varepsilon^{1/2}v\|.\end{aligned}$$

By using the Young inequality we get

$$\frac{d}{dt}\|v\|^2+\lambda_1\nu\|v\|^2 \le \frac{1}{\lambda_1\nu}(\|M\mathbb{P}_\varepsilon f\|_\infty^2 + C_2^2\varepsilon\|A_\varepsilon^{1/2}w\|^3\|A_\varepsilon w\|) \tag{3.56}$$

and

$$\frac{d}{dt}\|v\|^2+\nu\|A_\varepsilon^{1/2}v\|^2 \le \frac{1}{\lambda_1\nu}(\|M\mathbb{P}_\varepsilon f\|_\infty^2 + C_2^2\varepsilon\|A_\varepsilon^{1/2}w\|^3\|A_\varepsilon w\|). \tag{3.57}$$

By using the Gronwall inequality for (3.56) one finds that

$$\begin{aligned}\|v(t)\|^2 \le e^{-\nu\lambda_1 t}\|v_0\|^2 + \lambda_1^{-2}\nu^{-2}\|M\mathbb{P}_\varepsilon f\|_\infty^2\\ + \lambda_1^{-1}\nu^{-1}C_2^2\varepsilon\int_0^t \|A_\varepsilon^{1/2}w\|^3\|A_\varepsilon w\|\,ds\end{aligned} \tag{3.58}$$

for $0 \le t < T^N$. Next by using (2.6), (3.47), and (3.54) we find

$$\|v(t)\|^2 \le D_{12}\gamma(\varepsilon, t), \qquad 0 \le t < T^N, \tag{3.59}$$

where $D_{12} = \max(\lambda_1^{-1}, \lambda_1^{-2}\nu^{-2}, \lambda_1^{-1}\nu^{-1}C_2^2 D_9 D_{11})$ and

$$\gamma = \gamma(\varepsilon, t) \stackrel{\text{def}}{=} (e^{-\nu\lambda_1 t}\eta^{-2} + \eta_2^{-2} + [\varepsilon^2 + t]\varepsilon^{4+2r}\eta_4^{-4}). \tag{3.60}$$

Similarly by integrating (3.57) and using (3.47) and (3.54) we obtain

$$\begin{aligned}\int_0^t \|A_\varepsilon^{1/2}v\|^2\,ds &\le \nu^{-1}\|v_0\|^2 + \lambda_1^{-1}\nu^{-2}t\|M\mathbb{P}_\varepsilon f\|_\infty^2\\ &\quad + \lambda_1^{-1}\nu^{-2}C_2^2\varepsilon\int_0^t \|A_\varepsilon^{1/2}w\|^3\|A_\varepsilon w\|\,ds\\ &\le \lambda_1^{-1}\nu^{-1}(4\eta_1^{-2} + k_1^2\eta_3^{-4}) + \lambda_1^{-1}\nu^{-2}t\eta_2^{-2}\\ &\quad + \lambda_1^{-1}\nu^{-2}C_2^2 D_9 D_{11}(\varepsilon^2 + t)\varepsilon^{4+2r}\eta_4^{-4}\end{aligned} \tag{3.61}$$

for $0 \le t < T^N$. It follows that

$$\int_0^t \|A_\varepsilon^{1/2}v\|^2\,ds \le D_{13}\gamma(\varepsilon, t), \qquad 0 \le t < \min(1, T^N), \tag{3.62}$$

where $D_{13} = \max(\lambda_1^{-1}\nu^{-1}e^{\nu\lambda_1}, \lambda_1^{-1}\nu^{-2}, \lambda_1^{-1}\nu^{-2}C_2^2 D_9 D_{11})$. Furthermore, by integrating (3.57) once again we find

$$\begin{aligned}\int_{t-1}^t \|A_\varepsilon^{1/2}v\|^2\,ds \le \nu^{-1}\|v(t-1)\|^2 + \lambda_1^{-1}\nu^{-2}\|M\mathbb{P}_\varepsilon f\|_\infty^2\\ + \lambda_1^{-1}\nu^{-2}C_2^2\varepsilon\int_{t-1}^t \|A_\varepsilon^{1/2}w\|^3\|A_\varepsilon w\|\,ds.\end{aligned} \tag{3.63}$$

From (3.47), (3.55), (3.59), and (3.60) we get

$$\int_{t-1}^{t} \|A_\varepsilon^{1/2} v\|^2 \, ds \leq D_{14} \gamma(\varepsilon, t), \qquad 1 \leq t < T^N, \tag{3.64}$$

where

$$D_{14} = \nu^{-1} \max(D_{12} + \lambda_1^{-1}\nu^{-1}, D_{12} e^{\nu\lambda_1} + 2D_{10} D_{11} \lambda_1^{-1} \nu^{-1} C_2^2).$$

By combining (3.59), (3.60), (3.62), and (3.64) we find

$$\begin{aligned} \int_0^t \|v\|^2 \|A_\varepsilon^{1/2} v\|^2 \, ds &\leq \sup_{0 \leq s \leq t} \|v(s)\|^2 \int_0^t \|A_\varepsilon^{1/2} v\|^2 \, ds \\ &\leq e^{\nu\lambda_1} D_{12} D_{13} \gamma(\varepsilon, t)^2 \end{aligned} \tag{3.65}$$

for $0 \leq t < \min(1, T^N)$ and

$$\int_{t-1}^{t} \|v\|^2 \|A_\varepsilon^{1/2} v\|^2 \, ds \leq e^{\nu\lambda_1} D_{12} D_{14} \gamma(\varepsilon, t)^2, \qquad 1 \leq t < T^N. \tag{3.66}$$

Next by taking the scalar product of (3.24) with $A_\varepsilon v$ we obtain (3.25), (3.26) (3.27), and (3.28). For $0 \leq t < \min(1, T^N)$ we apply the Gronwall inequality to (3.28) to obtain

$$\|A_\varepsilon^{1/2} v(t)\|^2 \leq e^{G(t)} (\|A_\varepsilon^{1/2} v_0\|^2 + H(t)), \qquad 0 \leq t < \min(1, T^N),$$

where $H(t)$ and $G(t)$ are given by (3.30), (3.31), and (3.32). From (3.47), (3.54), and (3.65) we find

$$\|A_\varepsilon^{1/2} v(t)\|^2 \leq D_{16} \gamma(\varepsilon, t) \exp(D_{17} \gamma(\varepsilon, t)^2), \qquad 0 \leq t < \min(1, T^N), \tag{3.67}$$

where

$$D_{16} = \max(e^{\nu\lambda_1}, \nu^{-1}, 2\nu^{-1} C_2^2 D_9 D_{11}) \quad \text{and} \quad D_{17} = 2^{-1} 27 \nu^{-3} C_1^4 e^{\nu\lambda_1} D_{12} D_{13}.$$

For $1 \leq t < T^N$ we use the uniform Gronwall inequality (see Foias, et al. (1987))[5] on (3.28) to obtain

$$\|A_\varepsilon^{1/2} v(t)\|^2 \leq \left(\int_{t-1}^{t} \|A_\varepsilon^{1/2} v\|^2 \, ds + \int_{t-1}^{t} h(s) \, ds \right) \exp\left(\int_{t-1}^{t} g(s) \, ds \right), \tag{3.68}$$

where h and g are given by (3.31) and (3.32). For $t \geq 1$ we use (3.55), (3.66), and (3.64) to derive an inequality similar to (3.67). This can be combined with (3.67) to obtain

$$\|A_\varepsilon^{1/2} v(t)\|^2 \leq D_{18} \gamma(\varepsilon, t) \exp(D_{19} \gamma(\varepsilon, t)^2), \qquad 0 \leq t < T^N, \tag{3.69}$$

[5]Let y, g, h be nonnegative locally integrable functions on $(0, \infty)$, where y is absolutely continuous on $(0, \infty)$, and which satisfy $y' \leq gy + h$, $0 < t < \infty$. Then one has

$$y(t) \leq \left(\frac{1}{t - \tau} \int_\tau^t y(s) \, ds + \int_\tau^t h(s) \, ds \right) \exp\left(\int_\tau^t g(s) \, ds \right), \qquad 0 < t < \infty,$$

where $\tau = \max(0, t-1)$.

where

$$\begin{cases} D_{18} = \max(D_{16}, D_{14} + \max(\nu^{-1}, 4C_2^2 D_{10} D_{11} \nu^{-1})), \\ D_{19} = 27 C_1^4 e^{\nu\lambda_1} 2^{-1} \nu^{-3} \max(D_{13}, D_{14}) D_{12}. \end{cases}$$

The next step is to define $T_0 = T_0(\varepsilon)$. Since $\eta^{-4} \to \infty$ as $\varepsilon \to 0^+$, there is no loss in generality in assuming that

$$2D_{19}\eta^{-4} > 1.$$

We then define T_0 by requiring

$$(3.70) \qquad 2D_{19} e^{-2\nu\lambda_1 t} \eta^{-4} \le 1, \qquad T_0 \le t,$$

that is, set

$$(3.71) \qquad T_0 \stackrel{\text{def}}{=} \frac{1}{2\nu\lambda_1} \log(2D_{19}\eta^{-4}).$$

Also define

$$(3.72) \qquad E_3(\varepsilon) \stackrel{\text{def}}{=} (\varepsilon^2 + 2T_0(\varepsilon))\varepsilon^{4+2r}\eta_4^{-4}.$$

It follows from Hypothesis H2(a, b) that there are constants D_{20} and D_{21} such that

$$(3.73) \qquad E_3(\varepsilon) \le D_{20}, \qquad \tfrac{3}{2}k_2^2 \varepsilon^{2+r} \eta_4^{-2} \le D_{21}$$

for $0 < \varepsilon \le 1$. The term $R_0^2 = R_0^2(\varepsilon, d_1, d_2)$ is now fixed so that

$$(3.74) \qquad R_0^2 = R_0^2(\varepsilon, D_{20}, 2D_{19}),$$

and the term N is fixed so that

$$(3.75) \qquad N > \max(1, D_{21}, D_{22}),$$

where $D_{22} = D_{18} \exp(4D_{19}D_{20}^2)$. Finally we fix $\varepsilon_5 = \varepsilon_5(N, D_{20}, 2D_{19})$ for these choices of N, d_1, and d_2. Furthermore, we require that the constants a and b in Hypothesis H2(a, b) satisfy $a \ge 2D_{19}$ and $b \ge 4D_{19}$.

Let us return to the function γ, which we will write as

$$\gamma = \gamma(\varepsilon, t) = e^{-\nu\lambda_1 t}\eta^{-2} + \beta,$$

where

$$\beta = \beta(\varepsilon, t) \stackrel{\text{def}}{=} \eta_2^{-2} + (\varepsilon^2 + t)\varepsilon^{4+2r}\eta_4^{-4}.$$

Note that

$$(3.76) \qquad \gamma^2 \le 2e^{-2\nu\lambda_1 t}\eta^{-4} + 2\beta^2,$$

and from (3.72) and (3.73) we have

$$(3.77) \qquad \beta(\varepsilon, t) \le \eta_2^{-2} + D_{20}, \qquad 0 \le t \le 2T_0.$$

From (3.70) we see that

$$(3.78) \qquad D_{18}\gamma \exp(D_{19}\gamma^2) \le D_{22}(e^{-\nu\lambda_1 t}\eta^{-2} + \eta_2^{-2} + D_{20}) \exp(4D_{19}\eta_2^{-4} + 1)$$

for $T_0 \le t \le 2T_0$.

From (3.53) and (3.73) we find

$$\|A_\varepsilon^{1/2} w(t)\|^2 \le D_{21}, \qquad 0 \le t \le T^N. \tag{3.79}$$

By using (3.69), (3.76), and (3.77) we find that

$$\|A_\varepsilon^{1/2} v(t)\|^2 \le D_{22}(e^{-\nu\lambda_1 t}\eta^{-2} + \eta_2^{-2} + D_{20}) \exp(2D_{19}\eta^{-4}) \exp(4D_{19}\eta_2^{-4}) \tag{3.80}$$

for $0 \le t \le \min(2T_0, T^N)$.

We claim that $2T_0 \le T^N$. In order to prove this, we assume on the contrary that $T^N < 2T_0$. From the Pythagorean relation (2.21) and (3.79) and (3.80) it follows that

$$\|A_\varepsilon^{1/2} u(T^N)\|^2 \le D_{21} + D_{22}(\eta^{-2} + \eta_2^{-2} + D_{20}) \exp(2D_{19}\eta^{-4}) \exp(4D_{19}\eta_2^{-4}).$$

From the definition of R_0^2 in (3.49) and (3.74) and the characterization of N in (3.75) we obtain

$$\|A_\varepsilon^{1/2} u(T^N)\|^2 < NR_0^2,$$

which contradicts (3.51). Hence one has $2T_0 \le T^N$.

Finally we turn to the verification of (3.48). For this purpose we restrict t to the interval $[T_0, 2T_0]$. From (3.53) we have

$$\|A_\varepsilon^{1/2} w(t)\|^2 \le \left[k_2^2 \exp\left(\frac{-\nu C_5^{-2}\varepsilon^{-2}}{2} T_0\right) + \frac{1}{2}k_2^2\right] \varepsilon^{2+r}\eta_4^{-2}, \qquad T_0 \le t \le 2T_0.$$

From the definition of T_0 in (3.71), we see that there is an ε_6, $0 < \varepsilon_6 \le \varepsilon_5$, such that

$$2\nu^{-1} C_5^2 \varepsilon^2 \log 2 \le T_0(\varepsilon), \qquad 0 < \varepsilon \le \varepsilon_6. \tag{3.81}$$

Now (3.81) implies that $\exp(-\nu C_5^{-2}\varepsilon^{-2} T_0/2) \le \frac{1}{2}$, and consequently,

$$\|A_\varepsilon^{1/2} w(t)\|^2 \le k_2^2 \varepsilon^{2+r} \eta_4^{-2}, \qquad T_0 \le t \le 2T_0, \tag{3.82}$$

for $0 < \varepsilon \le \varepsilon_6$. From (3.69), (3.70), and (3.78) one finds

$$\|A_\varepsilon^{1/2} v(t)\|^2 \le \Gamma(\eta_2^{-2}), \qquad T_0 \le t \le 2T_0, \tag{3.83}$$

where

$$\Gamma(r) \overset{\text{def}}{=} D_{22}(r + D_{24}) \exp(4D_{19}r^2 + 1) \tag{3.84}$$

where $D_{24} = (2D_{19})^{-1/2} + D_{20}$. From Hypothesis H2(a, b) we see that there is an ε_7, $0 < \varepsilon_7 \le \varepsilon_6$, such that for $0 < \varepsilon \le \varepsilon_7$ one has

$$\Gamma(\eta_2^{-2}) \le \tfrac{1}{2}(4\eta_1^{-2} + k_1^2\eta_3^{-4}), \qquad 0 < \varepsilon \le \varepsilon_7.$$

It then follows that for $0 < \varepsilon \le \varepsilon_7$ one has

$$\|A_\varepsilon^{1/2} v(t)\|^2 \le \tfrac{1}{2}(4\eta_1^{-2} + k_1^2\eta_3^{-4}), \qquad T_0 \le t \le 2T_0.$$

By setting $\varepsilon_0 = \varepsilon_7$, we complete the proof of the lemma. □

Proof of Theorem 1. Since $0 < \varepsilon_0 \le \varepsilon_1$, where ε_1 and ε_0 are given by Lemmas 3.1 and 3.2, the proof of Theorem 1, for $0 < \varepsilon \le \varepsilon_0$, now follows by first applying Lemma 3.1 and then using Lemma 3.2 with induction, with $\widehat{T}_1 = T_0 + T_1$. The estimate for L_5^2 appearing in (2.33) follows from the Pythagorean identity (2.21) together with (3.82) and (3.83). The fact that $u(\cdot)$ belongs to $C^0([0, \infty), V_\varepsilon^1)$ is now a direct consequence of the local result contained in Lemma 3.0. □

4. H^2-Regularity: Theorems 2 and 3

In this section, we will prove Theorems 2 and 3.

Proof of Theorem 2. It is known that if u_0 belongs to $V_\varepsilon^2 = D(A_\varepsilon)$ and $\mathbb{P}_\varepsilon f$ belongs to $C^0([0, \infty), H_\varepsilon) \cap W^{1,\infty}((0, \infty), D(A_\varepsilon^{-1/2}))$, then u is in the space $C^0([0, \infty), V_\varepsilon^2)$, and the time-derivative u' belongs to $C^0([0, \infty); H_\varepsilon)$ and is the solution of the equation

$$du'/dt + \nu A_\varepsilon u' + B_\varepsilon(u', u) + B_\varepsilon(u, u') = \mathbb{P}_\varepsilon f' \tag{4.1}$$

with initial condition

$$u'(0) = \mathbb{P}_\varepsilon f(0) - B_\varepsilon(u_0, u_0) - \nu A_\varepsilon u_0;$$

see Temam (1982, 1983). On the other hand, if u_0 belongs only to V_ε^1, we will show that one can choose $t_0 > 0$, arbitrarily close to 0, such that $u(t_0)$ belongs to $D(A_\varepsilon)$. It then follows that $u'(t_0)$ belongs to H_ε for every such t_0, that u belongs to $C^0([t_0, \infty), V_\varepsilon^2)$, and that u' is in the space $C^0([t_0, \infty); H_\varepsilon)$ and is the solution of the equation (4.1) on (t_0, ∞). Our main objective here is to prove the estimates (2.35), (2.36), and (2.37) of Theorem 2. The proof will be given in four steps. We will not use here the decomposition $u = v + w$. The results of Steps 1 and 2 are already contained in Lemma 3.0. However, we will reproduce the proof here because we require the precise estimates of $\int_0^t \|A_\varepsilon u(s)\|^2\, ds$ and $\int_0^t \|u'(s)\|^2\, ds$ used to obtain (2.35), (2.36), and (2.37). As is usual, the formal estimates given here can be rigorously justified by using the Bubnov-Galerkin approximation method.

Step 1. First we derive an estimate for $\int_\tau^t \|A_\varepsilon u(s)\|^2\, ds$. Taking the scalar product of (2.5) with $A_\varepsilon u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{1/2} u\|^2 + \nu\|A_\varepsilon u\|^2 \le \frac{1}{\nu}\|\mathbb{P}_\varepsilon f\|_\infty^2 + \frac{\nu}{4}\|A_\varepsilon u\|^2 + |b_\varepsilon(u, u, A_\varepsilon u)|, \qquad t \ge 0. \tag{4.2}$$

From the inequalities (8.8) and (2.18), we deduce that

$$|b_\varepsilon(u^1, u^2, u^3)| \le C_9 \|A_\varepsilon^{1/2} u^1\|\, \|A_\varepsilon^{1/2} u^2\|^{1/2} \|A_\varepsilon u^2\|^{1/2} \|u^3\| \tag{4.3}$$

for any $u^1 \in D(A_\varepsilon^{1/2})$, $u^2 \in D(A_\varepsilon)$, $u^3 \in H_\varepsilon$, where C_9 is a positive constant, independent on ε. This in turn implies that

$$|b_\varepsilon(u, u, A_\varepsilon u)| \le C_9 \|A_\varepsilon^{1/2} u\|^{3/2} \|A_\varepsilon u\|^{3/2}. \tag{4.4}$$

Using (4.4) and the Young inequality (3.4), we deduce from (4.2) that

$$\frac{d}{dt}\|A_\varepsilon^{1/2}u\|^2 + \nu\|A_\varepsilon u\|^2 \le \frac{2}{\nu}\|\mathbb{P}_\varepsilon f\|_\infty^2 + \frac{27C_9^4}{2\nu^3}\|A_\varepsilon^{1/2}u\|^6, \qquad t \ge 0. \tag{4.5}$$

By integrating (4.5) we infer that for $\max(0, t-1) \le \tau \le t$ one has

(4.6)

$$\int_\tau^t \|A_\varepsilon u(s)\|^2\,ds \le \frac{2}{\nu^2}(t-\tau)\|\mathbb{P}_\varepsilon f\|_\infty^2 + D_{25}(t-\tau)\sup_{\tau\le s\le t}\|A_\varepsilon^{1/2}u(s)\|^6 + \frac{1}{\nu}\|A_\varepsilon^{1/2}u(\tau)\|^2,$$

where $D_{25} = 27C_9^4 2^{-1}\nu^{-4}$. Since the right-hand side of (4.6) is bounded for any $t > 0$, it follows that the integrand on the left-hand side is finite almost everywhere. Therefore, there exists arbitrarily small $t_0 > 0$ such that $u(t_0) \in D(A_\varepsilon)$.

Step 2. Next we derive an estimate of $\int_\tau^t \|u'(s)\|^2\,ds$ for $0 \le \tau \le t$. First we observe that (2.5) yields the identity

$$\langle u', u'\rangle = \langle \mathbb{P}_\varepsilon f, u'\rangle - \langle \nu A_\varepsilon u, \mathbb{P}_\varepsilon f - \nu A_\varepsilon u - B_\varepsilon(u, u)\rangle - \langle B_\varepsilon(u, u), u'\rangle,$$

and consequently one finds that

$$\|u'\|^2 \le \|\mathbb{P}_\varepsilon f\|\,\|u'\| + \nu\|\mathbb{P}_\varepsilon f\|\,\|A_\varepsilon u\| + \nu^2\|A_\varepsilon u\|^2 + \nu|b_\varepsilon(u, u, A_\varepsilon u)| + |b_\varepsilon(u, u, u')|.$$

By applying the Young inequality (3.4) several times and using (4.2) and (4.4), we obtain

$$\|u'\|^2 \le 3(\nu^2+1)\|A_\varepsilon u\|^2 + \nu^4 C_9^4\|A_\varepsilon^{1/2}u\|^6 + 3\|\mathbb{P}_\varepsilon f\|^2. \tag{4.7}$$

As a result one has

$$\int_\tau^t \|u'(s)\|^2\,ds \le \int_\tau^t (3(\nu^2+1)\|A_\varepsilon u(s)\|^2 + 3\|\mathbb{P}_\varepsilon f\|_\infty^2 + \nu^4 C_9^4\|A_\varepsilon^{1/2}u(s)\|^6)\,ds \tag{4.8}$$

for $t \ge \tau \ge 0$. The inequalities (4.6) and (4.8) imply that for $\max(0, t-1) \le \tau \le t$ one has

(4.9)

$$\int_\tau^t \|u'(s)\|^2\,ds \le D_{26}(t-\tau)\|\mathbb{P}_\varepsilon f\|_\infty^2 + D_{27}(t-\tau)\sup_{\tau\le s\le t}\|A_\varepsilon^{1/2}u(s)\|^6 + D_{28}\|A_\varepsilon^{1/2}u(\tau)\|^2,$$

where $D_{26} = 3 + 6(\nu^2+1)\nu^{-2}$, $D_{27} = 3(\nu^2+1)D_{25} + \nu^4 C_9^4$, and $D_{28} = 3(\nu^2+1)\nu^{-1}$.

Step 3. Next we shall derive an estimate of $\|u'(t)\|^2$ for $t > 0$. Let t_0 be fixed so that $0 \le t_0 < \frac{1}{2}$, and t_0 is close to 0 with $u(t_0) \in D(A_\varepsilon)$. By taking the scalar product of (4.1) with $u'(t)$, we obtain

(4.10)

$$\frac{1}{2}\frac{d}{dt}\|u'\|^2 + \nu\|A_\varepsilon^{1/2}u'\|^2 \le \frac{1}{\nu}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2 + \frac{\nu}{4}\|A_\varepsilon^{1/2}u'\|^2 + |b_\varepsilon(u', u, u')|,$$

for $t \geq t_0$. However,

$$
\begin{aligned}
|b_\varepsilon(u', u, u')| &= \left| \sum_{i,j=1}^{3} \int_{Q_3} u_i' \varepsilon^{-\{i\}} (D_i u_j) u_j' \, dx \right| \\
&\leq \sum_{i,j=1}^{3} \left(\int_{Q_3} \varepsilon^{-2\{i\}} (D_i u_j)^2 \, dx \right)^{1/2} \left(\int_{Q_3} (u_i')^2 (u_j')^2 \, dx \right)^{1/2} \\
&\leq C \|A_\varepsilon^{1/2} u\| \left(\sum_{i,j=1}^{3} \left(\int_{Q_3} (u_i')^2 \, dx \right)^{1/4} \left(\int_{Q_3} (u_j')^6 \right)^{1/4} \right),
\end{aligned}
$$

which gives

$$
|b_\varepsilon(u', u, u')| \leq C_{10} \|A_\varepsilon^{1/2} u\| \, \|u'\|^{1/2} \|A_\varepsilon^{1/2} u'\|^{3/2}. \tag{4.11}
$$

Once again by using the Young inequality (3.4), we infer from (4.10) and (4.11) that

$$
\frac{d}{dt} \|u'(t)\|^2 + \nu \|A_\varepsilon^{1/2} u'(t)\|^2 \leq \frac{2}{\nu} \|A_\varepsilon^{-1/2} \mathbb{P}_\varepsilon f'\|_\infty^2 + D_{29} \|A_\varepsilon^{1/2} u(t)\|^4 \|u'(t)\|^2 \tag{4.12}
$$

for $t \geq t_0$, where $D_{29} = 27 C_{10}^4 2^{-1} \nu^{-3}$. Next we apply the uniform Gronwall inequality, as in (3.68), on (4.12) to obtain

$$
\begin{aligned}
\|u'(t)\|^2 \leq & \left(\frac{1}{t - \tau_0} \int_{\tau_0}^{t} \|u'(s)\|^2 \, ds + \int_{\tau_0}^{t} \frac{2}{\nu} \|A_\varepsilon^{-1/2} \mathbb{P}_\varepsilon f'\|_\infty^2 \right) \\
& \times \exp\left(\int_{\tau_0}^{t} D_{29} \|A_\varepsilon^{1/2} u(s)\|^4 \, ds \right),
\end{aligned} \tag{4.13}
$$

where $\tau_0 = \max(t_0, t - 1)$. Therefore, by using (4.9) for $t_0 \leq t \leq 1$ one has

$$
\begin{aligned}
\|u'(t)\|^2 \leq & \left(\frac{2}{\nu} \|A_\varepsilon^{-1/2} \mathbb{P}_\varepsilon f'\|_\infty^2 + D_{26} \|\mathbb{P}_\varepsilon f\|_\infty^2 + D_{27} \sup_{t_0 \leq s \leq t} \|A_\varepsilon^{1/2} u(s)\|^6 \right. \\
& \left. + \frac{D_{28}}{t - t_0} \|A_\varepsilon^{1/2} u(t_0)\|^2 \right) \\
& \times \exp\left(D_{29} \sup_{t_0 \leq s \leq t} \|A_\varepsilon^{1/2} u(s)\|^4 \right).
\end{aligned} \tag{4.14}
$$

Likewise for $t \geq 1$ one has $(t - t_0) \geq \frac{1}{2}$ and (4.13), together with (4.9), implies that

$$
\begin{aligned}
\|u'(t)\|^2 \leq & \left(\frac{2}{\nu} \|A_\varepsilon^{-1/2} \mathbb{P}_\varepsilon f'\|_\infty^2 + D_{26} \|\mathbb{P}_\varepsilon f\|_\infty^2 + D_{27} \sup_{t-1 \leq s \leq t} \|A_\varepsilon^{1/2} u(s)\|^6 \right. \\
& \left. + 2 D_{28} \sup_{t-1 \leq s \leq t} \|A_\varepsilon^{1/2} u(s)\|^2 \right) \\
& \times \exp\left(D_{29} \sup_{t-1 \leq s \leq t} \|A_\varepsilon^{1/2} u(s)\|^4 \right).
\end{aligned} \tag{4.15}
$$

Let us now assume that u_0 belongs to $D(A_\varepsilon)$. Thanks to the Gronwall inequality, we deduce from (4.12) that one has

$$\|u'(t)\|^2 \le \left(\|u'(0)\|^2 + \frac{2}{\nu}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2\right) \times \exp\left(D_{29}\sup_{0\le s\le t}\|A_\varepsilon^{1/2}u(s)\|^4\right), \qquad 0 \le t \le 1. \tag{4.16}$$

However, (4.7) implies that

$$\|u'(0)\|^2 \le 3(\nu^2+1)\|A_\varepsilon u_0\|^2 + 3\|\mathbb{P}_\varepsilon f\|_\infty^2 + 3\nu^4 C_9^4\|A_\varepsilon^{1/2}u_0\|^2.$$

By combining the last two inequalities, we find that for $t \le 1$ one has

$$\begin{aligned}\|u'(t)\|^2 \le \Big(&3(\nu^2+1)\|A_\varepsilon u_0\|^2 + 3\nu^4 C_9^4\|A_\varepsilon^{1/2}u_0\|^2 \\ &+ 3\|\mathbb{P}_\varepsilon f\|_\infty^2 + \frac{2}{\nu}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2\Big) \\ &\times \exp\left(D_{29}\sup_{0\le s\le t}\|A_\varepsilon^{1/2}u(s)\|^4\right).\end{aligned} \tag{4.17}$$

Step 4. In this last step we shall verify inequalities (2.35), (2.36), and (2.37). By taking the scalar product of (2.5) with $A_\varepsilon u$ we obtain

$$\nu\|A_\varepsilon u\|^2 \le \|u'\|\,\|A_\varepsilon u\| + \|\mathbb{P}_\varepsilon f\|\,\|A_\varepsilon u\| + |b_\varepsilon(u, u, A_\varepsilon u)|.$$

By using the Young inequality (3.4) with (4.4) we find that

$$\|A_\varepsilon u(t)\|^2 \le \frac{3}{\nu^2}\|u'(t)\|^2 + \frac{3}{\nu^2}\|\mathbb{P}_\varepsilon f\|_\infty^2 + \frac{9^3 C_9^4}{16\nu^4}\|A_\varepsilon^{1/2}u(t)\|^6, \qquad t \ge t_0, \tag{4.18}$$

and consequently $u(t) \in D(A_\varepsilon)$ for all $t \ge t_0$. Since t_0 can be chosen arbitrarily small, one has $u(t) \in D(A_\varepsilon)$ for all $t > 0$. Inequalities (4.14) and (4.18) then imply that for $0 < t_0 \le t \le 1$ one has

$$\begin{aligned}\|A_\varepsilon u(t)\|^2 \le \Big(&D_{30}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2 + D_{31}\|\mathbb{P}_\varepsilon f\|_\infty^2 + D_{32}\sup_{0\le s\le t}\|A_\varepsilon^{1/2}u(s)\|^6 \\ &+\frac{D_{33}}{t-t_0}\|A_\varepsilon^{1/2}u(t_0)\|^2\Big) \\ &\times \exp\left(D_{29}\sup_{0\le s\le t}\|A_\varepsilon^{1/2}u(s)\|^4\right),\end{aligned} \tag{4.19}$$

where $D_{30} = 6\nu^{-3}$, $D_{31} = 3\nu^{-2}(D_{26}+1)$, $D_{32} = 3\nu^{-2}D_{27} + 9^3 C_9^4(2\nu)^{-4}$, and $D_{33} = 3\nu^{-2}D_{28}$. Since (4.19) is valid for any t_0 satisfying $0 < t_0 < t \le 1$, we can replace t_0 with its limit value $t_0 = 0$ to obtain

$$\begin{aligned}\|A_\varepsilon u(t)\|^2 \le \Big(&D_{30}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2 + D_{31}\|\mathbb{P}_\varepsilon f\|_\infty^2 + D_{32}\sup_{0\le s\le t}\|A_\varepsilon^{1/2}u(s)\|^6 \\ &+ D_{33}t^{-1}\|A_\varepsilon^{1/2}u_0\|^2\Big) \\ &\times \exp\left(D_{29}\sup_{0\le s\le t}\|A_\varepsilon^{1/2}u(s)\|^4\right)\end{aligned} \tag{4.20}$$

for $0 < t \leq 1$. For $t \geq 1$ one obtains from (4.15) instead that
(4.21)

$$\|A_\varepsilon u(t)\|^2 \leq \Big(D_{30}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2 + D_{31}\|\mathbb{P}_\varepsilon f\|_\infty^2 + D_{32}\sup_{t-1\leq s\leq t}\|A_\varepsilon^{1/2}u(s)\|^6 + 2D_{33}\sup_{t-1\leq s\leq t}\|A_\varepsilon^{1/2}u(s)\|^2\Big) \times \exp\Big(D_{29}\sup_{t-1\leq s\leq t}\|A_\varepsilon^{1/2}u(s)\|^4\Big).$$

The quantities K_2^2, K_3^2, and K_4^2 appearing in (2.35) are now readily identified from (4.20), (4.21), and (2.31). In the case that $t \geq \widehat{T}_1 + 1$, where $\widehat{T}_1$ is given by Theorem 1, we are able to use the bound (2.33) for $\|A_\varepsilon^{1/2}u\|^2$. As a result (4.21) implies that

$$\|A_\varepsilon u(t)\|^2 \leq L_6^2 \stackrel{\text{def}}{=} \Gamma_2(L_5^2), \qquad t \geq \widehat{T}_1 + 1,$$

where

$$\text{(4.22)}\quad \Gamma_2(\rho) = (D_{30}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2 + D_{31}\|\mathbb{P}_\varepsilon f\|_\infty^2 + 2D_{33}\rho + D_{32}\rho^3)\exp(D_{29}\rho^2).$$

Note that since L_5^2 does not depend on the initial condition u_0, it follows from (4.22) that L_6^2 is independent of u_0 as well. This completes the proof of (2.35) and (2.36).

Let us now assume that u_0 belongs to $D(A_\varepsilon)$. Then we deduce from (4.18) and (4.17) that for $t \leq 1$ one has

$$\text{(4.23)}\quad \begin{aligned}\|A_\varepsilon u(t)\|^2 \leq{}& (D_{30}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2 + D_{34}\|\mathbb{P}_\varepsilon f\|_\infty^2 + D_{35}\|A_\varepsilon^{1/2}u(t)\|^6 \\ &+ D_{36}\|A_\varepsilon^{1/2}u_0\|^2 + D_{37}\|A_\varepsilon u_0\|^2) \\ &\times \exp\Big(D_{29}\sup_{0\leq s\leq t}\|A_\varepsilon^{1/2}u(s)\|^4\Big),\end{aligned}$$

where $D_{34} = 12\nu^{-2}$, $D_{35} = 9^3C_9^4(2\nu)^{-4}$, $D_{36} = 9C_9^4\nu^2$, and $D_{37} = 9(\nu^2+1)\nu^{-2}$. The quantities K_5^2, K_6^2, and K_7^2 appearing in (2.37) are now readily identified in (4.23). This completes the proof of Theorem 2. □

Remarks. 1. Depending on the choice of η_2, η_4, and r, one could have $L_5^2 \to \infty$ as $\varepsilon \to 0^+$. If this happens, then one finds that $L_6^2 \to \infty$ as well. On the other hand, one can easily give conditions whereby both L_5^2 and L_6^2 are bounded for $0 < \varepsilon \leq 1$. We will be treating the latter situation in detail in §6 wherein we prove the upper semicontinuity of the attractors at $\varepsilon = 0$.

2. The decomposition $u = v + w$, as used in §3, together with the arguments used here, may lead to slight improvements in the estimates appearing in Theorem 2.

3. The proof of the Theorem 3, which we give next, is similar to the argument in Babin and Vishik (1989, Theorem 2, §6, Chapter 1) for the 2DNS.

Proof of Theorem 3. Let f satisfy the hypothesis of Theorem 3, and let u_0 be any point in $\mathscr{B}_\varepsilon^0 \cup \mathscr{B}_\varepsilon^1$. Let $\tau > 0$ be fixed. Without loss of generality we will assume that $0 < \tau \leq 1$.

Assume for the moment that there is a compact set $\mathscr{K}^0(\tau)$ in $L^2(Q_3)$ such that

$$A_\varepsilon S_\varepsilon(f, t)u_0 \in \mathscr{K}^0(\tau), \qquad t \geq \tau, \; u_0 \in \mathscr{B}_\varepsilon^0 \cup \mathscr{B}_\varepsilon^1, \tag{4.24}$$

or equivalently

$$S_\varepsilon(f, t)(\mathscr{B}_\varepsilon^0 \cup \mathscr{B}_\varepsilon^1) \subset A_\varepsilon^{-1}(\mathscr{K}^0(\tau)), \qquad t \geq \tau.$$

The continuity of A_ε^{-1} assures us that $\mathscr{K}(\tau) \stackrel{\text{def}}{=} A_\varepsilon^{-1}(\mathscr{K}^0(\tau))$ is a compact set in V_ε^2. In order to prove (4.24) we will use the fact that if $\mathscr{K}_1$, $\mathscr{K}_2$ are compact sets in $L^2(Q_3)$, then $\mathscr{K}_1 + \mathscr{K}_2$ is compact in $L^2(Q_3)$.

Since $H^+(f)$ is assumed to be compact, the sets $\mathrm{Ev}(H^+(f))$ and $\mathscr{K}_1 \stackrel{\text{def}}{=} \mathrm{Ev}(\mathbb{P}_\varepsilon H^+(f))$ are compact sets in $L^2(Q_3)$ that satisfy

$$\mathbb{P}_\varepsilon g(t) \in \mathscr{K}_1 \quad \text{for all } g \in H^+(f), \; t \geq 0; \tag{4.25}$$

see §2.11. Now the equation (2.5) can be rewritten as

$$\nu A_\varepsilon u(t) = \mathbb{P}_\varepsilon f(t) - u'(t) - B_\varepsilon(u(t), u(t)), \qquad t > 0. \tag{4.26}$$

Assume next that there are functions $L_1(\tau)$ and $L_2(\tau)$, defined for $\tau > 0$, which depend only on ν, λ_1, and η_i, $i = 1, 2, 3, 4$, such that

$$\|A_\varepsilon^{1/2} u'(t)\|^2 \leq L_1(\tau), \qquad t \geq \tau, \tag{4.27}$$

and

$$\|A_\varepsilon^{1/2} B_\varepsilon(u(t), u(t))\|^2 \leq L_2(\tau), \qquad t \geq \tau. \tag{4.28}$$

In this case there is a set $\mathscr{K}_2(\tau)$, which is bounded in V_ε^1 and compact in $H_\varepsilon \subset L^2(Q_3)$, such that

$$-(u'(t) + B_\varepsilon(u(t), u(t))) \in \mathscr{K}_2(\tau), \qquad t \geq \tau. \tag{4.29}$$

By combining (4.25), (4.26), and (4.29) one has

$$\nu A_\varepsilon u(t) \in \mathscr{K}_1 + \mathscr{K}_2(\tau), \qquad t \geq \tau,$$

which implies (4.24).

From Constantin and Foiaş (1988) we note that there is a constant E_3 such that[6]

(4.30)
$$|b_\varepsilon(u^1, u^2, A_\varepsilon^{1/2} u^3)| \leq E_3 \|A_\varepsilon u^1\| \, \|A_\varepsilon u^2\| \, \|u^3\|, \qquad u^1, u^2 \in V_\varepsilon^2, \; u^3 \in V_\varepsilon^1.$$

Now (4.30) implies that $\|A_\varepsilon^{1/2} B_\varepsilon(u, u)\| \leq E_3 \|A_\varepsilon u\|^2$, when $u \in V_\varepsilon^2$. Hence (2.35) implies that (4.28) holds. In order to prove (4.27), we note that from (2.30), (2.35), (2.41), (4.14), and (4.15) one has

$$\|u'(t)\|^2 + \|A_\varepsilon u(t)\|^2 \leq K_8^2 \tau^{-1}, \qquad t \geq \tau, \tag{4.31}$$

[6] One can show that E_3 is independent of ε.

where K_8^2 is a positive constant depending only on ν, λ_1, and η_i, $i = 1, 2, 3, 4$.

In order to prove (4.27) we derive once again some formal estimates, which can be justified rigorously by using the Bubnov-Galerkin approximation method. Let us now take the scalar product of (4.1) with $A_\varepsilon u'$ for $t > 0$. We obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{1/2}u'\|^2 + \nu\|A_\varepsilon u'\|^2 \\ \leq \nu^{-1}\|\mathbb{P}_\varepsilon f'\|_\infty^2 + \frac{\nu}{4}\|A_\varepsilon u'\|^2 + |b_\varepsilon(u', u, A_\varepsilon u')| + |b_\varepsilon(u, u', A_\varepsilon u')|.\end{aligned}$$

However, using (4.3) and (4.30) one then obtains

$$\frac{d}{dt}\|A_\varepsilon^{1/2}u'(t)\|^2+\nu\|A_\varepsilon u'(t)\|^2 \leq 2\nu^{-1}(\|\mathbb{P}_\varepsilon f'\|_\infty^2+8E_4^2K_8^2\tau^{-1}\|A_\varepsilon^{1/2}u'(t)\|^2) \tag{4.32}$$

for $t \geq \tau$, where $E_4^2 = (C_9^2 + E_3^2)(\lambda_1^{-1} + 1)$. Now apply the uniform Gronwall inequality to (4.32) to obtain

$$\begin{aligned}\|A_\varepsilon^{1/2}u'(t)\|^2 \leq \left(\frac{1}{t-\tau_1}\int_{\tau_1}^t \|A_\varepsilon^{1/2}u'(s)\|^2\,ds + 2\nu^{-1}\|\mathbb{P}_\varepsilon f'\|_\infty^2(t-\tau_1)\right) \\ \times \exp(8\nu^{-1}E_4^2K_8^2\tau^{-1}(t-\tau_1))\end{aligned} \tag{4.33}$$

for $t \geq \tau$, where $\tau_1 = \max(\tau, t-1)$.

It remains to estimate the term $\int_{\tau_1}^t \|A_\varepsilon^{1/2}u'(s)\|^2\,ds$. Integrating the inequality (4.12) between τ_1 and t and using the estimates (2.30) and (4.31), we obtain

$$\frac{1}{t-\tau_1}\int_{\tau_1}^t \|A_\varepsilon^{1/2}u'(s)\|^2\,ds \leq \frac{2}{\nu^2}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2 + \frac{1}{\nu}K_8^2\tau^{-1}(1+D_{29}K_1^4). \tag{4.34}$$

By combining (4.33) and (4.34) we deduce that (4.27) holds where

$$\begin{aligned}L_1(\tau) = \left(\frac{2}{\nu^2}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f'\|_\infty^2 + \frac{1}{\nu}K_8^2\tau^{-1}(1+D_{29}K_1^4) + \frac{2}{\nu}\|\mathbb{P}_\varepsilon f'\|_\infty^2\right) \\ \times \exp(8\nu^{-1}E_4^2K_8^2\tau^{-1}),\end{aligned}$$

which completes the proof of Theorem 3. □

5. The reduced 3-dimensional theory: Theorem 6

We return to the study of the reduced 3D Navier-Stokes evolutionary equation

$$\overline{v}' + \nu A_\varepsilon\overline{v} + B_\varepsilon(\overline{v}, \overline{v}) = M\mathbb{P}_\varepsilon f, \tag{5.1}$$

where $(I-M)\mathbb{P}_\varepsilon f = 0$. In this case, $\{w = 0\}$ is a positively invariant set for (2.5), the dilated Navier-Stokes evolutionary equation. Since $\overline{v}$ and $M\mathbb{P}_\varepsilon f$ do not depend on x_3, the terms in (5.1) do not depend on ε. Nevertheless the estimates derived in §§3 and 4 are valid for (5.1); see, for example, Ladyzhenskaya (1969, 1972).

We define $\alpha \stackrel{\text{def}}{=} \|A_\varepsilon^{1/2}\overline{v}_0\|$ and $\beta \stackrel{\text{def}}{=} \|M\mathbb{P}_\varepsilon f\|_\infty$. Let $L_1, L_2, \dots$ denote functions of ν, λ_1, and β which are independent of α and $M_1, M_2, \dots$ denote functions of ν, λ_1, α, and β. Let $D_1, D_2, \dots$ be defined as in §3.

Instead of applying directly the results of Ladyzhenskaya (1969, 1972) we can use the estimates derived in §3 and take into account that $\overline{v}$ does not depend on x_3. Thus by defining $L_1 = L_1(\beta)$ and $M_1 = M_1(\alpha, \beta)$ as

(5.2)
$$\begin{cases} L_1 \stackrel{\text{def}}{=} D_{18}\beta^2 \exp(D_{19}\beta^4), \\ M_1 \stackrel{\text{def}}{=} D_{18}(\alpha^2 \exp(D_{19}\beta^4) + D_{19}(\alpha^4 + 2\alpha^2\beta^2)(\alpha^2+\beta^2))\exp(D_{19}(\alpha^2+\beta^2)^2), \end{cases}$$

we obtain the following result.

Theorem 6 (Part 1). *Let $\overline{v}(t)$ be a solution of the reduced 3D Navier-Stokes evolutionary equation (5.1) with $\overline{v}_0 \in MD(A_\varepsilon^{1/2})$. Then there are functions*

$$M_1 = M_1(\|A_\varepsilon^{1/2}\overline{v}_0\|^2, \|M\mathbb{P}_\varepsilon f\|_\infty^2), \qquad L_1 = L_1(\|M\mathbb{P}_\varepsilon f\|_\infty^2)$$

given by (5.2), such that

$$\|A_\varepsilon^{1/2}\overline{v}(t)\|^2 \le M_1 e^{-\nu\lambda_1 t} + L_1, \qquad t \ge 0. \tag{5.3}$$

Notice that, by the definition (3.84), one has

$$L_1 \le \Gamma(\beta^2). \tag{5.4}$$

Moreover, there exists a time $\tau_1 > 0$, such that one has $M_1 e^{-\nu\lambda_1 t} \le L_1$ for $t \ge \tau_1$. Combining this with (5.3) and (5.4) we get

$$\|A_\varepsilon^{1/2}\overline{v}(t)\|^2 \le 2\Gamma(\beta^2), \qquad t \ge \tau_1.$$

If $\beta^2 \le \eta_2^{-2}$ and if $0 < \varepsilon \le \varepsilon_0$, we then have

$$\|A_\varepsilon^{1/2}\overline{v}(t)\|^2 \le 2\Gamma(\eta_2^{-2}) \le (4\eta_1^{-2} + k_1^2\eta_3^{-4}), \qquad t \ge \tau_1,$$

that is, for $0 < \varepsilon \le \varepsilon_0$ and $t \ge \tau_1$, $\overline{v}(t)$ belongs to $\mathscr{B}_\varepsilon^1$.

Let us denote by $S_0(g, t)$ the mapping generated on $MD(A_\varepsilon^{1/2})$ by the strong solutions of equation (5.1), where $g = M\mathbb{P}_\varepsilon f$. Arguing as in §4, one has the following regularity result.

Theorem 6 (Part 2). *If*

$$M\mathbb{P}_\varepsilon f \in C^0([0,\infty); MH_\varepsilon) \cap L^\infty((0,\infty); MH_\varepsilon) \cap W^{1,\infty}((0,\infty); MD(A_\varepsilon^{-1/2})),$$

then there exist six positive functions $K_i^* = K_i^*(\|A_\varepsilon^{1/2}\overline{v}_0\|, \|M\mathbb{P}_\varepsilon f\|_\infty)$ such that

$$\begin{cases} \|A_\varepsilon\overline{v}(t)\|^2 \le K_1^{*2} + K_2^{*2}\|A_\varepsilon^{-1/2}M\mathbb{P}_\varepsilon f'\|_\infty^2 + K_3^{*2}t^{-1} & \textit{for } 0 < t \le 1, \\ \|A_\varepsilon\overline{v}(t)\|^2 \le K_1^{*2} + K_2^{*2}\|A_\varepsilon^{-1/2}M\mathbb{P}_\varepsilon f'\|_\infty^2 & \textit{for } t \ge 1. \end{cases} \tag{5.5}$$

Moreover, if $\overline{v}_0$ belongs to $MD(A_\varepsilon)$, then one has

$$\|A_\varepsilon\overline{v}(t)\|^2 \le K_4^{*2} + K_5^{*2}\|A_\varepsilon\overline{v}_0\|^2 + K_6^{*2}\|A_\varepsilon^{-1/2}M\mathbb{P}_\varepsilon f'\|_\infty^2, \qquad 0 \le t \le 1.$$

Moreover, if f belongs to $W(Q_3)$, then, for any $\tau > 0$, $S_0(g, t)\overline{v}_0$ belongs to a compact set of MV_ε^2 for $t \geq \tau$, provided $\overline{v}_0$ belongs to a bounded set of MV_ε^1. Furthermore, if f belongs to $W(Q_3) \cap W^{1,\infty}((0, \infty); L^2(Q_3))$ and is chosen so that $H^+(f)$ is compact, then, for any bounded set $\mathscr{B}$ of MV_ε^1 and any $\tau > 0$, $S_0(g, t)\mathscr{B}$ is included in a compact set $K_0(\tau, \mathscr{B})$ of MV_ε^2, for $t \geq \tau$.

If $H^+(f)$ is no longer compact, then, under the above hypotheses, we can prove that, for $t > 0$, $S_0(g, t)\mathscr{B}$ is included in a compact set $\widetilde{K}_0(t, \mathscr{B})$ which may depend on t.

Assume now that $f \in W(Q_3) \cap W^{1,\infty}((0, \infty); L^2(Q_3))$ is chosen so that $H^+(f)$ is compact. Due to Theorem 6 (Parts 1 and 2), $S_0(g, t)$ maps $MD(A_\varepsilon^{1/2})$ into itself, is bounded dissipative in $MD(A_\varepsilon^{1/2})$, and for $t \geq t_1 > 0$ is completely continuous in $MD(A_\varepsilon^{1/2})$. Therefore, the skew-product semiflow $\pi_0(\overline{v}, g, t) = (S_0(g, t)\overline{v}_0, g_t)$ defined in §2.11 admits a global compact attractor $\mathfrak{A}_0(g)$ in $MD(A_\varepsilon^{1/2}) \times H^+(g)$; see, for example, Hale (1988, Theorem 2.4.7). Since, by Theorem 6 (Part 2), $S_0(g, t)$ is also bounded dissipative in $MD(A_\varepsilon)$ and for $t \geq t_1 > 0$ completely continuous in $MD(A_\varepsilon)$, $\mathfrak{A}_0(g)$ is also the global compact attractor in $MD(A_\varepsilon) \times H^+(g)$. By the estimates (5.3) and (5.4), we have

$$\begin{aligned} \mathfrak{A}_0(g) &\subset \{u = \overline{v} + w : \|A_\varepsilon^{1/2}\overline{v}\|^2 \leq L_1 \leq \Gamma(\|M\mathbb{P}_\varepsilon f\|^2),\ w = 0\} \times \omega(g) \\ &\subset \mathscr{B}_\varepsilon^1 \times \omega(M\mathbb{P}_\varepsilon f). \end{aligned} \tag{5.6}$$

6. Properties of attractors: Theorems 4 and 5

We turn next to the proofs of Theorems 4 and 5 concerning the attractors for the Navier-Stokes equations. Let $\mathscr{B}_\varepsilon^0$, $\mathscr{B}_\varepsilon^1$, and $\mathscr{B}_\varepsilon^2$ be given by (2.38), (2.39), and (2.40). By Lemmas 3.1 and 3.2, $\mathscr{B}_\varepsilon^2$ is well defined and is a bounded set in V_ε^1.

Proof of Theorem 4. Set $\mathscr{U}_\varepsilon^2 = \mathscr{B}_\varepsilon^2 \times H^+(\mathbb{P}_\varepsilon f)$. For $u_0 \in V_\varepsilon^1$ and $f \in W(Q_3)$ with $\mathbb{P}_\varepsilon f \in W^{1,\infty}((0, \infty); L^2(Q_3))$ we let

$$\pi_\varepsilon(u_0, \mathbb{P}_\varepsilon f, \tau) = (S_\varepsilon(\mathbb{P}_\varepsilon f, \tau)u_0, (\mathbb{P}_\varepsilon f)_\tau)$$

denote the skew-product semiflow generated by the strong solutions of the dilated Navier-Stokes evolutionary equation (2.5); see §2.11. Let $\mathfrak{A}_\varepsilon = \omega(\mathscr{U}_\varepsilon^2)$ be the ω-limit set of $\mathscr{U}_\varepsilon^2$ in $V_\varepsilon^1 \times \mathbb{P}_\varepsilon W(Q_3)$, i.e.,

$$\mathfrak{A}_\varepsilon \stackrel{\text{def}}{=} \bigcap_{\tau \geq 0} \text{Closure}_{V_\varepsilon^1 \times \mathbb{P}_\varepsilon W(Q_3)} \left(\bigcup_{t \geq \tau} \pi_\varepsilon(\mathscr{U}_\varepsilon^2, t) \right).$$

It follows from (2.36) in Theorem 2 that for $\tau \geq \widehat{T}_1 + 1$ the set

$$\bigcup_{t \geq \tau} S_\varepsilon(\mathbb{P}_\varepsilon f, t)\mathscr{B}_\varepsilon^2$$

lies in a bounded set in V_ε^2 and, thus, a compact set in V_ε^1. Since $H^+(\mathbb{P}_\varepsilon f)$ is compact, it then follows that

$$\text{Closure}_{V_\varepsilon^1 \times \mathbb{P}_\varepsilon W(Q_3)} \left(\bigcup_{t \geq \tau} \pi_\varepsilon(\mathscr{U}_\varepsilon^2, t) \right)$$

is a nonempty compact set in $V_\varepsilon^1 \times H^+(\mathbb{P}_\varepsilon f)$ for each $\tau \geq \widehat{T}_1 + 1$. Consequently $\mathfrak{A}_\varepsilon$ is a nonempty compact invariant set in $V_\varepsilon^1 \times H^+(\mathbb{P}_\varepsilon f)$. Since

$$S_\varepsilon(\mathbb{P}_\varepsilon f, t)\mathscr{B}_\varepsilon^2 \subset \mathscr{B}_\varepsilon^2, \qquad t \geq 0,$$

$\mathscr{U}_\varepsilon^2$ is a positively invariant neighborhood of $\mathfrak{A}_\varepsilon$. Therefore, $\mathfrak{A}_\varepsilon$ is a local attractor for the strong solutions of (2.5) in $V_\varepsilon^1 \times H^+(\mathbb{P}_\varepsilon f)$, and the basin of attraction satisfies $\mathscr{B}_\varepsilon^2 \times H^+(\mathbb{P}_\varepsilon f) \subset B(\mathfrak{A}_\varepsilon)$. □

Remarks. 1. While the basin of attraction $B(\mathfrak{A}_\varepsilon)$ is a *large* set in $V_\varepsilon^1 \times H^+(\mathbb{P}_\varepsilon f)$, we do not know whether $B(\mathfrak{A}_\varepsilon) = V_\varepsilon^1 \times H^+(\mathbb{P}_\varepsilon f)$. As a result we do not know whether $\mathfrak{A}_\varepsilon$ is the global attractor of π_ε. The reason for this is that there may exist $u_0 \in V_\varepsilon^1$ such that the solution $S_\varepsilon(\mathbb{P}_\varepsilon f, t)u_0$ is not globally regular. Because of this, the fact that Corollary 4.1 allows us to conclude that $\mathfrak{A}_\varepsilon$ is the global attractor in the space of Leray solutions and $B(\mathfrak{A}_\varepsilon) = H_\varepsilon \times H^+(\mathbb{P}_\varepsilon f)$ is all the more surprising.

2. The fact that the Leray solutions of (2.5) may not be unique is not a concern from the point of view of the dynamics. One can overcome this problem by using the Bebutov flow; see Sell (1973).

Proof of Corollary 4.1. For any Leray solution of (2.5) we use (3.35) to obtain

$$\frac{1}{t}\int_0^t \|A_\varepsilon^{1/2}u\|^2\, ds \leq \frac{\nu^{-1}}{t}\|u_0\|^2 + 2\nu^{-2}(\|A_\varepsilon^{-1/2}M\mathbb{P}_\varepsilon f\|_\infty^2 + \|A_\varepsilon^{-1/2}(I-M)\mathbb{P}_\varepsilon f\|_\infty^2)$$

for all $t > 0$. From (2.6), (2.22), (3.1), and (2.55) with $\lambda^{-1} > 2\nu^{-2}\max(\lambda_1^{-1}, C_5^2)$ we obtain

$$\begin{aligned}\frac{1}{t}\int_0^t \|A_\varepsilon^{1/2}u\|^2\, ds &\leq \frac{\nu^{-1}}{t}\|u_0\|^2 + 2\nu^{-2}(\lambda_1^{-1}\|M\mathbb{P}_\varepsilon f\|_\infty^2 + C_5^2\varepsilon^2\|(I-M)\mathbb{P}_\varepsilon f\|_\infty^2) \\ &\leq \frac{\nu^{-1}}{t}\|u_0\|^2 + 2\nu^{-2}(\lambda_1^{-1}\eta_2^{-2} + C_5^2\varepsilon^{2+r}\eta_4^{-2}) \\ &\leq \frac{\nu^{-1}}{t}\|u_0\|^2 + k\min(\eta_1^{-2}, \varepsilon^p\eta_3^{-2})\end{aligned}$$

for all $t > 0$ and $0 < \varepsilon \leq \varepsilon_{10}(\lambda)$, where $0 < k < 1$. Therefore, for

$$T = \frac{2\nu^{-1}\|u_0\|^2}{(1-k)\min(\eta_1^{-2}, \varepsilon^p\eta_3^{-2})}$$

there is a t_0, $0 \leq t_0 < T$, such that

$$\|A_\varepsilon^{1/2}u(t_0)\|^2 \leq \min(\eta_1^{-2}, \varepsilon^p\eta_3^{-2}), \qquad 0 < \varepsilon \leq \varepsilon_{10}.$$

For this t_0, it follows from (2.21) that

$$\|A_\varepsilon^{1/2}v(t_0)\|^2 \le \|A_\varepsilon^{1/2}u(t_0)\|^2 \le \eta_1^{-2},$$
$$\|A_\varepsilon^{1/2}w(t_0)\|^2 \le \|A_\varepsilon^{1/2}u(t_0)\|^2 \le \varepsilon^p\eta_3^{-2}.$$

Consequently for $0 < \varepsilon \le \min(\varepsilon_0, \varepsilon_{10})$, where ε_0 is given by Theorem 1, one has $u(t_0) \in \mathscr{B}_\varepsilon^1$. Theorem 1 then implies that $u(t)$ is regular for all $t \ge t_0$. It follows that for the Leray solutions the basin of attraction $B(\mathfrak{A}_\varepsilon)$ is $H_\varepsilon \times H^+(f)$. Consequently $\mathfrak{A}_\varepsilon$ is the global attractor for the Leray solutions. □

Remark. The concept of a *weak attractor* for the 3DNS was studied in Foiaş and Temam (1987). It follows from Corollary 4.1 that the weak attractor coincides with $\mathfrak{A}_\varepsilon$ for thin domains.

Proof of Corollary 4.2. The proof of the existence of a global attractor $\mathfrak{A}_0$ for the skew-product semiflow $\pi_0(\cdot, g, t)$ in $MD(A_\varepsilon^{1/2}) \times H^+(g)$, where $g = M\mathbb{P}_\varepsilon f$, has been given in §5. The proof of (2.56) follows from (5.6) and (5.4).

Assume now that $(I - M)\mathbb{P}_\varepsilon f = 0$ and that $0 < \varepsilon \le \varepsilon_0$. Clearly, any solution $\overline{v}(t)$ of (2.24) with initial data $\overline{v}_0 \in MD(A_\varepsilon^{1/2})$ is a solution of the dilated Navier-Stokes evolutionary equation (2.5). As $\mathfrak{A}_0(g)$ is included in $\mathscr{B}_\varepsilon^1 \times \omega(M\mathbb{P}_\varepsilon f) \subset B(\mathfrak{A}_\varepsilon)$ and is an invariant set for equation (2.5), it follows that $\mathfrak{A}_0(g) \subset \mathfrak{A}_\varepsilon$. Let us now show that $\mathfrak{A}_\varepsilon \subset \mathfrak{A}_0(g)$. Since $(I - M)\mathbb{P}_\varepsilon f = 0$, inequality (3.53) takes on the form

$$\|A_\varepsilon^{1/2}w(t)\|^2 \le k_2^2\varepsilon^{2+r}\eta_4^{-2}\exp(-\nu C_5^{-2}\varepsilon^{-2}t/2), \qquad t \ge 0,$$

provided $u_0 \in \mathscr{B}_\varepsilon^1$. This implies that the u-component of the ω-limit set of $\mathscr{B}_\varepsilon^1 \times H^+(\mathbb{P}_\varepsilon f)$ belongs to the set of functions in $\mathscr{B}_\varepsilon^1$ which are independent of the variable x_3, i.e., $\mathfrak{A}_\varepsilon \subset \mathfrak{A}_0(g)$. □

Proof of Theorem 5 *and Corollary* 5.1. We begin with Theorem 5. Let us consider a sequence of positive numbers $\varepsilon_n \to 0$ as $n \to \infty$. Let $\mathscr{F}$ be any positively invariant compact subset of $W(Q_3) \cap W^{1,\infty}((0, \infty), L^2(Q_3))$, and let f_n be a sequence of functions $f_n \in \mathscr{F}$ that satisfies

$$\lim_{n\to\infty} \|f_n - f_0\|_\infty = 0, \tag{6.1}$$

where $f_0 \in M\mathscr{F}$. Then each of the positive hulls $H^+(f_n)$ and $H^+(f_0)$ are compact sets in $\mathscr{F}$. We set $g_n = \mathbb{P}_{\varepsilon_n} f_n$ and $g_0 = M\mathbb{P}_{\varepsilon_n} f_0$. According to the comments made in §2.6, $\mathbb{P}_{\varepsilon_n} f_0(t)$ belongs to MH_{ε_n} for every t, and consequently

$$g_0 = M\mathbb{P}_{\varepsilon_n} f_0 = \mathbb{P}_{\varepsilon_n} f_0 = \begin{pmatrix} \mathbb{P}_2 \binom{f_{01}}{f_{02}} \\ f_{03} \end{pmatrix}, \tag{6.2}$$

where $f_0 = (f_{01}, f_{02}, f_{03})$. It follows from (6.1) and (6.2) and the fact that $\mathbb{P}_{\varepsilon_n}$ is a projection that

$$\lim_{n\to\infty} \|M\mathbb{P}_{\varepsilon_n} f_n - g_0\|_\infty = 0 \tag{6.3}$$

and

$$\lim_{n\to\infty} \|(I-M)\mathbb{P}_{\varepsilon_n} f_n\|_\infty = 0.$$

The last two conditions can be written as

$$\lim_{n\to\infty} \|\mathbb{P}_{\varepsilon_n} f_n - g_0\|_\infty = 0. \tag{6.4}$$

For every n, we consider the dilated Navier-Stokes evolutionary equation, i.e.,

$$u' + \nu A_{\varepsilon_n} u + B_{\varepsilon_n}(u, u) = g_n. \tag{6.5}$$

Let $S_{\varepsilon_n}(g_n, t)u_{0n} = u_n(t) = v_n(t) + w_n(t)$ denote the strong solution of the equation (6.5) with initial data u_{0n} in $V^1_{\varepsilon_n}$. We also consider the reduced 3D Navier-Stokes evolutionary equation

$$\overline{v}' + \nu A_0 \overline{v} + B_0(\overline{v}, \overline{v}) = g_0, \tag{6.6}$$

with initial data $\overline{v}(0) = \overline{v}_0$ in V^1_0. Let $S_0(g_0, t)\overline{v}_0$ denote the strong solution of (6.6) with initial condition $\overline{v}_0 \in V^1_0$. It follows from (6.4) that there exist an integer n_1 and a positive constant E_0 such that

$$\max(\|g_0\|^2_\infty, \|Mg_n\|^2_\infty, \|g_n\|^2_\infty) \le E_0, \qquad n \ge n_1.$$

According to Theorem 1 and Lemmas 3.1 and 3.2, every solution of (6.5) with initial data in the bounded set $\mathscr{B}^0_{\varepsilon_n}$ ultimately enters into the bounded set $\mathscr{B}^1_{\varepsilon_n}$ as well as the bounded set $\mathscr{B}^3_{\varepsilon_n}$ where

$$\mathscr{B}^3_{\varepsilon_n} \stackrel{\text{def}}{=} \{u = v + w \in \mathscr{B}^1_{\varepsilon_n} : \|A^{1/2}_{\varepsilon_n} v\|^2 \le \Gamma(E_0), \ \|A^{1/2}_{\varepsilon_n} w\|^2 \le k_2^2 \varepsilon_n^{2+r} \eta_4^{-2}\}.$$

In particular, the (local) attractor $\mathfrak{A}_{\varepsilon_n}$ of (6.5), see Theorem 4, is included in $\mathscr{B}^3_{\varepsilon_n} \times \omega(g_n)$, for $n \ge n_1$. Likewise, due to the property (5.6), the global attractor $\mathfrak{A}_0 = \mathfrak{A}_0(g_0)$ of (6.6) is included in the bounded set $\mathscr{B}^3_0 \times \omega(g_0)$, where

$$\mathscr{B}^3_0 = \{u = v + w \in V^1_{\varepsilon_n} : \|A^{1/2}_0 v\|^2 \le \Gamma(E_0), \ w = 0\}.$$

Note that, for every n one has $M\mathscr{B}^3_{\varepsilon_n} = \mathscr{B}^3_0$. Now define $E_1 = \Gamma(E_0)$.

For $\tau \in \mathbb{R}$, we let $f_{n,\tau}$, $g_{n,\tau}$, and $g_{0,\tau}$, denote the translate of f_n, g_n, and g_0; see §2.11 Then from (6.4) it follows that for every $\delta > 0$ there is an integer $n_2 \ge n_1$ such that

$$\|g_{n,\tau} - g_{0,\tau}\|_\infty \le \delta/2, \qquad n \ge n_2, \ \tau \ge 0.$$

Furthermore, there is a $T \ge 0$ such that

$$\operatorname{dist}_{W(Q_3)}(g_{0,\tau}, \omega(g_0)) < \delta/2, \qquad \tau > T.$$

It then follows that

$$\operatorname{dist}_{W(Q_3)}(g_{n,\tau}, \omega(g_0)) \le \delta, \qquad n \ge n_2, \ \tau \ge T, \tag{6.7}$$

which implies that the attractors $\omega(g_n)$ are upper semicontinuous as $n \to \infty$.

In the remainder of the argument we shall use the weaker condition (6.3) in place of (6.1). As a result the argument now applies both to Theorem 5 and Corollary 5.1.

We claim that there exists an integer $n_3 \geq n_2$ and two positive constants k_3 and E_2, with $E_2 \geq \max(E_0, E_1)$, such that

$$\|A_\varepsilon^{1/2} v_n(t)\|^2 \leq E_2, \qquad \|A_\varepsilon^{1/2} w_n(t)\|^2 \leq k_3^2 \varepsilon^{2+r} \eta_4^{-2}, \tag{6.8}$$

for $\varepsilon = \varepsilon_n$, $n \geq n_3$, and $t \geq 0$, provided $u_{0_n} \in \mathscr{B}_\varepsilon^3$. Furthermore, one has

$$\|A_0^{1/2} \overline{v}(t)\|^2 \leq E_2, \tag{6.9}$$

for $t \geq 0$, provided $(\overline{v}_0, 0) \in \mathscr{B}_0^3$. Indeed (6.8) and (6.9) are immediate consequences of (3.82), (3.84), and Theorem 6 (Part 1).

Now we want to compare the orbits of the dilated Navier-Stokes equation (6.5) with those of the reduced 3D Navier-Stokes equation (6.6) when u_{0n} belongs to $\mathscr{B}_{\varepsilon_n}^3$. To this end, we consider the equation satisfied by $z_n(t) \stackrel{\text{def}}{=} v_n(t) - \overline{v}(t)$ where $z_n(0) = 0$ (i.e., $v_n(0) = \overline{v}(0) = v_{0n}$), $w_n(0) = w_{0n}$, and $u_{0n} = v_{0n} + w_{0n}$ belongs to $\mathscr{B}_{\varepsilon_n}^3$. We have

$$z_n' + \nu A_{\varepsilon_n} z_n = (M g_n - g_0) - M(B_{\varepsilon_n}(u_n, u_n) - B_{\varepsilon_n}(\overline{v}, \overline{v})), \tag{6.10}$$

and $M z_n = z_n$. Taking the inner product of (6.10) by $A_{\varepsilon_n} z_n$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|A_{\varepsilon_n}^{1/2} z_n\|^2 + \nu \|A_{\varepsilon_n} z_n\|^2 \\ &\quad \leq \|M g_n - g_0\|_\infty \|A_{\varepsilon_n} z_n\| \\ &\qquad + |b_{\varepsilon_n}(v_n, v_n, A_{\varepsilon_n} z_n) - b_{\varepsilon_n}(\overline{v}, \overline{v}, A_{\varepsilon_n} z_n)| + |b_{\varepsilon_n}(w_n, w_n, A_{\varepsilon_n} z_n)| \end{aligned} \tag{6.11}$$

for $t \geq 0$. However, we can write

$$|b_{\varepsilon_n}(v_n, v_n, A_{\varepsilon_n} z_n) - b_{\varepsilon_n}(\overline{v}, \overline{v}, A_{\varepsilon_n} z_n)| = |b_{\varepsilon_n}(z_n, \overline{v}, A_{\varepsilon_n} z_n) + b_{\varepsilon_n}(v_n, z_n, A_{\varepsilon_n} z_n)|. \tag{6.12}$$

From inequality (8.13) we obtain

$$|b_{\varepsilon_n}(z_n, \overline{v}, A_{\varepsilon_n} z_n)| \leq c_{12} \|z_n\|_{L^\infty(Q_3)} \|A_{\varepsilon_n}^{1/2} \overline{v}\| \, \|A_{\varepsilon_n} z_n\|. \tag{6.13}$$

Since z_n does not depend on the variable x_3, we can apply the following Gagliardo-Nirenberg type inequality:

$$\|z_n\|_{L^\infty(Q_3)} \leq c \|z_n\|_{H^2(Q_3)}^{1/2} \|z_n\|_{L^2(Q_3)}^{1/2}; \tag{6.14}$$

see Friedman (1964). The estimates (6.13), (6.14), and (2.18) imply that

$$|b_{\varepsilon_n}(z_n, \overline{v}, A_{\varepsilon_n} z_n)| \leq C_{10} \|z_n\|^{1/2} \|A_{\varepsilon_n}^{1/2} \overline{v}\| \, \|A_{\varepsilon_n} z_n\|^{3/2} \tag{6.15}$$

for some constant C_{10}. From (6.11), (6.12), (6.15), (3.2), and (3.3), we find that

$$\begin{aligned}(6.16)\qquad &\frac{1}{2}\frac{d}{dt}\|A_{\varepsilon_n}^{1/2}z_n\|^2+\nu\|A_{\varepsilon_n}z_n\|^2\\ &\le \|A_{\varepsilon_n}z_n\|(\|Mg_n-g_0\|_\infty+C_2\varepsilon_n^{1/2}\|A_{\varepsilon_n}^{1/2}w\|^{3/2}\|A_{\varepsilon_n}w\|^{1/2})\\ &\quad+C_1\|v_n\|^{1/2}\|A_{\varepsilon_n}^{1/2}v_n\|^{1/2}\|A_{\varepsilon_n}^{1/2}z_n\|^{1/2}\|A_{\varepsilon_n}z_n\|^{3/2}\\ &\quad+C_{10}\|z_n\|^{1/2}\|A_{\varepsilon_n}^{1/2}\overline{v}\|\,\|A_{\varepsilon_n}z_n\|^{3/2}\end{aligned}$$

for $t\ge 0$. Using the Young inequality we derive from (6.16) that one has

$$\begin{aligned}&\frac{d}{dt}\|A_{\varepsilon_n}^{1/2}z_n\|^2+\nu\|A_{\varepsilon_n}z_n\|^2\\ &\le\frac{4}{\nu}\|Mg_n-g_0\|_\infty^2+\frac{4}{\nu}C_2^2\varepsilon_n\|A_{\varepsilon_n}^{1/2}w\|^3\|A_{\varepsilon_n}w\|\\ &\quad+\frac{108}{\nu^3}(C_1^4\|v_n\|^2\|A_{\varepsilon_n}^{1/2}v_n\|^2\|A_{\varepsilon_n}^{1/2}z_n\|^2+C_{10}^4\|z_n\|^2\|A_{\varepsilon_n}^{1/2}\overline{v}\|^4)\end{aligned}$$

for $t\ge 0$, or, by (6.8) and (6.9),

$$\begin{aligned}(6.17)\qquad \frac{d}{dt}\|A_{\varepsilon_n}^{1/2}z_n\|^2+\nu\|A_{\varepsilon_n}z_n\|^2&\le\frac{4}{\nu}\|Mg_n-g_0\|_\infty^2+\frac{4}{\nu}C_2^2\varepsilon_n\|A_{\varepsilon_n}^{1/2}w\|^3\|A_{\varepsilon_n}w\|\\ &\quad+D_{24}E_2^2\|A_{\varepsilon_n}^{1/2}z_n\|^2,\end{aligned}$$

where $D_{24}=108\lambda_1^{-1}\nu^{-3}(C_1^4+C_{10}^4)$.

Integrating the inequality (6.17) from 0 to t and using a Gronwall inequality, we deduce that

$$\begin{aligned}\|A_{\varepsilon_n}^{1/2}z_n(t)\|^2\le\frac{4}{\nu}&\left(t\|Mg_n-g_0\|_\infty^2+C_2^2\varepsilon_n\int_0^t\|A_{\varepsilon_n}^{1/2}w(s)\|^3\|A_{\varepsilon_n}w(s)\|\,ds\right)\\ &\times\exp(D_{24}E_2^2t)\end{aligned}$$

for $t\ge 0$. Arguing as in the proof of (3.54), we see that there exists a positive constant D_{25} such that

$$\frac{4}{\nu}C_2^2\varepsilon_n\int_0^t\|A_{\varepsilon_n}^{1/2}w(s)\|^3\|A_{\varepsilon_n}w(s)\|\,ds\le D_{25}(1+t)\varepsilon_n^{4+2r}\eta_4^{-4},\qquad t\ge 0.$$

Finally, we obtain

$$(6.18)\qquad \|A_{\varepsilon_n}^{1/2}z_n(t)\|^2\le\left(\frac{4}{\nu}t\|Mg_n-g_0\|^2+D_{25}(1+t)\varepsilon_n^{4+2r}\eta_4^{-4}\right)\exp(D_{24}E_2^2t).$$

Thanks to hypothesis (2.59) and condition (6.3), we infer, from (6.18) that, for any positive numbers δ and T, there exists an integer n_4, $n_4\ge n_3\ge 0$, such that

$$(6.19)\qquad \|A_{\varepsilon_n}^{1/2}z_n(T)\|^2\le\delta/3,\qquad n\ge n_4.$$

Let δ be a positive number. Since $\mathfrak{A}_0$ is the global attractor of (6.6), there exists a positive time $\tau_0\equiv\tau_0(\delta)$ such that

$$(6.20)\qquad \pi_0(\mathscr{B}_0^3,H^+(g_0),t)\subset\mathscr{N}_{V_0^1\times W(Q_3)}(\mathfrak{A}_0,\delta/3),\quad t\ge\tau_0,$$

where $\mathscr{N}_{V_0^1\times W(Q_3)}(\mathfrak{A}_0, \alpha)$ denotes the α-neighborhood of $\mathfrak{A}_0$ in $V_0^1 \times W(Q_3)$. Using the properties (6.8) and (6.19), as well as the hypothesis (2.59), we see that exists an integer n_0, $n_0 \geq n_4$, such that

$$\|A_{\varepsilon_n}^{1/2}(v_n(\tau_0) - \overline{v}(\tau_0))\|^2 + \|A_{\varepsilon_n}^{1/2} w_n(\tau_0)\|^2 \leq 2\delta/3, \qquad n \geq n_0, \tag{6.21}$$

where $u_n(t) = v_n(t) + w_n(t) = S_{\varepsilon_n}(g_n, t)u_{0n}$, $\overline{v}(t) = S_0(g_0, t)Mu_{0n}$, and $u_{0n} \in \mathscr{B}^3_{\varepsilon_n}$. From (6.7), (6.20), and (6.21), we infer that

$$\pi_{\varepsilon_n}(\mathscr{B}^3_{\varepsilon_n}, H^+(g_n), \tau_0) \subset \mathscr{N}_{V^1_{\varepsilon_n}\times W(Q_3)}(\mathfrak{A}_0, \delta), \qquad n \geq n_0,$$

and, in particular,

$$\pi_{\varepsilon_n}(\mathfrak{A}_{\varepsilon_n}, \tau_0) \subset \mathscr{N}_{V^1_{\varepsilon_n}\times W(Q_3)}(\mathfrak{A}_0, \delta), \qquad n \geq n_0. \tag{6.22}$$

Due to the invariance property of the attractors $\mathfrak{A}_{\varepsilon_n}$, we at once deduce the upper semicontinuity result (2.61) from (6.22). This completes the proof of Theorem 5 and Corollary 5.1. □

Proof of Corollary 5.2. We shall only give a sketch of the proof of Corollary 5.2. We keep here the notation of the proof of Theorem 5.

According to Theorems 2 and 6 (Part 2), every solution of (6.5), for $n \geq n_1$ (resp. of (6.6)), with initial data in the bounded set $\mathscr{B}^0_{\varepsilon_n}$ (resp. in any bounded set of V_0^1) ultimately enters into the bounded set $\mathscr{B}^4_{\varepsilon_n}$ (resp. $\mathscr{B}^4_0$) where

$$\mathscr{B}^4_{\varepsilon_n} \overset{\text{def}}{=} \{u \in V^2_{\varepsilon_n} : \|A_{\varepsilon_n} u\|^2 \leq E_4\}$$

(resp. $\mathscr{B}^4_0 = \{v \in V_0^2 : \|A_0 v\|^2 \leq E_4\}$) where E_4 is a positive constant independent of n. Note that, for every n, $M\mathscr{B}^4_{\varepsilon_n} = \mathscr{B}^4_0$. In particular, the (local) attractor $\mathfrak{A}_{\varepsilon_n}$ of (6.5) (resp. the global attractor $\mathfrak{A}_0$ of (6.6)) is included in the bounded set $\mathscr{B}^4_{\varepsilon_n} \times \omega(g_n)$ (resp. $\mathscr{B}^4_0 \times \omega(g_0)$). Furthermore, due to Theorems 2 and 6 (Part 2), there exist an integer n_5, $n_5 \geq n_4$, and a positive constant E_5, with $E_5 \geq \max(E_4, E_2)$ such that, for $t \geq 0$,

$$\|A_{\varepsilon_n} u_n(t)\|^2 \leq E_5 \quad \text{for } n \geq n_5,\ u_{0n} \in \mathscr{B}^3_{\varepsilon_n} \cap \mathscr{B}^4_{\varepsilon_n},$$

and

$$\|A_0 \overline{v}(t)\|^2 \leq E_5 \quad \text{for } \overline{v}_0 \in \mathscr{B}^3_0 \cap \mathscr{B}^4_0.$$

Let δ be a positive number. Since $\mathfrak{A}_0$ is the global attractor of (6.6) in $V_0^2 \times W(Q_3)$, there exists a positive time $\tau_1 \equiv \tau_1(\delta)$, with $\tau_1 > 1$ for instance, such that

$$\pi_0(\mathscr{B}^3_0 \cap \mathscr{B}^4_0, H^+(g_0), t) \subset \mathscr{N}_{V_0^2\times W(Q_3)}(\mathfrak{A}_0, \delta/3), \qquad t \geq \tau_1. \tag{6.23}$$

As in the proof of Theorem 5, due to the properties (6.7) and (6.23), the upper semicontinuity result (2.63) is valid if we show that there exists an integer n_6, $n_6 \geq n_5$, such that, for $n \geq n_6$, one has

$$\|A_{\varepsilon_n}(u_n(\tau_1) - \overline{v}(\tau_1))\|^2 \leq 2\delta/3 \tag{6.24}$$

where $u_n(t) = S_{\varepsilon_n}(g_n, t)u_{0n}$, $\overline{v}(t) = S_0(g_0, t)\overline{v}_0$, and $u_{0n} \in \mathscr{B}^3_{\varepsilon_n} \cap \mathscr{B}^4_{\varepsilon_n}$.

Note that $z_n \stackrel{\text{def}}{=} u_n - \overline{v}$ is the solution of the equation

$$z_n' + \nu A_{\varepsilon_n} z_n = (g_n - g_0) - (B_{\varepsilon_n}(u_n, u_n) - B_{\varepsilon_n}(\overline{v}, \overline{v})),$$

with initial condition $z_n(0) = u_{0n} - Mu_{0n} = w_{0n}$. The proof of the estimate (6.24) follows the lines of the proof of the estimates (2.35), (2.36) of Theorem 2 (see §4, Steps 1 to 4). As the proof of (6.24) is rather long and completely similar to the proof of Theorem 2, we omit the details. Let us just point out that, as in §4, we use the auxiliary equation

$$\frac{d}{dt}z_n' + \nu A_{\varepsilon_n} z_n' = (g_n' - g_0') - (B_{\varepsilon_n}(u_n', u_n) + B_{\varepsilon_n}(u_n, u_n') - B_{\varepsilon_n}(\overline{v}', \overline{v}) - B_{\varepsilon_n}(\overline{v}, \overline{v}')),$$

with initial condition

$$z_n'(0) = (g_n - g_0)(0) - (B_{\varepsilon_n}(u_{0n}, u_{0n}) - B_{\varepsilon_n}(v_{0n}, v_{0n})) - \nu A_{\varepsilon_n}(u_{0n} - v_{0n}). \quad \square$$

7. Remarks on other boundary conditions: Theorem 7

In this section, we assume that $\Omega_\varepsilon = Q_2 \times (0, \varepsilon)$, where Q_2 is a bounded domain in $\mathbb{R}^2$ with a boundary of class C^s, $s \geq 2$. The smoothness hypothesis $s \geq 2$ is made to avoid any problem of regularity of the solutions of the corresponding stationary Stokes equation. As in §2, we set $Q_3 = Q_2 \times (0, 1)$ and use the change of variables $(y_1, y_2, y_3) \mapsto (x_1, x_2, x_3)$, where $x_i = y_i$, $i = 1, 2$, and $x_3 = \varepsilon^{-1}y_3$. This change of variables sends Ω_ε onto Q_3.

7.1. **Mixed periodic-Dirichlet boundary conditions.** We are interested here in solutions of the Navier-Stokes evolutionary equation (2.5) that satisfy periodic boundary conditions on $\Gamma_1 = Q_2 \times \{0\} \cup Q_2 \times \{\varepsilon\}$ and Dirichlet boundary conditions on $\Gamma_2 = \partial Q_2 \times (0, \varepsilon)$. As before we use the operator J_ε of §2.1. Let H_ε (respectively V_ε^1) denote the closure in $L^2(Q_3)$ (respectively $H^1(Q_3)$) of those smooth functions u that satisfy periodic boundary conditions on Γ_1, Dirichlet boundary conditions on Γ_2, and $\nabla_\varepsilon \cdot u = 0$ in Q_3. We denote by $\mathbb{P}_\varepsilon$ the orthogonal projection of $L^2(Q_3)$ onto H_ε. By applying $\mathbb{P}_\varepsilon$ to (2.4), we obtain (as in §2.2) the nonlinear evolutionary equation (2.5) on H_ε, where $u \in H_\varepsilon$, $A_\varepsilon u = -\mathbb{P}_\varepsilon \Delta_\varepsilon u$. We set $V_\varepsilon^2 = D(A_\varepsilon)$. Using regularity results (see Dauge (1984) and the references therein), one can show that $V_\varepsilon^2 = V_\varepsilon^1 \cap H^2(Q_3)$. One also has $V_\varepsilon^1 = D(A_\varepsilon^{1/2})$. Using the classical Poincaré inequality, one easily shows that the inequalities (2.17) still hold. Likewise, thanks to the estimates (2.17) and to regularity results in Dauge (1984), one can prove the inequalities (2.18). Like in §2.4, we introduce the projection M. All the properties given in §2.4 are still true. In particular, if $u \in D(A_\varepsilon)$, we have

$$(I - M)A_\varepsilon u = A_\varepsilon(I - M)u. \tag{7.1}$$

The crucial estimate (2.22) still holds (see Hale and Raugel (1989)). The above property (7.1) allows us to write the equation (2.5) as the system (2.23) of two equations in $v = Mu$ and $w = (I - M)u$.

As in §2.6, we obtain a reduced 3D Navier-Stokes evolutionary equation which is given by (2.24). The reduced 3D Navier-Stokes evolutionary equation incorporates the 2DNS equation on Q_2 with homogeneous Dirichlet boundary conditions. In order to see this, we let $L^2(Q_2, \mathbb{R}^2)$ denote the L^2-space of 2-dimensional vector fields $m = (m_1, m_2)$ which depend on $(x_1, x_2) \in Q_2$ and let $H(Q_2)$ denote the closure in $L^2(Q_2, \mathbb{R}^2)$ of the smooth functions u that satisfy $D_1 m_1 + D_2 m_2 = 0$ on Q_2. Finally we let $\mathbb{P}_2$ denote the orthogonal projection of $L^2(Q_2, \mathbb{R}^2)$ onto the space $H(Q_2)$. Then $\mathbb{P}_\varepsilon$ and $\mathbb{P}_2$ satisfy the relations described in §2.6. Furthermore, $\overline{v}$ is a solution of the reduced 3D Navier-Stokes evolutionary equation (2.24) if and only if $m = (\overline{v}_1, \overline{v}_2)$ is a solution of the 2D Navier-Stokes evolutionary equation

$$\frac{d}{dt}m - \nu \mathbb{P}_2(D_1^2 + D_2^2)m + \mathbb{P}_2(m \cdot \nabla)m = (g_1, g_2)$$

and $\overline{v}_3$ is a solution of the linear equation

$$\frac{d}{dt}\overline{v}_3 - \nu(D_1^2 + D_2^2)\overline{v}_3 + (\overline{v}_1 D_1 + \overline{v}_2 D_2)\overline{v}_3 = g_3,$$

where $g = (g_1, g_2, g_3) = M\mathbb{P}_\varepsilon f$. With the changes made above in the definitions of the spaces H_ε, V_ε^1, V_ε^2 and the operators $\mathbb{P}_\varepsilon$ and $\mathbb{P}_2$, all the results given in §§2–6 (see also §8) are still true in the case where we have periodic boundary conditions on $\Gamma_0 \cup \Gamma_1$ and homogeneous Dirichlet boundary conditions on Γ_2. Moreover, the proofs given in §§3–6 are exactly the same.

7.2. **Homogeneous Dirichlet boundary conditions.** This case is quite different from the cases previously studied. Here we consider the Navier-Stokes equations (2.1) on Ω_ε (resp. (2.4) on Q_3) with homogeneous Dirichlet boundary conditions on $\partial\Omega_\varepsilon$ (resp. on ∂Q_3). Here we introduce the spaces

$$H_\varepsilon = \{u \in L^2(Q_3) \colon \nabla_\varepsilon \cdot u = 0,\ u \cdot n|_{\partial Q_3} = 0\}$$

and

$$V_\varepsilon^1 = \{u \in H_0^1(Q_3) \colon \nabla_\varepsilon \cdot u = 0\},$$

and we denote by $\mathbb{P}_\varepsilon$ the orthogonal projection of $L^2(Q_3)$ onto H_ε. By applying $\mathbb{P}_\varepsilon$ to (2.4), we obtain (as in §2.2) the dilated Navier-Stokes evolutionary equation (2.5) where $u = \mathbb{P}_\varepsilon u \in H_\varepsilon$, $A_\varepsilon u = -\mathbb{P}_\varepsilon \Delta_\varepsilon u$ (with homogenous Dirichlet boundary conditions). One has $V_\varepsilon^1 = D(A_\varepsilon^{1/2})$ and we set $V_\varepsilon^2 = D(A_\varepsilon)$. Using the regularity results given in Dauge (1984, 1989), one obtains that $V_\varepsilon^2 = V_\varepsilon^1 \cap H^2(Q_3)$. Using the classical Poincaré inequality, one shows at once that the estimates (2.17) still hold. Arguing as in Hale and Raugel (1992a, Corollary 2.8) one shows that

$$\|A_\varepsilon^i u\| \le C_{11}\varepsilon \|A_\varepsilon^{i+1/2} u\| \quad \text{for } i = 0, 1, \tag{7.2}$$

where C_{11} is a positive constant that does not depend on ε. Using the inequality (7.2) several times and the regularity results of Dauge (1984, 1989), one proves that

(7.3)

$$C_6(\|u\|_{H^2(Q)} + \varepsilon^{-1}\|D_3 u\| + \varepsilon^{-1}\|D_1 D_3 u\| + \varepsilon^{-1}\|D_2 D_3 u\| + \varepsilon^{-2}\|D_3^2 u\|) \le \varepsilon^{-1}\|A_\varepsilon u\|$$

and

$$\|A_\varepsilon u\| \le C_7(\|u\|_{H^2(Q_3)} + \varepsilon^{-1}\|D_3 u\|_{H^1(Q_3)} + \varepsilon^{-2}\|D_3^2 u\|).$$

The inequalities (7.3) and (8.20) imply that

$$|b_\varepsilon(u^1, u^2, u^3)| \le C_{12}\|A_\varepsilon^{1/2}u^1\|\,\|A_\varepsilon^{1/2}u^2\|^{1/2}\|A_\varepsilon u^2\|^{1/2}\|u^3\|, \tag{7.4}$$

for $u^1 \in D(A_\varepsilon^{1/2})$, $u^2 \in D(A_\varepsilon)$, and $u^3 \in H_\varepsilon$. We now state the following results which do not use the decomposition $u = Mu + (I - M)u$. We assume that $0 < \varepsilon \le 1$.

Theorem 7. *Let p and r be two real numbers satisfying $-1 < p < 0$ and $r > -3$, and let $\widetilde{C}_1$ and $\widetilde{C}_2$ be two positive constants. Then there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \le \varepsilon_0$, whenever $u_0 \in D(A_\varepsilon^{1/2})$, $f \in L^\infty((0, \infty), L^2(Q_3))$ satisfy*

$$\|A_\varepsilon^{1/2}u_0\|^2 \le \widetilde{C}_1\varepsilon^p, \qquad \|\mathbb{P}_\varepsilon f\|_\infty^2 \le \widetilde{C}_2\varepsilon^r,$$

then (2.5) *has a solution u that belongs to $C^0([0, \infty), V_\varepsilon^1)$ and we have*

$$\|A_\varepsilon^{1/2}u(t)\|^2 \le \exp(-\nu C_{11}^{-2}\varepsilon^{-2}t/2)\widetilde{C}_1\varepsilon^p + 2C_{11}^2\widetilde{C}_2\nu^{-2}\varepsilon^{2+r}, \qquad t \ge 0.$$

Proof. [7] We set

$$R_0^2 = \widetilde{C}_1\varepsilon^p + 2C_{11}^2\widetilde{C}_2\nu^{-2}\varepsilon^{2+r}.$$

Since $R_0^2 \ge \|A_\varepsilon^{1/2}u_0\|^2$, it follows from Lemma 3.0 that there is a time $T^0 > 0$ such that

$$\|A_\varepsilon^{1/2}u(t)\|^2 \le 2R_0^2, \qquad 0 \le t < T^0. \tag{7.5}$$

Without loss of generality, we let $[0, T^0)$ denote the maximal time interval for which (7.5) is valid. If $T^0 < \infty$, then we must have

$$\|A_\varepsilon^{1/2}u(T^0)\|^2 = 2R_0^2. \tag{7.6}$$

By taking the scalar product of (2.5) with $A_\varepsilon u$ and using (7.2) and (7.4), we obtain, for $0 \le t \le T^0$,

$$\frac{d}{dt}\|A_\varepsilon^{1/2}u\|^2 + \nu\|A_\varepsilon u\|^2 \le \frac{1}{\nu}\|\mathbb{P}_\varepsilon f\|_\infty^2 + 2C_{12}C_{11}^{1/2}\varepsilon^{1/2}\|A_\varepsilon^{1/2}u\|\,\|A_\varepsilon u\|^2. \tag{7.7}$$

For $0 \le t \le T^0$, we have

$$2C_{12}C_{11}^{1/2}\varepsilon^{1/2}\|A_\varepsilon^{1/2}u\| \le 2\sqrt{2}C_{12}C_{11}^{1/2}\varepsilon^{1/2}(\widetilde{C}_1^{1/2}\varepsilon^{p/2} + \sqrt{2}C_{11}\widetilde{C}_2^{1/2}\nu^{-1}\varepsilon^{1+r/2}),$$

which goes to 0 as $\varepsilon \to 0^+$. Consequently there is a positive number ε_0 such that

$$2\sqrt{2}C_{12}C_{11}^{1/2}\varepsilon^{1/2}R_0 \le \frac{\nu}{2}. \tag{7.8}$$

For $0 < \varepsilon \le \varepsilon_0$, we deduce from (7.7), (7.8), and (7.2) that

$$\frac{d}{dt}\|A_\varepsilon^{1/2}u\|^2 + \frac{\nu\varepsilon^{-2}}{2C_{11}^2}\|A_\varepsilon^{1/2}u\|^2 \le \frac{1}{\nu}\|\mathbb{P}_\varepsilon f\|_\infty^2, \qquad 0 \le t \le T^0,$$

[7] This proof of Theorem 7 is, in fact, a small data argument.

which, by the Gronwall inequality implies that

$$\|A_\varepsilon^{1/2}u\|^2 \le \exp(-\nu C_{11}^{-2}\varepsilon^{-2}t/2)\|A_\varepsilon^{1/2}u_0\|^2 + \frac{2C_{11}^2\varepsilon^2}{\nu^2}\|\mathbb{P}_\varepsilon f\|_\infty^2,$$

or

$$\|A_\varepsilon^{1/2}u\|^2 \le \exp(-\nu C_{11}^{-2}\varepsilon^{-2}t/2)\widetilde{C}_1\varepsilon^p + \frac{2C_{11}^2\widetilde{C}_2}{\nu^2}\varepsilon^{2+r} \tag{7.9}$$

for $0 \le t \le T^0$. From (7.9), it follows that

$$\|A_\varepsilon^{1/2}u(T^0)\|^2 \le R_0^2 < 2R_0^2,$$

which contradicts (7.6). Therefore $T^0 = \infty$. □

Remark. Like in §4 (see Theorem 2), one can show that, under the hypotheses of Theorem 7, if moreover $\mathbb{P}_\varepsilon f$ belongs to

$$C^0([0,\infty), H_\varepsilon) \cap W^{1,\infty}((0,\infty), D(A_\varepsilon^{-1/2})),$$

then the solution $u(t)$ of (2.5) belongs to $C^0((0,\infty), V_\varepsilon^2)$. Let $S_\varepsilon(\mathbb{P}_\varepsilon f, t)u_0$ denote the strong solution of (2.5) with initial data $u_0 \in V_\varepsilon^1$, and let $\mathscr{B}_\varepsilon^1 = \{u \in V_\varepsilon^1;\ \|A_\varepsilon^{1/2}u\|^2 \le \widetilde{C}_1\varepsilon^p + 2C_{11}^2\widetilde{C}_2\nu^{-2}\varepsilon^{2+r}\}$. As in §4 (see Theorem 2) one can show that, under the assumptions of Theorem 7, if in addition, $f \in W(Q_3)$ is chosen so that $\mathbb{P}_\varepsilon f \in W^{1,\infty}((0,\infty); H_\varepsilon)$ and $H^+(f)$ is compact, then for any $\tau > 0$ there is a compact subset $K(\tau)$ of V_ε^2 such that

$$S_\varepsilon(\mathbb{P}_\varepsilon f, t)\mathscr{B}_\varepsilon^1 \subset K(\tau), \qquad t \ge \tau.$$

The results below are more interesting than Theorem 7. We recall that $\pi_\varepsilon(u_0, \mathbb{P}_\varepsilon f, \tau) = (S_\varepsilon(\mathbb{P}_\varepsilon f, \tau)u_0, (\mathbb{P}_\varepsilon f)_\tau)$ denote the skew-product semiflow generated by the strong solutions of (2.5).

Corollary 7.1. *Assume that the hypotheses of Theorem 7 hold and that $f \in W(Q_3)$ is chosen so that $\mathbb{P}_\varepsilon f$ belongs to $W^{1,\infty}((0,\infty), H_\varepsilon)$ and $H^+(f)$ is compact. Let $\varepsilon_0 > 0$ be given by Theorem 7. Then, for $0 < \varepsilon \le \varepsilon_0$, the skew-product semiflow $\pi_\varepsilon(\cdot, \mathbb{P}_\varepsilon f, \tau)$ has a unique maximal compact (local) attractor $\mathfrak{A}_\varepsilon$ included in $\mathscr{B}_\varepsilon^1 \times \omega(\mathbb{P}_\varepsilon f)$ which attracts $\mathscr{B}_\varepsilon^1 \times H^+(\mathbb{P}_\varepsilon f)$ in the space $V_\varepsilon^1 \times \mathbb{P}_\varepsilon W(Q_3)$. Futhermore,*

$$\mathfrak{A}_\varepsilon \subset \{u \in V_\varepsilon^1 : \|A_\varepsilon^{1/2}u\|^2 \le 2C_{11}^2\widetilde{C}_2\nu^{-2}\varepsilon^{2+r}\} \times \omega(\mathbb{P}_\varepsilon f).$$

Moreover, $\mathfrak{A}_\varepsilon$ is bounded and compact in $V_\varepsilon^2 \times \omega(\mathbb{P}_\varepsilon f)$ and attracts the bounded set $(\mathscr{B}_\varepsilon^1 \cap V_\varepsilon^2) \times H^+(\mathbb{P}_\varepsilon f)$ in the space $V_\varepsilon^2 \times \mathbb{P}_\varepsilon W(Q_3)$. Finally, the attractor $\mathfrak{A}_\varepsilon$ is the global attractor for the Leray solutions of (2.5).

Proof. The first part of this theorem is proved in the same way as Theorem 4. We will only give the argument that $\mathfrak{A}_\varepsilon$ is the global attractor for the Leray solutions of (2.5), i.e., the weak solutions of (2.5) that satisfy the energy estimate

$$\|u(t)\|^2 - \|u(0)\|^2 + \nu\int_0^t \|A_\varepsilon^{1/2}u(s)\|^2\,ds \le \frac{t}{\nu}\|A_\varepsilon^{-1/2}\mathbb{P}_\varepsilon f\|_\infty^2, \qquad t > 0. \tag{7.10}$$

From (7.10) and (7.2) we infer that

$$\|u(t)\|^2 - \|u(0)\|^2 + \nu \int_0^t \|A_\varepsilon^{1/2} u(s)\|^2\, ds \le \frac{C_{11}^2 \varepsilon^2}{\nu} t \|A_\varepsilon^{-1/2} \mathbb{P}_\varepsilon f\|_\infty^2,$$

which implies that

$$\frac{1}{t}\int_0^t \|A_\varepsilon^{1/2} u(s)\|^2\, ds \le \frac{\nu^{-1}}{t}\|u_0\|^2 + C_{11}^2 \widetilde{C}_2 \nu^{-2} \varepsilon^{2+r}, \qquad t > 0. \tag{7.11}$$

Therefore, for $T = C_{11}^{-2} \widetilde{C}_2^{-1} \nu \varepsilon^{-(2+r)} \|u_0\|^2$, there is a time t_0, $0 \le t_0 \le T$, such that

$$\|A_\varepsilon^{1/2} u(t_0)\|^2 \le 2 C_{11}^2 \widetilde{C}_2 \nu^{-2} \varepsilon^{2+r},$$

that is, $u(t_0)$ belongs to $\mathscr{B}_\varepsilon^1$ and, according to the proof of Theorem 7, $u(t)$ is regular for all $t \ge t_0$. This implies that $\mathfrak{A}_\varepsilon$ is the global attractor for the Leray solutions of (2.5). □

Corollary 7.2. *Assume that the hypotheses of Corollary* 7.1 *hold. Then we have*

$$\sup_{(u,h)\in\mathfrak{A}_\varepsilon} \|u\| \to 0 \quad \text{as } \varepsilon \to 0. \tag{7.12}$$

If, in addition, $r > -2$, *then the first components of the attractors* $\mathfrak{A}_\varepsilon$ *converge to* 0 *in* $D(A_\varepsilon^{1/2})$, *i.e.,*

$$\sup_{(u,h)\in\mathfrak{A}_\varepsilon} \|A_\varepsilon^{1/2} u\| \to 0 \quad \text{as } \varepsilon \to 0. \tag{7.13}$$

Proof. Property (7.13) is an obvious consequence of Theorem 7 and Corollary 7.1, and property (7.12) is a direct consequence of Corollary 7.1 and (7.2). Indeed, we have

$$\frac{d}{dt}\|u\|^2 + \frac{\nu\varepsilon^{-2}}{C_{11}^2}\|u\|^2 \le \frac{C_{11}^2 \varepsilon^2}{\nu^2}\|\mathbb{P}_\varepsilon f\|^2, \qquad t > 0,$$

which by Gronwall inequality implies that

$$\|u\|^2 \le (\exp(-\nu C_{11}^{-2} \varepsilon^{-2} t))\|u_0\|^2 + C_{11}^4 \widetilde{C}_2 \nu^{-3} \varepsilon^{4+r}, \qquad t \ge 0. \tag{7.14}$$

Now (7.12) follows from (7.14), the fact that $r > -3$, and the invariance of $\mathfrak{A}_\varepsilon$. □

8. Appendix: Proofs of auxiliary estimates

In this section we give the proof of the estimates (3.3) in the case of periodic boundary conditions and of the corresponding estimates in the case of other boundary conditions. We use $c_1, c_2, \ldots$ to denote constants which do not depend on ε for $0 < \varepsilon \le 1$.

8.1. **Periodic boundary conditions.** We will keep the notation of §§2 and 3. Let us begin with the following lemma.

Lemma 8.1. *For any q, $2 \le q \le 6$, there exist two positive constants c_1, and c_2, such that for any w satisfying $Mw = 0$, one has*

$$\|w\|_{L^q(Q_3)} \le c_1 \varepsilon^{2q^{-1}} (\|w\|_{H^1(Q_3)} + \varepsilon^{-1} \|D_3 w\|_{L^2(Q_3)}) \tag{8.1}$$

and

$$\|w\|_{L^q(Q_3)} \le c_2 \varepsilon^{2q^{-1}} \|A_\varepsilon^{1/2} w\| \tag{8.2}$$

for $0 < \varepsilon \le 1$.

Proof. Inequality (8.2) is a direct consequence of (8.1) and (2.17). In order to prove (8.1) we will use two inequalities from Hale and Raugel, (1992b, Lemma 4.1 and Proposition 4.2), which can be written (in the notation of §2) as

$$\|w\|_{L^2(Q_3)} \le c_3 \varepsilon (\|w\|_{H^1(Q_3)} + \varepsilon^{-1} \|D_3 w\|_{L^2(Q_3)}) \tag{8.3}$$

and

$$\|w\|_{L^6(Q_3)} \le c_4 \varepsilon^{1/3} (\|w\|_{H^1(Q_3)} + \varepsilon^{-1} \|D_3 w\|_{L^2(Q_3)}) \tag{8.4}$$

whenever $Mw = 0$. Inequality (8.1) is then obtained by interpolation between (8.3) and (8.4). (Note that inequality (8.1) could also be derived by replacing $q = 6$ in the proof of Hale and Raugel (1992b, Proposition 4.2) by any q, $2 \le q \le 6$.) □

The next step is to prove the following result.

Lemma 8.2. *There exist positive constants c_5, c_6, and c_7 such that for all $u^1 \in D(A_\varepsilon^{1/2})$, $u^2 \in D(A_\varepsilon)$, and $u^3 \in H_\varepsilon$ the following hold*:

(1) *If $Mu^1 = 0$, then*

$$|b_\varepsilon(u^1, u^2, u^3)| \le c_5 \varepsilon^{1/3} \|A_\varepsilon^{1/2} u^1\| \, \|A_\varepsilon^{1/2} u^2\|^{1/2} \|A_\varepsilon u^2\| \, \|u^3\|. \tag{8.5}$$

(2) *If $Mu^2 = 0$, then*

$$|b_\varepsilon(u^1, u^2, u^3)| \le c_6 \varepsilon^{1/6} \|A_\varepsilon^{1/2} u^1\| \, \|A_\varepsilon^{1/2} u^2\|^{1/2} \|A_\varepsilon u^2\| \, \|u^3\|. \tag{8.6}$$

(3) *If $Mu^1 = Mu^2 = 0$, then*

$$|b_\varepsilon(u^1, u^2, u^3)| \le c_7 \varepsilon^{1/2} \|A_\varepsilon^{1/2} u^1\| \, \|A_\varepsilon^{1/2} u^2\|^{1/2} \|A_\varepsilon u^2\| \, \|u^3\|. \tag{8.7}$$

Proof. Let us recall that

$$b_\varepsilon(u^1, u^2, u^3) = \sum_{i,j=1}^{3} \int_{Q_3} \varepsilon^{-\{i\}} u_i^1 D_i u_j^2 u_j^3 \, dx,$$

where $\{1\} = \{2\} = 0$ and $\{3\} = 1$. Using the Hölder inequality several times, we obtain

(8.8)

$$|b_\varepsilon(u^1, u^2, u^3)| \le \sum_{i,j=1}^{3} \|u_i^1\|_{L^6(Q_3)} \|\varepsilon^{-\{i\}} D_i u_j^2\|_{L^2(Q_3)}^{1/2} \|\varepsilon^{-\{i\}} D_i u_j^2\|_{L^6(Q_3)}^{1/2} \|u_j^3\|_{L^2(Q_3)}.$$

Assume now that $Mu^1 = 0$. Then we deduce from (2.17), (8.2), and (8.8) that

(8.9)

$$|b_\varepsilon(u^1, u^2, u^3)| \leq c_8 \varepsilon^{1/3} \|A_\varepsilon^{1/2} u^1\| \, \|A_\varepsilon^{1/2} u^2\|^{1/2} \|u^3\| \left(\sum_{i=1}^{3} \|\varepsilon^{-\{i\}} D_i u^2\|_{H^1(Q_3)} \right),$$

and (8.5) is now a direct consequence of (8.9) and (2.18).

Assume next that $Mu^2 = 0$. Then obviously, $MD_i u^2 = 0$ for $i = 1, 2$. Since u^2 is periodic with respect to the third variable, we also have $MD_3 u^2 = 0$. Therefore, we can apply inequality (8.1) to $w = \varepsilon^{-\{i\}} D_i u_j^2$ to obtain

(8.10)

$$\|\varepsilon^{-\{i\}} D_i u_j^2\|_{L^6(Q_3)}^{1/2} \leq c_9 \varepsilon^{1/6} (\|\varepsilon^{-\{i\}} D_i u_j^2\|_{H^1(Q_3)} + \varepsilon^{-1} \|\varepsilon^{-\{i\}} D_3 D_i u_j^2\|_{L^2(Q_3)})^{1/2}.$$

The estimate (8.6) is a direct consequence of (8.8), (8.10), and (2.18).

The case $Mu^1 = Mu^2 = 0$ is a combination of the above, and (8.7) is a straightforward consequence of the inequalities (8.2), (8.8), (8.10), (2.17), and (2.18). □

Note that (8.7) establishes the first inequality in (3.3). In order to prove the other two inequalities in (3.3), we need the following results.

Lemma 8.3. *The following statements are valid:*

(1) *For any real numbers r and θ, satisfying $2 \leq r \leq 6$, $\frac{1}{2} \leq \theta \leq 1$, and $r\theta - 6(1-\theta) > 0$, there exists a positive constant $c_{10} = c_{10}(r, \theta)$ such that, for any $w \in D(A_\varepsilon)$ with $Mw = 0$, and any $u^2 \in D(A_\varepsilon^{1/2})$ and any u^3 in H_ε, one has*

$$|b_\varepsilon(w, u^2, u^3)| \leq c_{10} \varepsilon^{2(1-\theta)/r} \|A_\varepsilon w\|^\theta \|A_\varepsilon^{1/2} w\|^{1-\theta} \|A_\varepsilon^{1/2} u^2\| \, \|u^3\|. \tag{8.11}$$

(2) *For any real number q, $2 < q \leq 6$, there exists a positive constant $c_{11} = c_{11}(q)$ such that, for any $w \in D(A_\varepsilon)$ with $Mw = 0$, and any $v \in \mathscr{R}(M) \cap D(A_\varepsilon^{1/2})$ and any $u \in H_\varepsilon$, we have*

$$|b_\varepsilon(v, w, u)| \leq c_{11} \varepsilon^{1/q} \|A_\varepsilon^{1/2} v\| \, \|A_\varepsilon^{1/2} w\|^{1/2} \|A_\varepsilon w\|^{1/2} \|u\|. \tag{8.12}$$

Proof. Using the inequalities (2.17) and the Cauchy-Schwarz inequality, we obtain

$$|b_\varepsilon(w, u^2, u^3)| \leq \sum_{i,j=1}^{3} \|w_i\|_{L^\infty(Q_3)} \|\varepsilon^{-\{i\}} D_i u_j^2\|_{L^2(Q_3)} \|u_j^3\|_{L^2(Q_3)},$$

or

$$|b_\varepsilon(w, u^2, u^3)| \leq c_{12} \|w\|_{L^\infty(Q_3)} \|A_\varepsilon^{1/2} u^2\| \, \|u^3\|. \tag{8.13}$$

It is well known that, for any $p > 3$, there exists a positive constant c_{12} such that

$$\|w\|_{L^\infty(Q_3)} \leq c_{12} \|w\|_{W^{1,p}(Q_3)}. \tag{8.14}$$

Now, using a Gagliardo-Nirenberg inequality (see Friedman (1964, Theorem 10.1) for instance), we obtain, for $\frac{1}{2} \le \theta \le 1$ and $r(\theta - 2 + \frac{6}{p}) = 6(1-\theta)$, that

$$\|w\|_{W^{1,p}(Q_3)} \le c_{13} \|w\|_{H^2(Q_3)}^{\theta} \|w\|_{L^r(Q_3)}^{1-\theta}, \tag{8.15}$$

where c_{13} is a positive constant depending only on r, θ, p. Combining the inequalities (8.14) and (8.15), we see that actually, for any real numbers r, θ, satisfying $2 \le r \le 6$, $\frac{1}{2} \le \theta \le 1$, and $r\theta - 6(1-\theta) > 0$, we have

$$\|w\|_{L^\infty(Q_3)} \le c_{14} \|w\|_{H^2(Q_3)}^{\theta} \|w\|_{L^r(Q_3)}^{1-\theta}, \tag{8.16}$$

where c_{14} is a positive constant depending only on r, θ. Now the estimate (8.11) is a direct consequence of the inequalities (8.13), (8.16), (8.2), and (2.18).

Let us now prove the estimate (8.12). Using a Hölder inequality, we obtain, for any $1 < \tilde{q} \le 3$, that

(8.17)

$$|b_\varepsilon(v, w, u)| \le \sum_{i,j=1}^{3} \|v\|_{L^{4p}(Q_3)} \|\varepsilon^{-\{i\}} D_i w\|_{L^{2\tilde{q}}(Q_3)}^{1/2} \|\varepsilon^{-\{i\}} D_i w\|_{L^2(Q_3)}^{1/2} \|u_j\|_{L^2(Q_3)},$$

where $p = \frac{\tilde{q}}{\tilde{q}-1}$. Let us point out that the inequality (8.17) has a meaning since the vector v depends only on the variables x_1, x_2 and therefore belongs to any space $L^{4p}(Q_3)$, $\frac{1}{4} \le p < +\infty$, as soon as it belongs to $H^1(Q_3)$. As in the proof of Proposition 8.2, we remark that $MD_i w = 0$, $i = 1, 2, 3$; whence we may apply the inequality (8.1) to $w = \varepsilon^{-\{i\}} D_i w$. Using the estimate (2.18) in addition we obtain

$$|b_\varepsilon(v, w, u)| \le c_{15} \varepsilon^{1/2\tilde{q}} \|A_\varepsilon^{1/2} v\| \, \|A_\varepsilon w\|^{1/2} \|A_\varepsilon^{1/2} w\|^{1/2} \|u\|, \tag{8.18}$$

where c_{15} is a positive constant depending only on $\tilde{q}$. By replacing $2\tilde{q}$ with q we see that (8.12) follows from (8.18).

The second estimate in (3.3) is simply the estimate (8.11) in the particular case where $r = 6$, $\theta = \frac{17}{32}$. Likewise the third estimate (3.3) is derived from (8.12) by choosing $q = 4$. □

8.2. **Other boundary conditions.** In the proofs of §8.1, we never used the fact that the boundary conditions on $\partial Q_2 \times (0, 1)$ were periodic ones. In particular, the estimate (8.1) is independent of the boundary conditions. Therefore, by using (2.17) and (2.18), one easily checks that Lemma 8.1 and Propositions 8.2 and 8.3 still hold if we replace the periodic boundary conditions on ∂Q_3 by homogeneous Dirichlet boundary conditions on $\partial Q_2 \times (0, 1)$ and periodic boundary conditions on $(Q_2 \times \{0\}) \cup (Q_2 \times \{1\})$. Hence the estimates (3.3) are still true in this case.

Finally, let us consider the case where we have homogeneous Dirichlet boundary conditions on ∂Q_3. Arguing as in Hale and Raugel (1992b, Lemma 6.1) and in Lemma 8.1, one can prove the following result:

Lemma 8.4. *For any q, $2 \le q \le 6$, there exists a positive constant c_{16} such that, for any $u \in H^1(Q_3)$ with $u = 0$ on $(Q_2 \times \{0\}) \cup (Q_2 \times \{1\})$, one has*

$$\|u\|_{L^q(Q_3)} \le c_{16} \varepsilon^{2/q} (\|u\|_{H^1(Q_3)} + \varepsilon^{-1} \|D_3 u\|_{L^2(Q_3)}). \tag{8.19}$$

This lemma enables us to prove the following result.

Lemma 8.5. *There exists a positive constant* c_{17} *such that, for* $u^1 \in D(A_\varepsilon^{1/2})$, $u^2 \in D(A_\varepsilon)$ *and* $u^2 \in H_\varepsilon$, *one has*

$$\begin{aligned}|b_\varepsilon(u^1, u^2, u^3)| &\le c_{17}\varepsilon^{1/2}\|A_\varepsilon^{1/2}u^1\|\,\|A_\varepsilon^{1/2}u^2\|^{1/2}\|u^3\| \\ &\times\left(\sum_{i=1}^{3}\|\varepsilon^{-\{i\}}D_iu^2\| + \varepsilon^{-1}\|\varepsilon^{-\{i\}}D_3D_iu^2\|\right)^{1/2}.\end{aligned} \tag{8.20}$$

Proof. From (8.8), (8.19), and (2.17), we deduce that

$$|b_\varepsilon(u^1, u^2, u^3)| \le c_{18}\varepsilon^{1/3}\|A_\varepsilon^{1/2}u^1\|\,\|u^3\|\,\|A_\varepsilon^{1/2}u^2\|^{1/2}\left(\sum_{i,j=1}^{3}\|\varepsilon^{-\{i\}}D_iu_j^2\|_{L^6(Q_3)}^{1/2}\right). \tag{8.21}$$

It remains to estimate $\|\varepsilon^{-\{i\}}D_iu_j^2\|_{L^6(Q_3)}^{1/2}$ for $1 \le i, j \le 3$. Since D_iu_j is equal to zero on $(Q_2 \times \{0\}) \cup (Q_2 \times \{1\})$ if $i = 1, 2$ and since $MD_3u_j = 0$, for $1 \le j \le 3$, one can apply Lemmas 8.4 and 8.1 to D_iu_j, $i = 1, 2$, and D_3u_j, respectively, for $1 \le j \le 3$. From (8.1), (8.18), and (8.21), we at once infer the estimate (8.20). □

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ABSTRACT. We examine the Navier-Stokes equations (NS) on a thin 3-dimensional domain $\Omega_\varepsilon = Q_2 \times (0, \varepsilon)$, where Q_2 is a suitable bounded domain in $\mathbb{R}^2$ and ε is a small, positive, real parameter. We consider these equations with various homogeneous boundary conditions, especially spatially periodic boundary conditions. We show that there are *large* sets $\mathscr{R}(\varepsilon)$ in $H^1(\Omega_\varepsilon)$ and $\mathscr{S}(\varepsilon)$ in $W^{1,\infty}((0, \infty), L^2(\Omega_\varepsilon))$ such that if $U_0 \in \mathscr{R}(\varepsilon)$ and $F \in \mathscr{S}(\varepsilon)$, then (NS) has a strong solution $U(t)$ that remains in $H^1(\Omega_\varepsilon)$ for all $t \geq 0$ and in $H^2(\Omega_\varepsilon)$ for all $t > 0$. We show that the set of strong solutions of (NS) has a local attractor $\mathfrak{A}_\varepsilon$ in $H^1(\Omega_\varepsilon)$, which is compact in $H^2(\Omega_\varepsilon)$. Furthermore, this local attractor $\mathfrak{A}_\varepsilon$ turns out to be the global attractor for all the weak solutions (in the sense of Leray) of (NS). We also show that, under reasonable assumptions, $\mathfrak{A}_\varepsilon$ is upper semicontinuous at $\varepsilon = 0$.

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