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HILBERT TRANSFORMS AND MAXIMAL FUNCTIONS ASSOCIATED TO FLAT CURVES ON THE HEISENBERG GROUP

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INTRODUCTION

Suppose for each x in $\mathbb{R}^n$ $\gamma(x, t)$ is a smooth curve in $\mathbb{R}^n$ with $\gamma(x, 0) = x$. For $f \in C_0^\infty(\mathbb{R}^n)$, we define the Hilbert transform and maximal function associated to $\gamma(x, t)$ as

$$Hf(x) = \text{p.v.} \int_{-1}^{1} f(\gamma(x, t)) \tfrac{dt}{t}$$

and

$$\mathcal{M}f(x) = \sup_{0<h\leq 1} \tfrac{1}{h} \textstyle\int_0^h |f(\gamma(x, t))| dt,$$

respectively.

We are interested in L^p estimates for Hf and $\mathcal{M}f$. If $\gamma(x, t)$ satisfies an appropriate curvature condition then

$$\|Hf\|_{L^p} \leq A_p \|f\|_{L^p}, \qquad 1 < p < \infty, \tag{1}$$

and

$$\|\mathcal{M}f\|_{L^p} \leq A_p \|f\|_{L^p}, \qquad 1 < p \leq \infty. \tag{2}$$

See [C1] and [CNSW].

We are interested in obtaining estimates (1) and (2) above for curves $\gamma(x, t)$ for which the curvature condition fails. There are a number of papers dealing with this question if $\gamma(x, t)$ is of the form

$$\gamma(x, t) = x + \Gamma(t) \tag{3}$$

where $\Gamma(t)$ is a fixed curve. See [CVWWA] or [CZ], for example.

In this paper we shall consider certain curves $\gamma(x, t)$ which are not of the form (3). In fact these curves will be curves on the Heisenberg group, that is we take a fixed curve $\Gamma(t)$ in $\mathbb{R}^3$, and for x in $\mathbb{R}^3$ we set

$$\gamma(x, t) = x \cdot \Gamma^{-1}(t)$$

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where $\Gamma^{-1}(t) = -\Gamma(t)$, and

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \tfrac{1}{2}(x_1 y_2 - y_1 x_2)).$$

Then

$$Hf(x) = \text{p.v.} \int_{-1}^{1} f(x \cdot \Gamma^{-1}(x)) \tfrac{dt}{t}$$

and

$$\mathscr{M} f(x) = \sup_{0<h\leq 1} \tfrac{1}{h} \textstyle\int_0^h |f(x \cdot \Gamma^{-1}(t))| dt$$

where the multiplication is the Heisenberg group multiplication described above. We shall take $\Gamma(t)$ to be of the special form

$$\Gamma(t) = (t, \gamma(t), t\gamma(t)), \quad \text{for } t > 0.$$

This allows one to write $\Gamma(t)$ as

$$\Gamma(t) = \delta(t) v \quad \text{for } t > 0 \tag{4}$$

where $v = (1, 1, 1)$ and

$$\delta(t) = \operatorname{diag}(t, \gamma(t), t\gamma(t)) \tag{5}$$

are not only linear transformations on $\mathbb{R}^3$, but are also automorphisms of the Heisenberg group. It turns out that the appropriate curvature condition alluded to above will be satisfied exactly when $\gamma''(t)$ does not vanish to infinite order at $t = 0$. So we shall be interested in the case $\gamma^{(j)}(0) = 0$ for all j.

Our results are expressed in terms of the functional

$$\lambda(t) = \frac{t\gamma''(t)}{\gamma'(t)}. \tag{6}$$

Note that $\lambda(t)$ transforms well under scaling. That is if $\gamma_{a,b}(t) = a\gamma(bt)$,

$$\lambda_{a,b}(t) = \frac{t\gamma''_{a,b}(t)}{\gamma'_{a,b}(t)} = \lambda(bt).$$

Main Theorem. *Suppose for* $t > 0$

$$\Gamma(t) = (t, \gamma(t), t\gamma(t))$$

with $\gamma(0) = \gamma'(0) = 0$, *where* $\gamma(t)$ *is a convex curve in* $C^3([0, 1])$. *Assume* $\lambda(t)$ *is decreasing and positive on* $(0, 1]$.

Furthermore assume

$$\lim_{t \searrow 0} \frac{t\lambda'(t)}{\lambda^2(t)} \log^+ \lambda(t) = 0. \tag{7}$$

Then

$$\|\mathscr{M} f\|_{L^p} \leq C(p) \|f\|_{L^p}, \qquad 1 < p \leq \infty, \tag{8}$$

and if $\Gamma(t)$ *is extended for negative* t *to be an odd curve,*

$$\|Hf\|_{L^p} \leq C(p) \|f\|_{L^p}, \qquad 1 < p < \infty. \tag{9}$$

Remarks. The hypothesis of the main theorem seems rather technical, however it is straightforward to verify the hypothesis in examples such as

$$\gamma(t) = \exp\left(-\tfrac{1}{t}\right), \quad \text{or } \gamma(t) = \exp\left(-\exp\left(\tfrac{1}{t}\right)\right), \quad \text{etc.} \dots$$

We will actually prove the theorem for curves $\Gamma(t) = (t, \gamma(t), t\gamma(t))$ in a slightly more general setting. That is we will take the group law in $\mathbb{R}^3$ to be

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \beta(x_1 y_2 - y_1 x_2))$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\beta \in \mathbb{R}^1$ with $\beta \neq -1$, and consider $\gamma(x, t) = x \cdot \Gamma^{-1}(t)$ with the above multiplication. However for general β, we need to assume in addition that $\lambda(t)$ tends to infinity as t tends to 0. It will be evident that our proof does not directly apply to the case $\beta = -1$. Note that for $\beta = 0$, we are in the setting of (3), $\gamma(x, t) = x - \Gamma(t)$; the Euclidean translation invariant case. One reason for studying these more general curves is that results for these curves easily imply results in $\mathbb{R}^2$. More specifically when $\beta = 1$, we can use the diffeomorphism $\varphi(x, y, z) = (x, y, z + xy)$ on $\mathbb{R}^3$ to obtain L^p estimates for the Hilbert transform and maximal function associated to the curves

$$\gamma((x, y, z), t) = (x - t, y - \gamma(t), z - 2x\gamma(t)).$$

This in turn implies that the estimates (1) and (2) hold for the plane curves

$$\gamma(x, y, t) = (x - t, y - x\gamma(t))$$

when $\gamma(t)$ satisfies the hypothesis of the main theorem.

Since our basic operators do not commute with translations, the use of the Fourier transform does not seem to be a viable tool as in [CVWWA] or [CZ]. Instead we use ideas developed in [NSW], [C2], [RS] and [CVWWW]. Also we need to make use of a generalization of the space of homogeneous type as in [CW], developed in [CVWW].

1. Idea of the proof

We will only give the proof of the estimate for the maximal function. The proof of the estimate for the Hilbert transform is similar. See [CVWWW], where the necessary modifications in the argument are explained. We set

$$M_k f(x) = 2^k \int_{2^{-k}}^{2^{-k+1}} f(x \cdot \Gamma^{-1}(t))\,dt.$$

Let

$$Mf(x) = \sup_{k>0} |M_k f(x)|.$$

It is not hard to see that $\mathcal{M} f(x) \leq C M f(x)$ for $f \geq 0$, and hence it suffices to show

$$\|Mf\|_{L^p} \leq C(p)\|f\|_{L^p}, \qquad 1 < p \leq \infty.$$

Let A_k denote $\delta(2^{-k})$, and $f_k(x) = f(A_k x)$. Then

$$\begin{aligned} M_k f(x) &= 2^k \int_{2^{-k}}^{2^{-k+1}} f(x \cdot \Gamma^{-1}(t))dt \\ &= \int_1^2 f(x \cdot \Gamma^{-1}(2^{-k}t))dt \\ &= \int_1^2 f(A_k(A_k^{-1}x \cdot A_k^{-1}\Gamma^{-1}(2^{-k}t)))dt \\ &= f_k * d\mu_k(A_k^{-1}x). \end{aligned}$$

Here, for a test function g,

$$\begin{aligned} d\mu_k(g) &= \int_1^2 g(A_k^{-1}\Gamma(2^{-k}t))dt \\ &= \int_1^2 g(\Gamma_k(t))dt \end{aligned}$$

where

$$\begin{aligned} \Gamma_k(t) = A_k^{-1}\Gamma(2^{-k}t) &= \left(t, \frac{\gamma(2^{-k}t)}{\gamma(2^{-k})}, t\frac{\gamma(2^{-k}t)}{\gamma(2^{-k})}\right) \\ &= (t, \gamma_k(t), t\gamma_k(t)). \end{aligned}$$

Also for a Radon measure $d\mu$,

$$f * d\mu(x) = \int f(x \cdot y^{-1})d\mu(y).$$

(More generally if $d\mu$ and $d\nu$ are two Radon measures,

$$d\nu * d\mu(f) = \int\int f(z \cdot y)d\nu(z)d\mu(y).)$$

Let $d\mu_k^*$ be the measure defined on test functions g by

$$d\mu_k^*(g) = \int_1^2 g(\Gamma_k^{-1}(t))dt.$$

The operation $f \to f * d\mu_k^*$ is the adjoint of $f \to f * d\mu_k$.

The essence of the proof of the main theorem is to show that

$$d\mu_k * d\mu_k^* * d\mu_k * d\mu_k^*$$

has an L^1 density ρ_k with a certain amount of L^1 smoothness. Note that

$$M_k^* M_k M_k^* M_k f(x) = f_k * d\mu_k * d\mu_k^* * d\mu_k * d\mu_k^*(A_k^{-1}x).$$

In fact we shall show that $d\mu_k * d\mu_k^* * d\mu_k * d\mu_k^*$ has a density ρ_k and that

$$\rho_k = \rho_{k,1} + \rho_{k,2}, \tag{1.1}$$

where for some $\epsilon > 0$

$$\int |\rho_{k,1}(x \cdot y) - \rho_{k,1}(x)|dx \le Ch^\epsilon \tag{1.2}$$

whenever $\|y\| < h$, and

$$\int |\rho_{k,2}(y \cdot x) - \rho_{k,2}(x)| dx \le Ch^{\epsilon} \tag{1.3}$$

whenever $\|y\| < h$. Once we prove (1.2) and (1.3) we can follow the general ideas of [C2] or [RS] to complete the proof of the main theorem.

Now if f is a test function,

$$\begin{aligned} &d\mu_k * d\mu_k^* * d\mu_k * d\mu_k^*(f) \\ &\quad = \int_1^2 \int_1^2 \int_1^2 \int_1^2 f(\Gamma_k(s) \cdot \Gamma_k^{-1}(t) \cdot \Gamma_k(u) \cdot \Gamma_k^{-1}(w)) ds\,dt\,du\,dw. \end{aligned}$$

As in [C2] and [RS], we will divide the region of integration into a number of parts. In each part we will fix one variable and make a change of variables in the other three. Suppose for example we are in a region where we fix t. Consider the mapping

$$\varphi_t(s, u, w) = \Gamma_k(s) \cdot \Gamma_k^{-1}(t) \cdot \Gamma_k(u) \cdot \Gamma_k^{-1}(w)$$

which maps $\mathbb{R}^3$ into the Heisenberg group, and make the change of variables

$$x = \varphi_t(s, u, w) \tag{1.4}$$

in the integral. The new difficulty that arises is that we have no uniform control over the size of the derivatives of φ_t. So for example it becomes difficult to estimate the size of sets on which the transformation (1.4) is one to one. Also formally, one finds that the density $\rho(x)$ is $1/|J_{\varphi_t}(\varphi_t^{-1}(x))|$ where J_{φ_t} denotes the Jacobian of φ_t. Thus to estimate the smoothness of $\rho(x)$, one would like to have estimates on the derivatives of φ_t which are not available.

Our basic idea is to divide the cube

$$Q = \{(s, t, u, w) \in \mathbb{R}^4 \mid 1 \le s, t, u, w \le 2\}$$

into two parts. One part will have small area. On this part we will largely follow the argument of [RS] and use the fact that the area of this part is small to overcome the lack of uniformity in the control of the size of the derivatives of φ_t. On the other part we make a suitable approximation to the Jacobian of φ_t, J_{φ_t}, so that we may directly calculate that the argument of [CVWWW] applies.

2. The main estimate—some preliminary splitting

We shall assume $\lambda(t)$ tends to infinity as t tends to 0. If $\lambda(t)$ stays bounded the proof is much easier for certain β, including $\beta = 1/2$. A monotonicity argument is needed which is not valid for general β. Furthermore since we are only concerned with what happens for t small, we shall assume $\lambda(t)$ is arbitrarily large. We proceed to the proof of (1.1), (1.2) and (1.3) above.

For a test function f

$$\begin{aligned} d\mu_k * d\mu_k^* * d\mu_k * d\mu_k^*(f) &= \int_Q f(\Gamma_k(s)\cdot\Gamma_k^{-1}(t)\cdot\Gamma_k(u)\cdot\Gamma_k^{-1}(w))dsdtdudw \\ &= \int_{Q_1} f(\varphi(s,t,u,w))dsdtdudw \\ &\quad + \int_{Q_2} f(\varphi(s,t,u,w))dsdtdudw \\ &= \Theta_1(f)+\Theta_2(f). \end{aligned}$$

Here

$$\begin{aligned} Q &= \{(s,t,u,w)\in\mathbb{R}^4 \mid 1\le s,t,u,w\le 2\}, \\ Q_1 &= \{(s,t,u,w)\in Q \mid \max\{s,t,u,w\}=\max\{s,t\}\}, \\ Q_2 &= \{(s,t,u,w)\in Q \mid \max\{s,t,u,w\}=\max\{u,w\}\}, \end{aligned}$$

and

$$\varphi(s,t,u,w)=\Gamma_k(s)\cdot\Gamma_k^{-1}(t)\cdot\Gamma_k(u)\cdot\Gamma_k(w)^{-1}.$$

We shall show that Θ_1 has a density $\rho_{k,1}$ satisfying (1.2). Note that

$$\Theta_2(f)=\int_{Q_1} f(\varphi^{-1}(s,t,u,w))dsdtdudw$$

and so

$$\begin{aligned} \Theta_2(f) &= \int f(x^{-1})\rho_{k,1}(x)dx \\ &= \int f(x)\rho_{k,1}(x^{-1})dx. \end{aligned}$$

Thus $\rho_{k,2}(x)=\rho_{k,1}(x^{-1})$ and so (1.3) follows from (1.2). Therefore we shall only prove (1.2).

We now divide Q_1 into four parts, $Q_1=R_1\cup R_2\cup R_3\cup R_4$,

$$\begin{aligned} R_1 &= \{(s,t,u,w)\in Q_1 \mid \max\{s,t\}=s,\ \ u<w\}, \\ R_2 &= \{(s,t,u,w)\in Q_1 \mid \max\{s,t\}=s,\ \ w<u\}, \\ R_3 &= \{(s,t,u,w)\in Q_1 \mid \max\{s,t\}=t,\ \ u<w\}, \end{aligned}$$

and

$$R_4=\{(s,t,u,w)\in Q_1 \mid \max\{s,t\}=t,\ \ w<u\}.$$

We will discuss in detail the contribution to Θ_1 from integrating over R_1. The contributions from R_2, R_3 and R_4 are treated similarly. Roughly the difference is that in treating R_1 and R_2 we fix t, $1\le t\le 2$, and make a change of variables $x=\varphi_t(s,u,w)=\varphi(s,t,u,w)$ from $\mathbb{R}^3$ to $\mathbb{R}^3$. In R_3 and R_4 we fix s and consider the change of variables $x=\varphi_s(t,u,w)=\varphi(s,t,u,w)$. In R_1 and R_2 we make different approximations to the Jacobian of φ_t. Similarly we use different approximations to the Jacobian of φ_s in R_3 and R_4. It will be clear what modifications to make in the Jacobians.

We must now analyze

$$\int_{R_1} f(\varphi(s,t,u,w))dsdtdudw = \int_1^2 \left\{ \int_{R(t)} f(\varphi_t(s,u,w))dsdudw \right\} dt$$

where

$$R(t) = \{(s,u,w) \in \mathbb{R}^3 \mid (s,t,u,w) \in R_1\}.$$

Let

$$\Theta^t(f) = \int_{R(t)} f(\varphi_t(s,u,w))dsdudw.$$

Clearly it suffices to show that Θ^t has an L^1 density ρ_t such that 1.2) holds uniformly in t.

3. The balls

We begin by dividing $R(t)$ into three regions. We set

$$P_t(s,u,w) = s - w + \beta(w - s + 2t - 2u). \tag{3.1}$$

The significance of P_t is the set where our approximation to J_{φ_t} vanishes is precisely when $P_t = 0$. We should remark here that in the region where the variable t is the largest (and thus we fix s) and $w < u$, the above polynomial is

$$P_s(t,u,w) = t - u + \beta(t-u)$$

which vanishes identically when $\beta = -1$. Therefore our approximation to J_{φ_s} will not hold in this region and so a different argument is needed in this case.

Now let C be a large constant to be determined later. Set

$$\Omega_k^1 = \left\{ (s,u,w) \in R(t) \mid w - u \geq C\frac{\log\lambda(2^{-k}u)}{\lambda(2^{-k}u)}, \right. \\ \left. \text{and } |P_t(s,u,w)| \geq C\frac{1}{\lambda(2^{-k}s)} \right\}, \tag{3.2}$$

$$\Omega_k^2 = \left\{ (s,u,w) \in R(t) \mid w - u \leq C\frac{\log\lambda(2^{-k}u)}{\lambda(2^{-k}u)}, \right. \\ \left. \text{and } J_{\varphi_t}(s,u,w) \neq 0 \right\}, \tag{3.3}$$

and

$$\Omega_k^3 = \left\{ (s,u,w) \in R(t) \mid w - u > C\frac{\log\lambda(2^{-k}u)}{\lambda(2^{-k}u)} \right. \\ \left. |P_t(s,u,w)| \leq \frac{c}{\lambda(2^{-k}s)} \text{ and } J_{\varphi_t}(s,u,w) \neq 0 \right\}. \tag{3.4}$$

Let X denote the hypersurface

$$w - u = C\frac{\log \lambda(2^{-k}u)}{\lambda(2^{-k}u)},$$

and Y denote the hypersurface

$$|P_t(s, u, w)| = C\frac{1}{\lambda(2^{-k}s)}.$$

Once we show the measure of the zero set of J_{φ_t} is zero, we shall have an essential decomposition of $R(t)$,

$$R(t) = \Omega_k^1 \cup \Omega_k^2 \cup \Omega_k^3.$$

Our first type of ball is derived from a Whitney decomposition of Ω_k^1. As in Stein, [S], we find cubes $Q_{\ell,n}$ such that

(3.5) $$\Omega_k^1 = \cup Q_{\ell,n},$$

(3.6) $$\text{diameter}(Q_{\ell,n}) = \epsilon_0 2^{-\ell}$$

for a suitable $\epsilon_0 > 0$,

(3.7) $$2^{-\ell-1} \le \text{dist(center of } Q_{\ell,n}, \text{ boundary of } \Omega_k^1) \le 2^{-\ell}.$$

(3.8) There is a constant C so that no point is in more than C of the $Q_{\ell n}^*$. Here $Q_{\ell n}^*$ is the cube with the same center as $Q_{\ell n}$ and having a side length twice as large.

Since Ω_k^1 is bounded by a finite number of smooth hypersurfaces, it is clear that

(3.9) for a given ℓ, the number of cubes $Q_{\ell n}$ is at most $C2^{2\ell}$.

Our second type of ball will be contained in Ω_k^2. For $p = (s, u, w) \in \Omega_k^2$ and $\epsilon > 0$ small, set

(3.10) $$r_1(p) = \epsilon\frac{|J_{\varphi_t}(p)|}{\gamma_k''(s)\gamma_k'(w)}\Delta(p),$$

(3.11) $$r_2(p) = \epsilon\frac{|J_{\varphi_t}(p)|}{\gamma_k''(w)\gamma_k'(s)}\Delta(p),$$

and

(3.12) $$r_3(p) = \epsilon\frac{|J_{\varphi_t}(p)|}{\gamma_k''(w)\gamma_k'(s)}\Delta(p)$$

where

$$\Delta(p) = (s-t)(w-u)\operatorname{dist}(p, X)\operatorname{dist}(p, Y).$$

We then form balls

$$\begin{aligned} D(p_0) = \{(s, u, w) \mid s_0 - r_1(p_0) \le s \le s_0 + r_1(p_0), \\ w_0 - r_2(p_0) \le w \le w_0 + r_2(p_0), \\ u_0 - r_3(p_0) \le u \le u_0 + r_3(p_0)\}, \end{aligned} \tag{3.13}$$

and the corresponding doubles

$$\begin{aligned} D^*(p_0) = \{(s, u, w) \mid s_0 - 2r_1(p_0) \le s \le s_0 + 2r_1(p_0), \\ w_0 - 2r_2(p_0) \le w \le w_0 + 2r_2(p_0), \\ u_0 - 2r_3(p_0) \le u \le u_0 + 2r_3(p_0)\}. \end{aligned} \tag{3.14}$$

We define a third type of ball for Ω_k^3. For $p \in \Omega_k^3$, we set

$$\tilde{r}_1(p) = \epsilon \frac{|J_{\varphi_t}(p)|}{\gamma_k''(s)\gamma_k'(w)}\Delta(p), \tag{3.15}$$

$$\tilde{r}_2(p) = \epsilon \frac{|J_{\varphi_t}(p)|}{\gamma_k''(w)\gamma_k'(s)}\Delta(p), \tag{3.16}$$

and

$$\tilde{r}_3(p) = \epsilon \frac{|J_{\varphi_t}(p)|}{\gamma_k'(s)\gamma_k'(s)}\Delta(p). \tag{3.17}$$

If $p_0 = (s_0, u_0, w_0) \in \Omega_k^3$, set

$$\begin{aligned} G(p_0) = \{(s, u, w) \mid s_0 - \tilde{r}_1(p_0) \le s \le s_0 + \tilde{r}_1(p_0), \\ w_0 - \tilde{r}_2(p_0) \le w \le w_0 + \tilde{r}_2(p_0), \\ u_0 - \tilde{r}_3(p_0) \le u \le u_0 + \tilde{r}_3(p_0)\} \end{aligned} \tag{3.18}$$

and the corresponding doubles

$$\begin{aligned} G^*(p_0) = \{(s, u, w) \mid s_0 - 2\tilde{r}_1(p_0) \le s \le s_0 + 2\tilde{r}_1(p_0), \\ w_0 - 2\tilde{r}_2(p_0) \le w \le w_0 + 2\tilde{r}_2(p_0), \\ u_0 - 2\tilde{r}_3(p_0) \le u \le u_0 + 2\tilde{r}_3(p_0)\}. \end{aligned} \tag{3.19}$$

If $p_0 = (s_0, u_0, w_0) \in Q_{\ell n} \subset \Omega_k^1$, let

$$B(p_0) = Q_{\ell n}^*. \tag{3.20}$$

If $p_0 \in \Omega_k^2$, let

$$B(p_0) = D^*(p_0), \tag{3.21}$$

and finally if $p_0 \in \Omega_k^3$, let

$$B(p_0) = G^*(p_0). \tag{3.22}$$

So $\{B(p_0)\}$ covers all of $R(t)$ except perhaps for a set of measure zero. We shall prove that we can find a good partition of unity of $R(t)$ subordinate to a subcollection of the balls $\{B(p)\}$, and that φ_t is one to one on each ball $B(p)$. Note that if φ_t is one to one on $B(p)$ and ψ is supported in $B(p)$,

$$\int \psi(s, u, w) f(\varphi_t(s, u, w)) ds du dw$$
$$= \int \psi(\varphi_t^{-1}(x)) \frac{1}{|J_{\varphi_t}(\varphi_t^{-1}(x))|} f(x) dx.$$

Thus we will have a contribution of

$$K(x) = \psi(\varphi_t^{-1}(x)) \frac{1}{|J_{\varphi_t}(\varphi_t^{-1}(x))|}$$

to the density ρ_t. In order to prove (1.2) we will have to estimate $L_j K(x)$, where $L_1 = \frac{\partial}{\partial x_1} - \beta x_2 \frac{\partial}{\partial x_3}$, $L_2 = \frac{\partial}{\partial x_2} + \beta x_1 \frac{\partial}{\partial x_3}$ and $L_3 = \frac{\partial}{\partial x_3}$ are left invariant vector fields on the Heisenberg group. By left invariance we see that

$$L_j K(x) = \frac{\partial}{\partial y_j}(K(x \cdot y))|_{y=0}, \qquad j = 1, 2, 3.$$

Since we know explicitly what $K(x)$ is in the (s, u, w) coordinates, it will facilitate matters to express the above differential operators in the (s, u, w) coordinates. To do this let $(s_0, u_0, w_0) \in B(p)$ be a point in the support of ψ and consider the mapping

$$f(s, u, w) = \varphi_t^{-1}(s_0, u_0, w_0) \cdot \varphi_t(s, u, w)$$

which maps diffeomorphically a neighborhood of (s_0, u_0, w_0) onto a neighborhood of the identity of the Heisenberg group. Therefore the differential of f^{-1}, $d(f^{-1})$, maps the tangent space at the identity of the Heisenberg group to the tangent space at (s_0, u_0, w_0) of $\mathbb{R}^3$. We are interested in the image of the tangent vectors $\frac{\partial}{\partial y_j}|_{y=0}$, $j = 1, 2, 3$, under the mapping $d(f^{-1})$. By the chain rule we can express $d(f^{-1})\left(\frac{\partial}{\partial y_j}|_{y=0}\right)$ as

$$\langle (f^{-1})' e_j, \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u}, \frac{\partial}{\partial w}\right)|_{s_0, u_0, w_0}\rangle \tag{3.23}$$

where $(f^{-1})'$ denotes the Jacobian matrix of f^{-1}, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Note that

$$d(f)(s_0, u_0, w_0) = dL_{x^{-1}} \circ d(\varphi_t)(s_0, u_0, w_0)$$

where $L_{x^{-1}}$ is the operation of left multiplication by $x^{-1} = \varphi_t^{-1}(s_0, u_0, w_0)$ on the Heisenberg group. Therefore the Jacobian matrix for f at (s_0, u_0, w_0) is

$$\mathcal{G}(s_0, u_0, w_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta x_2 & -\beta x_1 & 1 \end{pmatrix} \varphi_t'(s_0, u_0, w_0).$$

Here (x_1, x_2, x_3) are the coordinates for the point $x = \varphi_t(s_0, u_0, w_0)$. Finally we see that $(f^{-1})' = \mathscr{G}^{-1}$ and so (3.23) may be expressed as

(3.24)
$$d(f^{-1})\left(\frac{\partial}{\partial y_j}\Big|_{y=0}\right) = \langle(\mathscr{G}^{-1})^*(s_0, u_0, w_0)\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u}, \frac{\partial}{\partial w}\right)\Big|_{(s_0, u_0, w_0)}, e_j\rangle.$$

Let us write
$$(\mathscr{G}^{-1})^* = \frac{1}{\det \mathscr{G}}(g_{ij}) = \frac{1}{J_{\varphi_t}} \cdot (g_{ij})$$
where g_{ij} is the (i, j) cofactor of the matrix $\mathscr{G}$. Thus we will need estimates on $\frac{g_{ij}}{\det \mathscr{G}}$ and the corresponding derivatives. In order to obtain these estimates we first need a few lemmas.

4. Some lemmas

In this section we derive some consequences of the hypothesis on λ.

Lemma 4.1. *Let C_0 and δ be positive. If $1 \le w \le 2$ and*
$$0 \le w - u \le C_0 \frac{\log \lambda(2^{-k}u)}{\lambda(2^{-k}u)},$$
then
$$\lambda(2^{-k}u) \le e^{2\delta C_0} \lambda(2^{-k}w).$$

Remark. Recall that we are assuming that λ is arbitrarily large and therefore we may assume that k is arbitrarily large in what follows.

Proof.
$$\log \frac{\lambda(2^{-k}u)}{\lambda(2^{-k}w)} = -\int_u^w \frac{2^{-k}r\lambda'(2^{-k}r)}{\lambda(2^{-k}r)} \frac{dr}{r}.$$
From (7)
$$\left|\frac{2^{-k}r\lambda'(2^{-k}r)}{\lambda(2^{-k}r)}\right| \le \delta \frac{\lambda(2^{-k}r)}{\log \lambda(2^{-k}r)}$$
for k large enough and $r \le 2$. Thus since $u \ge \frac{1}{2}$ for k large,
$$\log \frac{\lambda(2^{-k}u)}{\lambda(2^{-k}w)} \le 2\delta \int_u^w \frac{\lambda(2^{-k}r)}{\log \lambda(2^{-k}r)} dr$$
$$\le 2\delta \frac{\lambda(2^{-k}u)}{\log \lambda(2^{-k}u)}(w - u) \le 2C_0\delta.$$
The lemma follows by exponentiating the last inequality.

Lemma 4.2. *Let $1 \le w \le 2$, and assume $w - u \ge C_0 \frac{\log \lambda(2^{-k}u)}{\lambda(2^{-k}u)}$. Then for k large enough*
$$\frac{\gamma_k'(w)}{\gamma_k'(u)} \ge [\lambda(2^{-k}u)]^{C_0/4}.$$

Proof. Since γ_k' is increasing and λ is decreasing, we may assume $w - u = C_0 \frac{\log \lambda(2^{-k}u)}{\lambda(2^{-k}u)}$. Then

$$\log \frac{\gamma_k'(w)}{\gamma_k'(u)} = \int_u^w \lambda(2^{-k}r)\frac{dr}{r} \geq \frac{1}{2}\lambda(2^{-k}w)(w-u)$$
$$\geq \frac{1}{4}\lambda(2^{-k}u)(w-u) = \frac{C_0}{4}\log\lambda(2^{-k}u)$$

by Lemma 4.1 if k is large enough. The result follows now by exponentiating.

Lemma 4.3. *Suppose* $1 \leq w \leq 2$. *Then*

$$\frac{\gamma_k'(w)}{\gamma_k(w)} \geq \frac{1}{2}\lambda(2^{-k}w).$$

Proof.

$$\frac{\gamma_k(w)}{\gamma_k'(w)} = \int_0^w \frac{\gamma_k'(r)}{\gamma_k'(w)}dr.$$

But

$$\log \frac{\gamma_k'(w)}{\gamma_k'(r)} = \int_r^w \lambda(2^{-k}\sigma)\frac{d\sigma}{\sigma} \geq \frac{1}{2}\lambda(2^{-k}w)(w-r),$$

which implies

$$\frac{\gamma_k'(r)}{\gamma_k'(w)} \leq e^{-\frac{1}{2}\lambda(2^{-k}w)(w-r)}.$$

So

$$\frac{\gamma_k(w)}{\gamma_k'(w)} \leq \int_0^w e^{-\frac{1}{2}(w-r)\lambda(2^{-k}w)}dr$$
$$\leq \frac{2}{\lambda(2^{-k}w)}\int_0^\infty e^{-t}dt$$
$$= \frac{2}{\lambda(2^{-k}w)}$$

which proves the lemma.

Lemma 4.4. *There are positive constants* C_1 *and* C_2 *so that if* $1 \leq t \leq 2$ *and*

$$t - \frac{t}{\lambda(2^{-k}t)} \leq r, \quad s \leq t + \frac{t}{\lambda(2^{-k}t)},$$

then

$$C_1 \leq \frac{\lambda(2^{-k}r)}{\lambda(2^{-k}s)} \leq C_2, \tag{4.1}$$

$$C_1 \leq \frac{\gamma(2^{-k}r)}{\gamma(2^{-k}s)} \leq C_2, \tag{4.2}$$

$$C_1 \leq \frac{\gamma_k'(2^{-k}r)}{\gamma_k'(2^{-k}s)} \leq C_2, \tag{4.3}$$

and

$$C_1 \leq \frac{\gamma_k''(2^{-k}r)}{\gamma_k''(2^{-k}s)} \leq C_2. \tag{4.4}$$

Proof. To prove (4.1) note that it suffices to prove

$$\frac{\lambda\left(t-\frac{t}{\lambda(t)}\right)}{\lambda\left(t+\frac{t}{\lambda(t)}\right)} \leq C$$

for t small. First by (7),

$$\log \frac{\lambda(t)}{\lambda\left(t+\frac{t}{\lambda(t)}\right)} = -\int_t^{t+\frac{t}{\lambda(t)}} \frac{\lambda'(\sigma)}{\lambda(\sigma)} d\sigma$$

$$\leq \delta \int_t^{t+\frac{t}{\lambda(t)}} \frac{\lambda(\sigma)}{\sigma} d\sigma$$

where $\delta > 0$ is small if t is small. Since $\frac{\lambda(\sigma)}{\sigma}$ is decreasing, we may estimate the last integral by 1 and so

$$\frac{\lambda(t)}{\lambda\left(t+\frac{t}{\lambda(t)}\right)} \leq e^{\delta}$$

if t is small. Therefore it suffices to prove

$$\frac{\lambda\left(t-\frac{t}{\lambda(t)}\right)}{\lambda(t)} \leq C.$$

Set $s = \lambda(t)$ and $s' = \lambda\left(t-\frac{t}{\lambda(t)}\right) = \lambda\left(\lambda^{-1}(s)\left(1-\frac{1}{s}\right)\right)$. We will show that $s' \leq 2s$ which in turn implies (4.1). The estimate $s' \leq 2s$ will follow from the estimate $\lambda^{-1}(2s) \leq \lambda^{-1}(s')$ or

$$\lambda^{-1}(2s) \leq \lambda^{-1}(s)\left(1-\tfrac{1}{s}\right).$$

But from (7),

$$\log \frac{\lambda^{-1}(s)}{\lambda^{-1}(2s)} = -\int_s^{2s} \frac{1}{\lambda'(\lambda^{-1}(\sigma))} \frac{1}{\lambda^{-1}(\sigma)} d\sigma$$

$$\geq \frac{1}{\delta} \int_s^{2s} \frac{1}{\sigma^2} d\sigma = \frac{1}{2\delta s}$$

where $\delta > 0$ is small if t is small (hence s is large). Exponentiating this inequality gives us

$$\frac{\lambda^{-1}(2s)}{\lambda^{-1}(s)} \leq e^{-\frac{1}{2\delta s}} \leq 1 - \frac{1}{4\delta s} \leq 1 - \frac{1}{s}$$

if $\delta < \frac{1}{4}$ and s is large enough. This establishes (4.1).

If $r > s$,

$$\begin{aligned}\log\frac{\gamma'(2^{-k}r)}{\gamma'(2^{-k}s)} &= \int_s^r \frac{2^{-k}u\gamma''(2^{-k}u)}{\gamma'(2^{-k}u)}\frac{du}{u} \\ &= \int_s^r \lambda(2^{-k}u)\frac{du}{u} \le 2(r-s)\lambda(2^{-k}s) \\ &\le 8\frac{\lambda(2^{-k}s)}{\lambda(2^{-k}t)} \le C\end{aligned}$$

by (4.1) which implies (4.3). Next (4.4) follows from (4.1), (4.3) and the definition of λ. Finally to prove (4.2) for $r > s$ write

$$\gamma(2^{-k}r) - \gamma(2^{-k}s) = \int_s^r 2^{-k}\gamma'(2^{-k}u)du \le 2^{-k}\gamma'(2^{-k}r)(r-s).$$

Dividing by $\gamma(2^{-k}s)$ yields

$$\frac{\gamma(2^{-k}r)}{\gamma(2^{-k}s)} \le 1 + C\frac{\gamma'(2^{-k}r)}{\gamma'(2^{-k}s)}.$$

Then (4.2) follows from (4.3).

Lemma 4.5. *If* $1 \le u < s \le 2$,

$$\gamma_k'(s) - \gamma_k'(u) \ge \epsilon\gamma_k'(s)(s-u)$$

for some positive ϵ.

Proof. We may assume $\gamma_k'(u) \ge \frac{1}{2}\gamma_k'(s)$. Then

$$\begin{aligned}\gamma_k'(s) - \gamma_k'(u) &= \int_u^s \gamma_k''(r)dr \\ &\ge \tfrac{1}{2}\int_u^s \gamma_k'(r)\lambda(2^{-k}r)dr \ge \epsilon\gamma_k'(u)(s-u)\end{aligned}$$

since we are assuming that λ is bounded below. The proof is now complete since $\gamma_k'(u) \ge \frac{1}{2}\gamma_k'(s)$.

5. Some estimates for the Jacobian of φ_t, and applications to r_1, r_2, r_3, $\tilde{r}_1$, $\tilde{r}_2$, and $\tilde{r}_3$

If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are two vectors in $\mathbb{R}^3$, then we define $[a, b]$ as the vector

$$[a, b] = (0, 0, a_1b_2 - b_1a_2).$$

With this notation

$$\begin{aligned}\varphi_t(s, u, w) &= \Gamma_k(s) - \Gamma_k(t) + \Gamma_k(u) - \Gamma_k(w) \\ &\quad + \beta[\Gamma_k(s), -\Gamma_k(t) + \Gamma_k(u) - \Gamma_k(w)] \\ &\quad + \beta[-\Gamma_k(t), \Gamma_k(u) - \Gamma_k(w)] + \beta[\Gamma_k(u), -\Gamma_k(w)].\end{aligned} \tag{5.1}$$

Recall from the end of Section 3 that $J_{\varphi_t}(s, u, w) = \det\mathcal{G}(s, u, w)$ where

$$\mathcal{G}(s, u, w) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta x_2 & -\beta x_1 & 1 \end{pmatrix} \varphi_t'(s, u, w).$$

Here $x = (x_1, x_2, x_3) = \varphi_t(s, u, w)$. A calculation shows

$$\mathcal{G}(s, u, w) = \tag{5.2}$$

$$\begin{pmatrix} \Gamma_k'(s) & \Gamma_k'(u) & -\Gamma_k'(w) \\ +\beta[\Gamma_k'(s), \Gamma_k(s)] & +\beta[\Gamma_k'(u), \Gamma_k(u)] & +\beta[\Gamma_k'(w), \Gamma_k(w)] \\ +2\beta[\Gamma_k'(s), -\Gamma_k(t) + \Gamma_k(u) - \Gamma_k(w)] & +2\beta[\Gamma_k'(u), -\Gamma_k(w)] & \end{pmatrix}$$

and

$$\mathcal{G}(s, u, w) = \begin{pmatrix} 1 & 1 & -1 \\ \gamma_k'(s) & \gamma_k'(u) & -\gamma_k'(w) \\ A & C & -D \end{pmatrix} \tag{5.3}$$

where

$$\begin{aligned} A = {} & (s\gamma_k(s))' - \beta(s\gamma_k'(s) - \gamma_k(s)) + 2\beta(-\gamma_k(t) + \gamma_k(u) - \gamma_k(w)) \\ & - 2\beta(-t + u - w)\gamma_k'(s), \end{aligned} \tag{5.4}$$

$$C = (u\gamma_k(u))' - \beta(u\gamma_k'(u) - \gamma_k(u)) + 2\beta(w\gamma_k'(u) - \gamma_k(w)), \tag{5.5}$$

and

$$D = (w\gamma_k(w))' - \beta(\gamma_k(w) - w\gamma_k'(w)). \tag{5.6}$$

Using the formulas (5.3), (5.4), (5.5) and (5.6), a computation gives the following lemma.

Lemma 5.1. *For* $(s, u, w) \in R(t)$,

$$|J_{\varphi_t}(s, u, w)| \le C\gamma_k'(s)\gamma_k'(w), \tag{5.7}$$

$$\left|\frac{\partial}{\partial s}(J_{\varphi_t})\right| \le C\gamma_k''(s)\gamma_k'(w), \tag{5.8}$$

$$\left|\frac{\partial}{\partial u}(J_{\varphi_t})\right| \le C\gamma_k'(s)(\gamma_k''(u) + \gamma_k'(w)), \tag{5.9}$$

and

$$\left|\frac{\partial}{\partial w}(J_{\varphi_t})\right| \le C\gamma_k''(w)\gamma_k'(s). \tag{5.10}$$

We now turn our attention to estimating J_{φ_t} from below.

Lemma 5.2. *For $1 \le u < w \le 2$, there are three functions $s_1(u, w)$, $s_2(u, w)$ and $s_3(u, w)$ such that for $(s, u, w) \in R(t)$,*

$$|J_{\varphi_t}(s, u, w)| \ge \epsilon \gamma_k'(s)\gamma_k'(w)(w-u)|s-s_1||s-s_2||s-s_3|$$

for some $\epsilon > 0$.

Proof. For a fixed (u, w), $u < w$, consider

$$f(s) = J_{\varphi_t}(s, u, w).$$

Expanding (4.3) on the first column, we see that

$$f(s) = a(u, w) + b(u, w)\gamma_k'(s) + (\gamma_k'(w) - \gamma_k'(u))((\beta-1)s\gamma_k'(s) - (\beta+1)\gamma_k(s)).$$

Consider the case $\beta \ne 1$. Then

$$\left(\frac{f'(s)}{\gamma_k''(s)}\right)' = (\gamma_k'(w) - \gamma_k'(u))\left[(\beta-1) - 2\left(\frac{s}{\lambda(2^{-k}s)}\right)'\right]. \tag{5.11}$$

Since

$$\left(\frac{s}{\lambda(2^{-k}s)}\right)' = \frac{1}{\lambda(2^{-k}s)} - \frac{2^{-k}s\lambda'(2^{-k}s)}{\lambda^2(2^{-k}s)}$$

is arbitrarily small by (7) and the assumption that λ is large, we see that $f'(s)/\gamma_k''(s)$ is monotone and thus $f'(s)$ has at most one zero, $s_1 = s_1(u, w)$, and $f(s)$ has at most two zeros, $s_2 = s_2(u, w)$ and $s_3 = s_3(u, w)$. We will assume that f' has one zero and f has two zeros with $s_2 < s_1 < s_3$, otherwise the proof is easier. From (5.11), we have

$$|f'(s)| \ge \epsilon(\gamma_k'(w) - \gamma_t'(u))\gamma_k''(s)|s - s_1|$$

for some $\epsilon > 0$. Suppose $s_1 \le s \le s_3$. Then

$$\begin{aligned} |f(s)| &= \left|\int_s^{s_3} f'(r)ds\right| \\ &\ge \epsilon(\gamma_k'(w) - \gamma_k'(u))\int_s^{s_3} \gamma_k''(r)(r-s_1)dr \\ &\ge \epsilon(\gamma_k'(w) - \gamma_k'(u))(\gamma_k'(s_3) - \gamma_k'(s))(s-s_1) \end{aligned}$$

and so the conclusion follows from Lemma 4.5. Next suppose $s_3 \le s$. Then as before

$$\begin{aligned} |f(s)| &\ge \epsilon(\gamma_k'(w) - \gamma_k'(u))\int_{s_3}^{s} \gamma_k''(r)(r-s_1)dr \\ &\ge \epsilon(\gamma_k'(w) - \gamma_k'(u))\int_{\frac{s_3+s}{2}}^{s} \gamma_k''(r)(r-s_1)dr \\ &\ge \tfrac{\epsilon}{2}(\gamma_k'(w) - \gamma_k'(u))\left(\gamma_k'(s) - \gamma_k'\left(\tfrac{s+s_3}{2}\right)\right)(s-s_1) \end{aligned}$$

and so the conclusion follows from Lemma 4.5. The case $s \le s_1$ is treated similarly. When $\beta = 1$, one can consider

$$\left(\frac{f'(s)}{\gamma_k'(s)}\right)' = b(u, w)\left(\frac{\lambda(2^{-k}s)}{s}\right)'.$$

Then one can estimate the coefficient $b(u, w)$ from below and proceed as above. This concludes the proof of the lemma.

Now by Fubini's theorem we have

Lemma 5.3. *The zero set of* J_{φ_t} *has measure zero.*

We now obtain another estimate for J_{φ_t} from below on Ω_k^1.

Lemma 5.4. *For* $(s, u, w) \in \Omega_k^1$,

$$|J_{\varphi_t}(s, u, w)| \geq \tfrac{1}{2}\gamma_k'(s)\gamma_k'(w)|P_t(s, u, w)|.$$

Proof. Expanding (5.3), we see that for $(s, u, w) \in \Omega_k^1$,

$$\begin{aligned} J_{\varphi_t}(s, u, w) &= -\gamma_k'(s)\gamma_k'(w)P_t(s, u, w) \\ &\quad + \mathcal{O}(\gamma_k'(s)(\gamma_k'(u) + \gamma_k(w)) + \gamma_k(s)\gamma_k'(w)) \\ &= -\gamma_k'(s)\gamma_k'(w)P_t(s, u, w) + \mathcal{O}\left(\frac{\gamma_k'(s)\gamma_k'(w)}{\lambda(2^{-k}s)}\right). \end{aligned} \tag{5.12}$$

The last equality follows from Lemmas 4.2 and 4.3. Since $|P_t(s, u, w)| \geq C\frac{1}{\lambda(2^{-k}s)}$ for $(s, u, w) \in \Omega_k^1$, the desired estimate follows if C is chosen large enough.

One consequence of the lemmas in Sections 4 and 5 are certain estimates for the functions r_1, r_2, r_3, $\tilde{r}_1$, $\tilde{r}_2$ and $\tilde{r}_3$.

Lemma 5.5. *Let* $\delta > 0$. *Then if* ϵ *in the definition of* r_1, r_2, r_3, *etc. is sufficiently small, and* $p = (s, u, w) \in R(t)$,

$$r_1(p) = \tilde{r}_1(p) \leq \delta(s - t)\frac{1}{\lambda(2^{-k}s)}, \tag{5.13}$$

$$r_2(p) = r_3(p) = \tilde{r}_2(p) \leq \delta(w - u)\frac{1}{\lambda(2^{-k}w)}, \tag{5.14}$$

and for $p \in \Omega_k^3$,

$$\tilde{r}_3(p) \leq \delta(w - u)\frac{1}{\lambda(2^{-k}s)}. \tag{5.15}$$

Proof. The inequalities follow easily from (5.12). In particular for $p \in \Omega_k^3$, we have $|F_t(s, u, w)| \leq C/\lambda(2^{-k}s)$ and so (5.12) shows $|J_{\varphi_t}(p)| \leq C\frac{\gamma_k'(s)\gamma_k'(w)}{\lambda(2^{-k}s)}$ which implies (5.15).

Lemma 5.6. *There are positive constants* C_1 *and* C_2 *such that if* $p = (s, u, w) \in \Omega_k^2$ *and* $q \in D^*(p)$,

$$C_1 \leq \frac{|J_{\varphi_t}(q)|}{|J_{\varphi_t}(p)|} \leq C_2.$$

Proof.

$$\begin{aligned}|J_{\varphi_t}(q) - J_{\varphi_t}(p)| &= \left|\int_0^1 \nabla J_{\varphi_t}(\tau q + (1-\tau)p)\cdot(q-p)d\tau\right| \\ &\le Cr_1(p)\int_0^1 \left|\frac{\partial}{\partial s}J_{\varphi_t}(\tau q + (1-\tau)p)\right| d\tau \\ &\quad + Cr_2(p)\int_0^1 \left|\frac{\partial}{\partial w}J_{\varphi_t}(\tau q + (1-\tau)p)\right| d\tau \\ &\quad + Cr_3(p)\int_0^1 \left|\frac{\partial}{\partial u}J_{\varphi_t}(\tau q + (1-\tau)p)\right| d\tau .\end{aligned}$$

Using Lemmas 5.1, 5.5 and 4.4, we see that

$$\begin{aligned}r_1(p)\int_0^1 \left|\frac{\partial}{\partial s}J_{\varphi_t}(\tau q + (1-\tau)p)\right| d\tau &\le Cr_1(p)\gamma_k''(s)\gamma_k'(w) \\ &\le \epsilon|J_{\varphi_t}(p)|\end{aligned}$$

for $\epsilon > 0$ small. Similar remarks apply to the other partial derivatives of J_{φ_t} and we conclude that

$$|J_{\varphi_t}(q) - J_{\varphi_t}(p)| \le \tfrac{1}{2}|J_{\varphi_t}(p)|$$

which implies the desired result by taking $C_1 = \frac{1}{2}$ and $C_2 = \frac{3}{2}$.

Lemma 5.7. *There are positive constants C_1 and C_2 such that if $p \in \Omega_k^3$ and $q \in G^*(p)$,*

$$C_1 \le \frac{|J_{\varphi_t}(q)|}{|J_{\varphi_t}(p)|} \le C_2 .$$

Proof. The proof is similar to the proof of Lemma 5.6. We should note that for $p = (s, u, w) \in \Omega_k^3$,

$$w - u \ge C\frac{\log\lambda(2^{-k}u)}{\lambda(2^{-k}u)}$$

and so by Lemma 4.2,

$$\gamma_k''(u) \le C\gamma_k'(w).$$

Therefore from Lemmas 5.1, 5.5 and 4.4,

$$\begin{aligned}\tilde{r}_3(p)\left|\frac{\partial}{\partial u}J_{\varphi_t}(q)\right| &\le C\tilde{r}_3(p)\gamma_k'(s)\gamma_k'(w) \\ &\le \epsilon|J_{\varphi_t}(p)|\end{aligned}$$

for $\epsilon > 0$ small and for any $q \in G^*(p)$. The rest of the argument is the same as in Lemma 5.6.

Consequently, from the proofs of Lemmas 5.6 and 5.7 we have the following estimates. For $p \in \Omega_k^2$ and $q \in D^*(p)$,

$$\left|\frac{\partial}{\partial s} J_{\varphi_t}(q)\right| r_1(p) \leq C|J_{\varphi_t}(q)|, \tag{5.16}$$

$$\left|\frac{\partial}{\partial w} J_{\varphi_t}(q)\right| r_2(p) \leq C|J_{\varphi_t}(q)|, \tag{5.17}$$

$$\left|\frac{\partial}{\partial u} J_{\varphi_t}(q)\right| r_3(p) \leq C|J_{\varphi_t}(q)|. \tag{5.18}$$

Also for $p \in \Omega_k^3$ and $q \in G^*(p)$,

$$\left|\frac{\partial}{\partial s} J_{\varphi_t}(q)\right| \tilde{r}_1(p) \leq C|J_{\varphi_t}(q)|, \tag{5.19}$$

$$\left|\frac{\partial}{\partial w} J_{\varphi_t}(q)\right| \tilde{r}_2(p) \leq C|J_{\varphi_t}(q)|, \tag{5.20}$$

$$\left|\frac{\partial}{\partial u} J_{\varphi_t}(q)\right| \tilde{r}_3(p) \leq C|J_{\varphi_t}(q)|. \tag{5.21}$$

Lemma 5.8. *There are positive constants C_1 and C_2 such that if $p \in \Omega_k^2$ and $q \in D^*(p)$,*

$$C_1 \leq \frac{r_j(q)}{r_j(p)} \leq C_2, \qquad j = 1, 2, 3.$$

Also if $p \in \Omega_k^3$ and $q \in G^*(p)$, we have

$$C_1 \leq \frac{\tilde{r}_j(q)}{\tilde{r}_j(p)} \leq C_2, \qquad j = 1, 2, 3.$$

Proof. Let $p = (s, u, w)$ and $q = (s_1, u_1, w_1)$. Since $r_1(p) \leq \delta(s-t)$ by (5.13), it is clear that $C_1(s-t) \leq (s_1 - t) \leq C_2(s-t)$ for some positive constants C_1 and C_2. Similar remarks apply to $w - u$, $\operatorname{dist}(p, X)$ and $\operatorname{dist}(p, Y)$ and therefore we can find constants C_1 and C_2 such that

$$C_1 \leq \frac{\Delta(q)}{\Delta(p)} \leq C_2$$

for $q \in D^*(p)$ if $p \in \Omega_k^2$ and $q \in G^*(p)$ if $p \in \Omega_k^3$. Furthermore by Lemmas 5.5 and 4.4 we have similar estimates for the functions γ_k' and γ_k''. Now applying Lemmas 5.6 and 5.7 completes the proof of Lemma 5.8.

To end this section we will collect some of the estimates proved so far and put them in a form which will be useful in proving the estimate (1.2). Recall that to prove (1.2) we need to have estimates on $\frac{g_{ij}}{\det \mathscr{G}} = \frac{g_{ij}}{J_{\varphi_t}}$ where g_{ij} is the (i, j) cofactor of the matrix $\mathscr{G}$ introduced at the end of Section 3.

Lemma 5.9. *For* $p = (s, u, w) \in Q_{\ell n} \subset \Omega_k^1$,

$$\left|\frac{g_{1j}(p)}{\det \mathscr{G}(p)}\right| \le C2^{\ell}, \qquad j = 1, 2, 3, \tag{5.22}$$

$$\left|g_{11}(p)\frac{\partial}{\partial s}\left(\frac{1}{\det \mathscr{G}}\right)(p)\right| \le C2^{2\ell}\frac{\gamma_k''(s)}{\gamma_k'(s)^2}\gamma_k'(w), \tag{5.23}$$

$$\left|g_{12}(p)\frac{\partial}{\partial u}\left(\frac{1}{\det \mathscr{G}}\right)(p)\right| \le C2^{\ell}\left|\frac{g_{12}}{\det \mathscr{G}}(p)\right|, \tag{5.24}$$

and

$$\left|g_{13}(p)\frac{\partial}{\partial w}\left(\frac{1}{\det \mathscr{G}}\right)(p)\right| \le C2^{2\ell}. \tag{5.25}$$

For $p \in D^*(p_0) \subset \Omega_k^2$,

$$\left|\frac{g_{11}}{\det \mathscr{G}}(p)\right| \le C\frac{1}{r_3(p)\lambda(2^{-k}w)}, \tag{5.26}$$

$$\left|g_{11}(p)\frac{\partial}{\partial s}\left(\frac{1}{\det \mathscr{G}}\right)(p)\right| \le C\frac{1}{r_1(p)}\left|\frac{g_{11}}{\det \mathscr{G}}(p)\right|, \tag{5.27}$$

$$\left|\frac{g_{12}}{\det \mathscr{G}}(p)\right| \le C\frac{1}{r_1(p)\lambda(2^{-k}s)}, \tag{5.28}$$

$$\left|g_{12}(p)\frac{\partial}{\partial u}\left(\frac{1}{\det \mathscr{G}}\right)(p)\right| \le C\frac{1}{r_3(p)}\left|\frac{g_{12}}{\det \mathscr{G}}(p)\right|, \tag{5.29}$$

$$\left|\frac{g_{13}}{\det \mathscr{G}}(p)\right| \le C\frac{1}{r_1(p)}\frac{1}{\lambda(2^{-k}s)}, \tag{5.30}$$

and

$$\left|g_{13}(p)\frac{\partial}{\partial w}\left(\frac{1}{\det \mathscr{G}}\right)(p)\right| \le C\frac{1}{r_2(p)}\left|\frac{g_{13}}{\det \mathscr{G}}(p)\right|. \tag{5.31}$$

And for $p \in G^*(p_0) \subset \Omega_k^3$,

$$\left|\frac{g_{11}}{\det \mathscr{G}}(p)\right| \le C\frac{1}{\tilde{r}_3(p)}, \tag{5.32}$$

$$\left|g_{11}(p)\frac{\partial}{\partial s}\left(\frac{1}{\det \mathscr{G}}\right)(p)\right| \le C\frac{1}{\tilde{r}_1(p)}\left|\frac{g_{11}}{\det \mathscr{G}}(p)\right|, \tag{5.33}$$

$$\left|\frac{g_{12}}{\det \mathscr{G}}(p)\right| \le C\frac{1}{\tilde{r}_1(p)}, \tag{5.34}$$

$$\left|g_{12}(p)\frac{\partial}{\partial u}\left(\frac{1}{\det \mathscr{G}}\right)(p)\right| \le C\frac{1}{\tilde{r}_3(p)}\left|\frac{g_{12}}{\det \mathscr{G}}(p)\right|, \tag{5.35}$$

$$\left|\frac{g_{13}}{\det \mathscr{G}}(p)\right| \le C\frac{1}{\tilde{r}_3(p)}\frac{1}{\lambda(2^{-k}w)}, \tag{5.36}$$

and

$$\left|g_{13}(p)\frac{\partial}{\partial w}\left(\frac{1}{\det \mathscr{G}}\right)(p)\right| \le C\frac{1}{\tilde{r}_2(p)}\left|\frac{g_{13}}{\det \mathscr{G}}(p)\right|. \tag{5.37}$$

Proof. Using (5.3) we may write

$$g_{11}(p) = C\gamma_k'(w) - D\gamma_k'(u),$$
$$g_{12}(p) = D\gamma_k'(s) - A\gamma_k'(w),$$

and

$$g_{13}(p) = A\gamma_k'(u) - C\gamma_k'(s)$$

where A, C and D are written out in (5.5), (5.6) and (5.7). For $p = (s, u, w) \in R(t)$, we easily see that

$$|g_{1j}(p)| \le C\gamma_k'(s)\gamma_k'(w), \qquad j = 1, 2, 3. \tag{5.38}$$

However by examining $g_{13}(p)$, we actually have the estimate

$$|g_{13}(p)| \le C\gamma_k'(s)(\gamma_k'(u) + \gamma_k(w)).$$

Therefore if $p \in \Omega_k^1 \cup \Omega_k^3$,

$$|g_{13}(p)| \le C\frac{\gamma_k'(s)\gamma_k'(w)}{\lambda(2^{-k}w)}. \tag{5.39}$$

In fact for $p = (s, u, w) \in \Omega_k^1 \cup \Omega_k^3$,

$$w - u \ge C\frac{\log\lambda(2^{-k}u)}{\lambda(2^{-k}u)}$$

and so Lemmas 4.2 and 4.3 imply (5.39).

Now using the estimate from below on $\det\mathscr{G}(p) = J_{\varphi_t}(p)$ for $p \in \Omega_k^1$ obtained in Lemma 5.4, together with (5.38), gives us (5.22). In fact for $p = (s, u, w) \in Q_{\ell n}$, $|P_t(s, u, w)| \ge \epsilon 2^{-\ell}$ for some $\epsilon > 0$. With these observations, (5.23), (5.24) and (5.25) follow from a straightforward calculation, using (5.38), (5.39) and Lemma 5.1. Also from (5.38), the estimates contained in (5.26), (5.28), (5.30), (5.32) and (5.34) follow directly from the definitions of r_1, r_2, r_3, etc. Estimate (5.36) follows in a similar fashion from (5.39). Finally (5.27), (5.29), (5.31), (5.33) and (5.35) are consequences of the estimates (5.16)–(5.21). This completes the proof.

6. The one to one property of φ_t

In this section we prove that φ_t is one to one on each ball $B(p)$.

Lemma 6.1. *Let A, B, C and X be vectors in $\mathbb{R}^3$. Then*

$$\det(A, B, [X, C]) = \det([A, B], X, C)$$

and

$$\det([A, X], B, C) + \det(A, [B, X], C) + \det(A, B, [C, X]) = 0.$$

Proof. The proof can be established by direct computation.

Lemma 6.2. *φ_t is one to one on each $B(p)$.*

Proof. We shall show that $\varphi_t(s_1, u_1, w_1) \neq \varphi_t(s_2, u_2, w_2)$ for any two distinct points (s_1, u_1, w_1) and (s_2, u_2, w_2) in $B(p)$. We will further assume $s_1 \neq s_2$, $u_1 \neq u_2$ and $w_1 \neq w_2$. If two of the coordinates of the distinct points are the same, the proof follows trivially from the convexity of $\gamma(t)$. Now

$$\begin{aligned}\varphi_t(s_1, u_1, w_1) &- \varphi_t(s_2, u_2, w_2)\\ &= \varphi_t(s_1, u_1, w_1) - \varphi_t(s_2, u_1, w_1)\\ &\quad + \varphi_t(s_2, u_1, w_1) - \varphi_t(s_2, u_2, w_1)\\ &\quad + \varphi_t(s_2, u_2, w_1) - \varphi_t(s_2, u_2, w_2)\\ &= I + II + III\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}I = {}&\Gamma_k(s_1) - \Gamma_k(s_2)\\ &+ \beta[\Gamma_k(s_1) - \Gamma_k(s_2), -\Gamma_k(t) + \Gamma_k(u_1) - \Gamma_k(w_1)],\end{aligned} \tag{6.2}$$

$$\begin{aligned}II = {}&\Gamma_k(u_1) - \Gamma_k(u_2)\\ &+ \beta[\Gamma_k(u_1) - \Gamma_k(u_2), -\Gamma_k(s_2) + \Gamma_k(t) - \Gamma_k(w_1)],\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}III = {}&\Gamma_k(w_2) - \Gamma_k(w_1)\\ &+ \beta[\Gamma_k(w_2) - \Gamma_k(w_1), -\Gamma_k(s_2) + \Gamma_k(t) - \Gamma_k(u_2)].\end{aligned} \tag{6.4}$$

It suffices to prove I, II and III are linearly independent, and hence it suffices to show that $\det(I, II, III) \neq 0$. Let $\alpha(s_1, s_2, u_1, u_2, w_1, w_2) = \det(I, II, III)$. Then $\alpha(s, s, u_1, u_2, w_1, w_2) = 0$ and so we may write either

$$\begin{aligned}\alpha(s_1&, s_2, u_1, u_2, w_1, w_2)\\ &= -\int_{s_1}^{s_2} \frac{\partial \alpha}{\partial s_1}(s, s_2, u_1, u_2, w_1, w_2)\,ds\end{aligned} \tag{6.5}$$

or

$$\begin{aligned}\alpha(s_1&, s_2, u_1, u_2, w_1, w_2)\\ &= \int_{s_1}^{s_2} \frac{\partial \alpha}{\partial s_2}(s_1, s, u_1, u_2, w_1, w_2)\,ds.\end{aligned} \tag{6.6}$$

Furthermore, $\alpha(s_1, s_2, u, u, w_1, w_2) = 0$ so

$$\frac{\partial \alpha}{\partial s_1}(s, s_2, u, u, w_1, w_2) = 0$$

and

$$\frac{\partial \alpha}{\partial s_2}(s_1, s, u, u, w_1, w_2) = 0.$$

Thus we may use the reasoning establishing (6.5) and (6.6) to write α as a double integral involving

$$\frac{\partial^2 \alpha}{\partial s_1 \partial u_1}, \frac{\partial^2 \alpha}{\partial s_1 \partial u_2}, \frac{\partial^2 \alpha}{\partial s_2 \partial u_1} \quad \text{or} \quad \frac{\partial^2 \alpha}{\partial s_2 \partial u_2}.$$

After one more iteration we find

$$\alpha(s_1, s_2, u_1, u_2, w_1, w_2) = \pm \int_{s_1}^{s_2} \int_{u_1}^{u_2} \int_{w_1}^{w_2} \frac{\partial^3 \alpha}{\partial s_a \partial u_b \partial w_c} ds du dw, \tag{6.7}$$

where each of a, b and c can be chosen to be either 1 or 2. Thus it suffices to show that for some choice of (a, b, c)

$$\frac{\partial^3}{\partial s_a \partial u_b \partial w_c} \det(I, II, III) \neq 0$$

in $B(p)$.

The computation of $\frac{\partial^3}{\partial s_a \partial u_b \partial w_c} \det(I, II, III)$ is quite tedious, but can be somewhat simplified using Lemma 6.1. In order to express the result we use the notation that if $a = 2$, $a' = 1$ and if $a = 1$, $a' = 2$, and similarly with b and c. Then we find

$$\begin{aligned} &\pm \frac{\partial^3}{\partial s_a \partial u_b \partial w_c} \det(I, II, III) \\ &\quad = J_{\varphi_t}(s_a, u_b, w_c) \\ &\quad + \beta \det([\Gamma_k(s_a) - \Gamma_k(s_{a'}), \Gamma'_k(s_a)], \Gamma'_k(u_b), -\Gamma'_k(w_c)) \\ &\quad + \beta \det(\Gamma'_k(s_a), \Gamma'_k(u_b), [\Gamma'_k(w_c), \Gamma_k(w_c) - \Gamma_k(w_{c'})]). \end{aligned} \tag{6.8}$$

More explicitly,

$$\begin{aligned} &\pm \frac{\partial^3}{\partial s_a \partial u_b \partial w_c} \det(I, III, III) \\ &\quad = J_{\varphi_t}(s_a, u_b, w_c) \\ &\quad + \beta(\gamma'_k(s_a)(s_a - s_{a'}) - (\gamma_k(s_a) - \gamma_k(s_{a'})))(\gamma'_k(u_b) - \gamma'_k(w_c)) \\ &\quad + \beta(\gamma'_k(w_c)(w_c - w_{c'}) - (\gamma_k(w_c) - \gamma_k(w_{c'})))(\gamma'_k(s_a) - \gamma'_k(u_b)). \end{aligned} \tag{6.9}$$

Choose a and c so that $s_{a'} < s_a$ and $w_{c'} < w_c$, i.e., if $s_1 < s_2$ choose $a = 2$, and if $s_2 < s_1$ choose $a = 1$. Then if $p \in Q^*_{\ell n} \subset \Omega^1_k$,

$$\left| \frac{\partial^3 \det(I, II, III)}{\partial s_a \partial u_b \partial w_c} \right| = |J_{\varphi_t}(s_a, u_b, w_c)| + E, \tag{6.10}$$

where

$$|E| \leq C \operatorname{diam}(Q_{\ell n}) \gamma'_k(s_a) \gamma'_k(w_c). \tag{6.11}$$

Recall that for any point $p = (s, u, w) \in Q^*_{\ell n}$,

$$|P_t(s, u, w)| \geq \epsilon 2^{-\ell} \text{ for some } \epsilon > 0.$$

Therefore according to Lemma 5.4,

$$|J_{\varphi_t}(s_a, u_b, w_c)| \geq \epsilon 2^{-\ell} \gamma'_k(s_a) \gamma'_k(w_c).$$

Now if we choose ϵ_0 in the definition of $Q_{\ell n}$ sufficiently small, (6.10) and (6.11) imply $\det(I, II, III) \neq 0$ on $Q_{\ell n}^*$.

Next suppose $p = (s, u, w) \in \Omega_k^2$. Then we obtain from (6.9),

$$\left| \frac{\partial^3 \det(I, II, III)}{\partial s_a \partial u_b \partial w_c} \right| = |J_{\varphi_t}(s_a, u_b, w_c)| + E \tag{6.12}$$

where

$$|E| \leq C\gamma_k'(s_a)\gamma_k'(w_c)(r_1(p) + r_2(p)). \tag{6.13}$$

By Lemma 5.8,

$$|E| \leq c\gamma_k'(s_a)\gamma_k'(w_c)(r_1(s_a, u_b, w_c) + r_2(s_a, u_b, w_c)).$$

Hence from the definitions of r_1 and r_2, (3.10) and (3.11), we see that

$$\frac{\partial^3 \det(I, II, III)}{\partial s_a \partial u_b \partial w_c} \neq 0 \tag{6.14}$$

on $B(p)$ if ϵ in (3.10) and (3.11) is chosen sufficiently small. The same reasoning gives (6.14) on $B(p)$ where $p \in \Omega_k^3$. This completes the proof that φ_t is one to one on each ball $B(p)$.

7. A partition of unity

We wish to find a sequence of points $p_1, p_2, p_3, \ldots$ in $R(t)$, balls $B(p_1)$, $B(p_2), \ldots$, and corresponding functions $\psi_1, \psi_2, \ldots$ which form a suitable partition of unity.

Lemma 7.1. *There exists a sequence of points $\{p_j\}$ in $R(t)$ such that except for a set of measure zero (the zero set of J_{φ_t}),*

$$R(t) = \cup B(p_j). \tag{7.1}$$

(7.2) *There is a constant C so that no point x is in more than C of the balls $B(p_j)$.*

Moreover there exists nonnegative C^∞ functions ψ_j such that

(7.3) *support of $\psi_j \subset B(p_j)$.*

(7.4) *$\sum \psi_j \equiv 1$, except on a set of measure zero.*

(7.5) *If $p_j \in Q_{\ell n}^* \subset \Omega_k^1$,*

$$\|\nabla \psi_j\|_{L^\infty} \leq C 2^{-\ell}.$$

(7.6) *If $p_j \in \Omega_k^2$,*

$$\left\| \frac{\partial \psi_j}{\partial s} \right\|_{L^\infty} \leq C \frac{1}{r_1(p_j)}, \quad \left\| \frac{\partial \psi_j}{\partial w} \right\|_{L^\infty} \leq C \frac{1}{r_2(p_j)},$$

and

$$\left\| \frac{\partial \psi_j}{\partial u} \right\|_{L^\infty} \leq C \frac{1}{r_3(p_j)}.$$

(7.7) *If* $p_j \in \Omega_k^3$,

$$\left\|\frac{\partial \psi_j}{\partial s}\right\|_{L^\infty} \le C\frac{1}{\tilde r_1(p_j)}, \quad \left\|\frac{\partial \psi_j}{\partial w}\right\|_{L^\infty} \le C\frac{1}{\tilde r_2(p_j)},$$

and

$$\left\|\frac{\partial \psi_j}{\partial u}\right\|_{L^\infty} \le C\frac{1}{\tilde r_3(p_j)}.$$

Lemma 7.1 follows from the next lemma.

Lemma 7.2. *There exists a sequence of points p_j in Ω_k^2 such that $\cup D(p_j) = \Omega_k^2$ and every point is in at most C of the $D^*(p_j)$. Also there is a sequence of points p_j in Ω_k^3 such that $\cup G(p_j) = \Omega_k^3$ and no point is in more than C of the $G^*(p_j)$.*

Let us note that an analogous statement for the $Q_{\ell,n}$ is well known. The proof of Lemma 7.2 follows the lines of a similar argument due to Sogge and Stein [SS].

For $\eta > 0$ and $p_0 = (s_0, u_0, w_0) \in \Omega_k^2$, set

$$\begin{aligned} D_*(p_0) = \{(s, u, w)\,|s_0 - \eta r_1(p_0) < s < s_0 + \eta r_1(p_0),\\ w_0 - \eta r_2(p_0) < w < w_0 + \eta r_2(p_0),\\ u_0 - \eta r_3(p_0) < u < u_0 + \eta r_3(p_0)\},\end{aligned}$$

and for $p_0 \in \Omega_k^3$, let

$$\begin{aligned} G_*(p_0) = \{(s, u, w)\,|s_0 - \eta \tilde r_1(p_0) < s < s_0 + \eta \tilde r_1(p_0),\\ w_0 - \eta \tilde r_2(p_0) < w < w_0 + \eta \tilde r_2(p_0),\\ u_0 - \eta \tilde r_3(p_0) < u < u_0 + \eta \tilde r_3(p_0)\}.\end{aligned}$$

Lemma 7.2 follows from the following covering lemma.

Lemma 7.3. . *If $\eta > 0$ is sufficiently small, there exists a sequence $\{p_j\}$ in Ω_k^2 such that the $D_*(p_j)$ are disjoint, $\cup D(p_j)$ covers Ω_k^2 and no point is in more than C of the $D^*(p_j)$. Similarly there exists a sequence $\{p_j\}$ in Ω_k^3 such that the $G_*(p_j)$ are disjoint, $\cup G(p_j)$ covers Ω_k^3 and no point is in more than C of the $G^*(p_j)$.*

Proof. We shall just consider the first statement of the lemma. The second statement is proved in a similar fashion. We will use Lemma 5.8 which asserts that if q_1 and q_2 are in $D^*(p)$ for some $p \in \Omega_k^2$, then $r_j(q_1)$ and $r_j(q_2)$ are comparble for $j = 1, 2$, and 3. Taking this into account, we may use a Vitali type procedure to select balls, $D_*(p_j)$, according to the size of the measure of the $D_*(p_j)$, and find a sequence of points $\{p_j\}$ such that the $D_*(p_j)$ are disjoint and $\cup D(p_j)$ covers Ω_k^2. We claim that any point p is in at most C of the $D^*(p_j)$. Suppose p is in N of the $D^*(p_j)$. Then by Lemma 5.8, we have N disjoint rectangular parallelepipeds with volume $\approx \eta r_1(p) r_2(p) r_3(p)$

contained in a fixed rectangle of volume at most $Cr_1(p)r_2(p)r_3(p)$. Clearly this puts a bound on N.

8. The proof of (1.2)—the contribution from Ω_k^1

As was pointed out in Section 1, we must show

$$\int_{R(t)} f(\varphi_t(s, u, w))dsdudw = \int_{\mathbf{R}^2} f(x)\rho_t(x)dx$$

where for some $\epsilon > 0$ and constant C,

$$\int_{\mathbf{R}^2} |\rho_t(x \cdot y) - \rho_t(x)|dx \leq Ch^\epsilon \tag{8.1}$$

if $\|y\| < h$. We shall show for $\sigma = 1, 2, 3$,

$$\int_{\Omega_k^\sigma} f(\varphi_t(s, u, w))dsdudw = \int_{\mathbf{R}^2} f(x)\rho_t^\sigma(x)dx$$

where each ρ_t^σ satisfies (8.1). In this section we consider the case $\sigma = 1$.

Recall from Section 7 that we have functions $\psi_{\ell,n}$ such that $\operatorname{supp}(\psi_{\ell,n}) \subset Q_{\ell,n}^*$, $\sum \psi_{\ell,n} = 1$ on Ω_k^1, and

$$\|\nabla\psi_{\ell,n}\|_{L^\infty} \leq C2^{+\ell}. \tag{8.2}$$

Then

$$\begin{aligned}\int_{\Omega_k^1} & f(\varphi_t(s, u, w))dsdudw \\ &= \sum_{\ell,n} \int \psi_{\ell,n}(s, u, w)f(\varphi_t(s, u, w))dsdudw \\ &= \sum_{\ell,n} \int \rho_{\ell,n}(x)f(x)dx,\end{aligned}$$

where

$$\rho_{\ell,n}(x) = \frac{\psi_{\ell,n}(\varphi_t^{-1}(x))}{|J_{\varphi_t}(\varphi_t^{-1}(x))|}.$$

Here we have used Lemma 6.2 to justify the above change of variables. Thus

$$\rho_t^1 = \sum_{\ell,n} \rho_{\ell,n}.$$

To show (8.1), it suffices to show

$$\int |\rho_{\ell,n}(x \cdot y) - \rho_{\ell,n}(x)|dx \leq C2^{-3\ell} \tag{8.3}$$

and

$$\int |\rho_{\ell,n}(x \cdot y) - \rho_{\ell,n}(x)|dx \leq C\|y\|. \tag{8.4}$$

In fact if (8.3) and (8.4) are both valid, then

$$\begin{aligned}\int |\rho_t^1(x\cdot y)-\rho_t^1(x)|dx &\le \sum_{\ell,n}\int |\rho_{\ell,n}(x\cdot y)-\rho_{\ell,n}(x)|dx\\ &\le C\sum_{\ell\ge 0}\sum_{m\le C2^{2\ell}}\min(2^{-3\ell},\|y\|)\\ &\le C\left[\sum_{2^{-3\ell}\le\|y\|}2^{-\ell}+\|y\|\sum_{2^{-3\ell}\ge\|y\|}2^{2\ell}\right]\\ &= C\|y\|^{1/3}\end{aligned}$$

which gives (8.1) for $\sigma=1$.

The proof of (8.3) is trivial as the left-hand side is at most

$$\begin{aligned}2\int|\rho_{\ell,n}(x)|dx &= 2\int \psi_{\ell,n}(s,u,w)dsdudw\\ &\le 2|Q^*_{\ell n}|\le C2^{-3\ell}.\end{aligned}$$

We turn to the proof of (8.4). By writing

$$\rho_{\ell,n}(x\cdot y)-\rho_{\ell,n}(x)=\int_0^1\frac{d}{dt}\rho_{\ell,n}(x\cdot ty)dt,$$

and using the chain rule, we find that

$$\begin{aligned}&\int|\rho_{\ell,n}(x\cdot y)-\rho_{\ell,n}(x)|dx\\ &\le |y_1|\int_0^1\int\left|\frac{\partial\rho_{\ell,n}}{\partial x_1}(x\cdot ty)-\beta x_2\frac{\partial\rho_{\ell,n}}{\partial x_3}(x\cdot ty)\right|dxdt\\ &+|y_2|\int_0^1\int\left|\frac{\partial\rho_{\ell,n}}{\partial x_2}(x\cdot ty)+\beta x_1\frac{\partial\rho_{\ell,n}}{\partial x_3}(x\cdot ty)\right|dxdt\\ &+|y_3|\int_0^1\int\left|\frac{\partial\rho_{\ell,n}}{\partial x_3}(x\cdot ty)\right|dxdt.\end{aligned}$$

Replacing $x\cdot ty$ by x in the inner integrals, we find that

$$\int|\rho_{\ell,n}(x\cdot y)-\rho_{\ell,n}(x)|dx\le C\|y\|\sum_{j=1}^3\int|L_j(\rho_{\ell,n})(x)|dx$$

where

$$L_1=\frac{\partial}{\partial x_1}-\beta x_2\frac{\partial}{\partial x_3}$$
$$L_2=\frac{\partial}{\partial x_2}+\beta x_1\frac{\partial}{\partial x_3}$$

and

$$L_3=\frac{\partial}{\partial x_3}$$

are the left invariant vector fields discussed in Section 3. Of the three terms, the one corresponding to L_1 is the most difficult. To some extent this can be seen because the coordinate x_2 is unbounded in the support of $\rho_{\ell,n}$. Therefore we will restrict our attention to L_1 and in fact we will show

$$\int |L_1(\rho_{\ell,n})(x)|dx \leq C. \tag{8.5}$$

To prove (8.5) we will first make the change of variables $x = \varphi_t(s, u, w)$. To do this, let us define

$$K(s, u, w) = \frac{\psi_{\ell,n}(s, u, w)}{|J_{\varphi_t}(s, u, w)|}.$$

Note that $\rho_{\ell,n}(x) = K(\varphi_t^{-1}(x))$. Formula (3.24) shows us how L_1 transforms under the change of variables $x = \varphi_t(s, u, w)$ and so we find

$$\begin{aligned}\int &|L_1(\rho_{\ell,n})(x)|dx \\ &= \int |\det\mathscr{G}(s, u, w)||\langle(\mathscr{G}^{-1})^*\nabla K(s, u, w), e_1\rangle|dsdudw.\end{aligned}$$

Recall that $J_{\varphi_t}(s, u, w) = \det\mathscr{G}(s, u, w)$. By computing the first component of $(\mathscr{G}^{-1})^*\nabla K(s, u, w)$, we see that

$$\begin{aligned}\int |L_1(\rho_{\ell,n})(x)|dx &\leq \int \left|g_{11}(s, u, w)\frac{\partial K}{\partial s}\right| dsdudw \\ &\quad + \int \left|g_{12}(s, u, w)\frac{\partial K}{\partial u}\right| dsdudw \\ &\quad + \int \left|g_{13}(s, u, w)\frac{\partial K}{\partial w}\right| dsdudw \\ &= I + II + III.\end{aligned}$$

Using (8.2), (5.22) and (5.23), we have

$$\begin{aligned} I &\leq C\left[2^{\ell}\int_{Q^*_{\ell,n}} \left|\frac{g_{11}}{\det\mathscr{G}}(p)\right| dp \right.\\ &\qquad \left. + \int_{Q^*_{\ell,n}} \left|g_{11}(p)\frac{\partial}{\partial s}\left(\frac{1}{\det\mathscr{G}}\right)(p)\right| dp\right] \\ &\leq C\left[2^{2\ell}|Q^*_{\ell,n}| + 2^{2\ell}\int_{Q^*_{\ell,n}} \frac{\gamma_k''(s)}{\gamma_k'(s)^2}\gamma_k'(w)\right] \\ &\leq C[2^{-\ell} + 1] \leq C.\end{aligned}$$

The estimate for the second integral follows by integrating in s first and noting that every w value is smaller than any s value in $Q^*_{\ell,n}$.

To estimate II, we may use (8.2), (5.22) and (5.24) to see that

$$\begin{aligned} II &\le C\left[2^{\ell}\int_{Q^*_{\ell,n}}\left|\frac{g_{12}}{\det\mathcal{G}}(p)\right|dp\right. \\ &\quad\left.+\int_{Q^*_{\ell,n}}\left|g_{12}(p)\frac{\partial}{\partial u}\left(\frac{1}{\det\mathcal{G}}\right)(p)\right|dp\right] \\ &\le C2^{2\ell}|Q^*_{\ell,n}|\le C2^{-\ell}. \end{aligned}$$

The estimate for III is the same as for II. We simply use (5.25) instead of (5.24). This completes the proof of (8.5) and so we have shown that ρ^1_t satisfies (8.1).

9. The proof of (1.2)—the contribution from Ω^2_k

Let $\{\psi_\ell\}$ denote the functions from Lemma 7.1 which are supported in the rectangular parallelepipeds $D^*(p_\ell)$, $p_\ell\in\Omega^2_k$. Then

$$\int_{\Omega^2_k} f(\varphi_t(s,u,w))dsdudw=\int f(x)\rho^2_t(x)dx,$$

where

$$\rho^2_t(x)=\sum_\ell\frac{\psi_\ell(\varphi_t^{-1}(x))}{|J_{\varphi_t}(\varphi_t^{-1}(x))|}. \tag{9.1}$$

We have to show

$$\int|\rho^2_t(x\cdot y)-\rho^2_t(x)|dx\le Ch^{\epsilon} \tag{9.2}$$

for $h<\|y\|$.

We let

$$\rho_\ell(x)=\frac{\psi_\ell(\varphi_t^{-1}(x))}{|J_{\varphi_t}(\varphi_t^{-1}(x))|}$$

and we shall show

$$\int|\rho_\ell(x\cdot y)-\rho_\ell(x)|dx\le Cr_1(p_\ell)r_2(p_\ell)r_3(p_\ell), \tag{9.3}$$

$$\int|\rho_\ell(x\cdot y)-\rho_\ell(x)|dx\le C\frac{r_2(p_\ell)}{\lambda(2^{-k}s_\ell)}\|y\|, \tag{9.4}$$

where $p_\ell=(s_\ell,u_\ell,w_\ell)$, and ψ_ℓ is supported in $D^*(p_\ell)$.

The proof of (9.3) is immediate since

$$|D^*(p_\ell)|\le Cr_1(p_\ell)r_2(p_\ell)r_3(p_\ell).$$

We turn to the proof of (9.4). As before let

$$K(s,u,w)=\frac{\psi_\ell(s,u,w)}{|J_{\varphi_t}(s,u,w)|}.$$

Then as in the proof of (8.4), we see that the left-hand side of (9.4) is dominated by $\|y\|$ $(I + II + III)$ plus simpler integrals, where

$$I = \int \left| g_{11}(s, u, w) \frac{\partial K}{\partial s} \right| dsdudw,$$

$$II = \int \left| g_{12}(s, u, w) \frac{\partial K}{\partial u} \right| dsdudw,$$

and

$$III = \int \left| g_{13}(s, u, w) \frac{\partial K}{\partial w} \right| dsdudw.$$

Hence to show (9.4), it suffices to show

$$I + II + III \leq C \frac{r_2(p_\ell)}{\lambda(2^{-k} s_\ell)}. \tag{9.5}$$

Inequality (9.5) will follow from Lemma 5.9. First of all from (7.6), (5.27), (5.29), (5.31) and Lemma 5.8, we have

$$I \leq C \frac{1}{r_1(p_\ell)} \int_{D^*(p_\ell)} \left| \frac{g_{11}(p)}{\det \mathscr{G}(p)} \right| dp,$$

$$II \leq C \frac{1}{r_2(p_\ell)} \int_{D^*(p_\ell)} \left| \frac{g_{12}}{\det \mathscr{G}}(p) \right| dp,$$

and

$$III \leq C \frac{1}{r_3(p_\ell)} \int_{D^*(p_\ell)} \left| \frac{g_{13}}{\det \mathscr{G}}(p) \right| dp.$$

Now using (5.26), (5.28), (5.30) and Lemma 5.8, we see that the above integrals are each bounded by $\frac{r_2(p_\ell)}{\lambda(2^{-k} s_\ell)}$ and this gives (9.5). In the above analysis we used the fact that the function $\lambda(2^{-k} \cdot)$ does not change much in $D^*(p_\ell)$. This follows from Lemmas 4.4 and 5.5. This completes the proof of (9.4).

Inequalities (9.3) and (9.4) together imply for $0 \leq \delta \leq 1$,

$$\int |\rho_\ell(x \cdot y) - \rho_\ell(x)| dx \leq C |D_*(p_\ell)|^{1-\delta} \left[\frac{r_2(p_\ell)}{\lambda(2^{-k} s_\ell)} \right]^\delta \|y\|^\delta$$

$$\leq C \|y\|^\delta \int_{D_*(p_\ell)} \left[\frac{r_2(p_\ell)}{|D_*(p_\ell)| \lambda(2^{-k} s_\ell)} \right]^\delta dp$$

$$\leq C \|y\|^\delta \int_{D_*(p_\ell)} \left[\frac{1}{r_1(p) r_3(p) \lambda(2^{-k} s)} \right]^\delta dp$$

$$\leq C \|y\|^\delta \int_{D_*(p_\ell)} \lambda(2^{-k} w)^\delta \left[\frac{\gamma_k'(s) \gamma_k'(w)}{|J_{\varphi_\ell}(s, u, w)|} \right]^{2\delta} \left[\frac{1}{\Delta(p)} \right]^{2\delta} dsdudw.$$

Therefore

$$\int |\rho_t^2(x\cdot y) - \rho_t^2(x)|dx$$
$$\leq C\|y\|^\delta \int_{\Omega_k^2} \lambda(2^{-k}u)^\delta \left[\frac{\gamma_k'(s)\gamma_k'(w)}{|J_{\varphi_t}(s\,,\,u\,,\,w)|}\right]^{2\delta} \frac{1}{|\Delta(s\,,\,u\,,\,w)|^{2\delta}} ds du dw$$
$$\leq C\|y\|^\delta \int_{\Omega_k^2} \frac{1}{(w-u)^\delta} \frac{1}{|\Delta(s\,,\,u\,,\,w)|^{2\delta}} \left[\frac{\gamma_k'(s)\gamma_k'(w)}{|J_{\varphi_t}(s\,,\,u\,,\,w)|}\right]^{2\delta} ds du dw.$$

We used here the fact that for $p = (s\,,\,u\,,\,w) \in \Omega_k^2$,

$$w - u \geq C\frac{\log \lambda(2^{-k}u)}{\lambda(2^{-k}u)} \geq C\frac{1}{\lambda(2^{-k}u)}$$

if k is large enough. Now we do the s integration first and use Lemma 5.2 to estimate J_{φ_t} from below and we obtain

$$\int |\rho_t^2(x\cdot y) - \rho_t^2(x)|dx$$
$$\leq C\|y\|^\delta \int\int_{\Omega_k^2} \frac{1}{(w-u)^{5\delta}} du dw \leq C\|y\|^\delta$$

if δ is sufficiently small. This concludes the proof of (9.2).

10. The proof of (1.2)—the contribution from Ω_k^3

Let $\{\psi_\ell\}$ denote the functions from Lemma 7.1 which are supported in the rectangular parallelepipeds $G^*(p_\ell)$, $p_\ell \in \Omega_k^3$. Then we have

$$\int_{\Omega_k^3} f(\varphi_t(s\,,\,u\,,\,w)) ds du dw = \int f(x)\rho_t^3(x)dx$$

where

$$\rho_t^3(x) = \sum_\ell \frac{\psi_\ell(\varphi_t^{-1}(x))}{|J_{\varphi_t}(\varphi_t^{-1}(x))|}. \tag{10.1}$$

We must show

$$\int |\rho_t^3(x\cdot y) - \rho_t^3(x)|dx \leq Ch^\epsilon, \tag{10.2}$$

for $\|y\| < h$.

We let

$$\rho_\ell(x) = \frac{\psi_\ell(\varphi_t^{-1}(x))}{|J_{\varphi_t}(\varphi_t^{-1}(x))|}.$$

We wish to show

$$\int |\rho_\ell(x\cdot y) - \rho_\ell(x)|dx \leq C\tilde{r}_1(p_\ell)\tilde{r}_2(p_\ell)\tilde{r}_3(p_\ell), \tag{10.3}$$

and

$$\int |\rho_\ell(x\cdot y)-\rho_\ell(x)|dx \le C\|y\|\tilde{r}_2(p_\ell). \tag{10.4}$$

Again (10.3) is trivial. Furthermore, in analogy with Sections 8 and 9, the main work in proving (10.4) is the estimate

$$I+II+III\le C\tilde{r}_2(p_\ell), \tag{10.5}$$

where

$$I=\int_{G^*(p_\ell)}\left|g_{11}(p)\frac{\partial K}{\partial s}\right|dp,$$

$$II=\int_{G^*(p_\ell)}\left|g_{12}(p)\frac{\partial K}{\partial u}\right|dp,$$

and

$$III=\int_{G^*(p_\ell)}\left|g_{13}(p)\frac{\partial K}{\partial w}\right|dp.$$

As in Section 9, we can use (7.7), (5.33), (5.35), (5.37) and Lemma 5.8 to show

$$I\le C\frac{1}{\tilde{r}_1(p_\ell)}\int_{G^*(p_\ell)}\left|\frac{g_{11}(p)}{\det\mathcal{G}(p)}\right|dp,$$

$$II\le C\frac{1}{\tilde{r}_2(p_\ell)}\int_{G^*(p_\ell)}\left|\frac{g_{12}}{\det\mathcal{G}}(p)\right|dp,$$

and

$$III\le C\frac{1}{\tilde{r}_3(p_\ell)}\int_{G^*(p_\ell)}\left|\frac{g_{13}}{\det\mathcal{G}}(p)\right|dp.$$

Finally using (5.32), (5.34), (5.36) and Lemma 5.8 gives us (10.5). This completes the proof of (10.4).

We now complete the proof of (10.2) in analogy with Section 9. From (10.3) and (10.4), we see that for every $0<\delta\le 1$,

$$\begin{aligned}
\int_{G^*(p_\ell)}|\rho_\ell(x\cdot y)-\rho_\ell(x)|dx &\le C|G_*(p_\ell)|^{1-\delta}(\tilde{r}_2(p_\ell)\|y\|)^\delta\\
&\le C\|y\|^\delta\int_{G_*(p_\ell)}\left[\frac{1}{\tilde{r}_1(p_\ell)\tilde{r}_3(p_\ell)}\right]^\delta dp\\
&\le C\|y\|^\delta\int_{G_*(p_\ell)}\lambda(2^{-k}s)^\delta\left|\frac{\gamma_k'(s)\gamma_k'(w)}{J_{\varphi_t}(p)\Delta(p)}\right|^{2\delta}dp.
\end{aligned}$$

Summing on ℓ gives us

$$\begin{aligned}
&\int|\rho_t^3(x\cdot y)-\rho_t^3(x)|dx\\
&\le C\|y\|^\delta\int_{\Omega_k^3}\lambda(2^{-k}s)^\delta\left|\frac{\gamma_k'(s)\gamma_k'(w)}{J_{\varphi_t}(p)\Delta(p)}\right|^{2\delta}dp\\
&\le C\|y\|^\delta\int_{\Omega_k^3}\frac{1}{|P_t(s,u,w)|^\delta}\left|\frac{\gamma_k'(s)\gamma_k'(w)}{J_{\varphi_t}(p)\Delta(p)}\right|^{2\delta}dp
\end{aligned}$$

since for $p = (s, u, w) \in \Omega_k^3$, $|P_t(s, u, w)| \geq C\frac{1}{\lambda(2^{-k}s)}$. Doing the s integral first and using Lemma 5.2 to estimate J_{φ_t} from below now gives us (10.2) if we choose δ above to be sufficiently small. This completes the proof of (1.2) and (1.3).

11. The L^p boundedness of $\mathcal{M}f$

Let $d\nu_k$ be the measure which acts on a test function by

$$d\nu_k(f) = 2^k \int_{2^{-k}}^{2^{-k+1}} f(\Gamma(t))dt.$$

With the notation introduced in Section 1 we see that $d\nu_k$ is simply the measure such that $M_k f(x) = f * d\nu_k(x)$. Also in Section 1 we saw that the L^p estimates for $\mathcal{M}f$ follow from

$$\|Mf\|_{L^p} \leq C_p \|f\|_{L^p}, \qquad 1 < p \leq \infty, \tag{11.1}$$

where $Mf(x) = \sup_{k>0} |M_k f(x)|$.

We will prove (11.1) by following the bootstrap argument in [NSW1]. That is, we shall prove the following three lemmas.

Lemma 11.1. *M is bounded on $L^2(\mathbb{R}^3)$.*

Lemma 11.2. *Suppose that for some $p_0 < 2$,*

$$\left\| \left(\sum_k |f_k * d\nu_k|^2 \right)^{1/2} \right\|_{L^{p_0}} \leq C_{p_0} \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_{L^{p_0}};$$

then

$$\|Mf\|_{L^p} \leq C_p \|f\|_{L^p}, \qquad p_0 < p \leq 2.$$

Lemma 11.3. *Suppose that for some $p_0 \leq 2$,*

$$\|Mf\|_{L^{p_0}} \leq C_{p_0} \|f\|_{L^{p_0}};$$

then

$$\left\| \left(\sum_k |f_k * d\nu_k|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p}$$

for all p with $\frac{1}{p} \leq \frac{1}{2}\left(\frac{1}{p_0} + 1\right)$.

The proof of Lemma 11.3 is given in [NSW1], and so we concentrate on the proofs of Lemmas 11.1 and 11.2. We begin by recalling some known results. Let

$$\delta(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & \gamma(t) & 0 \\ 0 & 0 & t\gamma(t) \end{pmatrix}. \tag{11.2}$$

The convexity of γ implies

$$\|\delta^{-1}(t)\delta(s)\| \leq s/t \tag{11.3}$$

if $s < t$. We also put $A_k = \delta(2^{-k})$ and note that

$$\|A_k^{-1} A_{k+1}\| \le 1/2. \tag{11.4}$$

Let $\psi \in C_0^\infty(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} \psi = 1$ and $\psi(x) = \psi(-x)$. Set

$$\Psi_k(x) = \frac{1}{\det A_{k+1}} \psi(A_{k+1}^{-1} x) - \frac{1}{\det A_k} \psi(A_k^{-1} x).$$

Then we have the following Littlewood-Paley inequality,

$$\left\| \left(\sum_k |f * \Psi_k|^2 \right)^{1/2} \right\|_{L^p} \le C_p \|f\|_{L^p}, \quad 1 < p < \infty. \tag{11.5}$$

See [CVWW].

Now let φ be a second C_0^∞ function on $\mathbb{R}^3$ with $\varphi \ge 0$, $\int \varphi = 1$, and $\varphi(x) = \varphi(-x)$. Set

$$\varphi_k(x) = \frac{1}{\det A_k} \varphi(A_k^{-1} x),$$

and $Nf(x) = \sup_k |f * \varphi_k(x)|$. Then according to [CVWW],

$$\|Nf\|_{L^p} \le C_p \|f\|_{L^p}, \qquad 1 < p \le \infty. \tag{11.6}$$

Furthermore the argument in [NSW] proving Lemma 11.3 shows

$$\left\| \left(\sum_k |f_k * \varphi_k|^2 \right)^{1/2} \right\|_{L^p} \le C_p \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p}, \tag{11.7}$$

$1 < p \le 2$. In view of (11.6), (11.1) will follow from the inequality

$$\left\| \sup_{k>0} |f * (d\nu_k - \varphi_k)| \right\|_{L^p} \le C_p \|f\|_{L^p}.$$

Now

$$f * (d\nu_k - \varphi_k)(x) = \sum_\ell f * \Psi_{k+\ell} * (d\nu_k - \varphi_k)(x),$$

and so

$$\sup_{k>0} |f * (d\nu_k - \varphi_k)(x)| \le \sum_\ell \left(\sum_{k \ge 0} |f * \Psi_{k+\ell} * (d\nu_k - \varphi_k)|^2 \right)^{1/2}. \tag{11.8}$$

Set

$$G_\ell f(x) = \left(\sum_{k \ge 0} |f * \Psi_{k+\ell} * (d\nu_k - \varphi_k)(x)|^2 \right)^{1/2}.$$

We shall show that for some $\epsilon > 0$,

$$\|G_\ell f\|_{L^2} \le C 2^{-\epsilon|\ell|} \|f\|_{L^2}. \tag{11.9}$$

This will then prove Lemma 11.1. Then under the hypothesis of Lemma 11.2, using (11.7), we see

$$\|G_\ell f\|_{L^{p_0}} \leq C_{p_0} \left\| \left(\sum_{k\geq 0} |f * \Psi_{k+\ell}|^2 \right) \right\|_{L^{p_0}} .$$

Now using (11.5), we obtain

$$\|G_\ell f\|_{L^{p_0}} \leq C_{p_0} \|f\|_{L^{p_0}} .$$

Interpolating with (11.9), we find

$$\|G_\ell f\|_{L^p} \leq C 2^{-\epsilon(p)|\ell|} \|f\|_{L^p} , \qquad p_0 < p \leq 2, \tag{11.10}$$

for some $\epsilon(p) > 0$. Summing on ℓ then completes the proof of Lemma 11.2.

Thus matters are reduced to proving (11.9). Let $r_k(t)$ denote the standard Rademacher functions, and set

$$T_{\ell,t} f(x) = \sum_k r_k(t) f * \Psi_{k+\ell} * (d\nu_k - \varphi_k)(x).$$

A standard argument shows that (11.9) will follow from

$$\|T_{\ell,t} f\|_{L^2} \leq C 2^{-\epsilon|\ell|} \|f\|_{L^2} . \tag{11.11}$$

Let

$$S_k^\ell f(x) = f * \Psi_{k+\ell} * (d\nu_k - \varphi_k)(x).$$

We will show

$$\left\| S_k^\ell (S_j^\ell)^* \right\| \leq C 2^{-\epsilon|\ell|} 2^{-\delta|j-k|} \tag{11.12}$$

for some $\epsilon > 0$ and $\delta > 0$. Here $\|\cdot\|$ denotes the operator norm on L^2. A similar but somewhat more complicated argument shows

$$\|(S_k^\ell)^* S_j^\ell\| \leq C 2^{-\epsilon|\ell|} 2^{-\delta|j-k|} . \tag{11.13}$$

Then (11.12) and (11.13) imply (11.11) by the Cotlar-Stein lemma.

Note that

$$S_k^\ell (S_j^\ell)^* f = f * (d\nu_j^* - \varphi_j) * \Psi_{j+\ell} * \Psi_{k+\ell} * (d\nu_k - \varphi_k) ,$$

where $d\nu_j^*$ is the measure defined by

$$d\nu_j^*(g) = 2^j \int_{2^{-j}}^{2^{-j+1}} g(\Gamma^{-1}(t)) dt.$$

Since the total mass of $d\nu_k$ and φ_k is bounded, we have

$$\|S_k^\ell (S_j^\ell)^*\| \leq C \|\Psi_{j+\ell} * \Psi_{k+\ell}\|_{L^1} \leq C 2^{-\epsilon|j-k|} .$$

To verify the last inequality, we may assume $j \leq k$ and note that since $\Psi_{j+\ell}$ has mean value zero,

$$\|\Psi_{j+\ell} * \Psi_{k+\ell}\|_{L^1} \leq C \int_{\mathbf{R}^3} |\Psi_{j+\ell}(x)| |A_{k+\ell}^{-1} x| dx \leq C \|A_{k+\ell}^{-1} A_{j+\ell}\| \leq C 2^{-(k-j)}$$

by (11.3).

Therefore it remains to show

$$\|S_k^\ell (S_j^\ell)^*\| \le C2^{-\epsilon|\ell|}. \tag{11.14}$$

We shall first prove (11.14) when $\ell \le 0$. Note that

$$\|S_k^\ell\| \le C\|\Psi_{k+\ell} * (d\nu_k - \varphi_k)\|_{L^1}.$$

As before since $d\nu_k - \varphi_k$ has mean value zero, we have

$$\begin{aligned}
&\|\Psi_{k+\ell} * (d\nu_k - \varphi_k)\|_{L^1} \\
&\quad \le C\left[\int_{\mathbf{R}^3} |A_{k+\ell}^{-1}x|d\nu_k(x) + \int_{\mathbf{R}^3} |A_{k+\ell}^{-1}x|\varphi_k(x)dx\right] \\
&\quad \le C2^k \int_{2^{-k}}^{2^{-k+1}} |A_{k+\ell}^{-1}\Gamma(t)|dt + C\|A_{k+\ell}^{-1}A_k\| \\
&\quad = C\int_{1/2}^1 |A_{k+\ell}^{-1}A_{k-1}\Gamma_{k-1}(t)|dt + C\|A_{k+\ell}^{-1}A_k\| \\
&\quad \le C\|A_{k+\ell}^{-1}A_k\| \le C2^\ell.
\end{aligned}$$

Here we used the fact that the normalized curve $\Gamma_{k-1}(t)$ is bounded for $\frac{1}{2} \le t \le 1$. This proves (11.14) for $\ell < 0$.

We now suppose $\ell > 0$. Write

$$\begin{aligned}
S_k^\ell f &= f * \Psi_{k+\ell} * d\nu_k - f * \Psi_{k+\ell} * \varphi_k \\
&= R_k^\ell f - Q_k^\ell f.
\end{aligned}$$

We shall show

$$\|R_k^\ell f\|_{L^2} \le C2^{-\epsilon\ell}\|f\|_{L^2}.$$

The estimate for Q_k^ℓ is easier. Since

$$\begin{aligned}
(R_k^\ell)^* R_k^\ell f(x) &= f * \Psi_{k+\ell} * d\nu_k * d\nu_k^* * \Psi_{k+\ell}(x), \\
\|(R_k^\ell)^* R_t^\ell f\|_{L^2} &\le C\|f * \Psi_{k+\ell} * d\nu_k * d\nu_k^*\|_{L^2}.
\end{aligned}$$

Let $\overline{R}f(x) = f * \Psi_{k+\ell} * d\nu_k * d\nu_k^*(x)$.

$$\begin{aligned}
\|\overline{R}^*\overline{R}f\|_{L^2} &= \|f * \Psi_{k+\ell} * d\nu_k * d\nu_k^* * d\nu_k * d\nu_k^* * \Psi_{k+\ell}\|_{L^2} \\
&\le C\|\Psi_{k+\ell} * d\nu_k * d\nu_k^* * d\nu_k * d\nu_k^* * \Psi_{k+\ell}\|_{L^1}\|f\|_{L^2}.
\end{aligned}$$

Therefore since

$$\|R_k^\ell\| = \|(R_k^\ell)^* R_k^\ell\|^{1/2} \le \|\overline{R}\|^{1/2} \le \|\overline{R}^*\overline{R}\|^{1/4},$$

it suffices to show

$$\|\Psi_{k+\ell} * d\nu_k * d\nu_k^* * d\nu_k * d\nu_k^* * \Psi_{k+\ell}\|_{L^1} \le C2^{-\epsilon\ell}$$

for some $\epsilon > 0$. We know from Section 1 that

$$\begin{aligned}
&\Psi_{k+\ell} * d\nu_k * d\nu_k^* * d\nu_k * d\nu_k^*(x) \\
&\quad = \Psi_{k+\ell,k} * d\mu_k * d\mu_k^* * d\mu_k * d\mu_k^*(A_k^{-1}x)
\end{aligned}$$

where $\Psi_{k+\ell,k}(x) = \Psi_{k+\ell}(A_k x)$ and $d\mu_k$ is the normalized measure along the curve $\Gamma(t)$. Similarly we have

$$\begin{aligned} &d\nu_k * d\nu_k^* * d\nu_k * d\nu_k^* * \Psi_{k+\ell}(x) \\ &\quad = d\mu_k * d\mu_k^* * d\mu_k * d\mu_k^* * \Psi_{k+\ell,k}(A_k^{-1}x). \end{aligned}$$

Also from Sections 3 to 10, we know that

$$d\mu_k * d\mu_k^* * d\mu_k * d\mu_k^* = (\rho_{k,1}(x) + \rho_{k,2}(x))dx$$

where

$$\int |\rho_{k,1}(x \cdot y) - \rho_{k,1}(x)| dx \le Ch^\epsilon$$

if $\|y\| < h$, and

$$\int |\rho_{k,2}(y \cdot x) - \rho_{k,2}(x)| dx \le Ch^\epsilon$$

if $\|y\| < h$. Therefore we may write

$$\begin{aligned} &\Psi_{k+\ell} * d\nu_k * d\nu_k^* * d\nu_k * d\nu_k^* * \Psi_{k+\ell} \\ &\quad \Psi_{k+\ell} * A + B * \Psi_{k+\ell} \end{aligned}$$

where

$$A = \rho_{k,1} * \Psi_{k+\ell,k}(A_k^{-1}x)$$

and

$$B = \Psi_{k+\ell,k} * \rho_{k,2}(A_k^{-1}x).$$

Thus it suffices to show

$$\|A\|_{L^1} \le C2^{-\epsilon\ell} \tag{11.15}$$

and

$$\|B\|_{L^1} \le C2^{-\epsilon\ell}. \tag{11.16}$$

We will prove (11.16). The proof of (11.15) is similar.

$$\begin{aligned} B &= \int \Psi_{k+\ell}(A_k(A_k^{-1}x \cdot y^{-1}))\rho_{k,2}(y)dy \\ &= \int \Psi_{k+\ell}(x \cdot (A_k y)^{-1})\rho_{k,2}(y)dy \\ &= \frac{1}{\det A_k} \int \Psi_{k+\ell}(y)\rho_{k,2}(A_k^{-1}y \cdot A_k^{-1}x)dy \\ &= \frac{1}{\det A_k} \int [\rho_{k,2}(A_k^{-1}y \cdot A_k^{-1}x) - \rho_{k,2}(A_k^{-1}x)]\Psi_{k+\ell}(y)dy \end{aligned}$$

since $\Psi_{k+\ell}$ has mean value zero. Therefore

$$\begin{aligned}\|B\|_{L^1} &\leq \frac{1}{\det A_k}\int |\Psi_{k+\ell}(y)|\left[\int |\rho_{k,2}(A_k^{-1}y\cdot A_k^{-1}x)-\rho_{k,2}(A_k^{-1}x)|dx\right]dy\\ &= \int |\Psi_{k+\ell}(y)|\left[\int |\rho_{k,2}(A_k^{-1}y\cdot x)-\rho_{k,2}(x)|dx\right]dy\\ &\leq C\int |\Psi_{k+\ell}(y)|\;\|A_k^{-1}y\|^{\epsilon}dy\\ &\leq C\int |\psi(y)|\;\|A_k^{-1}A_{k+\ell+1}y\|^{\epsilon}dy\\ &\quad + C\int |\psi(y)|\;\|A_k^{-1}A_{k+\ell}y\|^{\epsilon}dy\\ &\leq C2^{-\epsilon\ell}.\end{aligned}$$

The last inequality follows from (11.3) and this completes the proof of (11.16), finishing the proof of the L^p boundedness of $\mathscr{M}f$.

Note added in proof

L^2 estimates for related, but different, singular Radon transforms were recently obtained by A. Seeger using completely different methods. His article is entitled *L^2-estimates for a class of singular oscillatory integrals*, and appears in Math. Res. Lett. I (1994), 65–73.

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