TENSOR STRUCTURES ARISING FROM AFFINE LIE ALGEBRAS. III

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This paper is a part of the series [KL]; however, it can be read independently of the first two parts.

In [D3], Drinfeld proved the existence of an equivalence between a tensor category of representations of a quantum group over $\mathbb{C}[[\varpi]]$ and a tensor category of representations of an undeformed enveloping algebra over $\mathbb{C}[[\varpi]]$, in which the associativity constraints are given by the Knizhnik-Zamolodchikov equations. However, Drinfeld's proof was nonconstructive, in the sense that he showed that the set of equivalences between the two tensor categories was nonempty, without describing an equivalence explicitly.

Our main result is an explicit construction of an equivalence between the tensor categories above. Its main virtue is, as we will show in Part IV, that it "extends analytically" to the case where ϖ is no longer a formal variable (where one of the two tensor categories becomes the category $\mathcal{O}_\kappa$ for an affine Lie algebra as in §2, and the other one becomes a category of representations of a quantum group for a parameter related to κ).

The relation between Drinfeld's result and ours could be perhaps compared with the relation between Tits's proof of the existence of an isomorphism between the group algebra of a Weyl group and the corresponding Iwahori-Hecke algebra, on the one hand, and the explicit construction [L1] of such an isomorphism, on the other hand.

The tensor categories of interest to us are defined in §§19 and 24. In §§20 and 22 we introduce three kinds of basic morphisms $T_{a,b}$, S_c , $\tau_{i;a,b}$ involving tensor products of "Weyl modules", and we prove various identities relating them. The most complicated such identities are derived from the classical theory of hypergeometric functions. These basic morphisms are combined to define other morphisms in §§21 and 23. They are used in §25 to construct the equivalence of tensor categories.

19. DRINFELD'S CATEGORY $\mathcal{D}$

19.1. Notation. Since the present treatment will include the nonsimply laced case, our notation will diverge from that in §1.

We fix a finite set I , a Cartan matrix (a_{ij}) indexed by $I \times I$, and integers $\delta_i \geq 1$ ($i \in I$) such that $\delta_i a_{ij} = \delta_j a_{ji}$ for all i, j . Recall that the matrix $(\delta_i a_{ij})$ is positive definite. We denote by (b_{ij}) the matrix inverse to (a_{ij}) .

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Let $\mathbf{Z}^I$ be the set of all maps $I \rightarrow \mathbf{Z}$; we regard $\mathbf{Z}^I$ as a free abelian group with basis $(\omega_i)_{i \in I}$ where $\omega_i(j) = \delta_{ij}$. Let $\alpha_j = \sum_{i \in I} a_{ij} \omega_i \in \mathbf{Z}^I$.

Given $i \in I$, let $s_i: \mathbf{Z}^I \rightarrow \mathbf{Z}^I$ be the homomorphism given by $s_i(\omega_j) = \omega_j - \delta_{ij} \alpha_i$ for all $j \in I$. Let W be the (finite) subgroup of $\text{Aut}(\mathbf{Z}^I)$ generated by the s_i ($i \in I$). There is a unique element $w_0 \in W$ and a unique involution $i \mapsto \bar{i}$ of I such that $w_0(\omega_i) = -\omega_{\bar{i}}$ for all i . For $a \in \mathbf{Z}^I$, let $\bar{a} \in \mathbf{Z}^I$ be defined by $\bar{a}(i) = a(\bar{i})$.

For $a, b \in \mathbf{Z}^I$, we set $\langle a, b \rangle = \sum_{i,j} \delta_i b_{ij} a(i) b(j)$. We have $\langle a, b \rangle = \langle b, a \rangle = \langle \bar{a}, \bar{b} \rangle$.

Let $\mathbf{A} = \mathbf{C}[[\varpi]]$ where ϖ is an indeterminate.

Let $\mathfrak{g}$ be the Lie algebra over $\mathbf{A}$ defined by the generators e_i, f_i, h_i ($i \in I$) and the Serre relations

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i, & [h_i, e_j] &= a_{ij} e_j, & [h_i, f_j] &= -a_{ij} f_j, & [h_i, h_j] &= 0, \\ [e_i, [e_i, \dots [e_i, e_j] \dots]] &= [f_i, [f_i, \dots [f_i, f_j] \dots]] = 0, \end{aligned}$$

(where e_i, f_i appear $1 - a_{ij}$ times) if $i \neq j$.

Let $\mathfrak{U}$ be the enveloping algebra (over $\mathbf{A}$) of $\mathfrak{g}$.

For $g \in \mathbf{A}$ we write $g(0)$ for the complex number such that $g - g(0) \in \varpi \mathbf{A}$.

We fix a square root $\sqrt{-1}$ of -1 in $\mathbf{C}$ and we set

$$v = e^{\pi \sqrt{-1} \varpi} \in \mathbf{A}.$$

Given $i \in I$, we set

$$\varpi_i = \delta_i \varpi, \quad v_i = v^{\delta_i} \in \mathbf{A}.$$

Given $i \in I$ and an integer n , we set

$$[n]_i = \frac{v_i^n - v_i^{-n}}{v_i - v_i^{-1}}.$$

For any $z \in \mathbf{C}$, we set

$$v^z = e^{\pi \sqrt{-1} z \varpi} \in \mathbf{A}.$$

19.2. The element t . Unless otherwise specified, the symbol $\otimes$ stands for $\otimes_{\mathbf{A}}$.

There is a unique element $t \in \mathfrak{g} \otimes \mathfrak{g}$ such that

- (a) $(x \otimes 1 + 1 \otimes x)t = t(x \otimes 1 + 1 \otimes x)$ for all $x \in \mathfrak{g}$ (equality in $\mathfrak{U} \otimes \mathfrak{U}$) and
- (b) $t = \sum_{i,j} \delta_i b_{ij} h_i \otimes h_j + \sum_i \delta_i (e_i \otimes f_i + f_i \otimes e_i) + \dots$;

the dots stand for a sum of terms of form $c \otimes c'$ where either c is in a (nonsimple) positive root space of $\mathfrak{g}$ and c' is in the corresponding negative root space, or c' is in a (nonsimple) positive root space of $\mathfrak{g}$ and c is in the corresponding negative root space.

The image of t under the multiplication map $\mathfrak{U} \otimes \mathfrak{U} \rightarrow \mathfrak{U}$ is the Casimir operator $\Omega \in \mathfrak{U}$; conversely, we have

(c) $t = \frac{1}{2}(\Delta(\Omega) - 1 \otimes \Omega - \Omega \otimes 1)$, where $\Delta: \mathfrak{U} \rightarrow \mathfrak{U} \otimes \mathfrak{U}$ is the standard comultiplication.

19.3. The category $\mathcal{D}$. Let $\mathcal{D}$ be the category whose objects are $\mathfrak{U}$ -modules V which are free of finite rank as $\mathbf{A}$ -modules; the morphisms are the $\mathfrak{U}$ -linear maps.

For any $a \in \mathbf{N}^I$, the $\mathfrak{U}$ -module

$$V_a = \mathfrak{U} / \left(\sum_i \mathfrak{U} f_i^{a(i)+1} + \sum_i \mathfrak{U} e_i + \sum_i \mathfrak{U} (h_i - a(i)) \right)$$

is an (indecomposable) object of $\mathcal{D}$. Moreover,

(a) any indecomposable object of $\mathcal{D}$ is isomorphic to V_a for a unique $a \in \mathbf{N}^I$.

Let y_a be the image of 1 in V_a . Note that y_a generates V_a as a $\mathfrak{U}$ -module and that $e_i y_a = 0$, $h_i y_a = a(i) y_a$ for all $i \in I$.

If $V \in \mathcal{D}$, then as an $\mathbf{A}$ -module, we have $V = \bigoplus_{\lambda} V^{\lambda}$ where λ runs through $\mathbf{Z}^I$ and $V^{\lambda} = \{x \in V \mid h_i(x) = \lambda(i)x \text{ for all } i \in I\}$.

If V, V' are objects of $\mathcal{D}$, then $V \otimes V'$ is naturally an object of $\mathcal{D}$ (via Δ).

We identify $V_0 = \mathbf{A}$ so that y_0 corresponds to $1 \in \mathbf{A}$.

19.4. For any object $V \in \mathcal{D}$ and any $i \in I$, we denote by $\theta_i: V \rightarrow V$ the $\mathbf{A}$ -linear map given by

$$\theta_i(x) = \sum_{p, q, r \in \mathbf{N}; p+q+r=\lambda(i)} (-1)^q (p!q!r!)^{-1} f_i^p e_i^q f_i^r x$$

for all $x \in V^{\lambda}$. The linear maps $\theta_i: V \rightarrow V$ define a representation of the braid group corresponding to W on V (see for example [L3, 39.4.3]). Hence, if $i_1, i_2, \dots, i_N$ is a sequence in I such that $s_{i_1} s_{i_2} \cdots s_{i_N}$ is a reduced expression for w_0 , then the linear map $\theta = \theta_{i_1} \theta_{i_2} \cdots \theta_{i_N}: V \rightarrow V$ is a linear isomorphism independent of the choice of the sequence $i_1, i_2, \dots, i_N$. We shall need the following known properties of the map θ .

(a) For any $x \in V$ and any $i \in I$, we have $\theta(f_i x) = -e_i \theta(x)$.

(b) If $f: V \rightarrow V'$ is a morphism in $\mathcal{D}$, then f is compatible with the operators θ on V, V' .

(c) If V, V' are objects of $\mathcal{D}$, then the operator θ on $V \otimes V'$ is equal to $\theta \otimes \theta$. (The two factors are the operators θ on V and V' .)

Let $a \in \mathbf{N}^I$. We set $x_a = \theta(y_a)$ where $\theta: V_a \rightarrow V_a$ is as above. Note that $x_a \in V_a$ is a vector satisfying $f_i x_a = 0$ for all i and $h_i x_a = -a(i)x_a$ for all i . (In fact, x_a is characterized by these properties, together with the property that it belongs to the canonical basis of V_a that contains y_a , see [L3, 28.1.4, 39.1.2]; we shall not use this fact here.)

Lemma 19.5. (a) Let $a, b \in \mathbf{N}^I$. The $\mathfrak{U}$ -module $V_a \otimes V_b$ is generated by $x_a \otimes y_b$.

(b) Let $V, V' \in \mathcal{D}$. The $\mathfrak{U}$ -module $V \otimes V'$ is generated by vectors of the form $(f \otimes f')(x_a \otimes y_b)$ for various $a, b \in \mathbf{N}^I$ and various morphisms $f: V_a \rightarrow V, f': V_b \rightarrow V'$ in $\mathcal{D}$.

(c) Let $V, V' \in \mathcal{D}$. The $\mathfrak{U}$ -module $V \otimes V'$ is generated by vectors of the form $x = \sum_k x_k \otimes x'_k$ such that the set $\{x_k\}$ is contained in a $\mathfrak{U}$ -submodule of V

isomorphic to V_a , the set $\{x'_k\}$ is contained in a $\mathfrak{U}$ -submodule of V' isomorphic to V_b , and x is contained in a $\mathfrak{U}$ -submodule of $V \otimes V'$ isomorphic to V_c (where $a, b, c \in \mathbf{N}^J$).

The proof of (a) is immediate. To prove (b), we may assume by 19.3(a) that $V = V_a$, $V' = V_b$; in that case, (b) follows from (a). (c) follows from 19.3(a).

19.6. If V, V' are objects of $\mathscr{D}$, then $V \otimes V'$ is a $\mathfrak{U} \otimes \mathfrak{U}$ -module in an obvious way; hence $t \in \mathfrak{U} \otimes \mathfrak{U}$ acts on $V \otimes V'$.

If $x \in V \otimes V'$ is a vector as in 19.5(c), then

$$t(x) = \frac{1}{2}(\langle c, c+2 \rangle - \langle a, a+2 \rangle - \langle b, b+2 \rangle)x.$$

This follows from 19.2(c) together with the well-known fact that Ω acts on V_a as multiplication by $\langle a, a+2 \rangle$. Hence, by Lemma 19.5(c),

(a) the $\mathbf{A}$ -module $V \otimes V'$ is generated by vectors x such that $tx = \zeta x$ for some $\zeta \in \mathbf{C}$.

19.7. If $V = V_a$, $V' = V_b$, then using 19.2(b), we have:

$$\begin{aligned} \text{(a)} \quad t(y_a \otimes y_b) &= \sum_{k,l} \delta_k b_{kl} a(k) b(l) y_a \otimes y_b = \langle a, b \rangle y_a \otimes y_b; \\ \text{(b)} \quad t(f_i y_a \otimes y_b) &= \sum_{k,l} \delta_k b_{kl} (a(k) - a_{ik}) b(l) f_i y_a \otimes y_b + \delta_i a(i) y_a \otimes f_i y_b \\ &= (\langle a, b \rangle - \delta_i b(i)) f_i y_a \otimes y_b + \delta_i a(i) y_a \otimes f_i y_b; \\ \text{(c)} \quad t(y_a \otimes f_i y_b) &= \sum_{k,l} \delta_k b_{kl} a(k) (b(l) - a_{il}) y_a \otimes f_i y_b + \delta_i b(i) f_i y_a \otimes y_b \\ &= (\langle a, b \rangle - \delta_i a(i)) y_a \otimes f_i y_b + \delta_i b(i) f_i y_a \otimes y_b. \end{aligned}$$

19.8. **Formal differential equations and asymptotics.** Let V be a free $\mathbf{A}$ -module of finite rank. A function f on the open real interval $(0, 1)$, with values in V , is said to be analytic if for some (or any) $\mathbf{C}$ -subspace ${}^0V \subset V$ such that $V = {}^0V \otimes_{\mathbf{C}} \mathbf{A}$, we have $f(z) = \sum_{n \geq 0} f_n(z) \varpi^n$ where $f_n: (0, 1) \rightarrow {}^0V$ is analytic for all $n \geq 0$. Then $\frac{df}{dz} := \sum_{n \geq 0} \frac{df_n(z)}{dz} \varpi^n$ is a well-defined analytic function.

Assume that we are given two endomorphisms $\Pi_0, \Pi_1: V \rightarrow V$. Following Drinfeld [D3], we consider the set $\mathscr{X}$ consisting of all analytic functions f on the open real interval $(0, 1)$ with values in V which are solutions of the differential equation

$$\text{(a)} \quad \frac{df}{dz} = \frac{\varpi \Pi_0 f}{z} + \frac{\varpi \Pi_1 f}{z-1}.$$

Note that $\mathscr{X}$ is in a natural way an $\mathbf{A}$ -module.

Let $f \in \mathscr{X}$. We set $\tilde{f}(z) = z^{-\varpi \Pi_0} f(z)$ where, by definition,

$$z^{-\varpi \Pi_0} = \sum_{n \geq 0} (-\log z)^n \varpi^n \Pi_0^n / n! \quad \text{for } z \in (0, 1).$$

Then $\tilde{f}$ satisfies the differential equation $\frac{d\tilde{f}}{dz} = \varpi(z-1)^{-1}z^{-\varpi\Pi_0}\Pi_1z^{\varpi\Pi_0}\tilde{f}$. If we write (with 0V as above) $\tilde{f} = \sum_{n \geq 0} \varpi^n \tilde{f}_n$ with $\tilde{f}_n: (0, 1) \rightarrow {}^0V$, the previous equation is equivalent to the sequence of equations

$$(b) \quad \frac{d\tilde{f}_n}{dz} = (z-1)^{-1} \sum_{k, k' \geq 0; k+k' \leq n-1} (-\log z)^{k'} \rho_{k', k} \tilde{f}_k$$

for $n = 0, 1, 2, \dots$. Here $\rho_{k', k}$ are endomorphisms of 0V depending only on Π_0, Π_1 .

We show by induction on n that $\tilde{f}_n$ extends to a continuous function $[0, 1) \rightarrow {}^0V$. This is clear for $n = 0$ since $\tilde{f}_0$ is a constant function. Assume that $n \geq 1$. Using the induction hypothesis and the fact that $\int_0^{1/2} (-\log z)^p dz < \infty$ for $p = 0, 1, 2, \dots$, we see from (b) that $\tilde{f}_n$ has the required property. Hence $\lim_{z \rightarrow 0} \tilde{f}_n(z) \in {}^0V$ is well defined.

By definition, $\lim_{z \rightarrow 0} \tilde{f} = \sum_{n \geq 0} \varpi^n \lim_{z \rightarrow 0} \tilde{f}_n(z) \in V$. Setting $R_0(f) = \lim_{z \rightarrow 0} \tilde{f}$, we obtain a well-defined map $R_0: \mathcal{X} \rightarrow V$. It is obvious that R_0 is an isomorphism of $\mathbf{A}$ -modules. Similarly, if $f \in \mathcal{X}$, then $\lim_{z \rightarrow 1} (1-z)^{-\varpi\Pi_1} f$ is a well-defined element of V and setting $R_1(f) = \lim_{z \rightarrow 1} (1-z)^{-\varpi\Pi_1} f$, we obtain an isomorphism of $\mathbf{A}$ -modules $R_1: \mathcal{X} \rightarrow V$.

Let $R(\Pi_0, \Pi_1): V \rightarrow V$ be the isomorphism of $\mathbf{A}$ -modules given by $R_1 R_0^{-1}$.

19.9. If M is an $\mathbf{A}$ -module and $m, m' \in M$, we shall write $m = m' \bmod \varpi$ instead of $m - m' \in \varpi M$.

For any $m \in V$, we have

$$(a) \quad R(\Pi_0, \Pi_1)m = m \bmod \varpi.$$

This follows from the fact that, if $f = \sum_n \varpi^n f_n$ is a solution of 19.8(a) (with $f_n: (0, 1) \rightarrow {}^0V$), then f_0 is constant.

If $x \in V$ is such that $\Pi_0(x) = \alpha x$, $\Pi_1(x) = \beta x$ for some $\alpha \in \mathbf{C}$, $\beta \in \mathbf{C}$, then

$$(b) \quad R(\Pi_0, \Pi_1)x = x.$$

Indeed, if f is defined by $z \mapsto z^{\varpi\alpha}(1-z)^{\varpi\beta}x$, then clearly $f \in \mathcal{X}$ and we have $R_0(f) = x$, $R_1(f) = x$; (b) follows.

Let V' be another free $\mathbf{A}$ -module of finite rank with two given endomorphisms Π'_0, Π'_1 and let $q: V \rightarrow V'$ be an $\mathbf{A}$ -linear map such that $q\Pi_0 = \Pi'_0 q$ and $q\Pi_1 = \Pi'_1 q$. Using the definitions, we see that

$$(c) \quad qR(\Pi_0, \Pi_1) = R(\Pi'_0, \Pi'_1)q \quad \text{as maps } V \rightarrow V'.$$

19.10. **The associativity isomorphisms.** Let V_1, V_2, V_3 be three objects of $\mathcal{D}$.

By the construction of 19.8 applied to $V = V_1 \otimes V_2 \otimes V_3$ and to the linear maps $\Pi_0 = t_{12}: V \rightarrow V$, $\Pi_1 = t_{23}: V \rightarrow V$, we obtain an isomorphism

$$R(\Pi_0, \Pi_1): V_1 \otimes V_2 \otimes V_3 \rightarrow V_1 \otimes V_2 \otimes V_3$$

that will be denoted by A_{V_1, V_2, V_3} and will be called the *associativity isomorphism*. Its inverse, A_{V_1, V_2, V_3}^{-1} is also called an associativity isomorphism. (Here,

$$\begin{aligned} t_{12}(x_1 \otimes x_2 \otimes x_3) &= \sum c' x_1 \otimes c'' x_2 \otimes x_3, \\ t_{23}(x_1 \otimes x_2 \otimes x_3) &= \sum x_1 \otimes c' x_2 \otimes c'' x_3 \end{aligned}$$

where $t = \sum c' \otimes c''$.)

Using 19.2(a) and 19.9(c), we see that

(a) A_{V_1, V_2, V_3} is a morphism in $\mathcal{D}$ and that it is functorial in V_1, V_3, V_3 .

Moreover,

(b) if one of V_1, V_2, V_3 is $V_0 = \mathbf{A}$, then A_{V_1, V_2, V_3} is the identity map.

Indeed, in this case, one of the maps t_{12}, t_{23} is zero and to the other one we may apply 19.6(a). We see that $V_1 \otimes V_2 \otimes V_3$ is generated by vectors x such that $t_{12}x = \alpha x, t_{23}x = \beta x$ for some $\alpha, \beta \in \mathbf{C}$. For such vectors we have $A_{V_1, V_2, V_3}(x) = x$, by 19.9(b); (b) follows.

Let V_1, V_2, V_3, V_4 be four objects of $\mathcal{D}$. The isomorphisms

$$B_{V_1, V_2, V_3, V_4} := (1 \otimes A_{V_2, V_3, V_4}^{-1}) A_{V_1, V_2, V_3 \otimes V_4},$$

$$B'_{V_1, V_2, V_3, V_4} := (A_{V_1, V_2, V_3} \otimes 1) A_{V_1 \otimes V_2, V_3, V_4}^{-1}$$

of $V_1 \otimes V_2 \otimes V_3 \otimes V_4$ with itself will be used many times in the sequel.

19.11. The pentagon identity. Given V_1, V_2, V_3, V_4 in $\mathcal{D}$, we have

$$A_{V_1, V_2, V_3 \otimes V_4} A_{V_1 \otimes V_2, V_3, V_4} = (A_{V_1, V_2, V_3} \otimes 1) A_{V_1, V_2 \otimes V_3, V_4} (1 \otimes A_{V_2, V_3, V_4})$$

as isomorphisms of $V_1 \otimes V_2 \otimes V_3 \otimes V_4$ with itself. (For a proof, see [D3].)

19.12. The commutativity isomorphism. Let V_1, V_2 be two objects of $\mathcal{D}$. Recall that t may be regarded as a linear map $V_2 \otimes V_1 \rightarrow V_2 \otimes V_1$. Hence $e^{\pi\sqrt{-1}\varpi t} =: \sum_{n \geq 0} \varpi^n (\pi\sqrt{-1}t)^n / n!$ is a well-defined (invertible) linear map $V_2 \otimes V_1 \rightarrow V_2 \otimes V_1$.

We define an isomorphism of $\mathbf{A}$ -modules $C_{V_1, V_2}: V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$ by $C_{V_1, V_2}(x_1 \otimes x_2) = e^{\sqrt{-1}\pi\varpi t}(x_2 \otimes x_1)$ for all $x_1 \in V_1, x_2 \in V_2$.

Using 19.2(a), we see that C_{V_1, V_2} is a morphism in $\mathcal{D}$; we call it the *commutativity isomorphism*. It is clearly functorial in V_1 and V_2 .

19.13. Hexagon identities. Let V_1, V_2, V_3 be objects of $\mathcal{D}$. The composition

$$\begin{aligned} V_1 \otimes V_2 \otimes V_3 &\xrightarrow{A_{V_1, V_2, V_3}} V_1 \otimes V_2 \otimes V_3 \xrightarrow{1 \otimes C_{V_2, V_3}} V_1 \otimes V_3 \otimes V_2 \\ &\xrightarrow{A_{V_1, V_3, V_2}^{-1}} V_1 \otimes V_3 \otimes V_2 \xrightarrow{C_{V_1, V_3} \otimes 1} V_3 \otimes V_1 \otimes V_2 \xrightarrow{A_{V_3, V_1, V_2}} V_3 \otimes V_1 \otimes V_2 \end{aligned}$$

coincides with $C_{V_1 \otimes V_2, V_3}$. The composition

$$\begin{aligned} V_1 \otimes V_2 \otimes V_3 &\xrightarrow{A_{V_1, V_2, V_3}} V_1 \otimes V_2 \otimes V_3 \xrightarrow{C_{V_1, V_2} \otimes 1} V_2 \otimes V_1 \otimes V_3 \\ &\xrightarrow{A_{V_2, V_1, V_3}} V_2 \otimes V_1 \otimes V_3 \xrightarrow{1 \otimes C_{V_1, V_3}} V_2 \otimes V_3 \otimes V_1 \xrightarrow{A_{V_2, V_3, V_1}^{-1}} V_2 \otimes V_3 \otimes V_1 \end{aligned}$$

coincides with $C_{V_1, V_2 \otimes V_3}$. (For a proof, see [D3].)

The category $\mathcal{D}$ with its tensor product $V, V' \mapsto V \otimes V'$ and with the associativity and commutativity isomorphisms described above is called *Drinfeld's category* $\mathcal{D}$.

19.14. To simplify notation, we shall write $A_{a,b,c}, A_{a \otimes b, c, d}, A_{a, b \otimes c, d}, C_{a,b}, C_{a \otimes b, c}, B_{a,b,c,d}, B_{a \otimes b, c, d, e}, B'_{a,b,c,d}$, etc., instead of $A_{V_a, V_b, V_c}, A_{V_a \otimes V_b, V_c, V_d}, A_{V_a, V_b \otimes V_c, V_d}, C_{V_a, V_b}, C_{V_a \otimes V_b, V_c}, B_{V_a, V_b, V_c, V_d}, B_{V_a \otimes V_b, V_c, V_d, V_e}, B'_{V_a, V_b, V_c, V_d}$, etc.

20. THE MORPHISMS $T_{a,b}$ AND S_c

20.1. **The morphism $T_{a,b}$.** Let $a, b \in \mathbf{N}^I$. The vector $y = y_a \otimes y_b \in V_a \otimes V_b$ satisfies $e_i y = 0$ and $h_i y = (a(i) + b(i))y$ for all $i \in I$; hence there is a unique morphism $T_{a,b}: V_{a+b} \rightarrow V_a \otimes V_b$ in $\mathcal{D}$ such that $T_{a,b}(y_{a+b}) = y_a \otimes y_b$. We have

$$(a) \quad T_{a,b}(x_{a+b}) = x_a \otimes x_b.$$

Indeed, using 19.4(b),(c), we have

$$T_{a,b}(x_{a+b}) = T_{a,b}(\theta(y_{a+b})) = \theta(T_{a,b}(y_{a+b})) = \theta(y_a \otimes y_b) = \theta y_a \otimes \theta y_b = x_a \otimes x_b.$$

Proposition 20.2 (*TT-identity*). Let $a, b, c \in \mathbf{N}^I$. Consider the compositions

$$(P) \quad V_{a+b+c} \xrightarrow{T_{a+b,c}} V_{a+b} \otimes V_c \xrightarrow{T_{a,b} \otimes 1} V_a \otimes V_b \otimes V_c,$$

$$(Q) \quad V_{a+b+c} \xrightarrow{T_{a,b+c}} V_a \otimes V_{b+c} \xrightarrow{1 \otimes T_{b,c}} V_a \otimes V_b \otimes V_c.$$

We have $Q = A_{a,b,c}P$.

Since $Q, A_{a,b,c}P$ are morphisms in $\mathcal{D}$ and y_{a+b+c} generates V_{a+b+c} as a $\mathfrak{U}$ -module, it suffices to show that $Q(y_{a+b+c}) = A_{a,b,c}P(y_{a+b+c})$. From the definitions, we have $Q(y_{a+b+c}) = P(y_{a+b+c}) = y_a \otimes y_b \otimes y_c$. Hence it suffices to show that $A_{a,b,c}(y_a \otimes y_b \otimes y_c) = y_a \otimes y_b \otimes y_c$.

This follows from 19.9(b), since, by 19.7(a), we have

$$t_{12}(y_a \otimes y_b \otimes y_c) = \langle a, b \rangle y_a \otimes y_b \otimes y_c,$$

$$t_{23}(y_a \otimes y_b \otimes y_c) = \langle b, c \rangle y_a \otimes y_b \otimes y_c.$$

The proposition is proved.

Proposition 20.3 (*TC-identity*). Let $a, b \in \mathbf{N}^I$. The composition

$$V_{a+b} \xrightarrow{T_{b,a}} V_b \otimes V_a \xrightarrow{C_{b,a}} V_a \otimes V_b$$

coincides with $v^{(a,b)}T_{a,b}$.

The composition takes y_{a+b} to

$$e^{\pi\sqrt{-1}\omega t}(y_a \otimes y_b) = v^{(a,b)}y_a \otimes y_b = v^{(a,b)}T_{a,b}(y_{a+b}).$$

The proposition follows since y_{a+b} generates V_{a+b} as a $\mathfrak{U}$ -module.

20.4. The morphism S'_c . The following statement is easily verified: given $c \in \mathbf{N}^I$, there is a unique morphism $S'_c: V_c \otimes V_c \rightarrow V_0 = \mathbf{A}$ in $\mathcal{D}$ such that $S'_c(x_c \otimes y_c) = 1$.

Lemma 20.5. *Let M, M' be two objects of $\mathcal{D}$ and let $c \in \mathbf{N}^I$. The compositions*

$$M \otimes V_c \otimes V_c \otimes M' \longrightarrow M \otimes V_c \otimes V_c \otimes M' \xrightarrow{1 \otimes S'_c \otimes 1} M \otimes M',$$

where the first map is either $B_{M, V_c, V_c, M'}$ or $B'_{M, V_c, V_c, M'}$ (see 19.10), coincide.

Using the pentagon identity, we are reduced to checking the equality

$$(1 \otimes S'_c \otimes 1)A_{M, V_c \otimes V_c, M'} = (1 \otimes S'_c \otimes 1).$$

This follows from the equality

$$(1 \otimes S'_c \otimes 1)A_{M, V_c \otimes V_c, M'} = A_{M, \mathbf{A}, M'}(1 \otimes S'_c \otimes 1)$$

(functoriality of the associativity isomorphisms 19.10(a)) and from the equality $A_{M, \mathbf{A}, M'} = 1$ (see 19.10(b)). The lemma is proved.

20.6. Let $a, b \in \mathbf{N}^I$. We consider the composition

$$(a) \quad V_{a+\bar{b}} \otimes V_{b+a} \xrightarrow{T_{a, \bar{b}} \otimes T_{b, a}} V_a \otimes V_{\bar{b}} \otimes V_b \otimes V_a \xrightarrow{(1 \otimes S'_b \otimes 1)B} V_a \otimes V_a \xrightarrow{S'_a} \mathbf{A},$$

where B is either $B_{a, \bar{b}, b, a}$ or $B'_{a, \bar{b}, b, a}$. (The composition is the same in both cases, by 20.5.)

Let $g_{a, b} \in \mathbf{A}$ be the image of $x_{a+\bar{b}} \otimes y_{b+a}$ under the composition (a).

Since the composition (a) is a morphism in $\mathcal{D}$ and $x_{a+\bar{b}} \otimes y_{b+a}$ generates $V_{a+\bar{b}} \otimes V_{b+a}$ as a $\mathfrak{U}$ -module (see 19.5(a)), it follows that

(b) the composition (a) coincides with $g_{a, b} S'_{a+b}$.

Lemma 20.7. (a) *We have $g_{a, b}(0) = 1$.*

(b) *If $a = 0$ or $b = 0$, then $g_{a, b} = 1$.*

The image of $x_{a+\bar{b}} \otimes y_{b+a}$ under the composition 20.6(a), modified by replacing B by the identity map, is equal to 1 (by 20.1(a) and the definitions). Since the associativity isomorphisms (and hence B) are the identity modulo ϖ (see 19.9(a)), it follows that the image of $x_{a+\bar{b}} \otimes y_{b+a}$ under the composition 20.6(a) is equal to 1 mod ϖ . This proves (a).

Assume now that we have $a = 0$ or $b = 0$. Then the map B in 20.6(a) is the identity map, by 19.10(b). As in the beginning of the proof, we see that $g_{a, b} = 1$. The lemma is proved.

Lemma 20.8. *Let $a, b, c \in \mathbf{N}^I$. Then $g_{a+b, c} g_{a, b} = g_{a, b+c} g_{b, c}$.*

We consider the commutative diagram

$$\begin{array}{ccc}
 V_{\bar{a}+\bar{b}+\bar{c}} \otimes V_{c+b+a} & \xrightarrow{f_1} & V_{\bar{a}} \otimes V_{\bar{b}+\bar{c}} \otimes V_{c+b} \otimes V_a \\
 f_2 \downarrow & & f_3 \downarrow \\
 V_{\bar{a}+\bar{b}} \otimes V_{\bar{c}} \otimes V_c \otimes V_{b+a} & \xrightarrow{f_4} & V_{\bar{a}} \otimes V_{\bar{b}} \otimes V_{\bar{c}} \otimes V_c \otimes V_b \otimes V_a \\
 s_1 \downarrow & & s_2 \downarrow \\
 V_{\bar{a}+\bar{b}} \otimes V_{b+a} & \xrightarrow{f_5} & V_{\bar{a}} \otimes V_{\bar{b}} \otimes V_b \otimes V_a \\
 & & s_3 \downarrow \\
 \mathbf{A} & \xleftarrow{s_4} & V_{\bar{a}} \otimes V_a
 \end{array}$$

where

$$\begin{aligned}
 f_1 &= T_{\bar{a}, \bar{b}+\bar{c}} \otimes T_{c+b, a}, & f_2 &= T_{\bar{a}+\bar{b}, \bar{c}} \otimes T_{c, b+a}, & f_3 &= 1 \otimes T_{\bar{b}, \bar{c}} \otimes T_{c, b} \otimes 1, \\
 f_4 &= T_{\bar{a}, \bar{b}} \otimes 1 \otimes 1 \otimes T_{b, a}, & f_5 &= T_{\bar{a}, \bar{b}} \otimes T_{b, a}, \\
 s_1 &= 1 \otimes S'_c \otimes 1, & s_2 &= 1 \otimes 1 \otimes S'_c \otimes 1 \otimes 1, & s_3 &= 1 \otimes S'_b \otimes 1, & s_4 &= S'_a.
 \end{aligned}$$

By 20.6(b), we have

$$g_{a+b, c} S'_{a+b+c} = S'_{a+b} s_1 A_1 f_2 \quad \text{and} \quad g_{a, b} S'_{a+b} = s_4 s_3 A_2 f_5,$$

where $A_1 = B_{\bar{a}+\bar{b}, \bar{c}, c, b+a}$ and $A_2 = B_{\bar{a}, \bar{b}, b, a}$. Hence,

$$g_{a+b, c} g_{a, b} S'_{a+b+c} = s_4 s_3 A_2 f_5 s_1 A_1 f_2.$$

We have $f_5 s_1 = s_2 f_4$; hence

$$g_{a+b, c} g_{a, b} S'_{a+b+c} = s_4 s_3 A_2 s_2 f_4 A_1 f_2.$$

Similarly, by 20.6(b), we have

$$g_{a, b+c} S'_{a+b+c} = s_4 (1 \otimes S'_{b+c} \otimes 1) A_3 f_1, \quad g_{b, c} (1 \otimes S'_{b+c} \otimes 1) = s_3 s_2 A_4 f_3,$$

where $A_3 = B_{\bar{a}, \bar{b}+\bar{c}, c+b, a}$ and $A_4 = 1 \otimes B_{\bar{b}, \bar{c}, c, b} \otimes 1$. Hence

$$g_{a, b+c} g_{b, c} S'_{a+b+c} = s_4 s_3 s_2 A_4 f_3 A_3 f_1.$$

Let A_5, A_6, A_7, A_8 be the isomorphisms of $V_{\bar{a}} \otimes V_{\bar{b}} \otimes V_{\bar{c}} \otimes V_c \otimes V_b \otimes V_a$ onto itself given by

$$\begin{aligned}
 A_5 &= B_{\bar{a}, \bar{b} \otimes \bar{c}, c \otimes b, a}, & A_6 &= B_{\bar{a} \otimes \bar{b}, \bar{c}, c, b \otimes a}, \\
 A_7 &= (1 \otimes A_{\bar{b}, \bar{c} \otimes c \otimes b, a}^{-1}) (1 \otimes 1 \otimes A_{\bar{c} \otimes c, b, a}^{-1}) A_{\bar{a}, \bar{b}, \bar{c} \otimes c \otimes b \otimes a}, \\
 A_8 &= A_{\bar{a}, \bar{b}, \bar{c}} \otimes A_{c, b, a}^{-1}.
 \end{aligned}$$

By the functoriality of the associativity isomorphisms, we have $f_3 A_3 = A_5 f_3$, $f_4 A_1 = A_6 f_4$, and $A_2 s_2 = s_2 A_7$ (for the last equality we are also using the fact

that $A_{0,b,a}^{-1} = 1$, see 19.10(b)). Moreover, by the TT -identity 20.2, we have $A_8 f_4 f_2 = f_3 f_1$. Hence

$$g_{a+b,c} g_{a,b} S'_{a+b+c} = s_4 s_3 s_2 A_7 A_6 f_4 f_2, \quad g_{a,b+c} g_{b,c} S'_{a+b+c} = s_4 s_3 s_2 A_4 A_5 A_8 f_4 f_2.$$

Using the pentagon identity several times, we see that $A_4 A_5 A_8 = A_7 A_6$. Therefore, we have

$$g_{a+b,c} g_{a,b} S'_{a+b+c} = g_{a,b+c} g_{b,c} S'_{a+b+c}.$$

Since $S'_{a+b+c} \neq 0$, it follows that $g_{a+b,c} g_{a,b} = g_{a,b+c} g_{b,c}$. The lemma is proved.

Lemma 20.9. *Let $a, b \in \mathbf{N}^I$. We have $g_{a,b} = g_{b,a}$.*

We consider the composition

$$(a) \quad V_{a+b} \otimes V_{b+a} \xrightarrow{T_{a,b} \otimes T_{b,a}} V_a \otimes V_b \otimes V_b \otimes V_a \xrightarrow{(1 \otimes S'_b \otimes 1) A_1} V_a \otimes V_a \xrightarrow{S'_a} \mathbf{A},$$

where $A_1 = B_{a,b,b,a}$, and the analogous composition

$$(b) \quad V_{a+b} \otimes V_{b+a} \xrightarrow{T_{b,a} \otimes T_{a,b}} V_b \otimes V_a \otimes V_a \otimes V_b \xrightarrow{(1 \otimes S'_a \otimes 1) A_2} V_b \otimes V_b \xrightarrow{S'_b} \mathbf{A},$$

where $A_2 = B'_{b,a,a,b}$.

By 20.6(b), the composition (a) is equal to $g_{a,b} S'_{a+b}$ and the composition (b) is equal to $g_{b,a} S'_{a+b}$. Since $S'_{a+b} \neq 0$, it is enough to prove that the composition (a) is equal to the composition (b).

Using the TC -identity 20.3 twice, we see that the composition

$$V_{a+b} \otimes V_{b+a} \xrightarrow{T_{a,b} \otimes T_{b,a}} V_a \otimes V_b \otimes V_b \otimes V_a \xrightarrow{C_{b,a}^{-1} \otimes C_{b,a}} V_b \otimes V_a \otimes V_a \otimes V_b$$

is equal to $T_{b,a} \otimes T_{a,b}$. (The two factors $v^{-\langle a, b \rangle}$ and $v^{\langle a, b \rangle}$ cancel each other.) The composition

$$\begin{aligned} V_a \otimes V_b \otimes V_b \otimes V_a &\xrightarrow{1 \otimes C_{b \otimes b, a}} V_a \otimes V_a \otimes V_b \otimes V_b \xrightarrow{A_{a,a,b \otimes b}^{-1}} V_a \otimes V_a \otimes V_b \otimes V_b \\ &\xrightarrow{1 \otimes 1 \otimes S'_b} V_a \otimes V_a \end{aligned}$$

is equal to $1 \otimes S'_b \otimes 1$. (This statement holds more generally if we replace $V_b \otimes V_b$ by any object M of $\mathcal{D}$ and S'_b by any morphism $M \rightarrow \mathbf{A}$ in $\mathcal{D}$; we use the functoriality of the associativity and commutativity isomorphisms together with the fact that these isomorphisms take a very simple form when one of the indices is $\mathbf{A}$.)

Similarly, the composition

$$\begin{aligned} V_a \otimes V_a \otimes V_b \otimes V_b &\xrightarrow{A_{a \otimes a, b, b}^{-1}} V_a \otimes V_a \otimes V_b \otimes V_b \xrightarrow{C_{b,a}^{-1} \otimes 1} V_b \otimes V_a \otimes V_a \otimes V_b \\ &\xrightarrow{1 \otimes S'_a \otimes 1} V_b \otimes V_b \end{aligned}$$

is equal to $S'_a \otimes 1 \otimes 1$. It is clear that the compositions

$$V_a \otimes V_a \otimes V_b \otimes V_b \xrightarrow{1 \otimes 1 \otimes S'_b} V_a \otimes V_a \xrightarrow{S'_a} \mathbf{A}$$

and

$$V_a \otimes V_a \otimes V_b \otimes V_b \xrightarrow{S'_a \otimes 1 \otimes 1} V_b \otimes V_b \xrightarrow{S'_b} \mathbf{A}$$

coincide.

Combining the statements above, we see that the equality of the compositions (a), (b) is a consequence of the equality

$$(C_{b,a \otimes a}^{-1} \otimes 1) A_{a \otimes a, b, b}^{-1} A_{a, a, b \otimes b}^{-1} (1 \otimes C_{b \otimes b, a}) A_1 = A_2 (C_{b, a}^{-1} \otimes C_{b, a})$$

between isomorphisms of $V_a \otimes V_b \otimes V_b \otimes V_a$ with itself.

This equality follows by applying each of the two hexagon identities once, the pentagon identity twice, and using the functoriality of the associativity isomorphisms, in each of the three variables.

(We use this functoriality in the following kind of situation. If X, Y, Z, U are objects of $\mathcal{D}$, then the diagram

$$\begin{array}{ccc} X \otimes Y \otimes Z \otimes U & \xrightarrow{C_{X,Y} \otimes 1 \otimes 1} & Y \otimes X \otimes Z \otimes U \\ A_{X \otimes Y, Z, U} \downarrow & & A_{Y \otimes X, Z, U} \downarrow \\ X \otimes Y \otimes Z \otimes U & \xrightarrow{C_{X,Y} \otimes 1 \otimes 1} & Y \otimes X \otimes Z \otimes U \end{array}$$

is commutative.) The lemma is proved.

In the following proposition, we use the following notation: $\nu(a) = \sum_i a(i)$ for $a \in \mathbf{N}^I$.

Proposition 20.10. *There is a unique collection of elements $g_a^0 \in \mathbf{A}$ ($a \in \mathbf{N}^I$) such that*

- (a) $g_a^0 = 1$ if $\nu(a) \in \{0, 1\}$ and
- (b) $g_{a+b}^0 = g_a^0 g_b^0 g_{a,b}$ for all $a, b \in \mathbf{N}^I$.

Moreover, we have $g_a^0(0) = 1$ for all $a \in \mathbf{N}^I$.

Assume that $n \geq 2$ and that g_a^0 are already defined for all a such $\nu(a) \leq n-1$, so that (a) holds and (b) holds when $\nu(a+b) \leq n-1$.

Let $a \in \mathbf{N}^I$ be such that $\nu(a) = n$. We can find $i \in I$ such that $a = \omega_i + a'$ where $a' \in \mathbf{N}^I$. Then $g_{a'}^0$ is defined and we set $g_a^0 = g_{a'}^0 g_{\omega_i, a'}$. We claim that this is independent of the choice of i . Assume that we have also $a = \omega_j + a''$ where $j \in I$ is different from i and $a'' \in \mathbf{N}^I$. We have $a = \omega_i + \omega_j + b$ where $b \in \mathbf{N}^I$ and $a' = \omega_j + b$, $a'' = \omega_i + b$. By Lemma 20.8, we have $g_{\omega_i, b + \omega_j} g_{b, \omega_j} = g_{\omega_i + b, \omega_j} g_{\omega_i, b}$. Substituting here $g_{b, \omega_j} = g_{a'}^0 / g_b^0$, $g_{\omega_i, b} = g_{a''}^0 / g_b^0$ (induction hypothesis), we obtain $g_{\omega_i, a'} g_{a'}^0 = g_{a''}^0 g_{\omega_j, a''}$. We now use $g_{a'', \omega_j} = g_{\omega_j, a''}$ (see Lemma 20.9) and we see that our claim is proved.

Now let $a, b \in \mathbf{N}^I$ be such that $\nu(a+b) \leq n$. We show that (a) holds for a, b . If $a = 0$ or $b = 0$, then $g_{a,b} = 1$ (see 20.7(b)) and the result is obvious. Assume that $\nu(a) \geq 1, \nu(b) \geq 1$. Write $a = \omega_i + a'$ with $a' \in \mathbf{N}^I$. Using the definitions and the induction hypothesis, we have

$$g_{a+b}^0 = g_{\omega_i, a'+b} g_{a'+b}^0 = g_{\omega_i, a'+b} g_{a'}^0 g_b^0 g_{a',b} = g_{\omega_i+a', b} g_{\omega_i, a'} g_{a'}^0 g_b^0 = g_a^0 g_b^0 g_{a,b}.$$

The last assertion of the proposition follows from the earlier ones, together with the equality $g_{a,b}(0) = 1$ (see 20.7(a)). The proposition is proved.

Definition 20.11. A collection of elements $g_a \in \mathbf{A}$ ($a \in \mathbf{N}^I$) is said to be admissible if

- (a) $g_0 = 1$,
- (b) $g_a(0) = 1$ for all $a \in \mathbf{N}^I$, and
- (c) $g_{a+b} = g_a g_b g_{a,b}$ for all $a, b \in \mathbf{N}^I$.

The previous proposition shows that an admissible collection (g_a) exists and that there is a canonical choice for it, namely (g_a^0) .

If (g_a) is an admissible collection, then we have $g_a = \chi_a g_a^0$ where (χ_a) is a collection of elements of $\mathbf{A}$ indexed by $a \in \mathbf{N}^I$ such that

- (d) $\chi_0 = 1$,
- (e) $\chi_a(0) = 1$ for all $a \in \mathbf{N}^I$, and
- (f) $\chi_{a+b} = \chi_a \chi_b$ for all $a, b \in \mathbf{N}^I$;

and conversely, if (χ_a) are given in $\mathbf{A}$ so that (d),(e),(f) hold, then $g_a = \chi_a g_a^0$ form an admissible collection. Now the conditions (d),(e),(f) for χ_a can be reformulated as follows:

- there exist elements $\sigma_i \in \mathbf{A}$ ($i \in I$) such that
- (g) $\sigma_i(0) = 1$ for all $i \in I$ and
- (h) $\chi_a = \prod_{i \in I} \sigma_i^{a(i)}$ for all $a \in \mathbf{N}^I$.

20.12. From now on, until the end of 23.7, we assume that an admissible collection (g_a) has been fixed (not necessarily the one given by Proposition 20.10).

Equivalently, we assume that a collection $(\sigma_i)_{i \in I}$ of elements of $\mathbf{A}$ has been fixed so that the condition 20.11(g) is satisfied; we set

$$(a) \quad g_a = \prod_{i \in I} \sigma_i^{a(i)} g_a^0 \quad \text{for all } a \in \mathbf{N}^I.$$

In terms of this fixed choice, we set, for any $a \in \mathbf{N}^I$,

$$(b) \quad S_a = g_a S'_a: V_a \otimes V_a \rightarrow \mathbf{A}.$$

Starting with the end of 23.7, a specific choice for (g_a) (or equivalently, for (σ_i)) will be made.

We can now reformulate 20.6(b) using S_a instead of S'_a , and the factor $g_{a,b}$ disappears:

Proposition 20.13 (*TS-identity*). *Let $a, b \in \mathbf{N}^I$. We consider the composition*

$$V_{a+b} \otimes V_{b+a} \xrightarrow{T_{a,b} \otimes T_{b,a}} V_a \otimes V_b \otimes V_b \otimes V_a \xrightarrow{(1 \otimes S_b \otimes 1)B} V_a \otimes V_a \xrightarrow{S_a} \mathbf{A}$$

where B is either $B_{\bar{a}, \bar{b}, b, a}$ or $B'_{\bar{a}, \bar{b}, b, a}$. This composition is equal to S_{a+b} .

21. THE MORPHISMS $tr_{a,b}^c$ AND m

21.1. The morphism $tr_{a,b}^c$. Let $a, b, c \in \mathbf{N}^I$. We define a morphism $tr_{a,b}^c: V_{a+c} \otimes V_{c+b} \rightarrow V_a \otimes V_b$ as the composition

$$V_{a+c} \otimes V_{c+b} \xrightarrow{T_{a,c} \otimes T_{c,b}} V_a \otimes V_c \otimes V_c \otimes V_b \rightarrow V_a \otimes V_c \otimes V_c \otimes V_b \xrightarrow{1 \otimes S_c \otimes 1} V_a \otimes V_b,$$

where the unnamed map is either $A_1 = B_{a, \bar{c}, c, b}$ or $A_2 = B'_{a, \bar{c}, c, b}$ (we get the same result by using either A_1 or A_2 ; see Lemma 20.5).

Lemma 21.2. *The morphism $tr_{a,b}^c$ carries $x_{a+c} \otimes y_{c+b}$ to a vector equal to $x_a \otimes y_b$ mod ϖ .*

By reduction modulo ϖ , the associativity isomorphisms become the identity and S_c becomes S'_c . Hence it suffices to show that

$$(a) \quad (1 \otimes S'_c \otimes 1)(T_{a,c} \otimes T_{c,b})(x_{a+c} \otimes y_{c+b}) = x_a \otimes y_b.$$

Using 20.1(a), we see that the left-hand side of (a) equals

$$(1 \otimes S'_c \otimes 1)(x_a \otimes x_c \otimes y_c \otimes y_b) = x_a \otimes y_b.$$

The lemma is proved.

Proposition 21.3 (*Transitivity of tr*). *For any $a, b, c, d \in \mathbf{N}^I$, the composition*

$$V_{a+c+d} \otimes V_{d+c+b} \xrightarrow{tr_{a+c, c+b}^d} V_{a+c} \otimes V_{c+b} \xrightarrow{tr_{a,b}^c} V_a \otimes V_b$$

coincides with $tr_{a,b}^{c+d}$.

The composition above is by definition equal to the composition

$$\begin{aligned} V_{a+c+d} \otimes V_{d+c+b} &\xrightarrow{T_{a+c, d} \otimes T_{d, c+b}} V_{a+c} \otimes V_d \otimes V_d \otimes V_{c+b} \\ &\xrightarrow{A'_1} V_{a+c} \otimes V_d \otimes V_d \otimes V_{c+b} \xrightarrow{1 \otimes S_d \otimes 1} V_{a+c} \otimes V_{c+b} \xrightarrow{T_{a,c} \otimes T_{c,d}} V_a \otimes V_c \otimes V_c \otimes V_b \\ &\xrightarrow{A_1} V_a \otimes V_c \otimes V_c \otimes V_b \xrightarrow{1 \otimes S_c \otimes 1} V_a \otimes V_b, \end{aligned}$$

where $A'_1 = B_{a+c, d, d, c+b}$ and $A_1 = B_{a, \bar{c}, c, b}$. We may replace here the composition

$$V_{a+c} \otimes V_d \otimes V_d \otimes V_{c+b} \xrightarrow{1 \otimes S_d \otimes 1} V_{a+c} \otimes V_{c+b} \xrightarrow{T_{a,c} \otimes T_{c,d}} V_a \otimes V_c \otimes V_c \otimes V_b$$

by the composition

$$V_{a+c} \otimes V_d \otimes V_d \otimes V_{c+b} \xrightarrow{T_{a,c} \otimes 1 \otimes 1 \otimes T_{c,b}} V_a \otimes V_c \otimes V_d \otimes V_d \otimes V_c \otimes V_b \\ \xrightarrow{1 \otimes 1 \otimes S_d \otimes 1 \otimes 1} V_a \otimes V_c \otimes V_c \otimes V_b;$$

then the composition

$$(T_{a,c} \otimes 1 \otimes 1 \otimes T_{c,b})A'_1 \quad \text{by} \quad A''_1(T_{a,c} \otimes 1 \otimes 1 \otimes T_{c,b}),$$

where $A''_1 = B_{a \otimes c, d, d, c \otimes b}$; and then the composition

$$(T_{a,c} \otimes 1 \otimes 1 \otimes T_{c,b})(T_{a+c, d} \otimes T_{d, c+b})$$

by

$$A_3(1 \otimes T_{c, d} \otimes T_{d, c})(T_{a, c+d} \otimes T_{d+c, b}),$$

where $A_3 = A_{a, c, d}^{-1} \otimes A_{d, c, b}$ (we have used twice the TT -identity 20.2).

Using the definition of $tr_{a,b}^{c+d}$ and the TS -identity 20.13, we see that we are reduced to proving that

$$(1 \otimes S_c \otimes 1)A_1(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A''_1A_3 = (1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5A_4,$$

where $A_4 = B_{a, c \otimes d, d \otimes c, b}$ and $A_5 = 1 \otimes B_{c, d, d, c} \otimes 1$. We have

$$(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1) = (1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_7, \\ A_1(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1) = (1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_6,$$

where $A_6 = B_{a, c, d \otimes d \otimes c, b}$ and $A_7 = 1 \otimes 1 \otimes A_{d \otimes d, c, b}^{-1}$.

Hence we are reduced to checking the equality $A_6A_7A''_1A_3 = A_5A_4$ (of isomorphisms of $V_a \otimes V_c \otimes V_d \otimes V_d \otimes V_c \otimes V_b$ with itself). But this follows from the pentagon identity. The proposition is proved.

21.4. The morphism m . For any $a, b, c, d \in \mathbf{N}^I$, we define a morphism

$$m: V_{a+c} \otimes V_{b+d} \rightarrow V_a \otimes V_b \otimes V_c \otimes V_d$$

as the composition

$$V_{a+c} \otimes V_{b+d} \xrightarrow{T_{a,c} \otimes T_{b,d}} V_a \otimes V_c \otimes V_b \otimes V_d \xrightarrow{v^{(b,c)} A' (1 \otimes C_{c,b} \otimes 1) A} V_a \otimes V_b \otimes V_c \otimes V_d,$$

where $A = B_{a, c, b, d}$ and $A' = B_{a, b, c, d}^{-1}$.

We show that we get the same morphism, if in the definition of m we replace A, A' respectively by $\tilde{A} = B'_{a, c, b, d}$, $\tilde{A}' = B'^{-1}_{a, b, c, d}$. Indeed, it suffices to show that

$$(a) \quad (1 \otimes C_{c,b} \otimes 1)A\tilde{A}^{-1} = A'^{-1}\tilde{A}'(1 \otimes C_{c,b} \otimes 1).$$

Now $A\tilde{A}^{-1} = A_{a, c \otimes b, d}$ and $A'^{-1}\tilde{A}' = A_{a, b \otimes c, d}$, so that (a) follows from the functoriality of the associativity isomorphisms.

Lemma 21.5. *We have $m(x_{a+c} \otimes y_{b+d}) = x_a \otimes y_b \otimes x_c \otimes y_d \bmod \varpi$.*

Using the equalities $\bmod \varpi$:

$$A = 1, \quad A' = 1, \quad v^{(b,c)} = 1, \quad C_{c,b}(x_c \otimes y_b) = y_b \otimes x_c,$$

we obtain the equalities $\bmod \varpi$:

$$m(x_{a+c} \otimes y_{b+d}) = x_a \otimes C_{c,b}(x_c \otimes y_b) \otimes y_d = x_a \otimes y_b \otimes x_c \otimes y_d.$$

The lemma is proved.

Proposition 21.6 (Left compatibility of m and tr). *Given $a, b, c, d, e \in \mathbb{N}^I$, the diagram*

$$\begin{array}{ccc} V_{a+\bar{c}+e} \otimes V_{e+b+d} & \xrightarrow{m} & V_{a+\bar{e}} \otimes V_{e+b} \otimes V_{\bar{c}} \otimes V_d \\ \text{\scriptsize } tr_{a+c, b+d}^e \downarrow & & \downarrow \text{\scriptsize } tr_{a,b}^e \otimes 1 \otimes 1 \\ V_{a+\bar{c}} \otimes V_{b+d} & \xrightarrow{m} & V_a \otimes V_b \otimes V_{\bar{c}} \otimes V_d \end{array}$$

is commutative.

Note that $(tr_{a,b}^e \otimes 1 \otimes 1)m$ is equal to the composition (1):

$$\begin{aligned} V_{a+\bar{c}+e} \otimes V_{e+b+d} & \xrightarrow{T_{a+e, \bar{c}} \otimes T_{e+b, d}} V_{a+\bar{e}} \otimes V_{\bar{c}} \otimes V_{e+b} \otimes V_d \\ & \xrightarrow{v^{(e+b, c)} A'_1 (1 \otimes C_{c, e+b} \otimes 1) A_1} V_{a+\bar{e}} \otimes V_{e+b} \otimes V_{\bar{c}} \otimes V_d \\ & \xrightarrow{T_{a, e} \otimes T_{e, b} \otimes 1 \otimes 1} V_a \otimes V_{\bar{e}} \otimes V_e \otimes V_b \otimes V_{\bar{c}} \otimes V_d \\ & \xrightarrow{(1 \otimes S_{\bar{e}} \otimes 1 \otimes 1 \otimes 1) A_2} V_a \otimes V_b \otimes V_{\bar{c}} \otimes V_d, \end{aligned}$$

where $A_1 = B_{a+\bar{e}, \bar{c}, e+b, d}$, $A'_1 = B_{a+e, e+b, \bar{c}, d}^{-1}$, and $A_2 = B_{a, \bar{e}, e, b} \otimes 1 \otimes 1$.

We replace here

$$(T_{a, \bar{e}} \otimes T_{e, b} \otimes 1 \otimes 1) A'_1 \quad \text{by} \quad A'_3 (T_{a, \bar{e}} \otimes T_{e, b} \otimes 1 \otimes 1),$$

where $A'_3 = B_{a \otimes \bar{e}, e \otimes b, \bar{c}, d}^{-1}$ (functoriality of associativity isomorphisms).

Next we replace

$$(T_{a, \bar{e}} \otimes T_{e, b} \otimes 1 \otimes 1) (v^{(e+b, c)} 1 \otimes C_{c, e+b} \otimes 1)$$

by

$$(v^{(e+b, c)} 1 \otimes 1 \otimes C_{c, e \otimes b} \otimes 1) (T_{a, \bar{e}} \otimes 1 \otimes T_{e, b} \otimes 1),$$

and

$$(T_{a, \bar{e}} \otimes 1 \otimes T_{e, b} \otimes 1) A_1 \quad \text{by} \quad A_3 (T_{a, \bar{e}} \otimes 1 \otimes T_{e, b} \otimes 1),$$

where $A_3 = B_{a \otimes \bar{e}, \bar{c}, e \otimes b, d}$.

Next, (using the TT -identity 20.2 twice) we replace

$$(T_{a, \bar{e}} \otimes 1 \otimes T_{e, b} \otimes 1) (T_{a+\bar{e}, \bar{c}} \otimes T_{e+b, d})$$

by

$$A_4 (1 \otimes T_{\bar{e}, \bar{c}} \otimes 1 \otimes T_{b, d}) (T_{a, \bar{c}+e} \otimes T_{e, b+d}),$$

where $A_4 = A_{a,e,c}^{-1} \otimes A_{e,b,d}^{-1}$. Using the TC -identity 20.3, we now replace

$$(1 \otimes T_{e,c} \otimes 1 \otimes T_{b,d})$$

by

$$v^{-(e,c)}(1 \otimes C_{\bar{c},e} \otimes 1 \otimes 1 \otimes 1)(1 \otimes T_{\bar{c},e} \otimes 1 \otimes T_{b,d})$$

and, using the TT -identity 20.2, we replace

$$(1 \otimes T_{\bar{c},e} \otimes 1 \otimes T_{b,d})(T_{a,\bar{c}+e} \otimes T_{e,b+d})$$

by

$$A_5(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{b,d})(T_{a+\bar{c},e} \otimes T_{e,b+d}),$$

where $A_5 = A_{a,e,c} \otimes 1 \otimes 1 \otimes 1$. Next, using the hexagon and pentagon identities, we replace

$$(1 \otimes 1 \otimes C_{\bar{c},e \otimes b} \otimes 1)A_3A_4(1 \otimes C_{\bar{c},e} \otimes 1 \otimes 1 \otimes 1)$$

by

$$A_6(1 \otimes C_{\bar{c},e \otimes e \otimes b} \otimes 1)A_7,$$

where

$$A_6 = A_{a,e,e \otimes b \otimes c \otimes d}^{-1}(1 \otimes A_{\bar{e},e \otimes b \otimes \bar{c},d})(1 \otimes A_{e,e \otimes b,\bar{c}} \otimes 1)(1 \otimes A_{e,e,b} \otimes 1 \otimes 1),$$

$$A_7 = (1 \otimes 1 \otimes A_{e,e,b}^{-1} \otimes 1)(1 \otimes A_{\bar{c},e,e \otimes b} \otimes 1)(1 \otimes A_{c \otimes e,e \otimes b,d}^{-1}) \\ \times (1 \otimes 1 \otimes 1 \otimes A_{e,b,d}^{-1})A_{a,\bar{c} \otimes e,e \otimes b \otimes d}.$$

In this way, the composition (1) has been transformed into

$$v^{(b,c)}(1 \otimes S_e \otimes 1 \otimes 1 \otimes 1)A_2A_3A_6(1 \otimes C_{\bar{c},e \otimes e \otimes b} \otimes 1) \\ \times A_7A_5(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{b,d})(T_{a+\bar{c},e} \otimes T_{e,b+d}).$$

(We have used that $v^{(e,c)} = v^{(e,c)}$ and the bilinearity of $\langle \cdot, \cdot \rangle$.)

Using the functoriality of the associativity isomorphisms, we replace here

$$(1 \otimes S_e \otimes 1 \otimes 1 \otimes 1)A_2A_3A_6 \quad \text{by} \quad A_8(1 \otimes S_e \otimes 1 \otimes 1 \otimes 1),$$

where $A_8 = B_{a,b,\bar{c},d}^{-1}$ and then, using the functoriality of the commutativity isomorphisms, we replace

$$(1 \otimes S_e \otimes 1 \otimes 1 \otimes 1)(1 \otimes C_{\bar{c},e \otimes e \otimes b} \otimes 1)$$

by

$$(1 \otimes C_{\bar{c},b} \otimes 1)(1 \otimes 1 \otimes S_e \otimes 1 \otimes 1).$$

We obtain the composition

$$v^{(b,c)}A_8(1 \otimes C_{\bar{c},b} \otimes 1)(1 \otimes 1 \otimes S_e \otimes 1 \otimes 1)A_7A_5(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{b,d})(T_{a+\bar{c},e} \otimes T_{e,b+d}).$$

Let $A_9 = B_{a+\bar{c},e,e,b+d}$, $A_{10} = B_{a \otimes \bar{c},e,e,b \otimes d}$, $A_{11} = B_{a,\bar{c},b,d}$.

Then

$$(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{b,d})A_9 = A_{10}(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{b,d})$$

and

$$A_{11}(1 \otimes 1 \otimes S_e \otimes 1 \otimes 1) = (1 \otimes 1 \otimes S_e \otimes 1 \otimes 1)A_7A_5A_{10}^{-1},$$

by the functoriality of the associativity isomorphisms. It follows that

$$\begin{aligned} & (1 \otimes 1 \otimes S_e \otimes 1 \otimes 1)A_7A_5(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{b,d}) \\ & = A_{11}(1 \otimes 1 \otimes S_e \otimes 1 \otimes 1)(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{b,d})A_9. \end{aligned}$$

We now replace

$$(1 \otimes 1 \otimes S_e \otimes 1 \otimes 1)(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{b,d})$$

by

$$(T_{a,\bar{c}} \otimes T_{b,d})(1 \otimes S_e \otimes 1)$$

and we see that our composition becomes

$$v^{(b,c)}A_8(1 \otimes C_{c,b} \otimes 1)A_{11}(T_{a,\bar{c}} \otimes T_{b,d})(1 \otimes S_e \otimes 1)A_9(T_{a+c,\bar{e}} \otimes T_{e,b+d}).$$

This is, by definition, $mtr_{a+c,b+d}^e$. The proposition is proved.

Proposition 21.7 (Right compatibility of m and tr). *Given $a, b, c, d, e \in \mathbb{N}^I$, the diagram*

$$\begin{array}{ccc} V_{a+\bar{c}+e} \otimes V_{e+b+d} & \xrightarrow{m} & V_a \otimes V_b \otimes V_{c+e} \otimes V_{e+d} \\ \text{\scriptsize } tr_{a+c,b+d}^e \downarrow & & \downarrow \text{\scriptsize } 1 \otimes 1 \otimes tr_{c,d}^e \\ V_{a+\bar{c}} \otimes V_{b+d} & \xrightarrow{m} & V_a \otimes V_b \otimes V_{\bar{c}} \otimes V_d \end{array}$$

is commutative.

The proof is entirely similar to that of the previous proposition.

Proposition 21.8 (mm -identity). *For any $a, b, c, d, e, f \in \mathbb{N}^I$, the diagram*

$$\begin{array}{ccc} V_{a+\bar{b}+\bar{c}} \otimes V_{d+e+f} & \xrightarrow{m} & V_{a+\bar{b}} \otimes V_{d+e} \otimes V_{\bar{c}} \otimes V_f \\ m \downarrow & & \downarrow A_0 m \\ V_a \otimes V_d \otimes V_{b+\bar{c}} \otimes V_{e+f} & \xrightarrow{m} & V_a \otimes V_d \otimes V_b \otimes V_e \otimes V_{\bar{c}} \otimes V_f \end{array}$$

where $A_0 = A_{a \otimes d, b \otimes e, c \otimes f}$, is commutative.

The composition (1) of the upper arrow and the right arrow is, by definition,

$$\begin{aligned} & V_{a+\bar{b}+\bar{c}} \otimes V_{d+e+f} \xrightarrow{T_{a+\bar{b},\bar{c}} \otimes T_{d+e},f} V_{a+\bar{b}} \otimes V_{\bar{c}} \otimes V_{d+e} \otimes V_f \\ & \xrightarrow{v^{(c,d+e)}A_2(1 \otimes C_{\bar{c},d+e} \otimes 1)A_1} V_{a+\bar{b}} \otimes V_{d+e} \otimes V_{\bar{c}} \otimes V_f \\ & \xrightarrow{T_{a,b} \otimes T_{d,e} \otimes 1 \otimes 1} V_a \otimes V_{\bar{b}} \otimes V_d \otimes V_e \otimes V_{\bar{c}} \otimes V_f \\ & \xrightarrow{v^{(b,d)}A_0A_4(1 \otimes C_{b,d} \otimes 1 \otimes 1 \otimes 1)A_3} V_a \otimes V_d \otimes V_b \otimes V_e \otimes V_{\bar{c}} \otimes V_f, \end{aligned}$$

where

$$\begin{aligned} A_1 &= B_{a+\bar{b},\bar{c},d+e,f}, & A_2 &= B_{a+\bar{b},d+e,\bar{c},f}^{-1}, \\ A_3 &= B_{a,b,d,e} \otimes 1 \otimes 1, & A_4 &= B_{a,d,b,e}^{-1} \otimes 1 \otimes 1. \end{aligned}$$

We replace here

$$(T_{a,b} \otimes T_{d,e} \otimes 1 \otimes 1)A_2 \quad \text{by} \quad A_5(T_{a,b} \otimes T_{d,e} \otimes 1 \otimes 1),$$

where $A_5 = B_{a \otimes b, d \otimes e, \bar{c}, f}^{-1}$ (functoriality of the associativity isomorphisms) and then we replace

$$(T_{a,b} \otimes T_{d,e} \otimes 1 \otimes 1)v^{(c, d+e)}(1 \otimes C_{\bar{c}, d+e} \otimes 1)$$

by

$$v^{(c, d+e)}(1 \otimes 1 \otimes C_{\bar{c}, d \otimes e} \otimes 1)(T_{a,b} \otimes 1 \otimes T_{d,e} \otimes 1 \otimes 1)$$

(functoriality of the commutativity isomorphisms). Next we replace

$$(T_{a,b} \otimes 1 \otimes T_{d,e} \otimes 1 \otimes 1)A_1 \quad \text{by} \quad A_6(T_{a,b} \otimes 1 \otimes T_{d,e} \otimes 1 \otimes 1),$$

where $A_6 = B_{a \otimes b, \bar{c}, d \otimes e, f}$ and, using the TT -identity 21.2 twice, we replace

$$(T_{a,b} \otimes 1 \otimes T_{d,e} \otimes 1 \otimes 1)(T_{a+b, \bar{c}} \otimes T_{d+e, f})$$

by

$$A_7(1 \otimes T_{b, \bar{c}} \otimes 1 \otimes T_{e, f})(T_{a, b+\bar{c}} \otimes T_{d, e+f}),$$

where $A_7 = A_{a, b, \bar{c}}^{-1} \otimes A_{d, e, f}^{-1}$. Thus, the composition (1) is equal to the composition

$$\begin{aligned} & v^{(b, d)} A_0 A_4 (1 \otimes C_{b, d} \otimes 1 \otimes 1 \otimes 1) A_3 A_5 v^{(c, d+e)} (1 \otimes 1 \otimes C_{\bar{c}, d \otimes e} \otimes 1) A_6 A_7 \\ & \times (1 \otimes T_{b, \bar{c}} \otimes 1 \otimes T_{e, f})(T_{a, b+\bar{c}} \otimes T_{d, e+f}). \end{aligned}$$

Similarly, the composition of the left arrow and the lower arrow in the proposition is equal to the composition

$$\begin{aligned} & v^{(c, e)} A'_4 (1 \otimes 1 \otimes 1 \otimes C_{\bar{c}, e} \otimes 1) A'_3 A'_2 v^{(b+c, d)} (1 \otimes C_{b \otimes c, d} \otimes 1 \otimes 1) A'_1 \\ & \times (1 \otimes T_{b, \bar{c}} \otimes 1 \otimes T_{e, f})(T_{a, b+\bar{c}} \otimes T_{d, e+f}), \end{aligned}$$

where

$$\begin{aligned} A'_1 &= B_{a, b \otimes c, d, e \otimes f}, & A'_2 &= B_{a, d, b \otimes c, e \otimes f}^{-1}, \\ A'_3 &= 1 \otimes 1 \otimes B_{b, \bar{c}, e, f}, & A'_4 &= 1 \otimes 1 \otimes B_{b, e, \bar{c}, f}^{-1}. \end{aligned}$$

Since $v^{(b, d)} v^{(c, d+e)} = v^{(c, e)} v^{(b+c, d)}$, it remains to check the following equality among isomorphisms $V_a \otimes V_b \otimes V_{\bar{c}} \otimes V_d \otimes V_e \otimes V_f \rightarrow V_a \otimes V_d \otimes V_b \otimes V_e \otimes V_{\bar{c}} \otimes V_f$:

$$\begin{aligned} & A_0 A_4 (1 \otimes C_{b, d} \otimes 1 \otimes 1 \otimes 1) A_3 A_5 (1 \otimes 1 \otimes C_{\bar{c}, d \otimes e} \otimes 1) A_6 A_7 \\ & = A'_4 (1 \otimes 1 \otimes 1 \otimes C_{\bar{c}, e} \otimes 1) A'_3 A'_2 (1 \otimes C_{b \otimes c, d} \otimes 1 \otimes 1) A'_1. \end{aligned}$$

This follows from the pentagon and hexagon identities. The proposition is proved.

22. THE MORPHISM $\tau_{i;a,b}$

22.1. **The morphisms $\tau'_{i;a,b}$.** In this section we fix $i \in I$.

Let $a, b \in \mathbf{N}^I$ be such that $a(i) \geq 1$, $b(i) \geq 1$. Then $a + b - \alpha_i \in \mathbf{N}^I$. The vector

$$y' = a(i)y_a \otimes f_i y_b - b(i)f_i y_a \otimes y_b \in V_a \otimes V_b$$

satisfies $e_j y' = 0$ and $h_j y' = (a(j) + b(j) - a_{ji})y'$ for all $j \in I$; hence there is a unique morphism

$$\tau'_{i;a,b}: V_{a+b-\alpha_i} \rightarrow V_a \otimes V_b$$

in $\mathcal{D}$ such that $\tau'_{i;a,b}(y_{a+b-\alpha_i}) = y'$. We have

$$(a) \quad \tau'_{i;a,b}(x_{a+b-\alpha_i}) = -a(i)x_a \otimes e_i x_b + b(i)e_i x_a \otimes x_b.$$

Indeed, using 19.4(a),(b),(c), we have

$$\begin{aligned} \tau'_{i;a,b}(x_{a+b-\alpha_i}) &= \tau'_{i;a,b}(\theta(y_{a+b-\alpha_i})) = \theta(\tau'_{i;a,b}(y_{a+b-\alpha_i})) \\ &= \theta(a(i)y_a \otimes f_i y_b - b(i)f_i y_a \otimes y_b) = a(i)\theta y_a \otimes \theta f_i y_b - b(i)\theta f_i y_a \otimes \theta y_b \\ &= -a(i)\theta y_a \otimes e_i \theta y_b + b(i)e_i \theta y_a \otimes \theta y_b = -a(i)x_a \otimes e_i x_b + b(i)e_i x_a \otimes x_b. \end{aligned}$$

22.2. For $a, b \in \mathbf{N}^I$ as above, we define $\tau_{i;a,b}: V_{a+b-\alpha_i} \rightarrow V_a \otimes V_b$ (a morphism in $\mathcal{D}$) by

$$\tau_{i;a,b} = \frac{\tau'_{i;a,b}}{\Gamma(1 + \varpi \delta_i a(i)) \Gamma(1 + \varpi \delta_i b(i)) \Gamma(1 - \varpi \delta_i (a(i) + b(i)))}.$$

Here, for any real number r , we denote by $\Gamma(1 + \varpi r)$ the power series expansion of the classical Gamma-function $z \mapsto \Gamma(1 + zr)$ at $z = 0$; thus,

(a) $\Gamma(1 + \varpi r) \in \mathbf{A}$ and $\Gamma(1 + \varpi r) = 1 \bmod \varpi$.

Later on we shall also have to consider the power series $\Gamma(\varpi r)$ and $\Gamma(-1 + \varpi r)$ for a real number $r \neq 0$. By definition,

(b) $\Gamma(\varpi r) = (\varpi r)^{-1} \Gamma(1 + \varpi r)$ and $\Gamma(-1 + \varpi r) = (-1 + \varpi r)^{-1} \Gamma(\varpi r)$ (elements of $\mathbf{C}((\varpi))$). Using (a), we see that

(c) $\tau_{i;a,b}(x) = \tau'_{i;a,b}(x) \bmod \varpi$ for any $x \in V_{a+b-\alpha_i}$.

Proposition 22.3 (τC -identity). *Let $a, b \in \mathbf{N}^I$ be as above. The composition*

$$V_{a+b-\alpha_i} \xrightarrow{\tau_{i;b,a}} V_b \otimes V_a \xrightarrow{C_{b,a}} V_a \otimes V_b$$

coincides with $-v^{(a,b)-\delta_i a(i)-\delta_i b(i)} \tau_{i;a,b}$

The composition $C_{b,a} \tau'_{i;b,a}$ takes $y_{a+b-\alpha_i}$ to

$$C_{b,a}(b(i)y_b \otimes f_i y_a - a(i)f_i y_b \otimes y_a) = e^{\pi \sqrt{-1} \varpi t} (b(i)f_i y_a \otimes y_b - a(i)y_a \otimes f_i y_b).$$

Using 19.7(b),(c), we see that

$$\begin{aligned} &t(b(i)f_i y_a \otimes y_b - a(i)y_a \otimes f_i y_b) \\ &= (\langle a, b \rangle - \delta_i a(i) - \delta_i b(i))(b(i)f_i y_a \otimes y_b - a(i)y_a \otimes f_i y_b). \end{aligned}$$

It follows that

$$\begin{aligned} & e^{\sqrt{-1}\pi\varpi t}(b(i)f_i y_a \otimes y_b - a(i)y_a \otimes f_i y_b) \\ &= v^{\langle a, b \rangle - \delta_i a(i) - \delta_i b(i)} (b(i)f_i y_a \otimes y_b - a(i)y_a \otimes f_i y_b). \end{aligned}$$

Thus,

$$C_{b,a} \tau'_{i;b,a} y_{a+b-\alpha_i} = -v^{\langle a, b \rangle - \delta_i a(i) - \delta_i b(i)} \tau'_{i;a,b} y_{a+b-\alpha_i}.$$

Now $y_{a+b-\alpha_i}$ generates $V_{a+b-\alpha_i}$ as a $\mathfrak{U}$ -module. Hence

$$C_{b,a} \tau'_{i;b,a} = -v^{\langle a, b \rangle - \delta_i a(i) - \delta_i b(i)} \tau'_{i;a,b}.$$

We divide both sides of the last equality by

$$\Gamma(1 + \varpi \delta_i a(i)) \Gamma(1 + \varpi \delta_i b(i)) \Gamma(1 - \varpi \delta_i (a(i) + b(i))).$$

The proposition follows.

Lemma 22.4. *Let $a, b \in \mathbf{N}^I$ be such that $a(i) \geq 1, b(i) \geq 1$. Let $B = B_{a,\bar{b},b,a}$.*

(a) *The composition*

$$V_{a+\bar{b}-\bar{\alpha}_i} \otimes V_{b+a} \xrightarrow{\tau_{i;a,b} \otimes T_{b,a}} V_a \otimes V_{\bar{b}} \otimes V_b \otimes V_a \xrightarrow{(1 \otimes S_b \otimes 1)B} V_a \otimes V_a \xrightarrow{S_a} \mathbf{A}$$

is zero.

(b) *The composition*

$$V_{a+\bar{b}} \otimes V_{b+a-\alpha_i} \xrightarrow{T_{a,b} \otimes \tau_{i;b,a}} V_a \otimes V_{\bar{b}} \otimes V_b \otimes V_a \xrightarrow{(1 \otimes S_b \otimes 1)B} V_a \otimes V_a \xrightarrow{S_a} \mathbf{A}$$

is zero.

(c) *The composition*

$$V_{a+\bar{b}-\bar{\alpha}_i} \otimes V_{b+a-\alpha_i} \xrightarrow{\tau_{i;a,b} \otimes \tau_{i;b,a}} V_a \otimes V_{\bar{b}} \otimes V_b \otimes V_a \xrightarrow{(1 \otimes S_b \otimes 1)B} V_a \otimes V_a \xrightarrow{S_a} \mathbf{A}$$

is equal to $-\rho_{i;a,b}[a(i)]_i[b(i)]_i[a(i) + b(i)]_i S_{a+b-\alpha_i}$ where $\rho_{i;a,b} \in \mathbf{A}$ satisfies $\rho_{i;a,b} = 1 \bmod \varpi$.

Any morphism $V_{a+\bar{b}-\bar{\alpha}_i} \otimes V_{b+a} \rightarrow \mathbf{A}$ (in $\mathscr{D}$) is easily seen to be zero; hence (a) follows. A similar argument gives (b).

Since any morphism $V_{a+\bar{b}-\bar{\alpha}_i} \otimes V_{b+a-\alpha_i} \rightarrow \mathbf{A}$ (in $\mathscr{D}$) is a scalar times $S'_{a+b-\alpha_i}$, we see that the composition in (c) is of the indicated form, for some $\rho_{i;a,b} \in \mathbf{A}$ and we only have to show that $\rho_{i;a,b} = 1 \bmod \varpi$. To do this, we must only compute the composition $\bmod \varpi$. We may therefore replace in our composition $\tau_{i;a,\bar{b}}, \tau_{i;b,a}, S_a, S_b, B$ by $\tau'_{i;a,\bar{b}}, \tau'_{i;b,a}, S'_a, S'_b, 1$ and we may replace $[a(i)]_i[b(i)]_i[a(i) + b(i)]_i S_{a+b-\alpha_i}$ by $a(i)b(i)(a(i) + b(i))S'_{a+b-\alpha_i}$. It is therefore enough to check the identity

$$(d) \quad S'_a(1 \otimes S'_b \otimes 1)(\tau'_{i;a,\bar{b}} \otimes \tau'_{i;b,a}) = -a(i)b(i)(a(i) + b(i))S'_{a+b-\alpha_i}.$$

Since $x := x_{a+\bar{b}-\bar{\alpha}_i} \otimes y_{b+a-\alpha_i}$ generates the $\mathfrak{U}$ -module $V_{a+\bar{b}-\bar{\alpha}_i} \otimes V_{b+a-\alpha_i}$, it suffices to show that the two sides of (d) coincide when evaluated at x . The left-hand side of (d) evaluated at x is

$$\begin{aligned} & S'_a(1 \otimes S'_b(1))((-a(i)x_a \otimes e_i x_b + b(i)e_i x_a \otimes x_b) \otimes (b(i)y_b \otimes f_i y_a - a(i)f_i y_b \otimes y_a)) \\ &= S'_a(-a(i)^2 b(i)x_a \otimes y_a + b(i)^2 e_i x_a \otimes f_i y_a) \\ &= -a(i)^2 b(i) - b(i)^2 a(i) = -a(i)b(i)(a(i) + b(i)). \end{aligned}$$

(We have used $S'_a(e_i x_a \otimes f_i y_a) = -a(i)$, $S'_b(e_i x_b \otimes f_i y_b) = -b(i)$.) This is equal to the right-hand side of (d) evaluated at x . The lemma is proved.

22.5. If α, β, γ are real numbers and γ is not an integer ≤ 0 , then the Euler hypergeometric series

$$F(\alpha, \beta, \gamma; z) = 1 + \frac{\alpha\beta}{\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}z^2 + \dots$$

converges for $z \in (-1, 1)$ and is the unique solution of the Gauss differential equation

$$(a) \quad z(1-z)\frac{d^2u}{dz^2} + (\gamma - (\alpha + \beta + 1)z)\frac{du}{dz} - \alpha\beta u = 0$$

such that $u|_{z=0} = 1$ and $\frac{du}{dz}|_{z=0} = \frac{\alpha\beta}{\gamma}$.

In a similar way we define $F(\alpha, \beta, \gamma; z)$ where α, β, γ are linear functions in ϖ with real coefficients (and again γ is not equal to an integer ≤ 0).

Proposition 22.6 (τT -identities). *Let $a, b, c \in \mathbf{N}^I$ be such that $a(i) > 0$, $b(i) > 0$, $c(i) > 0$. Consider the compositions*

$$(Q_1) \quad V_{a+b+c-\alpha_i} \xrightarrow{\tau_{i;a+b,c}} V_{a+b} \otimes V_c \xrightarrow{T_{a,b} \otimes 1} V_a \otimes V_b \otimes V_c,$$

$$(Q_2) \quad V_{a+b+c-\alpha_i} \xrightarrow{\tau_{i;a,b,c}} V_a \otimes V_{b+c} \xrightarrow{1 \otimes T_{b,c}} V_a \otimes V_b \otimes V_c,$$

$$(Q_3) \quad V_{a+b+c-\alpha_i} \xrightarrow{T_{a,b+c-\alpha_i}} V_a \otimes V_{b+c-\alpha_i} \xrightarrow{1 \otimes \tau_{i;b,c}} V_a \otimes V_b \otimes V_c,$$

$$(Q_4) \quad V_{a+b+c-\alpha_i} \xrightarrow{T_{a+b-\alpha_i,c}} V_{a+b-\alpha_i} \otimes V_c \xrightarrow{\tau_{i;a,b} \otimes 1} V_a \otimes V_b \otimes V_c.$$

We have

$$\begin{aligned} (1) \quad & [a(i) + b(i)]_i A^{-1} Q_3 + [c(i)]_i Q_4 = [b(i)]_i Q_1, \\ (2) \quad & [a(i)]_i A^{-1} Q_3 + [b(i) + c(i)]_i Q_4 = [b(i)]_i A^{-1} Q_2, \\ (3) \quad & [b(i) + c(i)]_i Q_1 - [c(i)]_i A^{-1} Q_2 = [a(i) + b(i) + c(i)]_i A^{-1} Q_3, \\ (4) \quad & -[a(i)]_i Q_1 + [a(i) + b(i)]_i A^{-1} Q_2 = [a(i) + b(i) + c(i)]_i Q_4, \end{aligned}$$

where $A = A_{a,b,c}$.

Let $y = y_{a+b+c-\alpha_i} \in V_{a+b+c-\alpha_i}$. We consider the vectors

$$\begin{aligned} r_1 &= (T_{a,b} \otimes 1) \tau'_{i;a+b,c} y \\ &= (a(i) + b(i)) y_a \otimes y_b \otimes f_i y_c - c(i) f_i y_a \otimes y_b \otimes y_c - c(i) y_a \otimes f_i y_b \otimes y_c; \\ r_2 &= (1 \otimes T_{b,c}) \tau'_{i;a,b+c} y \\ &= a(i) y_a \otimes f_i y_b \otimes y_c + a(i) y_a \otimes y_b \otimes f_i y_c - (b(i) + c(i)) f_i y_a \otimes y_b \otimes y_c; \\ r_3 &= (1 \otimes \tau'_{i;b,c}) T_{a,b+c-\alpha_i} y = b(i) y_a \otimes y_b \otimes f_i y_c - c(i) y_a \otimes f_i y_b \otimes y_c; \\ r_4 &= (\tau'_{i;a,b} \otimes 1) T_{a+b-\alpha_i,c} y = a(i) y_a \otimes f_i y_b \otimes y_c - b(i) f_i y_a \otimes y_b \otimes y_c \end{aligned}$$

of $V_a \otimes V_b \otimes V_c$.

Let Z be the $\mathbf{A}$ -submodule of $V_a \otimes V_b \otimes V_c$ spanned by r_3 and r_4 . We have $r_1, r_2 \in Z$, since

$$r_1 = \frac{a(i) + b(i)}{b(i)} r_3 + \frac{c(i)}{b(i)} r_4, \quad r_2 = \frac{a(i)}{b(i)} r_3 + \frac{b(i) + c(i)}{b(i)} r_4.$$

From the formulas in 19.7 we deduce

$$\begin{aligned} t_{12}(f_i y_a \otimes y_b \otimes y_c) &= (\langle a, b \rangle - \delta_i b(i)) f_i y_a \otimes y_b \otimes y_c + \delta_i a(i) y_a \otimes f_i y_b \otimes y_c; \\ t_{12}(y_a \otimes y_b \otimes f_i y_c) &= \langle a, b \rangle y_a \otimes y_b \otimes f_i y_c; \\ t_{12}(y_a \otimes f_i y_b \otimes y_c) &= (\langle a, b \rangle - \delta_i a(i)) y_a \otimes f_i y_b \otimes y_c + \delta_i b(i) f_i y_a \otimes y_b \otimes y_c; \\ t_{23}(f_i y_a \otimes y_b \otimes y_c) &= \langle b, c \rangle f_i y_a \otimes y_b \otimes y_c; \\ t_{23}(y_a \otimes y_b \otimes f_i y_c) &= (\langle b, c \rangle - \delta_i b(i)) y_a \otimes y_b \otimes f_i y_c + \delta_i c(i) y_a \otimes f_i y_b \otimes y_c; \\ t_{23}(y_a \otimes f_i y_b \otimes y_c) &= (\langle b, c \rangle - \delta_i c(i)) y_a \otimes f_i y_b \otimes y_c + \delta_i b(i) y_a \otimes y_b \otimes f_i y_c. \end{aligned}$$

It follows that

$$\begin{aligned} t_{12} r_3 &= \langle a, b \rangle r_3 + \delta_i c(i) r_4, & t_{12} r_4 &= (\langle a, b \rangle - \delta_i a(i) - \delta_i b(i)) r_4, \\ t_{23} r_3 &= (\langle b, c \rangle - \delta_i b(i) - \delta_i c(i)) r_3, & t_{23} r_4 &= \delta_i a(i) r_3 + \langle b, c \rangle r_4. \end{aligned}$$

Thus, Z is stable under t_{12}, t_{23} . Let $t'_{12}, t'_{23}: Z \rightarrow Z$ be the linear maps given by

$$t'_{12} = t_{12} - (\langle a, b \rangle - \delta_i a(i) - \delta_i b(i)) I, \quad t'_{23} = t_{23} - \langle b, c \rangle I,$$

where I is the identity map of Z . Then

$$\begin{aligned} t'_{12} r_3 &= (\delta_i a(i) + \delta_i b(i)) r_3 + \delta_i c(i) r_4, & t'_{12} r_4 &= 0, \\ t'_{23} r_3 &= (-\delta_i b(i) - \delta_i c(i)) r_3, & t'_{23} r_4 &= \delta_i a(i) r_3. \end{aligned}$$

We shall sometime represent a function $f: (0, 1) \rightarrow Z$ as a vector $\begin{pmatrix} f_3 \\ f_4 \end{pmatrix}$ where f_3, f_4 are functions $(0, 1) \rightarrow \mathbf{A}$ such that $f(z) = f_3(z) r_3 + f_4(z) r_4$. Then

$$\begin{aligned} t'_{12} \begin{pmatrix} f_3 \\ f_4 \end{pmatrix} &= \begin{pmatrix} \delta_i a(i) + \delta_i b(i) & 0 \\ \delta_i c(i) & 0 \end{pmatrix} \begin{pmatrix} f_3 \\ f_4 \end{pmatrix}, \\ t'_{23} \begin{pmatrix} f_3 \\ f_4 \end{pmatrix} &= \begin{pmatrix} -\delta_i b(i) - \delta_i c(i) & \delta_i a(i) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_3 \\ f_4 \end{pmatrix}. \end{aligned}$$

We interrupt the proof to state a lemma, which is easily verified.

Lemma 22.7. *Let z be a real number > 0 . If*

$$P = \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix},$$

then

$$z^P = \begin{pmatrix} z^p & 0 \\ q(z^p - 1)/p & 1 \end{pmatrix}, \quad z^Q = \begin{pmatrix} z^p & q(z^p - 1)/p \\ 0 & 1 \end{pmatrix}.$$

22.8. We return to the proof of Proposition 22.6. From the lemma it follows that for $z \in (0, 1)$, the operators $z^{-\varpi t'_{12}}$, $(1 - z)^{-\varpi t'_{23}}$ on Z are given by the 2×2 matrices

$$z^{-\varpi t'_{12}} = \begin{pmatrix} z^{-\varpi_i(a(i)+b(i))} & 0 \\ \frac{z^{-\varpi_i(a(i)+b(i))}-1}{a(i)+b(i)}c(i) & 1 \end{pmatrix}$$

$$(1 - z)^{-\varpi t'_{23}} = \begin{pmatrix} (1 - z)^{\varpi_i(b(i)+c(i))} & -\frac{(1-z)^{\varpi_i(b(i)+c(i))}-1}{b(i)+c(i)}a(i) \\ 0 & 1 \end{pmatrix}.$$

(Recall the notation $\varpi_i = \delta_i \varpi$.)

Let $\mathcal{X}'$ (resp. $\mathcal{X}''$) be the space of analytic functions f (resp. g) on the open real interval $(0, 1)$ with values in Z which are solutions of the differential equation

$$\frac{df}{dz} = \frac{\varpi t_{12}f}{z} + \frac{\varpi t_{23}f}{z-1} \quad \left(\text{resp. } \frac{dg}{dz} = \frac{\varpi t'_{12}g}{z} + \frac{\varpi t'_{23}g}{z-1} \right).$$

For $j = 0, 1$, let $R'_j: \mathcal{X}' \rightarrow Z$ be the isomorphisms defined by

$$R'_0(f) = \lim_{z \rightarrow 0} z^{-\varpi t_{12}} f, \quad R'_1(f) = \lim_{z \rightarrow 1} (1 - z)^{-\varpi t_{23}} f;$$

similarly, let $R''_j: \mathcal{X}'' \rightarrow Z$ be the isomorphisms defined by

$$R''_0(g) = \lim_{z \rightarrow 0} z^{-\varpi t'_{12}} g, \quad R''_1(g) = \lim_{z \rightarrow 1} (1 - z)^{-\varpi t'_{23}} g.$$

Let $\Phi: Z \rightarrow Z$ be the isomorphism defined by $\Phi = R'_1 R'^{-1}_0$. From the definitions we see that $R''_j \psi_1 = R'_j$ for $j = 0, 1$, where $\psi_1: \mathcal{X}' \rightarrow \mathcal{X}''$ is the isomorphism given by

$$f \mapsto g = z^{-\varpi(\langle a, b \rangle - \delta_i a(i) - \delta_i b(i))} (1 - z)^{-\varpi \langle b, c \rangle} f.$$

It follows that $\Phi = R''_1 R''^{-1}_0$. We set

$$\alpha = -\varpi \delta_i a(i), \quad \beta = \varpi \delta_i c(i), \quad \gamma = 1 - \varpi(\delta_i a(i) + \delta_i b(i)).$$

Let $\mathcal{U}$ be the vector space of analytic functions $u: (0, 1) \rightarrow \mathbf{A}$ which are solutions of the Gauss differential equation 22.5(a) with α, β, γ as above.

A simple computation shows that the map

$$u \mapsto \begin{pmatrix} z \frac{du}{dz} \\ \varpi \delta_i c(i) u \end{pmatrix}$$

is an isomorphism $\psi_2: \mathcal{Y} \rightarrow \mathcal{X}''$. For $j = 0, 1$, let $P_j: \mathcal{Y} \rightarrow Z$ be the isomorphisms defined by $P_j = R_j'' \psi_2$. Then from $\Phi = R_1'' R_0''^{-1}$, we deduce $\Phi = P_1 P_0^{-1}$.

Using the formulas above for the 2×2 matrices $z^{-\varpi i'_{12}}, (1-z)^{-\varpi i'_{23}}$, we see that

$$\begin{aligned}
 P_0(u) &= \lim_{z \rightarrow 0} z^{-\varpi i'_{12}} \begin{pmatrix} z \frac{du}{dz} \\ \varpi_i c(i) u \end{pmatrix} \\
 &= \lim_{z \rightarrow 0} \begin{pmatrix} z^{-\varpi_i a(i) - \varpi_i b(i)} z \frac{du}{dz} \\ \varpi_i c(i) u + \frac{z^{-\varpi_i a(i) - \varpi_i b(i)} - 1}{a(i) + b(i)} c(i) z \frac{du}{dz} \end{pmatrix}, \\
 P_1(u) &= \lim_{z \rightarrow 1} (1-z)^{-\varpi i'_{23}} \begin{pmatrix} z \frac{du}{dz} \\ \varpi \delta_i c(i) u \end{pmatrix} \\
 &= \lim_{z \rightarrow 1} \begin{pmatrix} (1-z)^{\varpi_i b(i) + \varpi_i c(i)} z \frac{du}{dz} - \frac{(1-z)^{\varpi_i b(i) + \varpi_i c(i)} - 1}{b(i) + c(i)} a(i) c(i) \varpi_i u \\ c(i) h_i u \end{pmatrix}
 \end{aligned}
 \tag{a}$$

for all $u \in \mathcal{Y}$. We set

$$\begin{aligned}
 u_1 &= z^{1-\gamma} (1-z)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 2-\gamma; z), \\
 u_2 &= F(\alpha, \beta, 1+\alpha+\beta-\gamma; 1-z), \\
 u_3 &= z^{1-\gamma} (1-z)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 1-\alpha-\beta+\gamma; 1-z), \\
 u_4 &= F(\alpha, \beta, \gamma; z).
 \end{aligned}$$

From [KF, 4.1, 4.2], we see that u_1, u_2, u_3, u_4 belong to $\mathcal{Y}$ and that

$$\begin{aligned}
 \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} u_4 &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} u_2 + \Gamma(\alpha+\beta-\gamma) u_3, \\
 \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta-\gamma+1)} u_2 &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)\Gamma(\alpha+1-\gamma)} u_4 + \Gamma(\gamma-1) u_1
 \end{aligned}$$

or, equivalently,

$$\begin{aligned}
 &\frac{\Gamma(-\varpi_i a(i))\Gamma(\varpi_i c(i))}{\Gamma(1-\varpi_i(a(i)+b(i)))} u_4 \\
 &= \frac{\Gamma(-\varpi_i a(i))\Gamma(\varpi_i c(i))\Gamma(1-\varpi_i(b(i)+c(i)))}{\Gamma(1-\varpi_i b(i))\Gamma(1-\varpi_i(a(i)+b(i)+c(i)))} u_2 \\
 &\quad + \Gamma(-1+\varpi_i(b(i)+c(i))) u_3, \\
 &\frac{\Gamma(-\varpi_i a(i))\Gamma(\varpi_i c(i))}{\Gamma(\varpi_i(b(i)+c(i)))} u_2 \\
 &= \frac{\Gamma(-\varpi_i a(i))\Gamma(\varpi_i c(i))\Gamma(\varpi_i(a(i)+b(i)))}{\Gamma(\varpi_i b(i))\Gamma(\varpi_i(a(i)+b(i)+c(i)))} u_4 \\
 &\quad + \Gamma(-\varpi_i(a(i)+b(i))) u_1.
 \end{aligned}
 \tag{b}$$

(Here and in subsequent formulas we extend scalars from $\mathbf{A}$ to $\mathbf{C}((\varpi))$.)

Using (a) and the definition of $F(\alpha, \beta, \gamma; z)$, we see that

$$\begin{aligned} P_0(u_1) &= \varpi_i b(i) r_1, \quad P_0(u_4) = \varpi_i c(i) r_4, \quad P_1(u_2) = \frac{\varpi_i b(i) c(i)}{b(i) + c(i)} r_2, \\ P_1(u_3) &= (1 - \varpi_i(b(i) + c(i))) r_3. \end{aligned}$$

From the equality $P_1 = \Phi P_0$ we deduce

$$P_0(u_2) = \frac{\varpi_i b(i) c(i)}{b(i) + c(i)} \Phi^{-1} r_2, \quad P_0(u_3) = (1 - \varpi_i(b(i) + c(i))) \Phi^{-1} r_3.$$

Applying P_0 to the identities (b) we therefore obtain

$$\begin{aligned} & \frac{\Gamma(-\varpi_i a(i)) \Gamma(\varpi_i c(i))}{\Gamma(1 - \varpi_i(a(i) + b(i)))} \varpi_i c(i) r_4 \\ &= \frac{\Gamma(-\varpi_i a(i)) \Gamma(\varpi_i c(i)) \Gamma(1 - \varpi_i(b(i) + c(i)))}{\Gamma(1 - \varpi_i b(i)) \Gamma(1 - \varpi_i(a(i) + b(i) + c(i)))} \frac{\varpi_i b(i) c(i)}{b(i) + c(i)} \Phi^{-1} r_2 \\ & \quad + \Gamma(-1 + \varpi_i(b(i) + c(i))) (1 - \varpi_i(b(i) + c(i))) \Phi^{-1} r_3, \\ & \frac{\Gamma(-\varpi_i a(i)) \Gamma(\varpi_i c(i))}{\Gamma(\varpi_i(b(i) + c(i)))} \frac{\varpi_i b(i) c(i)}{b(i) + c(i)} \Phi^{-1} r_2 \\ &= \frac{\Gamma(-\varpi_i a(i)) \Gamma(\varpi_i c(i)) \Gamma(\varpi_i(a(i) + b(i)))}{\Gamma(\varpi_i b(i)) \Gamma(\varpi_i(a(i) + b(i) + c(i)))} \varpi_i c(i) r_4 \\ & \quad + \Gamma(-\varpi_i(a(i) + b(i))) \varpi_i b(i) r_1, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \frac{\Gamma(-\varpi_i a(i)) \Gamma(1 - \varpi_i b(i))}{\Gamma(1 - \varpi_i(a(i) + b(i)))} \varpi_i c(i) r_4 \\ &= \frac{\Gamma(-\varpi_i a(i)) \Gamma(1 - \varpi_i(b(i) + c(i)))}{\Gamma(1 - \varpi_i(a(i) + b(i) + c(i)))} \frac{\varpi_i b(i) c(i)}{b(i) + c(i)} \Phi^{-1} r_2 \\ & \quad + \frac{\Gamma(-1 + \varpi_i(b(i) + c(i))) \Gamma(1 - \varpi_i b(i))}{\Gamma(\varpi_i c(i))} (1 - \varpi_i(b(i) + c(i))) \Phi^{-1} r_3, \\ & \frac{\Gamma(-\varpi_i a(i)) \Gamma(\varpi_i(a(i) + b(i) + c(i)))}{\Gamma(\varpi_i(b(i) + c(i)))} \frac{\varpi_i b(i) c(i)}{b(i) + c(i)} \Phi^{-1} r_2 \\ &= \frac{\Gamma(-\varpi_i a(i)) \Gamma(\varpi_i(a(i) + b(i)))}{\Gamma(\varpi_i b(i))} \varpi_i c(i) r_4 \\ & \quad + \frac{\Gamma(-\varpi_i(a(i) + b(i))) \Gamma(\varpi_i(a(i) + b(i) + c(i)))}{\Gamma(\varpi_i c(i))} \varpi_i b(i) r_1. \end{aligned}$$

Using the identity $\Gamma(x+1) = x\Gamma(x)$, we obtain

$$\begin{aligned} -\frac{\Gamma(-\varpi_i a(i)) \Gamma(-\varpi_i b(i))}{\Gamma(1 - \varpi_i(a(i) + b(i)))} r_4 &= -\frac{\Gamma(-\varpi_i a(i)) \Gamma(-\varpi_i(b(i) + c(i)))}{\Gamma(1 - \varpi_i(a(i) + b(i) + c(i)))} \Phi^{-1} r_2 \\ & \quad - \frac{\Gamma(\varpi_i(b(i) + c(i))) \Gamma(-\varpi_i b(i))}{\Gamma(1 + \varpi_i c(i))} \Phi^{-1} r_3, \end{aligned}$$

$$\begin{aligned}
& \frac{\Gamma(-\varpi_i a(i))\Gamma(\varpi_i(a(i) + b(i) + c(i)))}{\Gamma(1 + \varpi_i(b(i) + c(i)))} \Phi^{-1} r_2 \\
&= \frac{\Gamma(-\varpi_i a(i))\Gamma(\varpi_i(a(i) + b(i)))}{\Gamma(1 + \varpi_i b(i))} r_4 \\
&+ \frac{\Gamma(-\varpi_i(a(i) + b(i)))\Gamma(\varpi_i(a(i) + b(i) + c(i)))}{\Gamma(1 + \varpi_i c(i))} r_1.
\end{aligned}$$

Using the identity

$$\Gamma(\varpi_i n)\Gamma(1 - \varpi_i n)[n]_i = 2\pi\sqrt{-1}(v_i - v_i^{-1})$$

for $n \in \mathbf{Z} - \{0\}$ (notation of 19.1), we see that $\Gamma(\varpi_i n)\Gamma(1 - \varpi_i n)[n]_i$ is independent of n for $n \neq 0$. Hence we have

$$\begin{aligned}
& \frac{\Gamma(-\varpi_i a(i))\Gamma(-\varpi_i b(i))}{\Gamma(1 - \varpi_i(a(i) + b(i)))} \Gamma(-\varpi_i(b(i) + c(i))) \\
& \quad \times \Gamma(1 + \varpi_i(b(i) + c(i)))[-b(i) + c(i)]_i r_4 \\
&= \frac{\Gamma(-\varpi_i a(i))\Gamma(-\varpi_i(b(i) + c(i)))}{\Gamma(1 - \varpi_i(a(i) + b(i) + c(i)))} \Gamma(-\varpi_i b(i))\Gamma(1 + \varpi_i b(i))[-b(i)]_i \Phi^{-1} r_2 \\
& \quad + \frac{\Gamma(\varpi_i(b(i) + c(i)))\Gamma(-\varpi_i b(i))}{\Gamma(1 + \varpi_i c(i))} \Gamma(-\varpi_i a(i))\Gamma(1 + \varpi_i a(i))[-a(i)]_i \Phi^{-1} r_3, \\
& \frac{\Gamma(-\varpi_i a(i))\Gamma(\varpi_i(a(i) + b(i) + c(i)))}{\Gamma(1 + \varpi_i(b(i) + c(i)))} \Gamma(-\varpi_i(a(i) + b(i)))\Gamma(1 + \varpi_i(a(i) + b(i))) \\
& \quad \times [-(a(i) + b(i))]_i \Phi^{-1} r_2 \\
&= \frac{\Gamma(-\varpi_i a(i))\Gamma(\varpi_i(a(i) + b(i)))}{\Gamma(1 + \varpi_i b(i))} \Gamma(\varpi_i(a(i) + b(i) + c(i))) \\
& \quad \times \Gamma(1 - \varpi_i(a(i) + b(i) + c(i)))[a(i) + b(i) + c(i)]_i r_4 \\
& \quad + \frac{\Gamma(-\varpi_i(a(i) + b(i)))\Gamma(\varpi_i(a(i) + b(i) + c(i)))}{\Gamma(1 + \varpi_i c(i))} \\
& \quad \times \Gamma(-\varpi_i a(i))\Gamma(1 + \varpi_i a(i))[-a(i)]_i r_1.
\end{aligned}$$

Setting

$$\begin{aligned}
\tilde{r}_1 &= \frac{1}{\Gamma(1 + \varpi_i(a(i) + b(i)))\Gamma(1 + \varpi_i c(i))\Gamma(1 - \varpi_i(a(i) + b(i) + c(i)))} r_1, \\
\tilde{r}_2 &= \frac{1}{\Gamma(1 + \varpi_i a(i))\Gamma(1 + \varpi_i(b(i) + c(i)))\Gamma(1 - \varpi_i(a(i) + b(i) + c(i)))} \Phi^{-1} r_2, \\
\tilde{r}_3 &= \frac{1}{\Gamma(1 + \varpi_i b(i))\Gamma(1 + \varpi_i c(i))\Gamma(1 - \varpi_i(b(i) + c(i)))} \Phi^{-1} r_3, \\
\tilde{r}_4 &= \frac{1}{\Gamma(1 + \varpi_i a(i))\Gamma(1 + \varpi_i b(i))\Gamma(1 - \varpi_i(a(i) + b(i)))} r_4,
\end{aligned}$$

we deduce

$$\begin{aligned}
 & \Gamma(-\varpi_i(b(i) + c(i)))\Gamma(1 + \varpi_i(b(i) + c(i)))[-b(i) - c(i)]_i \tilde{r}_4 \\
 &= \Gamma(-\varpi_i(b(i) + c(i)))\Gamma(1 + \varpi_i(b(i) + c(i)))[-b(i)]_i \tilde{r}_2 \\
 & \quad + \Gamma(\varpi_i(b(i) + c(i)))\Gamma(1 - \varpi_i(b(i) + c(i)))[-a(i)]_i \tilde{r}_3, \\
 & \Gamma(-\varpi_i(a(i) + b(i)))\Gamma(1 + \varpi_i(a(i) + b(i)))[-a(i) - b(i)]_i \tilde{r}_2 \\
 &= \Gamma(\varpi_i(a(i) + b(i)))\Gamma(1 - \varpi_i(a(i) + b(i)))[a(i) + b(i) + c(i)]_i \tilde{r}_4 \\
 & \quad + \Gamma(-\varpi_i(a(i) + b(i)))\Gamma(1 + \varpi_i(a(i) + b(i)))[-a(i)]_i \tilde{r}_1,
 \end{aligned}$$

or, equivalently,

$$\begin{aligned}
 & [-b(i) - c(i)]_i^{-1}[-b(i) - c(i)]_i \tilde{r}_4 \\
 &= [-b(i) - c(i)]_i^{-1}[-b(i)]_i \tilde{r}_2 + [b(i) + c(i)]_i^{-1}[-a(i)]_i \tilde{r}_3, \\
 & [-a(i) - b(i)]_i^{-1}[-a(i) - b(i)]_i \tilde{r}_2 \\
 &= [a(i) + b(i)]_i^{-1}[a(i) + b(i) + c(i)]_i \tilde{r}_4 + [-a(i) - b(i)]_i^{-1}[-a(i)]_i \tilde{r}_1,
 \end{aligned}$$

or, equivalently,

$$\begin{aligned}
 & [b(i) + c(i)]_i \tilde{r}_4 = [b(i)]_i \tilde{r}_2 - [a(i)]_i \tilde{r}_3, \\
 & [a(i) + b(i)]_i \tilde{r}_2 = [a(i) + b(i) + c(i)]_i \tilde{r}_4 + [a(i)]_i \tilde{r}_1.
 \end{aligned}$$

Using the definitions, we deduce that, if ξ_k is the difference of the left-hand side and the right-hand side in the equation (k) of the proposition ($k = 1, 2, 3, 4$), then $\xi_2(y) = 0$, $\xi_4(y) = 0$. Since y generates $V_{a+b+c-\alpha_i}$ as a $\mathfrak{U}$ -module, it follows that $\xi_2 = 0$, $\xi_4 = 0$. Thus, the identities (2),(4) in the proposition are proved. The identities (1),(3) are formal consequences of the identities (2),(4). The proposition is proved.

Corollary 22.9. *Let $a, b, c \in \mathbf{N}^I$ be such that $a(i) > 0$, $b(i) > 0$, $c(i) > 0$. Let $C = C_{b,c}$, $A = A_{a,c,b}$, $A' = A_{a,b,c}$. We have*

$$\begin{aligned}
 & A'(T_{a,b} \otimes 1)\tau_{i;a+b,c} \\
 &= v^{\langle c,b \rangle}(1 \otimes C^{-1})A(\tau_{i;a,c} \otimes 1)T_{a+c-\alpha_i,b} + v_i^{-a(i)}(1 \otimes \tau_{i;b,c})T_{a,b+c-\alpha_i}
 \end{aligned}$$

as linear maps $V_{a+b+c-\alpha_i} \rightarrow V_a \otimes V_b \otimes V_c$.

Using Proposition 22.6 twice, we have

$$\begin{aligned}
 \text{(a)} \quad & [a(i)]_i(1 \otimes \tau_{i;c,b})T_{a,c+b-\alpha_i} + [c(i) + b(i)]_i A(\tau_{i;a,c} \otimes 1)T_{a+c-\alpha_i,b} \\
 &= [c(i)]_i(1 \otimes T_{c,b})\tau_{i;a,c+b}, \\
 \text{(b)} \quad & [b(i) + c(i)]_i A'(T_{a,b} \otimes 1)\tau_{i;a+b,c} - [c(i)]_i(1 \otimes T_{b,c})\tau_{i;a,b+c} \\
 &= [a(i) + b(i) + c(i)]_i(1 \otimes \tau_{i;b,c})T_{a,b+c-\alpha_i}.
 \end{aligned}$$

Using $T_{c,b} = v^{-\langle c,b \rangle} C T_{b,c}$ and $\tau_{i;c,b} = -v^{-\langle c,b \rangle + \delta_i c(i) + \delta_i b(i)} C \tau_{i;b,c}$ (the TC -identity and τC -identity) we may rewrite (a) as follows:

$$\begin{aligned} & -[a(i)]_i v^{\langle c,b \rangle} v^{-\langle c,b \rangle + \delta_i c(i) + \delta_i b(i)} (1 \otimes \tau_{i;b,c}) T_{a,c+b-\alpha_i} \\ & + [c(i) + b(i)]_i v^{\langle c,b \rangle} (1 \otimes C^{-1}) A(\tau_{i;a,c} \otimes 1) T_{a+c-\alpha_i, b} \\ & = [c(i)]_i (1 \otimes T_{b,c}) \tau_{i;a,c+b}. \end{aligned}$$

We substitute the right-hand side of the last equality in the left-hand side of (b); we obtain

$$\begin{aligned} & [b(i) + c(i)]_i A'(T_{a,b} \otimes 1) \tau_{i;a+b,c} \\ & + [a(i)]_i v^{\langle c,b \rangle} v^{-\langle c,b \rangle + \delta_i c(i) + \delta_i b(i)} (1 \otimes \tau_{i;b,c}) T_{a,c+b-\alpha_i} \\ & - [c(i) + b(i)]_i v^{\langle c,b \rangle} (1 \otimes C^{-1}) A(\tau_{i;a,c} \otimes 1) T_{a+c-\alpha_i, b} \\ & = [a(i) + b(i) + c(i)]_i (1 \otimes \tau_{i;b,c}) T_{a,b+c-\alpha_i}. \end{aligned}$$

We note that

$$[a(i) + b(i) + c(i)]_i - [a(i)]_i v^{\langle c,b \rangle} v^{-\langle c,b \rangle + \delta_i c(i) + \delta_i b(i)} = v_i^{-a(i)} [b(i) + c(i)]_i.$$

We cancel $[b(i) + c(i)]_i$; the corollary follows.

23. THE MORPHISMS $\Phi_{i;a,b}^c, \Psi_{i;a,b}^c$

23.1. Until the end of 23.3, we fix $i \in I$. Let $a, b, c \in \mathbf{N}^I$. If $a(i) \geq 1$ and $c(i) \geq 1$, we define a morphism

$$\Phi_{i;a,b}^c: V_{\bar{a}+\bar{c}-\bar{\alpha}_i} \otimes V_{c+b} \rightarrow V_{\bar{a}} \otimes V_b$$

as the composition

$$V_{\bar{a}+\bar{c}-\bar{\alpha}_i} \otimes V_{c+b} \xrightarrow{[c(i)]_i^{-1} \tau_{i;a,c} \otimes T_{c,b}} V_{\bar{a}} \otimes V_{\bar{c}} \otimes V_c \otimes V_b \xrightarrow{(1 \otimes S_{\bar{c}} \otimes 1)A} V_{\bar{a}} \otimes V_b,$$

where $A = B_{\bar{a}, \bar{c}, c, b}$.

Similarly, if $b(i) \geq 1$ and $c(i) \geq 1$, we define a morphism

$$\Psi_{i;a,b}^c: V_{\bar{a}+\bar{c}} \otimes V_{c+b-\alpha_i} \rightarrow V_{\bar{a}} \otimes V_b$$

as the composition

$$V_{\bar{a}+\bar{c}} \otimes V_{c+b-\alpha_i} \xrightarrow{[c(i)]_i^{-1} T_{\bar{a}, \bar{c}} \otimes \tau_{i;c,b}} V_{\bar{a}} \otimes V_{\bar{c}} \otimes V_c \otimes V_b \xrightarrow{(1 \otimes S_{\bar{c}} \otimes 1)A} V_{\bar{a}} \otimes V_b,$$

where $A = B_{\bar{a}, \bar{c}, c, b}$.

Proposition 23.2 (Φ tr-identities). *Let $a, b, c, d \in \mathbf{N}^I$.*

(a) *If $a(i) \geq 1$ and $d(i) \geq 1$, then the composition*

$$V_{\bar{a}+\bar{c}+\bar{d}-\bar{\alpha}_i} \otimes V_{d+c+b} \xrightarrow{\Phi_{i;a+c,b}^d} V_{\bar{a}+\bar{c}} \otimes V_{c+b} \xrightarrow{tr_{a,b}^c} V_{\bar{a}} \otimes V_b$$

is equal to $\Phi_{i;a,b}^{c+d}$.

(b) If $a(i) \geq 1$ and $c(i) \geq 1$, then the composition

$$V_{a+\bar{c}+\bar{d}-\bar{\alpha}_i} \otimes V_{d+c+b} \xrightarrow{tr_{a+c-\alpha_i, c+b}^d} V_{a+\bar{c}-\bar{\alpha}_i} \otimes V_{c+b} \xrightarrow{\Phi_{i;a,b}^c} V_a \otimes V_b$$

is equal to $\Phi_{i;a,b}^{c+d}$.

The composition in (a) is equal to the composition (1):

$$\begin{aligned} V_{a+\bar{c}+\bar{d}-\bar{\alpha}_i} \otimes V_{d+c+b} &\xrightarrow{[d(i)]_i^{-1} \tau_{i;a+\bar{c},d} \otimes T_{d,c+b}} V_{a+\bar{c}} \otimes V_{\bar{d}} \otimes V_d \otimes V_{c+b} \\ &\xrightarrow{(1 \otimes S_d \otimes 1)A_1} V_{a+\bar{c}} \otimes V_{c+b} \xrightarrow{T_{a,\bar{c}} \otimes T_{c,b}} V_a \otimes V_{\bar{c}} \otimes V_c \otimes V_b \xrightarrow{(1 \otimes S_c \otimes 1)A_2} V_a \otimes V_b, \end{aligned}$$

where $A_1 = B_{a+\bar{c},\bar{d},d,c+b}$ and $A_2 = B_{a,\bar{c},c,b}$. Here we replace

$$(T_{a,\bar{c}} \otimes T_{c,b})(1 \otimes S_d \otimes 1)A_1$$

by

$$\begin{aligned} (1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{c,b})A_1 \\ = (1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_3(T_{a,\bar{c}} \otimes 1 \otimes 1 \otimes T_{c,b}), \end{aligned}$$

where $A_3 = B_{a+\bar{c},\bar{d},d,c \otimes b}$. The composition (1) becomes

$$(1 \otimes S_c \otimes 1)A_2(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_3(F_1 \otimes F_2)$$

where

$$F_1 = [d(i)]_i^{-1}(T_{a,\bar{c}} \otimes 1)\tau_{i;a+\bar{c},d}, \quad F_2 = (1 \otimes T_{c,b})T_{d,c+b}.$$

By definition we have

$$\Phi_{i;a,b}^{c+d} = [c(i) + d(i)]_i^{-1}(1 \otimes S_{c+d} \otimes 1)A_4(\tau_{i;a,\bar{c}+d} \otimes T_{d+c,b}),$$

where $A_4 = B_{a,\bar{c}+d,d+c,b}$. In this expression for $\Phi_{i;a,b}^{c+d}$ we replace, using the *TS*-identity, $1 \otimes S_{c+d} \otimes 1$ by

$$(1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5(1 \otimes T_{\bar{c},d} \otimes T_{d,c} \otimes 1),$$

where $A_5 = 1 \otimes B_{\bar{c},\bar{d},d,c} \otimes 1$. Next we replace

$$(1 \otimes T_{\bar{c},d} \otimes T_{d,c} \otimes 1)A_4 \text{ by } A_6(1 \otimes T_{\bar{c},d} \otimes T_{d,c} \otimes 1),$$

where $A_6 = B_{a,\bar{c} \otimes \bar{d},d \otimes c,b}$; we obtain

$$\Phi_{i;a,b}^{c+d} = (1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5A_6(F_3 \otimes F_4),$$

where

$$F_3 = [c(i) + d(i)]_i^{-1}(1 \otimes T_{\bar{c},d})\tau_{i;a,\bar{c}+d}, \quad F_4 = (T_{d,c} \otimes 1)T_{d+c,b}.$$

We set $A_7 = A'_7 \otimes A''_7$, where $A'_7 = A_{a,\bar{c},\bar{d}}$ and $A''_7 = A_{d,c,b}^{-1}$. Using the pentagon identity and the functoriality of the associativity isomorphisms we see that

$$(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5A_6A_7 = A_2(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_3.$$

We are therefore reduced to checking the following equality

$$(1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5A_6A_7(F_1 \otimes F_2) \\ = (1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5A_6A_7(A_7'^{-1}F_3 \otimes A_7''^{-1}F_4).$$

Using $A_7''F_2 = F_4$ (the TT -identity) we see that this is equivalent to the equality

$$(1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5A_6(A_7'(F_1 - A_7'^{-1}F_3) \otimes F_4) = 0.$$

By the τT -identity (3) in 22.6, we have

$$F_1 - A_7'^{-1}F_3 = [a(i) + c(i) + d(i)]_i[d(i)]_i^{-1}[c(i) + d(i)]_i^{-1}A_7'^{-1}(1 \otimes \tau_{\bar{i}; \bar{c}, \bar{d}})T_{\bar{a}, \bar{c} + \bar{d}};$$

hence we are reduced to proving the equality

$$(1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5A_6(1 \otimes \tau_{\bar{i}; \bar{c}, \bar{d}} \otimes T_{d, c} \otimes 1)(T_{\bar{a}, \bar{c} + \bar{d} - \bar{\alpha}_i} \otimes T_{d + c, b}) = 0,$$

or, equivalently, the equality

$$(1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5(1 \otimes \tau_{\bar{i}; \bar{c}, \bar{d}} \otimes T_{d, c} \otimes 1)A_8(T_{\bar{a}, \bar{c} + \bar{d} - \bar{\alpha}_i} \otimes T_{d + c, b}) = 0,$$

where $A_8 = B_{\bar{a}, \bar{c} + \bar{d} - \bar{\alpha}_i, d + c, b}$. It is then enough to prove the equality

$$(1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5(1 \otimes \tau_{\bar{i}; \bar{c}, \bar{d}} \otimes T_{d, c} \otimes 1) = 0.$$

This follows from Lemma 22.4(a).

This proves (a). The proof of (b) is similar; instead of the τT -identity (3) in 22.6, we use the τT -identity (2) in 22.6:

$$([c(i) + d(i)]_i^{-1}A^{-1}(1 \otimes T_{\bar{c}, d})\tau_{\bar{i}; \bar{a}, \bar{c} + \bar{d}} - [c(i)]_i^{-1}(\tau_{\bar{i}}; \bar{a}, \bar{c} \otimes 1)T_{\bar{a} + \bar{c} - \bar{\alpha}_i, d}) \\ = [a(i)]_i[c(i)]_i^{-1}[c(i) + d(i)]_i^{-1}A^{-1}(1 \otimes \tau_{\bar{i}}; \bar{c}, \bar{d} \otimes 1)T_{\bar{a}, \bar{c} + \bar{d} - \bar{\alpha}_i},$$

where $A = A_{\bar{a}, \bar{c}, d}$. This completes the proof.

Proposition 23.3 (Ψtr -identities). *Let $a, b, c, d \in \mathbf{N}^I$.*

(a) *If $b(i) \geq 1$ and $d(i) \geq 1$, then the composition*

$$V_{\bar{a} + \bar{c} + \bar{d}} \otimes V_{d + c + b - \alpha_i} \xrightarrow{\Psi_{i; a + c, c + b}^d} V_{\bar{a} + \bar{c}} \otimes V_{c + b} \xrightarrow{tr_{a, b}^c} V_a \otimes V_b$$

is equal to $\Psi_{i; a, b}^{c + d}$.

(b) *If $b(i) \geq 1$ and $c(i) \geq 1$, then the composition*

$$V_{\bar{a} + \bar{c} + \bar{d}} \otimes V_{d + c + b - \alpha_i} \xrightarrow{tr_{a + c, c + b - \alpha_i}^d} V_{\bar{a} + \bar{c} - \bar{\alpha}_i} \otimes V_{c + b} \xrightarrow{\Psi_{i; a, b}^c} V_a \otimes V_b$$

is equal to $\Psi_{i; a, b}^{c + d}$.

The proof is entirely similar to that of the previous proposition; it is based on the identities

$$- [d(i)]_i^{-1}A^{-1}(1 \otimes T_{c, b})\tau_{i; d, c + b} + [c(i) + d(i)]_i^{-1}(T_{d, c} \otimes 1)\tau_{i; d + c, b} \\ = [d(i) + c(i) + b(i)]_i[d(i)]_i^{-1}[c(i) + d(i)]_i^{-1}(\tau_{i; d, c} \otimes 1)T_{d + c - \alpha_i, b}$$

and

$$\begin{aligned} & [c(i) + d(i)]_i^{-1} (T_{d,c} \otimes 1) \tau_{i;d+c,b} - [c(i)]_i^{-1} A^{-1} (1 \otimes \tau_{i;c,b}) T_{d,c+b-\alpha_i} \\ & = [b(i)]_i [c(i) + d(i)]_i^{-1} [c(i)]_i^{-1} (\tau_{i;d,c} \otimes 1) T_{d+c-\alpha_i,b} \end{aligned}$$

(see the τT -identities (4) and (1) in 22.6), where $A = A_{d,c,b}$.

23.4. Until the end of 23.5, we assume that $i, j \in I$ are given. Let $a, b, c, d \in \mathbf{N}^I$ be such that $a(i) \geq 1, c(i) \geq 1, d(i) \geq 1, b(j) \geq 1, c(j) \geq 1, d(j) \geq 1$.

We consider the morphisms

$$(P_1) \quad V_{a+c+d-\bar{\alpha}_i} \otimes V_{d+c+b-\alpha_j} \xrightarrow{\Psi_{j;a+c-\alpha_i,c+b}^d} V_{a+c-\bar{\alpha}_i} \otimes V_{c+b} \xrightarrow{\Phi_{i;a,b}^c} V_a \otimes V_b,$$

$$(P_2) \quad V_{a+c+d-\bar{\alpha}_i} \otimes V_{d+c+b-\alpha_j} \xrightarrow{\Phi_{i;a+c,c+b-\alpha_j}^d} V_{a+c} \otimes V_{c+b-\alpha_j} \xrightarrow{\Psi_{j;a,b}^c} V_a \otimes V_b.$$

We also consider the morphism

$$P_3: V_{a+c+d-\bar{\alpha}_i} \otimes V_{d+c+b-\alpha_j} \rightarrow V_a \otimes V_b$$

defined as 0 if $i \neq j$ and as $-\rho_{i;c,d}[b(i)-a(i)]_i \text{tr}_{a,b}^{c+d-\alpha_i}$ if $i = j$. Here $\rho_{i;c,d}$ is as in 22.4(c).

Proposition 23.5 ($\Phi\Psi$ -identity). *We have $P_1 - P_2 = P_3$.*

P_1 is equal to the composition

$$\begin{aligned} & V_{a+c+d-\bar{\alpha}_i} \otimes V_{d+c+b-\alpha_j} \xrightarrow{[d(j)]_j^{-1} T_{a+c-\bar{\alpha}_i,d} \otimes \tau_{j;d,c+b}} V_{a+c-\bar{\alpha}_i} \otimes V_d \otimes V_d \otimes V_{c+b} \\ & \xrightarrow{(1 \otimes S_d \otimes 1) A_1} V_{a+c-\bar{\alpha}_i} \otimes V_{c+b} \xrightarrow{[c(i)]_i^{-1} \tau_{i;a,c} \otimes T_{c,b}} V_a \otimes V_c \otimes V_c \otimes V_b \\ & \xrightarrow{(1 \otimes S_c \otimes 1) A_2} V_a \otimes V_b, \end{aligned}$$

where $A_1 = B_{a+c-\bar{\alpha}_i,d,d,c+b}$ and $A_2 = B_{a,c,c,b}$. We replace here

$$(\tau_{i;a,c} \otimes T_{c,b})(1 \otimes S_d \otimes 1) \quad \text{by} \quad (1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)(\tau_{i;a,c} \otimes 1 \otimes 1 \otimes T_{c,b}),$$

and

$$(\tau_{i;a,c} \otimes 1 \otimes 1 \otimes T_{c,b}) A_1 \quad \text{by} \quad A_3(\tau_{i;a,c} \otimes 1 \otimes 1 \otimes T_{c,b}),$$

where $A_3 = B_{a+c-\bar{\alpha}_i,d,d,c \otimes b}$. Thus,

$$\begin{aligned} P_1 &= [d(j)]_j^{-1} [c(i)]_i^{-1} (1 \otimes S_c \otimes 1) A_2 (1 \otimes 1 \otimes S_d \otimes 1 \otimes 1) A_3 \\ &\quad \times (\tau_{i;a,c} \otimes 1 \otimes 1 \otimes T_{c,b}) (T_{a+c-\bar{\alpha}_i,d} \otimes \tau_{j;d,c+b}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} P_2 &= [d(i)]_i^{-1} [c(j)]_j^{-1} (1 \otimes S_c \otimes 1) A_2 (1 \otimes 1 \otimes S_d \otimes 1 \otimes 1) A_3 \\ &\quad \times (T_{a,c} \otimes 1 \otimes 1 \otimes \tau_{j;c,b}) (\tau_{i;a+c,d} \otimes T_{d,c+b-\alpha_j}). \end{aligned}$$

Hence

$$P_1 - P_2 = (1 \otimes S_c \otimes 1)A_2(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_3 \\ \times ([c(i)]_i^{-1}[d(j)]_j^{-1}Q_4 \otimes Q'_2 - [c(j)]_j^{-1}[d(i)]_i^{-1}Q_1 \otimes Q'_3),$$

where

$$Q_4 = (\tau_{\bar{i}; a, \bar{c}} \otimes 1)T_{\bar{a} + \bar{c} - \bar{\alpha}_i, d}, \quad Q'_2 = (1 \otimes T_{c, b})\tau_{j; d, c+b}, \\ Q_1 = (T_{\bar{a}, \bar{c}} \otimes 1)\tau_{\bar{i}; \bar{a} + \bar{c}, \bar{d}}, \quad Q'_3 = (1 \otimes \tau_{j; c, b})T_{d, c+b-\alpha_j}.$$

Let

$$Q_2 = (1 \otimes T_{\bar{c}, d})\tau_{\bar{i}; \bar{a}, \bar{c} + \bar{d}}, \quad Q_3 = (1 \otimes \tau_{\bar{i}; \bar{c}, d})T_{\bar{a}, \bar{c} + \bar{d} - \bar{\alpha}_i}, \\ Q'_1 = (T_{d, c} \otimes 1)\tau_{j; d+c, b}, \quad Q'_4 = (\tau_{j; d, c} \otimes 1)T_{d+c-\alpha_j, b}.$$

By the τT -identities 22.6, we can write Q_1, Q_4 as linear combinations of $A_4^{-1}Q_2, A_4^{-1}Q_3$ and we can write Q'_2, Q'_3 as linear combinations of $A'_4Q'_1, A'_4Q'_4$, where $A_4 = A_{\bar{a}, \bar{c}, \bar{d}}, A'_4 = A_{d, c, b}$. Hence, we have

$$[c(i)]_i^{-1}[d(j)]_j^{-1}Q_4 \otimes Q'_2 - [c(j)]_j^{-1}[d(i)]_i^{-1}Q_1 \otimes Q'_3 \\ = [c(i)]_i^{-1}[d(j)]_j^{-1} \\ \times ([c(i)]_i[c(i) + d(i)]_i^{-1}A_4^{-1}Q_2 - [a(i)]_i[c(i) + d(i)]_i^{-1}A_4^{-1}Q_3) \\ \otimes ([d(j)]_j[c(j) + d(j)]_j^{-1}A'_4Q'_1 + [d(j) + c(j) + b(j)]_j[c(j) + d(j)]_j^{-1}A'_4Q'_4) \\ - [c(j)]_j^{-1}[d(i)]_i^{-1} \\ \times ([d(i)]_i[c(i) + d(i)]_i^{-1}A_4^{-1}Q_2 + [a(i) + c(i) + d(i)]_i[c(i) + d(i)]_i^{-1}A_4^{-1}Q_3) \\ \otimes ([c(j)]_j[c(j) + d(j)]_j^{-1}A'_4Q'_1 - [b(j)]_j[c(j) + d(j)]_j^{-1}A'_4Q'_4) \\ = f_{24}A_4^{-1}Q_2 \otimes A'_4Q'_4 + f_{31}A_4^{-1}Q_3 \otimes A'_4Q'_1 + f_{34}A_4^{-1}Q_3 \otimes A'_4Q'_4;$$

here f_{24}, f_{31}, f_{34} are suitable elements of $\mathbf{A}$ such that

$$f_{34} = [c(i)]_i^{-1}[d(i)]_i^{-1}[c(i) + d(i)]_i^{-1}[b(i) - a(i)]_i \quad \text{if } i = j.$$

It follows that $P_1 - P_2 = R_{24} + R_{31} + R_{34}$, where

$$R_{24} = f_{24}(1 \otimes S_c \otimes 1)A_2(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_3\tilde{A}_4(1 \otimes T_{\bar{c}, d} \otimes \tau_{j; d, c} \otimes 1) \\ \times (\tau_{\bar{i}; \bar{a}, \bar{c} + \bar{d}} \otimes T_{d+c-\alpha_j, b}), \\ R_{31} = f_{31}(1 \otimes S_c \otimes 1)A_2(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_3\tilde{A}_4(1 \otimes \tau_{\bar{i}; \bar{c}, d} \otimes T_{d, c} \otimes 1) \\ \times (T_{\bar{a}, \bar{c} + \bar{d} - \bar{\alpha}_i} \otimes \tau_{j; d+c, b}), \\ R_{34} = f_{34}(1 \otimes S_c \otimes 1)A_2(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_3\tilde{A}_4(1 \otimes \tau_{\bar{i}; \bar{c}, d} \otimes \tau_{j; d, c} \otimes 1) \\ \times (T_{\bar{a}, \bar{c} + \bar{d} - \bar{\alpha}_i} \otimes T_{d+c-\alpha_j, b}),$$

and $\tilde{A}_4 = A_4^{-1} \otimes A'_4$.

Using the pentagon identity and the functoriality of the associativity isomorphisms we see that

$$A_2(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_3\tilde{A}_4 = (1 \otimes 1 \otimes S_d \otimes 1 \otimes 1)A_5A_6,$$

where, as in the proof of 23.2, $A_5 = 1 \otimes B_{\bar{c}, d, d, c} \otimes 1$ and $A_6 = B_{a, \bar{c} \otimes d, d \otimes c, b}$. Moreover, we have

$$\begin{aligned} A_6(1 \otimes T_{\bar{c}, d} \otimes \tau_{j; d, c} \otimes 1) &= (1 \otimes T_{\bar{c}, d} \otimes \tau_{j; d, c} \otimes 1) B_{a, \bar{c} + d, d + c - \alpha_j, b}, \\ A_6(1 \otimes \tau_{\bar{i}; \bar{c}, d} \otimes T_{d, c} \otimes 1) &= (1 \otimes \tau_{\bar{i}; \bar{c}, d} \otimes T_{d, c} \otimes 1) B_{a, \bar{c} + d - \bar{\alpha}_i, d + c, b}, \\ A_6(1 \otimes \tau_{\bar{i}; \bar{c}, d} \otimes \tau_{j; d, c} \otimes 1) &= (1 \otimes \tau_{\bar{i}; \bar{c}, d} \otimes \tau_{j; d, c} \otimes 1) B_{a, \bar{c} + d - \bar{\alpha}_i, d + c - \alpha_j, b}. \end{aligned}$$

It follows that

$$\begin{aligned} R_{24} &= f_{24}(1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1) A_5(1 \otimes T_{\bar{c}, d} \otimes \tau_{j; d, c} \otimes 1) \\ &\quad \times B_{a, \bar{c} + d, d + c - \alpha_j, b}(\tau_{\bar{i}; a, \bar{c} + d} \otimes T_{d + c - \alpha_j, b}), \\ R_{31} &= f_{31}(1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1) A_5(1 \otimes \tau_{\bar{i}; \bar{c}, d} \otimes T_{d, c} \otimes 1) \\ &\quad \times B_{a, \bar{c} + d - \bar{\alpha}_i, d + c, b}(T_{a, \bar{c} + d - \bar{\alpha}_i} \otimes \tau_{j; d + c, b}), \\ R_{34} &= f_{34}(1 \otimes S_c \otimes 1)(1 \otimes 1 \otimes S_d \otimes 1 \otimes 1) A_5(1 \otimes \tau_{\bar{i}; \bar{c}, d} \otimes \tau_{j; d, c} \otimes 1) \\ &\quad \times B_{a, \bar{c} + d - \bar{\alpha}_i, d + c - \alpha_j, b}(T_{a, \bar{c} + d - \bar{\alpha}_i} \otimes T_{d + c - \alpha_j, b}). \end{aligned}$$

Using Lemma 22.4(a),(b), we see that $R_{24} = 0$, $R_{31} = 0$. A similar argument shows that, in the case where $i \neq j$, we have $R_{34} = 0$: we use the fact that any morphism $V_{\bar{c} + d - \bar{\alpha}_i} \otimes V_{d + c - \alpha_j} \rightarrow \mathbf{A}$ (in $\mathcal{D}$) is zero.

Hence we may assume that $i = j$ and we have $P_1 - P_2 = R_{34}$. Using our formula for f_{34} and Lemma 22.4(c), we see that R_{34} is equal to

$$\begin{aligned} &-\rho_{i; c, d}[b(i) - a(i)]_i(1 \otimes S_{c + d} \otimes 1) B_{a, \bar{c} + d - \bar{\alpha}_i, d + c - \alpha_i, b}(T_{a, \bar{c} + d - \bar{\alpha}_i} \otimes T_{d + c - \alpha_i, b}) \\ &= -\rho_{i; c, d}[b(i) - a(i)]_i tr_{a, b}^{c + d - \alpha_i}. \end{aligned}$$

The proposition is proved.

Proposition 23.6. *Given $i \in I$, there is a unique element $\rho_i \in \mathbf{A}$ such that $\rho_{i; c, d} = \rho_i$ whenever $c, d \in \mathbf{N}^I$ satisfy $c(i) \geq 1, d(i) \geq 1$. We have $\rho_i = 1 \bmod \varpi$.*

We consider $a, b, e \in \mathbf{N}^I$ such that $a(i) \geq 1, b(i) \geq 1$, and $a(i) \neq b(i)$. We also consider $c, d \in \mathbf{N}^I$ such that $c(i) \geq 1, d(i) \geq 1$. Using the Φtr -identity 23.2, the Ψtr -identity 23.3, the $\Phi\Psi$ -identity 23.5 and the transitivity of tr 21.3, we have

$$\begin{aligned} (a) \quad &\Phi_{i; a, b}^c \Psi_{i; a + c - \alpha_i, c + b}^{d + e} - \Psi_{i; a, b}^c \Phi_{i; a + c, c + b - \alpha_i}^{d + e} = -\rho_{i; c, d + e}[b(i) - a(i)]_i tr_{a, b}^{c + d + e - \alpha_i}, \\ (b) \quad &\Phi_{i; a, b}^c \Psi_{i; a + c - \alpha_i, c + b}^d - \Psi_{i; a, b}^c \Phi_{i; a + c, c + b - \alpha_i}^d = -\rho_{i; c, d}[b(i) - a(i)]_i tr_{a, b}^{c + d - \alpha_i}, \\ (c) \quad &\Psi_{i; a + c - \alpha_i, c + b}^{d + e} = \Psi_{i; a + c - \alpha_i, c + b}^d tr_{a + c + d - \alpha_i, d + c + b - \alpha_i}^e, \\ (d) \quad &\Phi_{i; a + c, c + b - \alpha_i}^{d + e} = \Phi_{i; a + c, c + b - \alpha_i}^d tr_{a + c + d - \alpha_i, d + c + b - \alpha_i}^e, \\ (e) \quad &tr_{a, b}^{c + d + e - \alpha_i} = tr_{a, b}^{c + d - \alpha_i} tr_{a + c + d - \alpha_i, d + c + b - \alpha_i}^e. \end{aligned}$$

Introducing (c), (d), (e) in (a), we obtain

$$\begin{aligned} & \Phi_{i;a,b}^c \Psi_{i;a+c-\alpha_i, c+b}^d tr_{a+c+d-\alpha_i, d+c+b-\alpha_i}^e \\ & - \Psi_{i;a,b}^c \Phi_{i;a+c, c+b-\alpha_i}^d tr_{a+c+d-\alpha_i, d+c+b-\alpha_i}^e \\ & = -\rho_{i;c, d+e} [b(i) - a(i)]_i tr_{a,b}^{c+d-\alpha_i} tr_{a+c+d-\alpha_i, d+c+b-\alpha_i}^e, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & (\Phi_{i;a,b}^c \Psi_{i;a+c-\alpha_i, c+b}^d - \Psi_{i;a,b}^c \Phi_{i;a+c, c+b-\alpha_i}^d + \omega_{i;c, d+e} [b(i) - a(i)] tr_{a,b}^{c+d-\alpha_i}) \\ & \times tr_{a+c+d-\alpha_i, d+c+b-\alpha_i}^e = 0. \end{aligned}$$

Using (b) here, we deduce

$$(-\rho_{i;c, d} + \rho_{i;c, d+e}) [b(i) - a(i)]_i tr_{a,b}^{c+d-\alpha_i} tr_{a+c+d-\alpha_i, d+c+b-\alpha_i}^e = 0.$$

Now $tr_{a,b}^{c+d-\alpha_i} tr_{a+c+d-\alpha_i, d+c+b-\alpha_i}^e$ is nonzero, since by applying it to $x_{a+c+d+e-\alpha_i} \otimes y_{e+d+c+b-\alpha_i}$ and using 21.2, we get a nonzero vector, namely one that is $x_a \otimes y_b \bmod \varpi$. It follows that $(-\rho_{i;c, d} + \rho_{i;c, d+e}) [b(i) - a(i)]_i = 0$. We have $[b(i) - a(i)]_i \neq 0$ (by assumption). Hence $\rho_{i;c, d} = \rho_{i;c, d+e}$. Thus, $\rho_{i;c, d}$ is independent of d . Similarly, it is independent of c . The proposition follows.

23.7. Recall that the morphisms S_a were defined in 20.12(b) in terms of a fixed admissible collection (g_a) (see 20.11). If (g_a) is replaced by another admissible collection $(\tilde{g}_a)$, then S_a are replaced by $\tilde{S}_a = \frac{\tilde{g}_a}{g_a} S_a$ and ρ_i in the previous proposition is replaced by

$$(a) \quad \tilde{\rho}_i = \rho_i \frac{\tilde{g}_c \tilde{g}_d g_{c+d-\alpha_i}}{g_c g_d \tilde{g}_{c+d-\alpha_i}},$$

where $c, d \in \mathbb{N}^I$ satisfy $c(i) \geq 1, d(i) \geq 1$.

Recall from 20.12(a), that $g_a = \prod_{i \in I} \sigma_i^{a(i)} g_a^0$ for all $a \in \mathbb{N}^I$ where $\sigma_i \in \mathbb{A}$ have constant term 1.

Similarly, $\tilde{g}_a = \prod_{i \in I} \tilde{\sigma}_i^{a(i)} g_a^0$ for all $a \in \mathbb{N}^I$, where $\tilde{\sigma}_i \in \mathbb{A}$ have constant term 1. Substituting in (a), we obtain

$$\tilde{\rho}_i = \rho_i \prod_{j \in I} \frac{\tilde{\sigma}_j^{\alpha_i(j)}}{\sigma_j^{\alpha_i(j)}}$$

or, equivalently,

$$l_i \tilde{\rho}_i = \prod_{j \in I} \tilde{\sigma}_j^{a_{ji}},$$

where $l_i = \rho_i^{-1} \prod_{j \in I} \sigma_j^{a_{ji}}$.

We now choose $(\tilde{\sigma}_j)$ as follows: $\tilde{\sigma}_j = \prod_{k \in I} l_k^{b_{kj}}$. (Recall that (b_{ij}) is the inverse of the Cartan matrix (a_{ij}) ; its entries are rational numbers. Since $l_k \in \mathbb{A}$

has constant term 1, the power $l_k^q \in \mathbf{A}$ with q rational is well defined if we require that it has constant term 1.) Then $\tilde{\sigma}_j \in \mathbf{A}$ has constant term 1 and we have $\tilde{\rho}_i = 1$ for all i .

The previous argument shows that *there is a unique collection $(\sigma_i)_{i \in I}$ of elements of $\mathbf{A}$ with constant term 1 such that, if (g_a) are defined in terms of (σ_i) as in 20.12(a) and S_a is defined in terms of (g_a) as in 20.12(b), then ρ_i (given by 23.6) is equal to 1 for any i .*

From now on we assume that S_a is defined in the way just described. Since the ρ_i are 1, we can restate Lemma 22.4(c) as follows.

Proposition (τS -identity). *Let $a, b \in \mathbf{N}^I$ be such that $a(i) \geq 1, b(i) \geq 1$. Let $B = B_{a, \bar{b}, b, a}$. The composition*

$$V_{a+b-\bar{\alpha}_i} \otimes V_{b+a-\alpha_i} \xrightarrow{\tau_{i; a, b} \otimes \tau_{i; b, a}} V_a \otimes V_b \otimes V_b \otimes V_a \xrightarrow{(1 \otimes S_b \otimes 1)B} V_a \otimes V_a \xrightarrow{S_a} \mathbf{A}$$

is equal to $-[a(i)]_i [b(i)]_i [a(i) + b(i)]_i S_{a+b-\alpha_i}$.

23.8. In the remainder of this section $i \in I$ is fixed. Let $a, a', b, b', c \in \mathbf{N}^I$ be such that $a(i) \geq 1, a'(i) \geq 1, c(i) \geq 1$. Let G_1 be the composition

$$V_{a+a'+c-\bar{\alpha}_i} \otimes V_{c+b+b'} \xrightarrow{\Phi_{i; a+a', b+b'}^c} V_{a+a'} \otimes V_{b+b'} \xrightarrow{m} V_a \otimes V_b \otimes V_{a'} \otimes V_{b'},$$

let G_2 be the composition

$$V_{a+a'+c-\bar{\alpha}_i} \otimes V_{c+b+b'} \xrightarrow{m} V_{a+c-\bar{\alpha}_i} \otimes V_{c+b} \otimes V_{a'} \otimes V_{b'} \xrightarrow{\Phi_{i; a, b}^c \otimes 1 \otimes 1} V_a \otimes V_b \otimes V_{a'} \otimes V_{b'},$$

and let G_3 be the composition

$$V_{a+a'+c-\bar{\alpha}_i} \otimes V_{c+b+b'} \xrightarrow{v_i^{b(i)-a(i)} m} V_a \otimes V_b \otimes V_{a'+c-\bar{\alpha}_i} \otimes V_{c+b'} \xrightarrow{1 \otimes 1 \otimes \Phi_{i; a', b'}^c} V_a \otimes V_b \otimes V_{a'} \otimes V_{b'}.$$

Proposition 23.9 (Φm -identity). $G_1 = G_2 + G_3$.

By definition, G_1 is equal to the composition

$$\begin{aligned} V_{a+a'+c-\bar{\alpha}_i} \otimes V_{c+b+b'} &\xrightarrow{[c(i)]_i^{-1} \tau_{i, a+a', c} \otimes T_{c, b+b'}} V_{a+a'} \otimes V_c \otimes V_c \otimes V_{b+b'} \\ &\xrightarrow{(1 \otimes S_c \otimes 1)A_1} V_{a+a'} \otimes V_{b+b'} \xrightarrow{T_{a, a'} \otimes T_{b, b'}} V_a \otimes V_{a'} \otimes V_b \otimes V_{b'} \\ &\xrightarrow{v^{(a', b)} A_3 (1 \otimes C_{a', b} \otimes 1) A_2} V_a \otimes V_b \otimes V_{a'} \otimes V_{b'}, \end{aligned}$$

where $A_1 = B_{a+a', c, c, b+b'}$, $A_2 = B_{a, a', b, b'}$, $A_3 = B_{a, b, a', b'}^{-1}$.

We substitute

$$\begin{aligned} (T_{a, a'} \otimes T_{b, b'}) (1 \otimes S_c \otimes 1) A_1 &= (1 \otimes 1 \otimes S_c \otimes 1 \otimes 1) (T_{a, a'} \otimes 1 \otimes 1 \otimes T_{b, b'}) A_1 \\ &= (1 \otimes 1 \otimes S_c \otimes 1 \otimes 1) A_4 (T_{a, a'} \otimes 1 \otimes 1 \otimes T_{b, b'}), \end{aligned}$$

where $A_4 = B_{a \otimes a', \bar{c}, c, b \otimes b'}$. Thus G_1 is equal to the composition

$$v^{\langle a', b \rangle} [c(i)]_i^{-1} A_3(1 \otimes C_{a', b} \otimes 1) A_2(1 \otimes 1 \otimes S_c \otimes 1 \otimes 1) A_4 \\ \times (((T_{\bar{a}, a'} \otimes 1) \tau_{\bar{i}; \bar{a} + \bar{a}', \bar{c}}) \otimes ((1 \otimes T_{b, b'}) T_{c, b + b'})).$$

Let $A = A_{\bar{a}, \bar{c}, a'}$, $A' = A_{a, a', \bar{c}}$. Substituting

$$(T_{\bar{a}, a'} \otimes 1) \tau_{\bar{i}; \bar{a} + \bar{a}', \bar{c}} = v^{\langle c, a' \rangle} A'^{-1} (1 \otimes C_{a', \bar{c}}^{-1}) A(\tau_{\bar{i}; \bar{a}, \bar{c}} \otimes 1) T_{\bar{a} + \bar{c} - \bar{\alpha}_i, a'} \\ + v_i^{-a(i)} A'^{-1} (1 \otimes \tau_{\bar{i}; a', \bar{c}}) T_{a, a' + \bar{c} - \bar{\alpha}_i}$$

(see Corollary 22.9) and

$$(1 \otimes T_{b, b'}) T_{c, b + b'} = A_{c, b, b'} (T_{c, b} \otimes 1) T_{c + b, b'},$$

we see that $G_1 = G'_2 + G'_3$ where G'_2 is the composition

$$v^{\langle a', b \rangle + \langle c, a' \rangle} [c(i)]_i^{-1} A_3(1 \otimes C_{a', b} \otimes 1) A_2(1 \otimes 1 \otimes S_c \otimes 1 \otimes 1) A_4 \\ \times ((A'^{-1} (1 \otimes C_{a', \bar{c}}^{-1}) A(\tau_{\bar{i}; \bar{a}, \bar{c}} \otimes 1) T_{\bar{a} + \bar{c} - \bar{\alpha}_i, a'}) \otimes (A_{c, b, b'} (T_{c, b} \otimes 1) T_{c + b, b'}))$$

and G'_3 is the composition

$$v^{\langle a', b \rangle} v_i^{-a(i)} [c(i)]_i^{-1} A_3(1 \otimes C_{a', b} \otimes 1) A_2(1 \otimes 1 \otimes S_c \otimes 1 \otimes 1) A_4 \\ \times ((A'^{-1} (1 \otimes \tau_{\bar{i}; a', \bar{c}}) T_{a, a' + \bar{c} - \bar{\alpha}_i}) \otimes (A_{c, b, b'} (T_{c, b} \otimes 1) T_{c + b, b'}))$$

It is therefore enough to prove that $G_2 = G'_2$ and $G_3 = G'_3$.

By definition, G_2 is equal to the composition

$$V_{\bar{a} + \bar{a}' + \bar{c} - \bar{\alpha}_i} \otimes V_{c + b + b'} \xrightarrow{T_{\bar{a} + \bar{c} - \bar{\alpha}_i, a'} \otimes T_{c + b, b'}} V_{\bar{a} + \bar{c} - \bar{\alpha}_i} \otimes V_{a'} \otimes V_{c + b} \otimes V_{b'} \\ \xrightarrow{v^{\langle a', c + b \rangle} A_6(1 \otimes C_{a', c + b} \otimes 1) A_5} V_{\bar{a} + \bar{c} - \bar{\alpha}_i} \otimes V_{c + b} \otimes V_{a'} \otimes V_{b'} \\ \xrightarrow{[c(i)]_i^{-1} \tau_{a, c} \otimes T_{c, b} \otimes 1 \otimes 1} V_{\bar{a}} \otimes V_{\bar{c}} \otimes V_c \otimes V_b \otimes V_{a'} \otimes V_{b'} \\ \xrightarrow{(1 \otimes S_c \otimes 1 \otimes 1 \otimes 1) A_7} V_{\bar{a}} \otimes V_b \otimes V_{a'} \otimes V_{b'},$$

where

$$A_5 = B_{\bar{a} + \bar{c} - \bar{\alpha}_i, a', c + b, b'}, \quad A_6 = B_{\bar{a} + \bar{c} - \bar{\alpha}_i, c + b, a', b'}, \quad A_7 = B_{\bar{a}, \bar{c}, c, b} \otimes 1 \otimes 1.$$

We replace here

$$(\tau_{a, c} \otimes T_{c, b} \otimes 1 \otimes 1) A_6(1 \otimes C_{a', c + b} \otimes 1) A_5 \\ = A'_6(1 \otimes 1 \otimes C_{a', c \otimes b} \otimes 1) A'_5(\tau_{a, c} \otimes 1 \otimes T_{c, b} \otimes 1),$$

where $A'_5 = B_{\bar{a} \otimes \bar{c}, a', c \otimes b, b'}$, $A'_6 = B_{\bar{a} \otimes \bar{c}, c \otimes b, a', b'}$. Hence G_2 is equal to the composition

$$v^{\langle a', c + b \rangle} [c(i)]_i^{-1} (1 \otimes S_c \otimes 1 \otimes 1 \otimes 1) A_7 A'_6(1 \otimes 1 \otimes C_{a', c \otimes b} \otimes 1) A'_5 \\ \times ((\tau_{\bar{i}; a, c} \otimes 1) T_{\bar{a} + \bar{c} - \bar{\alpha}_i, a'}) \otimes ((T_{c, b} \otimes 1) T_{c + b, b'}).$$

Thus, to prove that $G_2 = G'_2$, it suffices to prove that

$$A_3(1 \otimes C_{a',b} \otimes 1)A_2(1 \otimes 1 \otimes S_c \otimes 1 \otimes 1)A_4((A'^{-1}(1 \otimes C_{a',c}^{-1})A) \otimes A_{c,b,b'})$$

is equal to

$$(1 \otimes S_c \otimes 1 \otimes 1 \otimes 1)A_7A'_6(1 \otimes 1 \otimes C_{a',c \otimes b} \otimes 1)A'_5.$$

This follows from the pentagon, hexagon identities and functorialities.

By definition, G_3 is equal to the composition

$$\begin{aligned} & V_{a+a'+\bar{c}-\bar{\alpha}_i} \otimes V_{c+b+b'} \xrightarrow{v_i^{b(i)-a(i)} T_{a,a'+\bar{c}-\bar{\alpha}_i} \otimes T_{b,c+b'}} V_a \otimes V_{a'+\bar{c}-\bar{\alpha}_i} \otimes V_b \otimes V_{c+b'} \\ & \xrightarrow{v^{(a'+\bar{c}-\alpha_i,b)} A_9(1 \otimes C_{a'+\bar{c}-\bar{\alpha}_i,b} \otimes 1) A_8} V_a \otimes V_b \otimes V_{a'+\bar{c}-\bar{\alpha}_i} \otimes V_{c+b'} \\ & \xrightarrow{[c(i)]_i^{-1} 1 \otimes 1 \otimes \tau_{i,a',\bar{c}} \otimes T_{c,b'}} V_a \otimes V_b \otimes V_{a'} \otimes V_c \otimes V_c \otimes V_{b'} \\ & \xrightarrow{(1 \otimes 1 \otimes 1 \otimes S_c \otimes 1) A_{10}} V_a \otimes V_b \otimes V_{a'} \otimes V_{b'}, \end{aligned}$$

where

$$A_8 = B_{a,a'+\bar{c}-\bar{\alpha}_i,b,c+b'}, \quad A_9 = B_{a,b,a'+\bar{c}-\bar{\alpha}_i,c+b'}^{-1}, \quad A_{10} = 1 \otimes 1 \otimes B_{a',\bar{c},c,b'}.$$

We replace here

$$\begin{aligned} & (1 \otimes 1 \otimes \tau_{i,a',\bar{c}} \otimes T_{c,b'}) A_9(1 \otimes C_{a'+\bar{c}-\bar{\alpha}_i,b} \otimes 1) A_8 \\ & = A'_9(1 \otimes C_{a' \otimes \bar{c},b} \otimes 1 \otimes 1) A'_8(1 \otimes \tau_{i,a',\bar{c}} \otimes 1 \otimes T_{c,b'}), \end{aligned}$$

where $A'_8 = B_{a,a' \otimes \bar{c},b,c \otimes b'}$, $A'_9 = B_{a,b,a' \otimes \bar{c},c \otimes b'}^{-1}$. Hence G_3 is equal to the composition

$$\begin{aligned} & v^{(a'+\bar{c}-\alpha_i,b)} v_i^{b(i)-a(i)} [c(i)]_i^{-1} (1 \otimes 1 \otimes 1 \otimes S_c \otimes 1) A_{10} A'_9(1 \otimes C_{a' \otimes \bar{c},b} \otimes 1 \otimes 1) A'_8 \\ & \times (((1 \otimes \tau_{i,a',\bar{c}}) T_{a,a'+\bar{c}-\bar{\alpha}_i}) \otimes ((1 \otimes T_{c,b'}) T_{b,c+b'})). \end{aligned}$$

Here we replace

$$\begin{aligned} (1 \otimes T_{c,b'}) T_{b,c+b'} &= A_{b,c,b'}(T_{b,c} \otimes 1) T_{b+c,b'} \\ &= v^{-\langle b,c \rangle} A_{b,c,b'}((C_{c,b} \otimes 1)(T_{c,b} \otimes 1) T_{c+b,b'}). \end{aligned}$$

Using $v^{(\alpha_i,b)} = v_i^{b(i)}$, we see that, to prove the equality $G_3 = G'_3$, it suffices to prove that

$$(1 \otimes 1 \otimes 1 \otimes S_c \otimes 1) A_{10} A'_9(1 \otimes C_{a' \otimes \bar{c},b} \otimes 1 \otimes 1) A'_8(1 \otimes 1 \otimes 1 \otimes (A_{b,c,b'}(C_{c,b} \otimes 1)))$$

is equal to

$$A_3(1 \otimes C_{a',b} \otimes 1) A_2(1 \otimes 1 \otimes S_c \otimes 1 \otimes 1) A_4(A'^{-1} \otimes A_{c,b,b'}).$$

This follows from the pentagon, hexagon identities and functorialities.

The proposition is proved.

23.10. Let $a, a', b, b', c \in \mathbf{N}^I$ be such that $b(i) \geq 1, b'(i) \geq 1, c(i) \geq 1$. Let $\tilde{G}_1$ be the composition

$$V_{a+a'+c} \otimes V_{c+b+b'-\alpha_i} \xrightarrow{\Psi_{i;a+a',b+b'}^c} V_{a+a'} \otimes V_{b+b'} \xrightarrow{m} V_a \otimes V_b \otimes V_{a'} \otimes V_{b'},$$

let $\tilde{G}_2$ be the composition

$$V_{a+a'+c} \otimes V_{c+b+b'-\alpha_i} \xrightarrow{v_i^{a'(i)-b'(i)} m} V_{a+c} \otimes V_{c+b-\alpha_i} \otimes V_{a'} \otimes V_{b'} \\ \xrightarrow{\Psi_{i;a,b \otimes 1 \otimes 1}^c} V_a \otimes V_b \otimes V_{a'} \otimes V_{b'},$$

and let $\tilde{G}_3$ be the composition

$$V_{a+a'+c} \otimes V_{c+b+b'-\alpha_i} \xrightarrow{m} V_a \otimes V_b \otimes V_{a'+c} \otimes V_{c+b'-\alpha_i} \xrightarrow{1 \otimes 1 \otimes \Psi_{i;a',b'}^c} V_a \otimes V_b \otimes V_{a'} \otimes V_{b'}.$$

Proposition 23.11 (Ψm -identity). $\tilde{G}_1 = \tilde{G}_2 + \tilde{G}_3$.

The proof is entirely similar to that of the previous proposition.

24. DRINFELD'S CATEGORY $\mathcal{E}$

24.1. Following Drinfeld and Jimbo [D1, D2], we define a category $\mathcal{E}$ as follows. The objects of $\mathcal{E}$ are free $\mathbf{A}$ -modules $\mathbf{E}$ of finite rank with a given direct sum decomposition $\mathbf{E} = \bigoplus_{\lambda} \mathbf{E}^{\lambda}$ (where λ runs through $\mathbf{Z}^I$) and with given $\mathbf{A}$ -linear maps $E_i, F_i: \mathbf{E} \rightarrow \mathbf{E}$ ($i \in I$) such that

- (a) $E_i(\mathbf{E}^{\lambda}) \subset \mathbf{E}^{\lambda+\alpha_i}$ and $F_i(\mathbf{E}^{\lambda}) \subset \mathbf{E}^{\lambda-\alpha_i}$ for any i and any λ ;
- (b) $(E_i F_j - F_j E_i)(x) = \delta_{ij} [\lambda(i)]_i x$ for any i, j and any $x \in \mathbf{E}^{\lambda}$.

It is assumed that the *quantum Serre relations* are satisfied. In other words, for any $i \neq j$ in I , we have

$$\sum (-1)^{p'} ([p]_i^! [p']_i^!)^{-1} E_i^p E_j E_i^{p'} = 0, \quad \sum (-1)^{p'} ([p]_i^! [p']_i^!)^{-1} F_i^p F_j F_i^{p'} = 0$$

as linear maps $\mathbf{E} \rightarrow \mathbf{E}$ (in both sums p, p' run through $\mathbf{N}$ and are subject to $p + p' = 1 - a_{ij}$; by definition, $[p]_i^! = [1]_i [2]_i \cdots [p]_i$).

A morphism from $\mathbf{E}$ to $\mathbf{E}'$ in $\mathcal{E}$ is an $\mathbf{A}$ -linear map $f: \mathbf{E} \rightarrow \mathbf{E}'$ which is compatible with the direct sum decompositions and with the maps E_i, F_i on $\mathbf{E}, \mathbf{E}'$.

24.2. We recall some known properties of $\mathcal{E}$.

(a) For any $a \in \mathbf{N}^I$, there exists an object $\mathbf{E}_a$ (unique up to isomorphism) such that $\mathbf{E}_a^a$ is free of rank 1 over $\mathbf{A}$ and such that $\mathbf{E}_a = \sum F_{i_1} F_{i_2} \cdots F_{i_s} \mathbf{E}_a^a$ (sum over all sequences $i_1, i_2, \dots, i_s$ in I). This object is clearly indecomposable.

(b) We have $\text{Hom}_{\mathcal{E}}(\mathbf{E}_a, \mathbf{E}_b) = 0$ if $a \neq b$ and $\text{Hom}_{\mathcal{E}}(\mathbf{E}_a, \mathbf{E}_a) = \mathbf{A}$ as an $\mathbf{A}$ -algebra with 1.

(c) Any object of the category $\mathcal{E}$ is isomorphic to a direct sum of objects of the form $\mathbf{E}_a$.

24.3. If $\mathbf{E}', \mathbf{E}''$ are two objects of $\mathcal{E}$, the tensor product of $\mathbf{A}$ -modules $\mathbf{E}' \otimes \mathbf{E}''$ is regarded as an object of $\mathcal{E}$ with

$$(\mathbf{E}' \otimes \mathbf{E}'')^\lambda = \bigoplus_{\lambda' + \lambda'' = \lambda} (\mathbf{E}'^{\lambda'} \otimes \mathbf{E}''^{\lambda''})$$

and with $E_i, F_i: \mathbf{E}' \otimes \mathbf{E}'' \rightarrow \mathbf{E}' \otimes \mathbf{E}''$ defined by

$$\begin{aligned} E_i(x \otimes x') &= E_i(x) \otimes x' + v_i^{\lambda(i)} x \otimes E_i(x'), \\ F_i(x \otimes x') &= x \otimes F_i x' + v_i^{-\lambda'(i)} F_i x \otimes x' \end{aligned}$$

for $x \in \mathbf{E}'^\lambda, x' \in \mathbf{E}''^{\lambda'}$.

If $\mathbf{E}, \mathbf{E}', \mathbf{E}''$ are three objects of $\mathcal{E}$, the obvious isomorphism of $\mathbf{A}$ -modules $(\mathbf{E} \otimes \mathbf{E}') \otimes \mathbf{E}'' \rightarrow \mathbf{E} \otimes (\mathbf{E}' \otimes \mathbf{E}'')$ given by $(x \otimes x') \otimes x'' \mapsto x \otimes (x' \otimes x'')$ is an isomorphism in $\mathcal{E}$.

We shall regard $\mathbf{A}$ as an object of $\mathcal{E}$ with $\mathbf{A} = \mathbf{A}^0$ and with E_i, F_i acting as zero. For any $\mathbf{E} \in \mathcal{E}$, we have obvious isomorphisms $\mathbf{A} \otimes \mathbf{E} \rightarrow \mathbf{E}, \mathbf{E} \otimes \mathbf{A} \rightarrow \mathbf{E}$ in $\mathcal{E}$.

24.4. Drinfeld [D1, D2] has constructed by means of a “universal R -matrix” a collection of isomorphisms (in $\mathcal{E}$) $D_{\mathbf{E}, \mathbf{E}'}: \mathbf{E} \otimes \mathbf{E}' \rightarrow \mathbf{E} \otimes \mathbf{E}'$ defined for all $\mathbf{E}, \mathbf{E}'$ in $\mathcal{E}$. From his definition one can see that the properties (a), (b) below hold.

(a) $D_{\mathbf{E}, \mathbf{E}'}$ is functorial in $\mathbf{E}$ and $\mathbf{E}'$.

(b) If $x \in \mathbf{E}^\lambda$ is annihilated by all $F_i: \mathbf{E} \rightarrow \mathbf{E}$ and $x' \in \mathbf{E}'^{\lambda'}$ is annihilated by all $E_i: \mathbf{E}' \rightarrow \mathbf{E}'$, then

$$D_{\mathbf{E}, \mathbf{E}'}(x \otimes x') = v^{\langle \lambda, \lambda' \rangle} x' \otimes x.$$

The following uniqueness property of the isomorphisms $D_{\mathbf{E}, \mathbf{E}'}$ is similar to the characterization of the R -matrix given in [L3, 4.1.2]; it is simpler than Drinfeld’s characterization of his universal R -matrix [D2].

Lemma 24.5. $D_{\mathbf{E}, \mathbf{E}'}$ are uniquely determined by properties (a), (b) above.

If $\mathbf{E}, \mathbf{E}'$ are as above, one verifies easily that the $\mathbf{A}$ -module $\mathbf{E} \otimes \mathbf{E}'$ is generated by vectors of the form

$$E_{i_1} E_{i_2} \cdots E_{i_s} F_{j_1} F_{j_2} \cdots F_{j_{s'}}(x \otimes x')$$

for various sequences $i_1, i_2, \dots, i_s$ and $j_1, j_2, \dots, j_{s'}$ in I and various $x \in \mathbf{E}^\lambda$ and $x' \in \mathbf{E}'^{\lambda'}$ as in 24.4(b). For such x, x' , we have

$$\begin{aligned} D_{\mathbf{E}, \mathbf{E}'}(E_{i_1} E_{i_2} \cdots E_{i_s} F_{j_1} F_{j_2} \cdots F_{j_{s'}}(x \otimes x')) \\ = v^{\langle \lambda, \lambda' \rangle} E_{i_1} E_{i_2} \cdots E_{i_s} F_{j_1} F_{j_2} \cdots F_{j_{s'}}(x' \otimes x), \end{aligned}$$

and the lemma follows.

The category $\mathcal{E}$ with its tensor product $E, E' \mapsto E \otimes E'$, with the trivial associativity isomorphisms (see 24.3) and with the commutativity isomorphisms described above, is called *Drinfeld's category* $\mathcal{E}$.

25. THE EQUIVALENCE OF CATEGORIES $X: \mathcal{D} \rightarrow \mathcal{E}$

25.1. We fix $\lambda \in \mathbf{Z}^I$. Let V be an object of $\mathcal{D}$. For any $a, b \in \mathbf{N}^I$ such that $b - a = \lambda$, we set $V^{(a,b)} = \text{Hom}_{\mathcal{D}}(V_a \otimes V_b, V)$.

We shall regard the set $\{(a, b) \in \mathbf{N}^I \times \mathbf{N}^I \mid b - a = \lambda\}$ as a partially ordered set by setting $(a, b) \leq (a', b')$ whenever $a, b, a', b' \in \mathbf{N}^I$ satisfy $b - a = b' - a' = \lambda$ and $a' - a = b' - b \in \mathbf{N}^I$.

For $(a, b) \leq (a', b')$ as above, we consider the $\mathbf{A}$ -linear map $\psi_{a,b;a',b'}: V^{(a,b)} \rightarrow V^{(a',b')}$ defined by composition with $\text{tr}_{a,b}^{a'-a}: V_{a'} \otimes V_{b'} \rightarrow V_a \otimes V_b$.

From the transitivity of tr (see 21.3), we see that $(V^{(a,b)}, \psi_{a,b;a',b'})$, where $a, b \in \mathbf{N}^I$ vary subject to $b - a = \lambda$ and $\psi_{a,b;a',b'}$ are as above, form a direct system of $\mathbf{A}$ -modules.

For $a, b \in \mathbf{N}^I$ such that $b - a = \lambda$, we consider the $\mathbf{A}$ -linear map $\phi_{a,b}: V^{(a,b)} \rightarrow V^\lambda$ defined by $f \mapsto f(x_a \otimes y_b)$. (V^λ is as in 19.3.) These maps do not pass to the limit; we have only the following weaker result. (In the following lemma the notation $a \gg 0$ means that $a(i)$ is sufficiently large for each $i \in I$.)

Lemma 25.2. *Let a, b, a', b' be as above.*

- (a) *For any $x \in V^{(a,b)}$ we have $\phi_{a',b'}\psi_{a,b;a',b'}(x) = \phi_{a,b}(x) \bmod \varpi$.*
- (b) *If $a \gg 0$, then $\phi_{a,b}$ is an isomorphism of $\mathbf{A}$ -modules.*
- (c) *If $a \gg 0$, then $\psi_{a,b;a',b'}$ is an isomorphism of $\mathbf{A}$ -modules.*

(a) follows from the definitions, using Lemma 21.2.

(b) follows (for example) from [L3, 31.2.6].

If $a \gg 0$, then we also have $a' \gg 0$. Hence, by (a) and (b), $V^{(a,b)}$ and $V^{(a',b')}$ are free $\mathbf{A}$ -modules of finite rank and the $\mathbf{C}$ -linear map $V^{(a,b)}/\varpi V^{(a,b)} \rightarrow V^{(a',b')}/\varpi V^{(a',b')}$ induced by $\psi_{a,b;a',b'}$ is an isomorphism. It follows that $\psi_{a,b;a',b'}$ itself is an isomorphism. This proves (c).

25.3. From the previous lemma we see that the maps in our direct system are eventually isomorphisms. Hence, if we set

$$X_\lambda(V) = \varinjlim_{b-a=\lambda} V^{(a,b)},$$

then the natural homomorphism $V^{(a,b)} \rightarrow X_\lambda(V)$ is an isomorphism of $\mathbf{A}$ -modules, if $a \gg 0$.

The previous lemma shows also that

- (a) the $\mathbf{A}$ -module $X_\lambda(V)$ is (noncanonically) isomorphic to the $\mathbf{A}$ -module V^λ and that
- (b) there is a unique isomorphism of $\mathbf{C}$ -vector spaces $\phi_\lambda: X_\lambda(V)/\varpi X_\lambda(V) \rightarrow$

$V^\lambda/\varpi V^\lambda$ such that the diagram

$$\begin{array}{ccc} V^{(a,b)}/\varpi V^{(a,b)} & \xrightarrow{\phi_{a,b} \bmod \varpi} & V^\lambda/\varpi V^\lambda \\ \downarrow & & \downarrow \\ X_\lambda(V)/\varpi X_\lambda(V) & \xrightarrow{\phi_\lambda} & V^\lambda/\varpi V^\lambda \end{array}$$

is commutative. (The left vertical arrow is induced by the natural homomorphism $V^{(a,b)} \rightarrow X_\lambda(V)$.)

25.4. Let $g: V \rightarrow V'$ be a morphism in $\mathcal{D}$. For any $a, b \in \mathbb{N}^I$ such that $b - a = \lambda$, composition with g defines a homomorphism of $\mathbf{A}$ -modules $V^{(a,b)} \rightarrow V'^{(a,b)}$. By passage to limit, we obtain a homomorphism of $\mathbf{A}$ -modules $X_\lambda(g): X_\lambda(V) \rightarrow X_\lambda(V')$.

We have thus defined a functor X_λ from $\mathcal{D}$ to the category of free $\mathbf{A}$ -modules of finite rank.

Setting $X(V) = \bigoplus_\lambda X_\lambda(V)$ and $X(g) = \bigoplus_\lambda X_\lambda(g)$ (where λ runs through $\mathbb{Z}^I$), we obtain a functor X from $\mathcal{D}$ to the category of free $\mathbf{A}$ -modules of finite rank. (Indeed, from 25.3(a) we see that $X_\lambda(V) = 0$ for all λ such that $V^\lambda = 0$, hence for all but finitely many λ .)

For $V \in \mathcal{D}$, let

$$(a) \quad \phi: X(V)/\varpi X(V) \rightarrow V/\varpi V$$

be the isomorphism of $\mathbf{C}$ -vector spaces defined by $\bigoplus_\lambda \phi_\lambda$.

Using the definitions, we see that, if $g: V \rightarrow V'$ is as above, then the following diagram is commutative:

$$\begin{array}{ccc} X(V)/\varpi X(V) & \xrightarrow{\phi} & V/\varpi V \\ X(g) \bmod \varpi \downarrow & & \downarrow g \bmod \varpi \\ X(V')/\varpi X(V') & \xrightarrow{\phi} & V'/\varpi V' \end{array}$$

25.5. Let $\lambda, \lambda' \in \mathbb{Z}^I$. Let V, V' be two objects of $\mathcal{D}$ and let $f: V_a \otimes V_b \rightarrow V$ and $f': V_c \otimes V_d \rightarrow V'$ be morphisms in $\mathcal{D}$, where $a, b, c, d \in \mathbb{N}^I$ satisfy $b - a = \lambda, d - c = \lambda'$.

Then $f \otimes f': V_a \otimes V_b \otimes V_c \otimes V_d \rightarrow V \otimes V'$ is a morphism in $\mathcal{D}$; composing it with $m: V_{a+c} \otimes V_{b+d} \rightarrow V_a \otimes V_b \otimes V_c \otimes V_d$, we obtain a morphism

$$(f \otimes f')m: V_{a+c} \otimes V_{b+d} \rightarrow V \otimes V'$$

which may be regarded as an element of $(V \otimes V')^{(a+c, b+d)}$. Thus, $f \otimes f' \mapsto (f \otimes f')m$ defines an $\mathbf{A}$ -linear map

$$V^{(a,b)} \otimes V'^{(c,d)} \rightarrow (V \otimes V')^{(a+c, b+d)}.$$

This is compatible with the direct systems defining $X_\lambda(V), X_{\lambda'}(V'), X_{\lambda+\lambda'}(V \otimes V')$, by the left and right compatibility of m with tr (see 21.6, 21.7). Hence, by passage to limit, we obtain an $\mathbf{A}$ -linear map

$$(a) \quad X_\lambda(V) \otimes X_{\lambda'}(V') \rightarrow X_{\lambda+\lambda'}(V \otimes V').$$

Proposition 25.6. *Let $\mu \in \mathbf{Z}^I$. The homomorphism*

$$\bigoplus (\mathbf{X}_\lambda(V) \otimes \mathbf{X}_{\lambda'}(V')) \rightarrow \mathbf{X}_\mu(V \otimes V')$$

(direct sum over all $\lambda, \lambda' \in \mathbf{Z}^I$ such that $\lambda + \lambda' = \mu$) whose components are given by 25.5(a), is an isomorphism of $\mathbf{A}$ -modules.

This is a homomorphism between two free $\mathbf{A}$ -modules of finite rank; the induced homomorphism obtained by reduction mod ϖ may be identified via the isomorphisms $\phi_\lambda, \phi_{\lambda'}, \phi_\mu$, with the obvious isomorphism of $\mathbf{C}$ -vector spaces

$$\bigoplus ((V^\lambda / \varpi V^\lambda) \otimes_{\mathbf{C}} (V'^{\lambda'} / \varpi V'^{\lambda'})) \rightarrow (V \otimes V')^\mu / \varpi (V \otimes V')^\mu$$

(we use here Lemma 21.5). The proposition follows.

25.7. Taking direct sum over μ , we obtain an isomorphism of $\mathbf{A}$ -modules

$$(a) \quad M_{V, V'}: \mathbf{X}(V) \otimes \mathbf{X}(V') \rightarrow \mathbf{X}(V \otimes V').$$

We shall use the notation

$$x \hat{\otimes} y = M_{V, V'}(x \otimes y)$$

for $x \in \mathbf{X}(V)$, $y \in \mathbf{X}(V')$.

Proposition 25.8. *Let V_1, V_2, V_3 be objects of $\mathcal{D}$. The following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{X}(V_1) \otimes \mathbf{X}(V_2) \otimes \mathbf{X}(V_3) & \xrightarrow{1} & \mathbf{X}(V_1) \otimes \mathbf{X}(V_2) \otimes \mathbf{X}(V_3) \\ M_{V_1, V_2} \otimes 1 \downarrow & & 1 \otimes M_{V_2, V_3} \downarrow \\ \mathbf{X}(V_1 \otimes V_2) \otimes \mathbf{X}(V_3) & & \mathbf{X}(V_1) \otimes \mathbf{X}(V_2 \otimes V_3) \\ M_{V_1 \otimes V_2, V_3} \downarrow & & M_{V_1, V_2 \otimes V_3} \downarrow \\ \mathbf{X}(V_1 \otimes V_2 \otimes V_3) & \xrightarrow{\mathbf{X}(A_{V_1, V_2, V_3})} & \mathbf{X}(V_1 \otimes V_2 \otimes V_3) \end{array}$$

This follows from the definitions, using the *mm*-identity 21.8.

25.9. Given $a \in \mathbf{N}^I$, the obvious isomorphism $V_0 \otimes V_a \rightarrow V_a$ may be regarded as an element $\tilde{y}_a \in V_a^{(0, a)}$. Its image under the natural map $V_a^{(0, a)} \rightarrow \mathbf{X}_a(V_a)$ is denoted by $\mathbf{y}_a$.

Similarly, the obvious isomorphism $V_a \otimes V_0 \rightarrow V_a$ may be regarded as an element $\tilde{x}_a \in V_a^{(a, 0)}$. Its image under the natural map $V_a^{(a, 0)} \rightarrow \mathbf{X}_{-a}(V_a)$ is denoted by $\mathbf{x}_a$.

From the definitions we see immediately that

(a) under the isomorphism $\mathbf{X}(V_a) / \varpi \mathbf{X}(V_a) \rightarrow V_a / \varpi V_a$ (resp. $\mathbf{X}(V_a) / \varpi \mathbf{X}(V_a) \rightarrow V_a / \varpi V_a$) in 25.4(a), $\mathbf{y}_a \bmod \varpi$ corresponds to $y_a \bmod \varpi$ (resp. $\mathbf{x}_a \bmod \varpi$ corresponds to $x_a \bmod \varpi$).

Lemma 25.10. *Let $a, b \in \mathbf{N}^I$. Consider the diagram*

$$\begin{array}{ccc} \mathbf{X}_{-a}(V_a) \otimes \mathbf{X}_b(V_b) & \xrightarrow{M_{V_a, V_b}} & \mathbf{X}_{b-a}(V_a \otimes V_b) \\ p \downarrow & & \downarrow v^{(a,b)} \mathbf{X}(C_{a,b}) \\ \mathbf{X}_b(V_b) \otimes \mathbf{X}_{-a}(V_a) & \xrightarrow{M_{V_b, V_a}} & \mathbf{X}_{b-a}(V_b \otimes V_a) \end{array}$$

where $p(x \otimes y) = y \otimes x$ for all $x \in \mathbf{X}_{-a}(V_a)$ and $y \in \mathbf{X}_b(V_b)$. We have

$$v^{(a,b)} \mathbf{X}(C_{a,b}) M_{V_a, V_b}(\mathbf{x}_a \otimes \mathbf{y}_b) = M_{V_b, V_a} p(\mathbf{x}_a \otimes \mathbf{y}_b),$$

or, equivalently, $v^{(a,b)} \mathbf{X}(C_{a,b})(\mathbf{x}_a \widehat{\otimes} \mathbf{y}_b) = \mathbf{y}_b \widehat{\otimes} \mathbf{x}_a$.

From the definitions, $\mathbf{x}_a \widehat{\otimes} \mathbf{y}_b$ is represented by the element of $(V_a \otimes V_b)^{(a,b)}$ given by the identity map $V_a \otimes V_b \rightarrow V_a \otimes V_b$ (we use 19.10(b) and the equality $C_{0,0} = 1$). Again from the definitions, $\mathbf{y}_b \widehat{\otimes} \mathbf{x}_a$ is represented by the element of $(V_b \otimes V_a)^{(a,b)}$ given by the map $v^{(a,b)} C_{a,b}: V_a \otimes V_b \rightarrow V_b \otimes V_a$ (we use 19.10(b)). The lemma follows.

25.11. The maps E_i, F_i . Assume that $i \in I$ is given.

Let $\lambda \in \mathbf{Z}^I$ and let V be an object of $\mathcal{D}$. For any $c \in \mathbf{N}^I$ such that $c(i) \geq 1$, composition with the morphism

$$\Phi_{i;a,b}^c: V_{a+\bar{c}-\alpha_i} \otimes V_{c+b} \rightarrow V_a \otimes V_b$$

defines an $\mathbf{A}$ -linear map

$$(a) \quad V^{(a,b)} \rightarrow V^{(a+c-\alpha_i, c+b)}$$

for any $a, b \in \mathbf{N}^I$ such that $b - a = \lambda$ and $a(i) \geq 1$. The identity

$$tr_{a,b}^c \Phi_{i;a+c, c+b}^d = \Phi_{i;a,b}^c tr_{a+c-\alpha_i, c+b}^d = \Phi_{i;a,b}^{c+d}$$

(see Lemma 23.2) shows that the maps (a) define by passage to limit an $\mathbf{A}$ -linear map

$$\varinjlim_{a,b; b-a=\lambda} V^{(a,b)} \rightarrow \varinjlim_{a',b'; b'-a'=\lambda+\alpha_i} V^{(a',b')}$$

which depends only on λ and i , or equivalently, an $\mathbf{A}$ -linear map $E_i: \mathbf{X}_\lambda(V) \rightarrow \mathbf{X}_{\lambda+\alpha_i}(V)$. Taking direct sums over λ , we obtain an $\mathbf{A}$ -linear map $E_i: \mathbf{X}(V) \rightarrow \mathbf{X}(V)$.

Similarly, for any $c \in \mathbf{N}^I$ such that $c(i) \geq 1$, composition with the morphism

$$\Psi_{i;a,b}^c: V_{a+\bar{c}} \otimes V_{c+b-\alpha_i} \rightarrow V_a \otimes V_b$$

defines an $\mathbf{A}$ -linear map

$$(b) \quad V^{(a,b)} \rightarrow V^{(a+c, c+b-\alpha_i)}$$

for any $a, b \in \mathbf{N}^I$ such that $b - a = \lambda$ and $b(i) \geq 1$. The identity

$$tr_{a,b}^c \Psi_{i;a+c,c+b}^d = \Psi_{i;a,b}^c tr_{a+c,c+b-\alpha_i}^d = \Psi_{i;a,b}^{c+d}$$

(see Lemma 23.3) shows that the maps (b) define by passage to limit an $\mathbf{A}$ -linear map

$$\lim_{a,b;b-a=\lambda} V^{(a,b)} \rightarrow \lim_{a',b';b'-a'=\lambda-\alpha_i} V^{(a',b')}$$

which depends only on λ and i , or equivalently, an $\mathbf{A}$ -linear map $F_i: \mathbf{X}_\lambda(V) \rightarrow \mathbf{X}_{\lambda-\alpha_i}(V)$. Taking direct sums over λ , we obtain an $\mathbf{A}$ -linear map $F_i: \mathbf{X}(V) \rightarrow \mathbf{X}(V)$.

Lemma 25.12. *The diagrams*

$$\begin{array}{ccc} \mathbf{X}(V)/\varpi \mathbf{X}(V) & \xrightarrow{E_i \bmod \varpi} & \mathbf{X}(V)/\varpi \mathbf{X}(V) \\ \downarrow & & \downarrow \\ V/\varpi V & \xrightarrow{e_i \bmod \varpi} & V/\varpi V \\ \mathbf{X}(V)/\varpi \mathbf{X}(V) & \xrightarrow{F_i \bmod \varpi} & \mathbf{X}(V)/\varpi \mathbf{X}(V) \\ \downarrow & & \downarrow \\ V/\varpi V & \xrightarrow{f_i \bmod \varpi} & V/\varpi V \end{array}$$

(where the vertical arrows are isomorphisms as in 25.4(a)) are commutative.

It suffices to show that, for any morphism $f: V_a \otimes V_b \rightarrow V$ in $\mathscr{D}$, we have

(a) $f(\Phi_{i;a,b}^c(x_{a+\bar{c}-\bar{\alpha}_i} \otimes y_{c+b})) = e_i(f(x_a \otimes y_b)) \bmod \varpi$ and

(b) $f(\Psi_{i;a,b}^c(x_{a+\bar{c}} \otimes y_{c+b-\alpha_i})) = f_i(f(x_a \otimes y_b)) \bmod \varpi$.

We compute the left-hand side of (a) modulo ϖ by replacing the associativity isomorphisms by 1, the map S_c by S'_c , the map $\tau_{i,\bar{a},\bar{c}}$ by $\tau'_{i,\bar{a},\bar{c}}$, and the element $[c(i)]_i$ by $c(i)$; we obtain

$$\begin{aligned} & f(1 \otimes S'_c \otimes 1)(c(i)^{-1} \tau'_{i,\bar{a},\bar{c}}(x_{a+\bar{c}-\bar{\alpha}_i}) \otimes y_c \otimes y_b) \\ &= f(1 \otimes S'_c \otimes 1)(c(i)^{-1} (-a(i)x_a \otimes e_i x_{\bar{c}} + c(i)e_i x_a \otimes x_{\bar{c}}) \otimes y_c \otimes y_b) \\ &= f(e_i x_a \otimes y_b) = f(e_i(x_a \otimes y_b)) = e_i(f(x_a \otimes y_b)). \end{aligned}$$

This proves (a). The proof of (b) is entirely similar.

Lemma 25.13. *Let $i, j \in I$ be given. The $\mathbf{A}$ -linear maps $E_i, F_j: \mathbf{X}(V) \rightarrow \mathbf{X}(V)$ satisfy $(E_i F_j - F_j E_i)(x) = \delta_{ij}[\lambda(i)]_i x$ for any $x \in \mathbf{X}_\lambda(V)$.*

This follows from definitions, using the $\Phi\Psi$ -identity 23.5. (Note that the element $\rho_{i;c,d}$, which enters in the definition (23.4) of the map P_3 in the statement of 23.5, is equal to 1, see 23.7.)

25.14. In the following lemma we consider for fixed $i \neq j$ the $\mathbf{A}$ -linear maps G_{ij}^+ , G_{ij}^- of $\mathbf{X}(V)$ into itself given by

$$G_{ij}^+ = \sum (-1)^{p'} ([p]_i^! [p']_i^!)^{-1} E_i^p E_j E_i^{p'}, \quad G_{ij}^- = \sum (-1)^{p'} ([p]_i^! [p']_i^!)^{-1} F_i^p F_j F_i^{p'}$$

(in both sums p, p' run through $\mathbf{N}$ and are subject to $p + p' = 1 - a_{ij}$).

Lemma 25.15. $G_{ij}^+ = G_{ij}^- = 0$.

We shall assume, as we may, that $V = V_a$ for some $a \in \mathbf{N}^I$. By a direct computation, using Lemma 25.14, we see that $F_k G_{ij}^+ = G_{ij}^+ F_k$ for all $k \in I$ (equality of maps $\mathbf{X}(V_a) \rightarrow \mathbf{X}(V_a)$). (This kind of commutation formula has been used in [L2].) It follows that

$$F_{i_1} F_{i_2} \cdots F_{i_t} G_{ij}^+ y_a = G_{ij}^+ F_{i_1} F_{i_2} \cdots F_{i_t} y_a$$

for any sequence $i_1, i_2, \dots, i_t$ in I . We have $G_{ij}^+ y_a = 0$. Indeed, we have $G_{ij}^+ y_a \in \mathbf{X}_{a+\alpha_j+(1-a_{ij})\alpha_i}(V_a)$ and the last space is zero since $V_a^\lambda = 0$ for $\lambda = a + \alpha_j + (1 - a_{ij})\alpha_i$ (see 25.3(a)). Hence we have $G_{ij}^+ F_{i_1} F_{i_2} \cdots F_{i_t} y_a = 0$ for any sequence $i_1, i_2, \dots, i_t$ in I . To show that $G_{ij}^- = 0$ on $\mathbf{X}(V_a)$, it is therefore enough to show that the vectors $F_{i_1} F_{i_2} \cdots F_{i_t} y_a$, for various sequences $i_1, i_2, \dots, i_t$ in I , generate the $\mathbf{A}$ -module $\mathbf{X}(V_a)$.

By Nakayama's lemma, it is enough to show that the images of these vectors in $\mathbf{X}(V)/\varpi \mathbf{X}(V)$ generate that $\mathbf{C}$ -vector space; by 25.9(a) and 25.13, this is equivalent to the following obvious statement: the vectors $f_{i_1} f_{i_2} \cdots f_{i_t} y_a$, for various sequences $i_1, i_2, \dots, i_t$ in I , generate the $\mathbf{C}$ -vector space $V/\varpi V$.

Thus we have proved that $G_{ij}^+ = 0$. The equality $G_{ij}^- = 0$ is proved in a similar way.

In the following lemma we establish a connection between the endomorphisms E_i, F_i on $\mathbf{X}(V), \mathbf{X}(V'), \mathbf{X}(V \otimes V')$ where V, V' are two objects of $\mathcal{D}$. Recall the notation $x \widehat{\otimes} x'$ from 25.7.

Lemma 25.16. For any $x \in \mathbf{X}_\lambda(V), x' \in \mathbf{X}_{\lambda'}(V')$, we have

$$E_i(x \widehat{\otimes} x') = E_i(x) \widehat{\otimes} x' + v_i^{\lambda(i)} x \widehat{\otimes} E_i(x')$$

and

$$F_i(x \widehat{\otimes} x') = x \widehat{\otimes} F_i(x') + v_i^{-\lambda'(i)} F_i(x) \widehat{\otimes} x'.$$

We represent x by the element of $V^{(a,b)}$ given by $f: V_a \otimes V_b \rightarrow V$ (where $b - a = \lambda$ and $a \gg 0$) and we represent x' by the element of $V^{(a',b')}$ given by $f': V_{a'} \otimes V_{b'} \rightarrow V'$ (where $b' - a' = \lambda'$ and $a' \gg 0$).

Then $E_i(x \widehat{\otimes} x'), E_i(x) \widehat{\otimes} x', v_i^{\lambda(i)} x \widehat{\otimes} E_i(x')$ are represented respectively by the compositions $(f \otimes f')G_1, (f \otimes f')G_2, (f \otimes f')G_3$ (elements of $(V \otimes V')^{(a+a'+c-\alpha_i, c+b+b')}$), where G_1, G_2, G_3 are the morphisms

$$V_{a+a'+c-\alpha_i} \otimes V_{c+b+b'} \rightarrow V_a \otimes V_b \otimes V_{a'} \otimes V_{b'}$$

described in 23.8. Here $c \in \mathbf{N}^I$ is such that $c(i) \geq 1$. By the Φm -identity 23.9, we have $G_1 = G_2 + G_3$. The desired formula for E_i follows. The formula for F_i follows in symmetrical way from the Ψm -identity 23.11.

25.17. If V is an object of $\mathcal{D}$, then the $\mathbf{A}$ -module $\mathbf{X}(V)$ is free of finite rank and, by definition, has a natural direct sum decomposition $\mathbf{X}(V) = \bigoplus_{\lambda} \mathbf{X}_{\lambda}(V)$ (λ runs through $\mathbf{Z}^I$). It also has natural $\mathbf{A}$ -linear endomorphisms E_i, F_i which, by Lemmas 25.13, 25.15, make it into an object of $\mathcal{E}$. Note also that, if $g: V \rightarrow V'$ is a morphism in $\mathcal{D}$, then $\mathbf{X}(g): \mathbf{X}(V) \rightarrow \mathbf{X}(V')$ is clearly a morphism in $\mathcal{E}$. Thus, $\mathbf{X}$ may be regarded as a functor $\mathcal{D} \rightarrow \mathcal{E}$.

Lemma 25.18. *Let V, V' be objects of $\mathcal{D}$. The natural map $\text{Hom}_{\mathcal{D}}(V, V') \rightarrow \text{Hom}_{\mathcal{E}}(\mathbf{X}(V), \mathbf{X}(V'))$ given by $g \mapsto \mathbf{X}(g)$ is an isomorphism of $\mathbf{A}$ -modules.*

From the definition, it is clear that $\mathbf{X}$ commutes with taking direct sums. Hence we may assume that V, V' are indecomposable. Therefore, we may assume that $V = V_a, V' = V_b$ for some $a, b \in \mathbf{N}^I$. Now $\mathbf{X}_a(V_a)$ is a free $\mathbf{A}$ -module isomorphic to V_a^a ; hence it has rank 1 (with generator y_a). As we have seen in the proof of Lemma 25.15, we have $\mathbf{X}(V_a) = \sum F_{i_1} F_{i_2} \cdots F_{i_r} \mathbf{X}_a(V_a)$; hence $\mathbf{X}(V_a)$ is isomorphic to E_a (see 24.2(a)) in $\mathcal{E}$. Using 24.2(b), we see that $\text{Hom}_{\mathcal{E}}(\mathbf{X}(V_a), \mathbf{X}(V_b)) = 0$ if $a \neq b$ and $\text{Hom}_{\mathcal{E}}(\mathbf{X}(V_a), \mathbf{X}(V_a)) = \mathbf{C}$ as a $\mathbf{A}$ -algebra with 1. It is clear that $\text{Hom}_{\mathcal{D}}(V_a, V_b) = 0$ if $a \neq b$ and $\text{Hom}_{\mathcal{D}}(V_a, V_a) = \mathbf{A}$ as a $\mathbf{A}$ -algebra with 1. Hence the natural map $\text{Hom}_{\mathcal{D}}(V_a, V_b) \rightarrow \text{Hom}_{\mathcal{E}}(\mathbf{X}(V_a), \mathbf{X}(V_b))$ is the map $0 \rightarrow 0$ if $a \neq b$ and is the identity map $\mathbf{A} \rightarrow \mathbf{A}$ if $a = b$. The lemma is proved.

Lemma 25.19. *For any object E of $\mathcal{E}$, there exists an object V of $\mathcal{D}$ such that E is isomorphic to $\mathbf{X}(V)$ in $\mathcal{E}$.*

By 24.2(c), we may assume that E is isomorphic to E_a for some a . By the proof of the previous lemma, E is then isomorphic to $\mathbf{X}(V_a)$. The lemma is proved.

Combining Lemmas 25.18 and 25.19, we obtain the following result.

Lemma 25.20. *The functor $\mathbf{X}: \mathcal{D} \rightarrow \mathcal{E}$ is an equivalence of categories.*

25.21. By 25.16, the isomorphism of $\mathbf{A}$ -modules $M_{V, V'}: \mathbf{X}(V) \otimes \mathbf{X}(V') \rightarrow \mathbf{X}(V \otimes V')$ (see 25.7) is in fact an isomorphism of objects of $\mathcal{E}$ (where $\mathbf{X}(V) \otimes \mathbf{X}(V')$ is the tensor product of objects of $\mathcal{E}$, see 24.3). This isomorphism is clearly functorial.

Lemma 25.22. *$(\mathbf{X}, (M_{V, V'}))$ is a tensor functor.*

The compatibility of the associativity isomorphisms is contained in 25.8.

We have $\mathbf{X}(V_0) = \mathbf{A}$ canonically (where $\mathbf{A}$ is regarded as an object of $\mathcal{E}$ as in 24.3); the generator y_0 corresponds to $1 \in \mathbf{A}$.

It remains to show the compatibility of the commutativity isomorphisms. From 25.20, it follows that there is a well-defined functorial isomorphism $\tilde{D}_{E, E'}: E \otimes E' \rightarrow E' \otimes E$ defined for all E, E' in $\mathcal{E}$ such that for any $V, V' \in \mathcal{D}$

the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{X}(V) \otimes \mathbf{X}(V') & \xrightarrow{M_{V,V'}} & \mathbf{X}(V \otimes V') \\ \downarrow \tilde{D}_{\mathbf{X}(V), \mathbf{X}(V')} & & \downarrow \mathbf{X}(C_{V,V'}) \\ \mathbf{X}(V') \otimes \mathbf{X}(V) & \xrightarrow{M_{V',V}} & \mathbf{X}(V' \otimes V) \end{array}$$

We must show that $\tilde{D}_{\mathbf{E}, \mathbf{E}'} = D_{\mathbf{E}, \mathbf{E}'}$ for all $\mathbf{E}, \mathbf{E}'$.

By Lemma 24.5, it suffices to check that, for any $x \in \mathbf{E}^\lambda$ that is annihilated by all $F_i: \mathbf{E} \rightarrow \mathbf{E}$ and any $x' \in \mathbf{E}'^{\lambda'}$ that is annihilated by all $E_i: \mathbf{E}' \rightarrow \mathbf{E}'$, we have $\tilde{D}_{\mathbf{E}, \mathbf{E}'}(x \otimes x') = v^{(\lambda, \lambda')} x' \otimes x$.

By the functoriality of $\tilde{D}$, we may assume that $E = \mathbf{X}(V_a)$, $E' = \mathbf{X}(V_b)$, $x = \mathbf{x}_a$, $x' = \mathbf{y}_b$. Using the commutative diagram above, we are then reduced to proving the identity $v^{(a, b)} \mathbf{X}(C_{a, b})(\mathbf{x}_a \hat{\otimes} \mathbf{y}_b) = \mathbf{y}_b \hat{\otimes} \mathbf{x}_a$ (in our case, $\lambda = -a$, $\lambda' = b$). But this identity is proved in Lemma 25.10.

Combining Lemmas 25.20 and 25.22, we obtain the following result.

Theorem 25.23. *The functor $\mathbf{X}: \mathcal{D} \rightarrow \mathcal{E}$ together with the functorial isomorphism $M_{V, V'}: \mathbf{X}(V) \otimes \mathbf{X}(V') \rightarrow \mathbf{X}(V \otimes V')$, is an equivalence of tensor categories.*

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