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IMMANANT INEQUALITIES AND 0-WEIGHT SPACES

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1. Introduction

Let $n \in \mathbb{N}$, and let $A = (a_{ij})$ be an $n \times n$ complex matrix. Let S_n be the group of permutations of $\mathscr{N} = \{1, \ldots, n\}$, and let $\mathscr{P}_n$ be the set of all partitions of n. As one knows, to each $\lambda \in \mathscr{P}_n$ we may associate an irreducible complex representation $\nu_\lambda : S_n \to \operatorname{Aut} Y_\lambda$ such that, using standard notation, χ^λ is the character of ν_λ. Generalizing the determinant and permanent of A one defines the immanants of A by putting, for any $\lambda \in \mathscr{P}_n$,

$$\operatorname{Imm}_\lambda(A) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) a_{1\sigma 1} \cdots a_{n\sigma n}.$$

According to [GJ], although utilized much earlier by Schur, the term Immanant was introduced by Littlewood. In fact, in [Li], Littlewood uses immanants to define Schur functions. Indeed given any $g \in Gl(m, \mathbb{C})$ Littlewood (see (6.2;7) in [Li]) constructs a matrix $Z(g) \in M(n, \mathbb{C})$, using power sums of the eigenvalues of g, such that, for any $\lambda \in \mathscr{P}_n$,

$$\operatorname{Imm}_\lambda(Z(g)) = \frac{1}{n!} \operatorname{tr} \pi_\lambda(g), \tag{1}$$

where $\nu_\lambda \times \pi_\lambda$ occurs as an irreducible component of the reduction of $\otimes^n \mathbb{C}^m$ under the natural action of $S_n \times Gl(m, \mathbb{C})$.

A recent paper by Haiman [H] dealt with immanant inequality results and conjectures. Cited in particular were a result of Schur and conjectures and results of Stembridge. For any $\lambda \in \mathscr{P}_n$, let $f_\lambda = \dim \nu_\lambda$. Consider the validity of the statement: For any $\lambda \in \mathscr{P}_n$

$$\operatorname{Imm}_\lambda(A) \geq f_\lambda \tag{2}$$

for a matrix $A \in Sl(n, \mathbb{C})$ (to which one is readily reduced). Two of the results stated in [H] for $Sl(n, \mathbb{C})$ are as follows (see [Sc] and [St]):

Theorem 1 (Schur). *One has* (2) *if A is positive definite.*

A matrix A is called totally positive if all square minors are non-negative.

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Theorem 2 (Stembridge). *One has* (2) *if* A *is totally positive.*

In this paper spotlighting an interpretation of immanants based on 0-weight spaces, we give as an application a generalization, for all representations of $SL(n, \mathbb{C})$, of Theorems 1 and 2. In the case of Schur's theorem the generalization is straightforward. For the case of Stembridge's theorem we rely on a result of A. Whitney on the structure of totally positive matrices and a very deep result of George Lusztig on the coefficient non-negativity for the action of certain semigroups with respect to the canonical basis (see Theorem 22.1.7 in [Lu]).

The notation underlying the statement (2) deals with arbitrary $n, m \in \mathbb{N}$. Now assume that $m = n$. Furthermore regard π_λ as a representation of $Sl(n, \mathbb{C})$. Thus, as one knows, $\otimes^n \mathbb{C}^n$ is a multiplicity free $(S_n \times Sl(n, \mathbb{C}))$-module and its complete reduction into irreducible components can be written as

$$\otimes^n \mathbb{C}^n = \sum_{\lambda \in \mathscr{P}_n} Y_\lambda \otimes V_\lambda, \tag{3}$$

where $\pi_\lambda : Sl(n, \mathbb{C}) \to \operatorname{Aut} V_\lambda$ is an irreducible representation of $Sl(n, \mathbb{C})$. Let H be the Cartan subgroup of all diagonal matrices in $Sl(n, \mathbb{C})$ so that V_λ^H is the 0-weight space for the representation π_λ. Let $P_\lambda : V_\lambda \to V_\lambda^H$ be that projection operator on the 0-weight space which commutes the action of $\pi_\lambda(H)$.

We first establish Theorem 3 below—a 0-weight interpretation of immanants, for matrices $A \in Sl(n, \mathbb{C})$. If we extend the action of π_λ to be a multiplication preserving map of $M(n, \mathbb{C})$, then our proof of Theorem 3 in fact establishes the same characterization of immanants for any $n \times n$ matrix. The first statement of Theorem 3 is known. The second statement, although easy to prove (e.g., it is deducible from §II.2 in [B]), appears to be new.

Theorem 3. *Let* $\lambda \in \mathscr{P}_n$. *Then*

$$\dim V_\lambda^H = f_\lambda. \tag{4}$$

Now let $A \in Sl(n, \mathbb{C})$. *Then*

$$\operatorname{Imm}_\lambda(A) = \operatorname{tr} P_\lambda \pi_\lambda(A) P_\lambda. \tag{5}$$

Remark. If we extend π_λ to all of $M(n, \mathbb{C})$, then, as established in [Li], (1) is valid for any $g \in M(n, \mathbb{C})$ and hence the extended Theorem 3 yields the following somewhat startling trace equality for π_λ:

$$\operatorname{tr} P_\lambda \pi_\lambda(Z(g)) P_\lambda = \frac{1}{n!} \operatorname{tr} \pi_\lambda(g). \tag{6}$$

The representations $\{\pi_\lambda\}$, $\lambda \in \mathscr{P}_n$, appearing in Theorem 3 are of course only a finite subset of the set of all the irreducible representations of $Sl(n, \mathbb{C})$. Actually this finite set can be characterized by the condition, satisfied for example by the adjoint representation, that twice a root is not a weight.

By scaling, Theorem 1 follows immediately from the more restricted statement where A is assumed to lie in $Sl(n, \mathbb{C})$. The same is true of Theorem 2 by virtue of the non-singular approximation result—Theorem 1 in [W]—for totally positive matrices. In view of Theorem 3 the following result is then a generalization of Theorems 1 and 2.

Theorem 4. *Let $\pi : Sl(n, \mathbb{C}) \to \operatorname{Aut} V$ be any finite (holomorphic) dimensional representation of $Sl(n, \mathbb{C})$, and let $P : V \to V^H$ be the $\pi(H)$-projection on the 0-weight space V^H. Then*

$$\operatorname{tr} P\pi(A)P \geq \dim V^H \tag{7}$$

whenever $A \in Sl(n, \mathbb{C})$ is either positive definite or totally positive.

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2. The 0-weight space in $\otimes^n \mathbb{C}^n$

We retain the notation of the Introduction. The natural representation of $S_n \times Sl(n, \mathbb{C})$ on $\otimes^n \mathbb{C}^n$ will be denoted by ρ. Explicitly if $v_i \in \mathbb{C}^n$, $i = 1, \ldots, n$, $A \in Sl(n, \mathbb{C})$, and $\tau \in S_n$, then

$$\rho(A)(v_1 \otimes \cdots \otimes v_n) = Av_1 \otimes \cdots \otimes Av_n,$$
$$\rho(\tau)(v_1 \otimes \cdots \otimes v_n) = v_{\tau^{-1}1} \otimes \cdots \otimes v_{\tau^{-1}n}.$$

Let $e_i \in \mathbb{C}^n$, $i = 1, \ldots, n$, be the standard basis so that any pure monomial tensor product of the e_i in $\otimes^n \mathbb{C}^n$ is a weight vector with respect to $\rho(H)$. It is immediate then that for $\rho|Sl(n, \mathbb{C})$ the 0-weight space $(\otimes^n \mathbb{C}^n)^H$ is $n!$ dimensional and is given by

$$(\otimes^n \mathbb{C}^n)^H = \sum_{\tau \in S_n} \mathbb{C}\, e_{\tau^{-1}1} \otimes \cdots \otimes e_{\tau^{-1}n}.$$

Now if $A = \{a_{ij}\} \in Sl(n, \mathbb{C})$, then

$$A\, e_{\tau^{-1}j} = \sum_{i=1}^{n} a_{i\tau^{-1}j} e_i.$$

Thus

$$\rho(A)(e_{\tau^{-1}1} \otimes \cdots \otimes e_{\tau^{-1}n}) = \sum_{(i_1, \ldots, i_n) \in \mathcal{N}^n} a_{i_1\tau^{-1}1} \cdots a_{i_n\tau^{-1}n} e_{i_1} \otimes \cdots \otimes e_{i_n}.$$

Now let $Q : (\otimes^n \mathbb{C}^n) \to (\otimes^n \mathbb{C}^n)^H$ be the projection operator which commutes with the action of H. Then clearly

$$Q\rho(A)(e_{\tau^{-1}1} \otimes \cdots \otimes e_{\tau^{-1}n}) = \sum_{\sigma \in S_n} a_{\sigma^{-1}1\tau^{-1}1} \cdots a_{\sigma^{-1}n\tau^{-1}n} e_{\sigma^{-1}1} \otimes \cdots \otimes e_{\sigma^{-1}n};$$

that is, if $z = e_1 \otimes \cdots \otimes e_n$ and $\alpha(\sigma, \tau) = a_{\sigma^{-1}1\tau^{-1}1} \cdots a_{\sigma^{-1}n\tau^{-1}n}$, then

$$Q\rho(A)\rho(\tau)z = \sum_{\sigma \in S_n} \alpha(\sigma, \tau)\rho(\sigma)z. \tag{8}$$

On the other hand if W_n denotes the Weyl group of $Sl(n, \mathbb{C})$ with respect to H, then ρ induces a representation γ of W_n on the 0-weight space $(\otimes^n \mathbb{C}^n)^H$. Since elements of W_n may be represented, modulo H, by permutation matrices with signs—so as to have determinant 1—we may clearly identify W_n with S_n in such a fashion that for $\sigma, \tau \in S_n$ one has

$$\gamma(\sigma)\rho(\tau)z = sg(\sigma)\rho(\tau)\rho(\sigma^{-1})z. \tag{9}$$

The group algebra $\mathbb{C}[S_n]$ is an $(S_n \times S_n)$-module with respect to left and right multiplication so that if $g \in \mathbb{C}[S_n]$ and $(\tau, \sigma) \in S_n \times S_n$, then $(\tau, \sigma) \cdot g = \tau g \sigma^{-1}$. On the other hand $(\otimes^n \mathbb{C}^n)^H$ is an $(S_n \times S_n)$-module, where (τ, σ) operates by $sg(\sigma)\rho(\tau)\gamma(\sigma)$. It is clear then that there is a unique $S_n \times S_n$ isomorphism

$$\eta : (\otimes^n \mathbb{C}^n)^H \to \mathbb{C}[S_n] \tag{10}$$

such that $\gamma(z) = \epsilon$, where ϵ is the identity element of S_n. Now recalling the complete reduction (3) of ρ, let $\gamma_\lambda : W_n \to \operatorname{Aut} V_\lambda^H$ be the 0-weight space representation of the Weyl group corresponding to π_λ. By (10) and the Peter-Weyl theorem one has an identification

$$V_\lambda^H = Z_\lambda^*, \tag{11}$$

where Z_λ^* is the dual space to Z_λ, and an equivalence (since ν_λ is self-contragredient)

$$\gamma_\lambda \simeq \nu_\lambda \otimes sg.$$

With the identification (11) we may write

$$(\otimes^n \mathbb{C}^n)^H = \oplus_{\lambda \in \mathscr{P}_n} Z_\lambda \otimes Z_\lambda^*. \tag{12}$$

In particular recalling that $f_\lambda = \dim Z_\lambda$ so that $f_\lambda = \chi^\lambda(\epsilon)$ one has

$$\dim V_\lambda^H = f_\lambda, \tag{13}$$

establishing (4).

3. Proof of (5) in Theorem 3

One knows that for the extension of any representation ν of S_n to the group algebra $\mathbb{C}[S_n]$ the image under ν of the element $F_\lambda \in \mathbb{C}[S_n]$ given by

$$F_\lambda = \frac{f_\lambda}{n!} \sum_{\tau \in S_n} \chi^\lambda(\tau)\tau$$

is the projection operator on the ν_λ primary component of ν. But then, if $\{v_i\}$, $i = 1, \ldots, f_\lambda$, is a basis of Z_λ and $\{u_i\}$ is the dual basis of $Z_\lambda^* = V_\lambda^H$, one has, by the Peter-Weyl theorem, that

$$\frac{f_\lambda}{n!} \sum_{\tau \in S_n} \chi^\lambda(\tau)\rho(\tau)z = \sum_{i=1}^{f_\lambda} v_i \otimes u_i. \tag{14}$$

Then upon applying $Q\rho(A)$ to the left side of (14) it follows from (8) that

$$\sum_{i=1}^{f_\lambda} v_i \otimes P_\lambda \pi_\lambda(A) u_i = \frac{f_\lambda}{n!} \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \alpha(\sigma, \tau) \chi^\lambda(\tau) \rho(\sigma) z. \tag{15}$$

Now using the natural $*$-operation in the group algebra $\mathbb{C}[S_n]$ one defines a Hilbert space structure on $(\otimes^n \mathbb{C}^n)^H$ by putting, for $x, y \in (\otimes^n \mathbb{C}^n)^H$,

$$\{x, y\} = \operatorname{tr} L_{\eta(x)} L_{(\eta(y))^*}$$

recalling (10), where if $F \in \mathbb{C}[S_n]$, then L_F is the operator of left translation in $\mathbb{C}[S_n]$ by F. One notes then that

$$\{\rho(\sigma) z, \rho(\tau) z\} = n!\, \delta_{\sigma, \tau},$$

and if the v_i are chosen to be an orthonormal basis of Z_λ with respect to an S_n-invariant Hilbert space structure in Z_λ, then

$$\{v_i \otimes u_k, v_j \otimes u_l\} = f_\lambda \delta_{ij} \delta_{kl}.$$

But now the inner product of the right side of (14) with the left side of (15) is the same as the inner product of the right side of (15) with z. Thus

$$f_\lambda \operatorname{tr} P_\lambda \pi_\lambda(A) = f_\lambda \sum_{\tau \in S_n} \alpha(\epsilon, \tau) \chi^\lambda(\tau). \tag{16}$$

Division by f_λ yields (5). Q.E.D.

4. Proof of Theorem 4 when A is positive definite

Let $\langle u, v \rangle$ be a Hilbert space structure on V which is invariant under $\pi(SU(n))$. Let $A \in Sl(n, \mathbb{C})$ be positive definite, and let $\{w_i\}$ be an orthonormal basis of V^H. To prove Theorem 4 in this case it suffices to show that for all i

$$\langle \pi(A) w_i, w_i \rangle \geq 1. \tag{17}$$

But since A is positive definite, as one knows, we may write $A = U^* D^* D U$, where U is upper triangular and unipotent, $D \in H$, and the superscript $*$ denotes Hermitian adjoint. But clearly $\pi(U) w_i - w_i \in \operatorname{Ker} P$. Since $\pi(D) | V^H$ is the identity, one also has $\pi(DU) w_i - w_i \in \operatorname{Ker} P$. But then

$$\begin{aligned} \langle \pi(A) w_i, w_i \rangle &= \langle \pi(DU) w_i, \pi(DU) w_i \rangle \\ &\geq \langle w_i, w_i \rangle = 1. \quad \text{Q.E.D.} \end{aligned}$$

5. Proof of Theorem 4 when A is totally positive

Assume A is totally positive. Let α_i, $i = 1, \ldots, n-1$, be simple positive roots relative to $(H, Sl(n, \mathbb{C}))$ such that one can choose the matrix unit e_{ii+1} to be a corresponding root vector e_{α_i}. Let $\{x_j\}$ be the corresponding canonical basis of V (see [Lu]). In particular the $\{x_j\}$ are a weight basis. If $g \in$

$Sl(n, \mathbb{C})$, let $\mu(g)$ be the matrix of $\pi(g)$ with respect to the basis $\{x_j\}$. To prove the theorem it suffices to show that

$$\mu(A)_{jj} \geq 1 \quad \text{whenever } x_j \in V^H. \tag{18}$$

But now by Theorem 2 in [W] or as more clearly stated in [Lo], we can write $A = U_-DU_+$, where U_+ is a product of elements of the form $\exp te_{\alpha_i}$ with $t > 0$, U_- is a product of elements of the form $\exp te_{-\alpha_i}$ with $t > 0$, and $D \in H$ has positive diagonal entries. Since elements of the form U_+ are clearly stable under conjugation by elements of the form D, we can assume that $A = U_-U_+D$. But then if $x_j \in V^H$ one has

$$\pi(A)x_j = \pi(U_-)\pi(U_+)x_j. \tag{19}$$

But now by Theorem 22.1.7 in [Lu] all the entries of $\mu(\exp te_{\alpha_i})$ and $\mu(\exp te_{-\alpha_i})$ are non-negative for any α_i and $t > 0$. Thus all the entries of $\mu(U_-)$ and $\mu(U_+)$ are non-negative. On the other hand since U_- and U_+ are respectively lower and upper trianglar unipotent, one has that

$$\mu(U_-)_{jj} = \mu(U_+)_{jj} = 1$$

for any j. But now the product of two matrices each with non-negative entries and each having diagonal entries ≥ 1 still has these same two properties. But then (18) follows from (19). Q.E.D.

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