TENSOR STRUCTURES ARISING FROM AFFINE LIE ALGEBRAS. IV

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INTRODUCTION

In the earlier parts [1] and [2] we have defined the tensor product $\otimes$ on the category $\mathcal{O}_\kappa$ of representations of a simply-laced affine Lie algebra $\tilde{\mathfrak{g}}$ and associativity and commutativity isomorphisms a and $\mathcal{P}$. In this paper we will show that the data $(\otimes, a, \mathcal{P})$ defines a structure of a braided category on $\mathcal{O}_\kappa$ and construct an equivalence X between the braided category $\mathcal{O}_\kappa$ and the braided category $\mathcal{E}_\kappa$ of the representation of quantum groups for many $\kappa \in \mathbb{C} - \mathbb{Q}_+$. The results of Part III could be interpreted as the construction of such an equivalence when κ is generic. In this part we prove that $\otimes$ is coherent in κ , and therefore we can derive many results for specific κ_0 from the validity of corresponding results for generic κ . The usage of this technique would simplify proofs of some results from Part II (for example, the proof of Proposition 17.19).

We will use freely notations from the earlier parts and the language and results of the braided category theory. The exposition of those results is given in the Appendix. We thank J. Bernstein for useful discussions.

26. EXPLICIT FORMULAS FOR THE CONNECTIONS ∇ IN THE CASE OF WEYL MODULES

In this section we prove results which really belong to the first two parts of the paper. In particular, we give another definition of the submodule $T(W) \subset \widehat{W}$ (see §4) and write an explicit formula for the connection ∇ on $Y/\Delta Y$ (see §12) in the case when all V_s (see (11.2)) are Weyl modules.

Let $\gamma_1, \gamma_\infty, \gamma_0 \in \text{Aut}(P^1)$ be as in §13.5. We denote by $t \in R$ (see §4.5) the standard function on P^1 . So $t(0) = 0$, $t(1) = 1$, and $t(\infty) = \infty$. It is clear that t^{-1} lies in R . Let Γ be a Lie algebra as in §4.6. The imbedding $\mathbb{C}[t, t^{-1}] \hookrightarrow R$ induces the imbedding $\tilde{\mathfrak{g}} \hookrightarrow \Gamma$.

As follows from the definition (see §13.4) for any V, V' in $\mathcal{O}_\kappa$ the module $V \otimes V'$ is isomorphic to $T(W)^\sharp$, where $T(W) \stackrel{\text{def}}{=} \widehat{W}(-\infty)$ is a subspace of the completion $\widehat{W} = \varprojlim W/G_N W$ and $W = V \otimes V'$ (see §4).

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Let $V_+ \stackrel{\text{def}}{=} Q_1^\# V$, $V'_+ \stackrel{\text{def}}{=} Q_1^\# V'$, and $V_0 \subset V$, $V'_0 \subset V'$ be $\tilde{\mathfrak{g}}^+$ -invariant subspaces such that

$$V = V_0 + V_+, \quad V' = V'_0 + V'_+.$$

Lemma 26.1. *We have $W = V_0 \otimes V'_0 + G_1 W$.*

Proof. Follows immediately from the arguments of §7.8.

Let $V = \mathbf{V}_a$, $V' = \mathbf{V}_{a'}$ be Weyl modules, and $\mathcal{V}_a \subset \mathbf{V}_a$, $\mathcal{V}_{a'} \subset \mathbf{V}_{a'}$ the subspaces as in §2.4.

Lemma 26.2. *We have $W = (\mathcal{V}_a \otimes \mathcal{V}_{a'}) \oplus G_1 W$.*

Proof. Consider the natural map $\varphi: \mathcal{V}_a \otimes \mathcal{V}_{a'} \rightarrow W/G_1 W$. As follows from Lemma 26.1, φ is surjective. On the other hand, it follows from the arguments of §9.15 that $\dim W/G_1 W = \dim \mathcal{V}_a \cdot \dim \mathcal{V}_{a'}$. Therefore φ is an isomorphism. The lemma is proved.

Let c_p be a basis in $\mathfrak{g}$ as in §1.14. For any $M > 0$ we denote by $L_0(M) \in U(\tilde{\mathfrak{g}})$ the following truncation of the operator L_0 :

$$L_0(M) = \frac{1}{2\kappa} \sum_p c_p^2 + \frac{1}{\kappa} \sum_{1 \leq j \leq M} \sum_p (t^j c_p)(t^{-j} c_p).$$

Using the imbedding $\tilde{\mathfrak{g}} \hookrightarrow \Gamma$ we can consider $L_0(M)$ as an element of $U(\Gamma)$. Since W is a Γ -module we can also consider $L_0(M)$ as an operator on W .

Lemma 26.3. (a) *For any $M, N \in \mathbf{N}$, $M \geq N$, we have*

$$L_0(M)(G_N W) \subset G_N W.$$

(b) *For any $M \geq N$, $L_0(M)$ and $L_0(N)$ induce the same endomorphism of $W/G_N W$.*

(c) *If $V = \mathbf{V}_a$, $V' = \mathbf{V}_{a'}$ are Weyl modules and*

$$\Omega^{(2)} \stackrel{\text{def}}{=} \sum_p c_p \otimes c_p \in U(\mathfrak{g}) \otimes U(\mathfrak{g}),$$

then the diagram

$$\begin{array}{ccc} \mathcal{V}_a \otimes \mathcal{V}_{a'} & \xrightarrow{\frac{1}{2\kappa} \Omega^{(2)}} & \mathcal{V}_a \otimes \mathcal{V}_{a'} \\ \varphi \downarrow & & \downarrow \varphi \\ W/G_1 W & \xrightarrow{\ell_1} & W/G_1 W \end{array}$$

is commutative.

Proof. As follows from Lemma 4.9(b) for any $M \geq N$ we have

$$\text{Im}(L_0(M) - L_0(N)) \subset G_N W.$$

Therefore we can consider the operator $\hat{L}_0 \stackrel{\text{def}}{=} \varprojlim L_0(N)$ from W to $\hat{W}$. To prove (a) and (b) it is sufficient to show that $\hat{L}_0(G_N W) \subset \hat{L}_0(G_N \hat{W})$. Let δ be the unique differentiation of R such that $\delta(t) = t$ ($\delta = t \frac{\partial}{\partial t}$). The standard computations (see §1.14) show that for any $f \in R$, $c \in \mathfrak{g}$, we have

$[\hat{L}_0, fc] = \delta(f)c$. Since $\delta(R_1) \subset R_1$, we see that $\hat{L}_0(G_N W) \subset G_N \hat{W}$. Parts (a) and (b) are proved. Part (c) is easy. Lemma 26.3 is proved.

For any $N \in \mathbb{N}$ we denote by f_N the natural map $f_N: \widehat{W}(-\infty) \rightarrow W/G_N W$.

Lemma 26.4. (1) f_N is surjective for all $N \in \mathbb{N}$.

(2) The operator on $V_1 \otimes V_2$ induced by the restriction of $\hat{L}_0$ on $T(W)$ coincides with L_0 .

(3) The submodule $T(W) \subset \widehat{W}$ coincides with the submodule T_0 of $\hat{L}_0$ -finite vectors in $\widehat{W}$.

Proof. (1) It is sufficient to show that the dual map ${}^d f$ is injective. As follows from Theorem 7.9 the map 6.7(b) is an isomorphism and therefore (as follows from the remark after the Lemma 6.5) we can identify ${}^d f$ with the natural map $T'(W)(N) \rightarrow {}^d({}^d T'(W))(-\infty)$. By Proposition 2.26 this map is an imbedding.

(2) Obvious.

(3) It follows from Theorem 7.9 and Lemma 2.20 that $T(W) \subset T_0$ and the set of the real parts of eigenvalues of L_0 on $T(W)$ is bounded from above. Then as follows from parts (1) and (2) of the lemma, the set of the real parts of eigenvalues of $\hat{L}_0$ on T_0 is also bounded from above. Therefore for any $v \in T_0$ such that $\hat{L}_0 v = \lambda v$ there exists $N \in \mathbb{Z}^+$ such that $\lambda - N$ is not an eigenvalue of $\hat{L}_0$ on $\widehat{W}$. This implies $v \in \widehat{W}(-N)$. The lemma is proved.

Remark. Let $\mathcal{D} = \mathcal{D}(\mathbb{C})$ be the category of finite-dimensional complex vector spaces, and let $\mathcal{G}: \mathcal{O}_\kappa \rightarrow \mathcal{D}$ be the functor defined by $\mathcal{G}(V) = V/G_1 V$. As follows from the proof of Proposition 9.15, we have a canonical isomorphism $\text{Hom}_{\mathfrak{g}}(\mathcal{G}(V), \mathcal{V}_a) \xrightarrow{\sim} \text{Hom}_{\mathfrak{g}}(V, \mathbf{V}_a^*)$ for any $a \in \mathbb{N}^I$ and it follows from Lemma 26.2 that for any Weyl modules V_1, V_2 we have a natural isomorphism

$$\beta(V_1, V_2): \mathcal{G}(V_1) \otimes \mathcal{G}(V_2) \xrightarrow{\sim} \mathcal{G}(V_1 \otimes V_2).$$

Let $A = \mathbb{C}[\mathbb{Z}]$ be as in §9.3, and let $V_i = \mathbf{V}_{a_i}^\kappa$, $1 \leq i \leq n$, be Weyl modules, $\bar{Y} \stackrel{\text{def}}{=} \bigotimes_i \mathcal{V}_{a_i}$, $Y \stackrel{\text{def}}{=} \bigotimes_i \mathbf{V}_{a_i}^\kappa$. The natural imbedding $\mathcal{V}_a \hookrightarrow \mathbf{V}_a^\kappa$ (see §2.3) induces the imbedding $\iota: A \otimes \bar{Y} \hookrightarrow Y$ and therefore a map η from the free A -module $A \otimes \bar{Y}/\mathfrak{g}\bar{Y}$ to the A -module $Y/\Delta Y$, where the Lie algebra Δ is defined in §9.9. As follows from Proposition 9.15 this map is an isomorphism.

We will identify $A \otimes \bar{Y}$ with its image in Y . Let $\pi: Y \rightarrow Y/\Delta Y$ be the natural projection. It is clear that ι and π commute with the action of $\mathfrak{g}$.

For any pair $1 \leq i \neq j \leq n$ we associate a function $f_{ij} \in A$ on $\mathcal{V}$ such that for any $\vec{\gamma} = \{\gamma_s\} \in \mathcal{V}$, $1 \leq s \leq n$, we have $f_{ij}(\vec{\gamma}) = (\gamma_i^{-1} \circ \gamma_j(0))^{-1}$.

For any pair (i, F) , $1 \leq i \leq n$, $F \in A((\epsilon))$, we denote by $c_i(F): \bar{Y} \rightarrow Y/\Delta Y$ the linear operator such that

$$c_i(F)c(x' \otimes x_i \otimes x'') = \pi(x' \otimes (Fc)x_i \otimes x'')$$

for all $x' \in \bigotimes_{j < i} \mathcal{V}_{a_j}$, $x'' \in \bigotimes_{j > i} \mathcal{V}_{a_j}$, $x_i \in \mathcal{V}_{a_i}$. We write $c_i = c_i(1)$, $c'_i = c_i(\epsilon^{-1})$.

Lemma 26.5. *We have*

$$c'_i = - \sum_{j \neq i} f_{i,j} c_j.$$

Proof. Let f_s , $1 \leq s \leq n$, be functions as in §9.5. It is clear that $f_{ij} = {}^j f_i(0)$. We have

$$c'_i = f_i \cdot c - (f_i \cdot c - c'_i) = f_i c - \sum_{j \neq i} c_j ({}^j f_i).$$

Since $f_i c \in \Delta$ we have for any $\bar{y} \in \bar{Y}$

$$c_j ({}^j f_i)(\iota(\bar{y})) = {}^j f_i(0) c_j(\iota(\bar{y})) = f_{i,j} c_j(\iota(\bar{y})),$$

and the lemma is proved.

For any element $\alpha \in U(\tilde{\mathfrak{g}}_A)$ and i , $1 \leq i \leq n$, we denote by $\alpha_i \in \text{End } Y$ the linear operator such that $\alpha_i(\otimes x_j) = \otimes_{j < i} x_j \otimes (\alpha x_i) \otimes \otimes_{j > i} x_j$. Analogously for any $\bar{\alpha} \in U(\mathfrak{g})$ and i , $1 \leq i \leq n$, we denote by $\bar{\alpha}_i$ the corresponding endomorphism of $\bar{Y}$. In the same vein, for any $\bar{\beta} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and any i, j , $1 \leq i \neq j \leq n$, we denote by $\bar{\beta}_{i,j} \in \text{End}(\bar{Y})$ the operator acting as $\bar{\beta}$ on $\bar{V}_{a_i} \otimes \bar{V}_{a_j}$ and as identity on other factors. For any $\bar{\alpha} \in U(\mathfrak{g})$ and any i, j , $1 \leq i \neq j \leq n$, we write $\bar{\alpha}_{i,j} \stackrel{\text{def}}{=} (\hat{\alpha})_{i,j}$, where $\hat{\alpha} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the image of $\bar{\alpha}$ under the comultiplication $U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Let L_k , $k = \pm 1, 0$, be the Sugawara operators as in §1.14, and let $L_{k,s}: \bar{Y} \rightarrow Y/\Delta Y$, $1 \leq s \leq n$, be the composition $L_{k,s} = \pi \circ (L_k)_s \circ \iota$.

Proposition 26.1. (a) $L_{1,s} = 0$.

(b) $L_{0,s} = \frac{1}{\kappa} \Omega_s$.

(c) $L_{-1,s} = -\frac{1}{\kappa} \sum_{i \neq s} f_{s,i} \Omega_{s,i}^{(2)} \pmod{\Delta}$.

Proof. (a) and (b) follow immediately from the definition of the operators L_0 and L_1 .

To prove (c) we observe that for any $x \in \iota(\bar{Y})$ we have (see §1.14)

$$\begin{aligned} (L_{-1})_s x &\stackrel{\text{def}}{=} \frac{1}{2\kappa} \sum_{j \geq 1} \sum_p (\epsilon^{-j} c_p)_s (\epsilon^{-j} c_p)_s x + \frac{1}{2\kappa} \sum_{j \leq 0} \sum_p (\epsilon^{j-1} c_p)_s (\epsilon^{-j} c_p)_s x \\ &= \frac{1}{\kappa} \sum_p c_p(s)_s (c_p)_s x \equiv -\frac{1}{\kappa} \sum_p \sum_{i \neq s} f_{s,i} (c_p)_i (c_p)_s x \pmod{\Delta Y}. \end{aligned}$$

The proposition is proved.

We denote by $\mathcal{H} \simeq \mathfrak{sl}_2(\mathbf{R})$ the Lie algebra of regular vector fields on P^1 , and by $H = PGL_2(\mathbf{C})$ the group of automorphisms of P^1 . Let $\ell_1 = t^2 \frac{\partial}{\partial t}$, $\ell_0 = t \frac{\partial}{\partial t}$, $\ell_{-1} = \frac{\partial}{\partial t}$ be the standard basis in $\mathcal{H}$.

Let $\mathcal{V} \subset H^n$ be the open subset as in §9.1. The space of left invariant vector fields on $\mathcal{V}$ is canonically isomorphic to $\mathcal{H}^n = \mathcal{H} \otimes \mathbf{R}^n$.

For any pair (i, k) , $1 \leq i \leq n$, $-1 \leq k \leq 1$, we define $\ell_{i,k} \stackrel{\text{def}}{=} \ell_k \otimes e_i \in \mathcal{H}^n$, where (e_i) , $1 \leq i \leq n$, is the standard basis in $\mathbf{C}^n$. Let $\lambda_{i,k}$ the corresponding vector field on $\mathcal{V}$ and $\omega_{i,k}$ the 1-form on $\mathcal{V}$ such that $\omega_{i,k}(\lambda_{i',k'}) = \delta_{i,i'}^{\lambda} \delta_{k,k'}^k$.

Let $\hat{A}$ be the space of functions on $\mathcal{V}$, $\hat{A} \otimes Y$ the trivial vector bundle over $\mathcal{V}$, ∇_0 the trivial connection on $\hat{A} \otimes Y$, and $\tilde{\nabla} = \nabla_0 + \hat{\omega}$, where

$$\hat{\omega} = -\frac{1}{\kappa} \left(\sum_{i=1}^n \Omega_i \omega_{i,0} + \sum_{i=1}^n \sum_{\substack{j \neq i \\ 1 \leq j \leq n}} f_{i,j} \Omega_{i,j}^{(2)} \omega_{i,-1} \right).$$

It is clear that $\tilde{\nabla}$ commutes with the action of $\mathfrak{g}$ on $\bar{Y}$. Therefore it defines a connection $\hat{\nabla}$ on $\hat{A} \otimes \bar{Y}/\mathfrak{g}\bar{Y}$. It is easy to see that $\hat{\omega}$ is a preimage of the form $\omega \in \Omega^1(\mathcal{Z}, \text{End } \bar{Y}/\mathfrak{g}\bar{Y})$, where $\mathcal{Z} = H \setminus \mathcal{V}$ as in §9.1 and $\hat{\nabla}$ defines a connection on $A \otimes \bar{Y}/\mathfrak{g}\bar{Y} \approx Y/\Delta Y$.

Corollary 26.1. *The connection ∇ coincides with the connection defined in §12.11.*

Consider now the case $n = 4$ and let $\dot{A}$ be the quotient ring of A as in §17.2. The isomorphism $A \otimes \bar{Y}/\mathfrak{g}\bar{Y} \xrightarrow{\sim} Y/\Delta Y$ reduces to an isomorphism $\dot{A} \otimes \bar{Y}/\mathfrak{g}\bar{Y} \xrightarrow{\sim} Y/\Delta_{\dot{A}} Y$ which extends (see §15.11) to an isomorphism $A_{\infty} \otimes \bar{Y}/\mathfrak{g}\bar{Y} \xrightarrow{\sim} \mathcal{M}$ (see §17.10). Therefore the connection $\nabla_{t \frac{\partial}{\partial t}}$ on $\mathcal{M}$ can be described as a sum of the trivial connection ∇^{tr} on $A_{\infty} \otimes \bar{Y}/\mathfrak{g}\bar{Y}$ and a 1-form $\omega = \varphi(t)dt$ on $\mathbb{C} - \{1\}$ with values in $\text{End}(\bar{Y}/\mathfrak{g}\bar{Y})$. Since any endomorphism of $\bar{Y}$ commuting with $\mathfrak{g}$ induces an endomorphism of $\bar{Y}/\mathfrak{g}\bar{Y}$, we can represent $\varphi(t)$ by a function on $\mathbb{C} - \{1\}$ with values in $\text{End}_{\mathfrak{g}}(\bar{Y})$.

Lemma 26.6. *The connection $\nabla_{t \frac{\partial}{\partial t}}$ is equal to $\nabla^{\text{tr}} + \varphi(t)$, where*

$$\varphi(t) = -\frac{1}{\kappa} \left(-\frac{t}{1-t} \Omega_{13}^{(2)} + \Omega_{34} \right),$$

where $\Omega \in U(\mathfrak{g})$ is the Casimir operator corresponding to the scalar product $(\cdot, \cdot)$ on $\mathfrak{g}$ as in §1.2.

Proof. As follows from §17.8 and the corollary to Proposition 26.1 we have

$$\begin{aligned} \varphi(x_1 \otimes x_2 \otimes x_3 \otimes x_4) &= -\frac{1}{\kappa} \left(\Omega_3 + \Omega_4 + 2\Omega_{34}^{(2)} - \frac{t}{1-t} \Omega_{13}^{(2)} \right) \\ &= -\frac{1}{\kappa} \left(\Omega_{34} - \frac{t}{1-t} \Omega_{13}^{(2)} \right). \end{aligned}$$

The lemma is proved.

Let $B_{\infty}, B_n, W_n, \Delta_{2,n}$ be as in §§15.13–15.20, $\mathcal{M}$ the B_{∞} -module as in §17.17, and $\bar{M}(0) \stackrel{\text{def}}{=} \bar{M}/t\bar{M}$ the fiber of $\bar{M}$ at $t = 0$. By the definition $\mathcal{M}(0) = W_1/\Delta_{2,1}W$, where $W_1 = \bigotimes_{i=1}^4 V_{a_i}$. Therefore

$$\mathcal{M}(0) = (V_{a_1} \otimes V_{a_2} / Q_1^{\sharp}(V_{a_1} \otimes V_{a_2})) \otimes (V_{a_3} \otimes V_{a_4} / Q_1^{\sharp}(V_{a_3} \otimes V_{a_4})) / \text{Im } \mathfrak{g}$$

and the imbeddings $\mathcal{V}_{a_i} \hookrightarrow V_{a_i}$ induce an isomorphism $\bar{Y}/\mathfrak{g}\bar{Y} \xrightarrow{\sim} \mathcal{M}(0)$.

Let $\varphi_1: \mathcal{M}(0) \rightarrow \mathcal{M}(0)$ be the map as in §15.25. Since we have defined isomorphisms $\mathcal{M}(0) \simeq \bar{Y}/\mathfrak{g}\bar{Y} \simeq \mathcal{M}(0)$, we can consider φ_1 as an endomorphism of the space $\bar{Y}/\mathfrak{g}\bar{Y}$.

Lemma 26.7. $\varphi_1 = \text{Id}$.

Proof. Immediately follows from the definition of φ_1 .

Consider the A_∞ -module $\mathcal{M}' \stackrel{\text{def}}{=} A_\infty \otimes_{B_\infty} \mathcal{M}$ as in §17.24, and $\nabla_{t \frac{\partial}{\partial t}}, \nabla'_{t \frac{\partial}{\partial t}}$ be the connections on $\mathcal{M}, \mathcal{M}'$ defined in §§17.12 and 17.16.

Lemma 26.8. *There exists unique flat morphism $\varphi_{an}: \mathcal{M}'_{an} \rightarrow \mathcal{M}_{an}$ such that $\varphi_{an}(0) = \varphi_1$.*

Proof. The existence of φ_{an} is proven in §17.28 and the uniqueness is obvious.

27. THE CATEGORY $\mathcal{A}_\kappa$

In this section we prove some results about the structure of the category $\mathcal{O}_\kappa$ and its subcategory $\mathcal{A}_\kappa$ consisting of modules which have Weyl filtration. As before, we assume that $\kappa \notin \mathbf{Q}_{\geq 0}$. Let $[\mathcal{O}_\kappa]$ be the Grothendieck group of the category $\mathcal{O}_\kappa$. For any module V in $\mathcal{O}_\kappa$ we denote by $[V]$ the corresponding element in $[\mathcal{O}_\kappa]$. As is well known, $[\mathcal{O}_\kappa]$ is a free abelian group with generators $[V_a^\kappa]$, $a \in \mathbf{N}^I$, where, as before, I is the set of vertices of the Dynkin diagram of $\mathfrak{g}$. For any module V in $\mathcal{O}_\kappa$ we define integers $m(V, a)$, $a \in \mathbf{N}^I$, as the coefficients of $[V]$ in the basis $[V_a^\kappa]$. We denote by $(\mid)$ the scalar product on $[\mathcal{O}_\kappa]$ such that $([V_a^\kappa] \mid [V_b^\kappa]) = \delta_b^a$, $a, b \in \mathbf{N}^I$, where the map $a \rightarrow \bar{a}$ is the involution on $\mathbf{N}^I$ defined in §1.2.

We say that a module V in the category $\mathcal{O}_\kappa$ has a Weyl filtration if there exists an increasing filtration V_i , $0 \leq i \leq n$, such that $V_0 = (0)$, $V_n = V$, and the quotients V_i/V_{i-1} are isomorphic to Weyl modules. It is clear that, for such a module, $m(V, a)$ is the number of quotients isomorphic to V_a^κ . For such a module V we denote by $\text{supp } V \subset \mathbf{N}^I$ the set of $a \in \mathbf{N}^I$ such that $m(V, a) \neq 0$.

Let $\tilde{\mathfrak{g}}^- \stackrel{\text{def}}{=} \mathbf{C}[\epsilon^{-1}] \otimes \mathfrak{g} \subset \tilde{\mathfrak{g}}$, and let $\mathcal{E}^-$ be the category of representations of $\tilde{\mathfrak{g}}^-$ defined in the same way as the category $\mathcal{E}^+$ in §3.7. For any object M in $\mathcal{E}^-$ we define $\bar{M} \stackrel{\text{def}}{=} M/(\epsilon^{-1}\tilde{\mathfrak{g}}^-)M$. By the definition we have a direct sum decomposition $\bar{M} = \bigoplus_{\lambda \in \mathbf{C}^\lambda} \bar{M}^\lambda$. For any $a \in \mathbf{N}^I$ and $\lambda \in \mathbf{C}$ we denote by $\mathcal{V}_a^\kappa(\lambda)$ the object in the category $\mathcal{E}^-$ such that as a $\mathfrak{g}$ -module it is isomorphic to $\mathcal{V}_a^\kappa$, $\epsilon^{-1}\mathbf{C}[\epsilon^{-1}] \otimes \mathfrak{g}$ is acting as zero, $\mathbf{1}$ is acting as multiplication by $\kappa - h$, and ${}_\lambda \mathcal{V}_a^\kappa(\lambda) = \mathcal{V}_a^\kappa(\lambda)$. We also define an object $\mathcal{P}_a^\kappa(\lambda)$ in $\mathcal{E}^-$ by

$$\mathcal{P}_a^\kappa(\lambda) \stackrel{\text{def}}{=} U(\tilde{\mathfrak{g}}^-) \otimes_{U(\mathfrak{g})} \mathcal{V}_a^\kappa.$$

It is clear that $\mathcal{P}_a^\kappa(\lambda)$ is a projective object in $\mathcal{E}^-$. We denote by $\mathcal{E}_0^-$ the full subcategory of $\mathcal{E}^-$ consisting of objects V such that ${}_\lambda V = (0)$ for all λ with sufficiently small real part. It is clear that the modules $\mathcal{P}_a^\kappa(\lambda)$ lie in $\mathcal{E}_0^-$ for all $a \in \mathbf{N}^I$, $\lambda \in \mathbf{C}$. Let $\mathcal{E}$ be the category of $\tilde{\mathfrak{g}}$ -modules as in §3.7. As in §3.7, we will consider $\mathcal{O}_\kappa$ as a full subcategory of $\mathcal{E}$.

Proposition 27.1. *For any module V in $\mathcal{O}_\kappa$ the following conditions are equivalent:*

- (1) V has a Weyl filtration.
- (2) The restriction V^- of V on $\tilde{\mathfrak{g}}^-$ is a projective object in $\mathcal{E}_0^-$.
- (3) For any $a \in \mathbb{N}^I$, $\lambda \in \mathbb{C}$, and any $i > 0$ we have $\text{Ext}_{\mathcal{E}^-}^i(V^-, \mathcal{V}_a^\kappa(\lambda)) = 0$.
- (4) For any $a \in \mathbb{N}^I$ and any $i > 0$ we have $\text{Ext}_{\mathcal{E}}^i(V, D(\mathbf{V}_a^\kappa)) = 0$.
- (5) For any $a \in \mathbb{N}^I$ we have $\text{Ext}_{\mathcal{E}}^1(V, D(\mathbf{V}_a^\kappa)) = 0$.
- (6) For any $a \in \mathbb{N}^I$ we have $\text{Ext}_{\mathcal{O}_\kappa}^1(V, D(\mathbf{V}_a^\kappa)) = 0$.

Proof. For any object M in $\mathcal{E}_0^-$, $\lambda \in \mathbb{C}$, and any $a \in \mathbb{N}^I$ we define a vector space ${}_a L(M, a) \stackrel{\text{def}}{=} \text{Hom}_{\mathfrak{g}}(\mathcal{V}_a, {}_a \bar{M})$ and a projective module $P(M) \stackrel{\text{def}}{=} \bigoplus_{a \in \mathbb{N}^I} {}_a L(M, a) \otimes \mathcal{P}_a^\kappa(\lambda)$. Since $P(M)$ is projective, the natural morphism $\bar{f}: P(M) \rightarrow \bar{M}$ lifts to a morphism $f: P(M) \rightarrow M$. We begin the proof of the proposition with the following result.

Lemma 27.1. (1) *Any irreducible object in $\mathcal{E}^-$ is isomorphic to $\mathcal{V}_a^\kappa(\lambda)$ for some $a \in \mathbb{N}^I$, $\lambda \in \mathbb{C}$.*

- (2) *Let M be an object in $\mathcal{E}_0^-$ such that $\bar{M} = 0$. Then $M = 0$.*
- (3) *For any object M in $\mathcal{E}_0^-$ the morphism $f: P(M) \rightarrow M$ is surjective.*
- (4) *Any indecomposable projective object in $\mathcal{E}_0^-$ is isomorphic to $\mathcal{P}_a^\kappa(\lambda)$ for some $a \in \mathbb{N}^I$, $\lambda \in \mathbb{C}$.*
- (5) *Let P be a projective object in $\mathcal{E}_0^-$, λ a complex number such that ${}_a P \neq (0)$ but ${}_{(\lambda-1)} P = 0$, and $t: \mathcal{P}_a^\kappa(\lambda) \rightarrow P$ a nontrivial morphism. Then t is an imbedding of $\mathcal{P}_a^\kappa(\lambda)$ in P as a direct summand.*

Proof. Easy.

We can now prove Proposition 27.1.

(1) $\Rightarrow$ (2) It is clear that $(\mathbf{V}_a^\kappa)^- = \mathcal{P}_a^\kappa(\lambda)$ for $\lambda = \frac{1}{2\kappa} \langle a, a + 2 \rangle$. Therefore for any Weyl module V the module V^- in $\mathcal{E}_0^-$ has a filtration such that the successive quotients are projective. Therefore V^- is projective.

(2) $\Rightarrow$ (3) Clear.

(3) $\Leftrightarrow$ (4) As it is explained in §3.7 for any V in $\mathcal{O}_\kappa$, any $a \in \mathbb{N}^I$, and any $i > 0$ we have $\text{Ext}_{\mathcal{E}}^i(V, D(\mathbf{V}_a^\kappa)) = \text{Ext}_{\mathcal{E}^-}^i(V^-, \mathcal{V}_a^\kappa(\lambda))$.

(4) $\Rightarrow$ (5) Clear.

(5) $\Leftrightarrow$ (6) The group $\text{Ext}_{\mathcal{E}}^1(V, \mathbf{V}_a^\kappa)$ classifies the extensions of V by $\mathbf{V}_a^\kappa$ in $\mathcal{E}$. But it is clear that any such extension automatically lies in $\mathcal{O}_\kappa$.

(5) $\Rightarrow$ (2) Let $f: P(V^-) \rightarrow V$ be the surjective map as in Lemma 27.1 and $N \stackrel{\text{def}}{=} \ker(f)$. By the definition for any $a \in \mathbb{N}^I$, $\lambda \in \mathbb{C}$ we have $\text{Hom}(N, \mathcal{V}_a^\kappa(\lambda)) = \text{Ext}_{\mathcal{E}^-}^1(V^-, \mathcal{V}_a^\kappa(\lambda)) = (0)$. It follows from Lemma 27.1(2) that $N = (0)$ and therefore $V = P(V)$.

(2) $\Rightarrow$ (1) We prove this implication by the induction in the length of V . Let λ be a complex number such that ${}_a V \neq (0)$ but ${}_{(\lambda-1)} V = 0$, and let

$t: \mathcal{P}_a^\kappa(\lambda) \rightarrow V^-$ be a nontrivial morphism in $\mathcal{E}^-$. As follows from Lemma 27.1(5), t is an imbedding. It is clear there exists unique $\tilde{\mathbf{g}}^-$ -isomorphism $V_a^\kappa \xrightarrow{\sim} \mathcal{P}_a^\kappa(\lambda)$ and that t commutes with the action of $\tilde{\mathbf{g}}$. Therefore we can consider t as an imbedding of V_a^κ in V . Let V' be the quotient. As follows from Lemma 27.1(5) V' satisfies the condition (2). Therefore the inductive assumption implies that V' has a Weyl filtration. Therefore V also has a Weyl filtration. Proposition 27.1 is proved.

We denote by $\mathcal{A}_\kappa$ the full subcategory of $\mathcal{O}_\kappa$ of modules satisfying the conditions of the proposition.

Corollary 1. *For any two modules V, W in $\mathcal{A}_\kappa$ we have an equality $\dim\langle V, W \rangle = \sum_{a \in \mathbf{N}^I} m(V, a)m(W, a)$.*

Proof. As follows from §2.32(c), we have $\dim\langle V, W \rangle = \dim \operatorname{Hom}_{\mathcal{O}}(V, D(W))$. As follows from part (4) of the proposition we have $\dim \operatorname{Hom}_{\mathcal{O}_\kappa}(V, D(W)) = \sum_{a, b \in \mathbf{N}^I} m(V, a)m(W, b) \dim \operatorname{Hom}_{\mathcal{O}_\kappa}(V_a, D(V_b))$. But it follows from Proposition 2.33 that $\dim \operatorname{Hom}_{\mathcal{O}_\kappa}(V_a, D(V_b)) = \delta_b^a$. The corollary is proved.

Corollary 2. *For any two modules V, W in $\mathcal{A}_\kappa$ we have an equality*

$$\dim \operatorname{Hom}_{\mathcal{O}_\kappa}(V, D(W)) = ([V], [W]).$$

Lemma 27.2. *Any generalized Weyl module $\mathcal{N}^\kappa$ lies in $\mathcal{A}_\kappa$.*

Proof. Follows from Lemma 2.5(a).

We say that a module M in $\mathcal{O}_\kappa$ is tilting if M and $D(M)$ are in $\mathcal{A}_\kappa$.

Proposition 27.2. *For any $a \in \mathbf{N}^I$ there exists an indecomposable tilting module T_a^κ in $\mathcal{O}_\kappa$ such that*

$$[T_a^\kappa] = [V_a^\kappa] + \sum_{b < a} c_b [V_b^\kappa],$$

where the ordering $<$ on $\mathbf{N}^I$ is as in §3.3.

Proof. The proof of the proposition relies heavily on the results of [R]. Let $t = \langle a, a + 2 \rangle$ and $\prec$ be a total ordering on F^t compatible with $<$. Let $P = \bigoplus_{a \in F^t} P_a$, where P_a are projective objects of $\mathcal{O}_\kappa^t$ as in Proposition 3.9, $A \stackrel{\text{def}}{=} \operatorname{End}_{\mathcal{O}_\kappa}(P)$, and let $\varphi: M \rightarrow \operatorname{Hom}_{\mathcal{O}_\kappa}(P, M)$ be the functor from $\mathcal{O}_\kappa^t$ to the category of finitely generated A -modules. As is well known (see [B, Theorem 2.(1.3)]), φ is an equivalence of categories. For any $b \in F^t$ we denote by $E(b)$ the A -module $\varphi(L_b^\kappa)$ and by $\Delta(b)$ and $\nabla(b)$ the A -modules defined in §3 of [R].

Lemma 27.3. $\Delta(b) = \varphi(V_b^\kappa)$ and $\nabla(b) = \varphi(D(V_b^\kappa))$.

Proof. Let $\varphi: P_b \rightarrow V_b^\kappa$ be a morphism lifting a nontrivial morphism $\bar{\varphi}$ from P_b to L_{b_i} . Since L_{b_i} is the unique simple quotient of V_b^κ , we see that φ is surjective. Let $J \subset P_b$ be the kernel of φ . As follows from the long exact sequence, the module J satisfies condition (2) of Proposition 27.1. Therefore J lies in $\mathcal{A}_\kappa$. As follows from Proposition 3.9(b) the module J has a filtration

$D = J_0 \subset J_1 \subset \cdots \subset J_N = J$ with the quotients J_k/J_{k-1} isomorphic to $\mathbf{V}_c^\kappa$, where c is neither bigger nor equal to b . Therefore we have $V(i) = \varphi(I)$ and $\Delta(b) = \varphi(\mathbf{V}_b^\kappa)$. The proof of the equality $\nabla(b) = \varphi(D(\mathbf{V}_b^\kappa))$ is completely analogous. Lemma 27.3 is proved.

By the definition, the category $\mathcal{F}(\Delta)$ from §4 of [R] coincides with the image $\varphi(\mathcal{A}_\kappa^t)$, where $\mathcal{A}_\kappa^t \stackrel{\text{def}}{=} \mathcal{A}_\kappa \cap \mathcal{O}_\kappa^t$. Since P lies in $\mathcal{A}_\kappa^t$ and $\text{End}_{\mathcal{O}_\kappa}(\mathbf{V}_b^\kappa) = \mathbb{C}$ for all $b \in F^t$ we see that the algebra A is quasi-hereditary. Let T be the basic module as in §5 of [R] and $T(a)$ its direct summand as in Proposition 2 in [R], and let T_a^κ be an object in $\mathcal{O}_\kappa^t$ such that $T(a)$ is isomorphic to $\varphi(T_a^\kappa)$. As follows from Proposition 2 in [R] the module T_a^κ satisfies the conditions of Proposition 27.2. The proposition is proved.

We say that a system M_a , $a \in \mathbf{N}^t$, of subobjects of an object M in A_κ is an admissible filtration if the following conditions are satisfied.

- (a) For any $a, b \in \mathbf{N}^t$ such that $a \prec b$ we have $M_a \supset M_b$.
- (b) There exists $a \in \mathbf{N}^t$ such that $M_a = 0$.
- (c) $M_0 = M$.
- (d) For any $a \in \mathbf{N}^t$ the quotient $M_a / \bigcup_{b \succ a, b \neq a} M_b$ is isomorphic to a multiple of $\mathbf{V}_a^\kappa$.

Lemma 27.4. *For any object M in $\mathcal{A}_\kappa$ there exists unique admissible filtration.*

Proof. Choose $t \in \mathbf{Z}^+$ such that M lies in $\mathcal{O}_\kappa^t$, and for any $a \in \mathbf{N}^t$ we define $M(a)$ as the submodule generated by images of projective modules P_b in $\mathcal{O}_\kappa^t$ for all $b \succ a$. It is easy to see that $M(a)$ is the unique admissible filtration of M .

Let U^κ be the quotient of the algebra $U(\tilde{\mathfrak{g}})$ by the ideal generated by the central element $1 - (\kappa - h)$, where 1 is the element in the center of $\tilde{\mathfrak{g}}$ as in §1.3. Of course any element in the category $\mathcal{E}$ has a natural structure of a U^κ -module.

Lemma 27.5. *Let M, N be objects in $\mathcal{E}$ such that $\text{Ext}_{\mathcal{E}}^1(N, M) = \{0\}$. Then $\text{Ext}_{U^\kappa}^1(N, M) = \{0\}$.*

Proof. We say that a cocycle $\varphi: \mathfrak{g} \rightarrow \text{Hom}_{\mathbb{C}}(M, N)$ has degree γ , $\gamma \in \mathbb{C}$, if $\varphi(\epsilon^r \otimes c)(n) \subset {}_{\lambda-r+\gamma}M$ for all $c \in \mathfrak{g}$, $n \in {}_\lambda N$. For any $\gamma \in \mathbb{C}$ we denote by ${}_\gamma \text{Ext}^1(N, M)$ the subgroup of $\text{Ext}_{U^\kappa}^1(N, M)$ which can be realized by cocycles $\varphi: \mathfrak{g} \rightarrow \text{Hom}_{\mathbb{C}}(N, M)$ of degree γ . It is clear that $\text{Ext}_{U^\kappa}^1(N, M)$ is equal to the product $\prod_{\gamma \in \mathbb{C}} {}_\gamma \text{Ext}^1(N, M)$. So it is sufficient to prove that for any $\gamma \in \mathbb{C}$ we have ${}_\gamma \text{Ext}^1(N, M) = \{0\}$. Choose any cocycle φ and consider the U^κ -module X_φ which is isomorphic to $M \oplus N$ as a vector space and the action of $\tilde{\mathfrak{g}}$ is given by the rule

$$x(m, n) = (xm + \varphi(x)n, xn).$$

We have to show that X_φ is isomorphic to $M \oplus N$ as a $\tilde{\mathfrak{g}}$ -module. We define a grading on X_φ in such a way that ${}_\lambda(M \oplus N) = {}_\lambda M \oplus {}_{\lambda-\gamma}N$. It is clear that

the graded module X_φ belongs to $\mathcal{C}$. Since $\text{Ext}_{\mathcal{C}}^1(N, M) = \{0\}$, the object X_φ in $\mathcal{C}$ splits and there exists a morphism $s: N \rightarrow X_\varphi$ in $\mathcal{C}$ such that $X_\varphi = M \oplus s(N)$. Therefore φ is a coboundary of s . The lemma is proved.

Proposition 27.3. *For any $V_1, V_2 \in \mathcal{A}_\kappa$ we have*

$$\text{Ext}_{U^\kappa}^1(V_1, D(V_2)) = \text{Ext}_{U^\kappa}^2(V_1, D(V_2)) = \{0\}.$$

Proof. Since the subcategory $\mathcal{O}_\kappa$ of the category of representations of $U(\tilde{\mathfrak{g}})$ is closed under extensions, we have an equality $\text{Ext}_{U^\kappa}^1(M_1, M_2) = \text{Ext}_{\mathcal{O}_\kappa}^1(M_1, M_2)$ for all objects M_1 and M_2 in $\mathcal{O}_\kappa$. Therefore Proposition 27.1(6) implies that $\text{Ext}_{U^\kappa}^1(V_1, D(V_2)) = \{0\}$.

To prove that $\text{Ext}_{U^\kappa}^2(V_1, D(V_2)) = \{0\}$ we need an auxiliary construction. For any finite-dimensional $\mathfrak{g}$ -module $\mathcal{N}$ we denote by $\tilde{\mathcal{N}}$ the induced U^κ -module, $\tilde{\mathcal{N}} \stackrel{\text{def}}{=} U^\kappa \otimes_{U(\mathfrak{g})} \mathcal{N}$. In the case when $\mathcal{N}$ has a grading $\mathcal{N} = \bigoplus_{\lambda \in \mathbb{C}} \lambda \mathcal{N}$ we define the grading of $\tilde{\mathcal{N}}$, $\tilde{\mathcal{N}} = \bigoplus_{\lambda \in \mathbb{C}} \lambda \tilde{\mathcal{N}}$, as the unique extension from $1 \otimes \mathcal{N} \subset \tilde{\mathcal{N}}$ such that $(\epsilon^n \otimes c) \tilde{\mathcal{N}}_\lambda \subset \tilde{\mathcal{N}}_{\lambda-n}$. It is clear that the object $\tilde{\mathcal{N}}$ with this grading lies in the category $\mathcal{C}$.

Lemma 27.6. *For any finite-dimensional $\mathfrak{g}$ -module $\mathcal{N}$ and any object M in $\mathcal{O}_\kappa$ we have*

$$\text{Ext}_{U^\kappa}^1(\tilde{\mathcal{N}}, M) = \text{Ext}_{U^\kappa}^2(\tilde{\mathcal{N}}, M) = \{0\}.$$

Proof. As follows from Shapiro's lemma we have

$$\text{Ext}_{U^\kappa}^i(\tilde{\mathcal{N}}, M) = \text{Ext}_{U(\mathfrak{g})}^i(\mathcal{N}, M).$$

Since by the definition of $\mathcal{O}_\kappa$ the $\mathfrak{g}$ -module M is a direct sum of finite-dimensional $\mathfrak{g}$ -modules, it is sufficient to check that

$$\text{Ext}_{U(\mathfrak{g})}^1(\mathcal{N}, \mathcal{M}) = \text{Ext}_{U(\mathfrak{g})}^2(\mathcal{N}, \mathcal{M}) = \{0\}$$

for all finite-dimensional $\mathfrak{g}$ -modules $\mathcal{N}$ and $\mathcal{M}$, but this fact is well known. The lemma is proved.

We can now prove the proposition. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be numbers such that the subspace $\mathcal{N} \stackrel{\text{def}}{=} \bigoplus_{i=1}^n \lambda_i V_1$ generates V_1 as a $\tilde{\mathfrak{g}}$ -module, and let $\varphi: \tilde{\mathcal{N}} \rightarrow V_1$ be an isomorphism such that $\varphi(u \otimes n) = un$ for all $u \in U^\kappa$, $n \in \tilde{\mathcal{N}} \subset V_1$. It is clear that φ is a surjective morphism in the category $\mathcal{C}$. Let $I \stackrel{\text{def}}{=} \ker \varphi$. As follows from Lemma 27.4 we have $\text{Ext}_{U^\kappa}^2(V_1, D(V_2)) = \text{Ext}_{U^\kappa}^1(I, D(V_2))$. Since $\text{Ext}_{\mathcal{C}}^2(V_1, D(V_2)) = \{0\}$ we have $\text{Ext}_{\mathcal{C}}^1(I, D(V_2)) = \{0\}$. By Lemma 27.5 we have $\text{Ext}_{U^\kappa}^1(I, D(V_2)) = \{0\}$. Proposition 27.3 is proved.

For any $a \in \mathbb{N}^I$ we define

$$S_a^\kappa \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}_\kappa}(\mathbf{V}_a^\kappa, D(\mathbf{V}_a^\kappa)).$$

As follows from Proposition 2.33 the space S_a^κ is one-dimensional.

By the definition we can consider elements of S_a^κ as bilinear forms $s: \mathbf{V}_a^\kappa \times \mathbf{V}_a^\kappa \rightarrow \mathbb{C}$ such that $s(xv_a, v_a) + s(v_a, x^\sharp v_a) = 0$ for all $x \in \tilde{\mathfrak{g}}$, $v_a \in \mathbf{V}_a^\kappa$, $v_a \in \mathbf{V}_a^\kappa$, where $\sharp$ is the involution of $\tilde{\mathfrak{g}}$ defined in §1.5. Let $\bar{s}_a: \mathcal{V}_a \otimes \mathcal{V}_a \rightarrow \mathbb{C}$ be a nonzero $\mathfrak{g}$ -invariant bilinear form. We denote by $s_a^\kappa \in S_a^\kappa$ an element such that the restriction of the corresponding bilinear form to $\mathcal{V}_a \times \mathcal{V}_a$ coincides with $\bar{s}_a$. As follows from Lemma 2.32(c) s_a^κ defines a nonzero morphism $\bar{s}_a^\kappa: \mathbf{V}_a^\kappa \rightarrow D(\mathbf{V}_a^\kappa)$. For any $m \in \mathbb{N}$ we define $\lambda_a(n) = \frac{1}{2\kappa} \langle a, a+2 \rangle + m$ and $\mathbf{V}_{a,m}^\kappa = \lambda_a(m) \mathbf{V}_a^\kappa$. As follows from Proposition 2.7 we have $\mathbf{V}_a^\kappa = \bigoplus \mathbf{V}_{a,m}^\kappa$. Let $s_{a,m}^\kappa$ be the restriction of the bilinear form s_a^κ on $\mathbf{V}_{a,m}^\kappa \times \mathbf{V}_{a,m}^\kappa$. As is well known (see, for example, [J, Theorem 3]), the module $\mathbf{V}_a^\kappa$ is irreducible if and only if s_a^κ is an isomorphism. In other words, $\mathbf{V}_a^\kappa$ is irreducible if and only if the bilinear form $s_{a,m}^\kappa$ is nondegenerate for all $m \in \mathbb{N}$.

For any $a \in \mathbf{N}^I$ and $m \in \mathbb{N}$ we define $d_m(a) \stackrel{\text{def}}{=} \dim \mathbf{V}_{a,m}$ and we extend the function d_a from $\mathbf{N}^I$ to $\mathbf{Z}^I$ in the following way. Let W be the Weyl group of $\mathfrak{g}$. W acts naturally on $\mathbf{Z}^I$, $a \rightarrow wa$, and we define the modified action of W on $\mathbf{Z}^I$, $(w, a) \rightarrow w \cdot a$, by the rule $w \cdot a = w(a+1) - 1$. As is well known, for any $a \in \mathbf{Z}^I$ either there exists a nontrivial element $w \in W$ such that $w \cdot a = a$ or there exists unique $w \in W$ such that $w \cdot a \in \mathbf{N}^I$. In the first case we define $d_m(a) = 0$; in the second $d_m(a) = \epsilon(w) d_m(w \cdot a)$, where $\epsilon: W \rightarrow (\pm 1)$ is the sign character. If κ is irrational we define $W_\kappa = W$. If $\kappa = -p/q$, $p, q \in \mathbb{N}$, $(p, q) = 1$, we denote by W_κ the subgroup of affine automorphisms of $\mathbf{Z}^I$ generated by $a \rightarrow W \cdot a$, $w \in W$, and shifts by pa , $a \in \mathbf{Z}^I$. Let Z be the set of W_κ orbits on $\mathbf{Z}^I$. For any $z \in Z$ we denote by $\mathcal{O}_\kappa(z)$ the full subcategory of objects $\mathbf{V}$ in $\mathcal{O}_\kappa$ such that all irreducible subquotients of $\mathbf{V}$ are isomorphic to L_a^κ for some $a \in z \cap \mathbf{N}^I$.

Let Δ be the set of roots of $\mathfrak{g}$. For any $\alpha \in \Delta$ we denote by $X_\alpha(a)$ the subset of the set $\mathbf{N}^+$ of nonzero natural numbers n such that $-\kappa^{-1}((a+1, \alpha) - n) \in \mathbf{N}^+$ and by $Y_\alpha(a) \subset \mathbf{Z}^I$ the set of elements of the form $a - n\alpha$, $n \in X_\alpha(a)$. For any $m \in \mathbb{N}$ we define a number

$$D_{\alpha,m}(a) \stackrel{\text{def}}{=} \sum_{c \in Y_\alpha(a)} d_m(c).$$

Lemma 27.7. (a) The category $\mathcal{O}_\kappa$ is a direct sum of its subcategories $\mathcal{O}_\kappa(z)$, $z \in Z$.

(b) The bilinear form $s_{a,m}$ is nondegenerate if and only if $\sum_{\alpha \in \Delta} D_{\alpha,m}(a) = 0$.

Proof. (a) Follows from Theorem 2 in [KK] and Proposition 3.9.

(b) Follows from the last result in [KK].

Remark. The paper [KK] does not contain the proof of this result but the proof can easily be obtained along the lines of the proof of Theorem 2 in [J].

If κ is a rational number, $\kappa = -p/q$, $p, q \in \mathbb{N}$, $(p, q) = 1$ and denote by $\mathbf{N}_0 \subset \mathbf{N}^I$ the subset of weights a of the form $a = pb + (p-1)1$, $b \in \mathbf{N}^I$. If κ is irrational, we define $\mathbf{N}_0 = \mathbf{N}^I$.

Proposition 27.4. *For any $a \in \mathbf{N}_0$ the module $\mathbf{V}_a^\kappa$ is irreducible.*

Proof. As follows from Lemma 27.5 it is sufficient to show that for all $\alpha \in \Delta$ we have $D_{\alpha,m}(a) = 0$. If α is a negative root, then the set $X_\alpha(a)$ is empty. So we may assume that α is a positive root. Since a lies in $\mathbf{N}_0$ we have $a + 1 = p(b + 1)$ for some $b \in \mathbf{N}^I$. Therefore $X_\alpha(a) = \{p\ell | 0 < \ell < (b + 1\alpha)\}$ and $Y_\alpha(a) = \{a - p\ell\alpha | 0 < \ell < \frac{1}{p}(a + 1\alpha)\}$. Let $s_\alpha \in W$ be the reflection at α . It is clear that $s_\alpha \cdot (Y_\alpha(a)) = Y_\alpha(a)$. Since for all $c \in \mathbf{Z}^I$ we have $d_m(c) + d_m(s_\alpha \cdot c) = 0$, we see that $D_{\alpha,m}(a) = 0$. Proposition 27.4 is proved.

Corollary. *For any $a \in \mathbf{N}_0$ the module $\mathbf{V}_a^\kappa$ is projective.*

Proof. As follows from Proposition 3.9, it is sufficient to show that for any $c \in \mathbf{N}^I$ such that $\mathbf{V}_a^\kappa$ is isomorphic to a subquotient of $\mathbf{V}_c^\kappa$ we have $c = a$. Let $c \in \mathbf{N}^I$ be such a weight. As follows from Lemma 27.4(a), we have $c \in \mathbf{N}_0$. Therefore $\mathbf{V}_c^\kappa$ is irreducible and we have $a = c$.

28. $\dot{\otimes}$ ON $\mathcal{A}_\kappa$

In this section we study the operation $\dot{\otimes}$ on $\mathcal{O}_\kappa$. We show that the subcategory $\mathcal{A}_\kappa$ is closed under $\dot{\otimes}$ and prove that the restriction of $\dot{\otimes}$ on $\mathcal{A}_\kappa$ is exact.

Proposition 28.1. *For any modules V_1, V_2 in $\mathcal{A}_\kappa$ the product $V_1 \dot{\otimes} V_2$ lies in $\mathcal{A}_\kappa$.*

Proof. Let $Z \rightarrow Z^\infty$ be the functor from the category of $\hat{\mathfrak{g}}$ -modules to the category of smooth $\hat{\mathfrak{g}}$ -modules defined in §6.2. It is easy to see that it is left exact. Since the contravariant functor $W \rightarrow \text{Hom}_{\mathbf{C}}(W, \mathbf{C})$ from the category of $\mathbf{C}$ -vector spaces to itself and the contravariant functor D from $\mathcal{O}_\kappa$ to itself are exact it follows from Theorem 7.9(b) that the functor $\dot{\otimes}: (V_1, V_2) \rightarrow V_1 \dot{\otimes} V_2$ is right exact. As follows from §13.4 for any modules V_1, V_2, V_3 in the category $\mathcal{O}_\kappa$ we have $\dim \text{Hom}(V_1 \dot{\otimes} V_2, D(V_3)) = \dim \langle V_1, V_2, V_3 \rangle$. Therefore for any exact sequence $V_1' \rightarrow V_1'' \rightarrow V_1 \rightarrow 0$ in $\mathcal{O}_\kappa$ we have

$$\dim \langle V_1'', V_2, V_3 \rangle \leq \dim \langle V_1', V_2, V_3 \rangle + \dim \langle V_1, V_2, V_3 \rangle,$$

and the same is true for analogous exact sequences containing V_2 and V_3 . For any $a, b, c \in \mathbf{N}^I$ we define $C_{a,b,c} \stackrel{\text{def}}{=} \dim \text{Hom}(\mathcal{V}_a \otimes \mathcal{V}_b \otimes \mathcal{V}_c, \mathbf{C})$ (see §1.2).

Lemma 28.1. *For any modules V_1, V_2, V_3 in $\mathcal{A}_\kappa$ we have*

$$\dim \langle V_1, V_2, V_3 \rangle = \sum_{a,b,c \in \mathbf{N}^I} m(V_1, a)m(V_2, b)m(V_3, c)C_{a,b,c}.$$

Proof. To simplify notation we denote the right side of the equation by $R(V_1, V_2, V_3)$. It is clear that $R(V_1, V_2, V_3)$ can be considered as a trilinear function on $[\mathcal{O}_\kappa]$. In the case when V_1, V_2, V_3 are generalized Weyl modules the statement of the lemma follows immediately from 9.15 and 13.4(c). Since

by the definition any module in $\mathcal{A}_\kappa$ has a Weyl filtration, we see that for any modules V_1, V_2, V_3 in $\mathcal{A}_\kappa$ we have

$$\dim\langle V_1, V_2, V_3 \rangle \leq \sum_{a,b,c \in \mathbb{N}^l} m(V_1, a) \cdot m(V_2, b) \cdot m(V_3, c) \cdot C_{a,b,c}.$$

We can show now that the statement of the lemma is true if V_2, V_3 are generalized Weyl modules and V_1 is an arbitrary module in $\mathcal{A}_\kappa$. Really by Theorem 2.22 any module in $\mathcal{O}_\kappa$ is a quotient of a generalized Weyl module. Therefore there exists an exact sequence $0 \rightarrow V_1' \rightarrow V_1'' \rightarrow V_1 \rightarrow 0$, where V_1'' is a generalized Weyl module. As follows from Proposition 27.1 the module V_1' lies in $\mathcal{A}_\kappa$. Therefore we have $\dim\langle V_1', V_2, V_3 \rangle \leq \dim\langle V_1'', V_2, V_3 \rangle + \dim\langle V_1, V_2, V_3 \rangle$. Since by the construction V_1'' is a generalized Weyl module, we have $\dim\langle V_1'', V_2, V_3 \rangle = R(V_1'', V_2, V_3)$. On the other hand we have inequalities

$$\dim\langle V_1', V_2, V_3 \rangle \leq R(V_1', V_2, V_3) \quad \text{and} \quad \dim\langle V_1, V_2, V_3 \rangle \leq R(V_1, V_2, V_3).$$

This implies the equality $\dim\langle V_1, V_2, V_3 \rangle = R(V_1, V_2, V_3)$. The same arguments show that the equality $\dim\langle V_1, V_2, V_3 \rangle = R(V_1, V_2, V_3)$ is true for all modules V_1, V_2, V_3 in $\mathcal{A}_\kappa$. Lemma 28.1 is proved.

To prove Proposition 28.1 we have to show that for any modules V_1, V_2, V_3 in $\mathcal{A}_\kappa$

$$\text{Ext}_{\mathcal{O}_\kappa}^1(V_1 \otimes V_2, D(V_3)) = 0$$

Since the category $\mathcal{O}_\kappa$ is a direct limit of its full subcategories $\mathcal{O}_\kappa^t$, it is sufficient to show that the equality

$$\text{Ext}_{\mathcal{O}_\kappa^t}^1(V_1 \otimes V_2, D(\mathbf{V}_c^\kappa)) = 0$$

holds for all sufficiently big t . As follows from Proposition 3.9 there exists a surjective map $P \rightarrow V_3$, where P lies in $\mathcal{A}_\kappa$ and P is a projective module in $\mathcal{O}_\kappa^t$. Let V' be the kernel of the morphism $P \rightarrow V_3$. It follows from Proposition 26.1 that V' lies in $\mathcal{A}_\kappa$. From the exact sequence for the group Ext we see that $\dim \text{Ext}_{\mathcal{O}_\kappa}^1(V_1 \otimes V_2, D(V_3)) = R(V_1, V_2, P) - R(V_1, V_2, V') - R(V_1, V_2, V_3)$. But as follows from Lemma 27.1 this difference is equal to 0. Proposition 28.1 is proved.

Proposition 28.2. *For any $t \in \mathbb{Z}$ the category $\mathcal{O}_\kappa^t$ has finite cohomological dimension.*

Proof. We first consider the following special case. Let F^t be a partially ordered set as in §3.3.

Lemma 28.2. *For any $a \in F^t$ the module $\mathbf{V}_a^\kappa$ has a finite projective resolution in $\mathcal{O}_\kappa^t$.*

Proof. We prove this result by the reverse induction. As follows from Lemma 3.5 the statement is true if a is a maximal element in F^t . To prove the lemma we can assume that the statement is known for all elements in F^t bigger

than some element $a_0 \in F^t$ and show that it is true for a_0 . But this follows immediately from Proposition 3.9. The lemma is proved.

To prove Proposition 28.2 it is sufficient to show that any irreducible module L_a in $\mathcal{O}_\kappa^t$ has a finite cohomological dimension. We will prove this result by the induction in a . If $a = 0$ then $L_a = \mathbf{V}_a^\kappa$ and the result follows from Lemma 28.1. So we may assume that for all $a < a_0$ the module L_a has finite cohomological dimension, and we have to prove that L_{a_0} has also finite cohomological dimension. To prove this, it is sufficient to observe that there exists a surjective morphism from V_{a_0} to L_{a_0} such that the Jordan-Hölder components of the kernel are isomorphic to L_a for $a < a_0$. Proposition 28.2 is proved.

For any module V in $\mathcal{O}_\kappa^t$ and $a \in F^t$ we define a number $\chi^t(V, a)$ as the sum

$$\chi^t(V, a) = \sum_{i \in \mathbb{N}} (-1)^i \cdot \dim \operatorname{Ext}_{\mathcal{O}_\kappa^t}^i(V, D(\mathbf{V}_a^\kappa)).$$

As follows from Proposition 28.2 this sum is finite, and it is easy to see that $\chi^t(V, a)$ depends only on the image $[V]$ of V in the Grothendieck group.

Lemma 28.3. (a) *For any module V in $\mathcal{O}_\kappa^t$ we have*

$$[V] = \sum_{a \in F^t} \chi^t(V, a) \cdot [\mathbf{V}_a^\kappa].$$

(b) *For any projective module P in $\mathcal{O}_\kappa^t$ we have $\dim \operatorname{Hom}_{\mathcal{O}_\kappa^t}(P, DV) = ([P][V])$ for any module V in $\mathcal{O}_\kappa^t$.*

Proof. Since both sides depend only on the image $[V]$ of V in the Grothendieck group and are linear in $[V]$ and $[\mathbf{V}_a^\kappa]$ for $a \in F^t$ is a basis of $[\mathcal{O}_\kappa^t]$, it is sufficient to check the equality for $V = \mathbf{V}_a^\kappa$ when $a \in F^t$. But in this case the statement follows from Proposition 27.1(4). Part (a) is proved. The proof of part (b) follows from Corollary 2 to Proposition 27.1.

Theorem 28.1. *For any V_1, V_2 in $\mathcal{A}^k$ we have*

$$[V_1 \dot{\otimes} V_2] = \sum_{a, b, c \in \mathbb{N}^t} m(V_1, a) \cdot m(V_2, b) \cdot C_{a, b, c} [\mathbf{V}_c^\kappa].$$

Proof. Follows from Lemma 28.3 and Proposition 28.2.

For any module V in $\mathcal{O}_\kappa$ and $\lambda \in \mathbb{C}$ we define $d(V, \lambda) = \dim_\lambda V$ (see §2.6). It is clear that $d(V, \lambda)$ depends only on the image $[V]$ of V in the Grothendieck group.

Corollary. *For any two modules V_1, V_2 in $\mathcal{A}_\kappa$ and $\lambda \in \mathbb{C}$ we have*

$$d(V_1 \dot{\otimes} V_2, \lambda) = \sum_{a, b, c \in \mathbb{N}^t} m(V_1, a) \cdot m(V_2, b) \cdot C_{a, b, c} \cdot d(\mathbf{V}_c^\kappa, \lambda).$$

For any $a \in \mathbb{N}^t$ we denote by $\mathcal{O}_\kappa^a$, $\mathcal{O}_\kappa^{<a}$ full subcategories of $\mathcal{O}_\kappa$ of objects whose composition factors are of the form L_b^κ for some $b \not\leq a$, $b \not\leq a$ correspondingly. It is clear that $\mathcal{O}_\kappa^a$ is a subcategory of $\mathcal{O}_\kappa^t$ for $t = \langle a, a+2 \rangle$.

It is clear the $\mathcal{O}_\kappa^{<a}$ is a Serre subcategory of $\mathcal{O}_\kappa^a$ and any object in the quotient category $\bar{\mathcal{O}}_\kappa^a$ is isomorphic to a multiple of $\bar{L}_a^\kappa$, where $\bar{L}_a^\kappa$ is the image of V_a^κ in $\mathcal{O}_\kappa^a$.

Lemma 28.4. *For any V in $\mathcal{O}_\kappa^a$, W in $\mathcal{O}_\kappa^b$ the product $V \dot{\otimes} W$ lies in $\mathcal{O}_\kappa^{a+b}$.*

Proof. Since the functor $\dot{\otimes}$ is right exact and the categories $\mathcal{O}_\kappa^a$ are closed under extensions, it is sufficient to show that $L_c^\kappa \dot{\otimes} L_d^\kappa$ are in $\mathcal{O}_\kappa^{a+b}$ for all $c, d \in \mathbb{N}^I$, $c \leq a, d \leq b$. Since L_c^κ, L_d^κ are quotients of V_c^κ, V_d^κ it is sufficient to show that $V_c^\kappa \dot{\otimes} V_d^\kappa$ are in $\mathcal{O}_\kappa^{a+b}$. But this follows immediately from Theorem 28.1.

29. THE CATEGORY $\mathcal{O}_\kappa$ OVER A RING

A number of results about the categories $\mathcal{O}_\kappa$ for rational κ we obtain by the “analytic continuation” from the generic case. In this section we consider $\mathcal{O}_\kappa$ as a category over a ring R of functions of κ . Let $S \subset \mathbb{C}$ be an open unbounded subset, R the ring of analytic functions on S meromorphic at ∞ , K the field of fractions of R , and $x \in R$ the function $x(z) = z, z \in S$. For any R -module M and $\kappa \in S$ we define $M(\kappa) \stackrel{\text{def}}{=} M/(x - \kappa)M$. It is clear that $M(\kappa) = M \otimes R/(x - \kappa)R$. If $N \subset M$ is an R -submodule, $\kappa \in S$, we denote by $\varphi_\kappa^N: N \rightarrow M(\kappa)$ the composition of the imbedding $N \hookrightarrow M$ and the projection $M \rightarrow M(\kappa)$. We define $M_{\text{tor}}^\kappa = \{m \in M | (x - \kappa)^N m = 0 \text{ for some } N \in \mathbb{N}\}$. If A is an endomorphism of M , then for any $\kappa \in S$ we denote by $A_{\text{tor}}(\kappa)$ and $A(\kappa)$ the induced endomorphisms of vector spaces M_{tor}^κ and $M(\kappa)$. We say that A is locally nilpotent if for any $m \in M$ there exists a natural number n such that $A^n \cdot m = 0$.

Lemma 29.1. (a) *Let M be a free R -module and $A \in \text{End}_R(M)$ be such that for any $\kappa \in S$ the action of A on the quotient $\mathbb{C}$ -vector space $M(\kappa)$ is locally nilpotent. Then A is locally nilpotent.*

(b) *Let $N \subset M$ be a finitely generated R -submodule, $S' \subset S$ uncountable, and $d \in \mathbb{N}$ a number such that $\dim_{\mathbb{C}} \text{Im } \varphi_\kappa^N \leq d$ for all $\kappa \in S'$. Then N is a free R -module of rank $\leq d$.*

(c) *Let $N \subset M$ be a submodule such that $\dim_{\mathbb{C}} \text{Im } \varphi_\kappa^N < \infty$ for all $\kappa \in S$. Then N is finitely generated.*

Proof. (a) Let $\Sigma_n \subset S$ be the set of points κ such that the image of $A^n \cdot m$ in $M/(x - \kappa)M$ is 0. By our assumption we have $\bigcup_n \Sigma_n = S$. Therefore there exists a number n such that the set Σ_n is uncountable. We may assume that $n = 1$. Let $e_l, l \in L$, be a basis of M over R . Then we have $A \cdot m = \sum_{l \in L} t_l \cdot e_l$ for some $t_l \in R$. As follows from the definition of the set Σ_1 we have $t_l(\kappa) = 0$ for all $\kappa \in \Sigma_1$. Therefore $t_l = 0$ for all $l \in L$ and $A \cdot m = 0$.

(b) Since N is a torsion free R -module it is free of rank, say, $\tilde{d}$. We claim that $\tilde{d} \leq d$. Really, let $n_1, \dots, n_{\tilde{d}}$ be free generators of $\tilde{N}$, $n_i = \sum_{l \in L} a_i^l e_l$, $a_i^l \in R$. Since $n_i, 1 \leq i \leq \tilde{d}$, are linearly independent over K there exists a subset $\mathcal{J} \subset L$, $|\mathcal{J}| = \tilde{d}$, such that the minor $\Delta = |(a_i^j)| \in R, 1 \leq i \leq \tilde{d}$,

$j \in J$, is not equal to zero. Since the subset $S' \subset S$ is uncountable there exists $\kappa \in S'$ such that $\Delta(\kappa) \neq 0$. But this implies that elements $\varphi_\kappa(n_i) \in \text{Im } \varphi_\kappa^N$, $1 \leq i \leq \tilde{d}$, are linearly independent. Therefore $\tilde{d} \leq d$.

(c) It is clear that there exists a number $d \in \mathbb{N}$ and an uncountable subset $S' \subset S$ such that $\dim_{\mathbb{C}} \text{Im } \varphi_\kappa^N \leq d$ for all $\kappa \in S'$. It follows from (b) that any finitely generated R -submodule of N is free of rank $\leq d$. Let $\tilde{N} \subset N$ be a finitely generated submodule of the maximal possible rank, say $\tilde{d}$. We will prove the lemma by the induction in $\tilde{d}$. For any $n \in N$ we know that the rank of the submodule $\tilde{N} + Rn$ is equal to the rank of $\tilde{N}$. Therefore there exists a nonzero element $a \in R$ such that $an \in \tilde{N}$. So N is a submodule in $\tilde{N} \otimes_R K \approx K^{\tilde{d}}$. Since N is a submodule of a free module M and R is a principal ideal domain for any $n \in N$, the R -submodule $I_n = \{k \in K | kn \in N\}$ is of the form $I_n = fR$ for some $f \in K$. This immediately implies that N decomposes in a direct sum $N = R(fn) \oplus N'$. By the inductive assumption N' is a free finitely generated R -module. Lemma 29.1 is proved.

For any $\kappa \in S$ we denote by R_κ the completion of R at κ . We denote by R_∞ the completion of R at ∞ . It is clear that $R_\infty = \mathbb{C}((x^{-1}))$.

Lemma 29.2. *Let M be a finitely generated R -module and $\kappa \in S$ is such that $\dim_{\mathbb{C}} M(\kappa) = \dim_K M(K)$, where $M(K) \stackrel{\text{def}}{=} M \otimes_R K$. Then the R_κ -module $M \otimes_R R_\kappa$ is free.*

Proof. Clear.

Let F be a field, M a finite-dimensional F -vector space, and A an endomorphism of M such that the characteristic polynomial of A is a product of linear factors. For any root $\lambda \in F$ of the characteristic polynomial of A we denote by ${}_\lambda M \subset M$ the submodule of elements $m \in M$ such that $(A - \lambda)^N m = 0$ for some $N \in \mathbb{N}$ and define ${}_\lambda M^\perp \stackrel{\text{def}}{=} \bigcap_{N>0} \text{Im}(A - \lambda)^N$. Then $M = {}_\lambda M \oplus {}_\lambda M^\perp$ and $M = \bigoplus_{\lambda \in \mathbb{C}} {}_\lambda M$. We define $d_M(A, \lambda) \stackrel{\text{def}}{=} \dim_F({}_\lambda M)$. It is clear that $\dim M = \sum_\lambda d_M(A, \lambda)$.

Now let M be a finitely generated R -module and A an endomorphism of M . For any point $\kappa \in S$ we denote by $A(\kappa)$ the induced endomorphism of the linear space $M(\kappa)$ and define $d_M^\kappa(A, \lambda) \stackrel{\text{def}}{=} d_{M(\kappa)}(A(\kappa), \lambda)$. Analogously we denote by $A(K)$ the induced endomorphism of the K -linear space $M(K)$. We will always assume that the characteristic polynomial of the operator $A(K)$ is a product of linear factors. Let $T(A, M) \subset K$ be the set of roots of this characteristic polynomial. It is clear that $T(A, M) \subset R$. Therefore for any $\Lambda \in T(A, M)$ we can define the submodule ${}_\Lambda M \stackrel{\text{def}}{=} \bigcup_N \text{Ker}(A - \Lambda)^N \subset M$, and for any $\kappa \in S$ we can define $\Lambda(\kappa) \in \mathbb{C}$. For any $\kappa \in S$, $\lambda \in \mathbb{C}$, we define $T(A, M)(\lambda) \stackrel{\text{def}}{=} \{\Lambda \in T(A, M) | \Lambda(\kappa) = \lambda\}$. Let ${}_\lambda \tilde{M} \stackrel{\text{def}}{=} \bigoplus_{\Lambda \in T(A, M)(\lambda)} {}_\Lambda M$.

Lemma 29.3. *The imbedding ${}_\lambda \tilde{M} \rightarrow M$ induces an isomorphism ${}_\lambda \tilde{M}(\kappa) \rightarrow {}_\lambda(M(\kappa))$.*

Proof. Clear.

Corollary 1. Assume that $\Lambda \in T(A, M)$ is such that for any $\Lambda' \in T(A, M)$, $\Lambda' \neq \Lambda$ and for any $\kappa \in S$, we have $\Lambda'(\kappa) \neq \Lambda(\kappa)$. Then $M = {}_{\Lambda}M \oplus {}_{\Lambda}M^{\perp}$, where

$${}_{\Lambda}M^{\perp} \stackrel{\text{def}}{=} \bigoplus_{\substack{\Lambda' \in T(A, M) \\ \Lambda' \neq \Lambda}} {}_{\Lambda'}M.$$

For any $\Lambda \in T(A, M)$ we define $d_M^K(A, \Lambda) \stackrel{\text{def}}{=} d_{M(K)}(A_K, \Lambda)$.

Lemma 29.4. If $d_M^K(A, \lambda) = \sum_{\Lambda \in T(A, M)(\lambda)} d_M^K(A, \Lambda)$, then ${}_{\lambda}(M_{\text{tor}}^{\kappa}) = 0$.

Proof. It is clear that the conditions of the lemma imply that $\dim_{\mathbb{C}}({}_{\lambda}\widetilde{M}(\kappa)) = \dim_K({}_{\lambda}\widetilde{M}_K)$. Therefore it follows from Lemma 29.2 that ${}_{\lambda}\widetilde{M}$ does not have torsion at κ . Lemma 29.4 is proved.

Let $M_i, \phi_i: M_i \rightarrow M_{i-1}, i \in \mathbb{N}$, be a projective system of finitely generated R -modules such that all maps ϕ_i are surjective, $A_i \in \text{End}(M_i)$ a compatible system of endomorphisms, $M \stackrel{\text{def}}{=} \varprojlim M_i, A \stackrel{\text{def}}{=} \varprojlim A_i \in \text{End}(M)$. As follows from the surjectivity of the morphisms ϕ_i , the natural morphism $M(\kappa) \rightarrow \varprojlim M_i(\kappa)$ is an isomorphism and we will identify the module $\varprojlim M_i(\kappa)$ with $M(\kappa)$.

We denote by $M^{\infty} \subset M$ the submodule of elements $m \in M$ which lie in a finitely generated A -invariant R -submodule of M and denote by ξ_i the restriction to M^{∞} of the natural projection from M to M_i . Analogously for any $\kappa \in S$ we denote by $M^{\infty}(\kappa) \subset M(\kappa)$ the subspace of $A(\kappa)$ -finite vectors and by $\xi_i(\kappa)$ the corresponding map from $M^{\infty}(\kappa)$ to $M_i(\kappa)$. The natural projection $M \rightarrow M(\kappa)$ induces the map $\psi(\kappa): M^{\infty} \rightarrow M^{\infty}(\kappa)$. Of course the map $\psi(\kappa)$ is not necessarily surjective. We say that an endomorphism A is admissible if for any $\kappa \in S, \lambda \in \mathbb{C}$ the nondecreasing sequence $d_{M_i}^{\kappa}(A_i, \lambda)$ has a limit $d_M^{\kappa}(A, \lambda) < \infty$. It is easy to see that in this case for any $\lambda \in S$ we have $d_M^{\kappa}(A, \lambda) \stackrel{\text{def}}{=} \lim_i d_{M_i}^{\kappa}(A_i, \lambda) < \infty$. For any admissible endomorphism A and complex numbers $\kappa \in S, \lambda \in \mathbb{C}$ we define $T(A, M)(\lambda) \stackrel{\text{def}}{=} \bigcup T(A_i, M_i)(\lambda)$. It is clear that the subset $T(A, M)(\lambda) \subset R$ is finite.

Proposition 29.1. Let A be an admissible endomorphism of M . Then

- (1) For any $\kappa \in S, i \in \mathbb{N}$ the map $\xi_i(\kappa)$ is a surjection.
- (2) The map ξ_i is surjective for all $i \in \mathbb{N}$.
- (3) If for any $\kappa \in S, \lambda \in \mathbb{C}$ we have

$$d_M^{\kappa}(A, \lambda) = \sum_{\Lambda \in T(A, M)(\lambda)} d_M^{\kappa}(A, \Lambda)$$

then the module M^{∞} is free and for any $\kappa \in S$ the map $\psi(\kappa)$ is a surjection.

Proof. (1) We have to show that for any $\kappa \in S, i \in \mathbb{N}$, we have $\text{Im } \xi_i(\kappa) = M_i(\kappa)$. It is sufficient to show that for any $\lambda \in \mathbb{C}$ we have $\text{Im } \xi_i(\kappa) \supset {}_{\lambda}M_i$. Since A is admissible we can find an integer j such that $d_{M_j}^{\kappa}(A_j, \lambda) =$

$d_M^\kappa(A, \lambda)$. Then for any $i > j$ the natural morphism ${}_\lambda M_i(\kappa) \rightarrow {}_\lambda M_j(\kappa)$ is an isomorphism and we have ${}_\lambda M^\infty(\kappa) = \varprojlim {}_\lambda M_i(\kappa)$. So (1) is proved.

(2) Let $M'_i \subset M_i$ be the image of ξ_i . We want to prove that $M'_i = M_i$. Since M_i is a finitely generated R -module it is sufficient to show that for any $\kappa \in S$ the map $M'_i(\kappa) \rightarrow M_i(\kappa)$ is surjective. But this follows from (1) and Lemma 29.3.

(3) It is clear that (3) is an immediate consequence of the following result.

Lemma 29.5. *If the conditions of (3) are satisfied then for any $\Lambda \in R$ the submodule ${}_\Lambda M \subset M^\infty$ is a finitely generated free R -module and $M^\infty = \bigoplus_{\Lambda \in R} {}_\Lambda M$.*

Proof. For any $\Lambda \in R$ we define $\lambda \stackrel{\text{def}}{=} \Lambda(\kappa)$ and $N \stackrel{\text{def}}{=} \bigoplus_{\Lambda' \in T(A, M^\infty)(\lambda)} {}_{\Lambda'} M$. As follows from Lemma 29.4 the module N does not have torsion at κ . Therefore ${}_\Lambda M$ does not have torsion at κ . The same arguments as in (1) prove Lemma 29.5 and the proposition.

Let $S = \mathbf{C} - \mathbf{R}_{\geq 0}$, $R = R_S$, and $\hat{\mathbf{g}} \stackrel{\text{def}}{=} \hat{\mathbf{g}}_R$. We consider $\hat{\mathbf{g}}$ as a R -Lie algebra and denote by $\mathcal{O}$ be the category of $\hat{\mathbf{g}}$ -modules V such that the element $1 \in \hat{\mathbf{g}}$ acts as the multiplication by $x - h$. As in §1.9 we can define R -submodules $V(N)$ for all $N \in \mathbf{N}$ and a $\hat{\mathbf{g}}$ -submodule V^∞ of V . We denote by $\mathcal{O}$ the full subcategory of $\mathcal{O}$ of $\hat{\mathbf{g}}$ -modules which are free as R -modules and for any $\kappa \in S$ the module $V(\kappa)$ belongs to the category $\mathcal{O}_\kappa$. As follows from Lemma 29.1(c) the submodules $V(N)$ of V are finitely generated free R -modules. By Lemma 28.1 we have $V = \bigcup_{N \geq 0} V(N)$, and therefore we can define the contravariant exact functor D from $\mathcal{O}$ to $\mathcal{O}$ as in §1.16. It is clear that for any V in $\mathcal{O}$ and $\kappa \in S$ we have $D(V(\kappa)) = D(V)(\kappa)$. To any V, V' in $\mathcal{O}$ we associate the R -module $\langle V, V' \rangle = V \otimes_R V' / \tilde{\mathbf{g}}(V \otimes V')$.

We denote by $\mathcal{A}$ the full subcategory of $\mathcal{O}$ of modules V such that for any $\kappa \in S$ the module $V(\kappa)$ lies in $\mathcal{A}_\kappa$. Given a nil-module $\mathcal{N}$ as in §2.1 we extend $\mathcal{N}_R \stackrel{\text{def}}{=} \mathcal{N} \otimes_{\mathbf{C}} R$ to a $\hat{\mathbf{g}}_R^+$ -module by defining the action of $1 \in \hat{\mathbf{g}}_R^+$ to be the multiplication by $x - h$. Let $\hat{\mathcal{N}} = U(\hat{\mathbf{g}}) \otimes_{U(\hat{\mathbf{g}}^+)} \mathcal{N}$. We say that $\hat{\mathcal{N}}$ is a generalized Weyl module.

Lemma 29.6. (a) *Any generalized Weyl module lies in $\mathcal{A}$.*

(b) *Any module in $\mathcal{A}$ is a quotient of a generalized Weyl module.*

(c) *If $\mathcal{N}_1, \mathcal{N}_2$ are nil-modules, then the R -module $\langle \hat{\mathcal{N}}_1, \hat{\mathcal{N}}_2 \rangle$ is a free R -module canonically isomorphic to $R \otimes (\mathcal{N}_1 \otimes \mathcal{N}_2) / \mathbf{g}$.*

(d) *For any V, W in $\mathcal{A}$ the R -module $\text{Hom}_{\mathcal{O}}(V, W)$ is a finitely generated torsion free R -module.*

(e) *For any V, W in $\mathcal{A}$, $\kappa \in S$ the natural map $\langle V, W \rangle(\kappa) \rightarrow \langle V(\kappa), W(\kappa) \rangle$ is an isomorphism.*

Proof. (a) Follows from Lemma 27.2.

(b) Follows from Lemma 2.5(b).

(c) Follows from Proposition 9.15.

(d) The module $\text{Hom}_{\mathcal{O}}(V, W)$ is torsion free since V is torsion free. The finiteness property follows from (b) and (c).

(e) Follows from the corollary to Proposition 27.1.

Let γ be as in the beginning of §27. For any two modules V_1, V_2 in $\mathcal{A}$ we can define $W = V_1 \otimes_R V_2$ and consider the projective system of R -modules $M_i = W/G_i W$ as in §4.8. The Sugawara operator L_0 (see §1.14) defines a compatible family of endomorphisms on the projective system M_i and therefore an endomorphism A of $M \stackrel{\text{def}}{=} \varprojlim M_i$. We define the module $T(W)(V_1, V_2) \stackrel{\text{def}}{=} M^\infty$ as in §4.11 and define $V_1 \dot{\otimes} V_2 = T(W)^\sharp(V_1, V_2)$.

Lemma 29.7. *The endomorphism A is admissible and it satisfies the condition (3) of Proposition 29.1.*

Proof. Follows immediately from Theorem 27.1.

Theorem 29.1. *For any modules V_1, V_2 in $\mathcal{A}$ the module $V_1 \dot{\otimes} V_2$ lies in $\mathcal{A}$, and for any $\kappa \in S$ the module $(V_1 \dot{\otimes} V_2)(\kappa)$ is isomorphic to $V_1(\kappa) \dot{\otimes} V_2(\kappa)$.*

Proof. As follows from the corollary to Theorem 28.1, Proposition 29.1, and Lemma 29.6, $T(W)^\sharp(V_1, V_2)$ is a free R -module and for any $\kappa \in S$ we have an isomorphism

$$T(W)^\sharp(V_1, V_2)(\kappa) = T(W)^\sharp(V_1(\kappa), V_2(\kappa)).$$

Therefore $V_1 \dot{\otimes} V_2$ is a free R -module and $(V_1 \dot{\otimes} V_2)(\kappa) = V_1(\kappa) \dot{\otimes} V_2(\kappa)$. The theorem follows now from Proposition 27.1.

For any ring homomorphism $\varphi: R \rightarrow \tilde{R}$ we denote by $\mathcal{O}_\varphi$ the category of $\hat{\mathfrak{g}}_{\tilde{R}}$ -modules defined analogously to the category $\mathcal{O}$ and by $\mathcal{A}_\varphi$ the corresponding subcategory of $\mathcal{O}_\varphi$. For any nil-module $\mathcal{N}$ we denote by $\tilde{\mathcal{N}}$ the corresponding object in $\mathcal{O}_\varphi$. It is clear that in the case when $\tilde{R} = \mathbb{C}$ and the morphism $\varphi = \varphi_\kappa: R \rightarrow \mathbb{C}$ is given by the rule $\varphi_\kappa(f) = f(\kappa)$, $\kappa \in S$, we have $\mathcal{O}_{\varphi_\kappa} = \mathcal{O}_\kappa$. For any $\kappa \in S$ we denote by $\hat{\kappa}$ the natural imbedding $\hat{\kappa}: R \rightarrow R_\kappa$. Analogously we denote by $\kappa_\infty: R \rightarrow R_\infty$ the imbedding which is a composition of the natural imbedding $R \hookrightarrow R_\infty$ and an automorphism of R_∞ such that $x \rightarrow x/2\pi\sqrt{-1}$. We will denote the category $\mathcal{O}_{\kappa_\infty}$ by $\mathcal{O}_\infty$.

Lemma 29.8. (a) *For any V in $\mathcal{A}$ and W in $\mathcal{O}$ the R -module $H(V, W) \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}}(V, W)$ is a free R -module of finite rank.*

(b) *For any point $\kappa \in S$ the natural map*

$$r(\kappa): H(V, W)(\kappa) \rightarrow \text{Hom}_{\mathcal{O}_\kappa}(V(\kappa), W(\kappa))$$

is an imbedding and the morphism

$$r(\hat{\kappa}): H(V, W) \otimes_R R_\kappa \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_\kappa}(V \otimes_R R_\kappa, W \otimes_R R_\kappa)$$

is an isomorphism.

Proof. (a): (1) In the case when $W = D(W_1)$, and W_1 belongs to $\mathcal{A}$, the statement follows from Lemma 29.6.

(2) For arbitrary W in $\mathcal{O}$ we can find a surjective map $\mathcal{N}_0 \rightarrow D(W)$ where $\mathcal{N}_0$ is a generalized Weyl module. This gives us an imbedding $W \hookrightarrow D(\mathcal{N}_0)$ and therefore an imbedding $H(V, W) \hookrightarrow H(V, D(\mathcal{N}_0))$, and we see that $H(V, W)$

is a submodule of a free R -module of finite rank. Therefore $H(V, W)$ is itself a free R -module of finite rank.

(b): Let $h \in H(V, W)$ be such that $h(\kappa) \neq 0$. We want to show that $r(\kappa)(h(\kappa)) \neq 0$. Assume that $r(\kappa)h(\kappa) = 0$. Then for any $v \in V$ we have $h(v)(\kappa) = 0$ and therefore we can define a new element $\tilde{h} \in H(V, W)$ such that $\tilde{h}(v) = (x - \kappa)^{-1}h(v)$ for all $v \in V$. But in this case $\tilde{h} \in (x - \kappa)H(V, W)$ and therefore $h(\kappa) = 0$. This contradiction proves the first part of the lemma. The second part follows from the flatness of R_κ as an R -module.

Remark. The map $r(\kappa)$ is not necessarily an isomorphism.

For any $a \in \mathbb{N}^I$ we denote by $\mathcal{O}^a$ the full subcategory of $\mathcal{O}$ of modules V such that for any $\kappa \in S$, $V(\kappa)$ lies in $\mathcal{O}_\kappa^a$ (see §28).

Lemma 29.9. *For any V in $\mathcal{O}^a$ and any $\kappa \in S$ the natural morphism*

$$\mathrm{Hom}_{\mathcal{O}}(\mathbf{V}_a^\kappa, V)(\kappa) \longrightarrow \mathrm{Hom}_{\mathcal{O}_\kappa}(\mathbf{V}_a^\kappa, V(\kappa))$$

is an isomorphism.

Proof. By the definition of the category $\mathcal{O}$ the R -module $V(1)$ is finitely generated, and for any $M \in \mathbb{N}$ the action A of the operator $L_0(M)$ on $V(1)$ does not depend on a choice of a number $M > 1$. Take $\Lambda = \frac{1}{2\kappa}(a + 2, a) \in R$. As follows from Proposition 2.7 and the corollary to Lemma 29.3 we have a direct sum decomposition $V(1) = {}_\Lambda(V)(1) \otimes {}_\Lambda(V)(1)^\perp$. The same arguments as in the proof of Lemma 3.5 show that $\mathrm{Hom}_{\mathcal{O}}(\mathbf{V}_a^\kappa, V) = \mathrm{Hom}_{\mathbf{g}_R}(\mathcal{Z}_a, {}_\Lambda(V)(1))$ and $\mathrm{Hom}_{\mathcal{O}_\kappa}(\mathbf{V}_a^\kappa, V(\kappa)) = \mathrm{Hom}_{\mathbf{g}}(\mathcal{Z}_a, {}_\Lambda(V)(1)(\kappa))$. Since ${}_\Lambda(V)(1)$ is a finitely generated torsion free R -module and the algebra $\mathbf{g}$ is semisimple we see that the natural morphism

$$\mathrm{Hom}_{\mathbf{g}_R}(\mathcal{Z}_a, {}_\Lambda(V)(1))(\kappa) \longrightarrow \mathrm{Hom}_{\mathbf{g}}(\mathcal{Z}_a, {}_\Lambda(V)(1)(\kappa))$$

is an isomorphism. The lemma is proved.

Proposition 29.2. *For any tilting module T_κ in $\mathcal{O}_\kappa$ there exists an object T in $\mathcal{O}_\kappa$ such that T_κ is isomorphic to $T(\kappa)$. Moreover, T is unique up to an isomorphism.*

Proof. Let $R_\kappa^{(n)} \stackrel{\mathrm{def}}{=} R/(x - \kappa)^n$ and κ_n be the natural morphism $\kappa_n: R \rightarrow R_\kappa^{(n)}$. It is clear that Proposition 29.2 follows from the following inductive result.

Proposition 29.2⁽ⁿ⁾. *For any $n > 0$ and any object $T^{(n)}$ in $\mathcal{O}_{\kappa_n}$ such that $T^{(n)}$ is isomorphic to $T_\kappa \otimes R_\kappa^{(n)}$ as $R_\kappa^{(n)}$ -module and the $\widehat{\mathbf{g}}$ -module $T^{(n)}/(x - \kappa)$ is isomorphic to T_κ , there exists an object $T^{(n+1)}$ in $\mathcal{O}_{\kappa_{n+1}}$ such that the $\widehat{\mathbf{g}}_{R_\kappa^{(n)}}$ -module $T^{(n+1)}/(x - \kappa)^n T^{(n+1)}$ is isomorphic to $T^{(n)}$, and $T^{(n+1)}$ is isomorphic to $T_\kappa \otimes R_\kappa^{(n+1)}$ as $R_\kappa^{(n+1)}$ -module. Moreover such an object $T^{(n+1)}$ is unique up to an isomorphism.*

Proof. We fix an identification of the $R_\kappa^{(n)}$ -module $T^{(n)}$ with $T_\kappa \otimes R_\kappa^{(n)}$, denote by

$$\varphi_n: \widehat{\mathbf{g}}_{R_\kappa^{(n)}} \rightarrow R_\kappa^{(n)} \otimes \mathrm{End} T_\kappa$$

the representation of $\widehat{\mathfrak{g}}_{R_\kappa}^n$ on $T_\kappa^{(n)} = T_\kappa \otimes R_\kappa^{(n)}$, and consider any $R_\kappa^{(n+1)}$ -linear map

$$\varphi: \widehat{\mathfrak{g}}_{R_\kappa}^{(n+1)} \rightarrow \text{End}_{\mathbb{C}}(T_\kappa \otimes R_\kappa^{(n+1)})$$

such that $\varphi \equiv \varphi^{(n)} \pmod{(x - \kappa)^n}$. Let

$$\tilde{\theta}: \Lambda_{R_\kappa}^2(\widehat{\mathfrak{g}}_{R_\kappa}^{(n+1)}) \longrightarrow \text{End}_{R_\kappa^{(n+1)}}(T_\kappa \otimes R_\kappa^{(n+1)})$$

be the map

$$\begin{aligned} \tilde{\theta}(x, y)(m) &= \varphi(x)\varphi(y)m - \varphi(y)\varphi(x)m - \varphi([x, y])m, \\ x, y &\in \widehat{\mathfrak{g}}_{R_\kappa}^{(n+1)}, \quad m \in T_\kappa \otimes R_\kappa^{(n+1)}. \end{aligned}$$

It is easy to see that there exists a unique map $\theta: \Lambda^2 \widehat{\mathfrak{g}} \rightarrow \text{End}_{\mathbb{C}}(T_\kappa)$ such that

$$\tilde{\theta}(\tilde{x}, \tilde{y})(\tilde{m}) = (x - \kappa)^n \theta(x, y)m,$$

where x, y, m are images of $\tilde{x}, \tilde{y}$, and $\tilde{m}$ in $\widehat{\mathfrak{g}}$ and T_κ . It is also easy to see that θ is a 2-cocycle. Since T_κ is tilting, we know that both T_κ and $D(T_\kappa)$ are in $\mathcal{A}_\kappa$, and it follows from Proposition 27.2 that θ is a coboundary. Therefore there exists a linear map $\psi: \widehat{\mathfrak{g}} \rightarrow \text{End}_{\mathbb{C}}(T_\kappa)$ such that

$$\theta(x, y) = x\psi(y) - y\psi(x) - \psi([x, y]).$$

We define the map

$$\varphi_{n+1}: \widehat{\mathfrak{g}}_{R_\kappa}^{(n+1)} \rightarrow \text{End}_{\mathbb{C}}(T_\kappa) \otimes R_\kappa^{(n+1)}$$

by the rule

$$\varphi_{n+1}(\tilde{x}) = \varphi(\tilde{x}) + (x - \kappa)^n \psi(x)$$

for all $\tilde{x} \in \widehat{\mathfrak{g}}_{R_\kappa}^{(n+1)}$, where x is the image of $\tilde{x}$ in $\widehat{\mathfrak{g}}$. Proposition 29.2⁽ⁿ⁾ and therefore Proposition 29.2 are proved.

Proposition 29.3. *If M is an object in $\mathcal{O}_{\widehat{\kappa}}$ such that $M(\kappa)$ is projective, then M is projective.*

Proof. Let $M_1 \xrightarrow{\varphi} M_2$ be a surjective morphism in $\mathcal{O}_{\widehat{\kappa}}$ and α be an element in $\text{Hom}_{\mathcal{O}_{\widehat{\kappa}}}(M, M_2)$. We have to show the existence of $\beta \in \text{Hom}_{\mathcal{O}_{\widehat{\kappa}}}(M, M_1)$ such that $\alpha = \varphi \circ \beta$. Define

$$M_i^{(n)} \stackrel{\text{def}}{=} M_i / (x - \kappa)^n M_i, \quad i = 1, 2,$$

and denote by

$$\alpha^{(n)}: M \rightarrow M_2^{(n)}, \quad \varphi^{(n)}: M_1^{(n)} \rightarrow M_2^{(n)}$$

the induced morphisms of $\widehat{\mathfrak{g}}_{R_\kappa}$ -modules.

Lemma 29.10. *For any morphism $\beta^{(n)}: M \rightarrow M_1^{(n)}$ of $\widehat{\mathfrak{g}}_{R_\kappa}$ -modules such that $\alpha^{(n)} = \varphi^{(n)} \circ \beta^{(n)}$ there exists a morphism $\beta^{(n+1)}: M \rightarrow M_1^{(n+1)}$ such that $\alpha^{(n+1)} =$*

$\varphi^{(n+1)} \circ \beta^{(n+1)}$ and $\pi_n^{(n+1)} \circ \beta^{(n+1)} = \beta^{(n)}$, where $\pi_n^{(n+1)}: M_1^{(n+1)} \rightarrow M_1^{(n)}$ is the natural projection.

Proof. We choose any map $\widehat{\beta}^{(n+1)}: M \rightarrow M_1^{(n+1)}$ of R_κ -modules such that $\pi_n^{(n+1)} \circ \widehat{\beta}^{(n+1)} = \beta^{(n)}$ and define a map $\widehat{\gamma}: \widehat{\mathbf{g}}_{R_\kappa} \otimes_{R_\kappa} M \rightarrow M_1^{(n+1)}$ by the rule

$$\widehat{\gamma}(x \otimes m) \stackrel{\text{def}}{=} x \widehat{\beta}^{(n+1)}(m) - \widehat{\beta}^{(n+1)}(xm).$$

We have $\text{Im } \widehat{\gamma} \subset (x - \kappa)^n M_1^{(n+1)} \simeq M_1(\kappa)$ and $\widehat{\gamma}$ induces a map

$$\gamma: \widetilde{\mathbf{g}} \rightarrow \text{Hom}_{\mathbf{C}}(M(\kappa), M_1(\kappa)).$$

It is clear that γ is a 1-cocycle. Since $M(\kappa)$ is a projective object in $\mathcal{O}_\kappa$ any exact sequence $0 \rightarrow M_1(\kappa) \rightarrow ? \rightarrow M(\kappa) \rightarrow 0$ splits. Therefore there exists $\delta \in \text{Hom}_{\mathbf{C}}(M(\kappa), M_1(\kappa))$ such that $\gamma = \partial \delta$. It is clear that the map $\beta^{(n+1)} \stackrel{\text{def}}{=} \widehat{\beta}^{(n+1)} + (x - \kappa)^n \delta$ is a morphism of $\widehat{\mathbf{g}}_{R_\kappa}$ -modules. Lemma 29.10 is proved.

We can now prove the proposition. Since $M(\kappa)$ is projective we can find a morphism $\beta(\kappa): M(\kappa) \rightarrow M_1(\kappa)$ such that $\alpha(\kappa) = \varphi(\kappa) \circ \beta(\kappa)$. We can consider $\beta(\kappa)$ as a morphism $\beta^{(1)}$ from M to $M_1^{(1)}$. By the lemma we can lift $\beta^{(1)}$ to a family of morphisms $\beta^{(n)}: M \rightarrow M_1^{(n)}$ such that $\alpha^{(n)} = \varphi^{(n)} \circ \beta^{(n)}$ and $\beta^{(n)} = \pi^{(n+1)} \circ \beta^{(n+1)}$. Then the sequence $\{\beta^{(n)}\}$ defines a morphism $\beta = \varprojlim \beta^{(n)}: M \rightarrow M_1$ such that $\alpha = \varphi \circ \beta$. The proposition is proved.

We say that a system M_a , $a \in \mathbf{N}^I$, of subobjects M_a of an object M in $\mathcal{A}$ is an admissible filtration if for any $\kappa \in S$ the natural morphisms $M_a(\kappa) \rightarrow M(\kappa)$ are imbeddings and the system $M_a(\kappa)$ is an admissible filtration of $M(\kappa)$.

Lemma 29.11. *Any object M in $\mathcal{A}$ has unique admissible filtration.*

Proof. The uniqueness follows immediately from Lemma 27.4. To prove the existence we assume that M lies in the category $\mathcal{O}^a$ for some $a \in \mathbf{N}^I$ and prove the existence of an admissible filtration by the induction in a . Let $a' \in \mathbf{N}^I$ be a weight preceding a in the ordering $\prec$. We may assume that the result is known for all objects M' in $\mathcal{O}^{a'}$. Let M be an object in $\mathcal{A} \cap \mathcal{O}^a$. Define $N \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}}(\mathbf{V}_a, M)$ and let $\varphi: N \otimes_R \mathbf{V}_a \rightarrow M$ be the morphism such that $\varphi(n \otimes v) = n(v)$. As follows from Lemma 29.8 the morphism φ is an imbedding and for any $\kappa \in S$ the image of $\varphi(\kappa)$ in $M(\kappa)$ coincides with the submodule $M(\kappa)(a)$ defined in Lemma 27.4. Define subobject $M(a)$ of M as the image of φ . It is clear that $M(a)$ is isomorphic to a multiple of $\mathbf{V}_a$ and the quotient module $M/M(a)$ belongs to $\mathcal{A} \cap \mathcal{O}^{a'}$. By the inductive assumptions $M/M(a)$ has an admissible filtration. The lemma is proved.

Let F_κ be the field of fractions of R_κ and $\check{\kappa}: R \rightarrow F_\kappa$ the natural imbedding. For any M in $\mathcal{O}_{\check{\kappa}}$ we denote its image in $\mathcal{O}_{\check{\kappa}}$ by $\check{M}$.

Let M be an object in $\mathcal{A}_{\check{\kappa}}$ with a filtration $0 = M_0 \subset \dots \subset M_N = M$ such that $M_N/M_{N-1} = \mathbf{1}$ and all the quotients M_i/M_{i-1} , $1 \leq i < N$, are

isomorphic to direct sums of Weyl modules V_b , $b \neq 0$. Then $\check{M}$ is a direct sum $\check{M} = \mathbf{1} \oplus \bigoplus_{b \neq 0} V_b^{\check{\kappa}}$. Let $\check{p} \in \text{End } \check{M}$ be the projection of $\check{M}$ onto $\mathbf{1}$, and $\hat{p} = (x - \kappa)^{N-1} \check{p}$.

Proposition 29.4. $\check{p}(M) \subset M$.

The proof of Proposition 29.4 is based on the following result in commutative algebra.

Let Λ be the ring of formal power series $\Lambda = \mathbb{C}[[t]]$, F its field of fractions, M a free finitely generated Λ -module with a filtration $0 = M_0 \subset M_1 \subset \dots \subset M_N = M$ such that the quotients are torsion free, and A an endomorphism of M preserving the filtration and such that for all i , $1 \leq i \leq N$, the quotient module $\bar{M}_i \stackrel{\text{def}}{=} M_i/M_{i-1}$ decomposes to a direct sum of A -invariant submodules $\bar{M}_i = \bigoplus_{j=1}^{n_i} M^{\alpha_j^i}$, $\alpha_j^i \in \Lambda$, such that A acts on $M^{\alpha_j^i}$ as the multiplication by $\alpha_j^i \text{Id}$. Let $Q = \{(i, j) | 1 \leq i \leq N, 1 \leq j \leq n_i, (i, j) \neq (1, 1)\}$.

We assume that $\alpha_1^1 = 0$ and $\alpha_j^i \not\equiv 0 \pmod{t^2}$ for all $(i, j) \in Q$. Let $\check{M} \stackrel{\text{def}}{=} M \otimes_{\Lambda} F$. The endomorphism A defines an endomorphism $\check{A} \in \text{End}_F \check{M}$, and we have a direct sum decomposition $\check{M} = \ker \check{A} \oplus \text{Im } \check{A}$. Let $\check{p} \in \text{End}_F \check{M}$ be the projection on $\ker \check{A}$ and $\hat{p} = t^{N-1} \check{p}$.

Lemma 29.12. $\hat{p}(M) \in M$.

Proof. Let $Q_0 = \{(i, j) \in Q | \alpha_j^i \in t\Lambda\}$. There exists a polynomial $p(T) \in \Lambda[T]$ such that $p(0) = 1$, $p(\alpha_j^i) = 1$ if $(i, j) \in Q_0$, and $p(\alpha_j^i) = 0$ if $(i, j) \in Q - Q_0$. Let $P \stackrel{\text{def}}{=} p(A)^r \in \text{End } M$. It is clear that P is an idempotent, and we have an A -invariant decomposition $M = M_0 \oplus M^{\perp}$, where $M_0 = \ker P$, $M^{\perp} = \ker(\text{Id} - P)$. Let $\hat{p}_0 \in \text{End } \widetilde{M}_0$ be an endomorphism defined in the same way as $\hat{p}$. It is clear that $\hat{p} = \hat{p}_0 \circ P$. So it is sufficient to prove the lemma in the case when $M = M_0$ or equivalently when $Q = Q_0$.

We fix a decomposition of M in a direct sum of Λ -modules $M = \bigoplus_{j=1}^N N_j$ such that for all i , $1 \leq i \leq r$, we have $M_i = \bigoplus_{j=1}^i N_j$. Let $p_i \in \text{End}_{\Lambda} M$ be the projection of M on N_j , $B \stackrel{\text{def}}{=} \sum_{i=1}^r p_i A p_i$, and $C = A - B$. It is clear that B preserves the filtration M_i on M and $C(M_i) \subset M_{i-1}$. Since $Q_0 = Q$ we see that $BM \subset tM$.

We can write an explicit formula for $\hat{p}$ in terms of A . We have

$$\hat{p} = t^{N-1} \Delta^{-1} \prod_{(i,j) \in Q} (\alpha_j^i - A), \quad \Delta = \prod_{(i,j) \in Q} \alpha_j^i.$$

For any subset $Z \subset Q$ and a pair $(i, j) \in Q$ we define endomorphisms $m_{(i,j)}^Z$ of $\check{M}$ by the rule

$$m_{(i,j)}^Z = \begin{cases} \alpha_j^i - \check{B} & \text{if } (i, j) \notin Z, \\ -\check{C} & \text{if } (i, j) \in Z. \end{cases}$$

For any subset Z of Q we define endomorphisms $\hat{p}_Z$ of $\check{M}$ as the products

$$\hat{p}_Z = t^{N-1} \Delta^{-1} \prod_{(i,j) \in Q} m_{(i,j)}^Z.$$

It is clear that $\hat{p}$ is equal to the sum of $\hat{p}_Z$ over all subsets Z of Q . Therefore it is sufficient to prove that for any subset Z of Q we have $\hat{p}_Z(M) \subset M$. If the number $|Z|$ of elements in Z is $\geq N$ then $p_Z = 0$. On the other hand, for any $(i, j) \in Q - Z$ we have $m_{(i,j)}^Z M \subset tM$. Therefore for any $Z \subset Q$ we have $\hat{p}_Z(M) \subset t^{N-1-|Z|} M$. Lemma 29.12 and Proposition 29.4 are proved.

Corollary. *There exists an imbedding $\tilde{i}: \mathbf{1} \rightarrow M$ such that $e \circ \tilde{i} = (x - \kappa)^{N-1}$, where e is the natural projection $e: M = M_N \rightarrow M_N/M_{N-1} = \mathbf{1}$.*

Proof. Consider the image $P = \text{Im}(\hat{p}) \subset M$. It is clear that $P \otimes_{R_\kappa} F_\kappa$ is isomorphic to $\check{M}^0$. Since the $\tilde{\mathfrak{g}}$ -module V_0^κ is irreducible, we see that the $\tilde{\mathfrak{g}}_{R_\kappa}$ -module P is isomorphic to $\mathbf{1}$. The corollary is proved.

Remark. For any $a \in \mathbf{N}^I$ we define $\mathbf{V}_a^\kappa = \mathbf{V}_a \otimes_{R_\kappa} F_\kappa$, $\hat{s}_a^\kappa = \hat{s}_a \otimes 1$. It is clear that in the case when $\mathbf{V}_a^\kappa$ is irreducible $\hat{s}_a^\kappa$ defines an isomorphism $\hat{s}_a^\kappa: \mathbf{V}_a^\kappa \xrightarrow{\sim} D(\mathbf{V}_a^\kappa)$.

30. THE CATEGORY $\mathcal{O}_\kappa$ FOR “GENERIC” κ

In this section we study the category $\mathcal{O}_\kappa$ in the case when κ is “generic”.

Let $\kappa: R \rightarrow F$ be a ring homomorphism from R to a field F . We denote by $\mathcal{F}$ and $\mathcal{G}$ the functors from $\mathcal{O}_\kappa$ to the category $\mathcal{D} = \mathcal{D}(F)$ of finite-dimensional representations of the Lie algebra $\mathfrak{g}_F \stackrel{\text{def}}{=} \mathfrak{g} \otimes_{\mathbb{C}} F$ given by $\mathcal{F}(V) \stackrel{\text{def}}{=} V(1)$, $\mathcal{G}(V) \stackrel{\text{def}}{=} V/Q_1^\# V$. For any V in $\mathcal{O}_\kappa$ we have natural morphisms $p_V: \mathcal{F}(V) \rightarrow V$ and $q_V: V \rightarrow \mathcal{G}(V)$. We denote by α the composition $\alpha \stackrel{\text{def}}{=} q_V \circ p_V: \mathcal{F} \rightarrow \mathcal{G}$. Let $\Omega \in U(\mathfrak{g})$ be the Casimir element corresponding to the scalar product $(,)$ as in §1.2. We denote by l the endomorphism of the identity functor $1_{\mathcal{D}}$ which for any representation $\rho: \mathfrak{g}_F \rightarrow \text{End}(\check{V})$ in $\mathcal{D}$ acts as $\rho(\omega)$. We say that the κ is generic if $\kappa(x) \notin Q$. In this section we will assume that κ is generic.

Lemma 30.1. (1) *For any V in $\mathcal{O}_\kappa$ we have $L_0 \circ p_V = p_V \circ l$ and $q_V \circ L_0 = l \circ q_V$.*

(2) *The functors $\mathcal{F}$, $\mathcal{G}$ are equivalences of categories and α is a functorial isomorphism.*

Proof. (1) follows immediately from Proposition 2.7(d).

(2) For any $a \in \mathbf{N}^I$ we denote by $\mathbf{V}_a^\kappa$ the Weyl module in $\mathcal{O}_F$. As follows from Lemma 2.9 and Proposition 2.12 the modules $\mathbf{V}_a^\kappa$ are irreducible, all irreducible modules in $\mathcal{O}_F$ are isomorphic to $\mathbf{V}_a^\kappa$ for some $a \in \mathbf{N}^I$ and $\mathcal{F}(\mathbf{V}_a^\kappa) = \mathcal{V}_a$. As follows from Proposition 3.9 the modules $\mathbf{V}_a^\kappa$ are projective. Since all irreducible objects in $\mathcal{O}_F$ are projective this category is semisimple. It is clear now that the functor $\mathcal{F}$ is an equivalence of categories. The same

arguments show that $\mathcal{G}$ is also an equivalence of categories and that α is an isomorphism. Lemma 30.1 is proved.

Let $W \rightarrow \widetilde{W}$ be the functor from $\mathcal{D}$ to $\mathcal{O}_\kappa$ as in §29. The imbedding $W \hookrightarrow \widetilde{W}$ defines an isomorphism $W \xrightarrow{\sim} \mathcal{G}(\widetilde{W})$. We will identify $\mathcal{G}(\widetilde{W})$ with W .

Consider now two functors θ_1, θ_2 from $\mathcal{O}_\kappa \times \mathcal{O}_\kappa$ to $\mathcal{D}$.

$$\theta_1(V_1, V_2) \stackrel{\text{def}}{=} \mathcal{G}(V_1) \otimes \mathcal{G}(V_2), \quad \theta_2(V_1, V_2) \stackrel{\text{def}}{=} \mathcal{G}(V_1 \dot{\otimes} V_2).$$

For any Weyl modules V_1, V_2 in $\mathcal{O}_\kappa$ we denote by $\beta(V_1, V_2): \mathfrak{g}(V_1) \otimes \mathfrak{g}(V_2) \rightarrow \mathfrak{g}(V_1 \dot{\otimes} V_2)$ the isomorphism defined in the remark after Lemma 26.4. Since κ is generic and an object in $\mathcal{O}_\kappa$ is a direct sum of Weyl modules, β defines a natural isomorphism between functors θ_1 and θ_2 .

Lemma 30.2. *For any W_1, W_2 in $\mathcal{D}$, there exists a unique isomorphism*

$$\tilde{\beta}: \widetilde{W_1 \dot{\otimes} W_2} \xrightarrow{\sim} (W_1 \widetilde{\otimes} W_2)^\kappa$$

such that the composition

$$W_1 \otimes W_2 \rightarrow \mathcal{G}(\widetilde{W_1}) \otimes \mathcal{G}(\widetilde{W_2}) \xrightarrow{\beta} \mathcal{G}(\widetilde{W_1 \dot{\otimes} W_2}) \xrightarrow{\mathcal{G}(\tilde{\beta})} \mathcal{G}(W_1 \widetilde{\otimes} W_2) \xrightarrow{\sim} W_1 \otimes W_2$$

is the identity map.

Proof. Follows easily from the definition of $\dot{\otimes}$ using the equality 13.5(b).

31. A CRITERION OF RIGIDITY FOR WEYL MODULES IN $\mathcal{O}_\kappa$

In this section we prove that the category $\mathcal{O}_\kappa$ is a braided category and proves a criterion of the rigidity for Weyl modules in $\mathcal{O}_\kappa$. We assume in this section that F is a field of fractions of R .

For any entire function $\varphi = \sum_{n=0}^{\infty} a_n x^n$ on $\mathbf{C}$ and any $f \in R$ the series $\varphi = \sum_{n=0}^{\infty} a_n f^n$ is convergent and defines an element $\varphi(f)$ in R . Analogously, for any analytic function $\varphi(t)$ on $[0, 1]$ with values in the ring $M_n(R)$ of $(n \times n)$ -matrices over R and any vector $v_0 \in R^n$, there exists unique analytic function $v(t)$ on $[0, 1]$ with values in F^n such that $V(0) = V_0$ and $\frac{\partial V(t)}{\partial t} = \varphi(t)V(t)$. Therefore construction of §§14 and 17.28 are applicable, and we can define the commutativity isomorphism $\mathcal{P}: V_1 \dot{\otimes} V_2 \rightarrow V_2 \dot{\otimes} V_1$ (as in §14.6), the isomorphism $\varphi: \langle V_1 \dot{\otimes} V_2, V_3 \dot{\otimes} V_4 \rangle \xrightarrow{\sim} \langle V_1, V_2, V_3, V_4 \rangle$ as in §17.29, and the associativity isomorphism $a: (V_1 \dot{\otimes} V_2) \dot{\otimes} V_3 \xrightarrow{\sim} V_1 \dot{\otimes} (V_2 \dot{\otimes} V_3)$ as in §18.2 for all V_1, V_2, V_3, V_4 in $\mathcal{A}$. Analogously, for any $\widehat{V}_1, \widehat{V}_2, \widehat{V}_3$ in $\mathcal{O}_\infty$ we can define isomorphisms $\widehat{P}: \widehat{V}_1 \dot{\otimes} \widehat{V}_2 \rightarrow \widehat{V}_2 \dot{\otimes} \widehat{V}_1$ and $\widehat{a}: \widehat{V}_1 \dot{\otimes} (\widehat{V}_2 \dot{\otimes} \widehat{V}_3) \rightarrow \widehat{V}_1 \dot{\otimes} (\widehat{V}_2 \dot{\otimes} \widehat{V}_3)$.

Let R_∞ be the completion of R at ∞ as in §29, $\kappa_\infty: R \hookrightarrow R_\infty$ the imbedding, and $\kappa = \kappa_\infty(x) \in R_\infty$. We denote by A_{an} the ring of germs of analytic functions at $0 \in \mathbf{C}$ with values in R_∞ . Let M be a finitely generated torsion-free R_∞ -module, $M_{an} \stackrel{\text{def}}{=} M \otimes_{R_\infty} A_{an}$, $\nabla_{t \frac{\partial}{\partial t}}^0$ the trivial connection on M_{an} , and $\nabla_{t \frac{\partial}{\partial t}}$ a connection on M_{an} such that the difference $\nabla_{t \frac{\partial}{\partial t}} - \nabla_{t \frac{\partial}{\partial t}}^0$ has a form

$T(t) dt$, where $T(t)$ is in the maximal ideal of R_∞ and $\nabla'_{t \frac{\partial}{\partial t}} \stackrel{\text{def}}{=} \nabla_{t \frac{\partial}{\partial t}}^0 + T(0) dt$. Let $\langle M \rangle, \langle M' \rangle$ be the spaces of germs of flat sections of the connections ∇, ∇' on the intervals $(0, \alpha)$, $0 < \alpha \ll 1$.

Lemma 31.1. (a) *There exists a unique isomorphism $\varphi(t): (M_{an}, \nabla') \xrightarrow{\sim} (M_{an}, \nabla)$ such that $\varphi(0) = \text{Id}$.*

(b) *For any $\hat{V} = \hat{V}(t) \in \langle M \rangle$ there exists the limit $\hat{Q}(\hat{V}) = \lim_{t \rightarrow +0} t^{-T(0)} \hat{V}(t) \in M$.*

(c) *For any $V' = V'(t) \in \langle M' \rangle$ there exists $Q'(V') \in M$ such that*

$$t^{-T(0)} V'(t) = Q'(V') \quad \text{for any } t > 0.$$

(d) *The map $\varphi(t)$ induces an isomorphism $\varphi_M: \langle M \rangle \rightarrow \langle M' \rangle$ such that $Q' \circ \varphi_M = Q \circ \varphi_M$.*

Proof. Clear.

For any objects W_i in the category $\mathcal{D}$ of finite-dimensional $\mathfrak{g}_{R_\infty}$ -modules, $1 \leq i \leq 4$, we denote by M the space of $\mathfrak{g}_{R_\infty}$ -coinvariants of the tensor product $\bigotimes_{i=1}^4 W_i$. On the other hand, we denote by W_i^∞ the objects $(W_i)^{K_\infty}$ in $\mathcal{C}_\infty$ and consider two A_{an} -local systems with connections

$$\mathcal{M} = \langle W_1^\infty, W_2^\infty, W_3^\infty, W_4^\infty \rangle \quad \text{and} \quad \mathcal{M}' = \langle W_1^\infty \dot{\otimes} W_2^\infty, W_3^\infty \dot{\otimes} W_4^\infty \rangle$$

as in §17.28.

Let $\eta: M_{an} \xrightarrow{\sim} \mathcal{M}_{an}$ and $\eta': M_{an} \xrightarrow{\sim} \mathcal{M}'_{an}$ be the isomorphisms as in Lemma 26.7.

Let ∇ be the connection on M_{an} as in Lemma 26.6, and ∇' the connection on M_{an} as in Lemma 31.1.

Lemma 31.2. (a) $\eta: (M_{an}, \nabla) \xrightarrow{\sim} (\mathcal{M}_{an}, \nabla)$ and $\eta': (M_{an}, \nabla') \xrightarrow{\sim} (\mathcal{M}'_{an}, \nabla')$ are isomorphisms of modules with connections.

(b) *The induced diagram*

$$\begin{array}{ccc} \langle M \rangle & \xrightarrow{\varphi} & \langle M' \rangle \\ \eta \downarrow & & \downarrow \eta' \\ \langle \mathcal{M} \rangle & \xrightarrow{\varphi} & \langle \mathcal{M}' \rangle \end{array}$$

is commutative where φ is an isomorphism as in §17.28.

Proof. (a) Follows from Corollary 1 to Proposition 26.1.

(b) Follows from Lemma 26.8.

Let $\mathcal{D}$ be the category defined in §19, where we identify the ring A with R_∞ .

Given W_1, W_2, W_3 in $\mathcal{D}$ we denote by $(\tilde{\mathcal{N}}, \nabla_{\partial/\partial t})$ the pair of an R -vector bundle over $\mathbb{C} - \langle 0, 1 \rangle$ with a connection, where $\mathcal{N}$ is the trivial vector bundle over $\mathbb{C} - \langle 0, 1 \rangle$ with the fiber $\mathcal{N} = W_1 \otimes W_2 \otimes W_3$ and $\nabla_{\frac{\partial}{\partial t}}$ is the sum of the trivial connection $\nabla_{\frac{\partial}{\partial t}}^0$ and the 1-form $-\frac{1}{\kappa}(\frac{\Omega_{12}}{t} - \frac{\Omega_{23}}{1-t})dt$. Let $\langle \mathcal{N} \rangle$ be the

space of flat sections $n(t)$ of $\widetilde{\mathcal{N}}$ over the interval $(0, 1)$, $a_0: \langle \mathcal{N} \rangle \rightarrow \mathcal{N}$, $a_1: \langle \mathcal{N} \rangle \rightarrow \mathcal{N}$ the linear isomorphisms given by

$$a_0(n(t)) = \lim_{t \rightarrow 0} (t^{\Omega_{12}/\kappa} n(t)), \quad a_1(n(t)) = \lim_{t \rightarrow 1} ((1-t)^{\Omega_{23}/\kappa} n(t)),$$

and $a: (V_1^\infty \dot{\otimes} V_2^\infty) \dot{\otimes} V_3^\infty \xrightarrow{\sim} V_1^\infty \dot{\otimes} (V_2^\infty \dot{\otimes} V_3^\infty)$ an isomorphism as in §18.2. For any W_1, W_2, W_3 in $\mathcal{D}$, we define $a^\infty \in \text{Aut } \mathcal{N}$ as the composition

$$\begin{aligned} \mathcal{N} &= \mathcal{G}(((W_1 \otimes W_2) \otimes W_3)^\infty) \xrightarrow{\mathcal{G}(\tilde{\beta}^{-1})} \mathcal{G}((W_1^\infty \dot{\otimes} W_2^\infty) \dot{\otimes} W_3^\infty) \\ &\xrightarrow{\mathcal{G}(a)} \mathcal{G}(W_1^\infty \dot{\otimes} (W_2^\infty \dot{\otimes} W_3^\infty)) \xrightarrow{\mathcal{G}(\tilde{\beta})} \mathcal{G}(((W_1 \otimes W_2) \otimes W_3)^\infty) \xrightarrow{\sim} \mathcal{N}. \end{aligned}$$

Corollary. $a^\infty = a_1 \circ a_0^{-1}$.

For any W_1, W_2 in $\mathcal{D}$ we define the isomorphism $\mathcal{P}^\infty: W_1 \otimes W_2 \rightarrow W_2 \otimes W_1$ as the composition

$$\begin{aligned} W_1 \otimes W_2 &\xrightarrow{\sim} \mathcal{G}((W_1 \otimes W_2)^\infty) \xrightarrow{\mathcal{G}(\tilde{\beta}^{-1})} \mathcal{G}(W_1^\infty \dot{\otimes} W_2^\infty) \xrightarrow{\mathcal{G}(\mathcal{P})} \mathcal{G}(W_2^\infty \dot{\otimes} W_1^\infty) \\ &\xrightarrow{\mathcal{G}(\tilde{\beta})} \mathcal{G}((W_2 \otimes W_1)^\infty) = W_2 \otimes W_1. \end{aligned}$$

Proposition 31.1. *The data $(\mathcal{D}, \otimes, a^\infty, \mathcal{P}^\infty)$ of the category $\mathcal{D}$, a functor $\otimes: \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$, and functorial isomorphisms $a^\infty, \mathcal{P}^\infty$ define a braided category which is equal to the braided category $\mathcal{D}$ defined in [D1]. (See also the definition in §19, where $\varpi = -1/\kappa$.)*

Proof. Follows from the definition of $\mathcal{P}$, Proposition 26.1(b), and the corollary to Lemma 31.2.

Remark. For the definition of braided category see [JS] or the Appendix.

Let β be the natural isomorphism as in Lemma 30.2.

Lemma 31.3. *For any objects V_1, V_2, V_3 in $\mathcal{O}_\infty$ the diagrams*

$$\begin{array}{ccc} \mathcal{G}(V_1) \otimes \mathcal{G}(V_2) & \xrightarrow{\beta} & \mathcal{G}(V_1 \dot{\otimes} V_2) \\ \downarrow \mathcal{P}^\infty & & \downarrow \mathcal{G}(\mathcal{P}) \\ \mathcal{G}(V_2) \otimes \mathcal{G}(V_1) & \xrightarrow{\beta} & \mathcal{G}(V_2 \dot{\otimes} V_1) \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{G}(V_1) \otimes \mathcal{G}(V_2) \otimes \mathcal{G}(V_3) & \xrightarrow{\beta \otimes 1} & \mathcal{G}(V_1 \dot{\otimes} V_2) \otimes \mathcal{G}(V_3) & \xrightarrow{\beta} & \mathcal{G}(V_1 \dot{\otimes} (V_2 \dot{\otimes} V_3)) \\ \downarrow a^\infty & & & & \downarrow \mathcal{G}(a) \\ \mathcal{G}(V_1) \otimes \mathcal{G}(V_2) \otimes \mathcal{G}(V_3) & \xrightarrow{1 \otimes \beta} & \mathcal{G}(V_1) \otimes \mathcal{G}(V_2 \dot{\otimes} V_3) & \xrightarrow{\beta} & \mathcal{G}(V_1 \dot{\otimes} (V_2 \dot{\otimes} V_3)) \end{array}$$

are commutative.

Proof. Clear.

Corollary. *The category $(\mathcal{O}_\infty, \dot{\otimes}, a, \mathcal{P})$ is a braided category and the pair $(\mathcal{E}, \beta)$ defines an equivalence between braided categories $\mathcal{O}_\infty$ and $\mathcal{D}$.*

For any $\kappa \in \mathbf{C} - \mathbf{Q}_{\geq 0}$ and any object V of $\mathcal{O}_\kappa$ we denote by b_V the automorphism of V defined by $b_V = \exp(2\pi i L_0)$. We can consider b as an automorphism of the identity functor of $\mathcal{O}_\kappa$.

Proposition 31.2. *For any $\kappa_0 \in \mathbf{C} - \mathbf{Q}_{\geq 0}$ the category $(\mathcal{O}_{\kappa_0}, \dot{\otimes}, a^0, \mathcal{P}^0, b_{\kappa_0})$ is a semirigid braided balanced category, where $a^0, \mathcal{P}^0$ are functorial morphisms defined in §§18.2 and 14.6. (See Definitions A.6 and A.11 in the Appendix.)*

Proof. To prove that $(\mathcal{O}_{\kappa_0}, \dot{\otimes}, a^0, \mathcal{P}^0)$ is a braided category we have to show that $a^0, \mathcal{P}^0$ satisfy the pentagon and hexagon axioms. The pentagon axiom says that for any $V_1^0, V_2^0, V_3^0, V_4^0$ in $\mathcal{O}_{\kappa_0}$ two functorial morphisms $\phi_\pm^{\kappa_0}$ from $((V_1^0 \dot{\otimes} V_2^0) \dot{\otimes} V_3^0) \dot{\otimes} V_4^0$ to $V_1^0 \dot{\otimes} (V_2^0 \dot{\otimes} (V_3^0 \dot{\otimes} V_4^0))$ which are defined as the composition of isomorphisms coincide. Of course it is sufficient to check that for any V_5^0 in $\mathcal{O}_{\kappa_0}$ the induced maps $\phi_\pm^{\kappa_0}$ from $\langle ((V_1^0 \dot{\otimes} V_2^0) \dot{\otimes} V_3^0) \dot{\otimes} V_4^0, V_5^0 \rangle$ to $\langle V_1^0 \dot{\otimes} (V_2^0 \dot{\otimes} (V_3^0 \dot{\otimes} V_4^0)), V_5^0 \rangle$ coincide.

Since any object in $\mathcal{O}_{\kappa_0}$ is a quotient of a generalized Weyl module (see Lemma 2.5(a), it is sufficient to check the axioms in the case when $V_i^0 = \mathcal{N}_i^{\kappa_0}$ are generalized Weyl modules, $1 \leq i \leq 5$. Let $V_i^\kappa \stackrel{\text{def}}{=} \mathcal{N}_i^\kappa$, $\kappa \in \mathbf{C} - \mathbf{Q}_{\geq 0}$, let $\hat{V}_i = \hat{\mathcal{N}}_i$ be the corresponding objects in $\mathcal{A}_\kappa$ and $\mathcal{A}$, and define vector spaces

$$\begin{aligned} M_1^\kappa &= \langle ((V_1^\kappa \dot{\otimes} V_2^\kappa) \dot{\otimes} V_3^\kappa) \dot{\otimes} V_4^\kappa, V_5^\kappa \rangle, \\ M_2^\kappa &= \langle V_1^\kappa \dot{\otimes} (V_2^\kappa \dot{\otimes} (V_3^\kappa \dot{\otimes} V_4^\kappa)), V_5^\kappa \rangle, \end{aligned}$$

and R -modules

$$\begin{aligned} M_1 &\stackrel{\text{def}}{=} \langle ((V_1 \dot{\otimes} V_2) \dot{\otimes} V_3) \dot{\otimes} V_4, V_5 \rangle, \\ M_2 &\stackrel{\text{def}}{=} \langle (V_1 \dot{\otimes} (V_2 \dot{\otimes} (V_3 \dot{\otimes} V_4))), V_5 \rangle. \end{aligned}$$

As follows from Theorem 29.1 and Lemma 29.7 the R -modules M_1, M_2 are finitely generated torsion-free R -modules and we have natural isomorphisms $M_i(\kappa) \xrightarrow{\sim} M_i^\kappa$, $i = 1, 2$. It is easy to see that the isomorphisms $\phi_\pm^\kappa: M_1^\kappa \rightarrow M_2^\kappa$ define an analytic family of isomorphisms $\phi_\pm(\kappa): M_1 \rightarrow M_2$. We consider an imbedding $R \hookrightarrow R_\infty$ such that the isomorphisms $\phi_\pm(\kappa)$ extend to isomorphisms $\phi_\pm^\infty: M_1^\infty \rightarrow M_2^\infty$, where $M_i^\infty \stackrel{\text{def}}{=} M_i \dot{\otimes}_R R_\infty$ and the isomorphisms $\phi_\pm^\infty$ coincide with the isomorphisms $\phi_\pm$ in the category $\mathcal{O}_\infty$. As follows from the corollary to Lemma 31.3, the category $(\mathcal{O}_\infty, \dot{\otimes}, a^\infty)$ satisfies the pentagon axiom and therefore $\phi_+^\infty = \phi_-^\infty$. Since M_i is torsion free, this implies that $\phi_+(\kappa) = \phi_-(\kappa)$ for all $\kappa \in \mathbf{C} - \mathbf{Q}_{\geq 0}$. In particular, $\phi_+^{\kappa_0} = \phi_-^{\kappa_0}$. We see that the pentagon axiom in $\mathcal{O}_{\kappa_0}$ is satisfied.

The proof of the hexagon axioms is completely analogous.

We see that $(\mathcal{O}_{\kappa_0}, \dot{\otimes}, a^0, \mathcal{P}^0)$ is a braided category. The semirigidity of $\mathcal{O}_{\kappa_0}$ follows from the isomorphism 2.32(c), and the right exactness of $\dot{\otimes}$ proven

in the beginning of §28. It is easy to see that b defines a balancing on $\mathcal{O}_\kappa$. Proposition 31.2 is proved.

Corollary. *For any morphism $\kappa: R \rightarrow F$ the category $(\mathcal{O}_\kappa, \dot{\otimes}, a^\kappa, \mathcal{P}^\kappa)$ is a semirigid braided category.*

For any X, Y in $\mathcal{O}_\kappa$ we will denote by $\alpha_{X,Y}: D(Y) \dot{\otimes} D(X) \rightarrow D(X \dot{\otimes} Y)$ the morphism from Definition A.8 in the Appendix.

By the Mac Lane theorem, we know that the monoidal categories $(\mathcal{O}_\kappa, \dot{\otimes}, a)$ are equivalent to strict monoidal categories, and therefore we can assume that for any V_1, V_2, V_3 in $\mathcal{O}_\kappa$ the objects $V_1 \dot{\otimes} (V_2 \dot{\otimes} V_3)$, $V_1 \dot{\otimes} (V_2 \dot{\otimes} V_3)$ coincide and that the associativity isomorphism a is identity. Therefore we can and will omit brackets in tensor products of objects in $\mathcal{O}_\kappa$ and $\mathcal{A}$.

Since the category $\mathcal{O}_\infty$ is semisimple it follows from Theorem 28.1 that for any $a \in \mathbb{N}^I$ we have a direct sum decomposition $V_a^\infty \dot{\otimes} V_a^\infty = \bigoplus_{b \in \mathbb{N}^I} L_{a,a}^b \otimes V_b^\infty$, where $L_{a,a}^b \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}_\infty}(V_b^\infty, V_a^\infty \dot{\otimes} V_a^\infty)$ is a complex linear space of dimension $C_{a,a,b}$. It is clear that $\dim L_{a,a}^0 = 1$. Let i_a be an isomorphism $i_a: \mathbb{C} \xrightarrow{\sim} L_{a,a}^0$, $i_a^\infty: \mathbf{1} \hookrightarrow V_a^\infty \dot{\otimes} V_a^\infty$ the composition

$$i_a^\infty: \mathbf{1} \xrightarrow{\sim} \mathbb{C} \otimes \mathbf{1} \xrightarrow{i_a \otimes 1} L_{a,a}^0 \otimes \mathbf{1} = L_{a,a}^0 \otimes V_0^\infty \hookrightarrow V_a^\infty \dot{\otimes} V_a^\infty,$$

and $p_a^\infty: V_a^\infty \dot{\otimes} V_a^\infty \rightarrow \mathbf{1}$ the composition $p_a^\infty: V_a^\infty \dot{\otimes} V_a^\infty \xrightarrow{\pi} L_{a,a}^0 \otimes V_0^\infty \xrightarrow{i_a^{-1} \otimes 1} \mathbb{C} \otimes \mathbf{1} = \mathbf{1}$, where $p \in \text{End}_{\mathcal{O}_\infty}(V_a \dot{\otimes} V_a)$ is the projection on $L_{a,a}^0 \otimes V_0^\infty$. Let $\varphi_a \in \text{End } V_a^\infty$ be the composition

$$\begin{aligned} \varphi_a: V_a^\infty &= \mathbf{1} \dot{\otimes} V_a^\infty \xrightarrow{i_a^\infty \otimes 1} V_a^\infty \dot{\otimes} V_a^\infty \dot{\otimes} V_a^\infty \\ &\xrightarrow{1 \otimes p^\infty} V_a^\infty \otimes V_a^\infty \otimes V_a^\infty \xrightarrow{1 \otimes \pi_a} V_a^\infty \otimes \mathbf{1} = V_a^\infty. \end{aligned}$$

It is clear that φ_a does not depend on a choice of an isomorphism i_a . Since V_a^∞ is irreducible we have $\varphi_a = \delta_a \text{Id}$, $\delta_a \in R_\infty$. To describe the formula for δ_a we introduce the following notations.

Let G be the simply connected semisimple algebraic group over $\mathbb{C}$ with $\mathfrak{g}$ as its Lie algebra and $Z \subset G$ the center of G . For any $a \in \mathbb{N}^I$ we denote by $\rho_a: G \rightarrow \text{Aut}(\mathcal{V}_a)$ the unique representation of G compatible with the structure of $\mathfrak{g}$ -module on $\mathcal{V}_a$. Then there exists a character $\theta_a: Z \rightarrow \mathbb{C}^*$ of Z such that $\rho_a(z) = \theta_a(z) \text{Id}$.

Let $\rho = 1 \in \mathbb{N}^I$ be the sum of fundamental weights. Then 2ρ is a sum of simple roots $2\rho = \sum_{i \in I} c_i \alpha_i$, $c_i \in \mathbb{N}$. Let $\lambda_{2\rho}: \mathbb{C}^* \rightarrow G$ be the coroot corresponding to 2ρ , $\lambda_{2\rho} = \prod_{i \in I} \lambda_i^{c_i}$, where $\lambda_i: \mathbb{C}^* \rightarrow G$ are simple coroots.

Let $v = e^{-\pi\sqrt{-1}/\kappa}$, and let $\tau = \lambda_{2\rho}(v)$ be an R_∞ -point of G .

Lemma 31.4. $\delta_a = (\text{tr} \rho_a(\tau))^{-1} \cdot v^{(a, a+2)}$.

Proof. Follows from [D2, Proposition 5.1].

Let $d_a \stackrel{\text{def}}{=} \delta_a^{-1}$. It is clear that d_a lies in R and we can consider it as a function on S . For any $\kappa \in S$ we denote by $n_a(\kappa)$ the order of zero of d_a at κ .

As follows from Theorem 28.1 and Lemma 29.10 there exists a unique direct summand W_a^κ of $V_a^\kappa \otimes V_a^\kappa$ such that W_a^κ has a quotient isomorphic to $\mathbf{1}$. We denote by $n'_a(\kappa)$ the smallest number r such that there exists a filtration

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{r+1} = W_a^\kappa$$

such that the quotients W_i/W_{i-1} are isomorphic to direct sums of Weyl modules for all i , $1 \leq i \leq r+1$.

Proposition 31.3. *Assume that $a \in \mathbf{N}^I$ such that V_a^κ is irreducible. Then*

- (a) $n_a(\kappa) \leq n'_a(\kappa)$,
- (b) if $n_a(\kappa) = n'_a(\kappa)$ then V_a^κ is rigid.

Proof. Let $\hat{\kappa}: R \rightarrow R_\kappa$ be the completion as in §29, $V_a^\kappa \stackrel{\text{def}}{=} V_a \otimes_R R_\kappa$, $M^\kappa = V_a^\kappa \otimes V_a^\kappa$, and let $\hat{s}_a^\kappa: M^\kappa \rightarrow \mathbf{1}$ be the morphism as in the end of §29. As follows from the corollary to Proposition 29.4 there exists an imbedding $\tilde{i}: \mathbf{1}^\kappa \hookrightarrow M$ such that $\hat{s}_a^\kappa \circ \tilde{i} = (x - \kappa)^{n'_a(\kappa)} \text{Id}_1$. As follows from the remark in the end of §29, $\hat{s}_a^\kappa$ defines an isomorphism $\hat{s}_a^\kappa: V_a^\kappa \xrightarrow{\sim} D(V_a^\kappa)$. So we see that there exists an imbedding $\hat{i}: \mathbf{1} \rightarrow V_a^\kappa \otimes D(V_a^\kappa)$ such that

$$e_{V_a} \circ P^\kappa \circ \hat{i} = (x - \kappa)^{n'_a} \text{Id}_1.$$

If $n'_a(\kappa) \leq n_a(\kappa)$ then there exists a morphism $i_{V_a}: \mathbf{1} \rightarrow V_a^\kappa \otimes D(V_a^\kappa)$ such that

$$e_{V_a} \circ P^\kappa \circ i_{V_a} = d_a(\kappa) \text{Id}_1.$$

Then it follows from the definition of $d_a(\kappa)$ that the morphisms $i_{V_a}: \mathbf{1} \rightarrow V_a^\kappa \otimes D(V_a^\kappa)$ and $e_{V_a}: D(V_a^\kappa) \otimes V_a^\kappa \rightarrow \mathbf{1}$ satisfy the conditions of Definition A.5. Therefore V_a is a rigid object. It is easy to see that the strict inequality $n'_a(\kappa) < n_a(\kappa)$ is impossible. Proposition 31.3 is proved.

Let $P \subset \mathbf{Z}^I$ be the root lattice, and $\tilde{W} = W \rtimes P$ the affine Weyl group. We extend the action $(w, a) \rightarrow w \cdot a$ of W on $\mathbf{Z}^I$ to the action of $\tilde{W}$, where P acts by shifts.

For any $\kappa \in S$ we denote by $W_\kappa \subset \tilde{W}$ the subgroup of the affine Weyl group $\tilde{W}$ which is equal to W if κ is irrational and is equal to $W \rtimes pP$ if $\kappa = -p/q$, $p, q \in \mathbf{N}$, $(p, q) = 1$. We denote by $\Omega_\kappa \subset \mathbf{N}^I$ the intersection of the W_κ -orbit $W_\kappa \cdot 0$ of 0 with $\mathbf{N}^I$.

As follows from Theorem 2 in [KK] the module W_a^κ has a filtration with subquotients isomorphic to Weyl modules V_b^κ with $b \in \Omega_\kappa$.

Lemma 31.5. *Let $a \in \mathbf{N}^I$, $\kappa \in S$ be such that V_a^κ is irreducible and for any $b \in \Omega_\kappa - \{0\}$ we have $C_{a, a, b} = 0$. Then V_a^κ is rigid.*

Proof. In this case, $\mathbf{W}_a^\kappa = \mathbf{1}$ and $n'_a(\kappa) = 0$. So the conditions of Proposition 31.3 are satisfied.

Corollary 1. *If κ is irrational then the category $\mathcal{O}_\kappa$ is rigid.*

From now on we assume that κ is rational, $\kappa = -p/q$, $(p, q) = 1$.

Corollary 2. (a) *If $\mathfrak{g} = E_6$, $a \in \mathbf{N}^I$ is such that $\dim \mathcal{V}_a = 27$ and $p > 13$, then $\mathbf{V}_a^\kappa$ is rigid.*

(b) *If $\mathfrak{g} = E_7$, $a \in \mathbf{N}^I$ is such that $\dim \mathcal{V}_a = 56$ and $p > 19$, then $\mathbf{V}_a^\kappa$ is rigid.*

(c) *If $\mathfrak{g} = E_8$, $a \in \mathbf{N}^I$ is such that $\dim \mathcal{V}_a = 248$ and $p > 25$, then $\mathbf{V}_a^\kappa$ is rigid.*

Proof. It is clear that in the cases (a) and (b) there is no $P \in \mathbf{N}^I$ smaller than a and such that $\chi_a = \chi_b$. Therefore $\mathbf{V}_a^\kappa$ is irreducible for all $\kappa \in S$. It follows from Lemma 27.7 that in the case (c) the module $\mathbf{V}_a^\kappa$ is also irreducible for all $\kappa \in S$. It is not difficult to check that in all three cases we have $C_{a,a,b} = 0$ for all $b \in \Omega_\kappa - \{0\}$.

Lemma 31.6. *Let $\mathfrak{g} = \mathcal{S}\ell_n$, $a \in \mathbf{N}^I$ be such that $\dim \mathcal{V}_a = n$. Then $\mathbf{V}_a^\kappa$ is rigid for all $\kappa \in S$.*

Proof. It is easy to check that $\mathbf{V}_a^\kappa$ is irreducible. To show that $\mathbf{V}_a^\kappa$ satisfies the conditions of Proposition 31.4 we observe that $\mathcal{V}_a \otimes \mathcal{V}_a = \mathbf{C} \oplus \mathcal{V}_b$, where $\mathcal{V}_b$ is the adjoint representation of $\mathfrak{g}$, and therefore there exists an exact sequence

$$0 \rightarrow \mathbf{V}_b^\kappa \rightarrow \mathbf{V}_a^\kappa \otimes \mathbf{V}_a^\kappa \rightarrow \mathbf{1} \rightarrow 0.$$

So $n'_a(\kappa) \leq 1$. On the other hand, it follows from Lemma 31.4 that

$$d_a(\kappa) = \frac{v^n - v^{-n}}{v - v^{-1}}.$$

To prove the lemma we consider separately two cases: (a) $n \not\equiv 0 \pmod{p}$ and (b) $n \equiv 0 \pmod{p}$.

(a) It is easy to see that $e^{2\pi\sqrt{-1}L_0}|\mathbf{V}_b^\kappa = v^{2n} \text{Id}$. Since $n \not\equiv 0 \pmod{p}$ we see that $\mathbf{V}_a^\kappa \otimes \mathbf{V}_a^\kappa$ is isomorphic to a direct sum of $\mathbf{1}$ and $\mathbf{V}_b^\kappa$. Therefore $n'_a(\kappa) = 0$ and Lemma 31.6 follows from Proposition 31.3.

(b) In this case $n_a(\kappa) = 1 \geq n'_a(\kappa)$ and Lemma 31.6 also follows from Proposition 31.3.

Lemma 31.7. *Let $\mathfrak{g} = D_n$, $n > 3$, and let $a \in \mathbf{N}^I$ be such that $\dim \mathcal{V}_a = 2n$. Then $\mathbf{V}_a^\kappa$ is rigid for all $\kappa \in S$ if n is even and $\mathbf{V}_a^\kappa$ is rigid if $\kappa \neq -\frac{2}{q}$ for n odd.*

Proof. We consider only the case when n is even. The case when n is odd is completely analogous. It follows from Lemma 31.4 that

$$d_a(\kappa) = \left(\frac{v^n - v^{-n}}{v - v^{-1}} \right) (v^{n-1} + v^{1-n}).$$

It is easy to check that $\mathbf{V}_a^\kappa$ is irreducible for all $\kappa \in S$. To show that $\mathbf{V}_a^\kappa$ satisfies the conditions of Proposition 31.3 we observe that $\mathcal{V}_a \otimes \mathcal{V}_a = \mathbf{C} \oplus \mathcal{V}_{2a} \oplus \mathcal{V}_b$, where $\mathcal{V}_b$ is the adjoint representation of $\mathfrak{g}$. Therefore there exists a unique filtration

$$(*) \quad 0 = V_0 \subset V_1 \subset V_2 \subset V_3 = \mathbf{V}_a^\kappa \otimes \mathbf{V}_a^\kappa$$

such that $\bar{V}_3 = \mathbf{1}$, $\bar{V}_2 \simeq \mathbf{V}_b^\kappa$, $\bar{V}_1 = \mathbf{V}_{2a}^\kappa$, where $\bar{V}_i \stackrel{\text{def}}{=} V_i/V_{i-1}$, $1 \leq i \leq 3$.

Since $\bar{a} = a$ the commutativity constraint defines an automorphism P of $\mathbf{V}_a^\kappa \otimes \mathbf{V}_a^\kappa$. It is clear that the filtration $(*)$ is P -invariant and the induced automorphisms $\bar{P}_i$ of $\bar{V}_i$, $1 \leq i \leq 3$, are given by the rule $\bar{P}_1 = \text{Id}$, $\bar{P}_2 = -v^{2(n-1)} \text{Id}$, $\bar{P}_3 = v^{2n} \text{Id}$.

To prove the lemma we consider separately the following three cases:

(a) $v^{2(n-1)} \neq -1$, (b) $v^{2(n-1)} = -1$, but $v^2 \neq -1$, and (c) $v^{2(n-1)} = v^2 = -1$.

(a) In this case we have $n'_a(\kappa) \leq 1$. If $v^{2n} \neq 1$, then $W_a^\kappa = \mathbf{1}$ and $n'_a(\kappa) = 0$. If, on the other hand, $v^{2n} = 1$ then $n_a(\kappa) = 1$. Lemma 31.7 follows from Proposition 31.3.

(b) In this case $v^{2n} \neq 1$ and W_a^κ does not have $\mathbf{V}_{2a}^\kappa$ as a subquotient. So $n'_a(\kappa) \leq 1$. On the other hand, in this case $n_a(\kappa) = 1$. Case (b) is proved.

(c) In this case $n_a(\kappa) = 2 \geq n'_a(\kappa)$. Lemma 31.7 is proved.

32. PROOF OF THE RIGIDITY OF $\mathcal{O}_\kappa$ FOR "MOST" RATIONAL VALUES OF κ

In the last section we showed that in the case when $\mathfrak{g} = A_n, D_{2n}$ and κ is arbitrary $\mathfrak{g} = D_{2n+1}$, $\kappa \neq -\frac{2}{q}$ and in the case $\mathfrak{g} = E_6, E_7, E_8$ and κ is not of the form $\kappa = -p/q$ where p is "small" there exists $a \in \mathbf{N}^I - \{0\}$ such that the object $\mathbf{V}_a^\kappa$ is rigid and irreducible. In this section we show that the existence of such an object implies the rigidity of the category $\mathcal{O}_\kappa$.

The proof of the rigidity consists in a number of steps when we show that more and more objects in $\mathcal{O}_\kappa$ are rigid. We will constantly use the results of the Appendix. If κ is irrational, the rigidity of $\mathcal{O}_\kappa$ follows from Corollary 1 to Lemma 31.5. So from now on we assume that $\mathfrak{g}$ is a simple Lie algebra, and κ is a rational number $\kappa = -p/q$ (p, q are relatively prime natural numbers) such that there exists a nonzero weight $c \in \mathbf{N}^I - \{0\}$ such that $\mathbf{V}_c^\kappa$ is a rigid irreducible object in $\mathcal{O}_\kappa$.

Proposition 32.1. *Let T^κ be a tilting module in $\mathcal{O}_\kappa$, and let $a \in \mathbf{N}^I$ be such that $\mathbf{V}_a^\kappa$ is irreducible and both $\mathbf{V}_a^\kappa$ and $\mathbf{V}_a^\kappa \otimes T^\kappa$ are rigid. Then T^κ is rigid.*

Proof. By the definition there exists a morphism

$$\bar{i}: \mathbf{1} \rightarrow (\mathbf{V}_a^\kappa \otimes T^\kappa) \otimes (\mathbf{V}_a^\kappa \otimes T^\kappa)^*$$

which satisfies the conditions of Definition A.5. Let ${}_{\mathbf{V}_a^\kappa, T^\kappa} \beta: (\mathbf{V}_a^\kappa \otimes T^\kappa)^* \rightarrow T_\kappa^* \otimes \mathbf{V}_a^\kappa$ be the isomorphism as in Proposition A.1 and

$$i \stackrel{\text{def}}{=} (1 \otimes {}_{\mathbf{V}_a^\kappa, T^\kappa} \beta) \circ \bar{i}: \mathbf{1} \rightarrow \mathbf{V}_a^\kappa \otimes T^\kappa \otimes T_\kappa^* \otimes (\mathbf{V}_a^\kappa)^*.$$

Consider the morphism $j: (\mathbf{V}_a^\kappa)^* \dot{\otimes} \mathbf{V}_a^\kappa \rightarrow T^\kappa \dot{\otimes} T^{\kappa*}$ defined as the composition

$$j: (\mathbf{V}_a^\kappa)^* \dot{\otimes} \mathbf{V}_a^\kappa = (\mathbf{V}_a^\kappa)^* \dot{\otimes} \mathbf{1} \dot{\otimes} \mathbf{V}_a^\kappa \xrightarrow{1 \otimes i \otimes 1} (\mathbf{V}_a^\kappa)^* \dot{\otimes} \mathbf{V}_a^\kappa \dot{\otimes} T^\kappa \dot{\otimes} T^{\kappa*} \dot{\otimes} (\mathbf{V}_a^\kappa)^* \dot{\otimes} \mathbf{V}_a^\kappa \\ \xrightarrow{e_{\mathbf{V}_a^\kappa} \otimes 1 \otimes e_{\mathbf{V}_a^\kappa}} T^\kappa \otimes T^{\kappa*}.$$

Let $L \stackrel{\text{def}}{=} \ker e_{\mathbf{V}_a^\kappa} \subset (\mathbf{V}_a^\kappa)^* \otimes \mathbf{V}_a^\kappa$.

Lemma 32.1. $L \subset \ker j$.

Proof. By Proposition 29.2 there exists a lifting T of T^κ to the category $\mathcal{O}_{\hat{\kappa}}$. It is clear that the morphism i extends to a morphism $i^\vee: \mathbf{1} \rightarrow \mathbf{V}_a^{\hat{\kappa}} \dot{\otimes} T \dot{\otimes} T^* \dot{\otimes} (\mathbf{V}_a^{\hat{\kappa}})^*$ such that the composition

$$(1 \otimes \beta_{\mathbf{V}_a^{\hat{\kappa}}, T}) \circ i^\vee: \mathbf{1} \rightarrow (\mathbf{V}_a^{\hat{\kappa}} \dot{\otimes} T) \dot{\otimes} (\mathbf{V}_a^{\hat{\kappa}} \dot{\otimes} T)^*$$

satisfies the conditions of Definition A.5 and we can define a morphism

$$j^\vee: (\mathbf{V}_a^{\hat{\kappa}})^* \otimes \mathbf{V}_a^{\hat{\kappa}} \rightarrow T \otimes T^*.$$

Let $\hat{L} \stackrel{\text{def}}{=} \ker e_{\mathbf{V}_a^{\hat{\kappa}}} \subset (\mathbf{V}_a^{\hat{\kappa}})^* \otimes \mathbf{V}_a^{\hat{\kappa}}$. It is sufficient to check that $\hat{L} \subset \ker j^\vee$. Since $\mathbf{V}_a^{\hat{\kappa}}$ is irreducible, $(\mathbf{V}_a^{\hat{\kappa}})^*$ lies in $\mathcal{A}_{\hat{\kappa}}$ and therefore the $\mathcal{O}_{\hat{\kappa}}$ -module $(\mathbf{V}_a^{\hat{\kappa}})^* \otimes \mathbf{V}_a^{\hat{\kappa}}$ is torsion free. Therefore it is sufficient to check that $\tilde{L} \subset \ker \tilde{j}$, where $\tilde{L} = L \otimes_{R_\kappa} F_\kappa$, $\tilde{j} = j^\vee \otimes_{R_\kappa} F_\kappa$.

By the corollary to Lemma 31.5 the category $\mathcal{O}_{F_\kappa}$ is rigid and the inclusion $\tilde{L} \subset \ker \tilde{j}$ follows from Lemma A.11. Lemma 32.1 is proved.

Now we can prove Proposition 32.1. Since L lies in $\ker j$ we can consider j as a morphism i_{T^κ} from $\mathbf{1} = (\mathbf{V}_a^\kappa)^* \otimes \mathbf{V}_a^\kappa / L$ to $T^\kappa \otimes T^{\kappa*}$. It is easy to check that i_{T^κ} satisfies the conditions of Definition A.5. Proposition 32.1 is proved.

Let $\theta = \theta_c$ be the character of the center $Z \subset G$ defined in §31, $Z_\theta = \ker \theta \subset Z$, and $\mathbf{N}^I(\theta) = \{a \in \mathbf{N}^I \mid \theta_a|_{Z_\theta} \equiv 1\}$. We denote by $\mathcal{O}_\kappa^\theta$ the full subcategory of $\mathcal{O}_\kappa$ of objects whose irreducible subquotients are isomorphic to L_a for $a \in \mathbf{N}^I(\theta)$. As follows from Theorem 28.1 and Lemma 27.7(a) the subcategory $\mathcal{O}_\kappa^\theta$ is closed under the tensor product $\dot{\otimes}$.

Lemma 32.2. For any $a \in \mathbf{N}^I(\theta)$ there exists $N \in \mathbf{N}$ such that $\mathbf{V}_a^\kappa$ is isomorphic to a subquotient of $(\mathbf{V}_c^\kappa)^{\otimes N}$.

Proof. Follows from Theorem 28.1 and Proposition 28.1.

To shorten the notations we will write $\mathbf{V}^N$ instead of $(\mathbf{V}_c^\kappa)^{\otimes N}$.

Denote by $\mathbf{N}_0 \subset \mathbf{N}^I$ the subset of weights a of the form $a = pb + (p-1)\rho$, $b \in \mathbf{N}^I$, where $\rho = (1, \dots, 1)$ is the sum of fundamental weights. As follows from the corollary to Proposition 27.4, all the objects $\mathbf{V}_a^\kappa$ of $\mathcal{O}_\kappa$, $a \in \mathbf{N}_0$, are simultaneously projective and injective.

Lemma 32.3. *The objects V_a^κ , $a \in N_0 \cap N^I(\theta)$, are rigid.*

Proof. By Lemma 32.2, for any $a \in N_0$ there exists $N \subset N$ such that V_a^κ is isomorphic to a subquotient of V^N . Since V_a^κ is simultaneously projective and injective, it is isomorphic to a direct summand of V^N . On the other hand, it follows from Lemma A.3 that the object V^N is rigid. Therefore V_a^κ is also rigid. Lemma 32.2 is proved.

As follows from Corollary 2 to Lemma A.4, the products $V^N \otimes V_a^\kappa$ are simultaneously projective and injective for all $N \in N$, $a \in N_0(\theta)$. We call such products “special projective modules”.

Proposition 32.2. *For any $b \in N^I(\theta)$ there exists a special projective module P^κ such that $\text{Hom}_{\mathcal{O}_\kappa}(P^\kappa, L_b^\kappa) \neq \{0\}$, where L_b^κ is the irreducible quotient of V_b^κ .*

Proof. We start the proof with the following result. Let $G(\theta)$ be the quotient of G by Z_θ . We denote by D the ring of regular functions on the algebraic group $G(\theta)$, and for any $a \in N^I(\theta)$ we denote by χ_a the character of the representation $\rho_a: G \rightarrow \text{Aut}(\mathcal{Z})_a$. As is well known, the functions χ_a , $a \in N^I(\theta)$, form a basis of D . We define a scalar product on $(\ , \)$ on D in such a way that $(\chi_a, \chi_b) = \delta_b^a$, $a, b \in N^I(\theta)$.

Let $\varphi^-: D \rightarrow D$ be the involution such that $\overline{\chi}_a = \chi_{\bar{a}}$. It is clear that for any $d, d_1, d_2 \in D$ we have

$$(d_1 \bar{d}, d_2) = (d_1, dd_2).$$

Let $\tilde{D}_0$ be the subring of D generated by χ_c and χ_a for $a \in N_0(\theta)$, and denote by $D_0 \subset \tilde{D}_0$ its ideal generated by χ_a , $a \in N_0(\theta)$.

Lemma 32.4. *There exists a nonzero function ψ in D such that $D_0 \supset \psi D$.*

Proof. Let $T \subset G$ be a maximal torus, $\overline{T} = T/W$ the quotient of T by the action of the Weyl group, $\pi: T \rightarrow \overline{T}$ the map given by $\pi(t) = t^p$, and $\bar{\pi}: \overline{T} \rightarrow \overline{T}$ the induced map of the quotient spaces. It is clear that $\bar{\pi}$ is a finite morphism. We will identify D with the ring of regular functions on $\overline{T}$. Let Δ be the discriminant function on T and $\varphi \stackrel{\text{def}}{=} \pi^*(\Delta)/\Delta$. It is clear that φ is a regular W -invariant function on T which we will consider as an element in D . As follows from Weyl’s formula for characters of G , the functions χ_a , $a \in N_0$, span the $(\pi^*(D))$ -submodule of D generated by φ . It is easy to see that the function χ_c separates the points of the generic fiber of $\bar{\pi}$. Therefore we find a finite subset $J \subset N$ such that the restriction of χ_c^j , $j \in J$, on the generic fiber $\bar{\pi}^{-1}(\bar{t})$ of $\bar{\pi}$ gives a basis in the space of functions on $\bar{\pi}^{-1}(\bar{t})$. Let $T \subset D$ be the $\pi^*(D)$ -submodule generated by the functions $\chi_c^j \cdot \varphi$, $j > 0$.

It is clear that $T \subset D_0$. Since D is a finitely generated $\pi^*(D)$ -module, it follows from construction of T that there exists a nonzero function $\psi \in D$ such that $D_0 \supset \psi D$. The lemma is proved.

Corollary. *For any nonzero element $d \in D$ there exist $N \in N - \{0\}$ and an element $a \in N_0(\theta)$ such that $(d, \chi_c^N \chi_a) \neq 0$.*

Proof. Since the scalar product $(\ , \)$ on D is nondegenerate, there exists $\tilde{d} \in D$ such that $(d\bar{\psi}, \tilde{d}) \neq 0$. But then $(d, \psi\tilde{d}) \neq 0$. As follows from the lemma, $d_0 \stackrel{\text{def}}{=} \psi\tilde{d}$ lies in D_0 . Since T is spanned by the products $\chi_c^N \chi_a$, $N \in \mathbb{N}$, $a \in \mathbb{N}_0$, the corollary is proved.

Proof of Proposition 32.2 (continued). Consider the isomorphism $[\mathcal{O}_\kappa^\theta] \xrightarrow{\varphi} D$ such that $\varphi([\mathbf{V}_a^\kappa]) = \chi_a$. As follows from the corollary to Proposition 27.1 and Proposition 3.9 for any projective module P in $\mathcal{O}_\kappa$ and $a \in \mathbb{N}^I(\theta)$ we have $\dim \text{Hom}_{\mathcal{O}_\kappa}(P, D(\mathbf{V}_a^\kappa)) = (\varphi[P], \chi_a)$. Since the dimension of the space $\text{Hom}_{\mathcal{O}_\kappa}(P, T)$ depends only on the image $[M]$ of M in $[\mathcal{O}_\kappa]$ we see that for any projective module P in $\mathcal{O}_\kappa$ and any object M we have

$$\dim \text{Hom}_{\mathcal{O}_\kappa}(P, M) = (\varphi([P]), \varphi[M]).$$

For any $b \in \mathbb{N}^I(\theta)$ we define $d_b = \varphi([L_b]) \in D$. By the corollary to Lemma 32.2 there exists $N \subset \mathbb{N} - \{0\}$ and $a \in \mathbb{N}_0(\theta)$ such that $(\chi_c^N \chi_a, d_b) \neq 0$. Therefore $\text{Hom}_{\mathcal{O}_\kappa}(P, \mathbf{L}_b^\kappa) \neq \{0\}$ where P is the special projective module $P^\kappa = \mathbf{V}^N \dot{\otimes} \mathbf{V}_a^\kappa$. Proposition 32.2 is proved.

Corollary. (a) For any $b \in \mathbb{N}^I(\theta)$ there exists an indecomposable direct summand P_b^κ of a special projective module such that $\dim \text{Hom}_{\mathcal{O}_\kappa}(P_b^\kappa, L_a^\kappa) = \delta_b^a$ for all $a \in \mathbb{N}^I$ and P_b^κ is uniquely determined up to an isomorphism.

(b) The category $\mathcal{O}_\kappa(\theta)$ has enough projective objects and all those objects are also injective and rigid.

(c) Any projective object in $\mathcal{O}_\kappa(\theta)$ is isomorphic to a direct sum of P_b^κ , $b \in \mathbb{N}^I$.

(d) $\mathcal{O}_\kappa$ is a Frobenius category (see Definition A.9).

Proposition 32.3. Any tilting module T^κ in $\mathcal{O}_\kappa^\theta$ is rigid.

We start the proof with the following result.

Lemma 32.5. For any tilting module T^κ in $\mathcal{O}_\kappa$ and a projective module P^κ in $\mathcal{O}_\kappa$ the product $T^\kappa \dot{\otimes} P^\kappa$ is tilting.

Proof. As follows from Theorem 29.1 the product $T^\kappa \dot{\otimes} P^\kappa$ lies in $\mathcal{A}_\kappa$. So it is sufficient to check that $D(T^\kappa \dot{\otimes} P^\kappa)$ is also in $\mathcal{A}_\kappa$. Since P^κ is rigid, it follows from Proposition A.1 that the morphism $\alpha_{T^\kappa, P^\kappa}: D(P^\kappa) \dot{\otimes} D(T^\kappa) \rightarrow D(T^\kappa \dot{\otimes} P^\kappa)$ is an isomorphism. As follows from Proposition 32.1, P^κ is injective. So $D(P^\kappa)$ is projective and therefore lies in $\mathcal{A}_\kappa$. By Theorem 29.1 $D(P^\kappa) \dot{\otimes} D(T^\kappa)$ is also in $\mathcal{A}_\kappa$. Lemma 32.5 is proved.

For any $b \in \mathbb{N}^I(\theta)$ the object P_b^κ is projective and injective. Therefore P_b^κ is an indecomposable tilting module and therefore there exists $a = a(b) \in \mathbb{N}^I(\theta)$ such that $P_b^\kappa = T_a^\kappa$. It is clear that a is the maximal element in $\text{supp } P_b^\kappa$ (see the definition of supp in the beginning of §27). We denote by $\tilde{\mathbb{N}} \subset \mathbb{N}^I(\theta)$ the

set of all $a \in \mathbf{N}^I(\theta)$ such that T_a^κ is projective and by $\bar{\mathbf{N}}$ the image of $\tilde{\mathbf{N}}$ in $\mathbf{N}^I(\theta)/p\mathbf{N}^I(\theta)$.

Lemma 32.6. (a) For any $a, b \in \tilde{\mathbf{N}}$ we have $a + b \in \tilde{\mathbf{N}}$.

(b) There exists a finite set $Q \subset \mathbf{Z}^I$ such that for any $a \in \tilde{\mathbf{N}}$, such that $a - 2pp \in \mathbf{N}^I$, and any $b, c \in \text{supp } T_a^\kappa$ we have $b - c \in Q$.

(c) $\bar{\mathbf{N}} = \mathbf{N}^I(\theta)/p\mathbf{N}^I(\theta)$.

(d) There exists a number $m \in \mathbf{N}$ such that $\mathbf{N}^I(\theta) + 2mp \subset \tilde{\mathbf{N}}$.

Proof. (a) For any $a, b \in \tilde{\mathbf{N}}$ the objects $T_a^\kappa, T_b^\kappa, D(T_a^\kappa), D(T_b^\kappa)$ are rigid and therefore by Corollary 2 to Lemma A.4 the object $T_a^\kappa \otimes T_b^\kappa$ is projective. Since by Theorem 28.1, $a + b$ is the maximal element in $\text{supp}(T_a^\kappa \dot{\otimes} T_b^\kappa)$ the object T_{a+b}^κ is a direct summand of $T_a^\kappa \dot{\otimes} T_b^\kappa$, so the object T_{a+b}^κ is also projective.

(b) As follows from the corollary to Proposition 28.3, we have $\mathbf{N}_0(\theta) \subset \tilde{\mathbf{N}}$. It follows from part (a) that there exists a finite set $\Sigma \subset \tilde{\mathbf{N}}$ such that for any $a \in \tilde{\mathbf{N}}$ such that $a - 2pp \in \mathbf{N}^I(\theta)$ there exist $\sigma \in \Sigma$ and $a_0 \in \mathbf{N}_0$ such that $a = \sigma + a_0$. Let $P = \bigcup_{\sigma \in \Sigma} \text{supp } T_a^\kappa$, and let $Q' \subset \mathbf{Z}^I$ be the set of weights of representations $\mathbf{V}_a, a \in P$. Then the set $\{\lambda - \mu | \lambda, \mu \in Q'\}$ satisfies the conditions of (b). Really for any $a \in \tilde{\mathbf{N}}$ such that $a - 2pp \in \mathbf{N}^I(\theta)$ we choose $\sigma \in \Sigma$ and $a_0 \in \mathbf{N}_0(\theta)$ such that $a = \sigma + a_0$. Then T_a^κ is a direct summand of $T_\sigma^\kappa \dot{\otimes} T_{a_0}^\kappa = T_\sigma^\kappa \dot{\otimes} \mathbf{V}_{a_0}^\kappa$. Therefore $\text{supp } T_a^\kappa \subset \text{supp}(T_\sigma^\kappa \dot{\otimes} \mathbf{V}_{a_0}^\kappa)$. As follows from Theorem 28.1 we have $\text{supp}(T_\sigma^\kappa \dot{\otimes} \mathbf{V}_{a_0}^\kappa) \subset a_0 + Q'$. Part (b) is proved.

(c) As follows immediately from (a) the set $\bar{\mathbf{N}}$ is a subgroup. For any $m \in \mathbf{Z}^+$ we denote by $\mathbf{N}^I(m) \subset \mathbf{N}^I(\theta)$ the subset consisting of the elements $a = (a(i)), i \in I$, such that $a(i) \leq m$ for all $i \in I$. Let $L(m) \subset [\mathcal{O}_\kappa] \otimes_{\mathbf{Z}} \mathbf{Q}$ be a subspace generated by elements $[T_a^\kappa]$ for $a \in \mathbf{N}^I(m) \cap \tilde{\mathbf{N}}$. Then we have

$$\dim L(m) \leq \frac{\#\bar{\mathbf{N}} \cdot m^{\#(I)}}{\#(\mathbf{N}^I(\theta)/p\mathbf{N}^I(\theta))} \quad \text{for } m \gg 0.$$

On the other hand, it follows from part (b) that there exists a constant m_0 such that for any $b \in \mathbf{N}^I(m)$ such that $b - 2pp \in \mathbf{N}^I$ we have $P_b^\kappa = T_a^\kappa$ for $a \in \mathbf{N}^I(m + m_0)$. Since the elements $[P_b^\kappa]$ in $[\mathcal{O}]_\kappa \otimes \mathbf{Q}$ are linearly independent, we have

$$\dim L(m') \geq \frac{(m - m_0 - p)^{\#I_b}}{\#Z(\theta)}$$

for all $m \in \mathbf{Z}$. Therefore $\#\bar{\mathbf{N}}/\#(\mathbf{N}^I/p\mathbf{N}^I) \geq 1$. Part (c) is proved.

(d) Follows from (a), (c), and the inclusion $\mathbf{N}_0(\theta) \subset \tilde{\mathbf{N}}$.

Corollary. There exists $m_0 \in \mathbf{N}^I$ such that any tilting module T in $\mathcal{O}_\kappa^\theta$ such that $\text{supp}(T) \cap \mathbf{N}^I(m_0) = \emptyset$ is projective.

We can finish now the proof of Proposition 32.3. Let T be a tilting module in $\mathcal{O}_\kappa^\theta$. As follows from Theorem 28.1 and Proposition 28.1, we can find $a \in \mathbf{N}_0$

such that $\text{supp}(T \dot{\otimes} V_a^\kappa) \cap N^I(m_0) = \emptyset$. Therefore it follows from Lemma 32.5 and the corollary to Lemma 32.6 that $T \dot{\otimes} V_a^\kappa$ is projective and, as follows from the corollary to Proposition 32.2, $T \dot{\otimes} V_a^\kappa$ is rigid. Since V_a^κ is also rigid it follows from Proposition 32.1 that T is rigid. Proposition 32.3 is proved.

Proposition 32.4. *The category $\mathcal{O}_\kappa(\theta)$ is rigid.*

Proof. We prove by the induction in t that any object in $\mathcal{O}_\kappa^t$ is rigid. If $t = 0$, then any object in $\mathcal{O}_\kappa^t$ is a multiple of $\mathbf{1}$ and is therefore rigid. Suppose that we know already that any object in $\mathcal{O}_\kappa^{<t}$ is rigid. As follows from Proposition A.2 it is sufficient to show that for any $a \in N^I$ such that $\langle a, a + 2 \rangle = t$ the object L_a is rigid. As follows from Proposition 27.2 there exist submodules $M' \subset M''$ in T_a^κ such that M' and T_a^κ/M'' are in $\mathcal{O}_\kappa^{<t}$ and M''/M' is isomorphic to L_a . Since T_a^κ and T_a^κ/M'' are rigid, it follows from Proposition A.2 and part (d) of the corollary to Proposition 32.2 that M'' is rigid. Since M' is also rigid, it follows from the same proposition that L_a is rigid. Proposition 32.4 is proved.

Corollary. *Any object in $\mathcal{O}_\kappa$ is flat.*

Proof. Follows from Lemma A.11.

Proposition 32.5. *For any objects X, Y in $\mathcal{O}_\kappa$ the product $X \dot{\otimes} Y$ lies in $\mathcal{O}_\kappa$.*

Proof. It is sufficient to show that $X \dot{\otimes} Y$ is torsion free. As in the proof of Theorem 29.1 it is sufficient to show that for all $\lambda \in \mathbb{C}$ we have an equality

$$d(\bar{X} \dot{\otimes} \bar{Y}, \lambda) = d(\hat{X} \dot{\otimes} \hat{Y}, \lambda),$$

where $\bar{X}, \bar{Y}, \hat{X}, \hat{Y}$ are objects in categories $\mathcal{O}_\kappa$ and $\mathcal{O}_{F_\kappa}$ corresponding to X and Y . But this follows from Theorem 28.1 and the corollary to Proposition 32.4. Proposition 32.5 is proved. To formulate corollaries of this proposition we will use notations from Definition A.10.

Corollary 1. *$(\mathcal{O}_\kappa, \otimes, a_\kappa)$ is a monoidal category.*

Corollary 2. *The category $\mathcal{O}_\kappa$ is liftable and any tilting object in $\mathcal{O}_\kappa$ is liftable.*

Theorem 32.1. *Let $\kappa \in S$ be such that there exists $a \in N^I - \{0\}$ such that the object V_a^κ is $\mathcal{O}_\kappa$ is irreducible and rigid. Then the monoidal category $\mathcal{O}_\kappa$ is rigid.*

Proof. It follows from Proposition 32.4 that any object in the subcategory $\mathcal{O}_\kappa(\theta)$ of $\mathcal{O}_\kappa(\theta)$ of $\mathcal{O}_\kappa$ is rigid. Let $a \in N^I$ be any element which is minimal in the set $(a + Z^I(\theta)) \cap N^I$. Then it is clear that V_a^κ is irreducible and tilting. Let $r \in \mathbb{N} - \{0\}$ be an integer such that $r \cdot a$ lies in $N^I(\theta)$. Then $(V_a^\kappa)^{\otimes r}$ lies in $\mathcal{O}_\kappa(\theta)$ and it follows from Proposition 32.4 that $(V_a^\kappa)^{\otimes r}$ is rigid. Therefore it follows from Corollary 2 to Proposition 32.5 and Proposition A.3 that V_a^κ is also rigid. Therefore it follows from Proposition 32.4 that the category $\mathcal{O}_\kappa(\theta_a)$ is rigid. This easily implies the rigidity of $\mathcal{O}_\kappa$. Theorem 32.1 is proved.

33. THE MORPHISMS $t_{a,b}$ AND s_a

In this section we define morphisms $t_{a,b} \in \text{Hom}_{\mathcal{O}}(\mathbf{V}_{a+b}, \mathbf{V}_a \dot{\otimes} \mathbf{V}_b)$ and $s_a \in \text{Hom}(\mathbf{V}_a \dot{\otimes} \mathbf{V}_a, \mathbf{1})$ which we later use to construct a functor from the category $\mathcal{O}_\kappa$ to the category of representations of quantum groups.

Lemma 33.1. (a) For any $a, b \in \mathbf{N}^I$ the R -module $\mathcal{T}_{a,b} \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}}(\mathbf{V}_{a+b}, \mathbf{V}_a \dot{\otimes} \mathbf{V}_b)$ is free of rank 1, and for any $\kappa \in S$ the morphism $\mathcal{T}_{a,b}(\kappa) \longrightarrow \text{Hom}_{\mathcal{O}_\kappa}(\mathbf{V}_{a+b}^\kappa, \mathbf{V}_a^\kappa \dot{\otimes} \mathbf{V}_b^\kappa)$ is an isomorphism.

(b) For any $a \in \mathbf{N}^I$ the R -module

$$\mathcal{S}_a = \text{Hom}_{\mathcal{O}}(\mathbf{V}_a, D(\mathbf{V}_a)) = \text{Hom}_{\mathcal{O}}(\mathbf{V}_a \dot{\otimes} \mathbf{V}_a, \mathbf{1})$$

is free of rank 1 and the natural morphism $\mathcal{S}_a(\kappa) \longrightarrow \text{Hom}_{\mathcal{O}_\kappa}(\mathbf{V}_a^\kappa, D(\mathbf{V}_a^\kappa))$ is an isomorphism for all $\kappa \in S$.

Proof. It follows from Lemma 29.8 that the R -modules $\mathcal{T}_{a,b}$ are free of finite rank and the ranks of $\mathcal{T}_{a,b}$ are equal to dimensions of the linear spaces $\text{Hom}_{\mathcal{O}_\kappa}(\mathbf{V}_{a+b}^\kappa, \mathbf{V}_a^\kappa \dot{\otimes} \mathbf{V}_b^\kappa)$ for generic $\kappa \in \mathbf{C}$. For generic κ the category $\mathcal{O}_\kappa$ is semisimple and it follows from Lemma 28.1 that $\mathbf{V}_a^\kappa \dot{\otimes} \mathbf{V}_b^\kappa$ is isomorphic to the direct sum of $\mathbf{V}_c^\kappa$, $c \in \mathbf{N}^I$, where each $\mathbf{V}_c^\kappa$ appear $C_{a,b,c}$ times. Since $C_{a,b,\overline{a+b}} = 1$ we see that $\mathcal{T}_{a,b}$ is a free R -module of rank 1 for all $a, b \in \mathbf{N}^I$. The last part of (a) follows from Lemma 29.9. The proof of (b) is completely analogous.

Lemma 33.2. For any generator $t_{a,b}$ of $\mathcal{T}_{a,b}$ the quotient $\mathbf{V}_a \dot{\otimes} \mathbf{V}_b / \text{Im}(t_{a,b})$ lies in $\mathcal{A}$.

Proof. It is sufficient to check that the quotient $\mathbf{V}_a^\kappa \dot{\otimes} \mathbf{V}_b^\kappa / \text{Im}(t_{a,b}(\kappa))$ is in $\mathcal{A}_\kappa$ for any $\kappa \in S$. As follows from Proposition 27.1 it is sufficient to show that for any $c \in \mathbf{N}^I$ the induced map

$$\text{Hom}(\mathbf{V}_a^\kappa \dot{\otimes} \mathbf{V}_b^\kappa / \text{Im}(t_{a,b}(\kappa)), D(\mathbf{V}_c^\kappa)) \longrightarrow \text{Hom}(\mathbf{V}_{a+b}^\kappa, D(\mathbf{V}_c^\kappa))$$

is surjective. If $c \neq \bar{a} + \bar{b}$ then the right side is zero. On the other hand, it follows from Lemma 3.5 that $D(\mathbf{V}_c^\kappa)$ is an injective object in $\mathcal{O}_\kappa^t$ for $t = \langle a+b, a+b+2 \rangle$. Since $\mathbf{V}_{a+b}^\kappa, \mathbf{V}_a^\kappa \dot{\otimes} \mathbf{V}_b^\kappa$ are in $\mathcal{O}_\kappa^t$ we see that $t_{a,b}^*(\kappa)$ is surjective. Lemma 33.2 is proved.

Corollary 1. For any generator $\tilde{T}_{a,b}$ of $\mathcal{T}_{a,b}$ and V in $\mathcal{A}$ the morphism

$$t_{a,b} \otimes 1(\kappa): (\mathbf{V}_{a+b} \dot{\otimes} V)(\kappa) \longrightarrow (\mathbf{V}_a \dot{\otimes} \mathbf{V}_b \dot{\otimes} V)(\kappa)$$

is injective for all $\kappa \in S$.

Proof. Follows from Theorem 28.1.

Corollary 2. (a) For any $a, b \in \mathbf{N}^I$, $\kappa \in S$ the cokernel of $t_{a,b}^\kappa$ lies in $\mathcal{O}_\kappa^{<t}$.

(b) For any $a \in \mathbf{N}^I$ and any generator s'_a of S_a the kernel and cokernel of $s'_a(\kappa)$ are in $\mathcal{O}_\kappa^{<a}$.

(c) Let $\bar{t}_{a,b}(\kappa), \bar{s}'_a(\kappa)$ be the induced morphisms in quotient categories $\bar{\mathcal{O}}_\kappa^{a+b}$ and $\bar{\mathcal{O}}_\kappa^a$. Then $\bar{t}_{a,b}(\kappa), \bar{s}'_a(\kappa)$ are isomorphisms.

We denote by $\mathcal{D}_R$ the category of finitely generated R -modules. As before (see the beginning of §26) we denote by $\mathcal{G}$ the functor from $\mathcal{O}$ to $\mathcal{D}_R$ given by $\mathcal{G}(V) = V/G_1 V$. For any $a \in \mathbb{N}^I$ the imbedding $\mathcal{V}_a \otimes R \hookrightarrow \mathbf{V}_a$ induces an isomorphism $\mathcal{V}_a \otimes R \xrightarrow{\sim} \mathcal{G}(\mathbf{V}_a)$. We will identify $\mathcal{G}(\mathbf{V}_a)$ with $\mathcal{V}_a \otimes R$ using this isomorphism.

In the same way as in the remark to Lemma 33.4 we define for any $a, b \in \mathbb{N}^I$ an isomorphism $\beta: \mathcal{G}(\mathbf{V}_a) \otimes_R \mathcal{G}(\mathbf{V}_b) \xrightarrow{\sim} \mathcal{G}(\mathbf{V}_a \dot{\otimes} \mathbf{V}_b)$ and will identify $\mathcal{G}(\mathbf{V}_a \dot{\otimes} \mathbf{V}_b)$ with $\mathcal{G}(\mathbf{V}_a) \otimes_R \mathcal{G}(\mathbf{V}_b) = (\mathcal{V}_a \otimes_{\mathbb{C}} \mathcal{V}_b) \otimes R$ using the isomorphism β .

As is well known, there exists an imbedding $\mathcal{V}_{a+b} \xrightarrow{\varphi} \mathcal{V}_a \otimes \mathcal{V}_b$ such that $\varphi(y_{a+b}) = y_a \otimes y_b$ (see §1.3) and $\mathcal{V}_a \otimes \mathcal{V}_b$ is a direct sum of $\varphi(\mathcal{V}_{a+b})$ and submodules isomorphic to a direct sum of $\mathcal{V}_c$ for $c < a+b$. Let $\check{T}_{a,b}$ be a generator of $\mathcal{T}_{a,b}$. It is clear that the image of $\mathcal{G}(\check{T}_{a,b})$ in $\mathcal{G}(\mathbf{V}_a \dot{\otimes} \mathbf{V}_b) = \mathcal{V}_a \otimes \mathcal{V}_b$ lies in $\varphi(\mathcal{V}_{a+b}) \otimes R$. Therefore we can consider an endomorphism $(\varphi^{-1} \otimes 1) \circ t_{a,b}$ of $\mathcal{V}_{a+b} \otimes R$.

Lemma 33.3. $(\varphi^{-1} \otimes 1) \circ t_{a,b}$ is an automorphism of $\mathcal{V}_{a+b} \otimes R$.

Proof. It is sufficient to show that for any $\kappa \in S$ the endomorphism $\varphi^{-1} \circ t_{a,b}(\kappa)$ of $\mathcal{V}_{a+b}$ is an automorphism. As follows from Proposition 27.1 the restriction of the functor $\mathcal{G}$ to subcategory $\mathcal{A}_\kappa$ of $\mathcal{O}_\kappa$ is exact. Therefore this statement follows immediately from Lemma 33.2.

From now on we will fix $t_{a,b} \in \mathcal{T}_{a,b}$ in such a way that the automorphism $(\varphi^{-1} \otimes 1) \circ t_{a,b}$ is the identity.

Let s'_a be a generator of the free R -module $\mathcal{S}_a$. Then

$$\begin{aligned} \mathcal{G}(s'_a) &\in \text{Hom}_{\mathcal{G}}(\mathcal{G}(\mathbf{V}_a \dot{\otimes} \mathbf{V}_a), R) = \text{Hom}_{\mathcal{G}}(\mathcal{G}(\mathbf{V}_a) \otimes \mathcal{G}(\mathbf{V}_a), \mathbb{C}) \\ &= \text{Hom}_{\mathcal{G}}(\mathcal{V}_a \otimes \mathcal{V}_a, R) = \text{Hom}_{\mathcal{G}}(\mathcal{V}_a, \mathcal{V}_a \otimes R) = R. \end{aligned}$$

As follows from Lemma 33.1(b) $\mathcal{G}(s'_a)$ is an invertible element in R .

Let $r \in \mathbb{Q}^+$ be the smallest number such that the categories $\mathcal{O}_\kappa$ are rigid for all $\kappa < -r$. As follows from Theorem 32.1, $r = 0$ if $\mathcal{G}$ is of type A_n or D_n and we have explicit upper bounds on r for simple Lie algebras of type E_6, E_7, E_8 .

Let $Z_r = \{\kappa \in \mathbb{R} | \kappa \geq -r\}$, $Y = \mathbb{C} - Z_r$, let $\check{R}$ be the ring of analytic functions on Y which are meromorphic at ∞ , let $\check{\kappa}: R \rightarrow \check{R}$ be the natural imbedding, and $\check{\mathcal{O}} \stackrel{\text{def}}{=} \mathcal{O}_{\check{\kappa}}$. Since in this section we will work only with the category $\check{\mathcal{O}}$ we will denote by $\mathbf{V}_a$ the object $\mathbf{V}_a^{\check{\kappa}} \in \mathcal{O}_{\check{\kappa}}$. We denote by $V \hookrightarrow \check{V} \stackrel{\text{def}}{=} V \otimes_{\check{R}} R_\infty$ the imbedding of the braided category $\check{\mathcal{O}}$ to $\mathcal{O}_\infty$, and we denote by $(\check{\mathcal{G}}, \check{\beta})$ the morphism of the braided category $\check{\mathcal{O}}$ to the Drinfeld category $\mathcal{D}$ which is a composition of the imbedding $V \rightarrow \check{V}$ and the equivalence of categories $\mathcal{O}_\infty \rightarrow \mathcal{D}$ as in the corollary to Lemma 31.3.

Lemma 33.4. *The functor $\mathcal{G}: \mathcal{O} \rightarrow \mathcal{D}$ is faithful.*

Proof. Given two objects V, W in $\mathcal{O}$ we have to show that the map

$$\mathrm{Hom}_{\mathcal{O}}(V, W) \rightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}(V), \mathcal{G}(W))$$

is an embedding. Since V, W are torsion-free $\check{R}$ -modules it is clear that the map $\mathrm{Hom}_{\mathcal{O}}(V, W) \rightarrow \mathrm{Hom}_{\mathcal{O}_{\infty}}(\check{V}, \check{W})$ is an imbedding. Lemma 33.4 is proved.

For any $V \in \mathcal{O}$ we can identify $\mathcal{G}(V)$ with $\mathcal{G}(V) \otimes_{\check{R}} R_{\infty}$. Therefore for any $a \in \mathbf{N}^I$ we can and will identify $\mathcal{G}(V_a)$ with the object V_a in $\mathcal{D}$ (see §19.3). It is clear that under this identification we have $\mathcal{G}(t_{a,b}) = T_{a,b}$, where morphisms $T_{a,b} \in \mathrm{Hom}_{\mathcal{D}}(V_{a+b}, V_a \otimes V_b)$ are defined as in §20.1.

Let $s'_a, a \in \mathbf{N}^I$, be a generator of $\mathcal{S}_a$. For any $a, b \in \mathbf{N}^I$, we consider the composition

$$V_{a+b} \otimes V_{b+a} \xrightarrow{t_{a,b} \otimes t_{b,a}} V_a \otimes V_b \otimes V_b \otimes V_a \xrightarrow{1 \otimes s'_b \otimes 1} V_a \otimes V_a \xrightarrow{s'_a} 1.$$

Since $\mathcal{S}_{a+b}$ is a free $\check{R}$ -module with s'_{a+b} as a generator, there exists $g_{a,b} \in \check{R}$ such that this composition is equal to $g_{a,b} \cdot s'_{a+b}$.

Lemma 33.5. *$g_{a,b}$ is an invertible element of $\check{R}$.*

Proof. It is sufficient to show that for any $\kappa_0 \in Y$ we have $g_{a,b}(\kappa_0) \neq 0$. In other words it is sufficient to show that for any $\kappa \in \check{R}$ the composition

$$V_{a+b}^{\kappa_0} \otimes V_{b+a}^{\kappa_0} \xrightarrow{t_{a,b}(\kappa_0) \otimes t_{b,a}(\kappa_0)} V_a^{\kappa_0} \otimes V_b^{\kappa_0} \otimes V_b^{\kappa_0} \otimes V_a^{\kappa_0} \xrightarrow{1 \otimes s'_b(\kappa_0) \otimes 1} V_a^{\kappa_0} \otimes V_a^{\kappa_0} \xrightarrow{s'_a(\kappa_0)} 1$$

is not a zero map. Since $\mathcal{O}_{\kappa_0}$ is a rigid category it is sufficient to show that for any $\kappa \in \check{R}$ the composition

$$\Lambda: V_{a+b}^{\kappa_0} \xrightarrow{t_{a,b}(\kappa_0)} V_a^{\kappa_0} \otimes V_b^{\kappa_0} \xrightarrow{s'_a(\kappa_0) \otimes s'_b(\kappa_0)} (V_a^{\kappa_0})^* \otimes (V_b^{\kappa_0})^* \xrightarrow{\alpha} (V_b^{\kappa_0} \otimes V_a^{\kappa_0})^* \xrightarrow{t_{a,b}^*} (V_{b+a}^{\kappa_0})^*$$

is not a zero map. As follows from Corollary 2 to Lemma 33.2 the morphisms $t_{a,b}(\kappa_0)$, $t_{a,b}^*(\kappa_0)$, and $s'_a(\kappa) \otimes s'_b(\kappa)$ induce isomorphisms in the quotient category $\bar{\mathcal{O}}_{\kappa}^{a+b}$ (see the definition before Lemma 28.4). So we see that Λ induces an isomorphism $\bar{\Lambda}: \bar{L}_{a+b}^{\kappa} \rightarrow \bar{L}_{b+a}^*$ and therefore $\Lambda \neq 0$. Lemma 33.5 is proved.

Corollary. (a) *We can choose generators $s_a \in \mathcal{S}_a$ in such a way that $g_{a,b} \equiv 1$ for all $a, b \in \mathbf{N}^I$.*

(b) *For any two such choices s_a^1, s_a^2 of generators there exists a group homomorphism $\varphi: \mathbf{Z}^I \rightarrow \check{R}^*$ such that $s_a^2 = \varphi(a)s_a^1$ for all $a \in \mathbf{N}^I$.*

Proof. Let F be the field of fractions of $\check{R}$. As follows from the proof of Proposition 20.10 there exist elements $\tilde{\varphi}_a \in F^*$, $a \in \mathbf{N}^I$, such that for all $a, b \in \mathbf{N}^I$ we have $g_{a,b} = \tilde{\varphi}_{a+b} \tilde{\varphi}_a^{-1} \tilde{\varphi}_b^{-1}$. For any $a \in \mathbf{N}^I$, $\kappa_0 \in Y$, we denote by $\eta_a^{\kappa_0} \in \mathbf{Z}$ the order of pole of $\tilde{\varphi}_a$ at κ_0 . It is clear that for any $a \in \mathbf{N}^I$,

$\eta_a^{\kappa_0} = 0$ for almost all $\kappa_0 \in Y$. Since $g_{a,b} \in \check{R}^*$ we see that $\eta_{a+b}^{\kappa_0} = \eta_a^{\kappa_0} + \eta_b^{\kappa_0}$ for all $a, b \in \mathbf{N}^I$, $\kappa_0 \in Y$. For any $a \in \mathbf{N}^I$ we define functions $\check{\varphi}_a \stackrel{\text{def}}{=} \prod_{\kappa_0 \in Y} (\kappa - \kappa_0)^{\eta_a^{\kappa_0}} \tilde{\varphi}_a \in F^*$. It is clear that $\check{\varphi}_a$ lies in $\check{R}^*$ and $g_{a,b} = \check{\varphi}_{a+b} \check{\varphi}_a^{-1} \check{\varphi}_b^{-1}$. It is clear that elements $s_a \stackrel{\text{def}}{=} \varphi_a^{-1} s'_a$ satisfy the conditions of (a). Part (b) is obvious.

We fix a set of generators $\check{s}_a$ of $\mathcal{S}_a$ satisfying the conditions of the corollary.

For any $a, b \in \mathbf{N}^I$ and $i \in I$ such that $a(i), b(i) > 0$ we denote by $\theta_{i,a,b}$ the $\check{R}$ -module $\theta_{i,a,b} \stackrel{\text{def}}{=} \text{Hom}_{\check{\mathcal{S}}}(\mathbf{V}_{a+b-\alpha_i}, \mathbf{V}_a \dot{\otimes} \mathbf{V}_b)$. Since $C_{a,b,\overline{a+b-\alpha_i}} = 1$ it follows from the same arguments as in the proof of Lemma 33.1 that $\theta_{i,a,b}$ is a free $\check{R}$ -module of rank 1.

For any element $\theta \in \theta_{i,a,b}$ we denote by $\mathcal{G}(\theta)$ the image of θ in

$$\text{Hom}(\mathcal{G}(\mathbf{V}_{a+b-\alpha_i}), \mathcal{G}(\mathbf{V}_a \dot{\otimes} \mathbf{V}_b)) = \text{Hom}(V_{a+b-\alpha_i}, V_a \otimes V_b) \otimes \check{R},$$

and we define an element $\tau_{i,a,b}$ in $\text{Hom}(V_{a+b-\alpha_i}, V_a \otimes V_b) \otimes \check{R}$ by the rule

$$\tau_{i,a,b} = \frac{\tau'_{i,a,b}}{\Gamma(1 - \kappa^{-1}a(i))\Gamma(1 - \kappa^{-1}b(i))\Gamma(1 + \kappa^{-1}(a(i) + b(i)))},$$

where the morphism $\tau'_{i,a,b}$ is as in §22.1.

Proposition 33.1. *There exists an element $\check{\tau}_{i,a,b} \in \theta_{i,a,b}$ such that $\mathcal{G}(\check{\tau}_{i,a,b}) = \tau_{i,a,b}$.*

Proof. By Lemma 33.2 the quotient $Q \stackrel{\text{def}}{=} \mathbf{V}_a \dot{\otimes} \mathbf{V}_b / t_{a,b}(\mathbf{V}_{a+b})$ lies in $\mathcal{A}$, and it follows from Theorem 28.1 that there exists an imbedding $\mathbf{V}_{a+b-\alpha_i}$ in Q such that the quotient $Q' \stackrel{\text{def}}{=} Q / \text{Im}(\mathbf{V}_{a+b-\alpha_i})$ is also in $\mathcal{A}$. Let W be the pullback of the exact sequence $0 \rightarrow \mathbf{V}_{a+b} \rightarrow \mathbf{V}_a \dot{\otimes} \mathbf{V}_b \rightarrow Q \rightarrow 0$ under the imbedding $\mathbf{V}_{a+b-\alpha_i} \hookrightarrow Q$. Then we have an exact sequence $0 \rightarrow \mathbf{V}_{a+b} \xrightarrow{t} W \xrightarrow{j} \mathbf{V}_{a+b-\alpha_i} \rightarrow 0$.

Lemma 33.6. *There exists a morphism $\eta: \mathbf{V}_{a+b-\alpha_i} \rightarrow W$ such that $j \circ \eta = [(a+b)(i)]\text{Id}$, where as before for any $n \in \mathbf{N}$ we define as $[n] \in \check{R}$ by the rule $[n] = (v^n - v^{-n})/(v - v^{-1})$.*

Proof. As follows from Lemma 29.8 it is sufficient to show that for any $\kappa_0 \in Y$ there exists an element $\hat{\eta} \in \text{Hom}_{\mathcal{O}_{\kappa_0}}(\mathbf{V}_{a+b-\alpha_i}^{\hat{\kappa}_0}, \mathbf{W} \otimes_R R_{\kappa_0})$ such that $\hat{j} \circ \hat{\eta} = [(a+b)(i)]\text{Id}$, where $\hat{j} = j \otimes_R 1$. In other words, we have to show that there exists $\hat{\eta} \in \text{Hom}_{\mathcal{O}_{\kappa_0}}(\mathbf{V}_{a+b-\alpha_i}^{\hat{\kappa}_0}, \mathbf{W} \otimes_{\check{R}} R_{\kappa_0})$ such that the order of zero of $\hat{j} \circ \hat{\eta} \in R_{\hat{\kappa}_0}$ at κ_0 is not greater than the order of zero of $[a(i) + b(i)]$ at κ_0 . There are two cases: (a) $[a(i) + b(i)](\kappa_0) \neq 0$ and (b) $[a(i) + b(i)](\kappa_0) = 0$. In the first case it follows from Lemma 27.7(a) that $W(\kappa_0)$ is a direct sum of $\mathbf{V}_{a+b}^{\kappa_0}$ and $\mathbf{V}_{a+b-\alpha_i}^{\kappa_0}$ and therefore $W \otimes_{\check{R}} R_{\kappa_0}$ is a direct sum of $\mathbf{V}_{a+b}^{\hat{\kappa}_0}$ and $\mathbf{V}_{a+b-\alpha_i}^{\hat{\kappa}_0}$. In the

second case we have to show the existence of $\hat{\eta} \in \text{Hom}_{\mathcal{O}_{\kappa_0}}(V_{a+b-\alpha_i}^{\kappa_0}, W \otimes_R R_{\kappa_0})$ such that $\hat{j} \cdot \hat{\eta} \in R_{\kappa_0}$ has at most a simple zero at κ_0 . But this follows from Proposition 29.4. Lemma 33.6 is proved.

By the construction we have an imbedding $W \xrightarrow{\rho} V_a \dot{\otimes} V_b$. Let $\check{\tau} \in \theta_{i,a,b}$ be the composition of η and this imbedding. By Lemma 33.6 we know that $\mathcal{G}(\check{\tau}) = [(a+b)(i)] \cdot \tau'_{i,a,b}$. It is clear that the function

$$F_{a,b}(\kappa) \stackrel{\text{def}}{=} \frac{1}{\Gamma(1 - \kappa^{-1}a(i))\Gamma(1 - \kappa^{-1}b(i))\Gamma(1 + \kappa^{-1}(a(i) + b(i))) \times [(a+b)(i)]}$$

lies in $\check{R}$ and therefore we can define an element $\check{\tau}_{i,a,b} = F_{a,b} \cdot \check{\tau} \in \theta_{i,a,b}$. It is clear that $\mathcal{G}(\check{\tau}_{i,a,b}) = \tau_{i,a,b}$. Proposition 33.1 is proved.

Let $\check{K}$ be the field of fractions of $\check{R}$. For any $a, b \in \mathbf{N}^I$ we consider the composition

$$V_{a+b-\bar{\alpha}_i} \xrightarrow{\check{\tau}_{i,a,b} \dot{\otimes} \check{\tau}_{i,b,a}} V_a \dot{\otimes} V_b \otimes V_b \otimes V_a \xrightarrow{1 \otimes \check{s}_b \otimes 1} V_a \otimes V_a \xrightarrow{\check{s}_a} 1.$$

As follows from Proposition 23.6 the element $\rho_i \in \check{K}$ does not depend on $a, b \in \mathbf{N}^I$.

Proposition 33.2. ρ_i does not have zeros or poles in Y .

Proof. we have to show that for any $\kappa \in Y$, ρ_i is regular at κ_0 and $\rho_i(\kappa_0) \neq 0$. Assume first that κ is not of the form $\kappa_0 = -\frac{2}{q}$, where q is an odd integer. Then we can find $a, b \in \mathbf{N}^I$ such that $[a(i)][b(i)][(a+b)(i)](\kappa_0) \neq 0$, and it is sufficient to show that the composition

$$\begin{aligned} V_{a+b-\bar{\alpha}_i}^{\kappa_0} \dot{\otimes} V_{b+a-\alpha_i}^{\kappa_0} &\xrightarrow{\check{\tau}_{i,a,b}(\kappa_0) \dot{\otimes} \check{\tau}_{i,b,a}(\kappa_0)} V_a^{\kappa_0} \dot{\otimes} V_b^{\kappa_0} \dot{\otimes} V_b^{\kappa_0} \dot{\otimes} V_a^{\kappa_0} \\ &\xrightarrow{1 \otimes \check{s}_b(\kappa_0) \otimes 1} V_a^{\kappa_0} \dot{\otimes} V_a^{\kappa_0} \xrightarrow{\check{s}_a(\kappa_0)} 1 \end{aligned}$$

is not zero.

This statement is equivalent to the statement that the composition

$$\Lambda_i: V_{a+b-\bar{\alpha}_i}^{\kappa_0} \xrightarrow{\check{\tau}_{i,a,b}(\kappa_0)} V_a^{\kappa_0} \dot{\otimes}_{\check{b}}^{\kappa_0} \xrightarrow{\check{s}_a \dot{\otimes} \check{s}_b} (V_a^{\kappa_0})^* \dot{\otimes} (V_b^{\kappa_0})^* \xrightarrow{\check{\tau}_{i,a,b}^*(\kappa_0)} (V_{a+b-\alpha_i}^{\kappa_0})^*$$

is not zero.

To show that this composition is not zero it is convenient to introduce the following definitions. Given an object V in $\mathcal{O}_{\kappa_0}^{<a}$ we say that V is a -equivalent to zero and write $V \sim_a 0$ if $\text{Hom}_{\mathcal{O}_{\kappa_0}}(V_{a-\bar{\alpha}_i}^{\kappa_0}, V) = \langle 0 \rangle$. We say that a morphism $\varphi \in \text{Hom}_{\mathcal{O}_{\kappa_0}}(V, W)$, $V, W \in \mathcal{O}_{\kappa_0}$, is an a -isomorphism if $\ker \varphi, \text{coker } \varphi \sim_a 0$.

Lemma 33.7. (a) If φ is an a -isomorphism and ψ is a b -isomorphism, $a, b \in \mathbf{N}^I$, then $\varphi \dot{\otimes} \psi$ is an $(a+b)$ -isomorphism.

(b) If $a \in \mathbf{N}^I$ is such that $[a(i)] \neq 0$, then s_a is an a -isomorphism.

Proof. (a) follows from Lemma 28.4 and (b) follows from Lemma 27.7.

It is clear that for the proof of Proposition 33.2 it is sufficient to show that Λ_i is an $(a+b)$ -isomorphism. To show this it is sufficient to prove that the composition

$$\begin{aligned} \mathbf{V}_{a+b}^{\kappa_0} \oplus \mathbf{V}_{a+b-\alpha_i}^{\kappa_0} &\xrightarrow{\tilde{\mathbf{T}}_{a,b}(\kappa_0) \oplus \tilde{\mathbf{T}}_{i,a,b}(\kappa_0)} \mathbf{V}_a^{\kappa_0} \dot{\otimes} \mathbf{V}_b^{\kappa_0} \xrightarrow{\tilde{s}_a(\kappa_0) \dot{\otimes} \tilde{s}_b(\kappa_0)} (\mathbf{V}_a^{\kappa_0})^* \dot{\otimes} (\mathbf{V}_b^{\kappa_0})^* \\ &\xrightarrow{\sim} (\mathbf{V}_b^{\kappa_0} \dot{\otimes} \mathbf{V}_a^{\kappa_0})^* \xrightarrow{(\tilde{\mathbf{T}}_{i,b,a}(\kappa_0))^*} (\mathbf{V}_{a+b-\alpha_i}^{\kappa_0})^* \end{aligned}$$

is an $(a+b)$ -isomorphism. But it follows from Lemma 33.7(b) and the construction of $\tilde{\mathbf{T}}_{i,a,b}$ that all morphisms in this composition are $(a+b)$ -isomorphisms. So Proposition 33.2 is proved in the case when $\kappa_0 \neq -\frac{2}{q}$, where q is an odd number. The proof of Proposition 33.2 in the case when $\kappa_0 = -\frac{2}{q}$ is completely analogous (one has to consider $\rho_{i,a,b}(\kappa_0)$, where $a, b \in \mathbf{N}^I$ are such that $[a(i)][b(i)](\kappa_0) \neq 0$). Proposition 33.2 is proved.

Corollary. *There exists a unique choice of generators $s_a \in \mathcal{S}_a$ which satisfy conditions of the corollary to Lemma 33.4 in such a way that $\rho_i \equiv 1$ for all $i \in I$ and $\mathcal{G}(s_a)(\infty) = S'_a$, where morphisms S_a are defined in §23.7.*

Proof. Follows from §23.7 and the simple connectedness of the open set Y .

34. THE FUNCTOR $\mathbf{X}$ FROM $\mathcal{O}_\kappa$ TO THE CATEGORY OF R -MODULES

In this section we define a functor $\mathbf{X}$ from $\tilde{\mathcal{O}}$ to the category of $\tilde{R}$ -modules and endomorphisms E_i, F_i of this functor.

Using the morphisms $\tilde{\mathbf{T}}_{a,b}, \tilde{S}_a$ and $\tilde{\mathbf{T}}_{i,a,b}$ we can define morphisms

$$\begin{aligned} \tilde{tr}_{a,b}^c &: \mathbf{V}_{a+c} \dot{\otimes} \mathbf{V}_{c+b} \longrightarrow \mathbf{V}_a \dot{\otimes} \mathbf{V}_b, \\ \tilde{m} &: \mathbf{V}_{a+c} \dot{\otimes} \mathbf{V}_{b+d} \longrightarrow \mathbf{V}_a \dot{\otimes} \mathbf{V}_b \dot{\otimes} \mathbf{V}_c \dot{\otimes} \mathbf{V}_d, \\ \tilde{\phi}_{i;a,b}^e &: \mathbf{V}_{a+e-\alpha_i} \dot{\otimes} \mathbf{V}_{e+b} \longrightarrow \mathbf{V}_a \dot{\otimes} \mathbf{V}_b, \\ \tilde{\psi}_{i;a,b}^e &: \mathbf{V}_{a+e} \dot{\otimes} \mathbf{V}_{e+b-\alpha_i} \longrightarrow \mathbf{V}_a \dot{\otimes} \mathbf{V}_b \end{aligned}$$

for all $a, b, c, d, e \in \mathbf{N}^I$ such that $a(i) \geq 1$ and $c(i) = 1$ in the way completely analogous to the definition of morphisms $tr_{a,b}^c, m, \phi_{i;a,b}^e, \psi_{i;a,b}^e$ in the category $\mathcal{D}$ defined in §§21.1, 21.4, and 23.1.

It is clear from the construction that we have $\mathcal{G}(\tilde{tr}_{a,b}^c) = tr_{a,b}^c, \mathcal{G}(\tilde{m}) = m, \mathcal{G}(\tilde{\phi}_{i;a,b}^e) = \phi_{i;a,b}^e$, and $\mathcal{G}(\tilde{\psi}_{i;a,b}^e) = \psi_{i;a,b}^e$. As follows from Lemma 33.4 an existence of any relation between morphisms $tr_{a,b}^c, m, \phi_{i;a,b}^e$, and $\psi_{i;a,b}^e$ implies the validity of the analogous relation between morphisms $\tilde{tr}_{a,b}^c, \tilde{m}, \tilde{\phi}_{i;a,b}^e$ and $\tilde{\psi}_{i;a,b}^e$.

For any $\lambda \in \mathbf{Z}^I$, an object V in $\tilde{\mathcal{O}}$, and a pair $a, b \in \mathbf{N}^I$ such that $b-a = \lambda$, we set

$$V^{(a,b)} = \text{Hom}_{\tilde{\mathcal{O}}}(\mathbf{V}_a \dot{\otimes} \mathbf{V}_b, V).$$

As follows from Lemma 29.8, $V^{(a,b)}$ is a finitely generated free R -module.

We shall regard the set $\{(a, b) \in \mathbf{N}^I \times \mathbf{N}^I \mid b - a = \lambda\}$ as a partially ordered set by setting $(a, b) \leq (a', b')$ whenever $a, b, a', b' \in \mathbf{N}^I$ satisfy $b - a = b' - a' = \lambda$ and $a' - a = b' - b \in \mathbf{N}^I$.

For $(a, b) \leq (a', b')$ as above, we consider the $\check{R}$ -linear map $\psi_{a,b;a',b'}: V^{(a,b)} \rightarrow V^{(a',b')}$ defined by composition with $tr_{a,b}^{a'-a}: \mathbf{V}_{a'} \dot{\otimes} \mathbf{V}_{b'} \rightarrow \mathbf{V}_a \dot{\otimes} \mathbf{V}_b$.

From the transitivity of tr (see 21.3), we see that $(V^{(a,b)}, \psi_{a,b;a',b'})$, where $a, b \in \mathbf{N}^I$ vary subject to $b - a = \lambda$ and $\psi_{a,b;a',b'}$ are as above, form a direct system of $\check{R}$ -modules, and we define

$$\mathbf{X}_\lambda(V) = \varinjlim_{b-a=\lambda} V^{(a,b)} \quad \text{and} \quad \mathbf{X}(V) = \bigoplus_{\lambda \in \mathbf{Z}^I} \mathbf{X}_\lambda(V).$$

Remark. It is not obvious that the $\check{R}$ -module $\mathbf{X}(V)$ is finitely generated.

Let $\lambda, \lambda' \in \mathbf{Z}^I$, let V, V' be two objects of $\check{\mathcal{O}}$, and let $f: \mathbf{V}_a \otimes \mathbf{V}_b \rightarrow V$ and $f': \mathbf{V}_c \otimes \mathbf{V}_d \rightarrow V'$ be morphisms in $\check{\mathcal{O}}$, where $a, b, c, d \in \mathbf{N}^I$ satisfy $b - a = \lambda, d - c = \lambda'$.

Then $f \dot{\otimes} f': \mathbf{V}_a \dot{\otimes} \mathbf{V}_b \dot{\otimes} \mathbf{V}_c \dot{\otimes} \mathbf{V}_d \rightarrow V \dot{\otimes} V'$ is a morphism in $\check{\mathcal{O}}$; composing it with $\check{m}(\kappa_0): \mathbf{V}_{a+c} \dot{\otimes} \mathbf{V}_{b+d} \rightarrow \mathbf{V}_a \dot{\otimes} \mathbf{V}_b \dot{\otimes} \mathbf{V}_c \dot{\otimes} \mathbf{V}_d$, we obtain a morphism

$$(f \dot{\otimes} f') \check{m}: \mathbf{V}_{a+c} \dot{\otimes} \mathbf{V}_{b+d} \rightarrow V \dot{\otimes} V'$$

which may be regarded as an element of $(V \dot{\otimes} V')^{(a+c, b+d)}$. Thus, $f \dot{\otimes} f' \mapsto (f \dot{\otimes} f') \check{m}$ defines a map

$$V^{(a,b)} \dot{\otimes} V'^{(c,d)} \longrightarrow (V \dot{\otimes} V')^{(a+c, b+d)}.$$

This map is compatible with the direct systems defining $\mathbf{X}_\lambda(V), \mathbf{X}_{\lambda'}(V'), \mathbf{X}_{\lambda+\lambda'}(V \dot{\otimes} V')$, by the left and right compatibility of $\check{m}$ with $\check{tr}$ (see Propositions 21.6 and 21.7). Hence, by passage to limit, we obtain a $\check{R}$ -linear map

$$\mathbf{X}_\lambda(V) \dot{\otimes}_{\check{R}} \mathbf{X}_{\lambda'}(V') \longrightarrow \mathbf{X}_{\lambda+\lambda'}(V \dot{\otimes} V').$$

Taking the direct sum over μ , we obtain a morphism of $\check{R}$ -modules

$$M_{V, V'}: \mathbf{X}(V) \dot{\otimes}_{\check{R}} \mathbf{X}(V') \longrightarrow \mathbf{X}(V \dot{\otimes} V').$$

Assume that $i \in I$ is given. Fix $\lambda \in \mathbf{Z}^I$, $a, b \in \mathbf{N}^I$ such that $b - a = \lambda$ and $a(i) \geq 1$ and an object V of $\check{\mathcal{O}}$. For any $c \in \mathbf{N}^I$ such that $c(i) = 1$, we define a linear map $V^{(a,b)} \rightarrow V^{(a+c-\alpha_i, c+b)}$ as the composition with the morphism

$$\check{\Phi}_{i,a,b}^c: \mathbf{V}_{\bar{a}+\bar{c}-\bar{\alpha}_i} \dot{\otimes} \mathbf{V}_{c+b} \longrightarrow \mathbf{V}_{\bar{a}} \dot{\otimes} \mathbf{V}_b.$$

The identity

$$\check{tr}_{a,b}^c \check{\Phi}_{i,a+c,b}^d = \check{\Phi}_{i,a,b}^c \check{tr}_{a+c-\alpha_i, c+b}^d$$

(see Lemma 23.2) shows that these maps define by passage to limit a linear map

$$E_i^V: \mathbf{X}(V) \rightarrow \mathbf{X}(V).$$

Similarly, for any $c \in \mathbf{N}^I$ such that $c(i) = 1$, the composition with the morphism

$$\check{\Psi}_{i;a,b}^c: \mathbf{V}_{a+c} \otimes \mathbf{V}_{c+b-\alpha_i} \rightarrow \mathbf{V}_a \otimes \mathbf{V}_b$$

defines an $\check{R}$ -linear map

$$V^{(a,b)} \longrightarrow V^{(a+c, c+b-\alpha_i)}$$

for any $a, b \in \mathbf{N}^I$ such that $b - a = \lambda$ and $b(i) \geq 1$. The identity

$$\check{t}r_{a,b}^c \check{\Psi}_{i;a+c, c+b}^d = \check{\Psi}_{i;a,b}^c \check{t}r_{a+c, c+b-\alpha_i}^d$$

(see Lemma 23.3) shows that these maps define by passage to limit a $\check{R}$ -linear map

$$F_i^V: \mathbf{X}_\lambda(V) \longrightarrow \mathbf{X}_{\lambda-\alpha_i}(V).$$

We will consider $E_i, F_i, i \in I$, as endomorphisms of the functor $\mathbf{X}$ from $\check{\mathcal{O}}$ to the category of $\check{R}$ -modules.

Lemma 34.1. *For any V in $\check{\mathcal{O}}$ and $i, j \in I$ we have*

(a) $(E_i, F_j - F_j E_i)x = \delta_{ij}[\lambda(i)]x$ for all $x \in \mathbf{X}_\lambda(V)$.

As before, we denote by (a_{ij}) the Cartan matrix of $\mathfrak{g}$. Then

(b) If $a_{ij} = 0$, then

$$E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i.$$

(c) If $a_{ij} = -1$, then

$$\begin{aligned} E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 &= 0, \\ F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 &= 0, \end{aligned}$$

where we write E_i, F_i instead of E_i^V, F_i^V .

Proof. Follows from Lemmas 25.13 and 25.15.

For any $i \in I$ we denote by $\check{v}_i$ an automorphism of the functor $\mathbf{X}$ such that $\check{v}_i(x) = v^{\lambda(i)}$ for $x \in \mathbf{X}_i(V)$.

Lemma 34.2. *For any V, V' in $\mathcal{O}$ and $i \in I$ the diagrams*

$$\begin{array}{ccc} \mathbf{X}(V) \otimes_{\check{R}} \mathbf{X}(V') & \xrightarrow{M_{V,V'}} & \mathbf{X}(V \dot{\otimes} V') \\ \downarrow E^V \otimes \text{id} + \check{v}_i \otimes E_i^{V'} & & \downarrow E_i^{V \dot{\otimes} V'} \\ \mathbf{X}(V) \otimes_{\check{R}} \mathbf{X}(V') & \xrightarrow{M_{V,V'}} & \mathbf{X}(V \dot{\otimes} V') \end{array}$$

and

$$\begin{array}{ccc} \mathbf{X}(V) \otimes_{\check{R}} \mathbf{X}(V') & \xrightarrow{M_{V,V'}} & \mathbf{X}(V \dot{\otimes} V') \\ \downarrow \text{id} \otimes F_i^{V'} + F_i^V \otimes \check{v}_i^{-1} & & \downarrow F_i^{V \dot{\otimes} V'} \\ \mathbf{X}(V) \otimes_{\check{R}} \mathbf{X}(V') & \xrightarrow{M_{V,V'}} & \mathbf{X}(V \dot{\otimes} V') \end{array}$$

are commutative.

Proof. Follows from Lemma 25.16 and Lemma 33.4.

We fix now $\kappa \in Y$. For any object W in $\mathcal{O}_\kappa$ we define $\mathbf{C}$ -vector space $W^{(a,b)} = \text{Hom}_{\mathcal{O}_\kappa}(\mathbf{V}_a^\kappa \otimes \mathbf{V}_b^\kappa, W)$, morphisms $\bar{\psi}_{a,b,a',b'} : W^{(a,b)} \rightarrow W^{(a',b')}$, the direct limit $\bar{\mathbf{X}}_\lambda(W) = \varinjlim W^{(a,b)}$, $\bar{\mathbf{X}}(W) = \bigoplus_\lambda \bar{\mathbf{X}}_\lambda(W)$, and endomorphisms $\bar{E}_i^W, \bar{F}_i^W$ of $\bar{\mathbf{X}}(W)$ in the way analogous to one we used to define E_i^V and F_i^V , $i \in I$.

Given V in $\tilde{\mathcal{O}}$ we define the quotient $V^{(a,b)}(\kappa) = V^{(a,b)} / \mathfrak{m}_\kappa V^{(a,b)}$, where $\mathfrak{m}_\kappa \subset \check{R}$ is the maximal ideal of functions equal to zero at κ . As follows from Lemma 29.8(b) the natural maps

$$\mathbf{V}_r^{(a,b)}(\kappa) : V^{(a,b)}(\kappa) \longrightarrow \bar{V}^{(a,b)}$$

are imbeddings, where $\bar{V} \stackrel{\text{def}}{=} V / \mathfrak{m}_\kappa V$.

Let $N_0 \subset N^I$ be the subset defined at the end of §28.

Lemma 34.3. *If $a \in N_0$ then the map $r_V^{(a,b)}(\kappa)$ is an isomorphism.*

Proof. As follows from the corollary to Lemma 28.5 and Corollary 2 to Lemma A.4 the object $\mathbf{V}_a^\kappa \otimes \mathbf{V}_b^\kappa$ in $\mathcal{O}_\kappa$ is projective. Therefore the surjectivity of $r_V^{(a,b)}(\kappa)$ follows from Theorem 29.1 and Proposition 29.3.

We denote by r_V the induced map $r_V : \mathbf{X}(V)(\kappa) \rightarrow \bar{\mathbf{X}}(V(\kappa))$.

Corollary 1. *r_V is an isomorphism.*

It is clear that we have equalities $r_V E_i^V = \bar{E}_i^V r_V$, $r_V F_i^V = \bar{F}_i^V r_V$ for all $i \in I$. Therefore Lemma 34.1 implies the following result.

Corollary 2. *For any V in $\tilde{\mathcal{O}}$ we have the equalities*

- (a) $(\bar{E}_i^V \bar{F}_j^V - \bar{F}_j^V \bar{E}_i^V)(\bar{x}) = \delta_{ij}[\lambda(i)](\kappa)\bar{x}$ for all $\bar{x} \in \bar{\mathbf{X}}_\lambda(V)$.
- (b) If $a_{ij} = 0$, then $\bar{F}_i^V \bar{E}_j^V = \bar{E}_j^V \bar{E}_i^V$, $\bar{F}_i^V \bar{F}_j^V = \bar{F}_j^V \bar{F}_i^V$.
- (c) If $a_{ij} = 1$, then

$$\begin{aligned} (\bar{E}_i^V)^2 \bar{E}_j^V - (v(\kappa) + v(\kappa)^{-1}) \bar{E}_i^V \bar{E}_j^V \bar{E}_i^V + \bar{E}_j^V (\bar{E}_i^V)^2 &= 0, \\ (\bar{F}_i^V)^2 \bar{F}_j^V - (v(\kappa) + v(\kappa)^{-1}) \bar{F}_i^V \bar{F}_j^V \bar{F}_i^V + \bar{F}_j^V (\bar{F}_i^V)^2 &= 0. \end{aligned}$$

Corollary 3. *For any W in $\mathcal{O}_\kappa$ the endomorphisms $\bar{E}_i^W, \bar{F}_i^W$ of $\bar{\mathbf{X}}(W)$ satisfy the equalities of Corollary 2.*

Proof. By the corollary to Proposition 32.2 the category $\mathcal{O}_\kappa$ has enough projectives (see Definition A.9) it is sufficient to prove Corollary 3 in the case when W is projective. But in this case the result follows from Corollary 2 and Proposition 29.3.

Corollary 4. *There exists a unique functorial morphism*

$$M_{W, W'} : \bar{\mathbf{X}}(W) \otimes \bar{\mathbf{X}}(W') \rightarrow \bar{\mathbf{X}}(W \otimes W'), \quad W, W' \text{ in } \mathcal{O}_\kappa,$$

such that for any V, V' in $\mathcal{O}$ we have

$$r_{V \dot{\otimes} V'} \circ M_{V, V'} = \bar{M}_{V, V'} \circ r_V \dot{\otimes} r_{V'}.$$

Proof. The same argument as in the proof of Corollary 3.

Consider now the case when $V = (V_c)^*$ for some $c \in \mathbf{N}^I$. In this case $V(\kappa) = (V_c^\kappa)^*$ and it follows from Theorems 28.1 and 29.1 that $\dim_{\mathbf{C}} V^{(a,b)}(\kappa) = \dim_{\mathbf{C}} V^{(a,b)}(\kappa) = C_{a,b,c}$. As follows from Lemma 33.2 for sufficiently big (a, b) such that $b - a = \lambda$ we have an equality $C_{a,b,c} = \dim_{\mathbf{C}} (\mathcal{Z}_c^\lambda)$, where $\mathcal{Z}_c^\lambda = \{v \in \mathcal{Z}_c \mid h_i v = \lambda(i)v, i \in I\}$.

To simplify notations we denote the space $\bar{\mathbf{X}}((V_c^\kappa)^*)$ as $\mathbf{X}_c$.

Lemma 34.4. (a) $\dim_{\mathbf{C}} \mathbf{X}_c \leq \dim_{\mathbf{C}} (\mathcal{Z}_c^\lambda)$.

(b) The equality $\dim_{\mathbf{C}} \mathbf{X}_c = \dim_{\mathbf{C}} (\mathcal{Z}_c^\lambda)$ holds if and only if the linear maps $\bar{\psi}_{a,b,a',b'}$ are isomorphisms for all sufficiently big a, b, a', b' . In this case R_κ -morphisms $\psi_{a,b,a',b'} \otimes_{\bar{R}} R_\kappa$ are isomorphisms for sufficiently big a, b, a', b' , and $\mathbf{X}(V_c^\kappa) \otimes_{\bar{R}} R_\kappa$ is a free R_κ -module of rank $\dim_{\mathbf{C}} (\mathcal{Z}_c)$.

Proof. Clear.

Consider now the special case when $c = 0$. In this case $C_{a,a,0} = 1$ for all $a \in \mathbf{N}^I$ such that $b - a = c$ and therefore $\dim(V_0^\kappa)^{(a,a)} = 1$.

Moreover for any $a \in \mathbf{N}^I$ we have defined a nonzero element $s_a(\kappa) \in \text{Hom}_{\mathcal{O}_\kappa}(V_a^\kappa \dot{\otimes} V_a^\kappa, V_0^\kappa) = (V_0^\kappa)^{(a,a)}$.

Lemma 34.5. For any pair $a \leq a' \in \mathbf{N}^I$ we have $\bar{\psi}_{a,a,a',a'}(s_a(\kappa)) = s_{a'}(\kappa)$.

Proof. Follows from the definition of $s_a(\kappa)$ (see corollary to Lemma 33.5).

Corollary 1. There exists a unique morphism $\bar{\mathbf{X}}(V_\kappa^0) \xrightarrow{\sim} \mathbf{C}$ such that the image of $s_a(\kappa)$ under this isomorphism is equal to 1.

We will identify $\bar{\mathbf{X}}(V_\kappa^0)$ with $\mathbf{C}$.

Corollary 2. For any $c \in \mathbf{N}^I$ we have $\bar{\mathbf{X}}_c \neq \{0\}$.

Proof. Consider the pairing $\varphi_c: \mathbf{X}_c \otimes \mathbf{X}_c \rightarrow \mathbf{C}$ defined as the composition

$$\varphi_c: \mathbf{X}_c \otimes \mathbf{X}_c = \bar{\mathbf{X}}(V_c^\kappa) \otimes \bar{\mathbf{X}}(V_c^\kappa) \xrightarrow{M_{V_c^\kappa, V_c^\kappa}} \bar{\mathbf{X}}(V_c^\kappa \dot{\otimes} V_c^\kappa) \xrightarrow{\bar{\mathbf{X}}(s_c(\kappa))} \bar{\mathbf{X}}(V_0^\kappa) = \mathbf{C}.$$

It is sufficient to show that $\varphi \neq 0$. Consider elements

$$x_c^0 \in (V_c^\kappa)^{(c,0)} = \text{Hom}_{\mathcal{O}_\kappa}(V_c^\kappa \dot{\otimes} V_0^\kappa, V_c^\kappa) = \text{Hom}_{\mathcal{O}_\kappa}(V_c^\kappa, V_c^\kappa)$$

and

$$y_c^0 \in (V_c^\kappa)^{(0,c)} = \text{Hom}_{\mathcal{O}_\kappa}(V_0^\kappa \dot{\otimes} V_c^\kappa, V_c^\kappa) = \text{Hom}_{\mathcal{O}_\kappa}(V_c^\kappa, V_c^\kappa)$$

corresponding to the identity morphisms, and denote by x_c, y_c the images of x_c^0, y_c^0 in $\mathbf{X}_c$ and $\bar{\mathbf{X}}_c$ correspondingly. By the definition, $\varphi_c(x_c, y_c)$ is the

image in $\bar{X}(\mathbf{V}_0^\kappa)$ of the element $s_c(\kappa) \in (\mathbf{V}_0^\kappa)^{(c,c)}$. So $\varphi_c(x_c, y_c) = 1$. Corollary 6 is proved.

35. $\bar{X}$ AS A FUNCTOR FROM $\mathcal{O}_\kappa$ TO THE CATEGORY OF REPRESENTATIONS OF QUANTUM GROUPS

In this section we interpret the functor $\bar{X}$ as the functor from $\mathcal{O}_\kappa$ to the category of representations of the quantum group without divided powers.

For any $\kappa \in \mathbf{C}^*$ we define a category $\mathcal{E}(\kappa)$ as follows (compare §24). Objects of $\mathcal{E}(\kappa)$ are finite-dimensional vector spaces $\mathbf{E}$ with a given direct sum decomposition $\mathbf{E} = \bigoplus_\lambda \mathbf{E}^\lambda$ (λ runs through $\mathbf{Z}^I$) and given linear maps $E_i, F_i, i \in I$, such that

- (a) $E_i(\mathbf{E}^\lambda) \subset \mathbf{E}^{\lambda+\alpha_i}$ and $F_i(\mathbf{E}^\lambda) \subset \mathbf{E}^{\lambda-\alpha_i}$ for all $i \in I, \lambda \in \mathbf{Z}^I$.
- (b) $(E_i F_j - F_j E_i)(x) = \delta_{ij} [\lambda(i)](\kappa)x$ for all $i, j \in I$ and $x \in \mathbf{E}^\lambda$.
- (c) If $a_{ij} = 0$, then $E_i E_j - E_j E_i = F_i F_j - F_j F_i = 0$.
- (d) If $a_{ij} = -1$, then

$$E_i^2 E_j - (v(\kappa) + v(\kappa)^{-1}) E_i E_j E_i + E_j E_i^2 = F_i^2 F_j - (v(\kappa) + v(\kappa)^{-1}) F_i F_j F_i + F_j F_i^2 = 0.$$

For any object $\mathbf{E} = \bigoplus_\lambda \mathbf{E}^\lambda$ in $\mathcal{E}(\kappa)$, $i \in I$, we denote by $\hat{v}_i$ the automorphism of $\mathbf{E}$ such that $\hat{v}_i|_{\mathbf{E}^\lambda} = v(\kappa)^{\lambda(i)} \text{Id}$. If E', E'' are two objects in $\mathcal{E}(\kappa)$ the tensor product is regarded as an object in $\mathcal{E}(\kappa)$ with $\mathbf{E}$

$$(\mathbf{E}' \otimes \mathbf{E}'')^\lambda = \bigoplus_{\lambda' + \lambda'' = \lambda} (\mathbf{E}'^{\lambda'} \otimes \mathbf{E}''^{\lambda''})$$

and with $E_i, F_i: \mathbf{E} \rightarrow \mathbf{E}$ defined by $E_i = E'_i \otimes \text{id} + \hat{v}'_i \otimes E''_i$, $F_i = \text{id} \otimes F''_i + F'_i \otimes (\hat{v}''_i)^{-1}$.

Lemma 35.1. Assume that $\kappa = -p/q$, $p, q \in \mathbf{N}$, $(p, q) = 1$, $p \neq 1$, and $\mathbf{E}', \mathbf{E}''$ are objects in $\mathcal{E}(\kappa)$ such that $(E'_i)^p = (F'_i)^p = (E''_i)^p = (F''_i)^p = 0$ for all $i \in I$. Then $E_i^p = F_i^p = 0$, $i \in I$, where E_i, F_i are endomorphisms of $\mathbf{E}$ defined as above.

Proof. Follows from the equality $[p](\kappa) = 0$.

It is easy to see that for any $a \in \mathbf{N}^I$ there exists an irreducible object $(L_a = \bigoplus_\lambda L_a, E_i, F_i)$ such that $\dim L_a^a = 1$ and $E_i|_{L_a^a} = 0$ for all $i \in I$. Moreover such an object L^a is unique up to an isomorphism, it is finite dimensional, and we define $d(a, \kappa) \stackrel{\text{def}}{=} \dim L_a^a$. We denote by $N \subset \mathbf{N}^I \subset \mathbf{Z}^I$ the following subset. If $\mathfrak{g}$ is not of type E_8 we define N as the subset of miniscule fundamental weights, and if $\mathfrak{g}$ is of type E_8 we take N to be equal to $\langle a_0 \rangle$, where $a_0 \in \mathbf{N}^I$ is the highest root of $\mathfrak{g}$.

Proposition 35.1. (a) For any $a \in N \subset \mathbf{N}^I$ and $\kappa \in \mathbf{C}^*$ we have $d(a, \kappa) = \dim \mathcal{V}_a$, where $\mathcal{V}_a$ is as in §1.2.

(b) Assume that $\kappa = -p/q$, $p, q \in \mathbf{N}$, $(p, q) = 1$, $p \neq 1$. Then $E_i^p|_{L_a} = F_i^p|_{L_a} = 0$.

(c) Let $Z \subset G$ be the center of the simply connected algebraic group associated to $\mathfrak{g}$ and $z \in Z$ be an element such that $\theta_c(Z) = 1$ for all $c \in N$ (see the definition in §31). Then z is the identity element.

Proof. (a) Follows from Theorem 10.7 in [APK] and Proposition 7.1 in [L1].

(b) Clear.

(c) Easy.

36. $\bar{\mathbf{X}}$ AS A MONOIDAL FUNCTOR

In this section we define a functorial isomorphism $M_{V, V'}: \bar{\mathbf{X}}(V) \otimes \bar{\mathbf{X}}(V') \rightarrow \bar{\mathbf{X}}(V \dot{\otimes} V')$.

As follows from Corollary 3 to Lemma 34.3 we can consider $\bar{\mathbf{X}}$ as a functor from $\mathcal{O}_\kappa$ to $\mathcal{E}(\kappa)$

$$W \rightarrow \left(\bar{\mathbf{X}}(W) = \bigoplus_\lambda \bar{\mathbf{X}}_\lambda(W), \bar{E}_i^W, \bar{F}_i^W \right).$$

Let $N \subset \mathbf{N}^I$ be the subset as in Proposition 35.1. We fix $\kappa \in Y$, $c \in N$, and consider a $\mathbf{Z}^I$ -graded space $\mathbf{X}_c \stackrel{\text{def}}{=} \bar{\mathbf{X}}(\mathbf{V}_c^\kappa)$.

Lemma 36.1. (a) $\dim_{\mathbf{C}} \mathbf{X}_c = \dim_{\mathbf{C}} \mathcal{V}_c$.

(b) If $\kappa = -p/q$, $(p, q) = 1$, $p \neq 1$ then $(\bar{E}_i^{\mathbf{X}_c})^p = (\bar{F}_i^{\mathbf{X}_c})^p = 0$ for all $i \in I$.

Proof. It follows from Corollary 2 to Lemma 34.4 that $\mathbf{X}_c \neq \{0\}$. Since $\bar{E}_i|_{L_c} = 0$, for all $i \in I$ it follows from Proposition 35.1(a) that $\dim_{\mathbf{C}} \mathbf{X}_c \geq \dim_{\mathbf{C}} \mathcal{V}_c$. On the other hand, it follows from Lemma 34.2 that $\dim_{\mathbf{C}} \mathbf{X}_c \leq \dim_{\mathbf{C}} \mathcal{V}_c$. So $\dim_{\mathbf{C}} \mathbf{X}_c = \dim_{\mathbf{C}} \mathcal{V}_c$, and it follows from Proposition 35.1 that $\bar{\mathbf{X}}(\mathbf{V}_c^\kappa)$ is isomorphic to the object L_c . Therefore Lemma 36.1 follows from Proposition 35.1(b).

Consider now the object $\mathbf{X}_c^* = \bar{\mathbf{X}}((\mathbf{V}_c^\kappa)^*)$ in $\mathcal{E}(\kappa)$ and the pairing

$$\varphi: \mathbf{X}_c^* \otimes \mathbf{X}_c = \bar{\mathbf{X}}(\mathbf{V}_c^\kappa)^* \otimes \bar{\mathbf{X}}(\mathbf{V}_c^\kappa) \xrightarrow{(M_{\mathbf{V}_c^\kappa}^*)^*, \mathbf{V}_c^\kappa} \bar{\mathbf{X}}((\mathbf{V}_c^\kappa)^* \dot{\otimes} \mathbf{V}_c^\kappa) \xrightarrow{\bar{\mathbf{X}}(e_{\mathbf{V}_c^\kappa})} \bar{\mathbf{X}}(\mathbf{V}_c^\kappa) = \mathbf{C}.$$

Lemma 36.2. The pairing φ is nondegenerate.

Proof. Since $c \in N$ the object $(\mathbf{V}_c^\kappa)^*$ is isomorphic to $\mathbf{V}_c^\kappa$ it is sufficient to show that the pairing

$$\varphi: \mathbf{X}_c \otimes \mathbf{X}_c = \bar{\mathbf{X}}(\mathbf{V}_c^\kappa) \otimes \bar{\mathbf{X}}(\mathbf{V}_c^\kappa) \xrightarrow{M_{\mathbf{V}_c^\kappa, \mathbf{V}_c^\kappa}} \bar{\mathbf{X}}(\mathbf{V}_c^\kappa \dot{\otimes} \mathbf{V}_c^\kappa) \xrightarrow{\bar{\mathbf{X}}(\bar{\delta}_c)} \mathbf{C}$$

is nondegenerate.

It follows from the proof of Corollary 2 to Lemma 34.5 that $\varphi \neq 0$. To prove that φ is nondegenerate we observe that φ defines a morphism $\tilde{\varphi}: \bar{\mathbf{X}}_c \rightarrow (\bar{\mathbf{X}}_c)^*$ in $\mathcal{E}(\kappa)$ where $(\bar{\mathbf{X}}_c)^*$ is the object dual to $\bar{\mathbf{X}}_c$. Since by Proposition 35.1 the object $\mathbf{X}_c, \mathbf{X}_c^*$ are irreducible and $\varphi \neq 0$ we see that $\tilde{\varphi}$ is an isomorphism. Lemma 36.2 is proved.

Corollary. For any object V in $\mathcal{O}_\kappa$ and $c \in N$ the morphism

$$M_{\mathbf{V}_c^\kappa, V}: \bar{\mathbf{X}}(\mathbf{V}_c^\kappa) \otimes \bar{\mathbf{X}}(V) \rightarrow \bar{\mathbf{X}}(\mathbf{V}_c^\kappa \dot{\otimes} V)$$

is an isomorphism.

Proof. Consider the map $\psi: \mathbf{X}_c \otimes \mathbf{X}(\mathbf{V}_c^\kappa \dot{\otimes} V) \rightarrow \mathbf{X}(V)$ defined as the composition

$$\mathbf{X}(\mathbf{V}_c^\kappa) \otimes \mathbf{X}(\mathbf{V}_c^\kappa \dot{\otimes} V) \xrightarrow{M_{\mathbf{V}_c^\kappa, V}} \mathbf{X}(\mathbf{V}_c^\kappa \otimes \mathbf{V}_c^\kappa \otimes V) \xrightarrow{\mathbf{X}(\delta_c(\kappa) \otimes 1)} \mathbf{X}(V).$$

Then ψ defines a map $\hat{\psi}: \mathbf{X}(\mathbf{V}_c^\kappa \dot{\otimes} V) \rightarrow \mathbf{X}_c \otimes \mathbf{X}(V)$ and therefore we can define a map $M_{\mathbf{V}_c^\kappa, V}: \mathbf{X}(\mathbf{V}_c^\kappa \dot{\otimes} V) \rightarrow \mathbf{X}(\mathbf{V}_c^\kappa) \otimes \mathbf{X}(V)$ as the composition $M_{\mathbf{V}_c^\kappa, V} = (\check{\psi}^{-1} \otimes 1) \circ \hat{\psi}$. It is easy to see that the map $M_{\mathbf{V}_c^\kappa, V}$ is the inverse of $M_{\mathbf{V}_c^\kappa, V}$. The corollary is proved.

Proposition 36.1. (a) For any V, V' in $\mathcal{O}_\kappa$ the map $M_{V, V'}: \mathbf{X}(V) \otimes \mathbf{X}(V') \rightarrow \mathbf{X}(V \dot{\otimes} V')$ is an isomorphism.

(b) If $\kappa = -p/q$, $(p, q) = 1$, $p \neq 1$ then for any V in $\mathcal{O}_\kappa$ and $i \in I$ we have $(E_i^V)^p = (F_i^V)^p = 0$.

Proof. We say that an object V in $\mathcal{O}_\kappa$ is stable if for any V' in $\mathcal{O}_\kappa$ the map $M_{V, V'}$ is an isomorphism. As follows from Proposition 25.8 for any stable objects V_1, V_2 in $\mathcal{O}_\kappa$ the object $V_1 \dot{\otimes} V_2$ is also stable. It is clear that any direct summand of a stable object is stable. It follows from Corollary 2 to Lemma 36.2 that for any $c \in N$ the object $\mathbf{V}_c^\kappa$ is stable. Therefore it follows from the corollary to Proposition 32.2 that any injective object in $\mathcal{O}_\kappa$ is stable. Let V be any object in $\mathcal{O}_\kappa$. As follows from the corollary to Proposition 32.2 there exists an exact sequence $0 \rightarrow V \rightarrow I_0 \rightarrow I_1$, where I_0, I_1 are injective objects in $\mathcal{O}_\kappa$. Since the category $\mathcal{O}_\kappa$ is rigid, the tensor product $\dot{\otimes}$ is exact and therefore for any object V' in $\mathcal{O}_\kappa$ the sequence $0 \rightarrow V \dot{\otimes} V' \rightarrow I_0 \dot{\otimes} V' \rightarrow I_1 \dot{\otimes} V'$ is exact. Since the functor $\mathbf{X}$ from $\mathcal{O}_\kappa$ to vector spaces is left exact, the stability of V follows from the “5-lemma”. Part (a) is proved.

To prove part (b) we observe that it follows from Lemma 35.1 that for any objects V, V' in $\mathcal{O}_\kappa$ such that $(E_i^V)^p = (\bar{E}_i^{V'})^p$, $(F_i^V) = (\bar{F}_i^{V'})^p = 0$, $i \in I$, we have $(\bar{E}_i^{V \dot{\otimes} V'})^p = (\bar{F}_i^{V \dot{\otimes} V'})^p = 0$. Since for any $c \in N$ we have $(\bar{E}_i^{\mathbf{V}_c^\kappa})^p = (\bar{F}_i^{\mathbf{V}_c^\kappa})^p = 0$ for all $i \in I$ it follows from the same arguments as before that for any V in $\mathcal{O}_\kappa$ we have $(E_i^V)^p = (F_i^V)^p = 0$. Proposition 36.1 is proved.

Corollary. For any V in $\mathcal{O}$, $i \in I$, and $n \in \mathbb{N}$ the endomorphisms $(E_i^V)^n$, $(F_i^V)^n$ of the free $\check{R}$ -module $\mathbf{X}(V)$ are uniquely divisible by $[n]!$ where $[n]! \stackrel{\text{def}}{=} [1][2] \cdots [n]$.

Proof. It is sufficient to show that for any $\kappa \in Y$ the endomorphisms $(E_i^V)^\kappa$, $(F_i^V)^\kappa$ have zero at κ of order at least equal to the order of zero of $[n]!$ at κ . If κ is irrational or $\kappa = -1/q$, $q \in \mathbb{N}$, then $[n]!(\kappa) \neq 0$ for all $n \in \mathbb{N}$ and there is nothing to prove. So we may assume that $\kappa = -p/q$, $p, q \in \mathbb{N}$, $(p, q) = 1$, where $p > 1$. If $n < p$, then $[n]!(\kappa) \neq 0$. To prove the result in the case when $n = p$ we observe that the function $[p]!$ has a first order zero at κ . So to prove the corollary in the case when $n = p$ it is sufficient to show that the endomorphisms $(E_i^V(\kappa))^p$, $(F_i^V(\kappa))^p$ of the space $\mathbf{X}(V)(\kappa) \stackrel{\text{def}}{=} \mathbf{X}(V)/m_\kappa \mathbf{X}(V)$ are equal to zero. But this follows from Proposition 36.1(b) and Lemma 29.8.

As follows from [L, 35.2.2], for any $n \in \mathbf{N}$, we can write $(E_i^V)^n, (F_i^V)^n$ in the form

$$\begin{aligned} (E_i^V)^n &= [n]! P(E_i^V, (E_i^V)^p ([p]!)^{-1}), \\ (F_i^V)^n &= [n]! P(F_i^V, (F_i^V)^p ([p]!)^{-1}), \end{aligned}$$

where $P = P(x, y)$ is a polynomial with coefficients in the ring of rational functions on $\mathbf{C}$, regular at κ . The corollary is proved.

For any $i \in I$ and $n \in \mathbf{N}$ we denote by $E_i^{(n)}, F_i^{(n)}$ endomorphisms of the functor $\mathbf{X}$ such that $(E_i^{(n)})^V = \frac{1}{[n]!} (E_i^V)^n$, $(F_i^{(n)})^V = \frac{1}{[n]!} (F_i^V)^n$ for all $V \in \check{\mathcal{O}}$.

Lemma 36.3. *For any $\kappa \in Y$ there exist unique endomorphisms of the $E_i^{(n)}, F_i^{(n)}$ of the functor $\mathbf{X}$ such that for any V in $\check{\mathcal{O}}$*

$$r_V(E_i^{(n)})^V = (E_i^{(n)})^V r_V, \quad r_V(F_i^{(n)})^V = (F_i^{(n)})^V r_V,$$

where $\bar{V}$ and r_V are as in the corollary to Lemma 34.2.

Proof. Follows from Lemma 34.3 and the arguments used in the proof of Corollary 3 to Lemma 34.3.

37. REPRESENTATIONS OF QUANTUM GROUPS AS A BRAIDED CATEGORY

In this section we consider a category of modules over the quantized enveloping algebra associated to our Cartan matrix and to a $\mathbf{C}$ -algebra R with a given invertible element $v_0 \in R$. The results in this section are due to Drinfeld [D3] in the case where $R = \mathbf{C}((\epsilon))$ and $v_0 = \exp(\pi\sqrt{-1}\epsilon)$. The general case (including the case where v_0 is a root of 1) is treated in [L] to which the reader is referred for proofs.

Let v be an indeterminate and let $A = \mathbf{C}[v, v^{-1}]$. We shall use the following notation:

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}} \quad \text{for } n \in \mathbf{Z},$$

$$[n]^! = \prod_{s=1}^n [s] \quad \text{for } n \in \mathbf{N}.$$

For $a \in \mathbf{Z}$ and $t \in \mathbf{N}$ we set

$$\left[\begin{matrix} a \\ t \end{matrix} \right] = \frac{\prod_{s=0}^{t-1} (v^{a-s} - v^{-a+s})}{\prod_{s=1}^t (v^s - v^{-s})} \in A.$$

Recall that the symmetric Cartan matrix (a_{ij}) is given. Let $X = \mathbf{Z}^I$. For $i \in I$ we define $i' \in X$ by $i'(j) = a_{ij}$ for all $j \in I$.

Let $\mathbf{f}$ be the $\mathbf{C}(v)$ -algebra defined by the generators κ_i ($i \in I$) and the relations

$$\sum_{p, p' \in \mathbf{N}; p+p'=1-a_{ij}} (-1)^{p'} (\kappa_i^p / [p]^!) \kappa_j (\kappa_i^{p'} / [p']^!) = 0 \quad \text{if } i \neq j.$$

We have a direct sum decomposition $\mathbf{f} = \bigoplus_{\nu} \mathbf{f}_{\nu}$, where ν runs over $\mathbf{N}^I$ and $\mathbf{f}_{\nu}$ is spanned by products $\kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_p}$ for various sequences $i_1, i_2, \dots, i_p$ in I in which i appears $\nu(i)$ times for all $i \in I$.

Let ${}_A\mathbf{f}$ be the A -subalgebra of $\mathbf{f}$ generated by the elements $\kappa_i^{(p)} = \kappa_i^p/[p]^!$, where $i \in I$ and $p \in \mathbf{N}$. For any $\nu \in \mathbf{N}^I$, let ${}_A\mathbf{f}_{\nu}$ be the A -subalgebra of $\mathbf{f}$ generated by the elements $\kappa_i^{(p)} = \kappa_i^p/[p]^!$, where $i \in I$ and $p \in \mathbf{N}$. For any $\nu \in \mathbf{N}^I$, let ${}_A\mathbf{f} = {}_A\mathbf{f}_{\nu} = \mathbf{f}_{\nu}$.

Let R be a commutative A -algebra with 1. By change of scalars we obtain an R -algebra ${}_R\mathbf{f} = R \otimes_A ({}_A\mathbf{f})$. We have a direct sum decomposition ${}_R\mathbf{f} = \bigoplus_{\nu} ({}_R\mathbf{f}_{\nu})$.

The category ${}_R\mathcal{E}$. We define an abelian category ${}_R\mathcal{E}$ as follows. An object of ${}_R\mathcal{E}$ is a finitely generated R -module M with a given direct sum decomposition $M = \bigoplus_{\lambda \in X} M^{\lambda}$, a given ${}_R\mathbf{f}$ -module structure $(x, m \mapsto x^+m)$ on M , compatible with the R -module structure and a given ${}_R\mathbf{f}$ -module structure $(x, m \mapsto x^-m)$ on M , compatible with the R -module structure; it is assumed that the following properties are satisfied:

- (a) $E_i^{(n)} M^{\lambda} \subset M^{\lambda+ni'}$, $F_i^{(n)} M^{\lambda} \subset M^{\lambda-ni'}$ for any $i \in I$, $n \in \mathbf{Z}$, and $\lambda \in X$;
- (b1) $E_i^{(a)} F_j^{(b)} m = F_j^{(b)} E_i^{(a)} m$ if $i \neq j$;
- (b2) $E_i^{(a)} F_i^{(b)} m = \sum_{t \geq 0} \begin{bmatrix} a-b+\zeta(i) \\ t \end{bmatrix} F_i^{(b-t)} E_i^{(a-t)} m$;
- (b3) $F_i^{(b)} E_i^{(a)} m = \sum_{t \geq 0} \begin{bmatrix} -a+b-\zeta(i) \\ t \end{bmatrix} E_i^{(a-t)} F_i^{(b-t)} m$

for $m \in M^{\lambda}$. Here we use the notation $E_i^{(a)} = \kappa_i^{(a)+} : M \rightarrow M$, $F_i^{(a)} = \kappa_i^{(a)-} : M \rightarrow M$. The coefficients in (b2), (b3) are elements of A regarded as elements of R .

A morphism in ${}_R\mathcal{E}$ is an R -linear map which respects the direct sum decompositions, and the two ${}_R\mathbf{f}$ -module structures. Clearly, ${}_R\mathcal{E}$ is an abelian category.

When v is not a root of 1 in R , then the properties (a), (b1), (b2), (b3) in the previous definition are equivalent to the following apparently weaker ones:

- (a) $E_i M^{\lambda} \subset M^{\lambda+i'}$, $F_i M^{\lambda} \subset M^{\lambda-i'}$ for any $i \in I$ and $\lambda \in X$;
- (b) $(E_i F_j - F_j E_i) m = \delta_{i,j} [\lambda(i)] m$ for any $m \in M^{\lambda}$ and $i, j \in I$.

(Here $E_i = E_i^{(1)}$, $F_i = F_i^{(1)}$; in our case we have $E_i^{(p)} = E_i^p/[p]^!$ and $F_i^{(p)} = F_i^p/[p]^!$.)

Tensor products. Given two objects $M, M' \in {}_R\mathcal{E}$, the tensor product of R -modules $M \otimes_R M'$ can be regarded as an object of ${}_R\mathcal{E}$ in a unique way, so that the following requirements are satisfied:

- (a) we have $(M \otimes_R M')^{\lambda} = \bigoplus_{\lambda'+\lambda''=\lambda} M^{\lambda'} \otimes_R M'^{\lambda''}$;
- (b) we have $E_i^{(p)}(m \otimes m') = \sum_{p'+p''=p} v^{p'p''+p''\lambda(i)} E_i^{(p')}(m) \otimes E_i^{(p'')}(m')$, and
- (c) $F_i^{(p)}(m \otimes m') = \sum_{p'+p''=p} v^{p'p''-p'\lambda(i)} F_i^{(p')}(m) \otimes F_i^{(p'')}(m')$,

for $m \in M^\lambda$, $m' \in M^{\lambda'}$, $i \in I$, $p \in \mathbb{N}$.

Now let M, M', M'' be three objects of ${}_R\mathcal{E}$. By the previous construction, the R -module $M \otimes_R M' \otimes_R M''$ can be regarded as an object of ${}_R\mathcal{E}$ in two ways, $(M \otimes_R M') \otimes_R M''$ and $M \otimes_R (M' \otimes_R M'')$. In fact these two ways coincide.

Unit object. Let $\mathbf{1}$ be the object of ${}_R\mathcal{E}$ defined by $\mathbf{1}^0 = R$, $\mathbf{1}^\lambda = 0$ for $\lambda \neq 0$; the maps $E_i^{(n)}, F_i^{(n)}$ are zero for $n > 0$ and the identity for $n = 0$.

For any $M \in {}_R\mathcal{E}$ we have obvious isomorphisms $M \otimes_R \mathbf{1} \cong M$, $\mathbf{1} \otimes_R M \cong M$ (in ${}_R\mathcal{E}$).

Commutativity isomorphism. Let $i \in I$. There is a unique $\mathbb{C}(v)$ -linear map ${}_i r: \mathbf{f} \rightarrow \mathbf{f}$ such that ${}_i r(1) = 0$ and ${}_i r(\kappa_j y) = \delta_{i,j} y + v^{a_{i,j}} \kappa_j({}_i r(y))$ for all $y \in \mathbf{f}$ and $j \in I$.

There is a unique bilinear form $(\ , \) : \mathbf{f} \times \mathbf{f} \rightarrow \mathbb{C}(v)$ such that $(1, 1) = 0$, $(\mathbf{f}_\nu, \mathbf{f}_{\nu'}) = 0$ for $\nu \neq \nu'$, and $(\kappa_i y, x) = (1 - v_i^{-2})^{-1} (y, {}_i r(x))$ for all x, y and all $i \in I$. This restricts to a nonsingular symmetric bilinear form $(\ , \) : \mathbf{f}_\nu \times \mathbf{f}_\nu \rightarrow \mathbb{C}(v)$.

Let (b) be a $\mathbb{C}(v)$ -basis of $\mathbf{f}_\nu$ and let (b^*) be the dual basis with respect to $(\ , \)$. Then the sum $\sum_b b \otimes b^* \in \mathbf{f}_\nu \otimes \mathbf{f}_\nu$ is clearly independent of the choice of (b) . One shows that it is the image of a well-defined element $T_\nu \in {}_A \mathbf{f}_\nu \otimes_A ({}_A \mathbf{f}_\nu)$.

By change of scalars, this can be regarded as an element ${}_R T_\nu \in {}_R \mathbf{f}_\nu \otimes_R ({}_R \mathbf{f}_\nu)$; if now M, M' are two objects in ${}_R\mathcal{E}$, we may regard $M \otimes_R M'$ as a $({}_R \mathbf{f} \otimes_R ({}_R \mathbf{f}))$ -module via $(x \otimes x')(m \otimes m') = x^- m \otimes x'^+ m'$; hence the operator ${}_R T_\nu : M \otimes_R M' \rightarrow M \otimes_R M'$ is well defined in this module structure.

Let M, M' be two objects in ${}_R\mathcal{E}$. Let $\delta \geq 1$ be the determinant of the Cartan matrix $(a_{i,j})$. Let $\phi: X \times X \rightarrow \frac{1}{\delta} \mathbb{Z}$ be the unique symmetric bilinear pairing such that $\phi(i', j') = -a_{i,j}$ for all $i, j \in I$.

Assume that we are given an element $\zeta \in R$ such that $\zeta^\delta = v$ in R . For any rational number $q \in \frac{1}{\delta} \mathbb{Z}$ we will write v^q instead of $\zeta^{\delta q}$. (This is the usual power of V , when q is an integer.)

Define an (invertible) linear operator $\Pi_\phi: M \otimes M' \rightarrow M \otimes M'$ by $\Pi_\phi(m \otimes m') = v^{\phi(\lambda, \lambda')} m \otimes m'$ for $m \in M^\lambda$, $m' \in M^{\lambda'}$, $m' \in M^{\lambda'}$. Let $s: M' \otimes M \rightarrow M \otimes M'$ be the isomorphism of R -modules given by $s(m' \otimes m) = m \otimes m'$.

We define the R -linear map $\Theta: M \otimes_R M' \rightarrow M \otimes_R M'$ by $\Theta(m \otimes m') = \sum_\nu T_\nu(m \otimes m')$ for any $m \in M$, $m' \in M'$. (Only finitely many terms in the sum are nonzero.) Let $\mathcal{R}_{M', M} = \Theta \Pi_\phi s: M' \otimes M \rightarrow M \otimes M'$. One shows that $\mathcal{R}_{M', M}$ is an isomorphism in ${}_R\mathcal{E}$.

Hexagon property. Let M, M', M'' be three objects of ${}_R\mathcal{E}$. The map

$$\mathcal{R}_{M'', M \otimes M'}: M \otimes M' \otimes M'' \rightarrow M'' \otimes M \otimes M'$$

coincides with the composition

$$M \otimes M' \otimes M'' \xrightarrow{1_M \otimes \mathcal{R}_{M'', M'}} M \otimes M'' \otimes M' \xrightarrow{\mathcal{R}_{M'', M} \otimes 1_{M'}} M'' \otimes M \otimes M'.$$

The map $\mathcal{R}_{M \otimes M' \otimes M''}: M'' \otimes M \otimes M' \rightarrow M \otimes M' \otimes M''$ coincides with the composition

$$M'' \otimes M \otimes M' \xrightarrow{\mathcal{R}_{M, M''} \otimes 1_{M'}} M \otimes M'' \otimes M' \xrightarrow{1_M \otimes \mathcal{R}_{M', M''}} M \otimes M' \otimes M''.$$

Duality. Assume that R is a field. Let $M \in {}_R\mathcal{C}$. We consider the R -vector space M^* dual to M with the direct sum decomposition $M^* = \bigoplus_{\lambda \in X} (M^*)^\lambda$ given by $(M^*)^\lambda = (M^{-\lambda})^*$.

Define operators

$$E_i^{(p)}: (M^*)^\lambda \rightarrow (M^*)^{\lambda + p i'}, \quad F_i^{(p)}: (M^*)^\lambda \rightarrow (M^*)^{\lambda - p i'},$$

by

$$(E_i^{(p)} m^*)(m) = (-1)^p v^{p\lambda(i) + p(p-1)} m^*(E_i^{(p)} m)$$

for $m^*: M^{-\lambda} \rightarrow R$, $m \in M^{-\lambda - p i'}$ and

$$(F_i^{(p)} m^*)(m) = (-1)^p v^{-p\lambda(i) + p(p-1)} m^*(F_i^{(p)} m)$$

for $m^*: M^{-\lambda} \rightarrow R$, $m \in M^{-\lambda + p i'}$.

In this way, M^* becomes an object of ${}_R\mathcal{C}$.

Let $e_M: M^* \otimes_R M \rightarrow \mathbf{1}$ be the R -linear map given by $e_M(m^* \otimes m) = m^*(m)$ for $m^*: M^{-\lambda} \rightarrow R$, $m \in M^{-\lambda}$. This is a morphism in ${}_R\mathcal{C}$.

Let $i_M: \mathbf{1} \rightarrow M \otimes_R M^*$ be the R -linear map given by

$$i_M(1) = \sum_{\lambda \in X} \sum_h (m_{\lambda, h} \otimes m_{\lambda, h}^*),$$

where for each $\lambda \in X$, $(m_{\lambda, h})$ is an R -basis of M^λ and $(m_{\lambda, h}^*)$ is the corresponding dual basis of $(M^*)^{-\lambda}$. This is a morphism in ${}_R\mathcal{C}$.

The compositions

$$M = \mathbf{1} \otimes M \xrightarrow{i_M \otimes \mathbf{1}} M \otimes M^* \otimes M \xrightarrow{\mathbf{1} \otimes e_M} M$$

and

$$M^* = M^* \otimes \mathbf{1} \xrightarrow{\mathbf{1} \otimes i_M} M^* \otimes M \otimes M^* \xrightarrow{e_M \otimes \mathbf{1}} M^*$$

are the identity maps.

So we see that in the case when R is a field the category ${}_R\mathcal{C}$ has a natural structure of a braided rigid category.

Now fix $\kappa \in Y$ and consider the A -algebra which is isomorphic as an algebra to $\mathbf{C}$ and the morphism $A \rightarrow R$ is given by $v \rightarrow e^{-\sqrt{-1}\pi/\kappa}$. We take $\zeta = e^{-\sqrt{-1}\pi/\delta\kappa}$ and denote by $\mathcal{C}_\kappa$ the corresponding braided category.

Lemma 37.1. (a) For any $a \in \mathbf{N}^I$ there exists a unique irreducible object $\mathcal{L}^a$ in $\mathcal{E}_\kappa$,

$$\mathcal{L}^a = \left(L^a = \bigoplus_\lambda L_\lambda^a, E_i^{(n)}, F_i^{(n)} \right)$$

such that $L_a^a \neq \{0\}$ and for any $i \in I, n \in \mathbf{N}, \ell \in L_a^a, E_i^{(n)}\ell = 0$.

(b) If $a \in \mathbf{N}_0 \subset \mathbf{N}^I$, then $\mathcal{L}^a$ is a projective object in $\mathcal{E}_\kappa$ and $\dim L^a = \dim \mathcal{V}^a$.

Proof. Follows from [L1].

Remark. This result is not true if the Lie algebra $\mathfrak{g}$ is not simply laced.

38. THE FUNCTOR $\tilde{\mathbf{X}}$

In this section we show that for any $\kappa \in Y$ the braided functor $(\tilde{\mathbf{X}}, M)$ from $\mathcal{O}_\kappa$ to $\mathcal{E}_\kappa$ is an equivalence of categories.

As before, we fix $\kappa \in Y$ and we denote by $\tilde{\mathbf{X}}$ the functor

$$W \rightarrow \tilde{\mathbf{X}}(W) = \left(\bar{\mathbf{X}}(W) = \bigoplus_\lambda \bar{\mathbf{X}}_\lambda(W), (\bar{E}_i^W)^{(n)}, (\bar{F}_i^W)^{(n)} \right),$$

where the endomorphisms $(\bar{E}_i^W)^{(n)}, (\bar{F}_i^W)^{(n)}$ of $\bar{\mathbf{X}}(W)$ are defined in the corollary to Proposition 36.1. It is clear from the definition that the identities (b1)–(b3) from §37 are satisfied. Therefore we may consider $\tilde{\mathbf{X}}$ as a functor from $\mathcal{O}_\kappa$ to $\mathcal{E}_\kappa$. For any W, W' in $\mathcal{O}_\kappa$ we denote by $\bar{M}_{W, W'}$ the map $\bar{\mathbf{X}}(W) \otimes \bar{\mathbf{X}}(W')$ as in the corollary to Lemma 34.3.

As follows from that corollary and the definition of $M_{V, V'}, \bar{M}_{W, W'}$ defines a morphism in the category $\mathcal{E}_\kappa$.

Lemma 38.1. The pair $\tilde{\mathbf{X}} = (\bar{\mathbf{X}}, \bar{M})$ is a braided functor from $\mathcal{O}_\kappa$ to $\mathcal{E}_\kappa$ (see Definition A.12).

Proof. It follows from Proposition 36.1 and Lemma 34.3 that the morphism $\bar{M}_{W, W'}$ is an isomorphism for all $W, W' \in \mathcal{O}_\kappa$. The compatibility $\bar{M}$ with the commutativity and associativity constraints follows from Proposition 25.8 and Lemma 25.10. Lemma 38.1 is proved. Let $\mathbf{N}_0 \subset \mathbf{N}^I$ be the subset as in the end of §28.

Lemma 38.2. For any $c \in \mathbf{N}_0 \subset \mathbf{N}^I$, $\tilde{\mathbf{X}}(\mathbf{V}_c^\kappa)$ is isomorphic to $\mathcal{L}^c$.

Proof. Since $\mathbf{V}_c^\kappa$ is irreducible, the morphism $\delta_a(\kappa): \mathbf{V}_c^\kappa \rightarrow (\mathbf{V}_c^\kappa)^*$ is an isomorphism. It is sufficient to show that $\tilde{\mathbf{X}}(\mathbf{V}_c^\kappa)^*$ is isomorphic to $\mathcal{L}^c$. Let $\bar{\mathbf{X}}^c = \bar{\mathbf{X}}((\mathbf{V}_c^\kappa)^*) = \bigoplus_\lambda \bar{\mathbf{X}}_\lambda^c$. It follows from Lemma 34.4 that $\dim \bar{\mathbf{X}} \leq \dim \mathcal{V}_c$. On the other hand, it follows from Corollary 2 to Lemma 34.5 that $\bar{\mathbf{X}}_c^c \neq \{0\}$. Since for any $i \in I, n \in \mathbf{N}$ we have $\bar{\mathbf{X}}_{c+n\alpha_i}^c = 0$ it follows that $\tilde{\mathbf{X}}$ contains $\mathcal{L}^c$ as a quotient. By Lemma 37.1 we have $\dim \mathcal{L}^c = \dim \mathcal{V}_c$. Therefore $\bar{\mathbf{X}}^c$ is isomorphic to $\mathcal{L}^c$. Lemma 38.2 is proved.

Theorem 38.1. *The braided functor $\tilde{\mathbf{X}} = (\tilde{\mathbf{X}}, M)$, $\mathcal{O}_\kappa \rightarrow \mathcal{E}_\kappa$ is an equivalence of categories.*

Proof. In the proof we use results and notations from Lemma A.15. Let P_γ^κ , $\gamma \in \Gamma$, be the system of objects in $\mathcal{O}_\kappa$ isomorphic to objects of the form $\mathbf{V}_{c_1}^\kappa \dot{\otimes} \cdots \dot{\otimes} \mathbf{V}_{c_k}^\kappa \dot{\otimes} \mathbf{V}_a^\kappa$, where $c_1, \dots, c_k \subset N$, $a \in N_0$. Since $\mathbf{V}_a$ is projective, it follows from the rigidity of the category $\mathcal{O}_\kappa$ that P_γ is projective. Define $Q_\gamma^\kappa = \mathbf{X}(P_\gamma^\kappa)$. Since $\tilde{\mathbf{X}}$ is a braided functor and the category $\mathcal{E}_\kappa$ is rigid, it follows from Lemmas 38.2 and 37.1 that objects Q_γ in $\mathcal{E}_\kappa$ are projective. It follows then from the corollary to Proposition 32.2 that the $\{P_\gamma\}$ is an ample system of projective objects in $\mathcal{O}_\kappa$. The analogous arguments show that $\{Q_\gamma\}$ is an ample system of projective objects in $\mathcal{E}_\kappa$. By Lemma A.15 it is sufficient to check that for any $\gamma, \gamma' \in P$ the natural map $\text{Hom}_{\mathcal{O}_\kappa}(P_\gamma, P_{\gamma'}) \xrightarrow{\tilde{\mathbf{X}}} \text{Hom}_{\mathcal{E}_\kappa}(Q_\gamma, Q_{\gamma'})$ is an isomorphism.

As follows from Lemma A.14(d) this map is an imbedding. Therefore it is sufficient to prove that $\dim \text{Hom}_{\mathcal{O}_\kappa}(P_\gamma, P_{\gamma'}) = \dim \text{Hom}_{\mathcal{E}_\kappa}(Q_\gamma, Q_{\gamma'})$. We can write $P_\gamma^\kappa = \mathbf{V}_{c_1}^\kappa \dot{\otimes} \cdots \dot{\otimes} \mathbf{V}_{c_k}^\kappa \dot{\otimes} \mathbf{V}_a^\kappa$, $P_{\gamma'}^\kappa = \mathbf{V}_{c'_1}^\kappa \dot{\otimes} \cdots \dot{\otimes} \mathbf{V}_{c'_k}^\kappa \dot{\otimes} \mathbf{V}_{a'}^\kappa$, for some $c_i, c'_i \in N$, $a \in N_0$. We claim that (a)

$$\dim \text{Hom}_{\mathcal{O}_\kappa}(P_\gamma, P_{\gamma'}) = (\chi_{c_1} \cdots \chi_{c_k} \chi_a, \chi_{c'_1} \cdots \chi_{c'_k} \chi_{a'})$$

and (b)

$$\dim \text{Hom}_{\mathcal{E}_\kappa}(Q_\gamma, Q_{\gamma'}) = (\chi_{c_1} \cdots \chi_{c_k} \chi_a, \chi_{c'_1} \cdots \chi_{c'_k} \chi_{a'}),$$

where the scalar product is as in §32.2. We prove (a). As follows from Lemma 27.7 and Proposition 27.4 the objects $\mathbf{V}_{c_i}^\kappa, \mathbf{V}_a^\kappa$ are irreducible and therefore we can identify $(\mathbf{V}_{c_i}^\kappa)^*$ with $\mathbf{V}_{c_i}^\kappa$, $(\mathbf{V}_a^\kappa)^*$ with $\mathbf{V}_a^\kappa$. Since the category $\mathcal{O}_\kappa$ is rigid, this implies that the objects P_γ are tilting. Therefore the equality (a) follows from Theorem 28.1 and Corollary 2 to Proposition 27.1. The proof of the equality (b) is completely analogous. Theorem 38.1 is proved.

Remark. The validity of Theorem 38.1 was conjectured in [L2] (see Conjecture 2.3).

APPENDIX

In this Appendix we present the theory of tensor categories (see [JS]).

Braided categories.

Definition A.1. A monoidal category $\mathcal{E}$ is a triple $\mathcal{E} = (\tilde{\mathcal{E}}, \otimes, a)$, where $\tilde{\mathcal{E}}$ is a category, $\otimes: \tilde{\mathcal{E}} \times \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ is a functor, and a is the natural transformation between the functors $\otimes(\otimes \times \text{id})$ and $\otimes(\text{id} \times \otimes)$ from $\tilde{\mathcal{E}} \times \tilde{\mathcal{E}} \times \tilde{\mathcal{E}}$ to $\tilde{\mathcal{E}}$, $a = \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)\}$, X, Y, Z in $\tilde{\mathcal{E}}$ such that all $a_{X,Y,Z}$ are isomorphisms, satisfy the pentagon axiom, and there exists an object U in $\tilde{\mathcal{E}}$ and an isomorphism $u: U \otimes U \xrightarrow{\sim} U$ such that the functors

$X \rightarrow X \otimes U$ and $X \rightarrow U \otimes X$ are autoequivalences of $\tilde{\mathcal{C}}$. We call such a pair (U, u) the identity object of $\mathcal{C}$. It is clear that the identity object is determined uniquely up to a unique isomorphism.

Let $\mathcal{C} = (\tilde{\mathcal{C}}, \otimes, a)$ be a monoidal category, and (U, u) the identity object in $\mathcal{C}$.

Lemma A.1. (a) For any object X in $\tilde{\mathcal{C}}$ there exist unique isomorphisms $r_X: X \xrightarrow{\sim} X \otimes U$, $e_X: X \rightarrow U \otimes X$ such that

$$a_{X,U,U} \circ (r_X \otimes 1_U) = 1_X \otimes u, \quad 1_U \otimes e_X = a_{U,U,X} \circ (1_X \otimes u).$$

(b) $r_U = u$.

(c) For any morphism $\alpha: Y \rightarrow X$ in $\tilde{\mathcal{C}}$ the diagrams

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & X \\ \downarrow r_Y & & \downarrow r_X \\ Y \otimes U & \xrightarrow{\alpha \otimes 1} & X \otimes U \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{\alpha} & X \\ \downarrow e_Y & & \downarrow e_X \\ U \otimes Y & \xrightarrow{1 \otimes \alpha} & U \otimes X \end{array}$$

are commutative.

Proof. Easy.

Definition A.2. We say that a monoidal category $\mathcal{C} = (\tilde{\mathcal{C}}, \otimes, a)$ is strict if for any X, Y, Z in $\tilde{\mathcal{C}}$ we have $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, $a_{X,Y,Z} = 1$, there exists an object $\mathbf{1}$ in $\tilde{\mathcal{C}}$ such that $\mathbf{1} \otimes X = X \otimes \mathbf{1} = X$ for all X in $\tilde{\mathcal{C}}$, and for any X, Y in $\tilde{\mathcal{C}}$ the composition

$$X \otimes Y = (X \otimes \mathbf{1}) \otimes Y = X \otimes (\mathbf{1} \otimes Y) = X \otimes Y$$

is equal to $1_{X \otimes Y}$.

Mac Lane's theorem says that any monoidal category is equivalent to a strict one (see [M, Chapter 7]). To simplify formulas we will often assume that our monoidal categories are strict. Let $\mathcal{C}$ be a strict monoidal category.

For any object X in $\tilde{\mathcal{C}}$ we denote by F_X the functor from $\tilde{\mathcal{C}}$ to sets defined by $F_X(Y) \stackrel{\text{def}}{=} \text{Hom}_{\tilde{\mathcal{C}}}(Y \otimes X, \mathbf{1})$. It is clear that the correspondence $X \rightarrow F_X$ has a natural structure of a contravariant functor from $\tilde{\mathcal{C}}$ to the category $\mathcal{F}$ of functors from $\tilde{\mathcal{C}}$ to sets. Let $\mathcal{F}_0 \subset \mathcal{F}$ be the full subcategory of representable functors. As is well known the natural functor $\varphi: X \rightarrow \text{Hom}(\cdot, X)$ defines an equivalence between the categories $\tilde{\mathcal{C}}$ and $\mathcal{F}_0$. Therefore there exists a functor $\varphi: \mathcal{F}_0 \rightarrow \tilde{\mathcal{C}}$ such that there exists a natural isomorphism between $\varphi \circ \varphi$ and the identity functor on $\tilde{\mathcal{C}}$.

Definition A.3. We say that an object X in $\tilde{\mathcal{C}}$ is weakly rigid if the functor F_X is representable.

In this case we define $X^* = \varphi(F_X)$. By the definition there exists a morphism $e_X: X^* \otimes X \rightarrow \mathbf{1}$ such that for any Y in $\tilde{\mathcal{C}}$ the map

$$\Lambda_Y^X: \text{Hom}_{\tilde{\mathcal{C}}}(Y, X^*) \rightarrow \text{Hom}_{\tilde{\mathcal{C}}}(Y \otimes X, \mathbf{1})$$

given by $\varphi \rightarrow e_X(\varphi \otimes 1_X)$ is an isomorphism.

Definition A.4. We say that a monoidal category $\mathcal{C}$ is weakly rigid if F_X lies in $\mathcal{F}_0$ for all X in $\tilde{\mathcal{C}}$ and the contravariant functor $X \rightarrow X^*$ from $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}$ defined by $X^* \stackrel{\text{def}}{=} \varphi(F_X)$ is an anti-equivalence of categories.

Let $\mathcal{C}$ be a weakly rigid category. By the definition for any X, Y in $\mathcal{C}$ we have a map $\alpha \rightarrow \alpha^*$ from $\text{Hom}_{\mathcal{C}}(X, Y)$ to $\text{Hom}_{\mathcal{C}}(Y^*, X^*)$. On the other hand, for any X, Y in $\tilde{\mathcal{C}}$ we define the map $\gamma: \text{Hom}(X, Y) \rightarrow \text{Hom}(Y^*, X^*)$ as the composition

$$\begin{aligned} \text{Hom}(X, Y) &\xrightarrow{Y^* \otimes} \text{Hom}(Y^* \otimes X, Y^* \otimes Y) \\ &\xrightarrow{e_Y} \text{Hom}(Y^* \otimes X, \mathbf{1}) \xrightarrow{(\Lambda_Y^*)^{-1}} \text{Hom}(Y^*, X^*). \end{aligned}$$

Lemma A.2. For any $\alpha \in \text{Hom}(X, Y)$ we have $\gamma(\alpha) = \alpha^*$.

Proof. Easy.

Corollary 1. For any X, Y in $\tilde{\mathcal{C}}$ the natural map

$$\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(Y^* \otimes X, Y^* \otimes Y) \xrightarrow{e_Y} \text{Hom}_{\mathcal{C}}(Y^* \otimes X, \mathbf{1})$$

is an isomorphism.

Corollary 2. Let X, Y be objects in $\tilde{\mathcal{C}}$, and $\varphi \in \text{Hom}_{\mathcal{C}}(X, Y)$ a morphism such that for all Z in $\tilde{\mathcal{C}}$ the induced map

$${}^t\varphi: \text{Hom}_{\mathcal{C}}(Y \otimes Z, \mathbf{1}) \rightarrow \text{Hom}_{\mathcal{C}}(X \otimes Z, \mathbf{1})$$

is a bijection. Then φ is an isomorphism.

From now on we will always assume that the monoidal category $\mathcal{C}$ is weakly rigid.

Definition A.5. (a) An object X in $\tilde{\mathcal{C}}$ is rigid if there exists a morphism $i_X: \mathbf{1} \rightarrow X \otimes X^*$ such that the compositions

$$X = \mathbf{1} \otimes X \xrightarrow{i_X \otimes \mathbf{1}} X \otimes X^* \otimes X \xrightarrow{\mathbf{1} \otimes e_Y} X, \quad X^* = X^* \otimes \mathbf{1} \xrightarrow{\mathbf{1} \otimes i_Y} X^* \otimes X \otimes X^* \xrightarrow{e_X \otimes \mathbf{1}} X^*$$

are equal to 1_X and 1_{X^*} correspondingly.

(b) We say that a weakly rigid category $\mathcal{C}$ is rigid if every object in $\tilde{\mathcal{C}}$ is rigid.

Lemma A.3. (a) If X is a rigid object in $\tilde{\mathcal{C}}$ then for any $\alpha \in \text{Hom}_{\tilde{\mathcal{C}}}(X, Y)$ we can define a morphism $\alpha^* \in \text{Hom}_{\tilde{\mathcal{C}}}(Y^*, X^*)$ as the composition

$$Y^* = Y^* \otimes \mathbf{1} \xrightarrow{\mathbf{1} \otimes i_X} Y^* \otimes X \otimes X^* \xrightarrow{\mathbf{1} \otimes \alpha \otimes \mathbf{1}} Y^* \otimes Y \otimes X^* \xrightarrow{e_Y \otimes \mathbf{1}} \mathbf{1} \otimes X^* = X^*.$$

(b) If X, Y in $\tilde{\mathcal{C}}$ are rigid objects then the object $X \otimes Y$ is also rigid and the dual object $(X \otimes Y)^*$ is isomorphic to $Y^* \otimes X^*$.

Proof. (a) Is clear. To prove (b) we define the morphism $i_{X \otimes Y}: \mathbf{1} \rightarrow (X \otimes Y) \otimes (Y^* \otimes X^*)$, and $e_{X \otimes Y}: (Y^* \otimes X^*) \otimes (X \otimes Y) \rightarrow \mathbf{1}$ as the compositions

$$i_{X \otimes Y}: \mathbf{1} \xrightarrow{i_X} X \otimes X^* = X \otimes \mathbf{1} \otimes X^* \xrightarrow{\mathbf{1} \otimes i_Y} (X \otimes Y) \otimes (Y^* \otimes X^*),$$

$$e_{X \otimes Y}: (Y^* \otimes X^*) \otimes (X \otimes Y) \xrightarrow{1 \otimes e_X \otimes 1} Y^* \otimes \mathbf{1} \otimes Y = Y^* \otimes Y \xrightarrow{e_Y} \mathbf{1}.$$

It is easy to see that the triple $(Y^* \otimes X^*, i_{X \otimes Y}, e_{X \otimes Y})$ satisfies the conditions of Definition A.5.

Definition A.6. We say that an abelian weakly rigid monoidal category is semi-rigid if $\otimes$ is a biadditive right exact functor, the contravariant functor $*$ is exact, and $\mathbf{1}$ is a simple object in $\tilde{\mathcal{C}}$.

We will consider in this Appendix only semirigid abelian monoidal categories.

Definition A.7. We say that an object X in $\tilde{\mathcal{C}}$ is flat if the functor $Y \rightarrow Y \otimes X$ from $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}$ is exact.

For any objects A, B, C in $\tilde{\mathcal{C}}$ we denote by $\varphi_{A,B,C}: \text{Hom}(A, B \otimes C) \rightarrow \text{Hom}(B^* \otimes A, C)$ the composition

$$\text{Hom}(A, B \otimes C) \rightarrow \text{Hom}(B^* \otimes A, B^* \otimes B \otimes C) \xrightarrow{\text{id} \otimes \ell_B} \text{Hom}(B^* \otimes A, C).$$

Lemma A.4. If B is rigid then $\varphi_{A,B,C}$ is an isomorphism.

Proof. Let $\psi_{A,B,C}: \text{Hom}(B^* \otimes A, C) \rightarrow \text{Hom}(A, B \otimes C)$ be the composition

$$\text{Hom}(B^* \otimes A, C) \xrightarrow{B \otimes} \text{Hom}(B \otimes B^* \otimes A, B \otimes C) \xrightarrow{i_B} \text{Hom}(A, B \otimes C).$$

$$\text{Then } \varphi \circ \psi = \text{id}_{\text{Hom}(B^* \otimes A, C)}, \quad \psi \circ \varphi = \text{id}_{\text{Hom}(A, B \otimes C)}.$$

Corollary 1. If an object B in $\tilde{\mathcal{C}}$ is rigid, then it is flat.

Proof. Let $C_1 \xrightarrow{j} C_2$ be a monomorphism in $\tilde{\mathcal{C}}$. We have to prove that $B \otimes C_1 \xrightarrow{1 \otimes j} B \otimes C_2$ is a monomorphism. In other words, we have to show that for all $A \in \tilde{\mathcal{C}}$ the induced map $\text{Hom}(A, B \otimes C_1) \rightarrow \text{Hom}(A, B \otimes C_2)$ is a monomorphism. Since B is rigid, we have to show that $\text{Hom}(B^* \otimes A, C_1) \rightarrow \text{Hom}(B^* \otimes A, C_2)$ is a monomorphism. But this is clear since the morphism $C_1 \xrightarrow{j} C_2$ is a monomorphism.

Corollary 2. (a) If A is projective and B is rigid, then $B^* \otimes A$ is projective.

(b) If C is injective and B, B^* are rigid, then $B \otimes C$ is injective.

Proof. (a) Given any epimorphism $C_2 \rightarrow C_1$ we have to show that the natural map $\text{Hom}(B^* \otimes A, C_2) \rightarrow \text{Hom}(B^* \otimes A, C_1)$ is epimorphic. Since B is rigid, it is enough to show that the map $\text{Hom}(A, B \otimes C_2) \rightarrow \text{Hom}(A, B \otimes C_1)$ is epimorphic. This follows from the projectivity of A and the right exactness of $\otimes B$.

(b) Given a monomorphism $X \hookrightarrow Y$ in $\tilde{\mathcal{C}}$ we have to show that the map $\text{Hom}(Y, B \otimes C) \rightarrow \text{Hom}(X, B \otimes C)$ is surjective. Since B is rigid, it is sufficient to show that the map $\text{Hom}(B^* \otimes Y, C) \xrightarrow{\pi} \text{Hom}(B^* \otimes X, C)$ is surjective. Since B^* is rigid, we know from Corollary 1 that the map $B^* \otimes X \rightarrow B^* \otimes Y$ is a monomorphism and the surjectivity of π follows from the injectivity of C .

Corollary 3. An object B in $\tilde{\mathcal{C}}$ is rigid if and only if for all A, C , $\psi_{A,B,C}$ is an isomorphism.

Proof. The part “only if” is proven in Lemma A.3. To prove the “if” part it is sufficient to notice that we can construct $i_B: \mathbf{1} \rightarrow B \otimes B^*$ as $\psi_{\mathbf{1}, B, B^*}(\text{id}_{B^*})$.

Definition A.8. For any $A, B \in \tilde{\mathcal{C}}$ we define a morphism $\alpha = \alpha_{A, B}: B^* \otimes A^* \rightarrow (A \otimes B)^*$ as the preimage $\alpha = \left(\Lambda_{B^* \otimes A^*}^{A \otimes B} \right)^{-1}(\bar{\alpha})$, where $\bar{\alpha}: (B^* \otimes A^*) \otimes (A \otimes B) \rightarrow \mathbf{1}$ is defined as the composition

$$\bar{\alpha}: (B^* \otimes A^*) \otimes (A \otimes B) \xrightarrow{1 \otimes \ell_A \otimes 1} B^* \otimes \mathbf{1} \otimes B = B^* \otimes B \xrightarrow{\ell_B} \mathbf{1}.$$

Proposition A.1. If $B \in \tilde{\mathcal{C}}$ is such that B, B^* are rigid, then $\alpha_{A, B}$ is an isomorphism for all $A \in \tilde{\mathcal{C}}$.

Proof. We construct the inverse morphism $\beta_{A, B}: (A \otimes B)^* \rightarrow B^* \otimes A^*$ as the composition

$$\begin{aligned} \text{id}_{A^*} \in \text{Hom}(A, A) &\xrightarrow{i_B} \text{Hom}(A, A \otimes B \otimes B^*) \xrightarrow{\sim} \text{Hom}(((A \otimes B \otimes B^*)^*), A^*) \\ &\xrightarrow{\alpha_{A \otimes B, B^*}} \text{Hom}(B^{**} \otimes (A \otimes B)^*, A^*) \xrightarrow{\psi_{(A \otimes B)^*, B^*, A^*}} \text{Hom}((A \otimes B)^*, B^* \otimes A^*). \end{aligned}$$

It is easy to check that β is the inverse of α .

Definition A.9. (a) We say that an abelian category $\tilde{\mathcal{C}}$ has enough projective and injective objects if for any X in $\tilde{\mathcal{C}}$ there exists an epimorphism $P \twoheadrightarrow X$ and an imbedding $X \hookrightarrow I$ where P is projective and I is injective.

(b) We say that a semirigid abelian monoidal category $\mathcal{C}$ is a Frobenius category if

- (i) $\mathcal{C}$ has enough projective and injective objects.
- (ii) Any projective object is injective and any injective is projective.
- (iii) Any projective or injective object in $\mathcal{C}$ is rigid.

Proof. From now until the end of the proof of Proposition A.2 we assume that $\mathcal{C}$ is a Frobenius category.

Lemma A.5. If $X \in \tilde{\mathcal{C}}$ is rigid, then X^* is flat.

Proof. Let $A_1 \hookrightarrow A_2$ be a monomorphism. We have to show that the induced morphism from $X^* \otimes A_1$ to $X^* \otimes A_2$ is also a monomorphism. Since $\mathcal{C}$ has enough injectives, it is sufficient to show that for any injective C the map $\text{Hom}(X^* \otimes A_2, C) \rightarrow \text{Hom}(X^* \otimes A_1, C)$ is an epimorphism. Since X is rigid, it is sufficient to show that the map $\text{Hom}(A_2, X \otimes C) \xrightarrow{\pi} \text{Hom}(A_1, X \otimes C)$ is an epimorphism. We know by Corollary 2 to Lemma A.3 that the object $X \otimes C$ is injective, and therefore the surjectivity of π follows from the assumption that $A_1 \hookrightarrow A_2$ is a monomorphism. Lemma A.5 is proved.

Lemma A.6. Let $Z \in \tilde{\mathcal{C}}$ be a flat object in $\tilde{\mathcal{C}}$. Then for any exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ and any A in $\tilde{\mathcal{C}}$ the sequence $0 \rightarrow A \otimes X \rightarrow A \otimes Y \rightarrow A \otimes Z \rightarrow 0$ is exact.

Proof. Let $0 \rightarrow Q \rightarrow P \rightarrow A \rightarrow 0$ be an exact sequence in $\tilde{\mathcal{C}}$ such that P is projective. Then P is rigid and we have a diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & Q \otimes X & \xrightarrow{\partial_1^1} & Q \otimes Y & \xrightarrow{\partial_2^1} & Q \otimes Z & \longrightarrow 0 \\
 & \delta_1^1 \downarrow & & \downarrow \delta_2^1 & & \downarrow & \\
 0 \longrightarrow & P \otimes X & \xrightarrow{\partial_1^2} & P \otimes Y & \xrightarrow{\partial_2^2} & P \otimes Z & \longrightarrow 0 \\
 & \delta_1^2 \downarrow & & \downarrow \delta_2^2 & & \downarrow & \\
 & A \otimes X & \xrightarrow{\partial_1^3} & A \otimes Y & \xrightarrow{\partial_2^3} & A \otimes Z & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

To show that $\ker \partial_1^3 = 0$, it is sufficient to show that $\ker(\partial_1^3 \circ \delta_1^2) = \text{Im } \delta_1^1$. Let $C = \ker(\partial_1^3 \circ \delta_1^2) = \ker(\delta_2^2 \circ \partial_1^2) \subset P \otimes X$, $Q_X = Q \otimes X / \ker \delta_1^1 + \ker \partial_1^1$, $Q_Y = Q \otimes Y / \ker \delta_2^1$. We have the following diagram of exact sequences:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & Q_X & \xrightarrow{\bar{\partial}_1^1} & Q_Y & \xrightarrow{\bar{\partial}_2^1} & Q \otimes Z & \longrightarrow 0 \\
 & \bar{\delta}_1^1 \downarrow & & \downarrow \delta_2^1 & & \downarrow \delta_3^1 & \\
 0 \longrightarrow & P \otimes X & \xrightarrow{\partial_1^2} & P \otimes Y & \longrightarrow & P \otimes Z & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & A \otimes X & \longrightarrow & A \otimes Y & \longrightarrow & A \otimes Z & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

It is clear that there exists unique morphism $\varphi: C \rightarrow Q_Y$ such that $\delta_2^1 \circ \varphi = \partial_1^2|_C$. Since $\delta_3^1 \circ \bar{\partial}_2^1 \circ \varphi = 0$ we see that there exists a morphism $\bar{\varphi}: C \rightarrow Q_X$ such that $\varphi = \bar{\partial}_1^1 \circ \bar{\varphi}$. But then we have $\bar{\delta}_1^1 \circ \varphi = \text{Id}_C$. So $C = \text{Im } \bar{\delta}_1^1 = \text{Im } \delta_1^1$. Lemma A.6 is proved.

Corollary. Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be an exact sequence in $\tilde{\mathcal{C}}$.

- (a) If X and Z are flat, then Y is also flat.
- (b) If Y and Z are flat, then X is flat.

Proof. (a) We have to show that for any exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$

in $\tilde{\mathcal{E}}$ the map $A \otimes Y \rightarrow B \otimes Y$ is a monomorphism. Consider the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A \otimes X & \longrightarrow & B \otimes X & \longrightarrow & C \otimes X \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A \otimes Y & \xrightarrow{\partial} & B \otimes Y & \longrightarrow & C \otimes Y \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A \otimes Z & \longrightarrow & B \otimes Z & \longrightarrow & C \otimes Z \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where all vertical and horizontal lines are exact. Then it is clear that $\ker \partial = \{0\}$. Part (a) is proved. The proof of part (b) is completely analogous.

Proposition A.2. *If $\tilde{\mathcal{E}}$ is a Frobenius category and $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is an exact sequence in $\tilde{\mathcal{E}}$ such that two of its terms and the corresponding duals are rigid, then the same is true for the third term.*

Proof. (a) Assume that X, Z, X^*, Z^* are rigid objects in $\tilde{\mathcal{E}}$. Then we know that the objects X, X^*, Z, Z^* are flat. Therefore it follows from the corollary to Lemma A.6 that Y and Y^* also flat.

To prove that Y is rigid we have to show that for any $A, C \in \tilde{\mathcal{E}}$ the map

$$\mathrm{Hom}(A, Y \otimes C) \xrightarrow{\varphi_{A,Y,C}} \mathrm{Hom}(Y^* \otimes A, C)$$

is an isomorphism. By Lemma A.6 the sequences

$$\begin{aligned}
 0 \rightarrow X \otimes C \rightarrow Y \otimes C \rightarrow Z \otimes C \rightarrow 0, \\
 0 \leftarrow X^* \otimes A \leftarrow Y^* \otimes A \leftarrow Z^* \otimes A \leftarrow 0
 \end{aligned}$$

are exact. Therefore we have the following diagram:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(A, X \otimes C) & \xrightarrow{\sim} & \mathrm{Hom}(X^* \otimes A, C) \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(A, Y \otimes C) & \xrightarrow{\varphi_{A,Y,C}} & \mathrm{Hom}(Y^* \otimes A, C) \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(A, Z \otimes C) & \xrightarrow{\sim} & \mathrm{Hom}(Z^* \otimes A, C)
 \end{array}$$

This implies that $\varphi_{A,Y,C}$ is a monomorphism for all $A, C \in \tilde{\mathcal{E}}$ and is an isomorphism if A is projective.

In general, choose an exact sequence $0 \rightarrow Q \rightarrow P \rightarrow A \rightarrow 0$ such that P is

projective. Then we get:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \text{Hom}(A, Y \otimes C) & \xrightarrow{\varphi_{A,Y,C}} & \text{Hom}(Y^* \otimes A, C) \\
 \downarrow & & \downarrow \\
 \text{Hom}(P, Y \otimes C) & \xrightarrow[\sim]{\varphi_{A,Y,C}} & \text{Hom}(Y^* \otimes P, C) \\
 \downarrow & & \downarrow \\
 \text{Hom}(Q, Y \otimes C) & \xrightarrow[\hookrightarrow]{\varphi_{A,Q,C}} & \text{Hom}(Y^* \otimes Q, C)
 \end{array}$$

Since $\varphi_{A,P,C}$ is an isomorphism and $\varphi_{A,Q,C}$ is a monomorphism, we conclude that $\varphi_{A,Y,C}$ is an epimorphism. Therefore $\psi_{A,Y,C}$ is an isomorphism and Y is rigid. The same arguments show that Y^* is rigid.

(b) Assume that Y, Y^*, Z, Z^* are rigid.

First we show that X is rigid. By Lemma A.6 the exact sequences $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$, $0 \rightarrow Z^* \rightarrow Y^* \rightarrow X^* \rightarrow 0$ induce the exact sequences $0 \rightarrow X \otimes C \rightarrow Y \otimes C \rightarrow Z \otimes C \rightarrow 0$ and $Z^* \otimes A \rightarrow Y^* \otimes A \rightarrow X^* \otimes A \rightarrow 0$.

Therefore we have the diagram:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \text{Hom}(A, X \otimes C) & \xrightarrow{\varphi_{A,X,C}} & \text{Hom}(X^* \otimes A, C) \\
 \downarrow & & \downarrow \\
 \text{Hom}(A, Y \otimes C) & \xrightarrow{\sim} & \text{Hom}(Y^* \otimes A, C) \\
 \downarrow & & \downarrow \\
 \text{Hom}(A, Z \otimes C) & \xrightarrow{\sim} & \text{Hom}(Z^* \otimes A, C)
 \end{array}$$

This shows that the map $\varphi_{A,X,C}$ is an isomorphism, for all A and C in $\tilde{\mathcal{C}}$, and therefore X is rigid.

Now we have to show that X^* is rigid. We start with the proof of the following result.

Lemma A.7. (a) *The map $\varphi_{A,X^*,C}$ is a monomorphism if A is projective.*

(b) *The map $\varphi_{A,X^*,C}$ is an epimorphism if C is injective.*

Proof. (a) Assume that A is projective. Then it is rigid and flat and the exact sequences $0 \rightarrow X^{**} \rightarrow Y^{**} \rightarrow Z^{**} \rightarrow 0$ and $0 \rightarrow Z^* \rightarrow Y^* \rightarrow X^* \rightarrow 0$ induce exact sequences $0 \rightarrow X^{**} \otimes A \rightarrow Y^{**} \otimes A \rightarrow Z^{**} \otimes A \rightarrow 0$ and $Z^* \otimes C \rightarrow Y^* \otimes C \rightarrow X^* \otimes C \rightarrow 0$, and $Z^* \otimes C \rightarrow Y^* \otimes C$ is a monomorphism if C is injective.

Therefore we have a diagram:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(A, Z^* \otimes C) & \xrightarrow{\sim} & \mathrm{Hom}(Z^{**} \otimes A, C) \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(A, Y^* \otimes C) & \xrightarrow{\sim} & \mathrm{Hom}(Y^{**} \otimes A, C) \\
 \downarrow \delta_A & & \downarrow \delta_C \\
 \mathrm{Hom}(A, X^* \otimes C) & \xrightarrow{\varphi_{A, X^*, C}} & \mathrm{Hom}(X^{**} \otimes A, C)
 \end{array}$$

It follows from the “5-lemma” [CE, Proposition 1.1] that $\varphi_{A, X^*, C}$ is a monomorphism. Part (a) is proved. The proof of part (b) is analogous.

Corollary. *If A is projective and C is injective, then $\psi_{A, X^*, C}$ is an isomorphism.*

Lemma A.8. *If A is projective, then the map $\psi_{A, X^*, C}$ is an isomorphism.*

Proof. Let $0 \rightarrow C \rightarrow I \rightarrow J$ be an injective resolution of C .

Then the sequences $X^* \otimes C \rightarrow X^* \otimes I \rightarrow X^* \otimes J$ is exact and therefore we have a diagram:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(A, X^* \otimes C) & \xrightarrow{\varphi_{A, X^*, C}} & \mathrm{Hom}(X^{**} \otimes A, C) \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(A, X^* \otimes I) & \xrightarrow{\sim} & \mathrm{Hom}(X^{**} \otimes A, I) \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(A, X^* \otimes J) & \xrightarrow{\sim} & \mathrm{Hom}((X^{**} \otimes A), J)
 \end{array}$$

This implies that $\psi_{A, X^*, C}$ is an isomorphism. Lemma A.8 is proved.

Lemma A.9. *For all A, C in $\tilde{\mathcal{C}}$ the map $\varphi_{A, X^*, C}$ is an isomorphism.*

Proof. Let $Q \rightarrow P \rightarrow A \rightarrow 0$ be an exact sequence such that P, Q are projective. Since the object X^{**} is isomorphic to X , it is flat and we have a diagram

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(A, X^* \otimes C) & \xrightarrow{\psi_{A, X^*, C}} & \mathrm{Hom}(X^{**} \otimes A, C) \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(P, X^* \otimes C) & \xrightarrow{\sim} & \mathrm{Hom}(X^{**} \otimes P, C) \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}(Q, X^* \otimes C) & \xrightarrow{\sim} & \mathrm{Hom}(X^{**} \otimes Q, C)
 \end{array}$$

Therefore, $\psi_{A, X^*, C}$ is an isomorphism.

As follows from Corollary 3 to Lemma A.3, the object X^* is rigid. So part (b) of Proposition A.1 is proved. To finish the proof of Proposition A.1 we have to show that the objects X, X^*, Y, Y^* in $\tilde{\mathcal{C}}$ being rigid implies that Z and Z^* are also rigid. But $*$ is an exact functor and therefore the sequence $0 \rightarrow Z^* \rightarrow Y^* \rightarrow X^* \rightarrow 0$. So the rigidity of Z_* and Z^* follows from part (b). Proposition A.1 is proved.

As before, let $\mathcal{C}$ be a semirigid monoidal Frobenius category. Let X, Y be rigid objects in $\mathcal{C}$, and let

$$i_{X \otimes Y}: \mathbf{1} \longrightarrow X \otimes Y \otimes (X \otimes Y)^* \xrightarrow{\sim} X \otimes Y \otimes Y^* \otimes KX^*$$

be the canonical imbedding. We denote by $j \in \text{Hom}_{\mathcal{C}}(X^* \otimes X, Y \otimes Y^*)$ the morphism defined as the composition

$$X^* \otimes X \xrightarrow{\sim} X^* \otimes \mathbf{1} \otimes X \xrightarrow{1 \otimes i} X^* \otimes X \otimes Y \otimes Y^* \otimes X^* \otimes X \xrightarrow{e_X \otimes 1 \otimes e_X} Y \otimes Y^*.$$

Lemma A.10. $j = i_Y \circ e_X$.

Proof. Easy.

Let $\mathcal{C}$ be an arbitrary monoidal semirigid category, $\tilde{\mathcal{C}}^0 \subset \tilde{\mathcal{C}}$ a full subcategory closed under $\otimes$ such that $\mathbf{1}$ lies in $\tilde{\mathcal{C}}^0$. Then we have a natural structure of a monoidal category on $\tilde{\mathcal{C}}^0$. We denote this category by $\mathcal{C}^0$.

Lemma A.11. Assume that $\mathcal{C}^0$ is a rigid Frobenius category and for any object X in $\tilde{\mathcal{C}}$ there exists Y in $\tilde{\mathcal{C}}$ such that $X \otimes Y$ lies in $\tilde{\mathcal{C}}^0$. Then any object in $\tilde{\mathcal{C}}$ is flat.

Proof. We divide the proof into three steps.

(a) It is clear that for any nonzero object X in $\tilde{\mathcal{C}}$ the map $e_X: X^* \otimes X \rightarrow \mathbf{1}$ is not a zero map and is therefore surjective. The surjectivity of e_X implies the surjectivity of $e_X \otimes 1_Y: X^* \otimes (X \otimes Y) \rightarrow Y$ and therefore $X \otimes Y \neq \{0\}$.

(b) Let $A \hookrightarrow B$ be a monomorphism in $\tilde{\mathcal{C}}$. We want to show that for any X in $\tilde{\mathcal{C}}$ the map $\varphi: A \otimes X \rightarrow B \otimes X$ is nonzero. Let $Y \in \tilde{\mathcal{C}}$ be an object such that $X \otimes Y$ is in $\tilde{\mathcal{C}}^0$. Then $X \otimes Y$ is rigid. Therefore $X \otimes Y$ is flat and the morphism $(A \otimes X) \otimes Y \xrightarrow{\varphi \otimes 1} (B \otimes X) \otimes Y$ is an imbedding; since $A \otimes X \otimes Y \neq \{0\}$ we see that $\varphi \neq 0$.

(c) We want to show that φ is a monomorphism. Let $C = \ker \varphi$. Then we have an exact sequence

$$C \otimes Y \xrightarrow{\psi} A \otimes (X \otimes Y) \longrightarrow B \otimes (X \otimes Y).$$

Since $X \otimes Y$ is flat, we see that $\psi = 0$. It follows from b) that $C = \{0\}$. Lemma A.11 is proved.

Conjecture A.1. If X, Y are objects of $\tilde{\mathcal{C}}$ such that $X \otimes Y$ is rigid, then X and Y are rigid.

Unfortunately we do not know how to prove this conjecture in general, and we will prove it for the special class of liftable categories.

Let $R = \mathbb{C}[[t]]$ be the ring of formal power series, $F = \mathbb{C}((t))$ its field of fractions, and $\pi: R \rightarrow \mathbb{C}$, $j: R \hookrightarrow F$ the natural projection and surjection. We say that an additive monoidal category $\mathcal{D} = (\tilde{\mathcal{D}}, \otimes, \tilde{b})$ is given a structure of a monoidal R -category if for any X, Y in $\tilde{\mathcal{D}}$ we are given a structure of R -module on $\text{Hom}_{\tilde{\mathcal{D}}}(X, Y)$ in such a way that the compositions, $\otimes$ and $\tilde{b}$, are compatible with this R -module structure. We say that an object X of an R -category $\tilde{\mathcal{D}}$ is torsion free if the morphism $t1_X \in \text{Hom}_{\tilde{\mathcal{D}}}(X, X)$ is a monomorphism. Given an R -category $\mathcal{D} = (\tilde{\mathcal{D}}, \otimes, \tilde{b})$ we denote by $\mathcal{D}_\mathbb{C}$ and $\mathcal{D}_F$ the categories with the same objects as $\tilde{\mathcal{D}}$ and morphisms defined by the rules

$$\begin{aligned}\text{Hom}_{\mathcal{D}_\mathbb{C}}(X, Y) &= \text{Hom}_{\tilde{\mathcal{D}}}(X, Y) \otimes_R \mathbb{C}, \\ \text{Hom}_{\mathcal{D}_F}(X, Y) &= \text{Hom}_{\tilde{\mathcal{D}}}(X, Y) \otimes_R F.\end{aligned}$$

For any X in $\tilde{\mathcal{D}}$ we denote by $\bar{X}$ and $\hat{X}$ the corresponding objects in $\mathcal{D}_\mathbb{C}$ and $\mathcal{D}_F$.

It is clear that $\otimes, b$ define monoidal structures $\mathcal{D}_\mathbb{C} = (\tilde{\mathcal{D}}_\mathbb{C}, \otimes_\mathbb{C}, b_\mathbb{C})$ and $\mathcal{D}_F = (\tilde{\mathcal{D}}_F, \otimes_F, b_F)$ on categories $\tilde{\mathcal{D}}_\mathbb{C}$ and $\tilde{\mathcal{D}}_F$.

Definition A.10. Given a semirigid monoidal Frobenius category $\mathcal{C} = (\mathcal{C}, \otimes, a)$ we say that $\mathcal{C}$ is liftable if there exists a semirigid R -monoidal category $\mathcal{D} = (\tilde{\mathcal{D}}, \otimes, b)$ such that all objects in $\tilde{\mathcal{D}}$ are torsion free, the category $\mathcal{D}_F$ is rigid, and there exists equivalence between $\mathcal{D}_\mathbb{C}$ with a full subcategory of $\mathcal{C}$. We say that an object $\bar{X}$ in $\mathcal{C}$ is liftable if there exists an object X in $\tilde{\mathcal{D}}$ such that $\bar{X}$ is isomorphic to the image of X in $\mathcal{D}_\mathbb{C}$. We denote by $\hat{X}$ the corresponding object in $\mathcal{D}_F$.

For any object X in $\tilde{\mathcal{D}}$ we define $i_X \in \text{Hom}_{\mathcal{D}_F}(\mathbf{1}, \hat{X} \otimes \hat{X}^*)$ as in Definition A.5 and denote by $a_X \in \mathbb{N}$ the smallest number such that

$$t^{a_X} i_X \in \text{Hom}_{\tilde{\mathcal{D}}}(\mathbf{1}, X \otimes X^*) \subset \text{Hom}_{\mathcal{D}_F}(\mathbf{1}, \hat{X} \otimes \hat{X}^*).$$

We define $\check{i}_X \stackrel{\text{def}}{=} t^{a_X} i_X \in \text{Hom}_{\tilde{\mathcal{D}}}(\mathbf{1}, X \otimes X^*)$.

Let $\bar{i}_X: \mathbf{1} \rightarrow \bar{X} \otimes \bar{X}^*$ be the corresponding morphism in $\mathcal{D}_\mathbb{C}$. It is clear that $\bar{i}_X \neq 0$. Therefore $\bar{i}_X$ is a monomorphism.

For any Y in $\mathcal{D}$ we denote by $\alpha_{X,Y}: Y^* \otimes X^* \rightarrow (X \otimes Y)^*$ the morphisms as in Definition A.5. Since $\mathcal{D}_F$ is rigid, $\alpha_{X,Y}$ induces an isomorphism $\alpha_{X,Y}^F: \hat{Y}^* \otimes \hat{X}^* \rightarrow (X \hat{\otimes} Y)^*$ and we can define a map $\beta_{X,Y} \stackrel{\text{def}}{=} t^{a_X} (\alpha_{X,Y}^F)^{-1} \in \text{Hom}_{\mathcal{D}_F}((X \hat{\otimes} Y)^*, \hat{Y}^* \otimes \hat{X}^*)$.

Lemma A.12. *An object X in $\tilde{\mathcal{D}}$ is rigid if and only if $\bar{X}$ is a rigid object of $\mathcal{D}_\mathbb{C}$.*

Proof. Clear.

Lemma A.13.

$$\beta_{X,Y} \in \text{Hom}_{\mathcal{D}_F}((X \otimes Y)^*, Y^* \otimes X^*).$$

Proof. Since $\mathcal{D}$ is weakly rigid, for any Z in $\tilde{\mathcal{D}}$ we have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(Z, X \otimes Y) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}((X \otimes Y)^* \otimes Z, \mathbf{1}), \\ \mathrm{Hom}_{\mathcal{D}}(X^* \otimes Z, Y) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(Y^* \otimes X^* \otimes Z, \mathbf{1}) \end{aligned}$$

and it is easy to see that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(Z, X \otimes Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}}((X \otimes Y)^* \otimes Z, \mathbf{1}) \\ \downarrow \phi_{Z, X, Y} & & \downarrow {}^t\alpha_{X, Y} \\ \mathrm{Hom}_{\mathcal{D}}(X^* \otimes Z, Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}}(Y^* \otimes X^* \otimes Z, \mathbf{1}) \end{array}$$

is commutative.

Let

$$\psi_{Z, X, Y}^F: \mathrm{Hom}_{\mathcal{D}_F}(\hat{X}^* \otimes \hat{Z}, \hat{Y}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_F}(\hat{Z}, \hat{X} \otimes \hat{Y})$$

be the inverse of $\phi_{Z, X, Y}^F$ which exists by Lemma A.4. As follows immediately from the proof of Lemma A.4, $t^{a_X} \psi_{Z, X, Y}^F$ defines a morphism from $\mathrm{Hom}_{\mathcal{D}}(X^* \otimes Z, Y)$ to $\mathrm{Hom}_{\mathcal{D}}(Z, X \otimes Y)$. Therefore for any Z in $\mathcal{D}$ the image of

$${}^t\beta_{X, Y}(\mathrm{Hom}_{\mathcal{D}}(Y^* \otimes X^* \otimes Z, \mathbf{1})) \subset \mathrm{Hom}_{\mathcal{D}_F}((X \hat{\otimes} Y)^* \otimes \hat{Z}, \mathbf{1})$$

lies in the submodule $\mathrm{Hom}_{\mathcal{D}}((X \otimes Y)^* \otimes Z, \mathbf{1})$. Since Z is arbitrary it follows from Lemma A.2 that $\beta_{X, Y}$ lies in the submodule

$$\mathrm{Hom}_{\mathcal{D}}((X \otimes Y)^*, Y^* \otimes X^*) \subset \mathrm{Hom}_{\mathcal{D}_F}((X \hat{\otimes} Y)^*, \hat{Y}^* \otimes \hat{X}^*).$$

Lemma A.13 is proved.

Proposition A.3. *Let X, Y in $\tilde{\mathcal{D}}$ be such that $X \otimes Y$ is rigid. Then X and Y are rigid.*

Proof. Let $a_X, a_Y \in \mathbb{N}$ be defined as before. We have to show that $a_X = a_Y = 0$. We assume that $a_Y > 0$ and deduce the contradiction. Since the object $X \otimes Y$ is rigid there exists a morphism $i: \mathbf{1} \rightarrow (X \otimes Y) \otimes (X \otimes Y)^*$ satisfying the condition of Definition A.5. Consider the morphism $j \stackrel{\mathrm{def}}{=} (1 \otimes \beta_{X, Y}) \circ i$ from $\mathbf{1}$ to $X \otimes Y \otimes Y^* \otimes X^*$. On the other hand, we denote by $\tilde{j}$ the morphism from $\mathbf{1}$ to $X \otimes Y \otimes Y^* \otimes X^*$ defined as the composition

$$\mathbf{1} \xrightarrow{i_X} X \otimes X^* \otimes X \otimes \mathbf{1} \otimes X^* \xrightarrow{1 \otimes i_Y \otimes 1} X \otimes Y \otimes Y^* \otimes X^*.$$

It is clear that $\tilde{j} = t^{a_Y} j$. Since $a_Y > 0$ we have that $\tilde{j} = 0$ where $\tilde{j} \in \mathrm{Hom}_{\mathcal{D}_C}(\mathbf{1}, \bar{X} \otimes \bar{Y} \otimes \bar{Y}^* \otimes \bar{X}^*)$ is the morphism corresponding to $\tilde{j}$. On the other hand, $\tilde{j}$ is equal to the composition

$$\mathbf{1} \xrightarrow{\bar{i}_X} \bar{X} \otimes \bar{X}^* = \bar{X} \otimes \mathbf{1} \otimes \bar{X}^* \xrightarrow{1 \otimes \bar{i}_Y \otimes 1} \bar{X} \otimes \bar{Y} \otimes \bar{Y}^* \otimes \bar{X}^*.$$

Since $\bar{i}_X$ and $\bar{i}_Y$ are monomorphisms and $\bar{X} \otimes \bar{Y}$ is rigid, it follows from Lemma A.11 that $\tilde{j}$ is a monomorphism. This contradiction shows that $a_Y =$

0. The proof of the equality $a_X = 0$ is completely analogous. Proposition A.3 is proved.

Definition A.11. Let $\mathcal{E} = (\tilde{\mathcal{E}}, \otimes, \mathbf{1})$ be a strict monoidal category.

(1) A braiding s on $\mathcal{E}$ is a natural transformation between the functors $\otimes$ and $\otimes \circ P$, where $P: \tilde{\mathcal{E}} \times \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}} \times \tilde{\mathcal{E}}$ is the permutation $(X, Y) \rightarrow (Y, X)$ (so $s = (s_{X,Y})$ where for X, Y in $\tilde{\mathcal{E}}$, $s_{X,Y} \in \text{Hom}(X \otimes Y, Y \otimes X)$) such that (a) all $s_{X,Y}$ are isomorphisms, (b) for any X, Y, Z in $\tilde{\mathcal{E}}$ we have

$$s_{X \otimes Y, Z} = (s_{X,Z} \otimes 1_Y) \circ (1_X \otimes s_{Y,Z}), \quad s_{X, Y \otimes Z} = (1_X \otimes s_{X,Z}) \circ (s_{X,Y} \otimes 1_Z),$$

and (c) $s_{X, \mathbf{1}} = s_{\mathbf{1}, X} = 1_X$ for all X in $\tilde{\mathcal{E}}$.

(2) A balancing b on a braided category $\mathcal{E} = (\tilde{\mathcal{E}}, \otimes, \mathbf{1}, s)$ is an automorphism of the identity functor such that for any objects X, Y of $\tilde{\mathcal{E}}$ the automorphisms $s_{Y,X} \circ s_{X,Y}$ and $b_{X \otimes Y} \circ (b_X \otimes b_Y)^{-1}$ of $X \otimes Y$ coincide.

It is clear how to define a notion of a braiding and balancing for an arbitrary monoidal category.

Definition A.12. Let $\mathcal{E} = (\tilde{\mathcal{E}}, \otimes_{\mathcal{E}}, a^{\mathcal{E}}, s^{\mathcal{E}})$ and $\mathcal{D} = (\tilde{\mathcal{D}}, \otimes_{\mathcal{D}}, a^{\mathcal{D}}, s^{\mathcal{D}})$ be braided abelian categories. A braided functor from $\mathcal{D}$ to $\mathcal{E}$ is a pair $(\tilde{X}, M)$, where $\tilde{X}$ is a left exact functor from $\tilde{\mathcal{D}}$ to $\tilde{\mathcal{E}}$ and $M = M_{A,B}$ is a functorial isomorphism $M_{A,B}: \tilde{X}(A) \otimes_{\mathcal{E}} \tilde{X}(B) \xrightarrow{\sim} \tilde{X}(A \otimes_{\mathcal{D}} B)$ such that the diagrams

$$\begin{array}{ccc} \tilde{X}(A) \otimes_{\mathcal{E}} \tilde{X}(B) & \xrightarrow{M_{A,B}} & \tilde{X}(A \otimes_{\mathcal{D}} B) \\ \downarrow s_{\tilde{X}(A), \tilde{X}(B)}^{\mathcal{E}} & & \downarrow \tilde{X}(s_{A,B}^{\mathcal{D}}) \\ \tilde{X}(B) \otimes_{\mathcal{E}} \tilde{X}(A) & \xrightarrow{M_{B,A}} & \tilde{X}(B \otimes_{\mathcal{D}} A) \end{array}$$

and

$$\begin{array}{ccc} (\tilde{X}(A) \otimes_{\mathcal{E}} \tilde{X}(B)) \otimes_{\mathcal{E}} \tilde{X}(C) & \xrightarrow{a^{\mathcal{E}}} & \tilde{X}(A) \otimes_{\mathcal{E}} (\tilde{X}(B) \otimes_{\mathcal{E}} \tilde{X}(C)) \\ \downarrow M \otimes 1 & & \downarrow 1 \otimes M \\ \tilde{X}(A \otimes_{\mathcal{D}} B) \otimes_{\mathcal{E}} \tilde{X}(C) & & \tilde{X}(A) \otimes_{\mathcal{E}} \tilde{X}(B \otimes_{\mathcal{D}} C) \\ \downarrow M & & \downarrow M \\ \tilde{X}((A \otimes_{\mathcal{D}} B) \otimes_{\mathcal{D}} C) & \xrightarrow{\tilde{X}(a^{\mathcal{D}})} & \tilde{X}(A \otimes_{\mathcal{D}} (B \otimes_{\mathcal{D}} C)) \end{array}$$

are commutative for all A, B, C in $\mathcal{D}$ and $(\tilde{X}(U), \tilde{X}(u))$ is an identity object in $\mathcal{E}$ for any identity object (U, u) in $\mathcal{D}$. From now on we assume that braided categories $\mathcal{E}$ and $\mathcal{D}$ are strict as monoidal categories and $\tilde{X}(\mathbf{1}_{\mathcal{D}}) = \mathbf{1}_{\mathcal{E}}$ and $\mathbf{1}_{\mathcal{D}}, \mathbf{1}_{\mathcal{E}}$ are simple objects in $\mathcal{D}$ and $\mathcal{E}$.

Lemma A.14. Let $\mathcal{E}, \mathcal{D}$ be rigid braided categories and $X = (\tilde{X}, M)$ be a braided functor from $\mathcal{D}$ to $\mathcal{E}$. Then

(a) For any A in $\mathcal{D}$ the morphisms

$$\mathbf{1}_{\mathcal{E}} = \tilde{X}(\mathbf{1}_{\mathcal{D}}) \xrightarrow{\tilde{X}(i_A)} \tilde{X}(A \otimes_{\mathcal{D}} A^*) \xrightarrow{M^{-1}} \tilde{X}(A) \otimes_{\mathcal{E}} \tilde{X}(A^*)$$

and

$$\tilde{X}(A^*) \otimes_{\mathcal{E}} \tilde{X}(A) \xrightarrow{M} \tilde{X}(A^* \otimes_{\mathcal{D}} A) \xrightarrow{\tilde{X}(\ell_A)} \tilde{X}(\mathbf{1}_{\mathcal{D}}) = \mathbf{1}_{\mathcal{E}}$$

satisfy conditions of Definition A.5. Therefore we can identify $(\tilde{X}(A))^*$ with $\tilde{X}(A^*)$.

(b) The functor $\tilde{X}$ is exact.

(c) For any nonzero object A in $\mathcal{D}$, $\tilde{X}(A)$ is a nonzero object.

(d) The functor $\tilde{X}$ is faithful.

Proof. (a) Clear.

(b) We have to show that $\tilde{X}$ is right exact. In other words, we have to show that for any epimorphism $\alpha: A \rightarrow B$ in $\mathcal{D}$ the morphism $\tilde{X}(\alpha): \tilde{X}(A) \rightarrow \tilde{X}(B)$ is also an epimorphism. Since the morphism $*$: $\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}^{\mathfrak{g}}$ is an equivalence of categories, it is sufficient to prove that the morphism $(\tilde{X}(\alpha))^*: (\tilde{X}(B))^* \rightarrow (\tilde{X}(A))^*$ is a monomorphism. As follows from part (a),

$$(\tilde{X}(\alpha))^* = \tilde{X}(\alpha^*): \tilde{X}(B^*) \rightarrow \tilde{X}(A^*).$$

Since α is an epimorphism, α^* is a monomorphism and the left exactness of $\tilde{X}(\alpha)$ implies that $\tilde{X}(\alpha^*)$ is a monomorphism. Part (b) is proved.

(c) If A is a nonzero object in $\mathcal{D}$ then the morphism $i_A: \mathbf{1}_{\mathcal{D}} \rightarrow A \times_{\mathcal{D}} A^*$ is nonzero and therefore i_A is a monomorphism. Since $\tilde{X}$ is an exact functor we see that the morphism

$$\mathbf{1}_{\mathcal{E}} = \tilde{X}(\mathbf{1}_{\mathcal{D}}) \xrightarrow{\tilde{X}(i_A)} \tilde{X}(A \otimes_{\mathcal{D}} A^*) \xrightarrow{M^{-1}} \tilde{X}(A \otimes_{\mathcal{E}} A^*)$$

is also a monomorphism. Therefore $\tilde{X}(A)$ is a nonzero object.

(d) Let $\alpha: A \rightarrow B$ be a nonzero morphism in $\mathcal{D}$. Then we write α as a composition $\alpha = \gamma \circ \beta$, where $\beta: A \rightarrow C$ is an epimorphism in $\mathcal{D}$ and $\gamma: C \rightarrow B$ is a monomorphism. Then $\tilde{X}(\alpha) = \tilde{X}(\gamma) \circ \tilde{X}(\beta)$, where (by (b)) $\tilde{X}(\beta): \tilde{X}(A) \rightarrow \tilde{X}(C)$ is an epimorphism and $\tilde{X}(\gamma): \tilde{X}(C) \rightarrow \tilde{X}(B)$ is a monomorphism. Since (by (c)) $\tilde{X}(C)$ is a nonzero object in $\mathcal{E}$ we have $\tilde{X}(\alpha) \neq 0$. Lemma A.14 is proved.

Definition A.13. (a) We say that an abelian category $\tilde{\mathcal{D}}$ is Artinian if any object in $\tilde{\mathcal{D}}$ has a finite filtration with simple quotients W .

(b) Let $\tilde{\mathcal{D}}$ be an Artinian category and P_j , $j \in J$, be a system of projective objects in $\tilde{\mathcal{D}}$. We say that such a system P_j , $j \in J$, is ample if for any irreducible object L in $\tilde{\mathcal{D}}$ there exists $j_0 \in J$ such that $\text{Hom}_{\tilde{\mathcal{D}}}(P_{j_0}, L) \neq \langle 0 \rangle$.

Lemma A.15. Let $\tilde{\mathcal{D}}, \tilde{\mathcal{E}}$ be abelian categories, $\tilde{X}: \tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{E}}$ an exact faithful functor, P_j , $j \in J$, an ample system of projective objects in $\tilde{\mathcal{D}}$ such that $Q_j \stackrel{\text{def}}{=} \tilde{X}(P_j)$ is an ample system of projective objects in $\tilde{\mathcal{E}}$ and for all $j, j' \in J$, the

map $\tilde{X}: \text{Hom}_{\mathcal{D}}(P_j, P, \cdot) \rightarrow \text{Hom}_{\mathcal{E}}(Q_j, Q, \cdot)$ is an isomorphism. Then $\tilde{X}$ is an equivalence of categories.

Proof. We have to show that

(a) For any A, A' in $\tilde{\mathcal{D}}$ the map $\tilde{X}: \text{Hom}_{\mathcal{D}}(A, A') \rightarrow \text{Hom}_{\mathcal{E}}(\tilde{X}(A), \tilde{X}(A'))$ is an isomorphism.

(b) For any object B in $\mathcal{E}$ there exists an object A in $\tilde{\mathcal{D}}$ such that $\tilde{X}(A)$ is isomorphic to B .

Proof of part (a). Let $P_\gamma, \gamma \in P$, be the system of all objects in $\mathcal{D}$ which is isomorphic to finite direct sums of $P_j, j \in J$, and $Q_\gamma \stackrel{\text{def}}{=} \tilde{X}(P_\gamma), \gamma \in P$. Since P is an ample system of projective objects and the categories $\mathcal{E}$ and $\tilde{\mathcal{D}}$ are Artinian it is easy to see that for any object A in $\tilde{\mathcal{D}}$ there exist $\gamma \in \Gamma$ and a surjection from P_γ to A and for any object B in $\mathcal{E}$ there exist $\gamma \in \Gamma$ and a surjection from Q_γ to B .

We first prove part (a) in the case when $A = P_\gamma$ for some $\gamma \in P$.

As follows from the previous remark, we can construct an exact sequence $P_{\gamma''} \xrightarrow{\alpha''} P_{\gamma'} \xrightarrow{\alpha'} A' \rightarrow 0$ for some $\gamma', \gamma'' \in P$. Since P_γ and Q_γ are projective objects and $\tilde{X}$ is exact we have the diagram:

$$\begin{array}{ccccccc} \text{Hom}_{\mathcal{D}}(P_\gamma, P_{\gamma''}) & \rightarrow & \text{Hom}_{\mathcal{D}}(P_\gamma, P_{\gamma'}) & \rightarrow & \text{Hom}_{\mathcal{D}}(P_\gamma, A') & \rightarrow & 0 \\ \downarrow \tilde{X}_{\gamma''} & & \downarrow \tilde{X}_{\gamma'} & & \downarrow \tilde{X}_{A'} & & \\ \text{Hom}_{\mathcal{E}}(Q_\gamma, Q_{\gamma''}) & \rightarrow & \text{Hom}_{\mathcal{E}}(Q_\gamma, Q_{\gamma'}) & \rightarrow & \text{Hom}_{\mathcal{E}}(Q_\gamma, A') & \rightarrow & 0 \end{array}$$

Since $\tilde{X}_{\gamma''}, \tilde{X}_{\gamma'}$ are isomorphisms, it follows from “5-lemma” that $\tilde{X}_{A'}$ is also an isomorphism. Now we can prove part (a) for all A, A' in $\mathcal{D}$ by choosing a resolution $P_{\gamma'} \rightarrow P_\gamma \rightarrow A \rightarrow 0, \gamma, \gamma' \in P$, and using the same arguments as before. Part (a) is proved.

Proof of part (b). For any object B in $\mathcal{E}$ we can find an exact sequence $Q_{\gamma'} \xrightarrow{\beta} Q_\gamma \rightarrow B \rightarrow 0, \gamma, \gamma' \in \Gamma$. Let $\alpha \in \text{Hom}_{\mathcal{D}}(P_{\gamma'}, P_\gamma)$ be a morphism such that $\tilde{X}(\alpha) = \beta$ and $A = \text{Coker } \alpha$. Since $\tilde{X}$ is an exact functor, the isomorphisms $\tilde{X}(P_{\gamma'}) \xrightarrow{\sim} Q_{\gamma'}, \tilde{X}(P_\gamma) \xrightarrow{\sim} Q_\gamma$ define an isomorphism $\tilde{X}(A) \xrightarrow{\sim} B$. Lemma A.15 is proved.

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