JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 1, January 1995

VECTOR BUNDLES AND $SO(3)$-INVARIANTS FOR ELLIPTIC SURFACES

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INTRODUCTION

Beginning with Donaldson's paper on the failure of the h-cobordism theorem in dimension 4 [5], the techniques of gauge theory have proved to be highly successful in analyzing the smooth structure of simply connected elliptic surfaces. Recall that a relatively minimal simply connected elliptic surface S is specified up to deformation type by its geometric genus $p_g(S)$ and by two relatively prime integers m_1, m_2, the multiplicities of its multiple fibers. Here, if $p_g(S) = 0$, a surface S such that $m_i = 1$ for at least one i is rational, and thus all surfaces S with $p_g(S) = 0$ and $m_i = 1$ are deformation equivalent and in particular diffeomorphic. Moreover, if $p_g(S) = 1$ and $m_1 = m_2 = 1$, then S is a $K3$ surface. In all other cases, S is a surface with Kodaira dimension one.

Our goal in this paper is to prove the following result, which completes the smooth classification of elliptic surfaces:

Theorem. *Two possibly blown up simply connected elliptic surfaces are diffeomorphic if and only if they are deformation equivalent. More precisely, suppose that S and S' are relatively minimal simply connected elliptic surfaces. Suppose that S has multiple fibers of multiplicities m_1 and m_2, with $1 \leq m_1 \leq m_2$, and that S' has multiple fibers of multiplicities m_1' and m_2', with $1 \leq m_1' \leq m_2'$. Let $\tilde{S}$ be a blowup of S at r points and $\tilde{S}'$ a blowup of S' at r' points. Suppose that $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic. Then $r = r'$ and $p_g(S) = p_g(S')$, and moreover:*

(i) *If $p_g(S) > 0$, then $m_1 = m_1'$ and $m_2 = m_2'$.*

(ii) *If $p_g(S) = 0$, then S is rational, i.e. $m_1 = 1$, if and only if S' is rational if and only if $m_1' = 1$. If S and S' are not rational, then $m_1 = m_1'$ and $m_2 = m_2'$.*

There is also a routine generalization to the case of a finite cyclic fundamental group. The statements in the theorem that $r = r'$ and $p_g(S) = p_g(S')$ are easy consequences of the fact that $\tilde{S}$ and $\tilde{S}'$ are homotopy equivalent, and the main point is to determine the multiplicities. Before discussing the proof of the

Received by the editors September 9, 1993.

1991 *Mathematics Subject Classification.* Primary 14J60, 57R55; Secondary 14D20, 14F05, 14J27.

Research partially supported by National Science Foundation grant DMS-9203940.

theorem in more detail, we shall review some of the history of the classification of simply connected elliptic surfaces:

Theorem 0.1 [10]. *There is a function $f(m_1, m_2)$ defined on pairs of relatively prime positive integers (m_1, m_2) such that f is symmetric and finite-to-one provided that neither m_1 nor m_2 is 1, with the following property: Let S and S' be two simply connected surfaces with $p_g(S) = 0$. Denote the multiplicities of the multiple fibers of S by m_1, m_2 and the multiplicities for S' by m_1', m_2'. If S and S' are diffeomorphic, then $f(m_1, m_2) = f(m_1', m_2')$. Moreover, let $\tilde{S}$ and $\tilde{S}'$ be blowups of S and S' at r points. Then:*

(i) *Every diffeomorphism $\psi\colon \tilde{S} \to \tilde{S}'$ pulls back the cohomology class of an exceptional curve on $\tilde{S}'$ to $\pm$ the cohomology class of an exceptional curve on $\tilde{S}$.*
(ii) *Every diffeomorphism $\psi\colon \tilde{S} \to \tilde{S}'$ pulls back the cohomology class of a general fiber on $\tilde{S}'$ to a rational multiple of the cohomology class of a general fiber on $\tilde{S}$.*
(iii) *If $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic, then $f(m_1, m_2) = f(m_1', m_2')$.* □

The function $f(m_1, m_2)$ was then determined by S. Bauer [2] (the case $m_1 = 2$ is also in [10]):

Theorem 0.2. *In the above notation,*

$$f(m_1, m_2) = \frac{(m_1^2 - 1)(m_2^2 - 1)}{3} - 1. \quad \square$$

For the case $p_g(S) > 0$, there is the following result [11]:

Theorem 0.3. *Let S and S' be two simply connected surfaces with $p_g(S) > 0$. Denote the multiplicities of the multiple fibers of S by m_1, m_2 and the multiplicities for S' by m_1', m_2'. If S and S' are diffeomorphic, then $m_1m_2 = m_1'm_2'$. Moreover, let $\tilde{S}$ and $\tilde{S}'$ be blowups of S and S' at r points. Then:*

(i) *Every diffeomorphism $\psi\colon \tilde{S} \to \tilde{S}'$ pulls back the cohomology class of an exceptional curve on $\tilde{S}'$ to $\pm$ the cohomology class of an exceptional curve on $\tilde{S}$.*
(ii) *Except possibly for $p_g(S) = 1$, every diffeomorphism $\psi\colon \tilde{S} \to \tilde{S}'$ pulls back the cohomology class of a general fiber on $\tilde{S}'$ to a rational multiple of the cohomology class of a general fiber on $\tilde{S}$.*
(iii) *If $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic, then $m_1m_2 = m_1'm_2'$.* □

The crux of the argument involves calculating a coefficient of a suitable Donaldson polynomial invariant $\gamma_c(S)$. In fact, it is shown in [11] that $\gamma_c(S)$ can be written as a polynomial in the intersection form q_S and the primitive class $\kappa_S = \kappa$ such that the class of a general fiber $[f]$ of S is equal to $m_1m_2\kappa$, and that, for c sufficiently large, the first nonzero coefficient of this polynomial is given as follows: let $n = 2c - 2p_g(S) - 1$ and $d = 4c - 3p_g(S) - 3$. If

$\gamma_c(S) = \sum_{i=0}^{[d/2]} a_i q_S^i \kappa^{d-2i}$, then $a_i = 0$ for $i > n$ and

$$a_n = \frac{d!}{2^n n!}(m_1 m_2)^{p_g(S)}.$$

The proof of this statement involves showing that the moduli space of stable vector bundles V with $c_1(V) = 0$ and $c_2(V) = c$ fibers holomorphically over a Zariski open subset of a projective space, and that the fiber consists of $m_1 m_2$ copies of a complex torus. It is natural to wonder if the techniques of [11] can be pushed to determine some of the remaining terms. However, it seems to be difficult to use the vector bundle methods used in [11] to make the necessary calculations, even in the case of no multiple fibers. Thus, it is natural to look for other techniques to complete the C^∞ classification of elliptic surfaces.

Using a detailed analysis of certain moduli spaces of vector bundles, Morgan and O'Grady [21] together with Bauer [3] were able to calculate the coefficient a_{n-1} in case $p_g(S) = 1$ and $c = 3$. The calculation is long and involved for the following reason: the moduli spaces are nonreduced, not necessarily of the correct dimension, and (in the case of trivial determinant) the integer c is not in the "stable range". The final answer is that, up to a universal combinatorial factor, $a_{n-1} = m_1 m_2(2m_1^2 m_2^2 - m_1^2 - m_2^2)$. From this and from the knowledge of $m_1 m_2$, it is easy to determine the unordered pair $\{m_1, m_2\}$. In addition, the calculation shows that the class of a fiber of S is preserved up to rational multiples in case $p_g(S) = 1$ as well (the possible exception in (ii) of Theorem 0.3 above), provided that not both of m_1 and m_2 are 1.

Simultaneously with the research of this paper, Morgan and Mrowka [20] have independently determined the second coefficient a_{n-1} for all S such that $p_g(S) \geq 1$, for the case of the $SU(2)$-invariant $\gamma_c(S)$. The answer is that, up to combinatorial factors,

$$a_{n-1} = (m_1 m_2)^{p_g(S)}((m_1^2 m_2^2)(p_g(S) + 1) - m_1^2 - m_2^2).$$

From this, it is again easy to see that the diffeomorphism type of S determines the unordered pair $\{m_1, m_2\}$ in case $p_g(S) \geq 1$. The proof of this formula uses the knowledge of a_{n-1} for the case of $p_g(S) = 1$, together with the gauge theory gluing techniques developed by Mrowka in [22], to determine the coefficient a_{n-1} for $p_g(S) > 1$. Since the first draft of this paper was written, Kronheimer and Mrowka [17] conjectured a general formula for the $SU(2)$ Donaldson polynomial of an elliptic surface with $p_g > 0$ and verified this formula in the case of no multiple fibers. Subsequently Fintushel and Stern verified the formula of Kronheimer and Mrowka in general. Their proof, which does not use algebraic geometry or the results of [20], [22], gives another proof of the main theorem in this paper for the case $p_g > 0$.

In the proof given here of the main theorem, we shall use the following results. Aside from standard techniques in the theory of vector bundles, and the gauge theory results that are described in the book [11], we use only the results of this paper and of [11] to handle the case $p_g(S) > 0$. In case $p_g(S) = 0$, we use the results in this paper and in [11], as well as the calculation of Bauer described in Theorem 0.2 in case $m_1 m_2 \equiv 0 \bmod 2$. In case $m_1 m_2 \equiv 1 \bmod 2$, our proof does not depend on Bauer's results.

Next we outline the strategy of the argument. Following a well-established principle [7], [15], we shall work with $SO(3)$-invariants instead of $SU(2)$-invariants since these are often much easier to calculate. Moreover, in case $b_2^+ = 1$ a good choice of an $SO(3)$-invariant can simplify the problem that the invariant depends on the choice of a certain chamber. Thus we must choose a class $w \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ to be the second Stiefel-Whitney class of a principal $SO(3)$-bundle, although it will usually be more convenient to work with a lift of w to $\Delta \in H^2(S; \mathbb{Z})$. One possible choice of a lift of w would be the class κ, the primitive generator of $\mathbb{Z}^+ \cdot [f]$, or perhaps $c_1(S)$, or even $[f]$. All of these classes are rational multiples of $[f]$, and they do not fundamentally simplify the problem.

Instead we shall consider the case where Δ is transverse to f, more specifically where $\Delta \cdot \kappa = 1$. Of course, we shall need to choose Δ to be the class of a holomorphic divisor as well in order to be able to apply algebraic geometry. As we shall see in Section 1 of Part I, we can always make the necessary choices and the final calculation will show that the answer does not depend on the choices made. Note that Δ is well defined up to a multiple of κ, and that the choices Δ and $\Delta - \kappa$ correspond to different choices for $w^2 \equiv p \bmod 4$. Finally, as we shall show in Section 2 of Part I, in case $b_2^+(S) = 1$ or equivalently $p_g(S) = 0$, there is a special chamber $\mathscr{C}(w, p)$ which is natural in an appropriate sense under diffeomorphisms.

With this choice of Δ, the study of the relevant vector bundles divides into two very different cases, depending on whether $m_1 m_2 \equiv 0 \bmod 2$ or $m_1 m_2 \equiv 1 \bmod 2$. In the first part of this paper we shall collect results which are needed for both cases and show how the main theorems follow from the calculations in Parts II and III. In Part II, we shall consider the case where $\Delta \cdot \kappa = 1$ and $m_1 m_2 \equiv 0 \bmod 2$. In this case, m_1, say, is even (here we do not observe the convention that $m_1 \leq m_2$). Since $\Delta \cdot f = m_1 m_2 \Delta \cdot \kappa$, a vector bundle V with $c_1(V) = \Delta$ has even degree on a general fiber f. At first glance, then, it seems as if we are again in the situation of [8] and [11] and that there is no new information to be gained from the Donaldson polynomial. However, it turns out that the asymmetry between m_1 and m_2 appears in the moduli space as well. In this case, the moduli space again fibers holomorphically over a Zariski open subset of a projective space. But the fibers now consist of just m_2 copies of a complex torus. We then have, by an analysis that closely parallels [11], the following result (Theorem 4.1 of Part II):

Theorem 0.4. *Let w and p be as above, and set*

$$d = -p - 3(p_g(S) + 1) \quad \text{and} \quad n = (d - p_g(S))/2.$$

Suppose that $\gamma_{w,p}(S)$ is the Donaldson polynomial for the $SO(3)$-bundle P over S with $w_2(P) = w$ and $p_1(P) = p$ where if $p_g(S) = 0$ this polynomial is associated to the chamber $\mathscr{C}(w, p)$ defined in Definition 2.6 below. Then, assuming that m_1 is even and writing $\gamma_{w,p}(S) = \sum_{i=0}^{[d/2]} a_i q_S^i \kappa_S^{d-2i}$, we have, for all p such that $-p \geq 2(4p_g(S) + 2)$, $a_i = 0$ for $i > n$ and

$$a_n = \frac{d!}{2^n n!} (m_1 m_2)^{p_g(S)} m_2.$$

In particular, the leading coefficient contains an "extra" factor of m_2. Using this and either [11] in case $p_g(S) > 0$ or [2] in case $p_g(S) = 0$, we may then determine $\{m_1, m_2\}$. Note that in case $p_g(S) = 0$ and one of m_1, m_2 is 1, then it cannot be m_1 since m_1 is even. Thus $m_2 = 1$ and the leading coefficient does not determine m_1 (as well it cannot).

Finally, in Part III we shall discuss the case where $\Delta \cdot \kappa = 1$ and $m_1 m_2 \equiv 1 \bmod 2$. If $m_1 m_2 \equiv 1 \bmod 2$, then vector bundles V with $c_1(V) = \Delta$ have odd degree when restricted to a general fiber, and the general methods for studying vector bundles on elliptic surfaces described in [8] and [11], Chapter 7, do not apply. Thus we must develop new techniques for studying such bundles, and this is the subject of Part III. Fortunately, it turns out that this moduli problem is in many ways much simpler to study than the case of even degree on the general fiber. For example, as long as the expected dimension is nonnegative, for a suitable choice of ample line bundle the moduli space is always nonempty, irreducible, and smooth of the expected dimension. Moreover a Zariski open subset of the moduli space is independent of the multiplicities, and from this one can show easily that the leading coefficient of the Donaldson polynomial for the corresponding $SO(3)$-bundle is (up to the usual combinatorial factors) equal to 1. At first glance, this rather disappointing result suggests that no new information can easily be gleaned from the Donaldson polynomial. However, this suggestion is misleading: in some sense, the structure of the moduli space allows the contribution of the multiple fibers to be localized around the multiple fibers, enabling us to calculate the next two coefficients in the Donaldson polynomial. By contrast, in the case of trivial determinant, the moduli space for a surface with two multiple fibers of multiplicities m_1 and m_2 looks roughly like a branched cover of the corresponding moduli space for a surface without multiple fibers. A further simplification is that we can work with moduli spaces of small dimension, for example dimension two or four. Using the vector bundle results, we shall show (Corollary 6.4 and Corollary 9.5 of Part III):

Theorem 0.5. *Let S be a simply connected elliptic surface with two multiple fibers of multiplicities m_1 and m_2, with $m_1 m_2 \equiv 1 \bmod 2$. Let $w \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ satisfy $w \cdot \kappa = 1$. Suppose that $\gamma_{w,p}(S)$ is the Donaldson polynomial for the $SO(3)$-bundle P over S with $w_2(P) = w$ and $p_1(P) = p$ where if $p_g(S) = 0$ this polynomial is associated to the chamber $\mathcal{C}(w, p)$ defined in Definition 2.6 below.*

(i) *Suppose w and p are chosen so that the expected complex dimension of the moduli space $-p - 3(p_g(S) + 1)$ is 2. Then for all $\Sigma \in H_2(S; \mathbb{Z})$,*

$$\gamma_{w,p}(S)(\Sigma, \Sigma) = \Sigma^2 + ((m_1^2 m_2^2)(p_g(S) + 1) - m_1^2 - m_2^2)(\Sigma \cdot \kappa)^2.$$

(ii) *Suppose w and p are chosen so that the expected complex dimension of the moduli space $-p - 3(p_g(S) + 1)$ is 4. Then for all $\Sigma \in H_2(S; \mathbb{Z})$,*

$$\begin{aligned}\gamma_{w,p}(S)(\Sigma, \Sigma, \Sigma, \Sigma) = 3(\Sigma^2)^2 + 6C_1(\Sigma^2)(\Sigma \cdot \kappa)^2 \\ + (3C_1^2 - 2C_2)(\Sigma \cdot \kappa)^4,\end{aligned}$$

where

$$C_1 = (m_1^2 m_2^2)(p_g(S)+1) - m_1^2 - m_2^2; \quad C_2 = (m_1^4 m_2^4)(p_g(S)+1) - m_1^4 - m_2^4.$$

Here C_1 is the second coefficient of the degree two polynomial.

Note that the final answer has the following self-checking features. First, it is a polynomial in q_S and κ_S. If $p_g(S) = 1$ and $m_1 = m_2 = 1$, so that S is a $K3$ surface, then the term $(\Sigma \cdot \kappa)$ does not appear. This is in agreement with the general result that $\gamma_{w,p}(S)$ is a multiple of a power of q_S alone. If $p_g(S) = 0$ and $m_1 = 1$, then the answer is independent of m_2, since in this case all of the surfaces S for various choices of m_2 are diffeomorphic. In fact, we shall turn this remark around and use the knowledge of $\gamma_{w,p}(S)$ for $p_g = 0$, $m_1 = 1$ and m_2 arbitrary, to determine $\gamma_{w,p}(S)$ in general.

The techniques used to prove Theorem 0.5 should be capable of further generalization. For example, these methods should give in principle (that is, up to the knowledge of the multiplication table for divisors in $\mathrm{Hilb}^n S$) the full polynomial invariant in case m_1 and m_2 are odd. One might make a conjectural formula for $\gamma_{w,p}(S)$ in general along the lines suggested by Kronheimer and Mrowka in [17]. In our case the formula should conjecturally read as follows: let $\gamma_t(\Sigma)$ be the Donaldson polynomial $\gamma_{w,p}(S)(\Sigma, \dots, \Sigma)$ for $w = \Delta \bmod 2$ or $w = \Delta - \kappa \bmod 2$ and p chosen so that $w^2 \equiv p \bmod 4$ and $-p - 3\chi(\mathcal{O}_S) = 2t$, so that the complex dimension of the moduli space is $2t$. It follows from Proposition 1.1 below that γ_t depends only on t. Then the natural analogue of the conjectures in [17] is the conjecture that

$$\sum_{t\geq 0} \frac{\gamma_t(\Sigma)}{(2t)!} = \exp\left(\frac{q_S}{2}\right) \frac{(\cosh(m_1 m_2(\kappa \cdot \Sigma)))^{p_g+1}}{\cosh(m_1(\kappa \cdot \Sigma))\cosh(m_2(\kappa \cdot \Sigma))}.$$

It essentially follows from Theorem 0.5 that this formula is correct through the first three terms, including the case $p_g = 0$ where the quotient is not given by a finite sum of exponentials, and it is likely that a further extension of the methods in Part III and the knowledge of the multiplication in $\mathrm{Hilb}^n S$ can establish the general formula. This formula has also apparently been established by Fintushel and Stern in case $p_g > 0$. Finally we should add that many of the techniques used in Part III also have applications to the $SU(2)$ case.

Notation and conventions

All spaces are over $\mathbb{C}$, all sheaves are coherent sheaves in the classical topology unless otherwise specified. We do not distinguish between a vector bundle and its locally free sheaf of sections. Given a subvariety Y of a compact complex manifold X, we denote the associated cohomology class by $[Y]$.

If V is a rank two vector bundle on a complex manifold or smooth scheme X, we shall frequently need to consider the first Pontrjagin class of $\operatorname{ad} V$, which is $c_1^2(V) - 4c_2(V)$. We will denote this expression by $p_1(\operatorname{ad} V)$. We shall occasionally and incorrectly use the shorthand $p_1(\operatorname{ad} V)$, for an arbitrary coherent sheaf V, to denote $c_1^2(V) - 4c_2(V)$.

PART I: PRELIMINARIES

Let us describe the contents of Part I. In Section 1 we discuss the possible choices for $w = w_2(P)$ up to diffeomorphisms of S and show that there is always a generic elliptic surface for which w is the mod 2 reduction of a holomorphic divisor. There is also a discussion of certain elliptic surfaces which can be constructed from S. In Section 2 we introduce a class of ample line bundles which we shall use to define stability and which are well adapted to the geometry of S. Section 3 explains the meaning of stability of a vector bundle V with respect to such a line bundle: stability is equivalent to the assumption that the restriction of V to almost every fiber is semistable. Finally, in Section 4, we show how the main results concerning Donaldson polynomials lead to C^∞ classification results.

1. PRELIMINARIES ON ELLIPTIC SURFACES

Let S be a simply connected elliptic surface with at most two multiple fibers of multiplicities $m_1 \leq m_2$. Here we shall allow m_1 or both m_1 and m_2 to be one. Let $[f]$ denote the class in homology of a smooth nonmultiple fiber of S. There is a unique homology class $\kappa_S = \kappa$ such that $[f] = m_1 m_2 \kappa$, and κ is primitive [11]. Let P be a principal $SO(3)$-bundle over S with $w_2(P) = w$ and $p_1(P) = p$. Note that $w^2 \equiv p \bmod 4$. We shall be concerned with bundles P such that $w \cdot \kappa \bmod 2 = 1$. In this section, we shall show that, modulo diffeomorphism, the choice of w is not essential. Indeed, we shall prove that, given a class w, there is a diffeomorphism $\psi\colon S \to S'$, where S' is again a simply connected elliptic surface with two multiple fibers of multiplicities m_1 and m_2, such that $\psi^* \kappa_{S'} = \kappa_S$ and such that there exists a holomorphic divisor Δ with $w = \psi^*[\Delta] \bmod 2$. Thus we may always assume that w is the reduction of a $(1, 1)$ class. We begin with an arithmetic result, which is not in fact needed in what follows but which helps to clarify the role of the choice of w modulo diffeomorphisms. In the arguments below, we shall sometimes blur the distinction between $H_2(S)$ and $H^2(S)$ using the canonical identification between these two groups.

Proposition 1.1. *Let S be a simply connected elliptic surface.*

(i) *Suppose that $m_1 m_2 \equiv 1 \bmod 2$, and let $a \in \mathbb{Z}/4\mathbb{Z}$. Then the group of orientation-preserving diffeomorphisms $\psi\colon S \to S$ such that $\psi_*([f]) = [f]$ acts transitively on the set of $w \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ such that $w \cdot \kappa = 1$ and $w^2 \equiv a \bmod 4$.*

(ii) *If $m_1 m_2 \equiv 0 \bmod 2$ and $a \in \mathbb{Z}/4\mathbb{Z}$, then there are at most three orbits of the set $\{ w \in H^2(S; \mathbb{Z}/2\mathbb{Z}) : w \cdot \kappa = 1 \text{ and } w^2 \equiv a \bmod 4 \}$ under the group of diffeomorphisms of S which fix κ.*

Proof. Let L be the image of $H_2(S - \pi^{-1}(D))$ in $H_2(S)$, where D is a small disk in $\mathbb{P}^1$ which we may assume contains the multiple fibers and no other singular fiber. Thus $L \subseteq (\kappa^\perp)$, and in fact L has index $m_1 m_2$ in $(\kappa^\perp)$. Let φ be an automorphism of the lattice $H_2(S; \mathbb{Z})$ fixing κ. Thus by restriction

φ induces an automorphism of $(\kappa^{\perp})$. The method of proof of Theorem 6.5 of Chapter 2 of [11] shows that there is a diffeomorphism ψ, automatically orientation-preserving, inducing φ provided that $\varphi(L) \subseteq L$ and that φ has real spinor norm one.

Clearly we may write $L = \mathbb{Z}[f] \oplus W$, where W is an even unimodular lattice. Moreover $(\kappa^{\perp}) = \mathbb{Z} \cdot \kappa \oplus W$, with the inclusion $L \subseteq (\kappa^{\perp})$ the natural inclusion given by $[f] = m_1 m_2 \kappa$. If $W^{\perp}$ denotes the orthogonal complement of W in $H_2(S; \mathbb{Z})$, then $W^{\perp} = \operatorname{span}\{\kappa, x\}$ for some class x with $x \cdot \kappa = 1$. Given $a \bmod 4$, we can always assume after replacing x by $x + \kappa$ that $x^2 \equiv a \bmod 4$. Now it is easy to describe all automorphisms of $H_2(S; \mathbb{Z})$ fixing κ: choosing an isometry τ of W, φ is given by

$$\begin{aligned} \varphi(\kappa) &= \kappa; \\ \varphi(\alpha) &= \tau(\alpha) + \ell(\alpha)\kappa, \quad \alpha \in W; \\ \varphi(x) &= x + c\kappa + \beta. \end{aligned}$$

Here ℓ is an arbitrary homomorphism $W \to \mathbb{Z}$ and β is the unique element of the unimodular lattice W such that $-\beta \cdot \alpha = \ell(\tau(\alpha))$ for all $\alpha \in W$. Furthermore $c = -\beta^2/2$. It is clear that every choice of τ and ℓ (or equivalently β) produces an automorphism φ, and that $\varphi(L) = L$ if and only if $m_1 m_2$ divides ℓ or equivalently β. If x' is another class such that $x' \cdot \kappa \equiv 1 \bmod 2$ and $(x')^2 \equiv x^2 \bmod 4$, we can write $x' = nx + b\kappa + \beta$, where $\beta \in W$. Since we only care about $x' \bmod 2$, we may assume that $n = 1$. Note that $2b + \beta^2 \equiv 0 \bmod 4$ and thus $b \equiv \beta^2/2 \bmod 2$.

First assume that $m_1 m_2$ is odd. Then since $\beta \equiv m_1 m_2 \beta \bmod 2$, we may assume that β is divisible by $m_1 m_2$. Choosing $\tau = \mathrm{Id}$ and ℓ, c in the definition of φ as specified by β gives φ such that $\varphi(x) \equiv x' \bmod 2$. As φ is unipotent, it is easy to see that φ has spinor norm one, i.e. that φ is in the same connected component of the group of automorphisms of the quadratic form of $H_2(S; \mathbb{R})$ as the identity. Thus there is a diffeomorphism ψ realizing φ.

Next suppose that $2 | m_1 m_2$. In fact in this case the class x defined above is fixed mod 2 by every isometry φ as above which satisfies $\varphi(L) = L$: Since $m_1 m_2 | \beta$ and $c \cong \beta^2/2 \bmod 2$, it follows that $\varphi(x) \equiv x \bmod 2$. Now let x' be a class with $x' \cdot \kappa \equiv 1 \bmod 2$ and $(x')^2 \equiv x^2 \bmod 4$. We may assume that $x' \not\equiv x$. First consider the case where $x' = x + b\kappa + \alpha$ and $b \equiv 0 \bmod 2$. Thus we may replace x' by $x + \alpha$. By assumption $\alpha^2 \equiv 0 \bmod 4$. We may assume that α is primitive (otherwise $\alpha \equiv 0 \bmod 2$ or α is congruent to a primitive nonzero element mod 2). Replacing α by $\alpha + 2\beta$, where $\beta \in W^{\perp}$, replaces α^2 by $\alpha^2 + 4(\alpha \cdot \beta) + 4\beta^2$. Since α is primitive, it is easy to see that there is a choice of β so that $(\alpha + 2\beta)^2 = 0$. Thus we may assume that α is primitive and that $\alpha^2 = 0$. The group $SO(W)$ is a subgroup of the automorphism group of L. Every element of $SO(W)^*$, the set all elements of $SO(W)$ with spinor norm one, is realized by a diffeomorphism. Moreover an easy exercise shows that $SO(W)^*$ acts transitively on the set of primitive $\alpha \in W$ with $\alpha^2 = 0$.

Thus the set of all possible $x+\alpha$, with $\alpha \neq 0$, is contained in a single orbit under the diffeomorphism group.

In case $x' = x + \kappa + \alpha$ with $\alpha^2 \equiv 2 \bmod 4$, an argument similar to that given above shows that we may assume that $\alpha^2 = 2$ and that every two classes $x_1 = x+\kappa+\alpha_1$ and $x_2 = x+\kappa+\alpha_2$ with $\alpha_i^2 = 2$ are conjugate under the group of diffeomorphisms of S which fix κ. Thus there are at most three orbits in this case. □

The following result is really only needed in the case where $m_1 m_2$ is even, since in case $m_1 m_2$ is odd we can appeal to (i) of Proposition 1.1 above.

Proposition 1.2. *Let S be a simply connected elliptic surface and w be a class in $H^2(S; \mathbb{Z}/2\mathbb{Z})$ with $w \cdot \kappa = 1$. Then after replacing S with a deformation equivalent elliptic surface, we may assume that there is a divisor Δ on S with $\Delta \equiv w \bmod 2$ and $\Delta \cdot \kappa = 1$, and such that all singular fibers of S are irreducible rational curves with a singular ordinary double point, i.e. S is nodal.*

Proof. Fix a nodal simply connected elliptic surface with a section B such that $p_g(B) = p_g(S)$. Using [11], S is deformation equivalent through elliptic surfaces to a logarithmic transform of B at two smooth fibers, where the multiplicities of the logarithmic transforms are m_1 and m_2. Fix one such logarithmic transform S_0, and let $\psi: S \to S_0$ be a diffeomorphism preserving the class of the fiber. Using this diffeomorphism, we shall identify S and S_0. Let Δ be an element in $H^2(S; \mathbb{Z})$ whose mod 2 reduction is w and such that $\Delta \cdot \kappa = 1$. We shall show that, by further modifying the complex structure on S, we may assume that Δ is of type $(1,1)$.

Given Δ, we have the image $i_*([\Delta]) \in H^2(S; \mathcal{O}_S)$, where i_* is the map induced on sheaf cohomology by the inclusion $\mathbb{Z} \subset \mathcal{O}_S$. The set of all complex structures of an elliptic surface on S for which the associated Jacobian surface is B and which are locally isomorphic to S is a principal homogeneous space over $H^1(\mathbb{P}^1; \mathcal{B})$, where $\mathcal{B}$ is the sheaf of local holomorphic cross sections of B ([11], Chapter 1, Theorem 6.7). Moreover there is a surjective map $H^2(S; \mathcal{O}_S) \cong H^2(B; \mathcal{O}_B) \to H^1(\mathbb{P}^1; \mathcal{B})$ ([11], Chapter 1, Lemma 5.11). Thus given a cohomology class $\eta \in H^2(S; \mathcal{O}_S)$, we can form the associated surface S^η and consider the element $i_*^\eta([\Delta]) \in H^2(S^\eta; \mathcal{O}_{S^\eta}) \cong H^2(S; \mathcal{O}_S)$.

Lemma 1.3. *In the above notation, $i_*^\eta([\Delta]) = i_*([\Delta]) + m_1 m_2 \eta$.*

Proof. This presumably could be proved by a rather involved direct calculation. For another argument, note that the map $H^2(S; \mathcal{O}_S) \to H^2(S; \mathcal{O}_S)$ defined by

$$\eta \mapsto i_*^\eta([\Delta]) - i_*([\Delta]) - m_1 m_2 \eta$$

is holomorphic, since it arises from a variation of Hodge structure. The argument of Lemma 6.13 in Chapter 1 of [11] shows that $i_*^\eta([\Delta]) - i_*([\Delta]) - m_1 m_2 \eta$ lies in the countable (not necessarily discrete) subgroup $H^1(\mathbb{P}^1; R^1\pi_*\mathbb{Z})$ of $H^2(S; \mathcal{O}_S) = H^1(\mathbb{P}^1; R^1\pi_*\mathcal{O}_S)$. This is only possible if the image of the map is contained in a single point, and since the image contains the origin the map is identically zero. □

Returning to the proof of Proposition 1.2, since $H^2(S;\mathscr{O}_S)$ is divisible, there is a choice of η so that $i_*^\eta([\Delta])=0$. For the corresponding complex structure, $[\Delta]$ is then a $(1,1)$ class. □

Finally we shall describe a way to associate new elliptic surfaces to S which generalizes the construction of the Jacobian surface. Suppose that S is an elliptic surface over $\mathbb{P}^1$. Let $\eta=\operatorname{Spec} k$ be the generic point of $\mathbb{P}^1$, where $k=k(\mathbb{P}^1)$ is the function field of the base curve, let $\bar\eta=\operatorname{Spec}\bar k$, where $\bar k=\overline{k(\mathbb{P}^1)}$ is the algebraic closure of k, and let S_η and $S_{\bar\eta}$ be the restrictions of S to η and $\bar\eta$. Thus S_η is a curve of genus one over k.

Given an algebraic elliptic surface with a section $\pi\colon B\to\mathbb{P}^1$, it has an associated Weil-Chatelet group $WC(B)$ [4], which classifies all algebraic elliptic surfaces S whose Jacobian surface is B. As above we let B_η be the elliptic curve over k defined by the generic fiber of B. By definition $WC(B)$ is the Galois cohomology group $H^1(G,B_\eta(\bar k))$, where $G=\operatorname{Gal}(\bar k/k)$ and $B_\eta(\bar k)$ is the group of points of the elliptic curve B_η defined over $\bar k$. There is an exact sequence

$$0\to Ш(B)\to WC(B)\to\bigoplus_{t\in C}H_1(\pi^{-1}(t);\mathbb{Q}/\mathbb{Z})\to 0.$$

The subgroup $Ш(B)$ corresponds to algebraic elliptic surfaces without multiple fibers whose Jacobian surface is isomorphic to B, and the quotient describes the possible local forms for the multiple fibers. Thus if $\xi\in WC(B)$ corresponds to the surface S, then S has a multiple fiber of multiplicity m at $t\in\mathbb{P}^1$ if and only if the projection of ξ to $H_1(\pi^{-1}(t);\mathbb{Q}/\mathbb{Z})$ has order m.

The surface S is specified by an element ξ of $WC(J(S))$, where $J(S)$ is the Jacobian surface associated to S. Let us recall the recipe for ξ [25]: we have the curve S_η and its Jacobian $J(S_\eta)$ defined over k. The curve S_η is a principal homogeneous space over $J(S_\eta)$, and thus defines a class $\xi\in WC(J(S))$, by the following rule: let σ be a point of $S_{\bar\eta}$. Given $g\in\operatorname{Gal}(\bar k/k)$, the divisor $g(\sigma)-\sigma$ has degree zero on $S_{\bar\eta}$ and so defines an element of $J(S_{\bar\eta})$, which is easily checked to be a 1-cocycle. The induced cohomology class is ξ.

For every integer d there is an algebraic elliptic surface $J^d(S)$, whose restriction to the generic fiber η is the Picard scheme of divisors of degree d on the curve S_η. Thus $J^0(S)=J(S)$ and $J^1(S)=S$. We claim that, if S corresponds to the class $\xi\in WC(J(S))$, then $J^d(S)$ corresponds to the class $d\xi$. Indeed, using the above notation, if σ defines a point of $S_{\bar\eta}$, then $d\sigma$ is a point of $J^d(S_{\bar\eta})$. Thus, the corresponding cohomology class is represented by $d(g(\sigma)-\sigma)$ and so is equal to $d\xi$. In particular, if S has a multiple fiber of multiplicity m at t, then $J^d(S)$ has a multiple fiber of multiplicity $m/\gcd(m,d)$. Of course if $m|d$ then the multiplicity is one. Finally note that $J(S)$ is the Jacobian surface of $J^d(S)$ for every d and that $p_g(J^d(S))=p_g(S)$.

Ideally we would like there to be a Poincaré line bundle $\mathscr{P}_d$ over $S\times_{\mathbb{P}^1}J^d(S)$ such that the restriction of $\mathscr{P}_d$ to the slice $S\times_{\mathbb{P}^1}\{\lambda\}$ is the line bundle of degree

d on the fiber of S over $\pi(\lambda)$ corresponding to λ. In general this is too much to ask. However such a bundle exists locally around every smooth nonmultiple fiber: if X is the inverse image in S of a small disk D in $\mathbb{P}^1$ such that all fibers on X are smooth and nonmultiple, and X_d is the corresponding preimage in $J^d(S)$, then there is a Poincaré line bundle over $X \times_D X_d$. There is also an analogous statement where we replace a small classical open set in $\mathbb{P}^1$ with an étale open set. The proof for this result is essentially contained in the proof of Theorem 1.3 of Chapter 7 in [11]. Another construction is given in Section 7 of Part III.

2. Suitable line bundles

Suppose that we are given a class $w \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ with $w \cdot \kappa \bmod 2 = 1$ and an integer p with $w^2 \equiv p \bmod 4$. Choose once and for all a complex structure on S for which there is a divisor Δ with $w = \Delta \bmod 2$. Let c be the integer $(\Delta^2 - p)/4$. The principal $SO(3)$-bundle P over S with $w_2(P) = w$ and $p_1(P) = p$ lifts uniquely to a principal $U(2)$-bundle P' over S with $c_1(P') = \Delta$ and $c_2(P') = c$. Moreover, by Donaldson's theorem, if g is a Hodge metric on S corresponding to the ample line bundle L, we can identify the moduli space of gauge equivalence classes of g-anti-self-dual connections on P with the moduli space of L-stable rank two vector bundles V over S with $c_1(V) = \Delta$ and $c_2(V) = c$.

We shall also have to make a choice of the ample line bundle L. If $p_g(S) > 0$, then the resulting Donaldson polynomial invariant does not depend on the choice of L, whereas if $p_g(S) = 0$, then the invariant depends on the chamber containing $c_1(L)$ [15], [16]. We then make the following definition [24]:

Definition 2.1. A *wall of type* (Δ, c) is a class $\zeta \in H^2(S; \mathbb{Z})$ such that $\zeta \equiv \Delta \bmod 2$ and

$$\Delta^2 - 4c \leq \zeta^2 < 0.$$

In particular there are no such walls unless $\Delta^2 - 4c < 0$. Clearly this definition depends only on $\Delta \bmod 2 = w$ and $p = \Delta^2 - 4c$, and we shall also refer to walls of type (w, p).

Now suppose that $p_g(S) = 0$, i.e. that $b_2^+(S) = 1$. Let

$$\Omega_S = \{x \in H^2(S; \mathbb{R}) : x^2 > 0\}.$$

Let $W^\zeta = \Omega_S \cap (\zeta)^\perp$. A *chamber of type* (Δ, c) (or of type (w, p)) for S is a connected component of the set

$$\Omega_S - \bigcup \{W^\zeta : \zeta \text{ is a wall of type } (\Delta, c'),\ c' \leq c\}.$$

For the purposes of algebraic geometry, walls of type (Δ, c) arise as follows: let L be an ample line bundle and let V be a rank two bundle over S with $c_1(V) = \Delta$ and $c_2(V) = c$ which is strictly L-semistable. Let $\mathcal{O}_S(F)$ be a destabilizing sub-line bundle. Thus there is an exact sequence

$$0 \to \mathcal{O}_S(F) \to V \to \mathcal{O}_S(-F + \Delta) \otimes I_Z \to 0,$$

where I_Z is the ideal sheaf of a codimension two local complete intersection subscheme. Thus

$$c_2(V) = c = -F^2 + F \cdot \Delta + \ell(Z).$$

Since $\ell(Z)$ is nonnegative, we can rewrite this as

$$-F^2 + F \cdot \Delta \leq c.$$

Moreover

$$(2F - \Delta)^2 = -4(-F^2 + F \cdot \Delta) + \Delta^2 \leq \Delta^2 - 4c,$$

so that we can rewrite the last condition by

$$\Delta^2 - 4c \leq (2F - \Delta)^2.$$

Using the fact that $L \cdot F = (L \cdot \Delta)/2$, we have

$$L \cdot (2F - \Delta) = 0,$$

and so by the Hodge index theorem $(2F - \Delta)^2 \leq 0$, with equality holding if and only if $2F - \Delta = 0$ (recall that S is simply connected). This case cannot arise for us since $\Delta \cdot \kappa = 1$ and thus Δ is primitive. In particular $\zeta = 2F - \Delta$ is a wall of type (Δ, c). Of course, it is also the cohomology class of a divisor, and thus has type $(1, 1)$. It then follows easily that, if L_1 and L_2 are two ample line bundles such that $c_1(L_1)$ and $c_1(L_2)$ lie in the interior of the same chamber of type (Δ, c), then a rank two vector bundle V with $c_1(V) = \Delta$ and $c_2(V) = c$ is L_1-stable if and only if it is L_2-stable.

With this said, we can make the following definition:

Definition 2.2. Let c be an integer, and set $w = \Delta \bmod 2$ and $p = \Delta^2 - 4c$. An ample line bundle L is *(Δ, c)-suitable* or *(w, p)-suitable* if, for all walls ζ of type (Δ, c) which are the classes of divisors on S, we have $\operatorname{sign} f \cdot \zeta = \operatorname{sign} L \cdot \zeta$.

Remark. (1) Suppose that ζ is a $(1, 1)$ class satisfying $\zeta^2 \geq 0$. It follows from the Hodge index theorem that if $\zeta \cdot f > 0$, then $\zeta \cdot L > 0$ as well. Thus we can drop the requirement that $\zeta^2 < 0$.

(2) In our case $\zeta \equiv \Delta \bmod 2$ and thus $\zeta \cdot \kappa \equiv 1 \bmod 2$. It follows that $\zeta \cdot \kappa \neq 0$ and thus that $\zeta \cdot f \neq 0$. Thus the condition $\zeta \cdot f \neq 0$ (which was included as part of the definition in [11]) is always satisfied in our case.

(3) In case $b_2^+(S) = 1$, L is (Δ, c)-suitable if and only if the class κ lies in the closure of the chamber containing $c_1(L)$.

Lemma 2.3. *For every c, (Δ, c)-suitable ample line bundles exist.*

Proof. Let L_0 an ample line bundle. For $n \geq 0$, let $L_n = L_0 \otimes \mathcal{O}_S(nf)$. It follows from the Nakai-Moishezon criterion that L_n is ample as well. We claim that if $n > -p(L_0 \cdot f)/2$, then L_n is (Δ, c)-suitable.

To see this, let $\zeta = 2F - \Delta$ be a wall of type (Δ, c) with

$$\Delta^2 - 4c \leq \zeta^2 < 0.$$

We may assume that $a = \zeta \cdot f > 0$, and must show that $\zeta \cdot L_n > 0$ as well. The class $ac_1(L_0) - (L_0 \cdot f)\zeta$ is perpendicular to f. Since $f^2 = 0$, we may apply the Hodge index theorem to conclude:

$$0 \geq ac_1(L_0) - (L_0 \cdot f)\zeta = a^2 L_0^2 - 2a(L_0 \cdot f)(L_0 \cdot \zeta) + (L_0 \cdot f)^2 \zeta^2.$$

Using the fact that $\zeta^2 \geq \Delta^2 - 4c = p$, we find that

$$L_0 \cdot \zeta \geq \frac{a(L_0^2)}{2(L_0 \cdot f)} + \frac{\zeta^2}{2a}(L_0 \cdot f) > \frac{p}{2a}(L_0 \cdot f).$$

Thus

$$\begin{aligned} L_n \cdot \zeta &= (L_0 \cdot \zeta) + n(\zeta \cdot f) > \frac{p}{2a}(L_0 \cdot f) - \frac{pa}{2}(L_0 \cdot f) \\ &= -\frac{p}{2}(L_0 \cdot f)\left(a - \frac{1}{a}\right) \geq 0. \end{aligned}$$

Thus L_n is (Δ, c)-suitable. □

In the case where $b_2^+(S) = 1$, we have the following interpretation of (Δ, c)-suitability.

Lemma 2.4. *Suppose that $p_g(S) = 0$. If L_1 and L_2 are both (Δ, c)-suitable, then $c_1(L_1)$ and $c_1(L_2)$ lie in the same chamber of type (Δ, c). Thus there is a unique chamber $\mathscr{C}(w, p)$ of type (Δ, c) which contains the first Chern classes of (Δ, c)-suitable ample line bundles. Conversely, if L is ample and $c_1(L) \in \mathscr{C}(w, p)$, then L is (Δ, c)-suitable.*

Proof. Let L_1 and L_2 be (Δ, c)-suitable. Since $p_g(S) = 0$, every cohomology class is of type $(1, 1)$. Thus if ζ is a wall of type (Δ, c), then

$$\operatorname{sign} L_1 \cdot \zeta = \operatorname{sign} f \cdot \zeta = \operatorname{sign} L_2 \cdot \zeta.$$

This exactly implies that $c_1(L_1)$ and $c_1(L_2)$ are not separated by any wall $(\zeta)^\perp$.

Conversely suppose that $c_1(L) \in \mathscr{C}(w, p)$, where $\mathscr{C}(w, p)$ is the unique chamber containing the first Chern classes of (Δ, c)-suitable ample line bundles. This means in particular that $L \cdot \zeta \neq 0$ for every ζ of type (Δ, c). The proof of Lemma 2.4 shows that $c_1(L) + N[f] \in \mathscr{C}(w, p)$ for all sufficiently large N. Thus, for all ζ of type (Δ, c),

$$\operatorname{sign} L \cdot \zeta = \operatorname{sign} L \cdot \zeta + N \operatorname{sign} f \cdot \zeta.$$

Since $f \cdot \zeta \neq 0$, $\operatorname{sign} L \cdot \zeta + N \operatorname{sign} f \cdot \zeta = \operatorname{sign} f \cdot \zeta$ for all $N \gg 0$. Thus $\operatorname{sign} f \cdot \zeta = \operatorname{sign} L \cdot \zeta$, and L is (Δ, c)-suitable. □

Lemma 2.5. *Suppose that $p_g(S) = 0$. The chamber $\mathscr{C}(w, p)$ is the unique chamber of type (w, p) which contains κ in its closure. Thus every diffeomorphism ψ of S for which $\psi^*\kappa = \pm\kappa$ satisfies $\psi^*\mathscr{C}(w, p) = \pm\mathscr{C}(\psi^* w, p)$. More generally, if S and S' are two elliptic surfaces with $p_g = 0$ and $\psi\colon S \to S'$ is a diffeomorphism such that $\psi^*\kappa_{S'} = \pm\kappa_S$, then $\psi^*\mathscr{C}(w, p) = \pm\mathscr{C}(\psi^* w, p)$. In particular if S is not rational and $\psi\colon S \to S'$ is a diffeomorphism, then $\psi^*\mathscr{C}(w, p) = \pm\mathscr{C}(\psi^* w, p)$.*

Proof. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two distinct chambers which contain κ in their closures. Let ζ be a wall separating $\mathscr{C}_1$ and $\mathscr{C}_2$. We may assume that $\zeta \cdot x > 0$ for all $x \in \mathscr{C}_1$ and $\zeta \cdot x < 0$ for all $x \in \mathscr{C}_2$. Thus $0 \leq \zeta \cdot \kappa \leq 0$, so that $\zeta \cdot \kappa = 0$. However this contradicts the fact that $\zeta \cdot \kappa \neq 0$. Thus there is at most one chamber containing κ in its closure. We have seen in the proof of Lemma 2.3

that, for all ample line bundles L and integers $N \gg 0$, $c_1(L)+N\kappa \in \mathscr{C}(w, p)$. Thus $\kappa + (1/N)c_1(L) \in \mathscr{C}(w, p)$. It follows that κ indeed lies in the closure of $\mathscr{C}(w, p)$, so that $\mathscr{C}(w, p)$ is the unique chamber with this property.

Thus if ψ is a diffeomorphism of S for which $\psi^*\kappa = \pm\kappa$, then $\pm\kappa$ lies in the closure of $\psi^*\mathscr{C}(w, p)$. Clearly, if $\mathscr{C}$ is a chamber of type (w, p), then $\psi^*\mathscr{C}$ is a chamber of type $(\psi^* w, p)$. It follows that $\psi^*\mathscr{C}(w, p) = \pm\mathscr{C}(\psi^* w, p)$. The statement about two different surfaces S, S' is proved similarly. To see the final statement, we use [10] to see that every diffeomorphism $\psi: S \to S'$ satisfies $\psi^*\kappa_{S'} = \pm\kappa_S$ provided that S is not rational. □

Definition 2.6. The chamber described in Lemma 2.4 will be called the *suitable chamber* of type (Δ, c) or of type (w, p) or the (Δ, c)-*suitable* or (w, p)-*suitable chamber.*

3. The geometric meaning of suitability

The goal of this section is to describe the meaning of (Δ, c)-suitability. Given the bundle V on S, it defines by restriction a bundle $V|f$ on each fiber f. Our main result says essentially that V is stable for one, or equivalently all, (Δ, c)-suitable line bundles L if and only if $V|f$ is semistable for almost all f.

It will be more convenient to use the language of schemes to state this result. As in Section 2, let $k(\mathbb{P}^1)$ denote the function field of $\mathbb{P}^1$ and let $\overline{k(\mathbb{P}^1)}$ be the algebraic closure of $k(\mathbb{P}^1)$. Set $\eta = \operatorname{Spec} k(\mathbb{P}^1)$ and $\bar\eta = \operatorname{Spec} \overline{k(\mathbb{P}^1)}$. Thus η is the generic point of $\mathbb{P}^1$. Let $S_\eta = S \times_{\mathbb{P}^1} \eta$ be the generic fiber of π and let $S_{\bar\eta} = S \times_{\mathbb{P}^1} \bar\eta$. Here S_η is a curve of genus one over the field $k(\mathbb{P}^1)$ and $S_{\bar\eta}$ is the curve over $\overline{k(\mathbb{P}^1)}$ defined by extending scalars. Let V_η and $V_{\bar\eta}$ be the vector bundles over S_η and $S_{\bar\eta}$ respectively obtained by restricting V. We can then define stability and semistability for $V_{\bar\eta}$ and V_η; for V_η, a destabilizing subbundle must also be defined over $k(\mathbb{P}^1)$. Trivially, if V_η is unstable (resp. not stable) then $V_{\bar\eta}$ is unstable (resp. not stable). Thus if $V_{\bar\eta}$ is stable, then V_η is stable as well.

Lemma 3.1. *V_η is semistable if and only if $V_{\bar\eta}$ is semistable.*

Proof. We have seen that, if V_η is not semistable, then $V_{\bar\eta}$ is not semistable. Conversely suppose that $V_{\bar\eta}$ is not semistable. Then there is a canonically defined maximal destabilizing line subbundle of $V_{\bar\eta}$, which thus is fixed by every element of $\operatorname{Gal}(\overline{k(\mathbb{P}^1)}/k(\mathbb{P}^1))$. By standard descent theory this line subbundle must then be defined over $k(\mathbb{P}^1)$. Thus V_η is not semistable. □

Remark. If $V_{\bar\eta}$ is strictly semistable, it is typically the case that V_η is actually stable.

Lemma 3.2. *In case $\Delta \cdot \kappa = 1$, the bundle V_η is semistable if and only if it is stable.*

Proof. First assume that $m_1m_1 \equiv 1 \bmod 2$. In this case $\overline{V_\eta}$ has odd fiber degree, and so there are no strictly semistable bundles over $\overline{k(\mathbb{P}^1)}$. Hence, if V_η is semistable, then by Lemma 3.1 $V_{\bar\eta}$ is semistable and therefore stable. Thus V_η is stable by the remarks preceding Lemma 3.1.

In case $m_1m_1 \equiv 0 \bmod 2$, suppose that V_η is strictly semistable. Thus there is a line bundle on S_η of degree $m_1m_2/2$. There would thus exist a divisor D on S with $D\cdot f = m_1m_2/2$. Since $f = m_1m_2\kappa$, this possibility cannot occur. Thus V_η is stable. □

Here then is the theorem of this section:

Theorem 3.3. *Let V be a rank two vector bundle on S with $c_1(V) = \Delta$ and $c_2(V) = c$ and let L be a (Δ, c)-suitable ample line bundle. Then V is L-stable if and only if the restriction V_η of V to the generic fiber S_η is stable.*

Proof. First suppose that V is L-stable. Let F_η be a subbundle of V_η of rank one. Then there is a divisor F on S such that $\mathcal{O}_S(F)$ restricts to F_η and an inclusion $\mathcal{O}_S(F) \to V$. Hence there is an effective divisor D and an inclusion $\mathcal{O}_S(F+D) \to V$ such that the cokernel is torsion free. Since F_η is a subbundle of V_η, the divisor D cannot have positive intersection number with f. As D is effective it is supported in the fibers of π and so F and $F+D$ have the same restriction to the generic fiber. We may thus replace F by $F+D$. Then $V/\mathcal{O}_S(F)$ is torsion free. Hence there is an exact sequence

$$0 \to \mathcal{O}_S(F) \to V \to \mathcal{O}_S(\Delta - F) \otimes I_Z \to 0,$$

where Z is a codimension two subscheme of S. Thus

$$\Delta^2 - 4c \leq (2F - \Delta)^2.$$

Since V is L-stable, $L\cdot(2F - \Delta) < 0$. It follows from Definition 2.2 and (2) of the remark following it that $f\cdot(2F-\Delta) < 0$ as well. Thus $\deg F_\eta < \deg V_\eta/2$, which says that V_η is stable.

Conversely suppose that V_η is stable. Let $\mathcal{O}_S(F)$ be a sub-line bundle of V, where we may assume that $V/\mathcal{O}_S(F)$ is torsion free. Reversing the argument above shows that $f\cdot(2F-\Delta) < 0$ and therefore that $L\cdot(2F-\Delta) < 0$ as well. Thus V is L-stable. □

Corollary 3.4. *Let V be a rank two vector bundle on S with $c_1(V) = \Delta$ and $c_2(V) = c$. Then the following are equivalent:*

(i) *There exists a (Δ, c)-suitable ample line bundle L such that V is L-stable.*
(ii) *V is L-stable for every (Δ, c)-suitable ample line bundle L.*
(iii) *V_η is stable.*
(iv) *$V_{\bar\eta}$ is semistable.*
(v) *The restriction $V|\pi^{-1}(t)$ is semistable for almost all $t \in \mathbb{P}^1$.*
(vi) *There exists a $t \in \mathbb{P}^1$ such that $\pi^{-1}(t)$ is smooth and the restriction $V|\pi^{-1}(t)$ is semistable.*

Proof. By Lemmas 3.1 and 3.2, (iii) and (iv) are equivalent, and by Theorem 3.3 (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii). The implication (ii) $\Longrightarrow$ (i) is trivial. The implication (iv) $\Longrightarrow$ (v) follows from the openness of semistability in the Zariski topology in the sense of schemes, and the implication (v) $\Longrightarrow$ (vi) is trivial. To see that (vi) $\Longrightarrow$ (iv), suppose that $V_{\bar{\eta}}$ is not semistable. Then a destabilizing sub-line bundle extends to give a sub-line bundle over the pullback of S to some finite base change of $\mathbb{P}^1$. Thus $V|\pi^{-1}(t)$ is unstable for every $t \in \mathbb{P}^1$ such that $\pi^{-1}(t)$ is smooth, and so (vi) $\Longrightarrow$ (iv). □

Remark. In case $\Delta \cdot \kappa \equiv 0 \bmod 2$, there can exist strictly semistable bundles on S_η of degree $\Delta \cdot f$. There are examples of rank two bundles V on S with $c_1(V) = \Delta$ and V_η strictly semistable such that V is either stable, strictly semistable, or unstable (cf. [10]).

4. Donaldson polynomials and the main theorems

As above we let S denote a simply connected elliptic surface with $p_g(S) \geq 0$. Fix $w = \Delta \bmod 2$ and let p be an integer satisfying $w^2 \equiv p \bmod 4$. For $p_g(S) > 0$, there is the Donaldson polynomial $\gamma_{w,p}(S)$ corresponding to the $SO(3)$-bundle P with invariants w and p. Here for simplicity we shall always choose the orientation on the moduli space which agrees with the natural complex orientation. The polynomial $\gamma_{w,p}(S)$ is invariant up to sign under self-diffeomorphisms ψ of S such that $\psi^* w = w$. If $p_g(S) = 0$, then we have the distinguished chamber $\mathscr{C}(w, p)$ which contains κ in its closure. We shall then use $\gamma_{w,p}(S)$ to denote the Donaldson polynomial for S with respect to the chamber $\mathscr{C}(w, p)$, again with the orientation chosen to be the complex orientation. Since $\psi^*\mathscr{C}(w, p) = \pm\mathscr{C}(\psi^* w, p)$, the invariant $\gamma_{w,p}(S)$ is again natural up to sign under orientation-preserving self-diffeomorphisms which fix w and κ up to sign (and thus for all orientation-preserving self-diffeomorphisms which fix w if S is not rational). Of course, there are only finitely many choices for w, so that there is a subgroup of finite index in the full group of diffeomorphisms fixing κ which will also fix w.

Lemma 4.1. *For every choice of w and p, $\gamma_{w,p}(S)$ lies in $\mathbb{Q}[q_S, \kappa_S]$. Moreover, if for some choice of w and p, $\gamma_{w,p}(S)$ does not lie in $\mathbb{Q}[q_S]$, then every diffeomorphism ψ from S to another simply connected elliptic surface S' satisfies $\psi^*\kappa_{S'} = \pm\kappa_S$.*

Proof. The set of automorphisms of $H_2(S; \mathbb{Z})$ of the form ψ_*, where ψ is a diffeomorphism satisfying $\psi_*(\kappa) = \kappa$, $\psi^* w = w$, and $\psi^*\gamma_{w,p}(S) = \gamma_{w,p}(S)$, is a subgroup of finite index in the group of all isometries of $H_2(S; \mathbb{Z})$ preserving κ, by [10], Part I, Theorem 6, and [11], Chapter 2, Theorem 6.5. Thus by [11], Chapter 6, Theorem 2.12, $\gamma_{w,p}(S) \in \mathbb{Q}[q_S, \kappa_S]$. Moreover κ is the unique such class. The last statement of the lemma is then clear. □

Next let us discuss the effect of blowing up. Suppose that $\rho\colon \tilde{S} \to S$ is the r-fold blowup of S, and let the exceptional classes in $H_2(\tilde{S})$ be denoted by $e_1, \ldots, e_r$. Likewise let S' be another simply connected elliptic surface and let

$\rho' \colon \tilde{S}' \to S'$ be the r-fold blowup of S', with exceptional classes $e_1', \dots, e_r'$. If $\psi \colon \tilde{S} \to \tilde{S}'$ is a diffeomorphism, then $\psi^* e_i' = \pm e_j$ for a uniquely determined j, by [10], Part I, Theorem 7, and [11], Chapter 6, Corollary 3.8. It follows that, if $w' \in H^2(\tilde{S}'; \mathbb{Z}/2\mathbb{Z})$ is of the form $(\rho')^* w_0'$ for some $w_0' \in H^2(S'; \mathbb{Z}/2\mathbb{Z})$, then there is a $w_0 \in H^2(S; \mathbb{Z}/2\mathbb{Z})$, such that $\psi^* w' = \rho^* w_0$. Finally, we shall need the following extension of [11], Chapter 6, Theorem 3.1:

Proposition 4.2. *Let $w_0 \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ and let $\rho \colon \tilde{S} \to S$ be the r-fold blowup of S. If $b_2^+(S) = 1$, assume moreover that $\gamma_{\rho^* w_0, p}$ is defined with respect to some chamber $\mathscr{D}$. Let $\mathscr{C}$ be a chamber of type (w_0, p) on $H^2(S; \mathbb{R})$ such that $\mathscr{D}$ contains $\rho^* \mathscr{C}$ in its closure. Then*

$$\gamma_{\rho^* w_0, p} | \rho^* H_2(S; \mathbb{Z}) = \gamma_{w_0, p},$$

where if $b_2^+(S) = 1$, the polynomial $\gamma_{w_0, p}$ is defined with respect to the chamber $\mathscr{C}$.

Here, in case $p_g(S) = 0$, the chamber $\mathscr{C}$ does not in general determine a unique chamber $\mathscr{D}$ on $\tilde{S}$. However the conclusion of the proposition implies in particular that the value of $\gamma_{\rho^* w_0, p}$ on classes in $\rho^* H_2(S; \mathbb{Z})$ is independent of the chamber for $\tilde{S}$ of type $(\rho^* w_0, p)$ which contains $\mathscr{C}$ in its closure.

This result follows from standard gauge theory techniques [11]. It can also be proved in our case via algebraic geometry, using the blowup formulas for instance in [10]. Since it does not appear with an explicit proof in the literature, we shall outline a proof in the only case that concerns us, where the chamber $\mathscr{C}$ contains the first Chern class of an ample line bundle. We shall just write down the argument in the most interesting case, where $p_g(S) = 0$. We shall also assume here that the moduli spaces have the expected dimension for all choices of $p' \geq p$. The arguments given here can easily be extended to handle the case where $p \ll 0$, and with a little more effort will also cover the general case.

By induction we may assume that $\rho \colon \tilde{S} \to S$ is the blowup of S at a single point p. Let E be the exceptional curve and e be its cohomology class. We shall usually identify $H^2(S)$ with its image in $H^2(\tilde{S})$ under ρ^*. Let $\mathscr{D}$ be a chamber for $\tilde{S}$ of type $(\rho^* w_0, p)$ containing $\mathscr{C}$ in its closure and let ζ be a wall for $\mathscr{D}$. Then $\zeta = \zeta' + ae$, where $\zeta' \in H^2(S; \mathbb{Z})$ and $a \in \mathbb{Z}$ (in fact $2|a$ since $\zeta \equiv \Delta \bmod 2$). After possibly reflecting in e, which is realized by an orientation-preserving diffeomorphism r_e of $\tilde{S}$, we may assume that $a \geq 0$: Indeed, r_e^* switches the two possible chambers corresponding to $\pm a$, and so, if γ_1 and γ_2 are the two invariants corresponding to the two choices of chambers, then $r_e^* \gamma_1 = \gamma_2$. Since $r_e^* | H_2(S; \mathbb{Z})$ is the identity, it suffices to prove the result for either chamber. So we can assume that $a \geq 0$.

Since $\mathscr{C}$ is in the closure of $\mathscr{D}$, if $x \in \mathscr{C}$, then $x \cdot \zeta' = x \cdot \zeta \geq 0$. Conversely, if we start with an ample line bundle L on S such that $c_1(L) \in \mathscr{C}$, then for all $N \gg 0$, $Nc_1(L) - e$ is the first Chern class of an ample line bundle L_N on $\tilde{S}$. Moreover $(Nc_1(L) - e) \cdot \zeta \geq N(c_1(L) \cdot \zeta') \geq 0$. It follows from this that

$c_1(L_N)$ lies in $\mathscr{D}$, and if $c_1(L)$ is in the interior of $\mathscr{C}$ then $c_1(L_N)$ is in the interior of $\mathscr{D}$.

Consider rank two vector bundles $\tilde{V}$ on $\tilde{S}$ with $c_1(\tilde{V}) = \rho^*\Delta$ and $c_2(\tilde{V}) = c$. Set $V = (\rho_*\tilde{V})^{\vee\vee}$. Then V is a rank two vector bundle on S with $c_1(V) = \Delta$ and $c_2(V) \le c_2(\tilde{V})$, where equality holds if and only if $\tilde{V} = \rho^*V$. The arguments of the proof of Theorem 5.5 in Part II of [10], which essentially just depend on the determinant of $\tilde{V}$ being a pullback, show the following. There is a constant N_0, depending only on L and c, such that, for all $N \ge N_0$, if $\tilde{V}$ is L_N-stable then V is L-semistable, and conversely if V is L-stable then $\tilde{V}$ is L_N-stable. Moreover the map $V \mapsto \rho^*V$ defines an open immersion of schemes from the moduli space of L-stable rank two vector bundles on S with $c_1 = \Delta$ and $c_2 = c$ to the corresponding moduli space for $\tilde{S}$ and $c_1 = \rho^*\Delta$.

To evaluate the Donaldson polynomial on $\tilde{S}$ on a collection of classes of the form $\rho^*\alpha$, represent α by a smoothly embedded Riemann surface C on S which does not pass through p, the center of the blowup, and choose a theta characteristic on C. This choice leads to a divisor D_C on the moduli space. By definition $\tilde{V}$ lies in D_C if and only if the Dirac operator coupled to the ASD connection induced on C has a kernel. From this it is clear that $\tilde{V}$ lies in D_C if and only if V lies in the corresponding divisor on the moduli space for S of bundles with $c_1 = \Delta$ and $c_2 = c_2(V) \le c$. An easy counting argument then shows that, if d is the dimension of the moduli space for $\tilde{S}$ and we choose $C_1, \dots, C_d$ in general position and general theta characteristics on C_i, then $\tilde{V}$ lies in the intersection $D_{C_1} \cap \cdots \cap D_{C_d}$ if and only if $\tilde{V} = \rho^*V$ and V lies in the corresponding intersection for the moduli space of S. As $\#(D_{C_1} \cap \cdots \cap D_{C_d})$ calculates the value $\gamma_{\rho^* w_0, p}([C_1], \dots, [C_d])$, it is then clear that

$$\gamma_{\rho^* w_0, p} | \rho^* H_2(S; \mathbb{Z}) = \gamma_{w_0, p}. \quad \square$$

Assuming Theorems 0.4 and 0.5, we can now prove the main theorem of this paper, which we restate as a series of results.

Theorem 4.3. *Suppose that S and S' are two simply connected elliptic surfaces with $p_g(S) = p_g(S') = 1$. Suppose that neither S nor S' is a $K3$ surface. Let $\tilde{S}$ and $\tilde{S}'$ be two blowups of S and S', and let $\psi: \tilde{S} \to \tilde{S}'$ be a diffeomorphism. Identify $H^2(S; \mathbb{Z})$ with its image in $H^2(\tilde{S}; \mathbb{Z})$ under the natural map, and similarly for $H^2(S'; \mathbb{Z})$. Then $\psi^*\kappa_{S'} = \pm\kappa_S$.*

Proof. Arguing as in Corollary 3.6 of Chapter 6 of [11], we see that it suffices to show that some $\gamma_{w,p}(S)$ actually involves κ_S. If $m_1 m_2 \equiv 1 \bmod 2$, then the coefficient of κ_S^2 in $\gamma_{w,p}(S)$, for the choice of p given in Theorem 0.5 (i) corresponding to the two-dimensional moduli space, is $2m_1^2 m_2^2 - m_1^2 - m_2^2$. This number is zero if $m_1 = m_2 = 1$, in which case S is a $K3$ surface. Otherwise $2m_1^2 m_2^2 - m_1^2 - m_2^2 > 0$. Thus the coefficient of κ_S^2 is nonzero.

If $m_1 m_2 \equiv 0 \bmod 2$, then the expected dimension of the moduli space is $4c - \Delta^2 - 6 \equiv \Delta^2 \bmod 2$. Moreover $\Delta \cdot \kappa_S = 1$ and $K_S = (2m_1 m_2 - m_1 - m_2)\kappa$. Since exactly one of m_1, m_2 is even, $\Delta \cdot K_S \equiv 1 \bmod 2$. Thus by the Wu formula

$\Delta^2 \equiv 1 \bmod 2$, and the dimension of the moduli space is odd. It follows that every nonzero invariant must involve κ_S. Since nonzero invariants exist by Theorem 0.4 or more generally by Donaldson's theorem on the nonvanishing of the invariants, there are choices of p for which $\gamma_{w,p}(S)$ actually involves κ_S. $\square$

Theorem 4.4. *Suppose that S and S' are two elliptic surfaces with $p_g(S) = p_g(S') \geq 1$ with finite cyclic fundamental group and multiple fibers of multiplicities $\{m_1, m_2\}$ and $\{m_1', m_2'\}$, respectively, that $\tilde{S}$ and $\tilde{S}'$ are two blowups of S and S', and that $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic. Then $\{m_1, m_2\} = \{m_1', m_2'\}$. Hence S and S' are deformation equivalent.*

Proof. As in [11] we can reduce to the simply connected case. Using Theorem 0.3, we know that $m_1 m_2 = m_1' m_2'$ and that, if $\psi\colon \tilde{S} \to \tilde{S}'$ is a diffeomorphism, then $\psi_*(H_2(S; \mathbb{Z})) = H_2(S'; \mathbb{Z})$ under the identification of $H_2(S; \mathbb{Z})$ with a subspace of $H_2(\tilde{S}; \mathbb{Z})$ and likewise for S'. Fix a class $w \in H^2(S'; \mathbb{Z})$ with $w \cdot \kappa_{S'} = 1$. Thus

$$\gamma_{\psi^* w, p}(\tilde{S})|H_2(S; \mathbb{Z}) = \pm\gamma_{w,p}(\tilde{S}')|H_2(S'; \mathbb{Z})$$

under the natural identifications. Moreover $\psi^* w \cdot \kappa_S = w \cdot \kappa_{S'} = 1$. Thus the Donaldson polynomial invariants for the minimal surfaces S and S' for the values $\psi^* w$ and w respectively are equal. If $m_1 m_2 = m_1' m_2' \equiv 1 \bmod 2$, then

$$(p_g(S) + 1)m_1^2 m_2^2 - m_1^2 - m_2^2 = (p_g(S) + 1)(m_1')^2(m_2')^2 - (m_1')^2 - (m_2')^2.$$

Thus

$$m_1 m_2 = m_1' m_2' \quad \text{and} \quad m_1^2 + m_2^2 = (m_1')^2 + (m_2')^2.$$

It follows that $(m_1 + m_2)^2 = (m_1' + m_2')^2$ and so $m_1 + m_2 = m_1' + m_2'$. We can thus determine the elementary symmetric functions of m_1 and m_2 from the diffeomorphism type, and hence the unordered pair.

If $m_1 m_2 = m_1' m_2' \equiv 0 \bmod 2$, then, assuming that $2|m_1$, it follows from Theorem 0.4 that we can determine $m_1 m_2$ and m_2. Thus, we can determine m_1 as well. $\square$

Theorem 4.5. *Suppose that S and S' are two nonrational elliptic surfaces with finite cyclic fundamental group and with $p_g(S) = p_g(S') = 0$ with multiple fibers of multiplicities $\{m_1, m_2\}$ and $\{m_1', m_2'\}$, respectively, that $\tilde{S}$ and $\tilde{S}'$ are two blowups of S and S', and that $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic. Suppose further that $m_1 m_2 \equiv 0 \bmod 2$. Then $m_1' m_2' \equiv 0 \bmod 2$, and $\{m_1, m_2\} = \{m_1', m_2'\}$.*

Proof. We may again reduce to the simply connected case. Note that every diffeomorphism ψ from $\tilde{S}$ to $\tilde{S}'$ sends the subspace $H^2(S'; \mathbb{Z})$ to $H^2(S; \mathbb{Z})$. Choose a class $w \in H^2(S'; \mathbb{Z}/2\mathbb{Z})$ with $w \cdot \kappa_{S'} = 1$. We must have $\psi^* \mathscr{C}(w, p) = \pm\mathscr{C}(\psi^* w, p)$, by Lemma 2.5. As in the preceding argument, we are immediately reduced to comparing the Donaldson invariants for the surfaces S and S'. Let us first show that $m_1' m_2' \equiv 0 \bmod 2$. First note that the two-dimensional invariant corresponds to $-p = 5 > 4 = 2(4p_g(S) + 2)$. Thus we are in the stable range and can apply Theorem 0.4 to conclude that the leading coefficient of $\gamma_{w,-5}(S) = m_2$. Since S is not rational, $m_2 > 1$. But if $m_1'm_2' \equiv 1 \bmod 2$, then by (i) of Theorem 0.5 the leading coefficient of $\gamma_{w,-5}(S')$ is 1, a contradiction. Hence $m_1'm_2' \equiv 0 \bmod 2$ and the leading coefficient of $\gamma_{w,-5}(S')$ is just m_2'. It follows that $m_2 = m_2'$ and, by Bauer's result (Theorem 0.2), that $(m_1^2-1)(m_2^2-1) = ((m_1')^2-1)((m_2')^2-1)$. Thus $m_1 = m_1'$ as well. □

Theorem 4.6. *Suppose that S and S' are two nonrational elliptic surfaces with finite cyclic fundamental group and with $p_g(S) = p_g(S') = 0$ with multiple fibers of multiplicities $\{m_1, m_2\}$ and $\{m_1', m_2'\}$, respectively, that $\tilde{S}$ and $\tilde{S}'$ are two blowups of S and S', and that $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic. Suppose further that $m_1m_2 \equiv 1 \bmod 2$. Then $m_1'm_2' \equiv 1 \bmod 2$, and $\{m_1, m_2\} = \{m_1', m_2'\}$.*

Proof. As before we pass to the simply connected case. If $m_1'm_2' \equiv 0 \bmod 2$, then by Theorem 4.5 $m_1m_2 \equiv 0 \bmod 2$ as well, a contradiction. Thus $m_1'm_2' \equiv 1 \bmod 2$. Using (i) and (ii) of Theorem 0.5, we see that the Donaldson polynomials determine the quantities

$$A = m_1^2m_2^2 - m_1^2 - m_2^2 + 1 = (m_1^2-1)(m_2^2-1);$$
$$B = m_1^4m_2^4 - m_1^4 - m_2^4 + 1 = (m_1^4-1)(m_2^4-1).$$

We must show that A and B determine $\{m_1, m_2\}$ provided that both m_1 and m_2 are greater than one. This is just a matter of elementary algebra: let $\sigma_1 = m_1^2+m_2^2$ and $\sigma_2 = m_1^2m_2^2$. Then σ_1 and σ_2 are the elementary symmetric functions in m_1^2 and m_2^2 and thus determine $\{m_1^2, m_2^2\}$. As m_1 and m_2 are positive the knowledge of $\{m_1^2, m_2^2\}$ determines $\{m_1, m_2\}$.

To read off σ_1 and σ_2 from A and B, note that if $A \neq 0$ then

$$\frac{B}{A} = (m_1^2+1)(m_2^2+1) = m_1^2m_2^2 + m_1^2 + m_2^2 + 1.$$

Thus $2\sigma_2 = B/A + A - 2$ and $2\sigma_1 = B/A - A$ provided that $A \neq 0$. Now $A = 0$ if m_1 or m_2 is one, and otherwise $A \geq 1$. Thus, provided neither of m_1 or m_2 is one, A and B determine σ_2 and σ_1. □

PART II: THE CASE OF EVEN FIBER DEGREE

In this part, S shall always denote a simply connected elliptic surface over $\mathbb{P}^1$ with multiple fibers of multiplicities $2m_1$ and m_2, where m_2 is odd, and such that there exists a divisor Δ on S with $\Delta \cdot f = 2m_1m_2$, the minimum possible value. We shall describe certain moduli spaces of stable vector bundles V over S such that $c_1(V)$ and Δ have the same restriction to a general fiber. We then apply this study toward a partial calculation of the corresponding Donaldson polynomial invariants of S. Aside from quoting a few results from Part I, Part II can however be read independently. On the other hand, we draw heavily on the book [11], and many arguments which are very similar to arguments in [11] are sketched or simply omitted. Roughly speaking, the new ingredients in the proof consist of the algebraic geometry of certain elliptic surfaces associated to S, which have a single multiple fiber of multiplicity two and are birational to double covers of rational ruled surfaces. The vector bundle parts of the argument run more or less parallel to the arguments in [11], with a few new cases to analyze.

The outline of Part II is as follows. With notations and assumptions on S as above, there is an associated surface $J^{m_1m_2}(S)$ defined in Section 1 of Part I. The surface $J^{m_1m_2}(S)$ fibers over $\mathbb{P}^1$ and the fiber over a point t lying under a smooth fiber f of S is $J^{m_1m_2}(f)$, the set of line bundles of degree m_1m_2 on the fiber f of S. The surface $J^{m_1m_2}(S)$ has an involution defined by $\lambda \in J^{m_1m_2}(f) \mapsto \mathcal{O}_f(\Delta|f) \otimes \lambda^{-1}$. The quotient of $J^{m_1m_2}(S)$ by this involution is birational to a rational ruled surface $\mathbb{F}_N$, and we describe the geometry of the double cover in detail. In Section 2, we describe the rough classification of stable bundles V on S with $c_1(V) = \Delta$. To each such bundle there is an associated bisection C of $J^{m_1m_2}(S)$ which is invariant under the involution, and so defines a section of the quotient ruled surface. In Section 3, we show that for general bundles V, V is determined up to finite ambiguity by the section of the ruled surface and the choice of a certain line bundle on the associated bisection C of $J^{m_1m_2}(S)$. Thus, a Zariski open subset of the moduli space fibers over an open subset of the linear system of all sections on a certain rational ruled surface, and the fibers are a number of copies of the Jacobian $J(C)$ of the bisection C of $J^{m_1m_2}(S)$. It is here that the asymmetry between $2m_1$ and m_2 becomes apparent: the number of connected components of the fiber is just m_2. The reasons for this are explained following Lemma 2.4. Finally, in Section 4 we calculate the leading coefficient of a Donaldson polynomial invariant and show that it contains an "extra" factor of m_2. This calculation seems to be in general agreement with work of Kametani and Sato [13].

1. Geometry of certain elliptic surfaces

We fix the following notation. Let $\pi: S \to \mathbb{P}^1$ be an elliptic surface with two multiple fibers F_{2m_1} and F_{m_2} of multiplicities $2m_1$ and m_2, where $\gcd(2m_1, m_2) = 1$. We shall further assume that the reductions of the multiple fibers are smooth and that all other singular fibers are reduced and irreducible with just one ordinary double point. In other words, S is nodal in the terminology of [11]. Let $\kappa_S = \kappa \in H^2(S; \mathbb{Z})$ be the unique class such that $2m_1m_2\kappa = [f]$, where f is a general fiber of π. Finally we shall assume that there is a $2m_1m_2$-section Δ, i.e. a divisor Δ, not necessarily effective, with $\Delta \cdot f = 2m_1m_2$, or equivalently $\Delta \cdot \kappa = 1$. By [11], there always exist such nodal elliptic surfaces. In particular S is algebraic.

We shall be concerned with the associated elliptic surface $J^{m_1m_2}(S)$ defined in Section 1 of Part I. By the discussion there, $J^{m_1m_2}(S)$ has exactly one multiple fiber of multiplicity two, above the point of $\mathbb{P}^1$ corresponding to F_{2m_1}. We shall denote this fiber by F_2. Moreover, given the $2m_1m_2$-section Δ, there is an involution of $J^{m_1m_2}(S)$ defined on the generic fiber S_η by $\lambda \mapsto \mathcal{O}_{S_\eta}(\Delta) \otimes \lambda^{-1}$. Since $J^{m_1m_2}(S)$ is relatively minimal, this involution on the generic fiber extends to an involution on $J^{m_1m_2}(S)$, which we shall denote by ι. The data of $J^{m_1m_2}(S)$ and the involution ι do not depend on Δ, but only on the restriction of Δ to the generic fiber. A line bundle which has the same restriction as Δ to the generic fiber differs from Δ by a line bundle which is trivial on the generic fiber and thus is a multiple of κ. Up to twisting by a line bundle which is divisible by 2, the only possibilities are then Δ and $\Delta - \kappa$. Replacing Δ by $\Delta - \kappa$ replaces Δ^2 by $\Delta^2 - 2$ and thus changes $\Delta^2 \bmod 4$. Of course $\Delta^2 \equiv 1 \bmod 2$ since

$$\Delta \cdot K_S = 2m_1m_2(p_g + 1) - 2m_1 - m_2 \equiv 1 \bmod 2.$$

Let us determine the fixed point set of ι.

Lemma 1.1. *The fixed point set of ι consists of a smooth 4-section together with two isolated fixed points on F_2.*

Proof. The fixed point set of an involution on the smooth surface $J^{m_1m_2}(S)$ must consist of a smooth curve together with some isolated fixed points. For a smooth nonmultiple fiber f, there are four divisors λ such that $2\lambda = f$. Thus there is a component of the fixed point set which is a 4-section of $J^{m_1m_2}(S)$. Every other curve component of the fixed point set has trivial restriction to the generic fiber and (since the curve component is smooth) cannot meet the 4-section. Since all fibers are irreducible, there can be no other component, and the remaining fixed points are isolated.

Let us consider the possibilities for isolated fixed points away from F_2. In an analytic neighborhood of a nonmultiple fiber, there is a section of the elliptic fibration. Using this section to make the local identification of $J^{m_1m_2}(S)$ with $J^0(S)$, and since the group of local sections is divisible, it is easy to see that, after a translation, we can assume that ι corresponds to the involution $x \mapsto -x$. Direct inspection shows that this involution has no isolated fixed points, even at the nodal fibers. To handle the multiple fiber, the explicit description of a neighborhood X of a multiple fiber shows that after a base change of order two, say $\tilde{X} \to X$, we can assume that there is a local section of $\tilde{X}$. The induced map from the central fiber of $\tilde{X}$ to the central fiber of X corresponds to taking the quotient by a subgroup of order two. After a translation we can further assume that the pulled back involution on $\tilde{X}$ is given by $x \mapsto -x$. Since inverses commute with translations by a point of order two, the restriction of ι to F_2 again has four fixed points. Two of these lie on the 4-section (recall that a 4-section can meet F_2 in at most two distinct points) and the remaining two are isolated. $\square$

Thus there are two isolated fixed points of ι on F_2. If we blow these up and then take the quotient, the result will be a smooth surface $\mathbb{F}$ mapping to $\mathbb{P}^1$ whose fibers are smooth rational curves except over the point corresponding to F_2 (Figure 1). Over this point, the fiber is a curve $\mathfrak{d}_1 + 2\mathfrak{e} + \mathfrak{d}_2$, where $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are the images of the exceptional curves, $\mathfrak{e}$ is the image of F_2, and we have $(\mathfrak{d}_i)^2 = -2$ and $\mathfrak{e}^2 = -1$, $\mathfrak{d}_1 \cdot \mathfrak{e} = \mathfrak{d}_2 \cdot \mathfrak{e} = 1$ and $\mathfrak{d}_1 \cdot \mathfrak{d}_2 = 0$. In particular we may contract $\mathfrak{e}$ and then either $\mathfrak{d}_1$ or $\mathfrak{d}_2$ to obtain a rational ruled surface $\mathbb{F}_N$. We shall fix notation so that $\mathfrak{d}_2$ is the curve we contract and the resulting

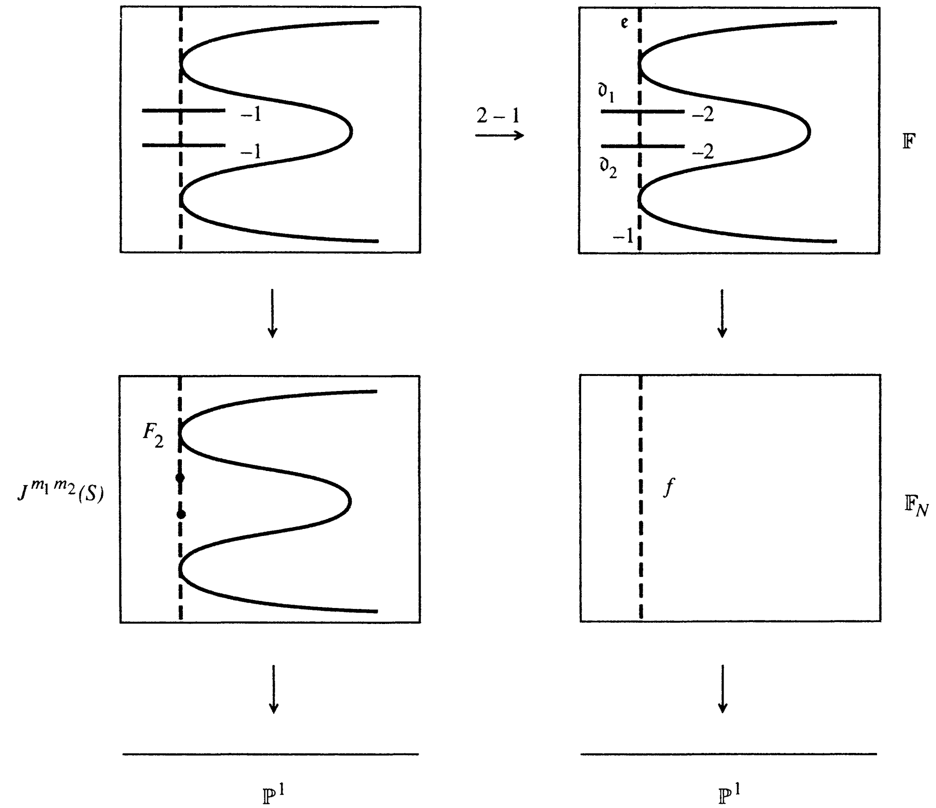

FIGURE 1

surface is $\mathbb{F}_N$. However, as we shall see, it is important to keep in mind the symmetry between $\mathfrak{d}_1$ and $\mathfrak{d}_2$. Contracting $\mathfrak{d}_1$ instead corresponds to making an elementary modification of $\mathbb{F}_N$ and thus replacing it by $\mathbb{F}_{N\pm 1}$. As we shall see, the symmetry between $\mathfrak{d}_1$ and $\mathfrak{d}_2$ corresponds to the choice of either Δ or $\Delta - \kappa$.

The branch divisor B' on the blowup $\mathbb{F}$ of $\mathbb{F}_N$ is of the form $B + \mathfrak{d}_1 + \mathfrak{d}_2$, where B is a smooth 4-section which does not meet $\mathfrak{d}_1$ or $\mathfrak{d}_2$ and hence $B \cdot \mathfrak{e} = 2$. Thus if we use the basis $\{\sigma, f, \mathfrak{d}_2, \mathfrak{e}\}$ for $\operatorname{Pic}(\mathbb{F})$, where σ is the negative section of $\mathbb{F}_N$ and f is the class of a fiber, viewed as curves on $\mathbb{F}$, it is easy to see that

$$B = 4\sigma + (2k+1)f - 4\mathfrak{e} - 2\mathfrak{d}_2$$

for some odd integer k and that

$$B + \mathfrak{d}_1 + \mathfrak{d}_2 = 4\sigma + (2k+2)f - 6\mathfrak{e} - 2\mathfrak{d}_2,$$

which is indeed divisible by 2. Note that we cannot say *a priori* that B is irreducible. However it cannot be a union of a 3-section and a section, since there are no sections of $J^{m_1 m_2}(S)$. In particular $B \cdot \sigma \geq 0$. Thus if $\mathfrak{d}_1$ is the proper transform of the fiber on $\mathbb{F}_N$, and we assume that the negative section σ does not pass through the point of the original fiber that was blown up, so that $\sigma \cdot \mathfrak{d}_2 = 0$, then $2k+1 \geq 4N$, or equivalently

$$k \geq 2N.$$

The same conclusion holds by a similar argument if σ does pass through the point that is blown up.

It is now easy to reverse this procedure. Begin with $\mathbb{F}_N$ and blow up a point in a fiber. Then blow up the point of intersection of the exceptional curve with the proper transform of the fiber. The result is a nonmiminal ruled surface $\mathbb{F}$ with a reducible fiber of the ruling of the form $\mathfrak{d}_1 + 2\mathfrak{e} + \mathfrak{d}_2$, where $\mathfrak{d}_1$ is the proper transform of the fiber, $\mathfrak{d}_2$ is the proper transform of the first exceptional curve, and $\mathfrak{e}$ is the second exceptional curve. Choose a smooth element B in the linear system $|4\sigma + (2k+1)f - 4\mathfrak{e} - 2\mathfrak{d}_2|$, if any exist, where σ is the negative section of $\mathbb{F}_N$ and f is a fiber. The double cover of $\mathbb{F}$ branched along $B + \mathfrak{d}_1 + \mathfrak{d}_2$ is then an elliptic surface with a multiple fiber of multiplicity 2, bisections corresponding to the pullbacks of sections of $\mathbb{F}_N$, and an involution ι.

Let us calculate $p_g(J^{m_1m_2}(S)) = p_g(S)$ in terms of $\mathbb{F}$ and B. The canonical bundle of $\mathbb{F}_N$ is given by $K_{\mathbb{F}_N} = -2\sigma - (N+2)f$. Thus, recalling that $\mathfrak{d}_2$ is the proper transform of the first exceptional curve and that $\mathfrak{e}$ is the second, we have

$$K_{\mathbb{F}} = -2\sigma - (N+2)f + \mathfrak{d}_2 + 2\mathfrak{e}.$$

As for the branch locus $B + \mathfrak{d}_1 + \mathfrak{d}_2$, we have

$$B + \mathfrak{d}_1 + \mathfrak{d}_2 = 2(2\sigma + (k+1)f - 3\mathfrak{e} - \mathfrak{d}_2).$$

By standard formulas for double covers,

$$H^0(J^{m_1m_2}(S); K_{J^{m_1m_2}(S)}) = H^0(\mathbb{F}; \mathcal{O}_{\mathbb{F}}(K_{\mathbb{F}} + 2\sigma + (k+1)f - 3\mathfrak{e} - \mathfrak{d}_2)).$$

Now, using the calculations above, we have

$$K_{\mathbb{F}} + 2\sigma + (k+1)f - 3\mathfrak{e} - \mathfrak{d}_2 = (k-N-1)f - \mathfrak{e}.$$

Recalling that f is linearly equivalent to $\mathfrak{d}_1 + 2\mathfrak{e} + \mathfrak{d}_2$, it is clear that

$$h^0((k-N-1)f - \mathfrak{e}) = \begin{cases} k-N-1, & \text{if } k-N \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Now if $k - N \leq 1$, since $k \geq 2N$ we must have $N \leq 1$. If $N = 1$, then $k = 2$ and so $k - N - 1 = 0$ in this case as well. If $N = 0$, then $k = 0$. In this case $B = 4\sigma + f - 4\mathfrak{e} - 2\mathfrak{d}_2 = 4\sigma + \mathfrak{d}_1 - 2\mathfrak{e} - \mathfrak{d}_2$, and it is easy to see that the effective curve $\sigma - \mathfrak{d}_2 - \mathfrak{e}$, which is the proper transform of the unique element of $|\sigma|$ passing through the point of the fiber which is blown up, satisfies

$$(\sigma - \mathfrak{d}_2 - \mathfrak{e}) \cdot (4\sigma + f - 4\mathfrak{e} - 2\mathfrak{d}_2) = -1.$$

Thus $|B|$ has the fixed component $\sigma - \mathfrak{d}_2 - \mathfrak{e}$, which is a section, and this case does not arise. So in all cases we have $h^0((k-N-1)f - \mathfrak{e}) = k - N - 1$.

Thus we may summarize this discussion as follows:

Lemma 1.2. *With notation and conventions as above,*

$$p_g(S) = p_g(J^{m_1 m_2}(S)) = k - N - 1. \quad \square$$

In particular, suppose that $p_g(S) = 0$. Since $k \geq 2N$, and the case $k = N = 0$ has been ruled out above, the only possibilities are $N = 0$, $k = 1$ or $N = 1$, $k = 2$. Thus $\mathbb{F}$ is the blowup of either $\mathbb{F}_0$ or $\mathbb{F}_1$, and of course the two cases are elementary transformations of each other. Thus we may assume that $k = 2$ and $N = 1$ in this case. Moreover the negative section of $\mathbb{F}_1$ does not pass through the exceptional point in the blowup.

2. Classification of stable bundles

We let S be a nodal elliptic surface over $\mathbb{P}^1$ with exactly two multiple fibers of multiplicities $2m_1$ and m_2 and let Δ be a divisor on S with $\Delta \cdot f = 2m_1 m_2$. Fix an integer c. In this section we shall study rank two vector bundles V with $c_1(V) = \Delta$ and $c_2(V) = c$. We shall also let $w = \Delta \bmod 2$ and $p = \Delta^2 - 4c$.

First recall the following standard definition from Part I:

Definition 2.1. An ample line bundle L on S is (Δ, c)-*suitable* or (w, p)-*suitable* if for all divisors D on S such that $-D^2 + D \cdot \Delta \leq c$, either $f \cdot (2D - \Delta) = 0$ or

$$\operatorname{sign} f \cdot (2D - \Delta) = \operatorname{sign} L \cdot (2D - \Delta).$$

The following is Lemma 2.3 of Part I:

Lemma 2.2. *For all pairs (Δ, c), (Δ, c)-suitable ample line bundles exist.* $\square$

With this said, here is the rough classification of rank two vector bundles V with $c_1(V) = \Delta$ and $c_2(V) = c$ which are stable with respect to a c-suitable line bundle L.

Theorem 2.3. *Let L be (Δ, c)-suitable, and let V be an L-stable rank two vector bundle with $c_1(V) = \Delta$ and $c_2(V) = c$. Then there exist:*

(i) *A smooth irreducible curve C and a birational map $C \to \overline{C} \subseteq J^{m_1 m_2}(S)$, where $\overline{C}$ is a bisection of $J^{m_1 m_2}(S)$ invariant under the involution ι;*

(ii) *A divisor D on T, the minimal desingularization of the normalization of $C \times_{\mathbb{P}^1} S$, such that $D \cdot f = m_1 m_2$, where f is a general fiber of $T \to C$, and moreover D has the same restriction to the generic fiber of T as the divisor induced by the section of $J^{m_1 m_2}(T)$ corresponding to the map $C \to J^{m_1 m_2}(S)$;*

(iii) *A codimension two local complete intersection Z and an exact sequence*

$$0 \to \mathcal{O}_T(D) \to \nu^* V \to \mathcal{O}_T(\nu^* \Delta - D) \otimes I_Z \to 0,$$

where $\nu\colon T \to S$ is the natural degree two map.

Moreover the bisection $\overline{C}$ and the double cover T are uniquely determined by the bundle V, and D is determined by the bundle V and the choice of a map $C \to J^{m_1 m_2}(S)$. Finally, every rank two vector bundle V with $c_1(V) = \Delta$ and

$c_2(V) = c$ satisfying (i)–(iii) *above is stable with respect to every* (Δ, c)*-suitable ample line bundle* L.

Proof. First suppose that V is L-stable. It follows from Corollary 3.4 of Part I that the restriction of V to the geometric generic fiber of π is semistable. More precisely, let $\eta = \operatorname{Spec} k(\mathbb{P}^1)$ and let $\bar{\eta} = \operatorname{Spec} \overline{k(\mathbb{P}^1)}$, where $\overline{k(\mathbb{P}^1)}$ denotes the algebraic closure of $k(\mathbb{P}^1)$. Let $V_{\bar{\eta}}$ denote the pullback of V to the curve $S_{\bar{\eta}}$ which is the geometric generic fiber of π. Then $V_{\bar{\eta}}$ is semistable. By the classification of rank two bundles on an elliptic curve, $V_{\bar{\eta}} = L_1 \oplus L_2$, where each L_i is a line bundle over $S_{\bar{\eta}}$ of degree $m_1 m_2$ and $L_1 \otimes L_2$ corresponds to the restriction of Δ to $S_{\bar{\eta}}$. The Galois group $\operatorname{Gal}(\overline{k(\mathbb{P}^1)}/k(\mathbb{P}^1))$ permutes the set $\{L_1, L_2\}$. This action cannot be trivial, since otherwise L_i would be rational over $k(\mathbb{P}^1)$ and then S would have an $m_1 m_2$-section. Thus the fixed field of the subgroup of $\operatorname{Gal}(\overline{k(\mathbb{P}^1)}/k(\mathbb{P}^1))$ which operates trivially on $\{L_1, L_2\}$ defines a degree two extension of $k(\mathbb{P}^1)$, corresponding to a morphism $C \to \mathbb{P}^1$. Setting T to be the minimal resolution of the normalization of $C \times_{\mathbb{P}^1} S$, there is a section of $J^{m_1 m_2}(T)$ defined by L_1, say. The image of this section in $J^{m_1 m_2}(S)$ is then the bisection $\overline{C}$. By construction $\overline{C}$ is invariant under the involution ι. Let $\nu \colon T \to S$ be the natural degree two map.

The inclusion $L_1 \to V_{\bar{\eta}}$ induces a sub-line bundle $\mathcal{O}_T(D) \to \nu^* V$, which we may assume to have torsion free cokernel. Since $L_2 \neq L_1$, it is clear that this sub-line bundle is unique. The quotient is then necessarily of the form $\mathcal{O}_T(\nu^*\Delta - D) \otimes I_Z$.

It remains to prove that every V satisfying the above description is indeed L-stable. It follows from (iii) that $V_{\bar{\eta}}$ is an extension of two line bundles of degree $m_1 m_2$ and is therefore semistable. Again using Corollary 3.4 of Part I, V is L-stable. □

Next we discuss the meaning of the scheme Z and the bisection $\overline{C}$. The following is the analogue of Lemma 1.11 in Chapter 7 of [11] and is proved in exactly the same way:

Lemma 2.4. *Let f be a smooth fiber of π and let g be a component of $\nu^{-1}(f)$. Then* $\operatorname{Supp} Z \cap g \neq \emptyset$ *if and only if $V|f$ is unstable. In particular, if ν is not branched over f, so that $\nu^{-1}(f) = g \cup g'$, then* $\operatorname{Supp} Z \cap g \neq \emptyset$ *if and only if* $\operatorname{Supp} Z \cap g' \neq \emptyset$. □

Next we turn to the section $\overline{C}$. Since $\overline{C}$ is invariant under ι, its proper transform on the blowup of $J^{m_1 m_2}(S)$ at the two isolated fixed points of ι is the pullback of a section A' of $\mathbb{F}$, which in turn induces a section A of $\mathbb{F}_N$. We shall use throughout the notation and conventions of the previous section. Notice that the section A' meets the reducible fiber either along $\mathfrak{d}_1$ or $\mathfrak{d}_2$, the two components of multiplicity one. Here $A' \cdot \mathfrak{d}_1 = 1$ and $A' \cdot \mathfrak{d}_2 = 0$ if A does not pass through the point of the corresponding fiber of $\mathbb{F}_N$ which was blown up, and $A' \cdot \mathfrak{d}_2 = 1$ and $A' \cdot \mathfrak{d}_1 = 0$ in the remaining case. Since the branch locus of the map $J^{m_1 m_2}(S) \to \mathbb{F}$ consists of $B + \mathfrak{d}_1 + \mathfrak{d}_2$, we see that A' always passes through the branch locus over the point corresponding to F_2. Of course, this is also clear from the picture on $J^{m_1 m_2}(S)$: since $\overline{C}$ is a bisection, $\overline{C} \cdot f = 2$ and therefore $\overline{C} \cdot F_2 = 1$. It follows that $\overline{C}$ is smooth at the point of intersection with F_2, that the intersection is transverse, and that the natural map $\overline{C} \to \mathbb{P}^1$ is always branched at the point corresponding to F_2. A similar statement will hold for the map $C \to \mathbb{P}^1$. This fact is the fundamental difference between the case studied here and the case of trivial determinant studied in [8] and [11].

Since A' always meets $\mathfrak{d}_1$ or $\mathfrak{d}_2$, it follows that the inverse image of A' in the blowup of $J^{m_1 m_2}(S)$ always meets exactly one of the two exceptional curves, and in fact meets it transversally at one point. Thus the inverse image of A' is the proper transform of $\overline{C}$, and therefore

$$(\overline{C})^2 = 2(A')^2 + 1.$$

Let us consider the section A of $\mathbb{F}_N$ in more detail. Either $A = \sigma$ or $A \in |\sigma + (N+s)f|$ for a uniquely specified nonnegative integer f. Moreover either A does not pass through the point on $\mathbb{F}_N$ which is the image of the exceptional divisor, in which case $A' \cdot \mathfrak{d}_2 = 0$, or it does, in which case $A' \cdot \mathfrak{d}_2 = 1$. The following lemma relates the odd integer $\Delta^2 - 4c = p$ to the invariants of V:

Lemma 2.5. *With notation as above, denote by the exceptional point the point of $\mathbb{F}_N$ which is blown up under the morphism $\mathbb{F} \to \mathbb{F}_N$. Then, if we set $p = p_1(\operatorname{ad} V) = \Delta^2 - 4c$,*

$$-p = \begin{cases} 4k - 6N + 1 + 2\ell(Z) + \delta, & \text{if } A = \sigma \text{ does not pass through the exceptional point;} \\ 4s + 4k - 2N + 1 + 2\ell(Z) + \delta, & \text{if } A \in |\sigma + (N+s)f| \text{ does not pass through the exceptional point;} \\ 4k - 6N - 1 + 2\ell(Z) + \delta, & \text{if } A = \sigma \text{ passes through the exceptional point;} \\ 4s + 4k - 2N - 1 + 2\ell(Z) + \delta, & \text{if } A \in |\sigma + (N+s)f| \text{ passes through the exceptional point.} \end{cases}$$

Here δ is a nonnegative integer which is zero if the map $C \to \mathbb{P}^1$ is not branched over any point corresponding to a singular nonmultiple fiber of $\pi\colon S \to \mathbb{P}^1$.

Proof. Since $c_1^2(\nu^* V) = 2c_1^2(V) = 2\Delta^2$ and $c_2(\nu^* V) = 2c_2(V) = 2c$, it will suffice to work with $\nu^* V$. Clearly

$$c_2(\nu^* V) = -D^2 + D \cdot \nu^* \Delta + \ell(Z).$$

Thus

$$2(4c - \Delta^2) = -(2D - \nu^* \Delta)^2 + 4\ell(Z).$$

Now we can write

$$2D - \nu^* \Delta = D - (\nu^* \Delta - D),$$

where both D and $\nu^*\Delta - D$ naturally correspond to sections of $J^{m_1m_2}(T)$. In fact, if $\varphi\colon J^{m_1m_2}(T) \to J^{m_1m_2}(S)$ is the obvious map, then the bisection $\overline{C}$ satisfies $\varphi^*\overline{C} = C_1 + C_2$, where C_1 and C_2 are sections of $J^{m_1m_2}(T)$ corresponding to the divisors D and $\nu^*\Delta - D$ on T. An argument essentially identical to the proofs of Claims 1.17 and 1.18 in Chapter 7 of [11] shows that there is a nonnegative integer δ such that

$$-(2D - \nu^*\Delta)^2 = -(C_1 - C_2)^2 + 2\delta.$$

Moreover $\delta = 0$ if the map $C \to \mathbb{P}^1$ is not branched over any point corresponding to a singular nonmultiple fiber of $\pi\colon S \to \mathbb{P}^1$. Thus we must calculate $(C_1 - C_2)^2$. But, using the fact that C_i is a section of $J^{m_1m_2}(T)$, we have

$$(C_i)^2 = -2(1 + p_g(S)) = -2(k - N).$$

Moreover $C_1 + C_2 = \varphi^*\overline{C}$. Thus

$$\begin{aligned}(C_1 - C_2)^2 &= 2(C_1)^2 + 2(C_2)^2 - (C_1 + C_2)^2\\ &= -8(k - N) - 2(\overline{C})^2\\ &= -8(k - N) - 4(A')^2 - 2.\end{aligned}$$

Clearly we have

$$(A')^2 = \begin{cases} -N, & \text{if } A = \sigma \text{ does not pass through the exceptional point;}\\ N + 2s, & \text{if } A \in |\sigma + (N+s)f| \text{ does not pass through}\\ & \text{the exceptional point;}\\ -N - 1, & \text{if } A = \sigma \text{ passes through the exceptional point;}\\ N + 2s - 1, & \text{if } A \in |\sigma + (N+s)f| \text{ passes through}\\ & \text{the exceptional point.}\end{cases}$$

Putting these formulas together gives the statement of the lemma. □

Using Lemma 2.5, the inequality $k \geq 2N$ and the fact that if $N = 0$ then $k \geq 1$, whereas if $N = 1$ and $k = 2$ then the section σ does not pass through the exceptional point, we can easily deduce the following slight strengthening of Bogomolov's inequality in our case:

Corollary 2.6. *We have the following inequality for* $-p$:

$$-p \geq \begin{cases} 4p_g(S) - 2N + 5, & \textit{if } A = \sigma \textit{ does not pass through}\\ & \textit{the exceptional point;}\\ 4p_g(S) + 2N + 5, & \textit{if } A \in |\sigma + (N+s)f| \textit{ does not pass through}\\ & \textit{the exceptional point;}\\ 4p_g(S) - 2N + 3, & \textit{if } A = \sigma \textit{ passes through the exceptional point;}\\ 4p_g(S) + 2N + 3, & \textit{if } A \in |\sigma + (N+s)f| \textit{ passes through}\\ & \textit{the exceptional point.}\end{cases}$$

In all cases $-p \geq 2p_g(S) + 1$, *and if* $p_g(S) = 0$, *then* $-p \geq 3$. □

3. A Zariski open subset of the moduli space

Our goal in this section is to prove the following theorem:

Theorem 3.1. *Let V be an L-stable rank two bundle on S. Suppose that*

(i) *The associated bisection $\overline{C}$ of $J^{m_1m_2}(S)$ is smooth, or equivalently that $\overline{C} = C$, and the image of $\overline{C}$ in $\mathbb{F}$ is not the proper transform of σ;*

(ii) *The map $C \to \mathbb{P}^1$ is not branched at any point corresponding to a singular fiber of π or at the multiple fiber of odd multiplicity m_2;*

(iii) *The scheme Z on the associated double cover T is empty, and thus there is an exact sequence*

$$0 \to \mathscr{O}_T(D) \to \nu^*V \to \mathscr{O}_T(\nu^*\Delta - D) \to 0.$$

Then $V = \nu_*\mathscr{O}_T(D+F) = \nu_*\mathscr{O}_T(\nu^*\Delta - D)$. In particular V is uniquely determined by the associated section A of $\mathbb{F}_N$ and the divisor D on T. Finally V is a smooth point of its moduli space, which is of dimension $-p - 3\chi(\mathscr{O}_S)$ at V.

It is clear that the conditions above are equivalent to assuming that A' meets the branch locus B transversally, and that no point of intersection lies over a point of $\mathbb{P}^1$ corresponding to a singular nonmultiple fiber or to the multiple fiber of multiplicity m_2, and that $Z = \emptyset$.

The proof of Theorem 3.1 will proceed along lines very similar to the proof of Theorem 1.12 in Chapter 7 of [11], and we shall simply sketch some of the details.

Let A be the section of $\mathbb{F}_N$ corresponding to A'. By assumption $A \neq \sigma$. Let r be the nonnegative integer such that $A \in |\sigma + (N+r)f|$. If the section A does not pass through the exceptional point of the blowup, then

$$(A')\cdot(B + \mathfrak{d}_1 + \mathfrak{d}_2) = (\sigma + (N+r)f)\cdot(4\sigma + (2k+2)f - 6\mathfrak{e} - 2\mathfrak{d}_2) = 4r + 2k + 2.$$

Of these points, one corresponds to the intersection $A' \cdot \mathfrak{d}_1$, and so the branch divisor of the map $T \to S$ is $(4r + 2k + 1)f$, where f is a general fiber of π. This divisor is even since f is divisible by 2, and we set $G = (4r+2k+1)f/2$. Likewise, if A does pass through the exceptional point of the blowup, then

$$(A')\cdot(B+\mathfrak{d}_1+\mathfrak{d}_2) = (\sigma+(N+r)f-\mathfrak{d}_2-\mathfrak{e})\cdot(4\sigma+(2k+2)f-6\mathfrak{e}-2\mathfrak{d}_2) = 4r+2k.$$

In this case we set $G = (4r + 2k - 1)f/2$. Let F be the branch divisor in T, so that $\nu^*G \equiv F$. Thus $F = (4r + 2k + 1)f$ or $(4r + 2k - 1)f$. For future reference, let us also record the genus of C:

Lemma 3.2. *Let C satisfy* (i) *and* (ii) *of Theorem* 3.1. *Then*

$$g(C) = \begin{cases} 2r + k, & \text{if } A \text{ does not pass through the exceptional point;} \\ 2r + k - 1, & \text{if } A \text{ passes through the exceptional point.} \end{cases}$$

Proof. The map $C \to \mathbb{P}^1$ is branched at $A'\cdot(B+\mathfrak{d}_1+\mathfrak{d}_2) = 4r+2k+2$ points if A does not pass through the exceptional point, and $4r + 2k$ points otherwise. The lemma now follows from the Riemann-Hurwitz formula. □

Now $\det \nu_*\mathscr{O}_T(D) = \nu_*D - G$. Clearly ν_*D and Δ have the same restriction to the generic fiber. Arguing as in Chapter 7, (1.20) of [11], there is an injective map

$$\nu_* \mathcal{O}_T(D+F) \to V.$$

Set $W = \nu_* \mathcal{O}_T(D+F)$. Our goal will be to show that $W = V$. Preliminary to this goal we shall analyze W and the map $W \to V$. As a divisor class $\det W = \nu_* D - G + 2G = \nu_* D + G$. In addition there is an effective divisor E such that $(\det W)^{-1} \otimes \det V = \mathcal{O}_S(E)$. Thus $\det W = \Delta - E$ and E has trivial restriction to the generic fiber, so that E is a union of fibers (possibly including the reductions of the multiple fibers). Moreover $\nu_* D = \Delta - G - E$. Set $E' = \nu^* E$. We have $D + \iota^* D = \nu^* \nu_* D = \nu^* \Delta - F - E'$, and therefore $\iota^* D = \nu^* \Delta - D - F - E'$. We can thus write

$$W = \nu_* \mathcal{O}_T(D+F) = \nu_* \mathcal{O}_T(\nu^* \Delta - D - E').$$

Using the fact that there is a surjection from $\nu^* W$ to $\mathcal{O}_T(\nu^* \Delta - D - E')$, we conclude that there is an exact sequence

$$0 \to \mathcal{O}_T(D) \to \nu^* W \to \mathcal{O}_T(\nu^* \Delta - D - E') \to 0.$$

Comparing this sequence to the defining exact sequence for $\nu^* V$ and arguing as in (1.24) of Chapter 7 of [11], we may conclude:

Lemma 3.3. *Let $Q = V/W$. Then $\nu^* Q \cong [\mathcal{O}_T / \mathcal{O}_T(-E')] \otimes \mathcal{O}_T(\nu^* \Delta - D)$.* □

Our goal now is to prove the following:

Lemma 3.4. *In the above notation, $E' = 0$. Thus $Q = 0$ and $V = W = \nu_* \mathcal{O}_T(D+F)$ where $\nu_* D = \Delta - G$.*

We begin with the following construction. Let e be a component of the support of E', and write $E' = ae + E''$, where E'' is effective and disjoint from e and $a > 0$. If e is not the multiple fiber of multiplicity m_1 on T, then either ν is unbranched over e or e is a smooth fiber. In either of these cases ν induces an isomorphism from e to $\nu(e) = f$, and we shall identify $\nu(e)$ with e. In the remaining case $e = F_{m_1}$ is the multiple fiber of multiplicity m_1 and ν is an étale double cover. There is then the following analogue of (1.25) of Chapter 7 of [11]:

Lemma 3.5. *There is a subsheaf Q_0 of $\nu_* Q$ which is isomorphic to*

(i) $\mathcal{O}_e(-(a-1)e + \nu^* \Delta - D)$, *viewed as a sheaf on $\nu(e) = f$, if $e \neq F_{m_1}$;*

(ii) *A line bundle on F_{2m_1} such that*

$$\nu^* Q_0 \cong \mathcal{O}_{F_{m_1}}(-(a-1)F_{m_1} + \nu^* \Delta - D),$$

in case $e = F_{m_1}$.

Proof. The argument in case $e \neq F_{m_1}$ runs as in (1.25) of Chapter 7 of [11]. If $e = F_{m_1}$, then $\nu^* Q$ contains the subsheaf

$$Q_0' = [\mathcal{O}_T(-(a-1)e) / \mathcal{O}_T(-ae)] \otimes \mathcal{O}_T(\nu^* \Delta - D),$$

which is a line bundle on F_{m_1}. The vector bundle $\nu_* Q_0'$ is a rank two vector bundle on F_{2m_1} with $\deg(\nu_* Q_0') = m_1 m_2$. Consider the rank two vector bundle $\nu^*\nu_* Q_0'$ on F_{m_1}. Its determinant is $Q_0' \otimes \iota^* Q_0' = (Q_0')^{\otimes 2}$ (recall that $\nu^*\Delta - D$ is fixed under the involution) and there is a surjective map $\nu^*\nu_* Q_0' \to Q_0'$. Thus there is an exact sequence

$$0 \to Q_0' \to \nu^*\nu_* Q_0' \to Q_0' \to 0.$$

It follows that $\nu^*\nu_* Q_0'$ is semistable and is either $Q_0' \oplus Q_0'$ or the nontrivial extension of Q_0' by Q_0'. On the other hand, $\nu_* Q_0'$ is either a direct sum of line bundles, say $Q_0 \oplus Q_1$ for two line bundles Q_i, or a nontrivial extension of a line bundle Q_0 by Q_0. In the first case we must have $\nu^* Q_i = Q_0'$ and in the second $\nu^* Q_0 = Q_0'$. In either case there is a subbundle Q_0 of $\nu_* Q$ as desired. □

To prove Lemma 3.4, we shall assume that $E' \neq 0$ and derive a contradiction. Again, the argument will be very similar to the argument given in [11]. It will suffice to show that $\dim \operatorname{Ext}^1(Q_0, W) \leq 1$. We also have that

$$\operatorname{Ext}^1(Q_0, W) = H^0(W \otimes \mathscr{O}_S(e) \otimes Q_0^{-1}).$$

The case where e does not lie over the branch locus of $C \to \mathbb{P}^1$ follows exactly as in [11]. The case where e is a smooth fiber in the branch locus also follows by these methods provided we can show that $R^1 \rho_* \mathscr{O}_T(2D - \nu^*\Delta)$ has length one at the point of C corresponding to e. This is a local calculation, which we shall leave to the reader; it uses the fact that A' meets the branch locus transversally and can in fact be deduced from the global argument in [11], proof of Lemma 1.19 of Chapter 7.

There remains the new case, where $e = F_{m_1}$. In this case, the natural map

$$W = \nu_* \mathscr{O}_T(\nu^*\Delta - D - E') \to \nu_* \mathscr{O}_{F_{m_1}}(\nu^*\Delta - D - aF_{m_1})$$

is surjective, as one can see from applying the surjective map ν_* to the exact sequence

$$0 \to \mathscr{O}_T(\nu^*\Delta - D - E' - F_{m_1}) \to \mathscr{O}_T(\nu^*\Delta - D - E') \to \mathscr{O}_{F_{m_1}}(\nu^*\Delta - D - aF_{m_1}) \to 0.$$

It follows that $W|F_{2m_1} = \nu_* \mathscr{O}_{F_{m_1}}(\nu^*\Delta - D - aF_{m_1})$. Thus

$$\begin{aligned} H^0(W \otimes \mathscr{O}_S(F_{2m_1}) \otimes Q_0^{-1}) &= H^0(\nu_* \mathscr{O}_{F_{m_1}}(\nu^*\Delta - D - aF_{m_1}) \otimes \mathscr{O}_S(F_{2m_1}) \otimes Q_0^{-1}) \\ &= H^0(\nu_*[\mathscr{O}_{F_{m_1}}(\nu^*\Delta - D - aF_{m_1}) \otimes \nu^*\mathscr{O}_S(F_{2m_1}) \otimes \nu^* Q_0^{-1}]) = H^0(\mathscr{O}_{F_{m_1}}), \end{aligned}$$

where we have used the fact that $\nu^* Q_0 = \mathscr{O}_{F_{m_1}}(\nu^*\Delta - D - (a-1)F_{m_1})$. Hence

$$\dim \operatorname{Ext}^1(Q_0, W) = h^0(W \otimes \mathscr{O}_S(F_{2m_1}) \otimes Q_0^{-1}) = 1$$

as desired. □

We see that we have proved all of Theorem 3.1 except the statement about the smoothness of the moduli space, which follows from:

Lemma 3.6. *Suppose that* $V = \nu_*\mathcal{O}_T(D+F)$ *as above. Then* V *is good, in the terminology of* [8]. *In other words,* $H^2(S; \operatorname{ad} V) = 0$.

Proof. It suffices to show that $\dim \operatorname{Hom}(V, V \otimes K_S) = h^0(K_S)$. Now

$$\begin{aligned}\operatorname{Hom}(V, V\otimes K_S) &= \operatorname{Hom}(V, \nu_*\left(\mathcal{O}_T(\nu^*\Delta - D)\otimes \nu^*K_S\right))\\ &= \operatorname{Hom}(\nu^*V, \mathcal{O}_T(\nu^*\Delta - D)\otimes \nu^*K_S).\end{aligned}$$

Using the defining exact sequence for ν^*, there is an exact sequence

$$0 \to H^0(\nu^*K_S) \to \operatorname{Hom}(\nu^*V, \mathcal{O}_T(\nu^*\Delta - D)\otimes\nu^*K_S) \to H^0(\mathcal{O}_T(\nu^*\Delta - 2D)\otimes\nu^*K_S).$$

Since K_S is a rational multiple of the fiber and $\nu^*\Delta - 2D$ is nontrivial on the generic fiber, the term $H^0(\mathcal{O}_T(\nu^*\Delta - 2D)\otimes \nu^*K_S)$ is zero. Thus

$$\dim \operatorname{Hom}(V, V\otimes K_S) = h^0(\nu^*K_S).$$

Using the isomorphism $H^0(\nu^*K_S) \cong H^0(K_S) \oplus H^0(K_S(-G))$, it suffices to show that $H^0(K_S(-G)) = 0$. Now

$$K_S = \mathcal{O}_S((k - N - 2)f + (2m_1 - 1)F_{2m_1} + (m_2 - 1)F_{m_2}).$$

Also $G = (2r + k \pm 1/2)f$. Thus it suffices to observe that the linear system

$$|(-N - 2r - 2 \pm 1/2)f + (2m_1 - 1)F_{2m_1} + (m_2 - 1)F_{m_2}|$$

is empty. □

Next let us describe the subset of the moduli space consisting of bundles V which satisfy the hypotheses of Theorem 3.1. We begin by reversing the procedure outlined above. Fix the section A', which is generic in the sense of Theorem 3.1: it meets B transversally and no point of intersection corresponds to a singular nonmultiple fiber or to the multiple fiber of odd multiplicity. The section A' determines the bisection $C = \overline{C}$, and thus a double cover $\nu\colon T \to S$ together with an elliptic fibration $\rho\colon T \to C$ and a divisor D_0, well-defined on the generic fiber. Moreover by construction $\nu_* D_0$ and Δ have the same restriction on the generic fiber, and thus differ by a multiple of κ. It is easy to see that changing D_0 by a sum of fiber components on T replaces ν_*D_0 by an arbitrary even multiple of κ. Thus we may assume that we have $\nu_*D_0 = \Delta - G$ or $\nu_*D_0 = \Delta - \kappa - G$. It is an exercise in the formulas of the preceding section to see that we have

$$\Delta^2 \equiv 2(A')^2 + 1 \bmod 4.$$

Additionally

$$2(A')^2 + 1 \equiv \begin{cases} 2N + 1 \bmod 4, & \text{if } A \text{ does not pass through the exceptional point;}\\ 2N - 1 \bmod 4, & \text{otherwise.}\end{cases}$$

Thus the symmetry between the possibility that A does or does not pass through the exceptional point, which is essentially the choice of blowing $\mathbb{F}$ down to $\mathbb{F}_N$

or $\mathbb{F}_{N\pm 1}$, reflects the choice of Δ or $\Delta - \kappa$, which in turn reflects $\Delta^2 \bmod 2$, or equivalently the dimension of the moduli space mod 2.

Having made one choice for a line bundle D_0 on the double cover T, where D_0 is specified on the generic fiber of T and satisfies $\nu_* D_0 = \Delta - G$, or $\nu_* D_0 = \Delta - \kappa - G$, the possibilities for D are given by the next lemma.

Lemma 3.7. *Given the double cover $T \to S$, the set of all D whose restriction to the generic fiber equals D_0 and which satisfy $\nu_* D = \Delta - G$, or $\nu_* D = \Delta - \kappa - G$ is a principal homogeneous space over* $\operatorname{Pic}^\tau T$, *which in turn is an extension of the Jacobian $J(C)$ by a cyclic group of order m_2. Moreover this principal homogeneous space is nonempty for exactly one of the two choices for $\nu_* D$ above.*

Proof. By the remarks preceding the lemma, there exists a D_0 with $\nu_* D_0 = \Delta - G$, or $\nu_* D_0 = \Delta - \kappa - G$, and only one of these possibilities can hold. If D has the same restriction to the generic fiber as D_0 and $\nu_* D = \nu_* D_0$, then $D - D_0$ has trivial restriction to the generic fiber and $\nu_*(D - D_0) = 0$. The first condition says that $D - D_0$ is of the form $\rho^* \lambda \otimes \mathcal{O}_T(aF_{m_1} + bF'_{m_2} + cF''_{m_2})$, where F_{m_1} is the multiple fiber of multiplicity m_1 lying above F_{2m_1} and F'_{m_2}, F''_{m_2} are the two multiple fibers of multiplicity m_2 lying over F_{m_2}. We may further assume that $0 \le a < m_1$ and that $0 \le b < m_2$, $0 \le c < m_2$. Here λ is a line bundle of degree d on C. Thus

$$\nu_*(D - D_0) = df + 2aF_{2m_1} + (b + c)F_{m_2}.$$

It is easy to see that this divisor is trivial if and only if $d = 0$, $a = 0$, and $b \equiv -c \bmod m_2$. Thus, there is a natural identification of the set of all D (given the fixed divisor D_0) with $J(C) \times \mathbb{Z}/m_2\mathbb{Z}$. □

We now assume that A is not the negative section of $\mathbb{F}_N$ and write $A \in |\sigma + (N + r)f|$ where $r \ge 0$. The dimension of the linear system A' is then equal to n, where

$$n = \begin{cases} N + 2r + 1, & \text{if } A \text{ does not pass through the exceptional point;} \\ N + 2r, & \text{otherwise.} \end{cases}$$

We also let $g = g(C)$ be the genus of the bisection C, as given in Lemma 3.2. Note that

$$n + g = \begin{cases} 4r + N + k + 1, & \text{if } A \text{ does not pass through the exceptional point;} \\ 4r + N + k - 1, & \text{otherwise.} \end{cases}$$

In both cases, comparing this with the formula for $-p$ given in Lemma 2.5 and using the fact that $1 + p_g(S) = k - N$, we see that

$$-p - 3\chi(\mathcal{O}_S) = g + n.$$

Note finally that the moduli space will be nonempty provided that $-p \ge 4p_g(S) + 2N + 3$. Since $p_g(S) = k - N - 1 \ge N - 1$, the moduli space will be nonempty as long as

$$-p \ge 6p_g(S) + 5.$$

Arguing as in Theorem 1.14 of Chapter 7 of [11], we obtain the following:

Theorem 3.8. *Let p be an odd negative integer, and choose $w \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ such that $w = \Delta \bmod 2$ or $w = \Delta - \kappa \bmod 2$, and that $w^2 \equiv p \bmod 4$. Let L be a (w, p)-suitable ample line bundle on S. Let $\mathfrak{M} = \mathfrak{M}(S, L; w, p)$ denote the moduli space of L-stable rank two bundles on S with $w_2(V) = w$ and $p_1(\operatorname{ad} V) = p$. Then for all p such that $-p \geq 6p_g(S) + 5$, $\mathfrak{M}$ contains a nonempty Zariski open subset M corresponding to vector bundles V satisfying the hypotheses of Theorem 3.1. The set M is smooth of dimension $-p - 3\chi(\mathcal{O}_S)$. Moreover, there is a holomorphic map from M to a Zariski open subset $U \subseteq \mathbb{P}^n$ and the fibers are isomorphic to m_2 copies of a complex torus of dimension g.* □

In fact, arguing as in [8], the component M is also Zariski dense. However we do not prove this result as we do not need it; the arguments in the proof of Theorem 4.3 show that any other components of $\mathfrak{M}$ do not contribute to the leading coefficient of the Donaldson polynomial.

Let us finally consider the case where $p_g(S) = 0$ and $-p = 3$, the case of a moduli space of expected dimension zero. In this case we fix $N = 1$ and $k = 2$, and the negative section does not pass through the exceptional point. The Chern class calculations of Lemma 2.5 show that we must have $\ell(Z) = \delta = 0$ and A must be the negative section σ of $\mathbb{F}_1$. Note that $\sigma \cdot (B + \mathfrak{d}_1 + \mathfrak{d}_2) = 2$, and the intersection must be transverse since σ cannot split into a union of two sections in the double cover. Thus $C = \overline{C} \to \mathbb{P}^1$ is branched at two points, so that $C = \mathbb{P}^1$ again. Assuming as we may that the multiple fiber of odd multiplicity does not correspond to a branch point, we see that the arguments of Theorem 3.1 go through to show that there are exactly m_2 vector bundles V whose associated section is σ. (Here, in case the intersection point of σ with B corresponds to a singular nonmultiple fiber, we must use the more detailed analysis of [8], (5.12) and (5.13), to see that the section σ and the line bundle on T determine V.) Each of these is a smooth point of the moduli space, by a slight modification of the proof of Lemma 3.6 (in this case $G = f/2$ and $K_S = \mathcal{O}_S(-f + (2m_1 - 1)F_{2m_1} + (m_2 - 1)F_{m_2}))$. Summarizing, then:

Theorem 3.9. *In case $p_g(S) = 0$, the moduli space corresponding to $-p = 3$ consists of m_2 reduced points.* □

4. Calculation of the leading coefficient

Fix w and p with $w^2 \equiv p \bmod 4$, and let $\mathfrak{M} = \mathfrak{M}(S, L; w, p)$ denote the moduli space of L-stable rank two bundles on S with $w_2(V) = w$ and $p_1(\operatorname{ad} V) = p$. Let d be the (complex) dimension of $\mathfrak{M}$:

$$d = \begin{cases} 4r + N + k + 1, & \text{if } A \text{ does not pass through the exceptional point;} \\ 4r + N + k - 1, & \text{otherwise,} \end{cases}$$

where the section $A \in |\sigma + (N + r)f|$. With this notation we see that $2n = d - p_g(S)$ and that $g = d - n$. Finally let us recall from Lemma 2.4 of Part I that in case $p_g(S) = 0$ there is a unique chamber of type (w, p) which contains κ in its closure, called the (w, p)-*suitable* chamber. We can now state the main result of Part II:

Theorem 4.1. *For $p_g(S) > 0$, let $\gamma_{w,p}(S, \beta)$ be the Donaldson polynomial corresponding to the $SO(3)$-bundle P over S with $w_2(P) = \Delta \bmod 2$ and $p_1(P) = p$, and β is a choice of orientation agreeing with the usual complex orientation for $\mathfrak{M}$. If $p_g(S) = 0$, let $\gamma_{w,p}(S, \beta)$ be the corresponding Donaldson polynomial for metrics whose associated self-dual harmonic 2-form lies in the (w, p)-suitable chamber. Then, writing $\gamma_{w,p}(S, \beta)$ as a polynomial in κ_S and q_S, say $\gamma_{w,p}(S, \beta) = \sum_{i=0}^{[d/2]} a_i q_S^i \kappa_S^{d-2i}$, we have, for all p with $-p \geq 2(4p_g + 2)$, $a_i = 0$ for $i > n$ and*

$$a_n = \frac{d!}{2^n n!}(2m_1 m_2)^{p_g(S)} m_2.$$

We first remark that the assumption that $-p \geq 2(4p_g + 2)$ implies that $\mathfrak{M}$ is nonempty and contains a smooth Zariski open subset as described in Theorem 3.8. Indeed $-p$ is an odd integer greater than $8p_g + 4$, and so $-p \geq 8p_g + 5 \geq 6p_g + 5$. Thus we are in the range of Theorem 3.8.

Let $X = X_{w,p}$ denote the Uhlenbeck compactification associated to $\mathfrak{M}$ [6], [11]. The orientation of $\mathfrak{M}$ induces a fundamental class of X. There is a μ-map $H_2(S) \to H^2(\mathfrak{M})$, which roughly speaking is given by taking the slant product with the class $-p_1(P)/4$, where P is the universal $SO(3)$-bundle over $S \times \mathfrak{M}$. If P lifts to a holomorphic bundle $\mathscr{V}$ over $S \times \mathfrak{M}$, then $p_1(P) = c_1^2(\mathscr{V}) - 4c_2(\mathscr{V}) = p_1(\operatorname{ad}\mathscr{V})$. The classes $\mu(\alpha) \in H^2(\mathfrak{M})$ extend uniquely to classes in $H^2(X)$, which we shall also denote by $\mu(\alpha)$. The Donaldson polynomial is then defined by taking cup products of the $\mu(\alpha)$ and evaluating on the fundamental cycle of X.

Arguing as in [11], it is a combinatorial exercise to deduce Theorem 4.1 from the following:

Theorem 4.2. *Let μ denote the μ-map associated to the Uhlenbeck compactification X of $\mathfrak{M}$. Then, using the complex orientation to identify $H^{2d}(X; \mathbb{Z}) \cong \mathbb{Z}$, we have, for all p with $-p \geq 2(4p_g + 2)$ and for all $\Sigma \in H_2(S)$,*

$$\mu(f)^m \cup \mu(\Sigma)^{d-m} = \begin{cases} 0, & \text{if } m > n, \\ (d-n)!(2m_1 m_2)^{d-n} m_2 (\Sigma \cdot \kappa_S)^{d-n}, & \text{if } m = n. \end{cases}$$

Here the factor $(2m_1 m_2)^{p_g(S)}$ appearing in Theorem 4.1 arises from the fact that $\mu(f)^n = (2m_1 m_2)^n \mu(\kappa)$ and that $d - 2n = p_g(S)$.

In order to prove Theorem 4.2, we shall introduce geometric divisors which will represent the cohomology class $\mu(f)$. Fix a smooth fiber f. Then there are exactly four line bundles θ of degree $m_1 m_2$ on f such that $\theta^{\otimes 2} = \mathscr{O}_f(\Delta)$. Each line bundle θ corresponds to a point of intersection of f with the branch divisor $B \subset \mathbb{F}$. If V is a rank two vector bundle over S with $c_1(V) = \Delta$, then $(V|f) \otimes (\theta)^{-1}$ is a vector bundle over f with trivial determinant. Thus by the Riemann-Roch theorem $\chi(f; (V|f) \otimes (\theta)^{-1}) = 0$.

For each integer c, fix a (Δ, c)-suitable line bundle L. Given an integer $b \leq c$, we define $\mathfrak{M}_b$ to be the moduli space of L-stable rank two bundles V

with $c_1(V) = \Delta$ and $c_2(V) = b$. Given f and θ as above, define the divisor $\mathcal{Z}_b(f, \theta)$ as a set by:

$$\mathcal{Z}_b(f, \theta) = \{ V \in \mathfrak{M}_b : h^0(f; (V|f) \otimes (\theta)^{-1}) \neq 0 \}.$$

A calculation with the Grothendieck-Riemann-Roch theorem as in the proof of Proposition 1.1 in Chapter 5 of [11] shows that we can use the divisors $\mathcal{Z}_b(f, \theta)$ to calculate the μ-map. More precisely, since the f_i are disjoint, suppose that, for all choices of $b > 0$ the intersection

$$\mathcal{Z}_{c-b}(f_{i_1}, \theta_{i_1}) \cap \cdots \cap \mathcal{Z}_{c-b}(f_{i_{n-b}}, \theta_{i_{n-b}}) = \emptyset,$$

that $\bigcap_{i=1}^n \mathcal{Z}_c(f_i, \theta_i) = J$ is compact and is contained in the Zariski open subset M of $\mathfrak{M}$, and that the $\mathcal{Z}_c(f_i, \theta_i)$ meet transversally along J. Here by Corollary 2.6, $\mathfrak{M}_{c-b} = \emptyset$ if $b > (-p-3)/4$ and so it is enough to check the above for $0 < b \leq (-p-3)/4$. In particular we must have $n > b$ for all $b \leq (-p-3)/4$. Then we can define $\mu([\Sigma])|J$ and arguments along the lines of the proof of Theorem 1.12 in Chapter 5 of [11] show that

$$\gamma_{w,p}(S)(f_1, \ldots, f_n, [\Sigma], \ldots, [\Sigma]) = (\mu([\Sigma])|J)^{d-n}.$$

Thus, Theorem 4.2 is a consequence of the following:

Theorem 4.3. *Suppose that $-p \geq 2(4p_g + 2)$. Let $f_1, \ldots, f_t$ denote distinct general fibers of π, and for each i choose θ_i a line bundle on f_i with $\theta_i^2 = \mathcal{O}_{f_i}(\Delta)$. Then:*

(i) *For all $t \geq n$ and for all choices of $b > 0$, the intersection*

$$\mathcal{Z}_{c-b}(f_{i_1}, \theta_{i_1}) \cap \cdots \cap \mathcal{Z}_{c-b}(f_{i_{t-b}}, \theta_{i_{t-b}}) = \emptyset.$$

(ii) *If $t > n$, then $\bigcap_{i=1}^t \mathcal{Z}_c(f_i, \theta_i) = \emptyset$, and moreover $\bigcap_{i=1}^n \mathcal{Z}_c(f_i, \theta_i)$ is compact and is contained in the Zariski open subset M of $\mathfrak{M}$. The intersection is transverse and is a fiber J of the map $M \to U \subseteq \mathbb{P}^n$.*

(iii) *If Σ is a smooth curve in S, then the restriction of $\mu([\Sigma])$ to each of the m_2 connected components of J is equal to $(\Sigma \cdot f) \cdot [\Theta]$, where Θ is the theta divisor of the component.*

We begin by determining the set-theoretic intersection of the $\mathcal{Z}_c(f_i, \theta_i)$. Recall that we have associated to V the section A and the scheme Z on the double cover T. The following is straightforward:

Lemma 4.4. *Let $V \in \mathfrak{M}$. Then $V \in \mathcal{Z}_c(f, \theta)$ if and only if either the section A' of $\mathbb{F}$ corresponding to V meets B transversally at the point corresponding to θ or* $\operatorname{Supp} Z \cap \nu^{-1}(f) \neq \emptyset$. $\square$

For the rest of the argument, we shall not worry about the case where the sections pass through the exceptional point, since this can be handled by symmetry. Arguing as in [11], Lemma 2.7 of Chapter 7, for a general choice of fibers $f_1, \ldots, f_t$ and line bundles θ_i on f_i, and, for all $s \leq r$, setting H_i to be the hyperplane of sections in $|\sigma + (N+s)f|$ passing through the point corresponding to θ_i, then $H_1 \cap \cdots \cap H_{N+2s+1} = \{A\}$ and the intersection of more than $N+2s+1$ of the H_i is empty. In particular, this means that the f_i are chosen so that the negative section σ does not meet the branch divisor at any of the f_i. Thus if V is a bundle whose associated section A is the negative section σ of $\mathbb{F}_N$ and V lies in the intersection $\bigcap_{i=1}^t \mathcal{Z}_c(f_i, \theta_i)$, then $\operatorname{Supp} Z$ meets the preimage in T of f_i for all i.

A counting argument as in the proof of Proposition 2.9 of Chapter 7 in [11] then establishes (i) and (ii) of Theorem 4.3, at least set-theoretically, except possibly for those V such that the corresponding section of $\mathbb{F}_N$ is the negative section. Likewise, arguments identical to the proof of (2.6) in Chapter 7 of [11] show that the intersection of the divisor $\mathcal{Z}_c(f_i, \theta_i)$ with the Zariski open subset M is reduced, so that the intersection in (ii) of Theorem 4.3 is transverse. Thus the only new case to consider is the possibility that the section associated to V is the negative section. We shall briefly outline the argument in this case; it is here that we must assume that $-p$ is sufficiently large. In particular, recalling that

$$n = \frac{d-p_g}{2} = \frac{-p-4p_g-3}{2},$$

and that we may assume that $b \le (-p-3)/4$, the condition $-p \ge 2(4p_g+2)$ insures that

$$\begin{aligned} n-b &\ge n + \frac{p+3}{4} = -\frac{p}{4} - \frac{8p_g+3}{4} \\ &\ge \frac{8p_g+4}{4} - \frac{8p_g+3}{4} > 0. \end{aligned}$$

Thus the intersection in (i) of Theorem 4.3 is not over the empty collection of divisors $\mathcal{Z}_{c-b}(f_i, \theta_i)$, i.e. the intersection is always contained in a divisor $\mathcal{Z}_{c-b}(f_i, \theta_i)$.

Given t general fibers $f_1, \dots, f_t$, we can assume that they do not correspond to intersections of the negative section σ with $B + \mathfrak{d}_1 + \mathfrak{d}_2$. Now let V be a bundle whose associated section is σ. Suppose further that $c_2(V) = c-b$ and that V lies in the intersection $\mathcal{Z}_{c-b}(f_{i_1}, \theta_{i_1}) \cap \cdots \cap \mathcal{Z}_{c-b}(f_{i_{t-b}}, \theta_{i_{t-b}})$ for $t \ge n$. We claim that, if $0 \le b < (-p-3)/4$, then $-p \le 4p_g + 2N - 2$. It clearly suffices to assume that $t = n$. Let $\nu\colon T \to S$ be the double cover corresponding to σ. Then ν is not branched over the f_i. Writing $\nu^* V$ as an extension

$$0 \to \mathcal{O}_T(D) \to \nu^* V \to \mathcal{O}_T(\nu^*\Delta - D) \otimes I_Z \to 0,$$

we have by Lemma 2.5 that $-p = -p_1(\operatorname{ad} V) + 4b = 4k - 6N + 1 + 2\ell(Z) + \delta + 4b$. Moreover $\operatorname{Supp} Z$ must meet each of the two components of $\nu^{-1}(f_i)$ for $i = 1, \dots, t$. Thus $\ell(Z) \ge 2n - 2b$. But we also have

$$\begin{aligned} 2n &= d - p_g = -p - 4p_g - 3 \\ &= 4k - 6N + 1 + 2\ell(Z) + \delta - 4(k-N-1) - 3 + 4b \\ &= -2N + 2 + 2\ell(Z) + \delta + 4b. \end{aligned}$$

Thus $2n \ge -2N + 2 + 4n - 4b + \delta + 4b$ and so $2n \le 2N - 2$. This says that

$$-p \le 4p_g + 2N - 2.$$

On the other hand, $2(4p_g+2) = 4p_g + 4(k-N) \geq 4p_g + 4N > 4p_g + 2N - 2$. Thus with our assumptions on p there can be no bundle V in the intersection with an associated section equal to σ. Taking $b > 0$ establishes (i) in the case where the section associated to V is σ. Taking $b = 0$ shows that no bundle with associated section σ lies in the intersection $\bigcap_{i=1}^{t} \mathcal{Z}_c(f_i, \theta_i)$ if $t \geq n$. This establishes (i) and (ii).

Finally, to prove (iii) of Theorem 4.3, we must determine $\mu([\Sigma])|J$, where J is a fiber of the map $M \to U$. The argument again closely parallels that of [11], Chapter 7. A fiber of $M \to U$ determines and is determined by a generic double cover $T \to S$. There is a divisor D_0 on T such that every V in the fiber is of the form $\nu_*\mathcal{O}_T(D_0 + F) \otimes \rho^*\lambda$, for a line bundle $\lambda \in \operatorname{Pic}^0 C$. Fix a smooth holomorphic multisection Σ of π, transverse to the double cover $T \to S$. Let Σ' be the inverse image of Σ under ν. There is a commutative diagram

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\nu} & \Sigma \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ C & \xrightarrow{g} & \mathbb{P}^1. \end{array}$$

Let $\mathcal{P}$ be the Poincaré bundle over $\operatorname{Pic}^0 C \times C$. Let E be the divisor on Σ' induced by $D_0 + F$. Then there is a universal bundle over $\operatorname{Pic}^0 C \times \Sigma$ of the form

$$(\operatorname{Id} \times \nu)_* \left[\pi_2^*\mathcal{O}_{\Sigma'}(E) \otimes (\operatorname{Id} \times \rho^*)\mathcal{P}\right].$$

Here $\pi_2\colon \operatorname{Pic}^0 C \times \Sigma' \to \Sigma'$ is the second projection. The Chern classes of $\mathcal{V}_E$ only depend on the numerical equivalence class of E. Moreover, $p_1(\operatorname{ad}\mathcal{V}_E) = p_1(\operatorname{ad}\mathcal{V}_E \otimes q_2^*\lambda)$ for every line bundle λ on Σ, where $q_2\colon \operatorname{Pic}^0 C \times \Sigma \to \Sigma$ is the projection. This has the effect of replacing $\mathcal{O}_{\Sigma'}(E)$ by $\mathcal{O}_{\Sigma'}(E) \otimes \nu^*\lambda$. Replacing Σ by 2Σ, replaces $\deg E$ by $2\deg E$. Thus we can assume that $\deg E$ is even, and then after twisting by an appropriate λ we can assume that $\deg E = 0$. So as far as the Chern classes are concerned we may as well assume that $E = 0$, and we need to calculate

$$p_1(\operatorname{ad}(\operatorname{Id} \times \nu)_*(\operatorname{Id} \times \rho)^*\mathcal{P}).$$

Now since cohomology commutes with flat base change, we have

$$(\operatorname{Id} \times \nu)_*(\operatorname{Id} \times \rho)^*\mathcal{P} = (\operatorname{Id} \times \pi)^*(\operatorname{Id} \times g)_*\mathcal{P}.$$

Thus we need to find

$$p_1(\operatorname{ad}(\operatorname{Id} \times \pi)^*(\operatorname{Id} \times g)_*\mathcal{P}) = (\operatorname{Id} \times \pi)^* p_1(\operatorname{ad}(\operatorname{Id} \times g)_*\mathcal{P}).$$

Let us first calculate $p_1(\operatorname{ad}(\operatorname{Id} \times g)_*\mathcal{P})$. A straightforward calculation using e.g. Lemma 2.14 of Chapter 7 of [11] shows the following:

Lemma 4.5. *Let $f\colon X \to Y$ be a double cover with X and Y smooth, and let Υ be the line bundle on Y defining the double cover, so that $\Upsilon^{\otimes 2} = \mathcal{O}_Y(B)$, where B is the branch locus of f. If D is a divisor on X, then*

$$p_1(\operatorname{ad} f_*\mathcal{O}_X(D)) = c_1(\Upsilon)^2 - (f_*D)^2 + 2f_*(D^2). \quad \square$$

Applying this in our situation, with $X = \mathrm{Pic}^0 C \times C$ and $f = \mathrm{Id} \times g$, we see that Υ is the pullback of a divisor on C and thus $c_1(\Upsilon)^2 = 0$. So we are left with calculating (in the sense of cycles) $(\mathrm{Id} \times g)_* \mathscr{P}$ and $(\mathrm{Id} \times g)_* [\mathscr{P}]^2$. Also, as in the proofs of (2.15) and (2.16) in Chapter 7 of [11], $(\mathrm{Id} \times g)_* \mathscr{P} = 0$ and $[\mathscr{P}]^2 = -2r_1^*[\Theta] \cup r_2^* x$, where x is the class of a point in C, Θ is the theta divisor on $\mathrm{Pic}^0 C$, and r_1, r_2 are the first and second projections on $\mathrm{Pic}^0 C \times C$. Thus

$$\begin{aligned} 2(\mathrm{Id} \times g)_* [\mathscr{P}]^2 &= 2(\mathrm{Id} \times g)_* (-2r_1^*[\Theta] \cup r_2^* x) \\ &= -4p_1^*[\Theta] \cup p_2^* y, \end{aligned}$$

where p_1, p_2 are the first and second projections on $\mathrm{Pic}^0 C \times \mathbb{P}^1$ and y is the class of a point on $\mathbb{P}^1$. It follows that $-p_1(\mathrm{ad}\, \mathscr{V}_E)/4 = q_1^* \Theta \cup q_2^* \pi^* y$.

Hence the slant product of $-p_1(\mathrm{ad}\, \mathscr{V}_E)/4$ with $[\Sigma]$ is $(\deg \pi) \cdot [\Theta]$. Since $\deg \pi = (\Sigma \cdot f)$, we have now established (iii) of Theorem 4.3. This concludes the proof of Theorem 4.3 and thus of Theorem 4.1. □

PART III: THE CASE OF ODD FIBER DEGREE

In this part, S shall denote a simply connected elliptic surface with at most two multiple fibers, of multiplicities m_1 and m_2, where one or both of the m_i are allowed to be 1. We shall study stable rank two vector bundles V on S such that $\det V \cdot f$ is odd, where f is a general fiber of S. Thus necessarily the multiplicities m_1 and m_2 are odd as well. As we shall see the case of odd fiber degree is fundamentally different from the case of even fiber degree and is in many ways simpler. Our goal will be to give a description of the moduli space of stable rank two bundles with odd fiber degree and then to use this information to calculate certain Donaldson polynomials. Before stating the main results of Part III, recall that, for an elliptic surface S, $J^d(S)$ denotes the elliptic surface whose general fiber is the set of line bundles of degree d on the general fiber of S. Let $w \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ be the mod 2 reduction of a divisor class Δ such that $\Delta \cdot f$ is odd. Given an integer $p \equiv \Delta^2 \bmod 4$, define the rational number t by the formula

$$-p - 3\chi(\mathscr{O}_S) \equiv 2t.$$

We note in fact that t is an integer: By the canonical bundle formula for an elliptic surface,

$$K_S = \mathscr{O}_S((p_g - 1)f + (m_1 - 1)F_1 + (m_2 - 1)F_2),$$

where F_i are the reductions of the multiple fibers. As m_1 and m_2 are odd,

$$K_S \cdot \Delta \equiv p_g - 1 \bmod 2.$$

By the Wu formula, $\Delta^2 \equiv p_g - 1 \bmod 2$ as well. Hence

$$4c - \Delta^2 - 3\chi(\mathscr{O}_S) = -p - 3\chi(\mathscr{O}_S) \equiv 0 \bmod 2.$$

Thus $-p - 3\chi(\mathcal{O}_S) = 2t$ is even. It is in fact the expected dimension of the moduli space $\mathfrak{M}_t = \mathfrak{M}(\Delta, c)$ of stable rank two vector bundles V with $c_1(V) = \Delta$ and $c_2(V) = c$, where stability is with respect to a suitable ample line bundle. We shall prove the following theorem:

Theorem. *Let $\mathfrak{M}_t$ be the moduli space of stable rank two bundles V on S (with respect to a suitable ample line bundle) with*

$$\det V \cdot f = 2e + 1 \quad \text{and} \quad 4c_2(V) - c_1^2(V) - 3\chi(\mathcal{O}_S) = 2t.$$

Then for all $t \geq 0$, $\mathfrak{M}_t$ is smooth and irreducible and is birational to $\operatorname{Sym}^t J^{e+1}(S)$.

Let us outline the basic ideas behind the proof of this theorem. Standard arguments show that, for a suitable choice of an ample line bundle L on S, a rank two vector bundle V with $c_1(V) \cdot f = 2e + 1$ is L-stable if and only if its restriction to a general fiber f is stable. A pleasant consequence of the assumption of odd fiber degree is that there is a unique stable bundle of a given determinant of odd degree on each smooth fiber f. Using this, it is easy to show that there exists a rank two vector bundle V_0 whose restriction to *every* fiber f is stable, and that V_0 is unique up to twisting by a line bundle. The bundle V_0 is the progenitor of all stable bundles on S, in the sense that every stable rank two vector bundle is obtained from V_0 by making elementary modifications along fibers. Generically, this involves choosing t smooth fibers f_i and line bundles λ_i of degree $e+1$ on f_i. These choices define the birational isomorphism from the moduli space to $\operatorname{Sym}^t J^{e+1}(S)$.

For w, p, and t as above, let $\gamma_{w,p}(S)$ be the corresponding Donaldson polynomial, where if $p_g(S) = 0$, $\gamma_{w,p}(S)$ is the polynomial for the (w, p)-suitable chamber defined in Part I, Definition 2.6. By Proposition 1.1 of Part I, $\gamma_{w,p}(S)$ depends only on p or equivalently t and we will denote it by γ_t. It follows from the theorem above that $\gamma_t \neq 0$ for all $t \geq 0$, and that $\gamma_0 = 1$. More generally one can show that the "leading coefficient" of γ_t in the sense of [11], Chapter 7, is always 1, although we shall not do so in this paper. Given the above analysis of stable bundles, the main problem in computing the remaining terms in the higher Donaldson polynomials γ_t is to fit together all of the various possible descriptions of stable bundles into a universal family whose Chern classes can be calculated. This is easier said than done! Even in the case where S has a section, the construction of the universal bundle for the four-dimensional moduli space, which just involves well-known techniques of extensions and elementary modifications, is already quite involved. We shall therefore proceed differently, and try to describe the moduli spaces and Chern classes involved up to contributions which only depend on the analytic type of a neighborhood of the multiple fibers. But we shall not try to analyze these contributions explicitly. Instead we shall repeatedly use the fact that an elliptic surface with $p_g = 0$ and just one multiple fiber is a rational surface, and thus its Donaldson polynomials are the same as those for an elliptic surface with $p_g = 0$ and with a section, or equivalently no multiple fibers. Thus if we know these, we can try to interpolate this knowledge into the general case. We shall use this idea twice. The first application will be to calculate the invariant γ_1. Here the moduli space is precisely $J^{e+1}(S)$ and a lengthy calculation with the Grothendieck-Riemann-Roch theorem identifies the divisor corresponding to the μ-map up to a rational multiple of the fiber, which depends only on the multiplicities. Appealing to the knowledge of the invariant for a rational surface enables us to determine this multiple. Of course, it is likely that the exact multiple could also be determined by a direct calculation. In order to calculate the polynomial γ_2, we shall use a variant of this idea. In this case, the divisor corresponding to the μ-map is essentially known from the corresponding calculation in the case of γ_1. However what changes is the moduli space itself: the presence of multiple fibers means that the birational map from the moduli space to $\mathrm{Hilb}^2 J^{e+1}(S)$ is not a morphism, and the actual moduli space differs from $\mathrm{Hilb}^2 J^{e+1}(S)$ in codimension two. Thus while the divisors are known, their top self-intersection is not. Again using the rational elliptic surfaces, we are able to determine the discrepancy between the self-intersection of the μ-divisors in $\mathrm{Hilb}^2 J^{e+1}(S)$ and in the actual moduli space. The methods used here are in a certain sense the analogue in algebraic geometry of gluing techniques for ASD connections.

Although the actual arguments are rather involved, the main point to emphasize here is that the coefficients of the Donaldson polynomial are quite formally determined by the knowledge of the polynomial for a rational surface. It is natural to wonder if the techniques in this paper can be pushed further to determine γ_t for all t. I believe that this should be possible, although one necessary and so far missing ingredient in this approach is the knowledge of the multiplication table for divisors in $\mathrm{Hilb}^n J^{e+1}(S)$.

Here is a rapid description of the contents of Part III. In Section 1 we describe some general results on rank two vector bundles on an elliptic curve. In Section 2 these results are extended to cover the case of an irreducible nodal curve of arithmetic genus one. In Section 3 we give the classification of stable bundles on an elliptic surface S and prove the theorem stated above. In Section 4 we specialize to the case of a surface with a section. Our purpose here is twofold: First, we would like to show how many of the results of the preceding section take a very concrete form in this case. Secondly, we shall make a model for a piece of the four-dimensional moduli space which we shall need to use later. In Section 5 we calculate the two-dimensional invariant γ_1 in case S has a section. This calculation has already been done by a different method in Section 4 and will be redone in full generality. However it seemed worthwhile to do this special case in order to make the general calculation more transparent. The next three sections are devoted to calculating γ_1 in general. The outline of the argument is given in Section 6. We construct a coherent sheaf which is an approximation to the universal bundle over the moduli space, over a branched cover T of S. We determine its Chern classes via a lengthy calculation using the Grothendieck-Riemann-Roch theorem, which is given in Section 7. The necessary correction terms are identified via the results in Section 8. In Sections 9 and 10 we deal with the invariant γ_2. Once again the outline of the calculation is given first and the technical details are postponed to Section 10. We conclude with an appendix which collects some general results about elementary modifications.

Preliminaries for Part III

Let V be a coherent sheaf on a smooth projective scheme. Recall that we denote by $p_1(\operatorname{ad} V)$ the expression $c_1^2(V) - 4c_2(V)$. Given a vector bundle V, we shall need to know how $p_1(\operatorname{ad} V)$ changes under elementary modifications. Recall that an elementary modification is defined as follows. Let X be a smooth scheme and let D be an effective divisor on X, not necessarily smooth, with $i\colon D \to X$ the inclusion. Let L be a line bundle on D. Then i_*L is a coherent sheaf on X, which we shall frequently just denote by L. Suppose that V_0 is a rank two vector bundle on X and that $V_0 \to i_*L$ is a surjective homomorphism. Let V be the kernel of the map $V \to i_*L$. Then V is again a rank two vector bundle on X (and in particular it is locally free). We call V an *elementary modification* of V_0. The change in p_1 is given as follows:

Lemma. *Let X be a smooth scheme and let D be an effective divisor on X, not necessarily smooth. Let L be a line bundle on D and V_0 a rank two vector bundle, and suppose that there is an exact sequence*

$$0 \to V \to V_0 \to i_*L \to 0,$$

where $i\colon D \to X$ is the inclusion. Then

$$p_1(\operatorname{ad} V) - p_1(\operatorname{ad} V_0) = 2c_1(V_0) \cdot [D] + [D]^2 - 4i_*c_1(L).$$

Proof. The proof follows easily from standard formulas for $c_1(V)$ and $c_2(V)$, cf. [12] or [9]. □

Next we will recall some properties of the scheme $\operatorname{Hilb}^2 S$, where S is an algebraic surface. In general, we denote by $\operatorname{Hilb}^n S$ the smooth projective scheme parametrizing 0-dimensional subschemes of S of length n. There is a universal codimension two subscheme $\mathcal{Z} \subset S \times \operatorname{Hilb}^n S$. We may describe the case $n = 2$ quite explicitly. Let $\tilde{H}$ be the blowup of $S \times S$ along the diagonal $\mathbb{D}$ and let $\tilde{\mathbb{D}}$ be the exceptional divisor. There is an involution ι of $\tilde{H}$ whose fixed set is $\tilde{\mathbb{D}}$. We claim that the quotient $\tilde{H}/\iota$ is naturally $\operatorname{Hilb}^2 S$. Indeed, if $\tilde{\mathbb{D}}_{12}$ and $\tilde{\mathbb{D}}_{13}$ are the proper transforms in $S \times \tilde{H}$ of the subsets

$$\mathbb{D}_{1j} = \{\, p \in S \times S \times S \mid \pi_1(p) = \pi_j(p) \,\},$$

then $\tilde{\mathcal{Z}} = \tilde{\mathbb{D}}_{12} + \tilde{\mathbb{D}}_{13}$ is a codimension two subscheme of $S \times \tilde{H}$ which is easily seen to be a local complete intersection. Thus it defines a flat family of subschemes of S and so a morphism $\pi\colon \tilde{H} \to \operatorname{Hilb}^2 S$. It is easy to see that the induced morphism $\tilde{H}/\iota \to \operatorname{Hilb}^2 S$ is an isomorphism. The projection $\mathcal{Z} \to \operatorname{Hilb}^2 S$ is a double cover which identifies $\mathcal{Z}$ with $\tilde{H}$.

Given $\alpha \in H_2(S)$, we can define the element $D_\alpha \in H^2(\operatorname{Hilb}^2 S)$ by taking the slant product with $[\mathcal{Z}] \in H^4(S \times \operatorname{Hilb}^2 S)$. If for example $\alpha = [C]$ where C is an irreducible curve on S, then D_α is represented by the effective divisor consisting of pairs $\{x, y\}$ of points of S such that either x or y lies on C. The inverse image $\pi^* D_\alpha \in H^2(\tilde{H})$ is the pullback of the class $1 \otimes \alpha + \alpha \otimes 1 \in H^2(S \times S)$. There is also the class in $H^2(\operatorname{Hilb}^2 S)$ represented by the divisor E of subschemes of S whose support is a single point. Since π is branched over E, the class $[E]$ is divisible by 2 and $\pi^*[E] = 2\tilde{\mathbb{D}}$. Using this it is easy to check that the map $\alpha \mapsto D_\alpha$ defines an injection $H_2(S) \to H^2(\mathrm{Hilb}^2 S)$ and that $H^2(\mathrm{Hilb}^2 S) = H_2(S) \oplus \mathbb{Z} \cdot [E/2]$. The multiplication table in $H^2(\mathrm{Hilb}^2 S)$ can be determined from the fact that $\tilde{H}$ is the blowup of $S \times S$ along the diagonal and that the normal bundle of the diagonal in $S \times S$ is the tangent bundle of S: we have

$$D_\alpha^4 = 3(\alpha^2)^2; \qquad D_\alpha^3 \cdot E = 0; \qquad D_\alpha^2 \cdot E^2 = -8(\alpha^2);$$
$$D_\alpha \cdot E^3 = -8(c_1(S) \cdot \alpha); \qquad E^4 = 8(c_2(S) - c_1(S)^2).$$

Finally we need to say a few words about calculating Donaldson polynomials. Let M be a closed oriented simply connected 4-manifold with a generic Riemannian metric g, and let P be a principal $SO(3)$-bundle over M with invariants $w_2(P) = w$ and $p_1(P) = p$. There is a Donaldson polynomial $\gamma_{w,p}(S)$ defined via the moduli space of g-ASD connections on P, together with a choice of orientation for this space. If $b_2^+(M) > 1$, then this polynomial is independent of g, whereas if $b_2^+(M) = 1$ then it only depends on a certain chamber in the positive cone of $H^2(M; \mathbb{R})$. If $M = S$ is a complex surface, Δ is a holomorphic line bundle such that $w = c_1(\Delta) \bmod 2$ and g is a Hodge metric corresponding to an ample line bundle L, then there is a diffeomorphism of real analytic spaces from the moduli space of g-ASD connections on P to the moduli space of L-stable rank two vector bundles V on S with $c_1(V) = \Delta$ and $c_2(V) = (\Delta^2 - p)/4$. We denote this moduli space for the moment by $\mathfrak{M}$. We shall always choose the orientation of the moduli space of g-ASD connections which agrees with the natural complex orientation of $\mathfrak{M}$.

If $\mathfrak{M}$ is smooth, compact, and of real dimension $2d$ and there is a universal bundle $\mathscr{V}$ over $S \times \mathfrak{M}$, then the slant product with $-p_1(\mathrm{ad}\,\mathscr{V})/4$ defines a homomorphism μ from $H_2(S)$ to $H^2(\mathfrak{M})$. In general we can define the holomorphic vector bundle $\mathrm{ad}\,\mathscr{V}$ even when the universal bundle $\mathscr{V}$ does not exist. To see this, note that there is always a universal $\mathbb{P}^1$-bundle $\pi\colon \mathbb{P}(\mathscr{V}) \to S \times \mathfrak{M}$, and taking π_* of the relative tangent bundle gives $\mathrm{ad}\,\mathscr{V}$. Thus given a class $\Sigma \in H_2(S)$, we can evaluate $\mu(\Sigma)^d$ on the fundamental class of $\mathfrak{M}$ and this gives the value $\gamma_{w,p}(\Sigma, \dots, \Sigma)$. For the applications here, since the moduli spaces always have the correct dimension and in particular are empty if $-p - 3\chi(\mathscr{O}_S) < 0$, the moduli spaces of complex dimension zero and two are compact. For the four-dimensional moduli space, we can calculate $\gamma_{w,p}$ by choosing an appropriate compactification of $\mathfrak{M}$. For the purposes of gauge theory, there is the Uhlenbeck compactification. For the purposes of algebraic geometry, there is the Gieseker compactification $\overline{\mathfrak{M}}$. Following O'Grady [23], the divisors $\mu(\Sigma)$ extend naturally to divisors $\nu(\Sigma)$ on $\overline{\mathfrak{M}}$, which we shall continue to denote by $\mu(\Sigma)$. If there is a universal sheaf $\mathscr{V}$ on the Gieseker compactification, then the μ-map is again defined by taking the slant product with $-p_1(\mathrm{ad}\,\mathscr{V})/4$. In general, for holomorphic curves Σ (which would suffice for the applications here) we can use determinant line bundles on the moduli functor. For a general $\Sigma \in H_2(S)$, we can define $\mu(\Sigma)$ for the moduli spaces that arise here (where there are no strictly semistable sheaves) as follows: there exists a universal coherent sheaf $\mathscr{E}$ over $S \times U$, where U is the open subset of an appropriate Quot scheme corresponding to stable torsion free sheaves with the appropriate Chern classes. Thus we can define an element of $H^2(U)$ by taking the slant product with $p_1(\operatorname{ad}\mathscr{E})$. As $\overline{\mathfrak{M}}$ is a quotient of U by a free action of $PGL(N)$ for some N, $H^2(\overline{\mathfrak{M}}) \cong H^2(U)$, and this defines $\mu(\Sigma)$ in general.

We can now evaluate $\mu(\Sigma)^d$ on the fundamental class of $\overline{\mathfrak{M}}$. By recent results of Li [18] and Morgan [19] the result is again $\gamma_{w,p}$. Strictly speaking, their results are stated with certain extra assumptions. However, the cases we will need here involve the following situation: all moduli spaces are smooth of the expected dimension and there are no strictly semistable torsion free sheaves. Under these assumptions, the proofs in e.g. [19] go over essentially unchanged.

1. Review of results on vector bundles over elliptic curves

We recall the following well-known result of Atiyah [1]:

Theorem 1.1. *Let V be a rank two vector bundle over a smooth curve C of genus 1. Then exactly one of the following holds:*

(i) *V is a direct sum of line bundles;*

(ii) *V is of the form $\mathscr{E} \otimes L$, where L is a line bundle on C and $\mathscr{E}$ is the (unique) extension of $\mathscr{O}_C$ by $\mathscr{O}_C$ which does not split into the direct sum $\mathscr{O}_C \oplus \mathscr{O}_C$;*

(iii) *V is of the form $\mathscr{F}_p \otimes L$, where L is a line bundle on C, $p \in C$, and $\mathscr{F}_p$ is the unique nonsplit extension of the form*

$$0 \to \mathscr{O}_C \to \mathscr{F}_p \to \mathscr{O}_C(p) \to 0. \quad \square$$

We shall not prove Theorem 1.1 but shall instead prove the analogous statement in the slightly more complicated case of a singular curve in Section 2.

Corollary 1.2. *Let V be a stable rank two bundle over a smooth curve C of genus 1. Then $\deg V$ is odd, say $\deg V = 2e + 1$. Moreover, for every line bundle λ of degree $e+1$ we have $\dim \operatorname{Hom}(V, \lambda) = 1$, $H^1(V^\vee \otimes \lambda) = 0$, and there is an exact sequence*

$$0 \to \mu \to V \to \lambda \to 0,$$

where μ is a line bundle of degree e on C, uniquely determined by the isomorphism

$$\mu \otimes \lambda = \det V,$$

and the surjection $V \to \lambda$ is unique mod scalars.

Proof. Clearly, if V is stable we must be in case (iii) of the theorem. Conversely, suppose that V is as in (iii). We shall show that V is stable. It suffices to show that $\mathscr{F}_p$ is stable. Let M be a line bundle on C of degree at least $\det \mathscr{F}_p/2 = 1/2$ such that there is a nonzero map $M \to \mathscr{F}_p$. Clearly $\deg M \leq 1$ and $\deg M = 1$ if and only if $M = \mathscr{O}_C(p)$. Since $\deg M \geq 1/2$, $\deg M = 1$

and $M = \mathscr{O}_C(p)$. But then $\mathscr{F}_p$ is the split extension, contradicting the definition of $\mathscr{F}_p$. Thus $\mathscr{F}_p$ is stable.

Now let V be a stable bundle of degree $2e+1$, so that there exists a line bundle L of degree e on C with $V = \mathscr{F}_p \otimes L$. Then, if λ is a line bundle of degree $e+1$, we have an exact sequence

$$0 \to \mathrm{Hom}(L \otimes \mathscr{O}_C(p), \lambda) \to \mathrm{Hom}(V, \lambda) \to \mathrm{Hom}(L, \lambda) \to H^1(\lambda \otimes L^{-1} \otimes \mathscr{O}_C(-p)).$$

If $\lambda = L \otimes \mathscr{O}_C(p)$, then by assumption there exists a surjection $V \to \lambda$. If φ_1 and φ_2 are two nonzero maps from V to λ, then for every $p \in C$ there is a scalar c such that $\varphi_1 - c\varphi_2$ vanishes at p, and thus defines a map $V \to \lambda \otimes \mathscr{O}_C(-p)$. By stability this map must be zero, so that $\varphi_1 = c\varphi_2$. Thus the surjection is unique mod scalars.

If $\lambda \neq L \otimes \mathscr{O}_C(p)$, then $\lambda \otimes L^{-1} \otimes \mathscr{O}_C(-p)$ is a line bundle of degree zero on C which is not trivial. Hence $H^1(\lambda \otimes L^{-1} \otimes \mathscr{O}_C(-p)) = 0$, and $\mathrm{Hom}(V, \lambda) \cong \mathrm{Hom}(L, \lambda)$. Moreover $\mathrm{Hom}(L, \lambda) = H^0(L^{-1} \otimes \lambda)$ has dimension one since $\deg(L^{-1} \otimes \lambda) = 1$. Thus there is a nontrivial map $V \to \lambda$, which is unique mod scalars. If it is not surjective, there is a factorization $V \to \lambda \otimes \mathscr{O}_C(-q) \subset \lambda$, and this contradicts the stability of V. Lastly we see that $H^1(V^\vee \otimes \lambda) \cong H^1(L^{-1} \otimes \lambda)$, and this last group is zero since $\deg(L^{-1} \otimes \lambda) = 1$. □

We can generalize the last statement of Corollary 1.2 as follows.

Lemma 1.3. *Let C be a smooth curve of genus one.*

(i) *Let V be a stable rank two vector bundle over C and suppose that $\deg V = 2e+1$. Let $d \geq e+1$, and let λ be a line bundle on V of degree d. Then $\dim \mathrm{Hom}(V, \lambda) = 2d - 2e - 1$, and there exists a surjection from V to λ. Conversely, with V as above, let λ be a line bundle such that there exists a nonzero map from V to λ. Then $\deg \lambda \geq e+1$.*

(ii) *Suppose that $V = L_1 \oplus L_2$ is a direct sum of line bundles L_i with $\deg V = 2e+1$ and $\deg L_1 \leq e < \deg L_2$. Let λ be a line bundle on C with $d = \deg \lambda > \deg L_2$. Then $\dim \mathrm{Hom}(V, \lambda) = 2d - 2e - 1$, and there exists a surjection from V to λ. Conversely, if λ is a line bundle and there exists a surjection from $L_1 \oplus L_2$ to λ, then either $\deg \lambda > \deg L_2$ or $\lambda = L_2$ or $\lambda = L_1$. If $\lambda = L_2$, then $\dim \mathrm{Hom}(V, \lambda) = 2d - 2e$, where $d = \deg L_2 = \deg \lambda$.*

Proof. We shall just prove (i), as the proof of (ii) is simpler. Let λ be a line bundle on C of degree $d \geq e+1$. We may assume that $\deg \lambda > e+1$, the case $\deg \lambda = e+1$ having been dealt with in Corollary 1.2. There is an exact sequence $0 \to L_1 \to V \to L_2 \to 0$, where $\deg L_1 = e$ and $\deg L_2 = e+1$. Thus there is an exact sequence

$$0 \to H^0(L_2^{-1} \otimes \lambda) \to \mathrm{Hom}(V, \lambda) \to H^0(L_1^{-1} \otimes \lambda) \to H^1(L_2^{-1} \otimes \lambda).$$

We have $\deg(L_1^{-1} \otimes \lambda) = d - e > 0$ and $\deg(L_2^{-1} \otimes \lambda) = d - e - 1 > 0$. Thus $H^1(L_2^{-1} \otimes \lambda) = 0$, $\dim H^0(L_1^{-1} \otimes \lambda) = d - e$, and $\dim H^0(L_2^{-1} \otimes \lambda) = d - e - 1$. So $\dim \mathrm{Hom}(V, \lambda) = 2d - 2e - 1$. To see the last statement, let Y be the set of elements φ of $\operatorname{Hom}(V, \lambda)$ such that φ is not surjective. Then Y is the union over $x \in C$ of the spaces $\operatorname{Hom}(V, \lambda \otimes \mathscr{O}_C(-x))$, each of which has dimension at most $2d - 2e - 3$. So the dimension of Y is at most $2d - 2e - 2$. Thus $\operatorname{Hom}(V, \lambda) - Y$ is nonempty, and every $\varphi \in \operatorname{Hom}(V, \lambda) - Y$ is a surjection. The final statement of (i) is then an immediate consequence of the stability of V. □

For future use let us also record the following lemmas:

Lemma 1.4. *Let C be a smooth curve of genus one and let ξ be a line bundle on C of degree zero such that $\xi^{\otimes 2} \neq 0$. Let V be a stable rank two vector bundle on C. Then* $\operatorname{Hom}(V, V \otimes \xi) = 0$.

Proof. Since $\deg V = \deg(V \otimes \xi)$ and both are stable, a nonzero map between them must be an isomorphism, by standard results on stable bundles. However $\det(V \otimes \xi) = \det V \otimes \xi^{\otimes 2} \neq \det V$, and so the bundles cannot be isomorphic. Thus there is no nonzero map from V to $V \otimes \xi$. □

Corollary 1.5. *Let $\mathbf{F} \subset X$ be a scheme-theoretic multiple fiber of odd multiplicity m of an elliptic surface, and let F be the reduction of $\mathbf{F}$. Let $\mathbf{V}$ be a rank two vector bundle on $\mathbf{F}$ whose restriction V to F is stable. Then* $\dim_{\mathbb{C}} \operatorname{Hom}(\mathbf{V}, \mathbf{V}) = 1$ *and every nonzero map from $\mathbf{V}$ to itself is an isomorphism.*

Proof. Let ξ be the normal bundle of F in X. Thus ξ has order m. For $a > 0$, let aF denote the subscheme of X defined by the ideal sheaf $\mathscr{O}_X(-aF)$. Thus $\mathbf{F} = mF$ and there is in general an exact sequence

$$0 \to \xi^{-a} \to \mathscr{O}_{(a+1)F} \to \mathscr{O}_{aF} \to 0.$$

Tensor the above exact sequence by $\mathcal{H}om(\mathbf{V}, \mathbf{V}) = \mathbf{V}^{\vee} \otimes \mathbf{V}$ and take global sections. This gives an exact sequence

$$0 \to \operatorname{Hom}(V, V \otimes \xi^{-a}) \to \operatorname{Hom}(\mathbf{V}|(a+1)F, \mathbf{V}|(a+1)F) \to \operatorname{Hom}(\mathbf{V}|aF, \mathbf{V}|aF).$$

For $a = 1$ we have $\dim_{\mathbb{C}} \operatorname{Hom}(\mathbf{V}|F, \mathbf{V}|F) = \dim_{\mathbb{C}} \operatorname{Hom}(V, V) = 1$. For $1 \leq a \leq m-1$, ξ^{-a} is a nontrivial line bundle of odd order. Thus by Lemma 1.4 $\operatorname{Hom}(V, V \otimes \xi^{-a}) = 0$. It follows that the map

$$\operatorname{Hom}(\mathbf{V}|(a+1)F, \mathbf{V}|(a+1)F) \to \operatorname{Hom}(\mathbf{V}|aF, \mathbf{V}|aF)$$

is an injection, so that by induction $\dim_{\mathbb{C}} \operatorname{Hom}(\mathbf{V}|(a+1)F, \mathbf{V}|(a+1)F) \leq 1$. On the other hand multiplication by an element of $H^0(\mathscr{O}_{(a+1)F}) = \mathbb{C}$ defines a nonzero element of $\operatorname{Hom}(\mathbf{V}|(a+1)F, \mathbf{V}|(a+1)F)$. Thus

$$\dim_{\mathbb{C}} \operatorname{Hom}(\mathbf{V}|(a+1)F, \mathbf{V}|(a+1)F) = 1 \quad \text{for all } a \leq m-1,$$

and in particular $\dim_{\mathbb{C}} \operatorname{Hom}(\mathbf{V}|mF, \mathbf{V}|mF) = 1$. □

2. The case of a singular curve

Our goal in this section will be to show that the statements of the previous section hold for rank two vector bundles on singular nodal curves C. Let C be an irreducible curve of arithmetic genus one, which has one node p as a singularity. Locally analytically, then, $\hat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x, y]]/(xy)$. Let $a: \tilde{C} \to C$ be the normalization map, and let p_1 and p_2 be the preimages of the singular point on $\tilde{C}$. We begin by giving a preliminary discussion concerning torsion free sheaves on C.

Definition 2.1. A *torsion free rank one sheaf* on C is a coherent sheaf which has rank one at the generic point of C and has no local sections vanishing on an open set. It is well known that every torsion free rank one sheaf on C is either a line bundle or of the form a_*L, where L is a line bundle on $\tilde{C}$. For example, the maximal ideal sheaf of the singular point p of C is a_*L, where $L = \mathcal{O}_{\tilde{C}}(-p_1 - p_2)$, where p_1 and p_2 are the preimages of p in $\tilde{C}$. Here the line bundle L has degree -2 on $\tilde{C}$. We define the *degree* of a torsion free rank one sheaf F on C by $\deg F = \chi(F)$. By the Riemann-Roch theorem on C, $\deg F$ is the usual degree in case F is a line bundle, whereas for $F = a_*L$, an easy calculation shows that $\chi(F) = \deg L + 1$. (Note that, in case $p_a(C)$ is arbitrary, we would have to correct by a term $p_a(C) - 1$, which is zero in our case, to get the usual answer for a line bundle.) It is easy to check that, if F is a rank one torsion free sheaf on C and L is a line bundle, then $\deg(F \otimes L) = \deg F + \deg L$.

Lemma 2.2. *If F_1 and F_2 are torsion free rank one sheaves on C, then so is $Hom(F_1, F_2)$, and*

$$\deg Hom(F_1, F_2) = \begin{cases} \deg F_2 - \deg F_1, & \text{if one of the } F_i \text{ is a line bundle}, \\ \deg F_2 - \deg F_1 + 1, & \text{if neither } F_1 \text{ nor } F_2 \text{ is a line bundle.} \end{cases}$$

Finally if $\deg F_2 > \deg F_1$ and neither is a line bundle, then the natural map from $\mathrm{Hom}(F_1, F_2) \otimes \mathcal{O}_{C,p}$ *to* $\mathrm{Hom}_{\mathcal{O}_{C,p}}(\mathfrak{m}_p, \mathfrak{m}_p)$ *is surjective, where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{C,p}$.*

Proof. The proof is clear if F_1 is a line bundle. Thus we may assume that F_1 is of the form a_*L for a line bundle L on $\tilde{C}$. First assume that F_2 is a line bundle. An easy calculation shows that

$$Hom(F_1, F_2) = a_*(L^{-1} \otimes \mathcal{O}_{\tilde{C}}(-p_1 - p_2)) \otimes F_2.$$

This is just the local calculation $Hom_{\mathcal{O}_{C,p}}(a_*\widetilde{\mathcal{O}_{C,p}}, \mathcal{O}_{C,p}) = \mathfrak{m}_p$, where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{C,p}$ and the isomorphism is canonical. Thus $Hom(F_1, F_2)$ is again a torsion free rank one sheaf and

$$\deg Hom(F_1, F_2) = -\deg L + 1 - 2 + \deg F_2 = \deg F_2 - \deg F_1.$$

Now assume that $F_1 = a_*L_1$ and $F_2 = a_*L_2$. Again using a local calculation $Hom_{\mathcal{O}_{C,p}}(a_*\widetilde{\mathcal{O}_{C,p}}, a_*\widetilde{\mathcal{O}_{C,p}}) = a_*\widetilde{\mathcal{O}_{C,p}}$, where the isomorphism is also canonical, it is easy to check that $Hom(a_*L_1, a_*L_2) = a_*(L_1^{-1} \otimes L_2)$, and so $Hom(F_1, F_2)$ is again a torsion free rank one sheaf. Moreover

$$\deg Hom(F_1, F_2) = \deg L_2 - \deg L_1 + 1 = \deg F_2 - \deg F_1 + 1.$$

To see the final statement, again writing $F_1 = a_*L_1$ and $F_2 = a_*L_2$, we have $Hom(a_*L_1, a_*L_2) = a_*(L_1^{-1} \otimes L_2)$. Moreover the global sections of $L_1^{-1} \otimes L_2$ separate the points p_1 and p_2. It is then easy to see that the map $\mathrm{Hom}(F_1, F_2) \otimes \mathscr{O}_{C,p} \to \mathrm{Hom}_{\mathscr{O}_{C,p}}(\mathfrak{m}_p, \mathfrak{m}_p) = \widetilde{\mathscr{O}_{C,p}}$ is surjective. □

Lemma 2.3. *Let F be a torsion free rank one sheaf on C. If $\deg F > 0$, or if $\deg F = 0$ and F is not trivial, then $h^0(F) = \deg F$ and $h^1(F) = 0$. If $\deg F < 0$, or if $\deg F = 0$ and $F \neq \mathscr{O}_C$, then $h^0(F) = 0$ and $h^1(F) = \deg F$.*

Proof. If $\deg F \geq 0$ and F is not $\mathscr{O}_C$, then the claim that $h^0(F) = \deg F$ is clear if F is a line bundle and follows from $h^0(a_*L) = \deg L + 1$ in case L is a line bundle of degree at least -1 on $\tilde{C} \cong \mathbb{P}^1$. In this case, since by definition $\deg F = \chi(F) = h^0(F)$, we must have $h^1(F) = 0$. The proof of the second statement is similar. □

Next let us consider extensions of torsion free sheaves. The maximal ideal $\mathfrak{m}_p$ has the following local resolution, where we set $R = \mathscr{O}_{C,p}$:

$$\cdots \to R \oplus R \to R \oplus R \to \mathfrak{m}_p \to 0,$$

where the maps $R \oplus R \to R \oplus R$ alternate between $(\alpha, \beta) \mapsto (x\alpha, y\beta)$ and $(\alpha, \beta) \mapsto (y\alpha, x\beta)$. A calculation shows that $\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$ has length two. More intrinsically it is isomorphic to $\mathscr{O}_{\tilde{C}}(-p_1 - p_2)/\mathscr{O}_{\tilde{C}}(-2p_1 - 2p_2)$. Thus as an R-module,

$$\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p) \cong \tilde{R}/\tilde{\mathfrak{m}}_p,$$

where $\tilde{R}$ is the normalization of R and $\tilde{\mathfrak{m}}_p = \mathfrak{m}_p\tilde{R}$. We can describe the $\tilde{R}$-action on $\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$ more invariantly as follows: multiplication by $r \in \tilde{R}$ gives an endomorphism $\mathfrak{m}_p \to \mathfrak{m}_p$, and hence an action of $\tilde{R}$ on $\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$. We leave to the reader the straightforward verification that this action is the same as the action on $\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$ implicit in the isomorphism $\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p) \cong \tilde{R}/\tilde{\mathfrak{m}}_p$ given above. There is an induced action of the invertible elements $\tilde{R}^*$ on $(\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p) - 0)/\mathbb{C}^* = \mathbb{P}^1$. Since R^* acts trivially, this induces an action of $\tilde{R}^*/R^* \cong \mathbb{C}^*$ on $(\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p) - 0)/\mathbb{C}^*$. It is easy to see that there are three orbits of this action: an open orbit isomorphic to $\mathbb{C}^*$ and two closed orbits which are points in $\mathbb{P}^1$, corresponding to the case of an element $e \in \mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$ such that $\tilde{R} \cdot e \neq \mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$.

Given an element $e \in \mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$, denote the corresponding extension of $\mathfrak{m}_p$ by $\mathfrak{m}_p$ by M_e. Note that two extensions M_e, $M_{e'}$ such that e, e' lie in the same $\tilde{R}^*$-orbit are abstractly isomorphic as R-modules, via a diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}_p \arrow[r] \arrow[d, "="'] & M_{re} \arrow[r] \arrow[d] & \mathfrak{m}_p \arrow[r] \arrow[d, "\times r"'] & 0 \\ 0 \arrow[r] & \mathfrak{m}_p \arrow[r] & M_e \arrow[r] & \mathfrak{m}_p \arrow[r] & 0, \end{tikzcd}$$

where $r \in \tilde{R}^*$ is such that $re = e'$.

Lemma 2.4. *M_e is locally free if and only if the image of e in*

$$\left(\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p) - 0\right)/\mathbb{C}^*$$

is not a closed orbit of $\tilde{R}$.

Proof. Consider the long exact Ext sequence

$$\mathrm{Hom}_R(\mathfrak{m}_p, \mathfrak{m}_p) \to \mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p) \to \mathrm{Ext}^1_R(M_e, \mathfrak{m}_p).$$

We see that $\mathrm{Ext}^1_R(M_e, \mathfrak{m}_p)$ contains as a submodule $\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)/\tilde{R}\cdot e$. Thus if $\tilde{R}\cdot e \neq \mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$, then $\mathrm{Ext}^1_R(M_e, \mathfrak{m}_p) \neq 0$ and so M_e is not locally free. Conversely suppose that the image of e does not lie in one of the closed orbits. Since every two extensions in the same orbit are abstractly isomorphic, it will suffice to exhibit one locally free extension of $\mathfrak{m}_p$ by $\mathfrak{m}_p$. However we have the obvious surjection $R \oplus R \to \mathfrak{m}_p$ given above, and its kernel is easily seen to be isomorphic to $\mathfrak{m}_p$ again. □

We leave as an exercise for the reader the description of the extensions corresponding to the closed orbits.

Let us also note that, using the resolution above, a short computation shows that $\mathrm{Ext}^1_R(\mathfrak{m}_p, R) = 0$. Thus there is no locally free R-module M which sits in an exact sequence

$$0 \to R \to M \to \mathfrak{m}_p \to 0.$$

Globally, we have the following:

Lemma 2.5. *Let n be a positive integer and let δ be a line bundle of degree one on C.*

(i) *There is a unique rank two vector bundle $V_{n,\delta}$ on C such that* $\det V_{n,\delta} = \delta$ *and such that there is an exact sequence*

$$0 \to F \to V_{n,\delta} \to F' \to 0,$$

where F and F' are torsion free rank one sheaves of degrees n and $1-n$ respectively, and F and F' are not locally free.

(ii) *Let G be a torsion free rank one subsheaf of $V_{n,\delta}$. Then either* $\deg G \leq -n$ *or G is contained in F.*

(iii) *The vector bundle $V_{n,\delta}$ is indecomposable for all n and δ and $V_{n,\delta} \cong V_{n',\delta'}$ if and only if $n = n'$ and $\delta = \delta'$.*

Proof. To see (i), let F and F' be the unique torsion free rank one sheaves of degrees n and $1-n$ respectively which are not locally free. Let us evaluate $\mathrm{Ext}^1(F', F)$. From the local to global Ext spectral sequence, there is an exact sequence

$$0 \to H^1(Hom(F', F)) \to \mathrm{Ext}^1(F', F) \to H^0(Ext^1(F', F)) \to 0.$$

Now $\chi(Hom(F', F)) = \deg Hom(F', F) = h^0(Hom(F', F))$, since

$$\deg Hom(F', F) = 2n > 0$$

by Lemmas 2.2 and 2.3. So $H^1(Hom(F', F)) = 0$. Thus $\operatorname{Ext}^1(F', F) \cong H^0(Ext^1(F', F))$. Moreover, the set of all locally free extensions is naturally a principal homogeneous space over $H^0(\mathcal{O}_{\tilde{C}}^*/\mathcal{O}_C^*) = \tilde{R}^*/R^*$. On the other hand, from the exact sequence

$$0 \to \mathcal{O}_C^* \to \mathcal{O}_{\tilde{C}}^* \to \mathcal{O}_{\tilde{C}}^*/\mathcal{O}_C^* \to 0,$$

we have a natural isomorphism $\operatorname{Pic}^0 C \cong H^0(\mathcal{O}_{\tilde{C}}^*/\mathcal{O}_C^*) = \tilde{R}^*/R^*$. Let $\partial\colon \tilde{R}^*/R^* \to \operatorname{Pic}^0 C$ be the coboundary map; it is an isomorphism. Given $e \in \operatorname{Ext}^1(F', F) \cong H^0(Ext^1(F', F))$, let V_e be the extension corresponding to e. A straightforward exercise in the definitions shows that, for $r \in \tilde{R}$,

$$\det V_{r\cdot e} = \partial(r) \otimes \det V_e.$$

From this it is clear that there is a unique extension $V_{n,\delta}$ with determinant δ.

Next we prove (ii). Let G be a torsion free rank one subsheaf, possibly a line bundle, of $V_{n,\delta}$ such that $\deg G > -n$. We have an exact sequence

$$0 \to \operatorname{Hom}(G, F) \to \operatorname{Hom}(G, V_{n,\delta}) \to \operatorname{Hom}(G, F').$$

Moreover $\operatorname{Hom}(G, F') = H^0(Hom(G, F'))$. First suppose that either $\deg G > 1-n$ or that G is locally free. The torsion free sheaf $Hom(G, F')$ has degree either $1 - n - \deg G$ or $2 - n - \deg G$, depending on whether G is or is not locally free. In any case it has degree ≤ 0 and is not locally free, so that $H^0(Hom(G, F')) = 0$, by Lemmas 2.2 and 2.3. So every such G is contained in F. In the remaining case where $\deg G = 1 - n$ and G is not locally free, then $G = F'$. Since $\operatorname{Hom}(F', F') \cong k^*$, every nonzero homomorphism from F' to itself is an isomorphism. Thus the exact sequence defining $V_{n,\delta}$ would be split, contrary to assumption. Hence this last case is impossible.

To see (iii), let G be a torsion free rank one subsheaf of degree at least $1 - n$ such that $V_{n,\delta}/G$ is torsion free. Then by (ii) $G = F$. This clearly implies that $V_{n,\delta} \cong V_{n',\delta'}$ if and only if $n = n'$ and $\delta = \delta'$ and that $V_{n,\delta}$ is indecomposable. □

Theorem 2.6. *Let C be an irreducible curve of arithmetic genus one, which has one node as a singularity. Let V be a rank two vector bundle on C and suppose that $\deg\det V = 2e + 1$. Then V is one of the following:*

(i) *A direct sum of line bundles;*

(ii) *$L\otimes\mathcal{F}_x$, where L is a line bundle of degree e, $x \in C$ is a smooth point, and $\mathcal{F}_x$ is the unique nontrivial extension*

$$0 \to \mathcal{O}_C \to \mathcal{F}_x \to \mathcal{O}_C(x) \to 0;$$

(iii) *$L \otimes V_{n,\delta}$, where L is a line bundle of degree e and $V_{n,\delta}$ is the rank two vector bundle described in Lemma 2.5. In this case, the subsheaf $L\otimes F$, where F is the subsheaf in the definition of $V_{n,\delta}$, is the maximal destabilizing subsheaf.*

Proof. Clearly we may assume that $\deg V = 1$. By the Riemann-Roch theorem, $h^0(V) \neq 0$. Thus there is a map $\mathscr{O}_C \to V$. If this map is the inclusion of a subbundle, then V is given as an extension

$$0 \to \mathscr{O}_C \to V \to \mathscr{O}_C(x) \to 0$$

for some smooth point $x \in C$. Either this extension splits, in which case we are in case (i), or it does not in which case we are in case (ii).

Now suppose that the map $\mathscr{O}_C \to V$ vanishes at some point. There is a largest rank one subsheaf F of V containing the image of $\mathscr{O}_C$, and $\deg F = n > 0$. The quotient $V/F = F'$ is torsion free. If F is a line bundle, then so is F', since locally $\mathrm{Ext}^1_R(\mathfrak{m}_p, R) = 0$. In this case $(F')^{-1} \otimes F$ has degree $2n - 1 > 0$, so that the extension splits and V is the direct sum of F and F'. Hence we are in case (i). Otherwise F and F' are not locally free. It follows that $V = V_{n,\delta}$ for $\delta = \det V$, and we are in case (iii). The last statement in (iii) then follows from the last paragraph of the proof of Lemma 2.5. □

Finally let us show that a statement analogous to Lemma 1.3 continues to hold for the case of a singular curve.

Lemma 2.7. *Let C be an irreducible nodal curve of arithmetic genus one.*

(i) *Let V be a stable rank two vector bundle over C and suppose that $\deg V = 2e + 1$. Let $d \geq e + 1$, and let λ be a torsion free rank one sheaf on V of degree d. Then $\dim \mathrm{Hom}(V, \lambda) = 2d - 2e - 1$, and there exists a surjection from V to λ. Moreover, if λ is a line bundle on C such that there exists a nonzero map from V to λ, then $\deg \lambda \geq e + 1$. Finally, if $d = e + 1$, then $H^1(V^\vee \otimes \lambda) = 0$.*

(ii) *Suppose that $V = L_1 \oplus L_2$ is a direct sum of line bundles L_i with $\deg V = 2e + 1$ and $\deg L_1 \leq e < \deg L_2$. Let λ be a rank one torsion free sheaf on C with $d = \deg \lambda > \deg L_2$. Then $\dim \mathrm{Hom}(V, \lambda) = 2d - 2e - 1$, and there exists a surjection from V to λ. Moreover, if λ is a rank one torsion free sheaf on C such that there exists a surjection from V to λ, then either $d = \deg \lambda > \deg L_2$ or $\lambda = L_2$ or $\lambda = L_1$ and $\dim \mathrm{Hom}(V, \lambda) = 2d - 2e$.*

(iii) *Suppose that $V = L \otimes V_{n,\delta}$ for some n, where L is a line bundle of degree e and that L_2 is the subsheaf $L \otimes F$ of V of degree $e + n$ corresponding to the subsheaf F of $V_{n,\delta}$ in the definition of $V_{n,\delta}$ and that L_1 is the quotient V/L_2. Let λ be a rank one torsion free sheaf on C with $d = \deg \lambda > \deg L_2 = e + n$. Then $\dim \mathrm{Hom}(V, \lambda) = 2d - 2e - 1$. Moreover, if there exists a surjection from V to λ then either $\deg \lambda > e + n$ or $\lambda = L_1$ and $\dim \mathrm{Hom}(V, \lambda) = 1$.*

Proof. The proof of (i) and (ii) follows the same lines as the proof of Lemma 1.3, with minor modifications, given Lemmas 2.2 and 2.3. Let us prove (iii) in the case where λ is not locally free (the proof in the other case is slightly simpler). By definition there is an exact sequence

$$0 \to L_2 \to V \to L_1 \to 0,$$

where L_1 and L_2 are not locally free and $\deg L_2 = e+n$, $\deg L_1 = 1-n+e$. There is a long exact sequence

$$0 \to \mathrm{Hom}(L_1, \lambda) \to \mathrm{Hom}(V, \lambda) \to \mathrm{Hom}(L_2, \lambda) \to \mathrm{Ext}^1(L_1, \lambda).$$

Moreover, by the long exact sequence for Ext, we have an exact sequence

$$0 \to H^1(\mathcal{H}om(L_1, \lambda)) \to \mathrm{Ext}^1(L_1, \lambda) \to H^0(\mathcal{E}xt^1(L_1, \lambda)).$$

Since $\deg \mathcal{H}om(L_1, \lambda) = d-e+n+1 \leq 0$ and $\mathcal{H}om(L_1, \lambda)$ is not locally free, $H^1(\mathcal{H}om(L_1, \lambda)) = 0$ by Lemma 2.3. Moreover $H^0(\mathcal{E}xt^1(L_1, \lambda)) = \mathbb{C}^2$. We claim that the composite map $\mathrm{Hom}(L_2, \lambda) \to \mathrm{Ext}^1(L_1, \lambda) \to H^0(\mathcal{E}xt^1(L_1, \lambda))$ is surjective. Since $\deg L_2 > \deg \mu$, the map $\mathrm{Hom}(L_2, \lambda) \to \mathrm{Hom}_R(\mathfrak{m}_p, \mathfrak{m}_p) \cong \tilde{R}$ is onto the quotient $\tilde{R}/\tilde{\mathfrak{m}}_p$ by the last statement in Lemma 2.2. Thus the image of the map $\mathrm{Hom}(L_2, \lambda) \to H^0(\mathcal{E}xt^1(L_1, \lambda))$ contains the orbit $\tilde{R} \cdot \xi \subseteq \mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$, where ξ is the extension class. Since V is locally free, this orbit is all of $\mathrm{Ext}^1_R(\mathfrak{m}_p, \mathfrak{m}_p)$ by the proof of Lemma 2.4, and so the map $\mathrm{Hom}(L_2, \lambda) \to H^0(\mathcal{E}xt^1(L_1, \lambda))$ is onto. It follows that

$$\begin{aligned} \dim \mathrm{Hom}(V, \lambda) &= \mathrm{Hom}(L_1, \lambda) + \mathrm{Hom}(L_2, \lambda) - 2 \\ &= d-(e+n)+1+d-(1-n+e)+1-2 = 2d-2e-1. \end{aligned}$$

Let us finally consider the case when there is a surjection from V to λ. Let the degree of λ be $d+e$. Thus there is a surjection from $V_{n,\delta}$ to $\lambda \otimes L^{-1}$, which is of degree d. Let G be the kernel of the map $V_{n,\delta} \to \lambda \otimes L^{-1}$. Then $\deg G = 1-d$. By Lemma 2.5(ii), either $1-d \leq -n$ or $G \subseteq F$. Thus either $d > e$ or $\lambda = L_1$. In the last case, there is a unique surjection from V to L_1 mod scalars, by the proof of Lemma 2.5(iii). □

3. A Zariski open subset of the moduli space

Let $\pi: S \to \mathbb{P}^1$ be an algebraic elliptic surface of geometric genus $p_g(S) = p_g$. We shall always assume that the only singular fibers of π are either irreducible nodal curves or multiple fibers with smooth reduction. Denote the multiple fibers by F_1 and F_2 and suppose that the multiplicity of F_i is m_i. We shall assume that the multiple fibers lie over points where the j-invariant of S is unramified. We denote by $J(S)$ the associated Jacobian elliptic surface or basic elliptic surface. For an integer n, $J^n(S)$ denotes the relative Picard scheme of line bundles on the fibers of degree n (see for example Section 1 in Part I). Hence $J(S) = J^0(S)$ and $S = J^1(S)$. If n is relatively prime to m_1m_2, then $J^n(S)$ again has two multiple fibers of multiplicities m_1 and m_2. We always have $p_g(J^n(S)) = p_g$. If Δ is a divisor on S, we let $f \cdot \Delta$ denote the *fiber degree*, i.e. the degree of the line bundle Δ on a smooth fiber f. Let $\mathrm{Pic}^{\mathrm{v}} S$ denote the set of *vertical* divisor classes, i.e. the set of divisor classes spanned by the class of a fiber and the classes of the reductions of the multiple fibers. With our assumptions $\mathrm{Pic}^{\mathrm{v}} S \cong \mathbb{Z} \cdot \kappa$, where $m_1m_2\kappa = f$ (see also [11], Chapter 2, Corollary 2.9). Clearly $\mathrm{Pic}^{\mathrm{v}} S$ is the kernel of the natural map from $\operatorname{Pic} S$ to the group of line bundles on the generic fiber. In general let $\eta = \operatorname{Spec} k(\mathbb{P}^1)$ be the generic point of $\mathbb{P}^1$ and let $\bar{\eta} = \operatorname{Spec} \overline{k(\mathbb{P}^1)}$, where $\overline{k(\mathbb{P}^1)}$ is the algebraic closure of $k(\mathbb{P}^1)$. Let S_η be the restriction of S to η and $S_{\bar{\eta}}$ be the pullback of S_η to $\bar{\eta}$. Define V_η to be the restriction of V to S_η and similarly for $V_{\bar{\eta}}$.

Definition 3.1. An ample line bundle L on S is (Δ, c)-*suitable* if for all divisors D on S such that $-D^2 + D \cdot \Delta \leq c$, either $f \cdot (2D - \Delta) = 0$ or

$$\operatorname{sign} f \cdot (2D - \Delta) = \operatorname{sign} L \cdot (2D - \Delta).$$

Given the pair (Δ, c), we set $w = \Delta \bmod 2 \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ and let $p = \Delta^2 - 4c$. Thus (Δ, c) and (Δ', c') correspond to the same values of w and p if and only if $\Delta' = \Delta + 2F$ for some divisor class F and $c' = c + \Delta \cdot F + F^2$. An easy calculation shows that the property of being (Δ, c)-suitable therefore only depends on the pair (w, p), and we will also say that L is (w, p)-suitable.

We have the following, which is Lemma 2.3 in Part I:

Lemma 3.2. *For all pairs* (Δ, c), (Δ, c)-*suitable ample line bundles exist.* □

Definition 3.3. Let Δ be a divisor on S and c an integer. Fix a (Δ, c)-suitable line bundle L. We denote by $\mathfrak{M}(\Delta, c)$ the moduli space of equivalence classes of L-stable rank two vector bundles V on S with $c_1(V) = \Delta$ and $c_2(V) = c$. Here V_1 and V_2 are *equivalent* if there exists a line bundle $\mathcal{O}_S(D)$ such that V_1 is isomorphic to $V_2 \otimes \mathcal{O}_S(D)$. In particular, since $\det V_1 = \det V_2$, the divisor $2D$ is linearly equivalent to zero, and in fact V_1 and V_2 must be isomorphic since there is no 2-torsion in $\operatorname{Pic} S$. As the notation suggests and as we shall shortly show, the scheme $\mathfrak{M}(\Delta, c)$ does not depend on the choice of the (Δ, c)-suitable line bundle L.

Given a divisor Δ on S and an integer c, we let $w = \Delta \bmod 2$ and $p = \Delta^2 - 4c$. The moduli space $\mathfrak{M}(\Delta, c)$ only depends on w and p and we shall also denote it by $\mathfrak{M}(w, p)$.

Now fix an odd integer $2e + 1$. We shall consider rank two vector bundles V such that the line bundle $\det V$ has fiber degree $2e + 1$. However, it will be convenient not to fix the determinant of V. In this section we shall show that the moduli space $\mathfrak{M}(w, p)$ is smooth and irreducible, and shall describe a Zariski open and dense subset of it explicitly. The basic idea is to show first that there is a largest integer p_0 such that $\mathfrak{M}(w, p_0)$ is nonempty and that there is a unique element in $\mathfrak{M}(w, p_0)$, corresponding to the bundle V_0. For all other $p < p_0$, the bundles in $\mathfrak{M}(w, p)$ are obtained by elementary modifications of V_0 along fibers. Let us begin by recalling the following result (Corollary 3.4 in Part I):

Theorem 3.4. *Let V be a rank two bundle with* $\det V = \Delta$ *and* $c_2(V) = c$. *Suppose that* $\det V$ *has fiber degree* $2e + 1$. *Let L be a* (Δ, c)-*suitable ample line bundle, and suppose that V is L-stable. Then there exists a Zariski open subset U of $\mathbb{P}^1$ such that, if f is a fiber of π corresponding to a point of U, then f is smooth and $V|f$ is stable. Conversely, if there exists a smooth fiber*

f such that $V|f$ is stable, then V is L-stable for every (Δ, c)-suitable ample line bundle L. $\square$

Next we show that there exist bundles satisfying the hypotheses of Theorem 3.4:

Lemma 3.5. *Let δ be a line bundle on the generic fiber S_η of odd degree $2e+1$. Then there exists a rank two vector bundle V such that the restriction of* $\det V$ *to S_η is δ and such that there exists a smooth fiber f for which the restriction $V|f$ is stable.*

Proof. Let Δ_0 be a line bundle on S restricting to δ on S_η. Fix a smooth fiber f. By Theorem 1.1 there exists a stable bundle E on f with determinant equal to $\Delta_0|f$. Let H be a line bundle on S such that $\deg(H|f) \geq e+1$. Then by Lemma 1.3 there is a surjection from E to $H|f$, and thus E is given as an extension

$$0 \to (H^{-1} \otimes \Delta_0)|f) \to E \to (H|f) \to 0.$$

This extension corresponds to a class in $H^1(f; (H^{\otimes -2} \otimes \Delta_0)|f)$. We would like to lift this exact sequence to an exact sequence on S. Of course, we can replace Δ_0 by $\Delta_0 + Nf$ for an integer N and get the same restriction to f. It suffices to show that, for some N, the map

$$H^1(S; H^{\otimes -2} \otimes \Delta_0 \otimes \mathcal{O}_S(Nf)) \to H^1(f; (H|f)^{\otimes -2} \otimes \Delta_0)$$

is surjective. The cokernel of this map is contained in

$$H^2(S; H^{-2} \otimes \Delta_0 \otimes \mathcal{O}_S((N-1)f)) = H^0(S; H^2 \otimes \Delta_0^{-1} \otimes \mathcal{O}_S((-N+1)f) \otimes K_S)^*.$$

Clearly $H^0(S; H^2 \otimes \Delta_0^{-1} \otimes \mathcal{O}_S((-N+1)f) \otimes K_S) = 0$ if $N \gg 0$, and thus there is an extension on S inducing E. $\square$

Note. We could also have proved Lemma 3.5 by descent theory.

Before we state the next lemma, recall that a stable vector bundle V on S is *good* if $H^2(S; \operatorname{ad} V) = 0$. This means that V is a smooth point of the moduli space, which has dimension $-p_1(\operatorname{ad} V) - 3\chi(\mathcal{O}_S)$ at V. Thus the content of the next lemma is that the moduli space is always smooth of the expected dimension.

Lemma 3.6. *Let V be a rank two bundle on S such that the restriction of V to the generic fiber is stable. Then V is good.*

Proof. By Serre duality, $H^2(S; \operatorname{ad} V) = 0$ if and only if $H^0(\operatorname{ad} V \otimes K_S) = 0$. A section φ of $H^0(\operatorname{ad} V \otimes K_S)$ gives a trace free endomorphism of $V_{\bar\eta}$ (since K_S has trivial restriction to the generic fiber). But $V_{\bar\eta}$ is simple, so that φ has trivial restriction to the generic fiber. Hence $\varphi = 0$. $\square$

Lemma 3.7. *Let V_1 and V_2 be rank two bundles on S whose restrictions to the generic fibers are stable and have the same determinant (as a line bundle on S_η). Then there exists a divisor D on S, lying in* $\operatorname{Pic}^{\vee} S$, *and an inclusion $V_1 \otimes \mathcal{O}_S(D) \subseteq V_2$. Moreover for an appropriate choice of D we have an exact sequence*

$$0 \to V_1 \otimes \mathcal{O}_S(D) \to V_2 \to Q \to 0,$$

where Q is supported on fibers or reductions of fibers and the map $V_1 \otimes \mathcal{O}_S(D) \to V_2$ does not vanish on any fiber.

Proof. By assumption V_1 and V_2 have isomorphic restrictions to S_η. An isomorphism between these extends to give a map $V_1 \otimes \mathcal{O}_S(D_1) \to V_2 \otimes \mathcal{O}_S(D_2)$, where D_i have trivial restriction to the generic fiber. Twisting gives a map $\varphi\colon V_1 \otimes \mathcal{O}_S(D') \to V_2$, where D' has trivial restriction to the generic fiber. By construction φ is an isomorphism on the generic fiber, so φ is an inclusion. The determinant $\det \varphi$ is a nonzero section of $\det V_1^{-1} \otimes \mathcal{O}_S(-2D') \otimes \det V_2$, which restricts trivially to the generic fiber. Thus $\det V_1^{-1} \otimes \mathcal{O}_S(-2D') \otimes \det V_2 = \mathcal{O}_S(\sum_i n_i F_i + nf)$, where the F_i are the multiple fibers, f is a general fiber and n_i, n are ≥ 0. Here Q has support whose reduction is the sum of the F_i for which $n_i \neq 0$ plus some smooth fibers. If φ vanishes identically on a fiber or fiber component F, then it factors:

$$V_1 \otimes \mathcal{O}_S(D') \subset V_1 \otimes \mathcal{O}_S(D' + F) \to V_2.$$

So after enlarging D' to a new divisor D we can assume that this doesn't happen. Thus D is as desired. □

Corollary 3.8. *Let V_1 and V_2 be two rank two bundles on S with the following property: for every curve F which is a reduced fiber or the reduction of a multiple fiber, the restriction of V_i to F is stable. Then there exists a divisor $D \in \operatorname{Pic}^{\vee} S$ such that $V_2 = V_1 \otimes \mathcal{O}_S(D)$.*

Proof. Find a nonzero map $\varphi\colon V_1 \otimes \mathcal{O}_S(D) \to V_2$ which does not vanish on F for every F the reduction of a fiber, via Lemma 3.7. For all F, $V_1 \otimes \mathcal{O}_S(D)|F$ and $V_2|F$ are stable bundles of the same degree and $\varphi|F$ is a nonzero map between them. Thus $\varphi|F$ is an isomorphism for all F and so φ is an isomorphism as well. □

Corollary 3.9. *Suppose that V_0 is a rank two vector bundle satisfying the hypotheses of* (3.8): *the restriction $V|F$ is stable for every reduction F of a fiber component. Let $\Delta = \det V_0$ and $c = c_2(V_0)$. Then $\mathfrak{M}(\Delta, c)$ consists of a single reduced point corresponding to the bundle V_0. Thus necessarily $p_1(\operatorname{ad} V_0) = -3\chi(\mathcal{O}_S)$.*

Proof. If V' is another such, $V' = V_0 \otimes \mathcal{O}_S(D)$, and so V' and V_0 are equivalent. By Lemma 3.6 V_0 is good. Thus $\mathfrak{M}(\Delta, c)$ is a single reduced point. Moreover the dimension of $\mathfrak{M}(\Delta, c)$ is $-p_1(\operatorname{ad} V_0) - 3\chi(\mathcal{O}_S) = 0$, and so $p_1(\operatorname{ad} V_0) = -3\chi(\mathcal{O}_S)$. □

Next we establish the existence of such a V_0. Before we do so let us pause to record the following lemma.

Lemma 3.10. *Let*

$$0 \to V_1 \to V_2 \to Q \to 0$$

be an exact sequence of coherent sheaves on S, where V_1 and V_2 are rank two vector bundles and $Q = i_* M$ where $i\colon F \to S$ is the inclusion of a reduced fiber or the reduction of a multiple fiber, and M is a torsion free rank one sheaf on

F. *Then*:

(i) *We have the following formula for* $p_1(\operatorname{ad} V_2)$:

$$\begin{aligned} p_1(\operatorname{ad} V_2) &= p_1(\operatorname{ad} V_1) + 4\Big(\deg M - \frac{\deg(V_2|F)}{2}\Big) \\ &= p_1(\operatorname{ad} V_1) + 4\Big(\deg M - \frac{\deg(V_1|F)}{2}\Big). \end{aligned}$$

(ii) *If we define* Q' *by the exact sequence*

$$0 \to V_2^\vee \to V_1^\vee \to Q' \to 0,$$

then $V_i^\vee \cong V_i \otimes (\det V_i)^{-1}$ *is a twist of* V_i *and* $Q' = Ext^1(Q, \mathcal{O}_X)$ *is of the form* $i_* M'$, *where* M' *is a torsion free rank one sheaf on* F *with* $\deg M' = -\deg M$. *Finally* M *is locally free if and only if* M' *is locally free.*

Proof. The first equality in (i) follows from the lemma on elementary modifications stated in the section on preliminaries in the introduction if M is locally free, with minor modifications in general. To see the second, since $\det V_2 = \det V_1 \otimes \mathcal{O}_S(F)$ and $F^2 = 0$, we have

$$\deg(V_1|F) = \deg(\det V_1|F) = \deg(\det V_2|F) = \deg(V_2|F).$$

To prove (ii), note that, after trivializing the bundles V_i in a Zariski open set U, the map $V_1 \to V_2$ is given by a 2×2 matrix A with coefficients in $\mathcal{O}_U$, and so the dual map corresponds to the matrix ${}^tA$. A local calculation shows that $Q' = i_* M'$, where M' is a torsion free rank one sheaf on F, where F is locally defined by $\det A$, and that M' is locally free if and only if M is locally free. To calculate $\deg M'$, use the formula in (i) for $\deg M'$, noting that $p_1(\operatorname{ad} V_i^\vee) = p_1(\operatorname{ad} V_i)$ and that $\deg(V_1^\vee|F) = -\deg(V_1|F)$. Putting this together gives

$$\begin{aligned} 4\deg M' &= p_1(\operatorname{ad} V_1) - p_1(\operatorname{ad} V_2) - 2\deg(V_1|F) \\ &= -4\deg M. \quad \square \end{aligned}$$

Using the above, we shall show the following:

Proposition 3.11. *Given a line bundle* δ *on* S_η *of odd degree, there exists a rank two bundle* V_0 *on* S *such that the restriction of* $\det V_0$ *to* S_η *is* δ *and such that the restriction* $V|F$ *is stable for every reduction* F *of a fiber component. The rank two bundle* V_0 *is unique up to equivalence: if* V_1 *is any other bundle with this property, then there exists a line bundle* $\mathcal{O}_S(D)$ *such that* $V_1 \cong V_0 \otimes \mathcal{O}_S(D)$. *Moreover* $p_1(\operatorname{ad} V_0) \geq p_1(\operatorname{ad} V)$ *for every rank two bundle* V *such that the restriction of* $\det V$ *to* S_η *is* δ *and such that there exists a smooth fiber* f *for which the restriction* $V|f$ *is stable, with equality if and only if* $V = V_0 \otimes \mathcal{O}_S(D)$.

Proof. Begin with V such that $\det V|S_\eta = \delta$ and such that there exists a smooth fiber f for which the restriction $V|f$ is stable. Such V exist by Lemma 3.5. If there exists an F such that $V|F$ is not stable, then there is a torsion free quotient Q of $V|F$ such that $\deg Q < (\deg(V|F))/2$. Define V' by the exact sequence

$$0 \to V' \to V \to Q \to 0,$$

where we abusively denote by Q the sheaf i_*Q, where i is the inclusion of F in S. Using (i) of Lemma 3.10,

$$p_1(\operatorname{ad} V) = p_1(\operatorname{ad} V') + 4\Big(\deg Q - \frac{\deg(V|F)}{2}\Big).$$

Thus $p_1(\operatorname{ad} V') > p_1(\operatorname{ad} V)$. If V' satisfies the conclusions of Proposition 3.11, we are done. Otherwise repeat this process. At each stage p_1 strictly increases. But p_1 is bounded from above, either from Bogomolov's inequality or using the fact that the dimension of the moduli space is always $-p_1 - 3\chi(\mathcal{O}_S) \geq 0$, by Lemma 3.6. Hence this process terminates and gives a V_0 as desired. By Corollary 3.8 V_0 is unique up to twisting by a line bundle, and the final statement is clear from the method of proof. □

Next we shall interpret the proof of Proposition 3.11 as saying that every stable bundle V is obtained from V_0 by an appropriate sequence of elementary modifications.

Definition 3.12. Let V be a rank two vector bundle on S whose restriction to the generic fiber is stable. Let F be a fiber on S and Q be a torsion free rank one sheaf on F, viewed as a sheaf on S. A surjection $V \to Q$ is *allowable* if

$$2 \deg Q > \deg(V|F).$$

Thus if $\deg(V|F) = 2e + 1$, then $\deg Q \geq e + 1$. If W is defined as an elementary modification

$$0 \to W \to V \to Q \to 0,$$

then we shall say that the elementary modification W is *allowable* if the surjection $V \to Q$ is allowable. It then follows from Lemma 3.10 that, if W is an allowable elementary modification of V, then $p_1(\operatorname{ad} W) < p_1(\operatorname{ad} V)$.

Let Q be a rank one torsion free sheaf on a fiber F, viewed also as a sheaf on S, and let $d = \deg Q$. It is an easy consequence of Lemmas 1.3 and 2.7 that if $V \to Q$ is allowable and $\deg(V|f) = 2e + 1$, then $d > e$ and either $\dim \operatorname{Hom}(V, Q) = 2d - 2e - 1$ or $\dim \operatorname{Hom}(V, Q) = 2d - 2e$ and Q is a uniquely specified rank one torsion free sheaf on F.

With this said, we have the following:

Proposition 3.13. *Let δ be a line bundle on S_η and let V be a stable rank two bundle on S such that the restriction of* $\det V$ *to S_η is δ. Then there is a sequence $V_0, V_1, \ldots, V_n = V$ such that V_{i+1} is an allowable elementary modification of V_i for $i = 1, \ldots, n-1$. Moreover $2n \leq p_1(\operatorname{ad} V_0) - p_1(\operatorname{ad} V)$. Finally if V is obtained from V_0 from a sequence of allowable elementary modifications then* $\dim \operatorname{Hom}(V, V_0) = 1$.

Proof. The construction given in the proof of Proposition 3.11 is the following: Begin with V. If $V \neq V_0$, then there is a fiber F, a rank one torsion free sheaf Q on F, and an elementary modification

$$0 \to V' \to V \to Q \to 0,$$

where $\deg Q < \deg(V|F)/2$. If $V' \neq V_0$, we repeat the process. So, noting that $V \cong V^\vee \otimes \det V$ and using the notation of Lemma 3.10(ii) it will suffice to show that the dual elementary modification

$$0 \to V^\vee \otimes \det V \to (V')^\vee \otimes \det V \to Q' \otimes \det V \to 0$$

is allowable, since then we have obtained V as an allowable elementary modification of $(V')^\vee \otimes \det V$. But we have $\deg Q' = -\deg Q$ by Lemma 3.10(ii), and so

$$\begin{aligned}\deg(Q' \otimes \det V) &= -\deg Q + \deg(V|F) \\ &> \frac{\deg(V|F)}{2}.\end{aligned}$$

Thus the surjection $(V')^\vee \otimes \det V \to Q' \otimes \det V$ is allowable. The statement about the number of elementary modifications follows since an allowable elementary modification always decreases p_1 by a quantity whose absolute value is at least 2.

Finally let us show that $\dim \operatorname{Hom}(V, V_0) = 1$. Since $\dim \operatorname{Hom}(V_0, V_0) = 1$, it is enough by induction on the number of elementary modifications to show the following: suppose given an exact sequence

$$0 \to V_2 \to V_1 \to Q \to 0,$$

where $\deg Q > \deg(V_i|F)/2$. Then $\operatorname{Hom}(V_1, V_0) \to \operatorname{Hom}(V_2, V_0)$ is an isomorphism. For simplicity we shall just give the argument in case Q is locally free on F. In any case $\operatorname{Hom}(Q, V_0) = 0$ since Q is a torsion sheaf and the cokernel of the map is

$$\operatorname{Ext}^1(Q, V_0) = H^0(Ext^1(Q, V_0)) = H^0(Q^\vee \otimes (V_0|F)) = \operatorname{Hom}(Q, V_0|F).$$

Since $V_0|F$ is stable and $\deg Q > \deg(V_i|F)/2 = \deg(V_0|F)/2$, this last group is zero. $\square$

Putting all this together, we shall describe a Zariski open subset of the moduli space. Let us first observe that the moduli space $\mathfrak{M}(\Delta, c)$ is always good and of dimension

$$4c - \Delta^2 - 3\chi(\mathcal{O}_S) = -p - 3\chi(\mathcal{O}_S).$$

As we have seen in the introduction, this dimension is always an even integer $2t$. Now suppose that δ is a line bundle on the generic fiber S_η of odd degree. Then there exists a divisor Δ on S which restricts to δ and Δ is determined up to a multiple of κ. Mod 2, the only possibilities are Δ and $\Delta+\kappa$. Note that $(\Delta+\kappa)^2 = \Delta^2 + 2(\Delta\cdot\kappa) \equiv \Delta^2 + 2 \bmod 4$. Thus if we also fix $\Delta^2 \bmod 4$, there is a unique choice of $w = \Delta \bmod 2$. Fix an integer $t \geq 0$ and let $-p = 2t + 3\chi(\mathcal{O}_S)$. There is then a unique class $w \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ with $w^2 \equiv p \bmod 4$ such that w is the mod two reduction of a divisor Δ which restricts to δ on S_η. Given δ and t, we shall denote the corresponding moduli space by $\mathfrak{M}_t$. The following theorem is a more precise version of the theorem stated in the Introduction:

Theorem 3.14. *In the above notation, $\mathfrak{M}_t$ is nonempty, smooth and irreducible, and is birational to* $\operatorname{Sym}^t J^{e+1}(S)$. *More precisely, there exists a Zariski open and dense subset U of $\mathfrak{M}_t$ which is isomorphic to the open subset of* $\operatorname{Sym}^t J^{e+1}(S)$ *consisting of t line bundles $\lambda_1, \dots, \lambda_t$ of degree $e+1$ lying on smooth (and reduced) fibers of π such that the images $\pi(\lambda_i)$ are distinct points of $\mathbb{P}^1$, where we continue to denote by π the projection from $J^{e+1}(S)$ to $\mathbb{P}^1$.*

Proof. Let us describe the set U. Given the line bundle δ on S_η, let $\deg\delta = 2e+1$. If f is a smooth fiber and λ is a line bundle of degree $e+1$ on f, the restriction $V_0|f$ sits in an exact sequence

$$0 \to \mu \to V_0|f \to \lambda \to 0,$$

where $\mu \otimes \lambda = \delta$. Once we have fixed λ, the surjection $V_0|f \to \lambda$ is unique mod scalars.

Now fix t distinct smooth fibers $f_1, \dots, f_t$ and line bundles λ_i of degree $e+1$ on f_i. Let Q_i be the sheaf λ_i viewed as a sheaf on S and let $Q = \bigoplus_i Q_i$. We shall consider the set of vector bundles V described by an exact sequence

$$0 \to V \to V_0 \to Q \to 0.$$

The set of all such vector bundles V is clearly parametrized by the open subset U of $\operatorname{Sym}^t J^{e+1}(S)$ consisting of t line bundles $\lambda_1, \dots, \lambda_t$ lying on smooth (reduced) fibers of π such that the images $\pi(\lambda_i)$ are distinct points of $\mathbb{P}^1$. For such a bundle V, we also have

$$p_1(\operatorname{ad} V) = p_1(\operatorname{ad} V_0) - 2t.$$

We shall first construct a family of bundles parametrized by U (more precisely, we shall construct "universal" bundles over the product of S with a finite cover of U), thereby giving a morphism from U to $\mathfrak{M}_t$ which is easily seen to be an open immersion. Finally we shall show that U is in fact dense in $\mathfrak{M}_t$.

Step I. Let U be the open subset of $\operatorname{Sym}^t J^{e+1}(S)$ described above, and let $\tilde{U}$ be defined as follows:

$$\tilde{U} = \{ (\lambda_1, \dots, \lambda_t) \in (J^{e+1}(S))^t : \{\lambda_1, \dots, \lambda_t\} \in U \}.$$

We shall try to construct a universal bundle $\mathcal{V}$ over $S \times \tilde{U}$ as follows. Let $\mathcal{Z} \subset S \times U$ be defined by

$$\mathcal{Z} = \{ (p, \{\lambda_1, \dots, \lambda_t\}) \in S \times U : \text{ for some } i,\ \pi(p) = \pi(\lambda_i) \}.$$

Thus given a point $u = \{\lambda_1, \dots, \lambda_t\} \in U$,

$$(S \times \{u\}) \cap \mathcal{Z} = \coprod_{i=1}^{t} (f_i \times \{u\}),$$

where f_i is the fiber corresponding to λ_i. Clearly $\mathcal{Z}$ is a smooth divisor in $S \times U$. Analogously, we have the pulled back divisor $\tilde{\mathcal{Z}} \subset S \times \tilde{U}$. In fact, $\tilde{\mathcal{Z}}$ breaks up into a disjoint union of divisors $\tilde{\mathcal{Z}}_i$, where for example

$$\tilde{\mathcal{Z}}_1 = (S \times_{\mathbb{P}^1} J^{e+1}(S)) \times J^{e+1}(S)^{t-1},$$

and the other $\tilde{\mathscr{Z}}_i$ are defined by taking the fiber product over $\mathbb{P}^1$ of S with the i^{th} factor of $J^{e+1}(S)^t$. Thus each $\tilde{\mathscr{Z}}_i$ fibers over $\tilde{U}$ and the fiber is an elliptic curve. Let $\rho_i\colon \tilde{\mathscr{Z}}_i \to S \times_{\mathbb{P}^1} J^{e+1}(S)$ be the projection. Over $S \times_{\mathbb{P}^1} J^{e+1}(S)$, there is a relative Poincaré bundle $\mathscr{P}_{e+1}$. Actually, $\mathscr{P}_{e+1}$ really just exists locally around sufficiently small neighborhoods of smooth nonmultiple fibers of $J^{e+1}(S)$, or in irreducible étale neighborhoods $\psi\colon \mathscr{U}_0 \to J^{e+1}(S)$ of smooth nonmultiple fibers, but we will write out all the arguments as if there were a global bundle. We shall return to this point in Section 7. So we should really replace $\tilde{U}$ by $\tilde{U}_0$ defined by

$$\tilde{U}_0 = \{ (x_1, \dots, x_t) \in \mathscr{U}_0^t : (\psi(x_1), \dots, \psi(x_t)) \in U \}.$$

We can define the divisors $\tilde{\mathscr{Z}}_i$ on $S \times \tilde{U}_0$ as well. Thus we have $\rho_i^*\mathscr{P}_{e+1}$, which is a line bundle on $\tilde{\mathscr{Z}}_i$. By extension, we can view $\rho_i^*\mathscr{P}_{e+1}$ as a coherent sheaf on $S \times \tilde{U}_0$.

Lemma 3.15. *For every i, there is a line bundle $\mathscr{L}_i$ on $\tilde{U}_0$ with the following property: There is a surjection*

$$\pi_1^* V_0 \to \bigoplus_{i=1}^{t} \left(\rho_i^*\mathscr{P}_{e+1} \otimes \pi_2^*\mathscr{L}_i\right),$$

and the surjection is unique up to multiplying by the pullback of a nowhere vanishing function on $\tilde{U}_0$.

Proof. We have

$$\begin{aligned}\operatorname{Hom}(\pi_1^* V_0, \bigoplus_{i=1}^{t} \rho_i^*\mathscr{P}_{e+1}) &= H^0((\pi_1^* V_0)^\vee \otimes \Big[\bigoplus_{i=1}^{t} \rho_i^*\mathscr{P}_{e+1}\Big]) \\ &= H^0\Big(\tilde{U}_0; \bigoplus_{i=1}^{t} R^0\pi_{2*}\big((\pi_1^* V_0)^\vee \otimes \rho_i^*\mathscr{P}_{e+1}\big)\Big).\end{aligned}$$

By base change and Corollary 1.2, the sheaf $R^0\pi_{2*}\big((\pi_1^* V_0)^\vee \otimes \rho_i^*\mathscr{P}_{e+1}\big)$ is a line bundle on $\tilde{U}_0$, which we denote by $\mathscr{L}_i^{-1}$. Choosing a nowhere vanishing section of $\mathscr{O}_{\tilde{U}_0}$ gives an element of

$$\begin{aligned}\operatorname{Hom}(\pi_1^* V_0, \rho_i^*\mathscr{P}_{e+1} \otimes \pi_2^*\mathscr{L}_i) &= H^0\Big(\tilde{U}_0; R^0\pi_{2*}\big((\pi_1^* V_0)^\vee \otimes \rho_i^*\mathscr{P}_{e+1} \otimes \pi_2^*\mathscr{L}_i\big)\Big) \\ &= H^0(\tilde{U}_0; \mathscr{L}_i^{-1} \otimes \mathscr{L}_i) = H^0(\tilde{U}_0; \mathscr{O}_{\tilde{U}_0}).\end{aligned}$$

Since the $\tilde{\mathscr{Z}}_i$ are disjoint, we can make such a choice for each i to obtain the desired surjection. □

Note. We shall essentially calculate $\mathscr{L}_i$ in Section 7.

Making a choice of a surjection from $\pi_1^* V_0$ to $\bigoplus_{i=1}^{t}\left(\rho_i^*\mathscr{P}_{e+1} \otimes \pi_2^*\mathscr{L}_i\right)$ gives a rank two vector bundle $\mathscr{V}$ over $S \times \tilde{U}_0$ defined by the exact sequence

$$0 \to \mathscr{V} \to \pi_1^* V_0 \to \bigoplus_{i=1}^{t}\left(\rho_i^*\mathscr{P}_{e+1} \otimes \pi_2^*\mathscr{L}_i\right) \to 0.$$

Thus there is a morphism $\tilde{U}_0 \to \mathfrak{M}_t$. It is easy to see that this morphism descends to a morphism of schemes $U \to \mathfrak{M}_t$ whose image is the set of bundles described at the beginning of the proof of Theorem 3.14. Clearly the morphism $U \to \mathfrak{M}_t$ is injective. By Zariski's Main Theorem it is an open immersion. This concludes the proof of Step I.

Step II. Now we must show that the open set U constructed above is Zariski dense. To do so, we shall make a standard moduli count which essentially shows that the closed subset $\mathfrak{M}_t - U$ may be parametrized by a scheme of dimension strictly smaller than $\dim \mathfrak{M}_t = 2t$. Consider the set of all allowable elementary modifications of a fixed vector bundle V' with $\deg(V'|F) = 2e+1$. Thus there is a reduced fiber or the reduction of a multiple fiber, say F, and a rank one torsion free sheaf Q on F with $\deg Q = d \geq e+1$. By Lemmas 1.3 and 2.7, there is a surjection from V' to Q, and the set of all such has dimension $2d-2e-1$ or $2d-2e$. Let V be the kernel of such a surjection. By Lemma 3.10,

$$p_1(\operatorname{ad} V') = p_1(\operatorname{ad} V) + 4d - 4e - 2.$$

Thus the number of moduli of all V is

$$-p_1(\operatorname{ad} V) - 3\chi(\mathcal{O}_S) = -p_1(\operatorname{ad} V') - 3\chi(\mathcal{O}_S) + 4d - 4e - 2.$$

On the other hand, for d and V' fixed, the above construction depends on $2d-2e$ parameters. If F is generic, there is one parameter to choose F. Next, either $\dim \operatorname{Hom}(V', Q) = 2d-2e-1$ or $2d-2e$, and in this last case Q is fixed. Taking the homomorphisms mod scalars, the number of moduli is either $2d-2e-2$ or $2d-2e-1$. In the first case the choice of Q is one more parameter, but not in the second case. Thus we always get $2d-2e-1$ parameters for the choice of the sheaf Q and the surjection $V' \to Q$. Adding in the choice of F gives $2d-2e$ moduli. For the above construction to account for a Zariski open subset of the moduli space, we clearly must have V' a general point of its moduli space, F a general fiber, and $2d-2e \geq 4d-4e-2$. It follows that $d \leq e+1$, and hence since $d > e$ that $d = e+1$. Arguing by induction, we may assume that V is obtained from V_0 by performing successive elementary modifications along distinct fibers F_i which are smooth and nonmultiple and with respect to line bundles μ_i on F_i of degree exactly $e+1$. In this case V is in the open set U described above. □

Notation 3.16. Given a line bundle δ on S_η and a nonnegative integer t, we let $\mathfrak{M}_t$ be the moduli space defined prior to (3.14) of equivalence classes of stable bundles V with $-p_1(\operatorname{ad} V) = 2t + 3\chi(\mathcal{O}_S)$, such that $w_2(V)$ is the mod two reduction of a divisor Δ with $\Delta|S_\eta = \delta$. Thus $\mathfrak{M}_t$ depends only on δ and t. Let $\overline{\mathfrak{M}}_t$ denote the Gieseker compactification of $\mathfrak{M}_t$.

4. The case where S has a section

In this section, we shall assume that there is a section σ on S, so that $m_1 = m_2 = 1$. In this case, $\sigma^2 = -(1 + p_g(S))$. Our goal is to give a very explicit description of the set of stable bundles on S such that $\det V$ has the same restriction to the generic fiber as σ. Thus $\det V = \sigma + nf$ for some integer n. We begin with a lemma on various cohomology groups which will be used often.

Lemma 4.1. *Let S be an elliptic surface with a section σ. Let $p_g = p_g(S)$.*

(i) *For all integers a, $h^0(-\sigma + af) = 0$.*
(ii) *For all integers a,*

$$h^1(-\sigma + (p_g + 1 - a)f) = \begin{cases} 0, & a > 0, \\ -a + 1, & a \leq 0. \end{cases}$$

(iii) *For all integers a,*

$$h^2(-\sigma + (p_g + 1 - a)f) = \begin{cases} a - 1, & a \geq 2, \\ 0, & a \leq 1. \end{cases}$$

Proof. Clearly $h^0(-\sigma + af) = 0$ for all integers a. Likewise

$$R^0\pi_*\mathcal{O}_S(-\sigma + af) = 0$$

for all a. In addition $R^2\pi_*\mathcal{O}_S(-\sigma + af) = 0$ for all a since π has relative dimension one. Thus, from the Leray spectral sequence, we see that

$$H^1(\mathcal{O}_S(-\sigma + (p_g + 1 - a)f)) = H^0(R^1\pi_*\mathcal{O}_S(-\sigma + (p_g + 1 - a)f)),$$
$$H^2(\mathcal{O}_S(-\sigma + (p_g + 1 - a)f)) = H^1(R^1\pi_*\mathcal{O}_S(-\sigma + (p_g + 1 - a)f)).$$

Thus we must determine the sheaf $R^1\pi_*\mathcal{O}_S(-\sigma + (p_g + 1 - a)f)$ on $\mathbb{P}^1$. Now $R^1\pi_*\mathcal{O}_S(-\sigma + (p_g + 1 - a)f) = R^1\pi_*\mathcal{O}_S(-\sigma) \otimes \mathcal{O}_{\mathbb{P}^1}(p_g + 1 - a)$. To calculate $R^1\pi_*\mathcal{O}_S(-\sigma)$, we use the exact sequence

$$0 \to \mathcal{O}_S(-\sigma) \to \mathcal{O}_S \to \mathcal{O}_\sigma \to 0.$$

Taking the long exact sequence for $R^i\pi_*$ gives $R^1\pi_*\mathcal{O}_S(-\sigma) \cong R^1\pi_*\mathcal{O}_S$, and, by e.g. [11], Chapter 1, (3.18), $R^1\pi_*\mathcal{O}_S \cong \mathcal{O}_{\mathbb{P}^1}(-p_g - 1)$. So

$$R^1\pi_*\mathcal{O}_S(-\sigma + (p_g + 1 - a)f) \cong \mathcal{O}_{\mathbb{P}^1}(-a),$$

and (ii) and (iii) follow from the usual calculations for $\mathbb{P}^1$. □

Next we shall determe the unique stable vector bundle V_0 (up to equivalence) which satisfies $-p_1(\mathrm{ad}\, V_0) = 3\chi(\mathcal{O}_S)$.

Proposition 4.2. *Let S be a nodal elliptic surface with a section σ.*

(i) *If $p_g(S)$ is odd, set $k = (1 + p_g(S))/2$. Then there is a unique nonsplit extension*

$$0 \to \mathcal{O}_S(kf) \to V_0 \to \mathcal{O}_S(\sigma - kf) \to 0,$$

and $\det V = \sigma$, $-p_1(\mathrm{ad}\, V_0) = 3\chi(\mathcal{O}_S)$, *and the restriction of V_0 to every fiber is stable.*

(ii) *If $p_g(S)$ is even, set $k = p_g(S)/2$. Then there is a unique nonsplit extension* $0 \to \mathcal{O}_S(kf) \to V_0 \to \mathcal{O}_S(\sigma - (k+1)f) \to 0$, *and* $\det V = \sigma - f$, $-p_1(\mathrm{ad}\, V_0) = 3\chi(\mathcal{O}_S)$, *and the restriction of V_0 to every fiber is stable.*

Proof. We shall just consider the case where p_g is odd; the other case is identical. First note that $H^1(S; \mathcal{O}_S(-\sigma+2kf)) = H^1(-\sigma+(p_g+1)f)$ has dimension one, by Lemma 4.1(ii). Thus there is a unique nonsplit extension up to isomorphism. Clearly $\det V_0 = \sigma$ and $-p_1(\operatorname{ad} V_0) = 4k - \sigma^2 = 3(1+p_g)$. Finally we claim that the restriction of V_0 to every fiber is stable. It suffices to show that the restriction of V_0 to every fiber f is the nontrivial extension of $\mathcal{O}_f(p)$ by $\mathcal{O}_f$, where p is the point $\sigma \cdot f$. Thus we must consider the restriction map

$$H^1(S; \mathcal{O}_S(-\sigma+2kf)) \to H^1(f; \mathcal{O}_S(-\sigma+2kf)|f).$$

Its kernel is $H^1(S; \mathcal{O}_S(-\sigma+(2k-1)f)) = H^1(S; \mathcal{O}_S(-\sigma+p_g f))$. Again by Lemma 4.1(ii) this group is zero, so that $H^1(S; \mathcal{O}_S(-\sigma+2kf) \to H^1(f; \mathcal{O}_S(-\sigma+2kf)|f)$ is an injection and hence an isomorphism since both spaces have dimension one. It follows that $V_0|f$ is stable for every f and is thus the unique bundle up to equivalence satisfying the hypotheses of Corollary 3.8. □

The bundle V_0 (with a slightly different normalization) has been described independently by Kametani and Sato [13].

Let us now consider the case where V is a stable bundle with $-p_1(\operatorname{ad} V) - 3\chi(\mathcal{O}_S) = 2t \geq 0$.

Proposition 4.3. *With S as above, let V be a stable rank two vector bundle over S such that $\det V = \sigma + nf$ for some n and $-p_1(\operatorname{ad} V) - 3\chi(\mathcal{O}_S) = 2t$.*

(i) *If p_g is odd and we set $k = (1+p_g)/2$, then, after twisting by a line bundle of the form $\mathcal{O}_S(af)$, there exist an integer s, $0 \leq s \leq t$, and an exact sequence*

$$0 \to \mathcal{O}_S((k-s)f) \to V \to \mathcal{O}_S(\sigma+(-k+s-t)f) \otimes I_Z \to 0.$$

Here Z is a codimension two local complete intersection subscheme of length s. Moreover the inclusion of $\mathcal{O}_S((k-s)f)$ into V is canonically given by the map $\pi^*\pi_* V \to V$. If $\varphi: \mathcal{O}_S(af) \to V$ is a sub-line bundle, then φ factors through the inclusion $\mathcal{O}_S((k-s)f) \to V$.

(ii) *If p_g is even and we set $k = p_g/2$, then, after twisting by a line bundle of the form $\mathcal{O}_S(af)$, there exist an integer s, $0 \leq s \leq t$, and an exact sequence*

$$0 \to \mathcal{O}_S((k-s)f) \to V \to \mathcal{O}_S(\sigma+(-k-1+s-t)f) \otimes I_Z \to 0.$$

Here Z is again a codimension two local complete intersection subscheme of length s. Finally the inclusion of $\mathcal{O}_S((k-s)f)$ into V is canonically given by the map $\pi^*\pi_* V \to V$, and every nonzero map $\mathcal{O}_S(af) \to V$ factors through $\mathcal{O}_S((k-s)f)$.

Proof. We shall just write down the argument in case p_g is odd. By Proposition 3.13, possibly after twisting, V is obtained from V_0 by a sequence of $r \leq t$ allowable elementary modifications. In particular V may be identified with a subsheaf of V_0, and $\det V = \sigma - rf$. There is the map from $\mathscr{O}_S(kf)$ to V_0, and clearly the image of the subsheaf $\mathscr{O}_S((k-r)f)$ lies in V. Of course, the map $\mathscr{O}_S((k-r)f) \to V$ may vanish along a divisor, but this divisor must necessarily be a union of at most r fibers. Thus there is an integer u with $0 \leq u \leq r$ and an exact sequence for V of the form

$$0 \to \mathscr{O}_S((k-r+u)f) \to V \to \mathscr{O}_S(\sigma + (-k-u)f) \otimes I_Z \to 0.$$

Using the condition that $-p_1(\operatorname{ad} V) - 3(p_g + 1) = 2t$ gives

$$4\ell(Z) + 4(k-r+u) + (1+p_g) - 2r - 3(1+p_g) = 2t.$$

Solving, we get

$$-r + 2u + 2\ell(Z) = t.$$

Let $s = \ell(Z)$. Twisting the exact sequence by $\mathscr{O}_S(bf)$, where $b = u + \ell(Z) - t$, gives a new exact sequence (where we rename V by $V \otimes \mathscr{O}_S(bf)$)

$$0 \to \mathscr{O}_S((k-s)f) \to V \to \mathscr{O}_S(\sigma + (-k+s-t)f) \otimes I_Z \to 0.$$

Clearly $s = \ell(Z) \geq 0$ and since $2s = t + r - 2u$ with $u \geq 0$, $r \leq t$, we have $s \leq t$. This gives the desired expression of V as an extension. Since the restriction of this extension to the generic fiber is not split, the map

$$R^0\pi_*(\mathscr{O}_S(\sigma + (-k+s-t)f) \otimes I_Z) \to R^1\pi_*\mathscr{O}_S((k-s)f)$$

is injective. Thus $\pi_* V = \pi_*\mathscr{O}_S((k-s)f) = \mathscr{O}_{\mathbb{P}^1}(k-s)$ and the map $\pi^*\pi_* V \to V$ is just the inclusion $\mathscr{O}_S((k-s)f) \to V$. Finally if $\mathscr{O}_S(af) \to V$ is nonzero then $\pi_*\mathscr{O}_S(af) \to \pi_* V = \pi_*\mathscr{O}_S((k-s)f)$ is nonzero as well, and the last assertion of the proposition is then clear. □

There is an analogue of Proposition 4.3 for Gieseker semistable torsion free sheaves:

Proposition 4.3′. *With S and k as above, suppose that V is a rank two torsion free sheaf with $c_1(V) = \Delta = \sigma + nf$ for some n and $c_2(V) = c$ such that V is Gieseker semistable with respect to a (Δ, c)-suitable line bundle. Suppose that $-p_1(\operatorname{ad} V) - 3\chi(\mathscr{O}_S) = 2t$. Then the restriction of V to a general fiber of S is stable, and after twisting by $\mathscr{O}_S(af)$ for some a there are zero-dimensional subschemes Z_1 and Z_2 of S, not necessarily local complete intersections, an integer s with $0 \leq s \leq t$, and an exact sequence*

$$0 \to \mathscr{O}_S((k-s)f) \otimes I_{Z_1} \to V \to \mathscr{O}_S(\sigma + (-k+s-t)f) \otimes I_{Z_2} \to 0$$

if $p_g = 2k-1$ is odd, and

$$0 \to \mathscr{O}_S((k-s)f) \otimes I_{Z_1} \to V \to \mathscr{O}_S(\sigma + (-k-1+s-t)f) \otimes I_{Z_2} \to 0$$

if $p_g = 2k$ is even. Moreover $\ell(Z_1) + \ell(Z_2) = s$.

Proof. The double dual $V^{\vee\vee}$ of V is a semistable rank two vector bundle. Thus it is stable and fits into an exact sequence as in (i) or (ii) of Proposition 4.3. Now (4.3′) follows from manipulations along the lines of the proof of Proposition 4.3. □

Next let us consider when an extension as in Proposition 4.3 can be unstable. For simplicity we shall just write out the case where p_g is odd.

Proposition 4.4. *Suppose that* $p_g = 2k - 1$ *is odd and that* V *is an extension of the form*

$$0 \to \mathcal{O}_S((k-s)f) \to V \to \mathcal{O}_S(\sigma + (-k+s-t)f) \otimes I_Z \to 0,$$

where $\ell(Z) = s$. *Let* s_0 *be the smallest integer such that* $h^0(\mathcal{O}_S(s_0 f) \otimes I_Z) \neq 0$. *Thus* $0 \leq s_0 \leq s$, *and* $s_0 = 0$ *if and only if* $s = 0$. *If* V *is unstable, then the maximal destabilizing subbundle is equal to* $\mathcal{O}_S(\sigma - af)$, *where*

$$t + k - (s - s_0) \leq a \leq t + k.$$

Thus if $s = s_0$ *the only possibility is* $\mathcal{O}_S(\sigma - (t+k)f)$.

Proof. The maximal destabilizing subbundle has a torsion free quotient. Clearly, it restricts to σ on the generic fiber, and thus must be of the form $\mathcal{O}_S(\sigma - af)$ for some integer a. Using the exact sequence

$$0 \to \mathcal{O}_S(\sigma - af) \to V \to \mathcal{O}_S((a-t)f) \otimes I_{Z'} \to 0,$$

where Z' is a codimension two subscheme, and the fact that

$$\begin{aligned} c_2(V) &= k - s + s = k \\ &= a - t + \ell(Z'), \end{aligned}$$

we see that $a \leq t + k$. On the other hand, there is a nonzero map from $\mathcal{O}_S(\sigma - af)$ to $\mathcal{O}_S(\sigma + (-k+s-t)f) \otimes I_Z$ and thus a nonzero section of $\mathcal{O}_S((-k+s-t+a)f) \otimes I_Z$. Thus

$$-k + s - t + a \geq s_0,$$

or in other words $a \geq t + k - (s - s_0)$. □

Corollary 4.5. *With assumptions as above, suppose that* $Z = \emptyset$, *so that* V *is an extension*

$$0 \to \mathcal{O}_S(kf) \to V \to \mathcal{O}_S(\sigma - (k+t)f) \to 0.$$

Then V *is stable if and only if it is not the split extension. In this case we can identify the set of all nonsplit extensions with* $\operatorname{Sym}^t \sigma$, *and an extension* V *corresponding to* $\{p_1, \ldots, p_t\} \in \operatorname{Sym}^t \sigma$ *has unstable restriction to a fiber* f *if and only if* $p_i \in f$ *for some* i.

Proof. As we are in the case $s = 0$ of Proposition 4.4, if V is unstable then the destabilizing line bundle is $\mathcal{O}_S(\sigma - (k+t)f)$, which splits the exact sequence. Conversely, if the sequence is not split, then V is stable.

The set of nonsplit extensions of $\mathscr{O}_S(\sigma-(k+t)f)$ by $\mathscr{O}_S(kf)$ is parametrized by $\mathbb{P}H^1(\mathscr{O}_S(-\sigma+(2k+t)f))$. By Lemma 4.1

$$H^1(\mathscr{O}_S(-\sigma+(2k+t)f)) \cong H^0(R^1\pi_*\mathscr{O}_S(-\sigma+(2k+t)f)) = H^0(\mathbb{P}^1; \mathscr{O}_{\mathbb{P}^1}(t)).$$

Moreover $\mathbb{P}H^0(\mathbb{P}^1; \mathscr{O}_{\mathbb{P}^1}(t)) = \operatorname{Sym}^t\sigma$ by associating to a section the set of points where it vanishes. This says that the extension V restricts to the split extension on a fiber f exactly when the corresponding section of $\mathscr{O}_{\mathbb{P}^1}(t)$ vanishes at the point of $\mathbb{P}^1$ under f. □

Next we analyze the generic case where $\ell(Z) = t$.

Proposition 4.6. *Suppose that $p_g = 2k-1$ is odd and that V is an extension of the form*

$$0 \to \mathscr{O}_S((k-t)f) \to V \to \mathscr{O}_S(\sigma - kf) \otimes I_Z \to 0,$$

where $\ell(Z) = t > 0$.

(i) *A locally free extension V as above exists if and only if Z has the Cayley-Bacharach property with respect to $|\sigma+(t-2)f|$.*
(ii) *Suppose that $s_0 = t$ or $t-1$ in the notation of Proposition 4.4, and that* $\operatorname{Supp} Z \cap \sigma = \varnothing$. *Then* $\dim \operatorname{Ext}^1(\mathscr{O}_S(\sigma-kf)\otimes I_Z, \mathscr{O}_S((k-t)f)) = 1$. *A locally free extension exists in this case if $s_0 = t$.*
(iii) *Suppose that Z consists of t points lying in distinct fibers, exactly one of which lies on σ. Then* $\dim \operatorname{Ext}^1(\mathscr{O}_S(\sigma-kf)\otimes I_Z, \mathscr{O}_S((k-t)f)) = 1$. *A locally free extension exists in this case if and only if $t = 1$.*
(iv) *If $s_0 \le t-1$, for example if Z contains two distinct points lying on the same fiber, then V is unstable.*
(v) *If $s_0 = t$, then V is stable if no point of Z lies on σ. Likewise if $t = 1$ and $Z \subset \sigma$, then V is not stable.*

Proof. The long exact sequence for Ext gives

$$\begin{aligned} H^1(-\sigma+(2k-t)f) &\to \operatorname{Ext}^1(\mathscr{O}_S(\sigma-kf)\otimes I_Z, \mathscr{O}_S((k-t)f)) \to \\ &\to H^0(\mathscr{O}_Z) \to H^2(-\sigma+(2k-t)f). \end{aligned}$$

By Lemma 4.1(ii)

$$H^1(-\sigma+(2k-t)f) = 0.$$

The map $H^0(\mathscr{O}_Z) \to H^2(-\sigma+(2k-t)f)$ is dual to the map

$$H^0(\mathscr{O}_S(\sigma+(t-2)f)) \to H^0(\mathscr{O}_Z)$$

defined by restriction. Thus (i) follows by definition. As for (ii), since $\operatorname{Supp} Z \cap \sigma = \varnothing$ and $H^0(\mathscr{O}_S(\sigma+(t-2)f)) = H^0(\mathscr{O}_S((t-2)f))$ under the natural inclusion, clearly $H^0(\mathscr{O}_S(\sigma+(t-2)f)\otimes I_Z) = H^0(\mathscr{O}_S((t-2)f)\otimes I_Z)$. By assumption $H^0(\mathscr{O}_S((t-2)f)\otimes I_Z) = 0$, so that the map $H^0(\mathscr{O}_S(\sigma+(t-2)f)) \to H^0(\mathscr{O}_Z)$ is an inclusion. But $h^0(\mathscr{O}_S(\sigma+(t-2)f)) = t-1$ and $h^0(\mathscr{O}_Z) = t$. Thus the cokernel has dimension one. It is clear that if $s_0 = t$ and Z is reduced, then it has the Cayley-Bacharach property with respect to $|\sigma+(t-2)f|$. A more involved argument left to the reader handles the nonreduced case. Thus a locally free extension exists. This proves (ii), and the proof of (iii) is similar.

To see (iv), note that if $s_0 \leq t-1$, then there is a section of $\mathcal{O}_S((t-1)f)\otimes I_Z$. Consider the exact sequence

$$0 \to \mathcal{O}_S(-\sigma+(2k-1)f) \to Hom(\mathcal{O}_S(\sigma-(k+t-1)f), V) \to \mathcal{O}_S((t-1)f)\otimes I_Z \to 0.$$

Since $H^1(\mathcal{O}_S(-\sigma+(2k-1)f)) = 0$ by Lemma 4.1, the section of $\mathcal{O}_S((t-1)f)\otimes I_Z$ lifts to define a nonzero homomorphism from $\mathcal{O}_S(\sigma-(k+t-1)f)$ to V. Thus V is unstable.

Finally we must prove (v). The bundle V is stable if and only if its restriction to a general fiber f is stable. Let f be a fiber not meeting $\operatorname{Supp} Z$. Then there is a natural map $\operatorname{Ext}^1(\mathcal{O}_S(\sigma-kf)\otimes I_Z, \mathcal{O}_S((k-t)f)) \to \operatorname{Ext}^1(\mathcal{O}_f(p), \mathcal{O}_f) = H^1(\mathcal{O}_f(-p))$. This fits into an exact sequence

$$\begin{aligned} H^0(\mathcal{O}_f(-p)) &\to \operatorname{Ext}^1(\mathcal{O}_S(\sigma-kf)\otimes I_Z, \mathcal{O}_S((k-t-1)f)) \to \\ &\to \operatorname{Ext}^1(\mathcal{O}_S(\sigma-kf)\otimes I_Z, \mathcal{O}_S((k-t)f)) \to H^1(\mathcal{O}_f(-p)). \end{aligned}$$

Since $H^0(\mathcal{O}_f(-p)) = 0$ and

$$h^1(\mathcal{O}_f(-p)) = \dim \operatorname{Ext}^1(\mathcal{O}_S(\sigma-kf)\otimes I_Z, \mathcal{O}_S((k-t)f)) = 1,$$

by (ii) and (iii), it will suffice to show that

$$\dim \operatorname{Ext}^1(\mathcal{O}_S(\sigma-kf)\otimes I_Z, \mathcal{O}_S((k-t-1)f)) \geq 1$$

if $\operatorname{Supp} Z \cap \sigma \neq \emptyset$. Now since $H^1(\mathcal{O}_S(-\sigma+(2k-t-1)f)) = 0$ by Lemma 4.1, $\operatorname{Ext}^1(\mathcal{O}_S(\sigma-kf)\otimes I_Z, \mathcal{O}_S((k-t-1)f))$ is dual to the cokernel of the restriction map $H^0(\mathcal{O}_S(\sigma+(t-1)f)) \to H^0(\mathcal{O}_Z)$. Since $s_0 = t$, by definition $h^0(\mathcal{O}_S((t-1)f)\otimes I_Z) = 0$. Thus if $\operatorname{Supp} Z \cap \sigma = \emptyset$, then $H^0(\mathcal{O}_S(\sigma+(t-1)f))$ and $H^0(\mathcal{O}_S((t-1)f))$ have the same image in $H^0(\mathcal{O}_Z)$ and $H^0(\mathcal{O}_S((t-1)f)) \to H^0(\mathcal{O}_Z)$ is injective. As both $H^0(\mathcal{O}_S((t-1)f))$ and $H^0(\mathcal{O}_Z)$ have dimension t, the map between them is an isomorphism and the cokernel is zero. It follows that V restricts to a stable bundle on f. Likewise, if $t=1$ and $Z \subset \sigma$, then clearly the map $H^0(\mathcal{O}_S(\sigma+(t-1)f)) \to H^0(\mathcal{O}_Z)$ cannot be surjective, and so the cokernel is nonzero. Thus V restricts on f to an unstable bundle for almost every fiber f, so that V is unstable. □

Let us give another proof for Proposition 4.6(v). Using Proposition 4.4 we know that the maximal destabilizing line bundle, if it exists, must necessarily be of the form $\mathcal{O}_S(\sigma-(t+k)f)$. There is an exact sequence

$$0 \to \mathcal{O}_S(-\sigma+2kf) \to Hom(\mathcal{O}_S(\sigma-(t+k)f), V) \to \mathcal{O}_S(tf)\otimes I_Z \to 0,$$

and V is unstable if and only if the nonzero section of $\mathcal{O}_S(tf)\otimes I_Z$ lifts to a homomorphism from $\mathcal{O}_S(\sigma-(t+k)f)$ to V. The nonzero section of $\mathcal{O}_S(tf)\otimes I_Z$ defines an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(tf)\otimes I_Z \to Q \to 0.$$

Here if Z consists of points z_i on distinct fibers f_i, then $Q = \bigoplus_i \mathcal{O}_{f_i}(-z_i)$. The coboundary map from $H^0(\mathcal{O}_S(tf) \otimes I_Z)$ to $H^1(\mathcal{O}_S(-\sigma + 2kf))$ is given by taking the cup product of the nonzero section with the extension class ξ in $\mathrm{Ext}^1(\mathcal{O}_S(tf) \otimes I_Z, \mathcal{O}_S(-\sigma + 2kf))$ corresponding to V. It is easy to see by the naturality of the pairing that this is the same as taking the image of ξ in $\mathrm{Ext}^1(\mathcal{O}_S, \mathcal{O}_S(-\sigma + 2kf)) = H^1(\mathcal{O}_S(-\sigma + 2kf))$ using the above exact sequence. Taking the long exact Ext sequence and using the fact that $H^0(\mathcal{O}_S(-\sigma+2kf)) = 0$, there is an exact sequence

$$0 \to \mathrm{Ext}^1(Q, \mathcal{O}_S(-\sigma + 2kf)) \to \mathrm{Ext}^1(\mathcal{O}_S(tf) \otimes I_Z, \mathcal{O}_S(-\sigma + 2kf)) \to \\ \to H^1(\mathcal{O}_S(-\sigma + 2kf)).$$

Since $\dim \mathrm{Ext}^1(\mathcal{O}_S(tf) \otimes I_Z, \mathcal{O}_S(-\sigma+2kf)) = 1$, we see that $\xi \mapsto 0$ if and only if $\mathrm{Ext}^1(Q, \mathcal{O}_S(-\sigma+2kf)) \neq 0$. So we shall show that $\mathrm{Ext}^1(Q, \mathcal{O}_S(-\sigma+2kf)) = 0$ if and only if the support of Z does not meet σ.

First consider the case where Z consists of points z_i on distinct fibers f_i. Then $Q = \bigoplus_i \mathcal{O}_{f_i}(-z_i)$, and standard arguments (cf. [11], Chapter 7, Lemma 1.27) show that $\mathrm{Ext}^1(\mathcal{O}_{f_i}(-z_i), \mathcal{O}_S(-\sigma + 2kf)) = H^0(\mathcal{O}_{f_i}(z_i - p_i))$, where $p_i = f_i \cap \sigma$. This group is then zero unless $z_i = p_i$.

We shall briefly outline the argument in the case where $\operatorname{Supp} Z$ is a single point z supported on a fiber f (the proof in the general case is then just a matter of notation). In this case $Q = (\mathcal{O}_S(tf) \otimes I_Z)/\mathcal{O}_S \cong I_Z/I_{tf}$, where I_{tf} is the ideal of the nonreduced subscheme tf. Moreover the assumption that $s_0 = t$ implies that t is the smallest integer s such that $x^s \in I_Z$, where x is a local defining function for the fiber f. Our goal now is again to prove that $\mathrm{Ext}^1(Q, \mathcal{O}_S(-\sigma + 2kf)) = 0$.

Now the sheaf Q has a filtration by subsheaves whose successive quotients are

$$Q_n = I_Z \cap I_{nf}/I_Z \cap I_{(n+1)f} \cong (I_Z \cap I_{nf} + I_{(n+1)f})/I_{(n+1)f},$$

for $0 \leq n \leq t-1$. It is easy to see that each such quotient is a torsion free rank one $\mathcal{O}_f$-module contained in $I_{nf}/I_{(n+1)f} \cong \mathcal{O}_f$. Thus it is a line bundle on f of strictly negative degree, necessarily of the form $\mathcal{O}_f(-a_n z)$, unless $(I_Z \cap I_{nf} + I_{(n+1)f})/I_{(n+1)f} = I_{nf}/I_{(n+1)f}$, or in other words $I_Z \cap I_{nf} + I_{(n+1)f} = I_{nf}$. In this case, in the local ring of z we would have $x^n = h + gx^{n+1}$, where x is a local defining function for f and $h \in I_Z$. But then $h = x^n(1 - gx)$, so that $x^n \in I_Z$, contradicting the fact that x^t is the smallest power of x which lies in I_Z. Hence $Q_n \cong \mathcal{O}_f(-a_n z)$ with $a_n \geq 1$.

A standard argument with Chern classes shows that

$$c_2(Q) = t[z] = \sum_{n=0}^{t-1} c_2(Q_n) = -\sum_{n=0}^{t-1} \deg Q_n,$$

where $c_2(Q)$, $c_2(Q_n)$ are taken in the sense of sheaves on S and $\deg Q_n$ is in the sense of line bundles on f. Thus $\deg Q_n = -1$ for all n and $Q_n = \mathcal{O}_f(-z)$.

It follows that $\mathrm{Ext}^1(Q_n, \mathcal{O}_S(-\sigma+2nf)) = H^0(\mathcal{O}_f(z-p))$ where $p = \sigma \cap f$. This group is zero if $z \neq p$ and is nonzero otherwise. Thus $\mathrm{Ext}^1(Q, \mathcal{O}_S(-\sigma+2nf)) = 0$ if $z \neq p$ and $\mathrm{Ext}^1(Q, \mathcal{O}_S(-\sigma+2nf)) \neq 0$ if $z = p$.

We shall now reverse the above constructions and try to find a universal bundle in the case where the dimension of the moduli space is 2 or 4. For simplicity we shall just consider the case where p_g is odd.

The two-dimensional invariant. Let $\mathfrak{M}_1$ denote the moduli space of equivalence classes of stable rank two bundles V for which $-p_1(\mathrm{ad}\, V) - 3\chi(\mathcal{O}_S) = 2$. Thus $\mathfrak{M}_1$ is compact. Since p_g is odd, we may fix the determinant of V to be $\sigma - f$. Our goal is to show the following:

Theorem 4.7. *$\mathfrak{M}_1 \cong S$. Moreover there is a universal bundle $\mathcal{V}$ over $S \times S$, and*

$$p_1(\mathrm{ad}\,\mathcal{V})/[\Sigma] = (2(\sigma \cdot \Sigma) - 2p_g(f \cdot \Sigma))f - 4(f \cdot \Sigma)\sigma - 4\Sigma.$$

Thus, as $-4\mu(\Sigma) = p_1(\mathrm{ad}\,\mathcal{V})/[\Sigma]$, we have

$$\mu(\Sigma)^2 = (\Sigma)^2 + (p_g - 1)(f \cdot \Sigma)^2.$$

Proof. It follows from Propositions 4.3 and 4.6(v) and Corollary 4.5 that if V is stable with $-p_1(\mathrm{ad}\, V) - 3\chi(\mathcal{O}_S) = 2$ and $c_1(V) = \sigma - f$, then either there is an exact sequence

$$0 \to \mathcal{O}_S((k-1)f) \to V \to \mathcal{O}_S(\sigma - kf) \otimes \mathfrak{m}_q \to 0$$

with $\mathfrak{m}_q$ the maximal ideal of a point $q \notin \sigma$ or there is a nonsplit exact sequence

$$0 \to \mathcal{O}_S(kf) \to V \to \mathcal{O}_S(\sigma + (-k-1)f) \to 0.$$

In this case the set of all nonsplit extensions is isomorphic to σ. Thus the moduli space $\mathfrak{M}_1$ is made up of $S - \sigma$, together with a copy of σ. To glue these two pieces, we shall construct a universal bundle over $S \times S$ by taking extensions and then making an elementary modification. To this end, let $\mathbb{D}$ be the diagonal in $S \times S$. Consider the extension $\mathcal{W}$ over $S \times S$ defined as follows:

$$0 \to \pi_1^*\mathcal{O}_S((k-1)f) \otimes \pi_2^*\mathcal{L} \to \mathcal{W} \to \pi_1^*\mathcal{O}_S(\sigma - kf) \otimes I_{\mathbb{D}} \to 0.$$

Here, using the relative Ext sheaves and standard exact sequences we should take

$$\begin{aligned}
\mathcal{L}^{-1} &= \mathcal{E}xt^1_{\pi_2}(\pi_1^*\mathcal{O}_S(\sigma - kf) \otimes I_{\mathbb{D}}, \pi_1^*\mathcal{O}_S((k-1)f)) \\
&= \pi_{2*}(\det N_{\mathbb{D}} \otimes \pi_1^*\mathcal{O}_S(-\sigma + (2k-1)f)) \\
&= \pi_{2*}(\mathcal{O}_{\mathbb{D}}(-(p_g - 1)f) \otimes \pi_1^*\mathcal{O}_S(-\sigma + p_g f)) \\
&= \mathcal{O}_S(-\sigma + f).
\end{aligned}$$

With this choice of $\mathcal{L}$, we find that

$$\mathrm{Ext}^1(\pi_1^*\mathcal{O}_S(\sigma - kf) \otimes I_{\mathbb{D}}, \pi_1^*\mathcal{O}_S((k-1)f) \otimes \pi_2^*\mathcal{L}) \cong H^0(\mathcal{O}_{\mathbb{D}})$$

and that the unique nontrivial extension is indeed locally free. This defines $\mathscr{W}$, and an easy computation gives

$$\begin{aligned} c_1(\mathscr{W}) &= \pi_1^*(\sigma - f) + \pi_2^* c_1(\mathscr{L}); \\ p_1(\operatorname{ad}\mathscr{W}) &= 2\pi_1^*(-\sigma + (2k-1)f) \cdot \pi_2^*(\sigma - f) - 4[\mathbb{D}] + \cdots, \end{aligned}$$

where the omitted terms do not affect the slant product.

The restriction W of $\mathscr{W}$ to the slice $S \times \{q\}$ is the unique nontrivial extension of $\mathscr{O}_S(\sigma - kf) \otimes \mathfrak{m}_q$ by $\mathscr{O}_S((k-1)f)$. By Proposition 4.6(v) W is stable if and only if q does not lie on σ. To remedy this problem, we shall make an elementary modification along $S \times \sigma$. Note that, if W is given as an extension

$$0 \to \mathscr{O}_S((k-1)f) \to W \to \mathscr{O}_S(\sigma - kf) \otimes \mathfrak{m}_q \to 0,$$

where $q \in \sigma$, then the maximal destabilizing sub-line bundle of W must be $\mathscr{O}_S(\sigma + (-k-1)f)$ by Proposition 4.4 and thus there is an exact sequence

$$0 \to \mathscr{O}_S(\sigma + (-k-1)f) \to W \to \mathscr{O}_S(kf) \to 0.$$

It follows that $\pi_{2*} Hom(\mathscr{W}|(S \times \sigma), \pi_1^* \mathscr{O}_S(kf))$ is a line bundle $\mathscr{M}$ and the natural map

$$\mathscr{W} \to i_*(\pi_1^* \mathscr{O}_S(kf) \otimes \pi_2^* \mathscr{M})$$

is surjective. Thus we can define $\mathscr{V}$ by taking the associated elementary modification. By construction there is an exact sequence

$$0 \to \mathscr{V} \to \mathscr{W} \to i_*(\pi_1^* \mathscr{O}_S(kf) \otimes \pi_2^* \mathscr{M}) \to 0.$$

Moreover for each $q \in \sigma$ there is an exact sequence

$$0 \to \mathscr{O}_S(kf) \to \mathscr{V}|S \times \{q\} \to \mathscr{O}_S(\sigma + (-k-1)f) \to 0.$$

Thus by Corollary 4.5 $\mathscr{V}|S \times \{q\}$ is stable provided that this extension does not split. We state this fact explicitly as a lemma, whose proof will be deferred until later:

Lemma 4.8. *In the above notation, the extension for $\mathscr{V}|S \times \{q\}$ does not split.*

Assuming the lemma, the restriction of $\mathscr{V}$ to each slice is stable and thus $\mathscr{V}$ defines a morphism from S to the moduli space $\mathfrak{M}_1$. It is clear that this morphism is a bijection between two smooth surfaces and is thus an isomorphism. Moreover, by the lemma on elementary modifications,

$$p_1(\operatorname{ad}\mathscr{V}) = p_1(\operatorname{ad}\mathscr{W}) + 2c_1(\mathscr{W}) \cdot [S \times \sigma] + [S \times \sigma]^2 - 4i_* c_1(\pi_1^* \mathscr{O}_S(kf) \otimes \pi_2^* \mathscr{M}).$$

Plugging in for $c_1(\mathscr{W})$ and $p_1(\operatorname{ad}\mathscr{W})$ gives

$$\begin{aligned} p_1(\operatorname{ad}\mathscr{V}) = {} & 2\pi_1^*(-\sigma + (2k-1)f) \cdot \pi_2^*(\sigma - f) - 4[\mathbb{D}] \\ & + 2\pi_1^*(\sigma - f) \cdot \pi_2^* \sigma - 4\pi_1^*(kf) \cdot \pi_2^* \sigma + \cdots, \end{aligned}$$

where as usual the omitted terms do not affect the slant product. Thus collecting terms and taking the slant product gives

$$-4\mu(\Sigma) = 2(\sigma \cdot \Sigma)f - 2p_g(f \cdot \Sigma)f - 4(f \cdot \Sigma)\sigma - 4\Sigma,$$

as claimed in the statement of Theorem 4.7. This concludes the proof of Theorem 4.7. □

Proof of Lemma 4.8. We shall use the criterion (A.4) of the Appendix and the discussion following it to see that the extension does not split. Given $q \in \sigma$, let $W = \mathscr{W}|S \times \{q\}$. We need to show:

(i) $\mathrm{Hom}(\mathscr{O}_S(\sigma + (-k-1)f), \mathscr{O}_S(kf)) = 0$.

(ii) The map (coming from the usual long exact sequences)

$$\begin{aligned} R^0\pi_{2*}(\pi_1^*\mathscr{O}_S(\sigma - kf) \otimes I_{\mathbb{D}} \otimes \pi_1^*\mathscr{O}_S(-\sigma + (k+1)f)) \\ = R^0\pi_{2*}(\pi_1^*\mathscr{O}_S(f) \otimes I_{\mathbb{D}}) \to R^1\pi_{2*}\pi_1^*(\mathscr{O}_S((k-1)f) \otimes \mathscr{O}_S(-\sigma + (k+1)f)) \\ = R^1\pi_{2*}\pi_1^*(\mathscr{O}_S(-\sigma + 2kf)) \end{aligned}$$

vanishes simply along σ.

(iii) $H^1(\mathscr{O}_S(f) \otimes \mathfrak{m}_q)$ is independent of q, and is nonzero only if $p_g = 0$. Moreover $H^2(\mathscr{O}_S(-\sigma + 2kf)) = 0$.

(iv) At each point of σ, the map

$$H^1(\mathscr{O}_S(-\sigma + 2kf)) \to H^1(\mathscr{O}_S(kf) \otimes \mathscr{O}_S(-\sigma + (k+1)f)) = H^1(-\sigma + (2k+1)f)$$

induced by the map

$$H^1(\mathscr{O}_S(-\sigma + 2kf)) \to H^1(W \otimes \mathscr{O}_S(-\sigma + (k+1)f))$$

followed by the natural map

$$H^1(W \otimes \mathscr{O}_S(-\sigma + (k+1)f)) \to H^1(\mathscr{O}_S(kf) \otimes \mathscr{O}_S(-\sigma + (k+1)f))$$

is injective.

The statement (i) is clear. To prove the statement (ii), we shall calculate $R^1\pi_{2*}(\mathscr{W} \otimes \mathscr{O}_S(\sigma + (-k-1)f))$ by an argument similar to the second proof of Proposition 4.6(v). By base change $R^0\pi_{2*}(\pi_1^*\mathscr{O}_S(f) \otimes I_{\mathbb{D}}) = \mathscr{L}_1$ is a line bundle on S. From the definition of $\mathscr{W}$ and $\mathscr{L}$ the sheaf $Ext^1_{\pi_2}(\pi_1^*\mathscr{O}_S(f) \otimes I_{\mathbb{D}}, \pi_1^*\mathscr{O}_S(-\sigma + 2kf)) \otimes \mathscr{L}$ is the trivial line bundle. A global section induces the map $\mathscr{L}_1 \to R^1\pi_{2*}\pi_1^*\mathscr{O}_S(\sigma + 2kf) \otimes \mathscr{L}$. The cokernel of this map is a subsheaf of $R^1\pi_{2*}(\mathscr{W} \otimes \mathscr{O}_S(-\sigma + (k+1)f))$. To determine where the map vanishes, use the exact sequence

$$0 \to \pi_2^*\mathscr{L}_1 \to \pi_1^*\mathscr{O}_S(f) \otimes I_{\mathbb{D}} \to \mathscr{P} \to 0.$$

Here the map $\pi_2^*\mathscr{L}_1 \to \pi_1^*\mathscr{O}_S(f) \otimes I_{\mathbb{D}}$ is the natural one and a calculation in local coordinates shows that it vanishes simply along $D = S \times_{\mathbb{P}^1} S \subset S \times S$. It follows that, up to a line bundle pulled back from the second factor $\mathscr{P} = \mathscr{O}_D(-\mathbb{D})$. Thus $\mathscr{P}$ is up to sign a Poincaré bundle.

Now $\mathrm{Ext}^2(\mathscr{O}_S(f) \otimes \mathfrak{m}_q, \mathscr{O}_S(-\sigma + 2kf)) = 0$ since $H^2(\mathscr{O}_S(-\sigma + (2k-1)f)) = 0$. Thus $Ext^2_{\pi_2}(\pi_1^*\mathscr{O}_S(f) \otimes I_{\mathbb{D}}, \pi_1^*\mathscr{O}_S(-\sigma + 2kf)) = 0$ and there is an exact sequence

$$\begin{aligned} Ext^1_{\pi_2}(\mathscr{P}, \pi_1^*\mathscr{O}_S(-\sigma + 2kf)) \to Ext^1_{\pi_2}(\pi_1^*\mathscr{O}_S(f) \otimes I_{\mathbb{D}}, \pi_1^*\mathscr{O}_S(-\sigma + 2kf)) \to \\ \to R^1\pi_{2*}\pi_1^*\mathscr{O}_S(\sigma + 2kf) \to Ext^2_{\pi_2}(\mathscr{P}, \pi_1^*\mathscr{O}_S(-\sigma + 2kf)) \to 0. \end{aligned}$$

It follows from the naturality of the pairings involved that the image of the map

$$Ext^1_{\pi_2}(\pi_1^*\mathcal{O}_S(f)\otimes I_{\mathbb{D}},\ \pi_1^*\mathcal{O}_S(-\sigma+2kf)) \to R^1\pi_{2*}\pi_1^*\mathcal{O}_S(\sigma+2kf)$$

is, up to a twist by the line bundle $\mathcal{L}$, the image of $\mathcal{L}_1$.

Note that the restriction of $\mathcal{P}^\vee\otimes\pi_1^*\mathcal{O}_S(-\sigma+2kf)$ to $\pi_2^{-1}(q)\subset D$, where q is any point except the singular point on a singular fiber, is $\mathcal{O}_f(p-q)$, where f is the fiber containing q and $p=f\cap\sigma$. Ignoring the possible double points of D, we have by standard arguments

$$Ext^1_{\pi_2}(\pi_1^*\mathcal{O}_S(f)\otimes I_{\mathbb{D}},\ \pi_1^*\mathcal{O}_S(-\sigma+2kf)) = \pi_{2*}(\mathcal{P}^\vee\otimes\pi_1^*\mathcal{O}_S(-\sigma+2kf))$$
$$Ext^2_{\pi_2}(\pi_1^*\mathcal{O}_S(f)\otimes I_{\mathbb{D}},\ \pi_1^*\mathcal{O}_S(-\sigma+2kf)) = R^1\pi_{2*}(\mathcal{P}^\vee\otimes\pi_1^*\mathcal{O}_S(-\sigma+2kf))$$

(where $\mathcal{P}^\vee$ means that the dual is taken as a line bundle on D). Thus $\pi_{2*}(\mathcal{P}^\vee\otimes\pi_1^*\mathcal{O}_S(-\sigma+2kf))=0$ and $R^1\pi_{2*}(\mathcal{P}^\vee\otimes\pi_1^*\mathcal{O}_S(-\sigma+2kf))$ is supported on σ. To calculate its length, we have (again ignoring the double points of D which will not cause trouble) an exact sequence

$$0\to\mathcal{O}_D(\mathbb{D}-\pi_1^*\sigma+\pi_1^*(2kf))\to\mathcal{O}_D(\mathbb{D}+\pi_1^*(2kf))\to\mathcal{O}_D(\mathbb{D}+\pi_1^*(2kf))|\pi_1^*\sigma\cap D\to 0.$$

Now $\pi_1^*\sigma\cap D\cong S$ via π_2 and under this isomorphism $\mathcal{O}_D(\mathbb{D}+\pi_1^*(2kf))|\pi_1^*\sigma\cap D\cong\mathcal{O}_S(\sigma+2kf)$. The map

$$R^0\pi_{2*}\mathcal{O}_D(\pi_1^*(2kf))\to R^0\pi_{2*}\mathcal{O}_D(\mathbb{D}+\pi_1^*(2kf))$$

is an isomorphism, since the induced map on H^0's for the restriction to each fiber of π_2 is an isomorphism. Using the exact sequence

$$0\to\mathcal{O}_D(\pi_1^*(-\sigma+2kf))\to\mathcal{O}_D(\pi_1^*(2kf))\to\mathcal{O}_S(2kf)\to 0,$$

it follows that the image of $R^0\pi_{2*}\mathcal{O}_D(\pi_1^*(2kf))=R^0\pi_{2*}\mathcal{O}_D(\mathbb{D}+\pi_1^*(2kf))$ in

$$R^0\pi_{2*}\mathcal{O}_D(\mathbb{D}+\pi_1^*(2kf))|\pi_1^*\sigma\cap D=\mathcal{O}_S(\sigma+2kf)$$

is just the image of $\mathcal{O}_S(2kf)$ in $\mathcal{O}_S(\sigma+2kf)$. Thus this map vanishes simply along σ, and its cokernel, which is

$$R^1\pi_{2*}\mathcal{O}_D(\mathbb{D}-\pi_1^*\sigma+\pi_1^*(2kf))=R^1\pi_{2*}(\mathcal{P}^\vee\otimes\pi_1^*\mathcal{O}_S(-\sigma+2kf)),$$

is a line bundle on σ. It follows that the map of line bundles

$$Ext^1_{\pi_2}(\pi_1^*\mathcal{O}_S(f)\otimes I_{\mathbb{D}},\ \pi_1^*\mathcal{O}_S(-\sigma+2kf))\otimes\mathcal{L}\to R^1\pi_{2*}\pi_1^*\mathcal{O}_S(\sigma+2kf)\otimes\mathcal{L}$$

vanishes simply along σ, so that we are in the situation of (A.4): the cokernel contributes torsion of length one.

To see that the above is exactly the torsion in $R^1\pi_{2*}(\mathcal{W}\otimes\mathcal{O}_S(-\sigma+(k+1)f))$ follows from (iii), as in the discussion after (A.4). To see (iii), use the exact sequence

$$0\to\mathcal{O}_S(f)\otimes\mathfrak{m}_q\to\mathcal{O}_S(f)\to\mathbb{C}_q\to 0.$$

The long exact cohomology sequence shows that $H^1(\mathcal{O}_S(f)\otimes\mathfrak{m}_q)\cong H^1(\mathcal{O}_S(f))$. It is easy to see that this last group is zero if $p_g>0$ and has dimension one if

$p_g = 0$ (and in any case its dimension is obviously independent of q). Finally $H^2(\mathscr{O}_S(-\sigma + 2kf)) = 0$ by Lemma 4.1. Thus we have identified the torsion in $R^1\pi_{2*}(\mathscr{W} \otimes \mathscr{O}_S(-\sigma + (k+1)f))$, compatibly with base change.

We finally need to check that the induced map

$$H^1(\mathscr{O}_S(-\sigma + 2kf)) \to H^1(\mathscr{O}_S(-\sigma + (2k+1)f))$$

is injective. But this map is induced from the composition of the map of sheaves

$$\mathscr{O}_S(-\sigma + 2kf) \to W \otimes \mathscr{O}_S(-\sigma + (k+1)f)$$

together with the map $W \otimes \mathscr{O}_S(-\sigma + (k+1)f) \to \mathscr{O}_S(-\sigma + (2k+1)f)$. This composition is then a nonzero map from $\mathscr{O}_S(-\sigma + 2kf)$ to $\mathscr{O}_S(-\sigma + (2k+1)f)$ and so fits into an exact sequence

$$0 \to \mathscr{O}_S(-\sigma + 2kf) \to \mathscr{O}_S(-\sigma + (2k+1)f) \to \mathscr{O}_f(-p) \to 0.$$

Since $H^0(\mathscr{O}_f(-p)) = 0$, the map $H^1(\mathscr{O}_S(-\sigma + 2kf)) \to H^1(\mathscr{O}_S(-\sigma + (2k+1)f))$ is injective. Thus the extension class for $\mathscr{V}|S \times \{q\}$ is nonzero, and we are done. □

The four-dimensional invariant. We again assume that p_g is odd and list the possible types of extensions for a stable bundle. The generic case (Type 1) is where there exists a codimension two subscheme Z with $\ell(Z) = 2$ and an exact sequence

(Type 1) $$0 \to \mathscr{O}_S((k-2)f) \to V \to \mathscr{O}_S(\sigma - kf) \otimes I_Z \to 0.$$

Other possibilities (Types 2 and 3 respectively) are

(Type 2) $$0 \to \mathscr{O}_S((k-1)f) \to V \to \mathscr{O}_S(\sigma + (-k-1)f) \otimes \mathfrak{m}_q \to 0;$$

(Type 3) $$0 \to \mathscr{O}_S(kf) \to V \to \mathscr{O}_S(\sigma + (-k-2)f) \to 0.$$

Here $\mathfrak{m}_q$ is the maximal ideal of a point q. Finally there is also the case where V is not locally free. In this case the double dual of V fits into an extension

$$0 \to \mathscr{O}_S((k-1)f) \to V^{\vee\vee} \to \mathscr{O}_S(\sigma + (-k-1)f) \to 0$$

which must be nonsplit if V is to be stable, in which case $V^{\vee\vee}$ is just a twist of V_0. One possibility is that V is given as the unique non-locally free extension of $\mathscr{O}_S(\sigma + (-k-1)f) \otimes \mathfrak{m}_q$ by $\mathscr{O}_S((k-1)f)$ as in the second exact sequence above. The remaining possibility (Type 4) is that V is given as an extension

(Type 4) $$0 \to \mathscr{O}_S((k-1)f) \otimes \mathfrak{m}_q \to V \to \mathscr{O}_S(\sigma + (-k-1)f) \to 0.$$

For a fixed q, the set of all such extensions is parametrized by a $\mathbb{P}^1$, one point of which correspond to a V such that $V^{\vee\vee}$ is unstable.

Our goal here is to give a very brief sketch of the following, where we use the notation of the introduction for divisors on $\mathrm{Hilb}^2 S$:

Theorem 4.9. *The moduli space* $\mathfrak{M}_2$ *of dimension* 4 *is isomorphic to* $\mathrm{Hilb}^2 S$, *and for all* $\Sigma \in H_2(S)$,

$$\mu(\Sigma) = D_{\alpha_2} - ((f \cdot \Sigma)/2)E,$$

where, setting $\alpha_1 = \mu_1(\Sigma)$ *to be the class computed by the* μ*-map for the two-dimensional invariant,*

$$\begin{aligned}\alpha_2 &= \Sigma + (-(\sigma \cdot \Sigma) + (p_g + 1)(f \cdot \Sigma)/2)f + (f \cdot \Sigma)\sigma \\ &= \alpha_1 + ((f \cdot \Sigma)/2)f.\end{aligned}$$

Thus an easy calculation using the multiplication table for $\mathrm{Hilb}^2 S$ gives the following:

Corollary 4.10. *In the above notation,*

$$\mu(\Sigma)^4 = 3(\Sigma^2)^2 + 6(p_g - 1)(\Sigma^2)(f \cdot \Sigma)^2 + [3(p_g + 1)(p_g - 1) - 8(p_g - 1)](f \cdot \Sigma)^4.$$

We shall not give a complete proof of Theorem 4.9 here, but shall outline the argument and prove some statements which will be used later. In Sections 9 and 10, we shall prove a more general statement which will imply Theorem 4.9.

We begin as before by analyzing the generic case, Type 1. Let Z be a codimension two subscheme of S with $\ell(Z) = 2$. Let D_σ be the effective divisor of $\mathrm{Hilb}^2 S$ which is the closure of the locus of pairs $\{z_1, z_2\}$ where $z_1 \in \sigma$. Then arguing as in the proof of Proposition 4.6(i)–(iii), we see that

$$\dim \mathrm{Ext}^1(\mathcal{O}_S(\sigma - kf) \otimes I_Z, \mathcal{O}_S((k-2)f)) = \begin{cases} 1, & \text{if } Z \notin \mathrm{Sym}^2\sigma \subset \mathrm{Hilb}^2 S, \\ 2, & \text{otherwise.}\end{cases}$$

In case $Z \notin D_\sigma$, the unique extension class mod scalars corresponds to a locally free extension. If $Z \in D_\sigma - \mathrm{Sym}^2\sigma$, then the unique nontrivial extension is not locally free. If $Z \in \mathrm{Sym}^2\sigma$, then there exist locally free extensions.

Next we must analyze when a locally free extension is stable. Let $\mathcal{D}$ be the irreducible divisor in $\mathrm{Hilb}^2 S$ corresponding to the divisor $S \times_{\mathbb{P}^1} S \subset S \times S$. Equivalently

$$\mathcal{D} = \{ Z \in \mathrm{Hilb}^2 S \mid h^0(\mathcal{O}_S(f) \otimes I_Z) = 1 \}.$$

The divisor $\mathcal{D}$ is smooth, although $S \times_{\mathbb{P}^1} S$ is singular at the finitely many pairs of points (x, x), where x is a double point of a singular fiber. One way to see this is as follows. The divisor $S \times_{\mathbb{P}^1} S$ has ordinary threefold double points at the singularities. Moreover it contains the diagonal $\mathbb{D} \subset S \times S$, which is smooth and passes through the double points. It is well known (and easy to check) that the blowup of a threefold double point $(xy - zw)$ along a subvariety of the form $(x - z, y - w)$ gives a small resolution of the singularity. Thus the proper transform of $S \times_{\mathbb{P}^1} S$ in the blowup of $S \times S$ along $\mathbb{D}$ is smooth, and $\mathcal{D}$ is the quotient of this proper transform by an involution whose fixed point set is smooth (it is $\mathbb{D}$). Thus $\mathcal{D}$ is smooth.

Lemma 4.11. *Let V be a vector bundle given by an extension*

$$0 \to \mathcal{O}_S((k-2)f) \to V \to \mathcal{O}_S(\sigma - kf) \otimes I_Z \to 0,$$

where $\ell(Z) = 2$. Then V is not stable if and only if either $Z \in \operatorname{Sym}^2 \sigma$ or $Z \in \mathcal{D}$. If $Z \in \mathcal{D}$, then the maximal destabilizing sub-line bundle is $\mathcal{O}_S(\sigma + (-k-1)f)$ and there is an exact sequence

$$0 \to \mathcal{O}_S(\sigma + (-k-1)f) \to V \to \mathcal{O}_S((k-1)f) \otimes \mathfrak{m}_q \to 0.$$

Here $q = z_1 + z_2 - p$, where f is the unique fiber containing $Z = \{z_1, z_2\}$, $p = \sigma \cap f$, and the addition is with respect to the group law on f (if f is singular and $\operatorname{Supp} Z$ *meets the singular point then q is the singular point as well). If $Z \in \operatorname{Sym}^2 \sigma - \mathcal{D}$, then the maximal destabilizing sub-line bundle is $\mathcal{O}_S(\sigma - (k+2)f)$.*

Proof. If $Z \notin D_\sigma$, then we have seen in Proposition 4.6(v) that V is unstable if and only if $Z \in \mathcal{D}$, and in this case the destabilizing sub-line bundle must be $\mathcal{O}_S(\sigma + (-k-1)f)$ by Proposition 4.4. The quotient is torsion free and by a Chern class calculation it must be $\mathcal{O}_S((k-1)f) \otimes \mathfrak{m}_q$ for some point q. To identify the point q, let us assume for simplicity that $\operatorname{Supp} Z$ does not meet the singular point of a singular fiber, we can restrict the two exact sequences for V to the fiber f containing Z. From these we see that there are surjective maps $V|f \to \mathcal{O}_f(p - z_1 - z_2)$ and $V|f \to \mathcal{O}_f(-q)$. Since $\deg V|f = 1$, it splits and the unique summand of negative degree is thus $\mathcal{O}_f(p - z_1 - z_2) \cong \mathcal{O}_f(-q)$. It follows that $q = z_1 + z_2 - p$. The case where $\operatorname{Supp} Z$ contains the singular point of a singular fiber is similar.

If $Z \in D_\sigma$, then since V is locally free $Z \in \operatorname{Sym}^2 \sigma$. Arguments as in Proposition 4.6 then show that V is unstable. If moreover $Z \notin \mathcal{D}$, then the maximal destabilizing sub-line bundle is $\mathcal{O}_S(\sigma - (k+2)f)$ by Proposition 4.4. □

Our next task will be to construct a universal sheaf $\mathcal{V}$ over $\operatorname{Hilb}^2 S - D_\sigma$. We begin by finding a sheaf $\mathcal{W}$ as follows: let $\mathcal{Z} \subset S \times \operatorname{Hilb}^2 S$ be the universal subscheme, and consider the relative extension sheaf $Ext^1_{\pi_2}(\pi_1^*\mathcal{O}_S(\sigma - kf) \otimes I_{\mathcal{Z}}, \pi_1^*\mathcal{O}_S((k-2)f))$. Since $H^1(\mathcal{O}_S(-\sigma + (2k-2)f)) = 0$, there is an exact sequence

$$0 \to Ext^1_{\pi_2}(\pi_1^*\mathcal{O}_S(\sigma - kf) \otimes I_{\mathcal{Z}}, \pi_1^*\mathcal{O}_S((k-2)f)) \to$$
$$\to R^0\pi_{2*}Ext^1(\pi_1^*\mathcal{O}_S(\sigma - kf) \otimes I_{\mathcal{Z}}, \pi_1^*\mathcal{O}_S((k-2)f)).$$

Over the complement of $\operatorname{Sym}^2 \sigma$, $Ext^1_{\pi_2}(\pi_1^*\mathcal{O}_S(\sigma - kf) \otimes I_{\mathcal{Z}}, \pi_1^*\mathcal{O}_S((k-2)f))$ is a line bundle on $\operatorname{Hilb}^2 S - \operatorname{Sym}^2 \sigma$ which we denote by $\mathcal{L}^{-1}$ and thus there is a coherent sheaf $\mathcal{W}$ defined by

$$0 \to \pi_1^*\mathcal{O}_S((k-2)f) \otimes \mathcal{L} \to \mathcal{W} \to \pi_1^*\mathcal{O}_S(\sigma - kf) \otimes I_{\mathcal{Z}} \to 0.$$

However, if $Z \in \mathcal{D}$, then $\mathcal{W}|S \times \{Z\}$ is not stable, and if $Z \in D_\sigma$ then $\mathcal{W}|S \times \{Z\}$ is neither locally free nor stable. We shall first study $\mathcal{W}|S \times (\operatorname{Hilb}^2 S - D_\sigma)$, and shall denote this for simplicity again by $\mathscr{W}$. There is a unique point $q = z_1 + z_2 - p$ such that $\mathscr{W}|S \times \{Z\}$ maps surjectively to $\mathscr{O}_S((k-1)f) \otimes \mathfrak{m}_q$, and so we expect to be able to make an elementary transformation along $\mathscr{D}$. Indeed, since $\dim \operatorname{Hom}(\mathscr{O}_S(\sigma + (-k-1)f), \mathscr{W}|S \times \{Z\}) = 1$ for all $Z \in \mathscr{D}$, there are line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ on $\mathscr{D}$ and an exact sequence

$$0 \to \pi_1^*\mathscr{O}_S(\sigma+(-k-1)f)\otimes\pi_2^*\mathscr{L}_1 \to \mathscr{W}|S\times\mathscr{D} \to \pi_1^*\mathscr{O}_S((k-1)f)\otimes\pi_2^*\mathscr{L}_2\otimes I_{\mathscr{Y}} \to 0,$$

where $\mathscr{Y}$ is the set

$$\{(q, z_1, z_2) \in S \times_{\mathbb{P}^1} \mathscr{D} \mid q = z_1 + z_2 - p\}.$$

It is easy to check from the definition that $\mathscr{Y}$ is smooth and that the map $\mathscr{Y} \to \mathscr{D}$ is an isomorphism. Thus we may define $\mathscr{V}$ by the exact sequence

$$0 \to \mathscr{V} \to \mathscr{W} \to i_*\pi_1^*\mathscr{O}_S((k-1)f) \otimes \pi_2^*\mathscr{L}_2 \otimes I_{\mathscr{Y}} \to 0,$$

where i is the inclusion of $S \times \mathscr{D}$ in $S \times (\operatorname{Hilb}^2 S - D_\sigma)$. We then have the following:

Proposition 4.12. *The sheaf $\mathscr{V}$ is a reflexive sheaf, flat over $\operatorname{Hilb}^2 S - D_\sigma$. The restriction of $\mathscr{V}$ to each slice $S \times \{Z\}$ is a stable torsion free sheaf, which is locally free if and only if $Z \notin \mathscr{D}$.*

Proof. By (A.2) of the Appendix, $\mathscr{V}$ is reflexive and flat over $\operatorname{Hilb}^2 S - D_\sigma$. For each $Z \in \mathscr{D}$, if V_Z is the restriction of $\mathscr{V}$ to the slice $S \times \{Z\}$, there is an exact sequence

$$0 \to \mathscr{O}_S((k-1)f) \otimes \mathfrak{m}_q \to V_Z \to \mathscr{O}_S(\sigma + (-k-1)f) \to 0,$$

by (A.2) again. If $Z \notin \mathscr{D}$ then $V_Z = \mathscr{V}|S \times \{Z\}$ is locally free and stable. Thus we need only check that the double dual of V_Z, for $Z \in \mathscr{D}$, is the unique nonsplit extension of $\mathscr{O}_S(\sigma + (-k-1)f)$ by $\mathscr{O}_S((k-1)f)$, which will imply that $V_Z^{\vee\vee}$ is up to a twist the stable bundle V_0.

To verify that the double dual of V_Z is a nonsplit extension amounts to the following: the extension class corresponding to V_Z lives in

$$\operatorname{Ext}^1(\mathscr{O}_S(\sigma + (-k-1)f), \mathscr{O}_S((k-1)f) \otimes \mathfrak{m}_q) = H^1(\mathscr{O}_S(-\sigma + 2kf) \otimes \mathfrak{m}_q),$$

and we must show that its image in $H^1(\mathscr{O}_S(-\sigma + 2kf))$ is nonzero. To do this we shall use the result (A.4) of the Appendix. Let $M = \mathscr{O}_S(\sigma - (k+1)f)$ and $L = \mathscr{O}_S((k-1)f)$. Clearly $\operatorname{Hom}(M, L) = 0$. By the definition of $\mathscr{W}$ there is an exact sequence

$$0 \to \pi_1^*\mathscr{O}_S(-\sigma + (2k-1)f) \otimes \pi_2^*\mathscr{L} \to \mathscr{W} \otimes \pi_1^*M^{-1} \to \pi_1^*\mathscr{O}_S(f) \otimes I_{\mathscr{Z}} \to 0.$$

By Lemma 4.1 $R^1\pi_{2*}\pi_1^*\mathscr{O}_S(-\sigma + (2k-1)f) = R^2\pi_{2*}\pi_1^*\mathscr{O}_S(-\sigma + (2k-1)f) = 0$. Thus

$$R^1\pi_{2*}\mathscr{W} \otimes \pi_1^*M^{-1} \cong R^1\pi_{2*}(\pi_1^*\mathscr{O}_S(f) \otimes I_{\mathscr{Z}}).$$

To analyze $R^1\pi_{2*}(\pi_1^*\mathscr{O}_S(f) \otimes I_{\mathscr{Z}})$, use the exact sequence

$$0 \to \pi_1^*\mathscr{O}_S(f) \otimes I_{\mathscr{Z}} \to \pi_1^*\mathscr{O}_S(f) \to \mathscr{O}_{\mathscr{Z}} \otimes \pi_1^*\mathscr{O}_S(f) \to 0.$$

It is easy to check that $R^1\pi_{2*}\pi_1^*\mathscr{O}_S(f)=0$ if $p_g>0$, and is a line bundle if $p_g=0$. Clearly $R^1\pi_{2*}(\mathscr{O}_{\mathscr{Z}}\otimes\pi_1^*\mathscr{O}_S(f))=0$. Thus the torsion in $R^1\pi_{2*}(\pi_1^*\mathscr{O}_S(f)\otimes I_{\mathscr{Z}})$ is the cokernel of the map between two rank two vector bundles on $\mathrm{Hilb}^2 S$

$$R^0\pi_{2*}\pi_1^*\mathscr{O}_S(f)\to R^0\pi_{2*}(\mathscr{O}_{\mathscr{Z}}\otimes\pi_1^*\mathscr{O}_S(f)).$$

Since $\mathscr{D}$ is a smooth divisor, by using elementary divisors for the vector bundle map we can describe this cokernel by describing what it looks like at the generic point. It is a simple exercise in local coordinates to identify the determinant of the vector bundle map with a local equation for $\mathscr{D}$ at the generic point. Thus the torsion in $R^1\pi_{2*}\mathscr{W}\otimes\pi_1^*M^{-1}$ is a line bundle on $\mathscr{D}$, which is identified with the torsion in $R^1\pi_{2*}(\pi_1^*\mathscr{O}_S(f)\otimes I_{\mathscr{Z}})$. Similar statements hold via standard base change results if we restrict to a first order neighborhood of $\mathscr{D}$, where torsion is to be interpreted in the sense of (A.4)(ii) of the appendix.

Next, let

$$Z=\{z_1,z_2\}\in\mathscr{D}$$

and let W be the extension corresponding to the restriction of $\mathscr{W}$ to the slice $S\times\{Z\}$; we must identify the corresponding extension class, i.e. the image of the one-dimensional vector space $H^1(\mathscr{O}_S(f)\otimes I_Z)$ in $H^1(M^{-1}\otimes L\otimes\mathfrak{m}_q)$ and its further image in $H^1(M^{-1}\otimes L)$. Using the two exact sequences

$$0\to\mathscr{O}_S\to W\otimes M^{-1}\to M^{-1}\otimes L\otimes\mathfrak{m}_q\to 0,$$
$$0\to\mathscr{O}_S(-\sigma+(2k-1)f)\to W\otimes M^{-1}\to\mathscr{O}_S(f)\otimes I_Z\to 0,$$

we see that the composite map $\mathscr{O}_S\to\mathscr{O}_S(f)\otimes I_Z$ is nonzero and gives the nontrivial section. Now the quotient of $\mathscr{O}_S(f)\otimes I_Z$ by $\mathscr{O}_S$ is $\mathscr{O}_f(-z_1-z_2)$. Thus there is an induced map $M^{-1}\otimes L\otimes\mathfrak{m}_q\to\mathscr{O}_f(-z_1-z_2)$ which must factor through the natural map $M^{-1}\otimes L\otimes\mathfrak{m}_q=\mathscr{O}_S(-\sigma+2kf)\otimes\mathfrak{m}_q\to\mathscr{O}_f(-p-q)$. (Here as usual $p=\sigma\cap f$.) As the induced map $\mathscr{O}_f(-p-q)\to\mathscr{O}_f(-z_1-z_2)$ is nonzero, it is an isomorphism, and we recover the fact that $q=z_1+z_2-p$. Using the commutativity of

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_S(f)\otimes I_Z \arrow[r]\arrow[d] & \mathscr{O}_S(f) \arrow[r]\arrow[d] & \mathscr{O}_Z \arrow[r]\arrow[d, equal] & 0\\
0 \arrow[r] & \mathscr{O}_f(-z_1-z_2) \arrow[r] & \mathscr{O}_f \arrow[r] & \mathscr{O}_Z \arrow[r] & 0,
\end{tikzcd}$$

we also see that the image of $H^1(\mathscr{O}_S(f)\otimes I_Z)$ in $H^1(\mathscr{O}_f(-z_1-z_2))$ is the same as the image of $H^0(\mathscr{O}_Z)$ in $H^1(\mathscr{O}_f(-z_1-z_2))$.

There is a commutative diagram

$$\begin{tikzcd}
 & H^1(\mathscr{O}_S(-\sigma+2kf)\otimes\mathfrak{m}_q) \arrow[r]\arrow[d] & H^1(\mathscr{O}_S(-\sigma+2kf))\arrow[d]\\
H^1(\mathscr{O}_f(-z_1-z_2)) \arrow[r, "\cong"] & H^1(\mathscr{O}_f(-p-q)) \arrow[r] & H^1(\mathscr{O}_f(-p)).
\end{tikzcd}$$

Moreover the map $H^1(\mathcal{O}_S(-\sigma + 2kf)) \to H^1(\mathcal{O}_f(-p))$ is an isomorphism. So the problem is the following: does the image of $H^0(\mathcal{O}_Z)$ in $H^1(\mathcal{O}_f(-z_1 - z_2))$ map to zero in $H^1(\mathcal{O}_f(-p))$? The image of $H^0(\mathcal{O}_Z)$ in $H^1(\mathcal{O}_f(-z_1 - z_2))$ is dual to the image of $H^0(\mathcal{O}_f)$ in $H^0(\mathcal{O}_f(z_1 + z_2))$, giving a section vanishing at z_1 and z_2. On the other hand the kernel of the map $H^1(\mathcal{O}_f(-p-q)) \to H^1(\mathcal{O}_f(-p))$ is dual to the image of the map $H^0(\mathcal{O}_f(p)) \to H^0(\mathcal{O}_f(p+q))$, and the corresponding section of $\mathcal{O}_f(p+q)$ vanishes at p and q. So the only way that this can equal the image of $H^0(\mathcal{O}_Z)$ is for z_1 or z_2 to equal p, i.e. $Z \in D_\sigma$. Conversely, if $Z \notin D_\sigma$, then the image of the extension class in $H^1(M^{-1} \otimes L)$ is not zero. Thus the double dual of the restriction of $\mathcal{V}$ to $S \times \{Z\}$ is a nonsplit extension and so it is stable. □

This is as far as we shall go in this section in calculating the four-dimensional invariant. But let us sketch here how to obtain the full formula in Theorem 4.9. We will prove a more general statement in Section 10, where we will use Proposition 4.12.

First, to deal with the fact that $\dim \operatorname{Ext}^1(\mathcal{O}_S(\sigma - kf) \otimes I_Z, \mathcal{O}_S((k-2)f))$ jumps along $\operatorname{Sym}^2 \sigma$, blow up $\operatorname{Sym}^2 \sigma$ inside $\operatorname{Hilb}^2 S$. Let the exceptional divisor be G. After blowing up, we can assume that the extension is not locally trivial along G. There is thus a universal extension of torsion free sheaves $\mathcal{W}$ over $S \times \operatorname{Bl}_{\operatorname{Sym}^2 \sigma} \operatorname{Hilb}^2 S$. Now make an elementary modification along $\mathcal{D}$, replacing unstable Type 1 extensions with $Z \in \mathcal{D} - \operatorname{Sym}^2 \sigma$ with stable Type 4 extensions. Next make an elementary modification along D_σ, replacing unstable Type 1 extensions with $Z \in D_\sigma$ with Type 2 extensions; this also fixes some of the unstable Type 4 extensions. Finally make an elementary modification along G to replace the remaining unstable extensions with Type 3 extensions. At this point every member of the family is a stable torsion free sheaf, and the induced morphism to $\overline{\mathfrak{M}}_2$ blows G back down again to $\operatorname{Sym}^2 \sigma$. The morphism $\operatorname{Hilb}^2 S \to \overline{\mathfrak{M}}_2$ is then an isomorphism. Keeping track of the Chern classes gives the formula in Theorem 4.9.

Finally, we state a general conjecture:

Conjecture 4.13. If S has a section, then the map of Theorem 3.14 extends to an isomorphism $\operatorname{Hilb}^t S \to \overline{\mathfrak{M}}_t$.

If the conjecture is true, then the method of test surfaces used in the proof of Lemma 9.2 can be used to show that the μ-map is given by the following formula (where we use the notation introduced in the section on preliminaries for Part III for divisors in $\operatorname{Hilb}^t S$ as well):

$$\mu(\Sigma) = D_{\alpha_t} - \big((f \cdot \Sigma)/2\big)E,$$

where

$$\begin{aligned}\alpha_t &= \Sigma + \big((-(\sigma \cdot \Sigma) + (p_g - 1 + t)(f \cdot \Sigma))/2\big)f + (f \cdot \Sigma)\sigma \\ &= \alpha_1 + (t-1)\big((f \cdot \Sigma)/2\big)f.\end{aligned}$$

5. Calculation of the invariant for dimension two and no multiple fibers

Our goal in this and the following three sections will be a complete calculation of the Donaldson polynomial invariant $\gamma_{w,p}$ in case $-p-3\chi(\mathcal{O}_S)=2$. In this case, the moduli space is compact of real dimension four and complex dimension two, and may be identified with the algebraic surface $J^{e+1}(S)$. We shall begin with the case where S has a section σ and $e=-2$. We have already described how to calculate the invariant in this case in the last section. However, we shall give another method for doing so here, since it will serve to explain the construction in the general case.

To describe the μ-map, we begin by describing a universal bundle over S. Recall that every bundle V with $-p_1(\operatorname{ad} V)-3\chi(\mathcal{O}_S)=2$ is obtained from the fixed bundle V_0 by a single allowable elementary modification. For convenience we will look at the case where $e=-2$. Thus we shall normalize V_0 to have $\det V_0\cdot f=-3$ and

$$-p_1(\operatorname{ad} V_0)-p=c_1(V_0)^2-4c_2(V_0)=3(1+p_g(S)).$$

As V_0 is well defined up to twisting, so that we can assume that $c_1(V_0)=-3\sigma$ if $p_g(S)$ is odd, and $c_1(V_0)=-3\sigma+f$ if $p_g(S)$ is even. (Here we could use the explicit description of V_0 from the preceding section, or use the congruence $p\equiv 1+p_g \bmod 4$ to see that these choices always give $c_1(V_0)^2\equiv p \bmod 4$.) We shall just consider the argument in case p_g is odd. Setting $c=c_2(V_0)$, we have $4c-(-3\sigma)^2=3(1+p_g)$ and thus

$$c=-\frac{3}{2}(1+p_g).$$

If V is stable, with $-p_1(\operatorname{ad} V)=3(1+p_g)+2$, then there is an exact sequence

$$0\to V\to V_0\to Q\to 0,$$

where Q is a rank one torsion free sheaf on a fiber f with $\deg Q=-1$ and $\det V_0\cdot f=-3$, and conversely every such V is stable. We need to parametrize such sheaves Q as a family over $S\times S$, where the first factor should be viewed as the surface and the second as the moduli space. To do so, let π_1 and π_2 be the projections of $S\times S$ to the first and second factors, let $\mathbb{D}$ denote the diagonal inside $S\times S$ and let $D=S\times_{\mathbb{P}^1}S$ be the fiber product. Thus D is a Cartier divisor, which is not however smooth at the images of pairs of double points. At such a point D has the local equation $xy=zw$, and thus D has an ordinary double point in dimension three. The diagonal $\mathbb{D}$ is of course contained as a hypersurface in D, but this hypersurface fails to be Cartier at the singular points of D. Let $\mathcal{P}=I_{\mathbb{D}}/I_D$. In local analytic coordinates, $\mathcal{P}$ looks like

$$(x-z,y-w)R/(xy-zw)R$$

near the double point, where $R=\mathbb{C}\{x,y,z,w\}$. We claim that the sheaf $\mathcal{P}$ is flat over S (the second factor). Indeed there is an exact sequence

$$0\to I_{\mathbb{D}}/I_D\to\mathcal{O}_D\to\mathcal{O}_{\mathbb{D}}\to 0.$$

Moreover $\mathscr{O}_{\mathbb{D}}$ is obviously flat over S and $\mathscr{O}_D$ is flat over S since D is a local complete intersection inside $S \times S$. Thus $\mathscr{P}$ is flat over S also. Given $q \in S$ denote $\mathscr{P}|\pi_2^{-1}(q)$ by $\mathscr{P}_q$, where we shall identify $\mathscr{P}_q$ with the corresponding torsion sheaf on S. If q is not a singular point of a nodal fiber, then $\mathscr{P}_q = \mathscr{O}_f(-q)$, where f is the fiber containing q and we have identified $\mathscr{O}_f(-q)$ with its direct image on S under the inclusion. If q is the singular point of a singular fiber, then in local analytic coordinates $\mathscr{P}_q$ is given by

$$(x-z, y-w)R/(xy-zw, z, w)R \cong (x, y)\mathbb{C}\{x, y\}/(xy)\mathbb{C}\{x, y\}.$$

Thus globally $\mathscr{P}_q$ is the maximal ideal of q, in other words it is the unique torsion free rank one sheaf of degree -1 on the singular fiber which is not locally free.

Fix as above V_0 to be a stable rank two vector bundle on S of fiber degree -3 such that the restriction of V_0 to every fiber is stable. Thus as we have seen in Corollary 1.2 and Lemma 2.7(i), $\dim \operatorname{Hom}(V_0, \mathscr{P}_q) = h^0(V_0^\vee \otimes \mathscr{P}_q) = 1$ and $h^1(V_0^\vee \otimes \mathscr{P}_q) = 0$. It follows via flat base change as in the proof of Lemma 3.15 that $\pi_{2*}((\pi_1^* V_0)^\vee \otimes \mathscr{P})$ is a line bundle on S. We let $\mathscr{L}$ denote the dual line bundle. Thus

$$\begin{aligned} \operatorname{Hom}(\pi_1^* V_0, \mathscr{P} \otimes \pi_2^* \mathscr{L}) &= H^0(S \times S; (\pi_1^* V_0)^\vee \otimes \mathscr{P} \otimes \pi_2^* \mathscr{L}) \\ &= H^0(S; \pi_{2*}((\pi_1^* V_0)^\vee \otimes \mathscr{P}) \otimes \mathscr{L}) \\ &= H^0(S; \mathscr{L}^{-1} \otimes \mathscr{L}) = H^0(S; \mathscr{O}_S). \end{aligned}$$

Thus there is a nonzero map $\pi_1^* V_0 \to \mathscr{P} \otimes \pi_2^* \mathscr{L}$, essentially unique, and its restriction to each fiber $\pi_2^{-1}(q)$ is also nonzero. We may then define a universal bundle $\mathscr{V}'$ by the exact sequence

$$0 \to \mathscr{V}' \to \pi_1^* V_0 \to \mathscr{P} \otimes \pi_2^* \mathscr{L} \to 0.$$

Lemma 5.1. *The sheaf $\mathscr{V}'$ is locally free and its restriction to each slice $S \times \{q\}$ is a stable rank two vector bundle V_q with $-p_1(\operatorname{ad} V_q) - 3\chi(\mathscr{O}_S) = 2$. The resulting morphism $S \to \mathfrak{M}_1$ is an isomorphism.*

Proof. There is an exact sequence

$$0 \to V_q \to V_0 \to \mathscr{P}_q \to 0.$$

Thus V_q is locally free for all q and so is $\mathscr{V}'$. By construction V_q has stable restriction to every fiber except the one containing q. Thus V_q is stable. The statement about $p_1(\operatorname{ad} V_q)$ is clear. Finally, examining the description of Proposition 3.13, we see that the map $S \to \mathfrak{M}_1$ is a bijection. Since $\mathfrak{M}_1$ is smooth, the map is therefore an isomorphism. □

We now turn to calculating the Chern classes of $\mathscr{V}'$. By the lemma on elementary modifications,

$$p_1(\operatorname{ad} \mathscr{V}') - p_1(\operatorname{ad} \pi^* V_0) = 2c_1(V_0) \cdot D + [D]^2 - 4i_* c_1(\mathscr{P} \otimes \pi_2^* \mathscr{L}),$$

where $i: D \to S \times S$ is the inclusion. Here the sheaf $\mathscr{P} \otimes \pi_2^*\mathscr{L}$ fails to be a line bundle exactly at the singular points of D, which does not affect the Chern classes c_1 and c_2. Thus we can simply define $i_*c_1(\mathscr{P} \otimes \pi_2^*\mathscr{L})$ to be the unique extension of the class $i_*c_1(\mathscr{P} \otimes \pi_2^*\mathscr{L}|D_{\text{reg}})$. Next we claim:

Lemma 5.2. *In $H^2(S \times S)$, we have $[D] = f \otimes 1 + 1 \otimes f$.*

Proof. Let C be a Riemann surface embedded in S, and consider $([C] \otimes [x]) \cup [D]$, where x is a point of S. This is the same as $\#((C \times \{x\}) \cap D)$, where the points are counted with signs. Clearly this intersection is the same as $\#(C \cap f)$. A similar argument holds for $([x] \otimes [C]) \cap [D]$. Thus $[D]$ and $f \otimes 1 + 1 \otimes f$ define the same element of $H^2(S \times S)$. □

It follows that, up to a term not affecting the slant product,

$$p_1(\operatorname{ad}\mathscr{V}') - p_1(\operatorname{ad}\pi^* V_0) = -6\sigma \otimes f + 2f \otimes f - 4i_*c_1(\mathscr{P} \otimes \pi_2^*\mathscr{L}).$$

Next we must calculate the most interesting term in the expression for $p_1(\operatorname{ad}\mathscr{V}')$ above, the term $c_1(\mathscr{P} \otimes \pi_2^*\mathscr{L})$, viewed as a coherent sheaf on D. As far as c_1 is concerned, we can ignore the singularities of D. Thus

$$\begin{aligned} c_1(\mathscr{P} \otimes \pi_2^*\mathscr{L}|D_{\text{reg}}) &= c_1(I_{\mathbb{D}}/I_D|D_{\text{reg}}) + \pi_2^*c_1(\mathscr{L}) \\ &= -[\mathbb{D}] + \pi_2^*c_1(\mathscr{L}). \end{aligned}$$

Here $[\mathbb{D}]$ is viewed as a divisor on D_{reg}. However the unique extension of $i_*[\mathbb{D}]$ to an element of $H^4(S \times S)$ is clearly again $[\mathbb{D}]$, where we now view $\mathbb{D}$ as a codimension two cycle on $S \times S$. Now let $\alpha = c_1(\mathscr{L}^{-1}) \in H^2(S)$. Then

$$\begin{aligned} i_*\pi_2^*c_1(\mathscr{L}^{-1}) &= i_*i^*(1 \otimes \alpha) \\ &= i_*i^*(1) \cup (1 \otimes \alpha) = [D] \cup (1 \otimes \alpha) \\ &= f \otimes \alpha + 1 \otimes [f \cdot \alpha]. \end{aligned}$$

Thus up to a term which does not affect the slant product, $i_*\pi_2^*c_1(\mathscr{L}^{-1}) = f \otimes \alpha$. To calculate this term, we shall use the following lemma:

Lemma 5.3. $\alpha = c_1(\mathscr{L}^{-1}) = -3\sigma - \frac{5}{2}(p_g + 1)f$.

Proof. We shall apply the Grothendieck-Riemann-Roch theorem to calculate the Chern classes of

$$\mathscr{L}^{-1} = \pi_{2*}((\pi_1^* V_0)^\vee \otimes \mathscr{P}).$$

We have

$$\operatorname{ch}((\pi_2)_!((\pi_1^* V_0)^\vee \otimes \mathscr{P})) \operatorname{Todd} S = \pi_{2*}\left[\operatorname{ch}((\pi_1^* V_0)^\vee \otimes \mathscr{P}) \cdot \operatorname{Todd}(S \times S)\right].$$

Now $H^1(V_0^\vee \otimes Q) = 0$ for all Q a torsion free rank one sheaf on a fiber f, so that $(\pi_2)_!((\pi_1^* V_0)^\vee \otimes \mathscr{P}) = \pi_{2*}((\pi_1^* V_0)^\vee \otimes \mathscr{P}) = \mathscr{L}^{-1}$ and the left-hand side above is just $c_1(\mathscr{L}^{-1})\,\mathrm{Todd}\,S$. Now we can also multiply by $(\mathrm{Todd}\,S)^{-1}$ to get

$$\begin{aligned} c_1(\mathscr{L}^{-1}) &= \pi_{2*}\Big[\mathrm{ch}((\pi_1^* V_0)^\vee \otimes \mathscr{P})\cdot \mathrm{Todd}(S\times S)\Big]\cdot(\mathrm{Todd}\,S)^{-1} \\ &= \pi_{2*}\Big[\mathrm{ch}((\pi_1^* V_0)^\vee \otimes \mathscr{P})\cdot \mathrm{Todd}(S\times S)\cdot \pi_2^*(\mathrm{Todd}\,S)^{-1}\Big] \\ &= \pi_{2*}\Big[\mathrm{ch}((\pi_1^* V_0)^\vee \otimes \mathscr{P})\cdot \pi_1^*\,\mathrm{Todd}\,S\Big] \\ &= \pi_{2*}\Big[\mathrm{ch}(\pi_1^* V_0)^\vee \cdot \pi_1^*\,\mathrm{Todd}\,S\cdot \mathrm{ch}(\mathscr{P})\Big], \end{aligned}$$

using the multiplicativity of the Todd class. Moreover

$$\mathrm{ch}(V_0^\vee) = 2 - c_1(V_0) + \frac{c_1(V_0)^2 - 2c_2(V_0}{2} = 2 + 3\sigma - 6(1+p_g)[\mathrm{pt}]$$

and

$$\mathrm{Todd}\,S = 1 - \frac{(p_g-1)}{2}f + (p_g+1)[\mathrm{pt}].$$

So

$$\pi_1^*\,\mathrm{ch}(V_0^\vee)\cdot \pi_1^*\,\mathrm{Todd}\,S = 2 + 3\sigma\otimes 1 - (p_g-1)f\otimes 1 + N[\mathrm{pt}]\otimes 1,$$

where

$$N = \frac{(3\sigma)^2 - 2c}{2} + 2(p_g+1) - \frac{3}{2}(p_g-1) = \frac{-5p_g+1}{2},$$

using the fact that $c = -\frac{3}{2}(1+p_g)$.

Next we compute $\mathrm{ch}\,\mathscr{P} = \mathrm{ch}(I_{\mathbb{D}}/I_D) = \mathrm{ch}\,I_{\mathbb{D}} - \mathrm{ch}\,I_D$. Now $I_D = \mathscr{O}_{S\times S}(-D)$, so that $\mathrm{ch}\,I_D = 1 - [D] + [D]^2/2 - \cdots$. As for $\mathrm{ch}\,I_{\mathbb{D}}$, we have $\mathrm{ch}\,I_{\mathbb{D}} = 1 - \mathrm{ch}\,\mathscr{O}_{\mathbb{D}}$. Applying the Grothendieck-Riemann-Roch formula to the inclusion $j\colon \mathbb{D}\to S\times S$ gives $\mathrm{ch}\,\mathscr{O}_{\mathbb{D}} = j_*((\mathrm{Todd}\,N_{\mathbb{D}/S\times S})^{-1})$, where $N_{\mathbb{D}/S\times S}$ is the normal bundle of $\mathbb{D}$ in $S\times S$, and so is equal to the tangent bundle T_S on $\mathbb{D}$. Thus

$$\begin{aligned} \mathrm{ch}\,\mathscr{O}_{\mathbb{D}} &= j_*\Big((1 - \frac{(p_g-1)}{2}f + (1+p_g)[\mathrm{pt}])^{-1}\Big) \\ &= j_*(1 + \frac{(p_g-1)}{2}f - (1+p_g)[\mathrm{pt}]) \\ &= [\mathbb{D}] + \frac{(p_g-1)}{2}j_*f - (1+p_g)j_*[\mathrm{pt}]. \end{aligned}$$

Collecting up the terms through degree 3 (which are the only ones which will contribute) gives

$$\mathrm{ch}\,\mathscr{P} = [D] - \frac{[D]^2}{2} - [\mathbb{D}] - \frac{p_g-1}{2}j_*f + \cdots.$$

Putting this together, we see that α is the degree one term in

$$\pi_{2*}\Big[(2 + 3\sigma\otimes 1 - (p_g-1)f\otimes 1 + N[\mathrm{pt}]\otimes 1)\cdot([D] - \frac{[D]^2}{2} - [\mathbb{D}] - \frac{p_g-1}{2}j_*f)\Big].$$

Recalling that $D = f\otimes 1 + 1\otimes f$ and that $[D]^2/2 = f\otimes f$, we must apply π_{2*} to

$$\pi_{2*}((p_g-1)j_*f - 3(\sigma\otimes 1)\cdot[\mathbb{D}] + (p_g-1)(f\otimes 1)\cdot[\mathbb{D}] - 3(\sigma\otimes 1)\cdot(f\otimes f) + N[\mathrm{pt}]\otimes f).$$

The result is then

$$-(p_g - 1)f - 3f - 3\sigma + (p_g - 1)f + Nf = -3\sigma + (N-3)f,$$

as claimed. □

The above lemma thus implies that

$$-4i_* c_1(\mathscr{P} \otimes \pi_2^* \mathscr{L}) = 4[\mathbb{D}] - 12f \otimes \sigma - 10(p_g + 1)f \otimes f.$$

Putting this together gives (neglecting all terms which do not affect the slant product)

$$p_1(\operatorname{ad} \mathscr{V}') = -6(\sigma \otimes f) + (-10p_g - 8)f \otimes f - 12f \otimes \sigma + 4[\mathbb{D}] + \cdots .$$

We may finally summarize our calculations as follows:

Lemma 5.4. *In the above notation, using the bundle $\mathscr{V}'$ to identify $\mathfrak{M}_1$ with S and denoting by μ' the corresponding μ-map, we have*

$$-4\mu'(\Sigma) = \left[-6(\sigma \cdot \Sigma) + (-10p_g - 8)(f \cdot \Sigma)\right]f - 12(f \cdot \Sigma)\sigma + 4\Sigma.$$

Thus $\mu'(\Sigma)^2 = (\Sigma)^2 + (p_g - 1)(f \cdot \Sigma)^2$. □

At first glance, this formula looks quite different from the previous formula

$$-4\mu(\Sigma) = (2(\sigma \cdot \Sigma) - 2p_g(f \cdot \Sigma))f - 4(f \cdot \Sigma)\sigma - 4\Sigma.$$

However, the surface S (viewed as the moduli space) has an involution ι, coming from taking $x \mapsto -x$ on each fiber using σ as the identity section. This involution corresponds to viewing S as the double cover of a rational ruled surface as in [11], Chapter 1. Since S has only nodal singular fibers, it follows that on $H^2(S)$, ι fixes σ and f and is equal to $-\mathrm{Id}$ on the orthogonal complement $\{f, \sigma\}^\perp$. It is then an easy exercise to see that for a general Σ we have

$$\iota^*(\Sigma) = -\Sigma + 2\left[(\sigma \cdot \Sigma) + (p_g + 1)(f \cdot \Sigma)\right]f + 2(f \cdot \Sigma)\sigma.$$

Applying ι then exchanges $\mu(\Sigma)$ and $\mu'(\Sigma)$. Clearly this discrepancy arose as follows. In the general scheme for identifying the moduli space implicit in Theorems 3.14 and 4.7 we used not $\mathscr{P}$ but its dual. However it was technically slightly simpler not to make this choice in the Riemann-Roch calculation above. Thus the identifications of the moduli space differ by ι.

6. The case of multiple fibers

Having done the rather tedious calculation in the preceding section in case S has a section, we must now move on to deal with the case where S has multiple fibers. Fortunately, it will turn out that much of the calculation in this case exactly follows the pattern of the previous calculation. Before getting into the nitty-gritty, let us fix notation. Let $\pi\colon S \to \mathbb{P}^1$ be a nodal surface with at most two multiple fibers of odd multiplicity. Fix a divisor on the generic fiber S_η of odd degree $2e+1$. Let V_0 be a rank two vector bundle on S with

$c_1(V_0) = \Delta$ and $c_2(V_0) = c$, whose restriction to the reduction of every fiber is stable. Thus $4c - \Delta^2 = 3(p_g + 1)$ and so

$$2c = \frac{\Delta^2 + 3(p_g + 1)}{2}.$$

We would like to construct a universal bundle using $J^{e+1}(S)$. Unfortunately, this is not in general possible, and we shall instead use a finite cover. Thus we fix an elliptic surface T together with a map $T \to S$, such that T has a section. We may further assume that T is obtained as follows: choose a smooth multisection C of π, for example a general hyperplane section of S in some projective embedding. For C sufficiently general, we may assume that C meets the multiple fibers transversally and that the map $C \to \mathbb{P}^1$ is not branched at any points corresponding to singular nonmultiple fibers of π. Then set T to be the normalization of $S \times_{\mathbb{P}^1} C$. It follows that the only singular fibers of T lie over singular nonmultiple fibers of S, and that T has a section σ. If d is the degree of $C \to \mathbb{P}^1$, then at the point of $\mathbb{P}^1$ lying under the multiple fiber F_i of multiplicity m_i, $C \to \mathbb{P}^1$ is branched to order m_i at exactly d/m_i points.

Let $\varphi\colon T \to S$ be the natural map and $\rho\colon T \to C$ be the elliptic fibration, so that we have a commutative diagram

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, "\rho"'] & S \arrow[d, "\pi"] \\ C \arrow[r] & \mathbb{P}^1. \end{tikzcd}$$

Now we can state the main result of this section:

Theorem 6.1. *There exists a vector bundle $\tilde{\mathscr{V}}$ over $S \times T$ with the following properties*:

(i) *The restriction of $\tilde{\mathscr{V}}$ to each slice $S \times \{p\}$ is a stable rank two vector bundle V with* $\det V = \Delta - f$ *and* $-p_1(\operatorname{ad} V) - 3\chi(\mathscr{O}_S) = 2$.

(ii) *The morphism $T \to \mathfrak{M}_1$ induced by $\tilde{\mathscr{V}}$ has degree d.*

(iii) *If $\tilde{\mu}\colon H_2(S) \to H^2(T)$ is the map induced by the slant product with the class $-p_1(\operatorname{ad}\tilde{\mathscr{V}})/4$, then, setting $\delta = [\Delta]$,*

$$\begin{aligned} -4\tilde{\mu}(\Sigma) &= \left[\delta^2 - (1+p_g) - 4(e+2)^2(1+p_g) + 2 + c(e, m_1) + c(e, m_2)\right](f\cdot\Sigma)df \\ &\quad - 4(e+2)(\varphi^*\delta\cdot\sigma)(f\cdot\Sigma)f - 4(e+2)(\varphi^*\Sigma\cdot\sigma)f + 2d(\delta\cdot\Sigma)f \\ &\quad + 4(f\cdot\Sigma)\varphi^*\delta - 8(e+2)(f\cdot\Sigma)\sigma + 4\varphi^*\Sigma, \end{aligned}$$

where $c(e, m_i)$ depends only on m_i and e and on an analytic neighborhood of the multiple fiber, and not on S or p_g, and where $c(e, 1) = 0$.

We shall defer the proof of Theorem 6.1 to the next two sections. The constant $c(e, m_i)$ in fact might depend *a priori* on the particular choice of the multiple fiber. However, as we shall see from Theorem 6.3, the choice of the fiber and of e does not matter. Let us begin with a calculation of $\mu(\Sigma)^2$:

Lemma 6.2. *With notation as in Theorem* 6.1, *we have*

$$16\tilde{\mu}(\Sigma)^2 = 16d(\Sigma)^2 + 16d(p_g - 1 - c(e, m_1) - c(e, m_2))(f \cdot \Sigma)^2.$$

Thus as $\tilde{\mu}(\Sigma)^2 = d\mu(\Sigma)^2$, *we have*

$$\begin{aligned}\mu(\Sigma)^2 &= (\Sigma)^2 + (p_g - 1 - c(e, m_1) - c(e, m_2))(f \cdot \Sigma)^2 \\ &= (m_1 m_2)^2 (p_g - 1 - c(e, m_1) - c(e, m_2))(\kappa \cdot \Sigma)^2,\end{aligned}$$

where κ *is the primitive class such that* $m_1 m_2 \kappa = f$.

Proof. This is a tedious calculation. □

Theorem 6.3. *With notation as in the statement of Theorem* 6.1, *we have*

$$c(e, m_i) = -1 + \frac{1}{m_i^2}.$$

Proof. By symmetry it suffices to consider $i = 1$. Choose a general nodal rational elliptic surface S_0 with a single multiple fiber of multiplicity m_1. We can assume that an analytic neighborhood of the multiple fiber in S_0 is analytically isomorphic to a neighborhood of F_1 in S, which is possible since we assumed that the multiple fibers did not lie over branch points of the j-function of S. Since $m_1 | 2e+1$, there exists a divisor Δ on S_0 with $\Delta \cdot f = 2e+1$. Thus we may use S_0 to calculate $c(e, m_1)$. Now setting $p_g = 0$ and $m_2 = 1$ in the formula of Lemma 6.2 gives the coefficient of $(\kappa \cdot \Sigma)^2$ in the Donaldson polynomial: it is $(m_1)^2(-1 - c(e, m_1))$. On the other hand, S_0 is orientation-preserving diffeomorphic to a rational elliptic surface S_1 with a section, by a diffeomorphism ψ which carries κ to the class of a fiber. Using Lemma 2.5 of Part I, this diffeomorphism must then carry a (w, p)-suitable chamber for S_1 to a $(\psi^* w, p)$-suitable chamber for S_0. The Donaldson polynomial for S_1 and a (w, p)-suitable chamber is then sent under ψ^* to the $\pm$ the Donaldson polynomial for S_0 and a $(\psi^* w, p)$-suitable chamber. Normalizing the orientations so that the leading coefficients agree (these are both (Σ^2)), the coefficients of $(\kappa \cdot \Sigma)^2$ must agree also. We have already calculated the coefficient of $(\kappa \cdot \Sigma)^2$ for S_1 (by two different methods): it is -1. Thus

$$(m_1)^2(-1 - c(e, m_1)) = -1.$$

Hence $c(e, m_1) = -1 + 1/m_1^2$, as claimed. □

Thus we get the formula for $\mu(\Sigma)^2$ stated in (i) of Theorem 0.5 of the Introduction:

Corollary 6.4. *The two-dimensional Donaldson polynomial is given by the formula*

$$\begin{aligned}\mu(\Sigma)^2 &= (\Sigma)^2 + (m_1 m_2)^2 (p_g - 1 + 1 - \frac{1}{m_1^2} + 1 - \frac{1}{m_2^2})(\kappa \cdot \Sigma)^2 \\ &= (\Sigma)^2 + [(m_1 m_2)^2 (p_g + 1) - m_1^2 - m_2^2](\kappa \cdot \Sigma)^2. \quad \square\end{aligned}$$

For future reference we note the following lemma:

Lemma 6.5. *If f denotes the general fiber of $\mathfrak{M}_1 = J^{e+1}(S) \to \mathbb{P}^1$, then*
$$\mu(\Sigma) \cdot f = 2(f \cdot \Sigma).$$

Proof. It suffices to calculate $\tilde{\mu}(\Sigma) \cdot f$, where $\tilde{\mu}$ is as defined in Theorem 6.1(iii) and here f denotes a general fiber on T. But using the formula in Theorem 6.1(iii) gives
$$\tilde{\mu}(\Sigma) \cdot f = -(f \cdot \Sigma)(2e+1) + 2(e+2)(f \cdot \Sigma) - (f \cdot \Sigma) = 2(f \cdot \Sigma). \quad \square$$

7. Proof of Theorem 6.1: A Riemann-Roch calculation

We return to the notation of the preceding section. Our goal in this section will be to approximate the universal bundle by a coherent sheaf which is essentially an elementary modification of $\pi_1^* V_0$, where V_0 is as described at the beginning of the preceding section and π_i denotes the i^{th} projection now on $S \times T$. We have the map $\varphi: T \to S$ of elliptic surfaces covering the map $\rho: C \to \mathbb{P}^1$ of the base curves. Let Γ be the graph of φ in $S \times T$ and let H be the graph of the composition $\psi: T \xrightarrow{\rho} C \xrightarrow{\sigma} T \xrightarrow{\varphi} S$, where we view σ temporarily not as a curve in T but rather as a morphism. Let $D = S \times_{\mathbb{P}^1} T \subset S \times T$ and let $\tilde{D}$ be the normalization of D. Let $i: \tilde{D} \to S \times T$ be the natural map. The singularities of D are of two types. The first type consists of points (p, q) where $\varphi(q) = p$ and p and q are the singular points on a nodal fiber. At such points D has an ordinary double point as in the case where S has a section. The second type of singularity is along a multiple fiber F_i. At a point of $\mathbb{P}^1$ lying under F_i, the map $C \to \mathbb{P}^1$ is branched to order m_i. Thus, in local analytic coordinates x, y, z, w on $S \times T$ the divisor D has the local equation $x^{m_i} = z^{m_i}$. If R is the local ring of D at such a point and $\tilde{R}$ is its normalization, then the inclusion $R \subseteq \tilde{R}$ is given by
$$\mathbb{C}\{x, y, z, w\}/(x^{m_i} - z^{m_i}) \hookrightarrow \bigoplus_k \mathbb{C}\{x, y, w\},$$
where the map from R to the k^{th} factor in the direct sum is given by setting $z = \zeta^k x$ for $\zeta = e^{2\pi\sqrt{-1}/m_i}$. It follows that i is an immersion of schemes.

Let $\tilde{F}_i = \varphi^{-1}(F_i)$ and let E_i be a component of $\tilde{F}_i$. There is thus an induced map $\nu_i: E_i \to F_i$ which is étale of degree m_i. We also have maps $D \to T$ and $\tilde{D} \to D$. Clearly D and $\tilde{D}$ are flat over T (note that $\tilde{D}$ is smooth away from the images of pairs of double points). The calculations above for R and $\tilde{R}$ show that the scheme-theoretic fiber of D at a point $q \in E_i$ is F_i as a multiple fiber and that $i_* \mathcal{O}_{\tilde{D}}$ restricted to this fiber is $\nu_{i*} \mathcal{O}_{E_i}$.

Since a section cannot pass through a singular point of a fiber, the graph H avoids the double point singularities of D. Denote also by H the pullback of H to $\tilde{D}$. Then H is a Cartier divisor on $\tilde{D}$. Define a torsion sheaf on $S \times T$, supported on D, by the formula
$$\mathscr{P} = i_* \mathcal{O}_{\tilde{D}}(-\Gamma + (e+2)H).$$
This notation does not define $\mathscr{P}$ near the double points of D, but as H does not pass through the double points and $\tilde{D} = D$ in a neighborhood of the double

points we can just glue $\mathscr{P}$ to I_Γ/I_D at the double points. Equivalently we could just take the push-forward of the restriction of $i_*\mathscr{O}_{\tilde{D}}(-\Gamma+(e+2)H)$ to D_{reg}. Finally we shall let π_1 and π_2 denote the first and second projections on $S\times T$.

Lemma 7.1. *The sheaf* $\pi_{2*}((\pi_1^* V_0)^\vee \otimes \mathscr{P})$ *is a line bundle on* T, *whose dual is denoted* $\mathscr{L}$. *Moreover*

$$R^1\pi_{2*}((\pi_1^* V_0)^\vee \otimes \mathscr{P}) = 0.$$

Proof. Letting $h\colon \tilde{D} \to T$ and $j\colon \tilde{D} \to S\times S \xrightarrow{\pi_1} S$ be the natural maps, it is clear that

$$\pi_{2*}((\pi_1^* V_0)^\vee \otimes \mathscr{P}) = h_*((j^* V_0)^\vee \otimes \mathscr{O}_{\tilde{D}}(-\Gamma+(e+2)H)).$$

So we must check that the restriction of $(j^*V_0)^\vee \otimes \mathscr{O}_{\tilde{D}}(-\Gamma+(e+2)H)$ to each fiber of h has $h^0=1$ and $h^1=0$. The only new case is the case corresponding to a multiple fiber. In this case the restriction to the fiber is $(\nu_i^* V_0)^\vee \otimes L$, where L is a line bundle of degree $e+1$ on E_i. The degree of $\nu_i^* V_0$ is $m_i(\deg V/m_i) = 2e+1$ and $\nu_i^* V_0$ is stable since it is the pullback of the stable bundle $V_0|F_i$. Thus by Corollary 1.2, $H^0(E_i; (\nu_i^* V_0)^\vee \otimes L)$ has dimension one and $H^1(E_i; (\nu_i^* V_0)^\vee \otimes L) = 0$. □

Thus arguing as in the case of a section there is a unique nonzero map (mod scalars)

$$\pi_1^* V_0 \to \mathscr{P} \otimes \pi_2^* \mathscr{L}.$$

Unfortunately, if there are multiple fibers this map is no longer surjective. We shall return to this point in the next section. Our remaining goal in this section is to calculate $\mathscr{L}$:

Lemma 7.2. *With* $\mathscr{L}^{-1} = \pi_{2*}((\pi_1^* V_0)^\vee \otimes \mathscr{P})$ *and* $\delta = [\Delta]$, *we have*

$$c_1(\mathscr{L}^{-1}) = \left[\frac{\delta^2}{4} - \frac{1+p_g}{4} - (e+2)^2(1+p_g)\right] df - (e+2)(\varphi^*\delta\cdot\sigma)f + \varphi^*\delta - 2(e+2)\sigma.$$

Proof. As before we shall apply the Grothendieck-Riemann-Roch theorem to find $c_1(\pi_{2*}((\pi_1^* V_0)^\vee \otimes \mathscr{P}))$: it is the degree one term in

$$\pi_{2*}(\pi_1^* \operatorname{ch} V_0^\vee \cdot \pi_1^* \operatorname{Todd} S \cdot \operatorname{ch} i_*\mathscr{O}_{\tilde{D}}(-\Gamma+(e+2)H)).$$

We have

$$\operatorname{ch} V_0^\vee = 2 - \delta + \left(\frac{\delta^2 - 2c}{2}\right)[\mathrm{pt}],$$

where $\delta = [\Delta]$, and

$$\operatorname{Todd} S = 1 + \frac{r}{2} f + (1+p_g)[\mathrm{pt}],$$

where

$$-r = (p_g+1) - \frac{1}{m_1} - \frac{1}{m_2}.$$

Thus the product of the first two terms above is $\pi_1^*(2-\delta+rf+M[\mathrm{pt}])$, where

$$M = \frac{\delta^2-2c}{2} + 2(1+p_g) - \frac{r}{2}(2e+1).$$

Since we have

$$\delta^2 - 2c = \frac{\delta^2-4c}{2} + \frac{\delta^2}{2} = -\frac{3}{2}(1+p_g) + \frac{\delta^2}{2},$$

we can rewrite this as

$$M = \frac{\delta^2}{4} + \frac{5}{4}(1+p_g) - \frac{r}{2}(2e+1).$$

Next we must calculate $\mathrm{ch}\, i_*\mathcal{O}_{\tilde{D}}(-\Gamma+(e+2)H)$. Again using the Grothendieck-Riemann-Roch theorem, and setting $G = -\Gamma + (e+2)H$ for notational simplicity, we have, at least in the complement of the double points of D,

$$\mathrm{ch}\, i_*\mathcal{O}_{\tilde{D}}(G) = i_*[\mathrm{ch}\,\mathcal{O}_{\tilde{D}}(G)\cdot(\mathrm{Todd}\, N_i)^{-1}],$$

where N_i is the normal bundle to the immersion i. Now $\mathrm{ch}\,\mathcal{O}_{\tilde{D}}(G) = 1 + G + G^2/2 + \cdots$. As for N_i, locally near the multiple fiber F_i, D is the union of m_i sheets, and so

$$N_i = \mathcal{O}_{\tilde{D}}(D - (m_1-1)B_1 - (m_2-1)B_2),$$

where $B_i = F_i \times \tilde{F}_i$. It follows that

$$(\mathrm{Todd}\, N_i)^{-1} = 1 - \frac{D-(m_1-1)B_1-(m_2-1)B_2}{2} + \cdots$$

and so

$$\begin{aligned}\mathrm{ch}\, i_*\mathcal{O}_{\tilde{D}}(G) = D + G &- i_*\left(\frac{D-(m_1-1)B_1-(m_2-1)B_2}{2}\right) + i_*\left(\frac{G^2}{2}\right)\\ &- i_*\left(\frac{G\cdot(D-(m_1-1)B_1-(m_2-1)B_2)}{2}\right) + \cdots.\end{aligned}$$

So we must take the degree three term in the product of the above expression with $\pi_1^*(2-\delta+rf+M[\mathrm{pt}])$ and then apply π_{2*}. First, a calculation along the lines of Lemma 5.2 shows that

$$[D] = f\otimes 1 + d(1\otimes f),$$

where f denotes either the class of a fiber in S or T, depending on the context. The degree three term above is then a sum of three terms: $T_1+T_2+T_3$, where

$$\begin{aligned}T_1 &= M([\mathrm{pt}]\otimes 1)\cdot D,\\ T_2 &= -G\cdot(\delta\otimes 1) + G\cdot(rf\otimes 1)\\ &\quad - \frac{1}{2}i_*(D-(m_1-1)B_1-(m_2-1)B_2)\cdot(-\delta\otimes 1 + rf\otimes 1),\\ T_3 &= i_*(G^2 - G\cdot i^*D + (m_1-1)(G\cdot B_1) + (m_2-1)(G\cdot B_2)).\end{aligned}$$

Let us now apply π_{2*} to these terms. First

$$\pi_{2*}T_1 = \pi_{2*}(Md)[\mathrm{pt}]\otimes f = (Md)f.$$

To calculate $\pi_{2*}T_2$, first note the following, whose proof is an easy verification:

Lemma 7.3. *For every* $\alpha \in H^2(S)$,

$$\pi_{2*}(\Gamma \cdot \alpha \otimes 1) = \varphi^* \alpha;$$
$$\pi_{2*}(H \cdot \alpha \otimes 1) = (\varphi^* \alpha \cdot \sigma) f. \quad \square$$

So the terms involving G in $\pi_{2*}T_2$ give

$$\begin{aligned} &-(e+2)(\varphi^*\delta \cdot \sigma)f + \varphi^*\delta + (e+2)r(\varphi^* f \cdot \sigma)f - r\varphi^* f \\ &= -(e+2)(\varphi^*\delta \cdot \sigma)f + \varphi^*\delta + (e+1)rdf, \end{aligned}$$

where we have used $\varphi^* f = df$.

To handle the terms involving B_i, note that $i_*[B_i] = m_i[F_i \times \tilde{F}_i]$. Also $[F_i] = (1/m_i)f$ and $\tilde{F}_i$ consists of d/m_i copies of f (the fiber on T) so that

$$[F_i \times \tilde{F}_i] = \left(\frac{d}{m_i^2}\right) f \otimes f; \qquad i_*[B_i] = \frac{d}{m_i} f \otimes f.$$

Also $i_*D = D^2 = 2d(f \otimes f)$. Thus

$$-\frac{1}{2} i_*(D - (m_1 - 1)B_1 - (m_2 - 1)B_2) = -\frac{d}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(f \otimes f).$$

The product of this term with $f \otimes 1$ is zero, and we are left with the product with $-\delta \otimes 1$, which contributes

$$\frac{d(2e+1)}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) f.$$

Combining these, we see that

$$\pi_{2*}T_2 = -(e+2)(\varphi^*\delta \cdot \sigma)f + \varphi^*\delta + (e+1)rdf + \frac{d(2e+1)}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) f.$$

We turn now to the term $\pi_{2*}T_3$. We have $G^2 = (e+2)^2 H^2 - 2(e+2)H \cdot \Gamma + \Gamma^2$. To calculate π_{2*} applied to these terms, we shall use the following lemma:

Lemma 7.4. (i) $\pi_{2*} i_* H^2 = \pi_{2*} i_* \Gamma^2 = -d(1 + p_g) f$.

(ii) $\pi_{2*} i_* H \cdot \Gamma = \sigma$.

Proof. To see (i), note that we have an exact sequence

$$0 \to N_{\Gamma/\tilde{D}} \to N_{\Gamma/S \times T} \to N_i \to 0.$$

Also $(\Gamma^2)_{\tilde{D}} = \phi_* c_1(N_{\Gamma/\tilde{D}})$, where $\phi\colon \Gamma \to \tilde{D}$ is the inclusion. Now

$$\begin{aligned} c_1(N_{\Gamma/\tilde{D}}) &= c_1(N_{\Gamma/S \times T}) - c_1(N_i) \\ &= c_1(\pi_1^* T_S | \Gamma) - (D - (m_1 - 1)B_1 - (m_2 - 1)B_2) | \Gamma \\ &= \varphi^*(rf) - ((f \otimes 1 + d(1 \otimes f)) \cdot \Gamma - (m_1 - 1)(B_1 \cdot \Gamma) - (m_2 - 1)(B_2 \cdot \Gamma)) \\ &= \left[-d\left(p_g + 1 - \frac{1}{m_1} - \frac{1}{m_2}\right) - \left(2d - \frac{d(m_1 - 1)}{m_1} - \frac{d(m_2 - 1)}{m_2}\right)\right] f \\ &= -d(p_g + 1)f. \end{aligned}$$

Thus $\pi_{2*}i_*\Gamma^2 = -d(1+p_g)f$. A similar calculation handles $\pi_{2*}i_*H^2$. The proof of (ii) is an easy calculation. □

Thus

$$\pi_{2*}G^2 = -d(p_g+1)((e+2)^2+1)f - 2(e+2)\sigma.$$

The remaining term is $-\pi_{2*}(G\cdot(D-(m_1-1)B_1-(m_2-1)B_2))$. We have seen in the course of the proof of Lemma 7.4 that

$$\begin{aligned}&\pi_{2*}\Gamma\cdot(D-(m_1-1)B_1-(m_2-1)B_2)\\ &\quad = \pi_{2*}H\cdot(D-(m_1-1)B_1-(m_2-1)B_2) = \left(\frac{1}{m_1}+\frac{1}{m_2}\right)df.\end{aligned}$$

Thus

$$\pi_{2*}G\cdot(D-(m_1-1)B_1-(m_2-1)B_2) = d(e+1)\left(\frac{1}{m_1}+\frac{1}{m_2}\right)f.$$

In all then,

$$\pi_{2*}T_3 = d\left[-(p_g+1)((e+2)^2+1)-(e+1)\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right]f - 2(e+2)\sigma.$$

Combining terms, we have

$$c_1(\mathscr{L}^{-1}) = \left[\frac{\delta^2}{4}-\frac{1+p_g}{4}-(e+2)^2(1+p_g)\right]df-(e+2)(\varphi^*\delta\cdot\sigma)f+\varphi^*\delta-2(e+2)\sigma,$$

as claimed. This concludes the proof of Lemma 7.2. □

8. Proof of Theorem 6.1: Conclusion

We keep the notation of the two previous sections. We begin by constructing a "universal bundle" $\tilde{\mathscr{V}}$ over $S\times T$. Begin with the morphism $\pi_1^*V_0\to\mathscr{P}\otimes\pi_2^*\mathscr{L}$ defined in the previous section, and let $\tilde{\mathscr{V}}$ be the kernel. By construction $\tilde{\mathscr{V}}$ is locally free away from $(F_1\times\tilde{F}_1)\amalg(F_2\times\tilde{F}_2)$. There is an exact sequence:

$$0\to\tilde{\mathscr{V}}\to\pi_1^*V_0\to\mathscr{P}\otimes\pi_2^*\mathscr{L}\to\mathscr{Q}_1\oplus\mathscr{Q}_2\to 0,$$

where $\mathscr{Q}_i$ is supported on $F_i\times\tilde{F}_i$. Now $\tilde{F}_i$ is a disjoint union of d/m_i fibers of T. Let $\mathscr{Q}=\mathscr{Q}_1\oplus\mathscr{Q}_2$ and let c denote the total Chern polynomial. Then

$$c(\tilde{\mathscr{V}}) = \pi_1^*c(V_0)\cdot c(\mathscr{P}\otimes\pi_2^*\mathscr{L})^{-1}\cdot c(\mathscr{Q}).$$

Thus if we let $\pi_1^*c(V_0)\cdot c(\mathscr{P}\otimes\pi_2^*\mathscr{L})^{-1} = 1+x_1+x_2+\cdots$, then

$$c_2(\tilde{\mathscr{V}}) = x_2+c_2(\mathscr{Q});$$
$$c_1(\tilde{\mathscr{V}})^2-4c_2(\tilde{\mathscr{V}}) = x_1^2-4x_2-4c_2(\mathscr{Q}).$$

Now we claim that Theorem 6.1 is a consequence of the following two results:

Theorem 8.1. *There exist integers $q(e, m_i)$ such that*

$$c_2(\mathscr{Q}_i) = dq(e, m_i)[F_i \times f].$$

Here the integer $q(e, m_i)$ depends only on an analytic neighborhood of F_i and e but not on S or $p_g(S)$.

Theorem 8.2. *The coherent sheaf $\tilde{\mathscr{V}}$ is locally free.*

Proof that Theorems 8.1 *and* 8.2 *imply Theorem* 6.1. Let us consider the restriction of $\tilde{\mathscr{V}}$ to a slice $S \times \{q\}$. In all cases this restriction is a vector bundle V whose restriction to every smooth fiber f of S not equal to the fiber containing $\varphi(q)$ is $V_0|f$. Thus the restriction of V to such a fiber f is stable, and so V is stable by Theorem 3.4. Now if $\varphi(q)$ does not lie on a multiple fiber, there is an exact sequence

$$0 \to V \to V_0 \to Q \to 0,$$

where Q is the direct image of the line bundle on f corresponding to the divisor $(e+2)\psi(q) - \varphi(q)$, which has degree $e+1$. Thus $c_1(V) = \Delta - f$ and $p_1(\operatorname{ad} V) = p_1(\operatorname{ad} V_0) - 2$. This establishes (i) of Theorem 6.1. Note also that the map $q \mapsto (e+2)\psi(q) - \varphi(q)$ defines a rational map from T to $J^{e+1}(S)$ (which in fact is a morphism) and the map $T \to \mathfrak{M}_1$ factors through the map $T \to J^{e+1}(S)$, compatibly with the identification of a dense open subset of $J^{e+1}(S)$ with a dense open subset of $\mathfrak{M}_1$ given in Theorem 3.14.

Next let us calculate the degree of the induced morphism $T \to \mathfrak{M}_1$. Fix a general smooth fiber f of S, a line bundle L on f of degree $e+1$ and a vector bundle V which is uniquely specified by an exact sequence

$$0 \to V \to V_0 \to i_*L \to 0,$$

where $i\colon f \to S$ is the inclusion. We shall count the preimage of V in T. If f is general, then $T \to S$ is unbranched over f and the preimage of f consists of d distinct fibers $f_1, \dots, f_d$. Moreover φ restricts to an isomorphism from f_i to f for each i. The image of f_i under ψ is a single point $p_i \in f$ corresponding to the point $\sigma \cap f_i$. Now clearly there is a unique point $q_i \in f_i$ such that

$$L = \mathscr{O}_f((e+2)p_i - \varphi(q_i)).$$

Thus the preimage of V consists of d distinct points, and so the map $T \to \mathfrak{M}_1$ has degree d.

Lastly we must calculate $p_1(\operatorname{ad}\tilde{\mathscr{V}})$. We begin by calculating $\pi_1^* c(V_0) \cdot c(\mathscr{P} \otimes \pi_2^* \mathscr{L})^{-1}$. Here $\pi_1^* c(V_0) = 1 + \pi_1^*\delta + \pi_1^* c[\mathrm{pt}]$. As for the term $c(\mathscr{P} \otimes \pi_2^* \mathscr{L})$, we clearly have $c_1(\mathscr{P} \otimes \pi_2^* \mathscr{L}) = D = (f \otimes 1) + d(1 \otimes f)$. On the other hand, with the notation of Section 7 we may apply the Grothendieck-Riemann-Roch theorem to the immersion $i\colon \tilde{D} \to S \times T$ to obtain

$$\operatorname{ch}(\mathscr{P} \otimes \pi_2^* \mathscr{L}) = i_*\left[\operatorname{ch}\mathscr{O}_{\tilde{D}}((e+2)H - \Gamma)(\operatorname{Todd} N_i)^{-1} \cdot \pi_2^* \operatorname{ch}\mathscr{L})\right].$$

A calculation similar to those in Section 7 shows that this is equal to

$$i_*\left[1 + (e+2)H - \Gamma - \pi_2^*\alpha - \frac{D - (m_1 - 1)B_1 - (m_2 - 1)B_2}{2} + \cdots\right],$$

where $\alpha = c_1(\mathscr{L}^{-1})$ has been calculated in Lemma 7.2, and further manipulation gives

$$-2c_2(\mathscr{P}\otimes\pi_2^*\mathscr{L}) = -2[D]^2+2\left[(e+2)H-\Gamma-\pi_2^*\alpha\cdot[D]-d\left(1-\frac{1}{m_1}+1-\frac{1}{m_2}\right)(f\otimes f)\right].$$

Recalling that $\pi_1^*c(V_0)\cdot c(\mathscr{P}\otimes\pi_2^*\mathscr{L})^{-1} = 1+x_1+x_2+\cdots$, we have

$$(1+x_1+x_2+\cdots)(1+[D]+c_2(\mathscr{P}\otimes\pi_2^*\mathscr{L})) = 1+\pi_1^*\delta+\pi_1^*c[\mathrm{pt}].$$

Thus $x_1 = \pi_1^*\delta-[D]$ and

$$x_2 = \pi_1^*c[\mathrm{pt}]-\pi_1^*\delta\cdot[D]+[D]^2-c_2(\mathscr{P}\otimes\pi_2^*\mathscr{L}).$$

A calculation then shows that

$$x_1^2-4x_2 = \pi_1^*p_1(\mathrm{ad}\, V_0)+2\pi_1^*\delta\cdot[D]+[D]^2-4(e+2)[H]+4[\Gamma]+4\pi_2^*\alpha\cdot[D] + 4\left(1-\frac{1}{m_1}+1-\frac{1}{m_2}\right)d(f\otimes f).$$

There are correction terms $b(m_i) = 1-1/m_i$ depending on the multiple fibers. Now

$$\begin{aligned} p_1(\mathrm{ad}\,\tilde{\mathscr{V}}) &= x_1^2-4x_2-4c_2(\mathscr{Q}) \\ &= \pi_1^*p_1(\mathrm{ad}\, V_0)+2\pi_1^*\delta\cdot[D]+[D]^2-4(e+2)[H]+4[\Gamma]+4\pi_2^*\alpha\cdot[D] \\ &\quad + 4(b(m_1)-q(e, m_1)/m_1+b(m_2)-q(e, m_2)/m_2)d(f\otimes f), \end{aligned}$$

where the terms $b(m_i)$, $q(e, m_i)$ depend only on an analytic neighborhood of the multiple fiber and are both 0 if $m_i = 1$. Let

$$c(e, m_i) = 4(b(m_i)-q(e, m_i)/m_i).$$

Taking the slant product of this expression with $[\Sigma]$, using the fact that $[\Gamma]\backslash[\Sigma] = \varphi^*\Sigma$ and $[H]\backslash[\Sigma] = (\varphi^*\Sigma\cdot\sigma)f$, and plugging in the expression for α given by Lemma 7.2 gives the final formula in Theorem 6.1(iii). □

Proof of Theorem 8.1. Choose an analytic neighborhood X of F_i. We may assume that X fibers over the unit disk in $\mathbb{C}$. Then $\varphi^{-1}(X)$ consists of d/m_i copies of $\tilde{X}$, which is the normalization of the pullback of X by the map from the disk to itself defined by $z = w^{m_i}$. Restrict φ and V_0 to this local situation, and let D now denote the fiber product inside $X\times\tilde{X}$ and $\tilde{D}$ its normalization. We can similarly define the codimension two subsets Γ and H. Let us examine the dependence of the terms V_0 and $\mathscr{O}_{\tilde{D}}((e+2)H-\Gamma)$ on the various choices.

First, suppose that V_0 and V_0' are two different choices of a bundle over X whose determinants have fiber degree $2e+1$ and whose restrictions to the reduction of every fiber are stable. Then $\det V_0\otimes(\det V_0')^{-1}$ has fiber degree zero. On the other hand, from the exponential sheaf sequence

$$H^1(X;\mathscr{O}_X)\to \mathrm{Pic}\, X\to H^2(X;\mathbb{Z})$$

and the identification $H^2(X;\mathbb{Z}) \cong H^2(F_i;\mathbb{Z}) \cong \mathbb{Z}$, it follows that the group of line bundles of fiber degree zero is divisible. Thus there is a line bundle L on X such that $\det V_0' = \det(V_0 \otimes L)$. The proof of Corollary 3.8 shows that V_0' and $V_0 \otimes L$ differ by twisting by a line bundle pulled back from the disk, which is necessarily trivial. Thus $V_0' \cong V_0 \otimes L$.

The remaining choice was the choice of a section σ of $\tilde{X}$. Given two such choices σ_1 and σ_2, we have two divisors H_1 and H_2 on $\tilde{D}$, and two line bundles $\mathcal{O}_{\tilde{D}}((e+2)H_1 - \Gamma)$ and $\mathcal{O}_{\tilde{D}}((e+2)H_2 - \Gamma)$. Their difference is the line bundle $\mathcal{O}_{\tilde{D}}((e+2)(H_1 - H_2))$. The restriction of $\mathcal{O}_{\tilde{D}}((e+2)H_i - \Gamma)$ to each fiber f of the map $\tilde{D} \to \tilde{X}$ over $q \in \tilde{X}$ is the line bundle $\mathcal{O}_f((e+2)p_i - q)$, where $p_i = \sigma_i \cap f$ and we can identify the fiber over q with the fiber on $\tilde{X}$ containing q via φ. Let $\Psi\colon \tilde{X} \to \tilde{X}$ be the inverse of the map given by translation by the divisor of fiber degree zero $(e+2)(\sigma_1 - \sigma_2) - c_1(\tilde{L})$, where $\tilde{L}$ is the pullback to $\tilde{X}$ of L. Thus $\Psi^{-1}(q) = q + (e+2)(p_1 - p_2) - \lambda$, where $q \in f$ and λ is the line bundle $L|f$. Now $\mathrm{Id} \times \Psi$ acts on $X \times \tilde{X}$, preserving the divisor D and acting as well on the normalization $\tilde{D}$. Clearly the line bundles $\mathcal{O}_{\tilde{D}}((e+2)H_2 - \Gamma) \otimes \pi_1^* L$ and $(\mathrm{Id} \times \Psi)^* \mathcal{O}_{\tilde{D}}((e+2)H_1 - \Gamma)$ have isomorphic restrictions to each fiber of the map $\tilde{D} \to \tilde{X}$. Thus they differ by the pullback of a line bundle L' on $\tilde{X}$. Thus we have an isomorphism

$$(\mathrm{Id} \times \Psi)^*(\pi_1^* V_0^\vee \otimes i_* \mathcal{O}_{\tilde{D}}((e+2)H_1 - \Gamma)) \cong (\pi_1^* V_0)^\vee \otimes i_* \mathcal{O}_{\tilde{D}}((e+2)H_2 - \Gamma) \otimes \pi_1^* L \otimes \pi_2^* L'$$

and a similar isomorphism when we apply $R^0\pi_{2*}$. Lastly every map $\pi^* V_0 \to i_* \mathcal{O}_{\tilde{D}}((e+2)H_1 - \Gamma)$ which corresponds to an everywhere generating section of the line bundle $\pi_{2*} Hom(\pi_1^* V_0, i_* \mathcal{O}_{\tilde{D}}((e+2)H_1 - \Gamma))$ under the natural map

$$\pi_2^* \pi_{2*} Hom(\pi_1^* V_0, i_* \mathcal{O}_{\tilde{D}}((e+2)H_1 - \Gamma)) \to Hom(\pi_1^* V_0, i_* \mathcal{O}_{\tilde{D}}((e+2)H_1 - \Gamma))$$

is determined up to multiplication by a nowhere vanishing function on $\tilde{X}$. It now follows that, up to twisting by the pullback of the line bundle L' on $\tilde{X}$, we may identify the map $\pi_1^* V_0' \to i_* \mathcal{O}_{\tilde{D}}((e+2)H_2 - \Gamma)$, up to a nowhere vanishing function on $\tilde{X}$ and up to twisting by the pullback of a line bundle on $\tilde{X}$, with the pullback under $(\mathrm{Id} \times \Psi)^*$ of the corresponding map from $\pi_1^* V_0$ to $i_* \mathcal{O}_{\tilde{D}}((e+2)H_1 - \Gamma)$. In particular the cokernels of these maps, viewed as sheaves supported on $F_i \times E_i$, have the same length. But the lengths of the cokernels are exactly what is needed to calculate $c_2(\mathcal{Q}_i)$, in the notation of the beginning of this section. Thus we have established Theorem 8.1. □

Remark. We could easily show directly by a slight modification of the proof above that the integers $q(e, m_i)$ defined above are independent of e.

Proof of Theorem 8.2. We begin with the following (see also (A.2)(i)):

Lemma 8.3. *The sheaf $\tilde{\mathcal{V}}$ is reflexive.*

Proof. Since $\tilde{\mathcal{V}}$ is a subsheaf of the locally free sheaf $\pi_1^* V_0$, it is torsion free. Thus it will suffice to show that every section τ of $\tilde{\mathcal{V}}$ defined on an open set of the form $W - Z$, where W is an open subset of $S \times T$ and Z is a closed subvariety of W of codimension at least two, extends to a section of $\tilde{\mathscr{V}}$ over W. Now locally (after possibly shrinking W) $\tilde{\mathscr{V}}$ is given by an exact sequence

$$0 \to \tilde{\mathscr{V}}|W \to \mathscr{O}_W^2 \to i_*\mathscr{O}_{\tilde{D}}|W.$$

Now viewing the section τ as a section of $\mathscr{O}_W^2$ over $W - Z$, it extends as a section of $\mathscr{O}_W^2$ by Hartogs' theorem. Let $\tilde{\tau}$ be the unique extension. Then the image of $\tilde{\tau}$ in $i_*\mathscr{O}_{\tilde{D}}|W$ vanishes on $D - Z$, which is nonempty. Clearly then it is zero. Thus the extension $\tilde{\tau}$ defines a section of $\tilde{\mathscr{V}}$ extending τ, so that $\tilde{\mathscr{V}}$ is reflexive. □

Returning to the proof of Theorem 8.2, let $U = S \times T - (F_1 \times \tilde{F}_1) - (F_2 \times \tilde{F}_2)$. By Lemma 8.3, $\tilde{\mathscr{V}}$ is a reflexive sheaf which is locally free on U. We claim that $\tilde{\mathscr{V}}$ is everywhere locally free. The problem is local around each point (x, y) of $F_i \times \tilde{F}_i$. Since $\tilde{\mathscr{V}}$ is reflexive, it will suffice to show the following: each point y of $\tilde{F}_i$ has a neighborhood $\mathscr{N}$ such that $\tilde{\mathscr{V}}|(S \times \mathscr{N}) \cap U$ has an extension to a locally free sheaf over $S \times \mathscr{N}$.

Let $T_0 = T - \tilde{F}_1 - \tilde{F}_2$. Clearly T_0 is the inverse image of $J^{e+1}(S) - F_1 - F_2$ under the natural morphism from T to $J^{e+1}(S)$. The restriction of $\mathscr{V}_U$ to $S \times T_0$ is a bundle over $S \times T_0$ in the sense of schemes since it is the restriction of a coherent sheaf over $S \times T$. Thus it induces a morphism of schemes from T_0 to $\mathfrak{M}_1$. If we denote the points of $\mathfrak{M}_1$ corresponding to multiple fibers by F_1 and F_2 again, then it is easy to see that the map of Theorem 3.14 extends to an embedding $J^{e+1}(S) - F_1 - F_2 \to \mathfrak{M}_1 - (F_1 \cup F_2)$. Thus the map $T_0 \to \mathfrak{M}_1 - (F_1 \cup F_2)$ is proper. This map extends to a rational map from T to $\mathfrak{M}_1$. After blowing up T, there is a morphism from the blowup $\tilde{T}$ to $\mathfrak{M}_1$. The image of $\tilde{T} - T_0$ must clearly lie inside the two elliptic curves in $\mathfrak{M}_1$ corresponding to elementary modifications along F_1 or F_2. Since there are no nonconstant maps from $\mathbb{P}^1$ to an elliptic curve, every exceptional curve on $\tilde{T}$ is mapped to a point, and the map $T_0 \to \mathfrak{M}_1$ extends to a morphism $\Phi: T \to \mathfrak{M}_1$. Clearly the morphism $\Phi: T \to \mathfrak{M}_1$ identifies $\mathfrak{M}_1$ with $J^{e+1}(S)$.

Given $y \in \tilde{F}_i$, choose a neighborhood N_0 of $\Phi(y)$ in $\mathfrak{M}_1$ such that there exists a universal bundle over $S \times N_0$, and let $\mathscr{N}$ be the component of $\Phi^{-1}(N_0)$ containing y. Thus there is a universal vector bundle $\mathscr{W}$ over $S \times \mathscr{N}$. By construction $\mathscr{W}|S \times (\mathscr{N} - \tilde{F}_i)$ and $\tilde{\mathscr{V}}|S \times (\mathscr{N} - \tilde{F}_i)$ have isomorphic restrictions to every slice $S \times \{z\}$. Thus $\pi_{2*}Hom(\mathscr{W}, \tilde{\mathscr{V}})$ is a torsion free rank one sheaf on $\mathscr{N}$, which is thus an ideal sheaf on $\mathscr{N}$ if $\mathscr{N}$ is small enough. We may assume that $\pi_{2*}Hom(\mathscr{W}, \tilde{\mathscr{V}})|\mathscr{N} - \{y\}$ is just the structure sheaf. Choosing an everywhere generating section of $\pi_{2*}Hom(\mathscr{W}, \tilde{\mathscr{V}})|\mathscr{N} - \{y\}$ gives a homomorphism $\mathscr{W}|S \times (\mathscr{N} - \{y\}) \to \tilde{\mathscr{V}}|S \times (\mathscr{N} - \{y\})$. This homomorphism is an isomorphism over $S \times (\mathscr{N} - \tilde{F}_i)$ and is nonzero at a general point of $S \times ((\mathscr{N} - \{y\}) \cap \tilde{F}_i)$. As both $\mathscr{W}$ and $\tilde{\mathscr{V}}$ are vector bundles away from $F_i \times (\mathscr{N} \cap \tilde{F}_i)$ whose restrictions to every smooth fiber of S in every slice are stable, it follows that $\mathscr{W}|S \times (\mathscr{N} - \{y\}) \to \tilde{\mathscr{V}}|S \times (\mathscr{N} - \{y\})$ is an isomorphism in codimension one. Since both sheaves are reflexive, they are isomorphic. Finally $\mathscr{W}$ and $\tilde{\mathscr{V}}$ are two reflexive sheaves which are isomorphic on the complement of the codimension two set $S \times \{y\} \subset S \times \mathcal{N}$, so they are isomorphic. Thus $\tilde{\mathscr{V}}$ is locally free. $\square$

9. The four-dimensional invariant

Our goal in this section will be to calculate the four-dimensional invariant. What follows is an outline of the calculation. Let $\overline{\mathfrak{M}}_2$ denote the moduli space of Gieseker stable torsion free sheaves on S of dimension four. As we have seen, $\overline{\mathfrak{M}}_2$ is smooth and irreducible and birational to $\mathrm{Hilb}^2 J^{e+1}(S)$. In fact, we shall begin by establishing a more precise statement. Let $Y_i \subset \mathrm{Hilb}^2 J^{e+1}(S)$ be the subset of codimension two consisting of subschemes of $J^{e+1}(S)$ whose support has reduction contained in the multiple fiber F_i on $J^{e+1}(S)$. Clearly Y_i has two components: one component is just $\mathrm{Sym}^2 F_i$, the closure of the locus of two distinct points lying on F_i, and the other is a $\mathbb{P}^1$-bundle over F_i corresponding to nonreduced subschemes whose support is a point on F_i. There is a similar subscheme Y_i' of $\overline{\mathfrak{M}}_2$, consisting of torsion free sheaves V on S such that either V is not locally free and the unique point where V is not locally free lies on F_i or V is a bundle obtained from V_0 up to equivalence by taking two elementary modifications along line bundles on F_i. We claim:

Lemma 9.1. *The isomorphism defined in Theorem* 3.14 *from a Zariski open subset of* $\mathrm{Sym}^2 J^{e+1}(S)$ *to an open subset of* $\mathfrak{M}_2$ *extends to an isomorphism* $\mathrm{Hilb}^2 J^{e+1}(S) - Y_1 - Y_2 \to \overline{\mathfrak{M}}_2 - Y_1' - Y_2'$.

Let us remark that, in case there are multiple fibers, the birational map above does not extend to a morphism. This follows from the identification of the function $d(e, m_i)$ below, and can also be seen directly as follows. The moduli space $\overline{\mathfrak{M}}_2$ contains the set of nonlocally free sheaves, which is a smooth $\mathbb{P}^1$-bundle over S. The corresponding subset of $\mathrm{Hilb}^2 J^{e+1}(S)$ is the image of the blowup of $J^{e+1}(S) \times_{\mathbb{P}^1} J^{e+1}(S)$ along the diagonal (which is not a Cartier divisor) under the involution. It is easy to see that this image is not normal along the image of $F_i \times F_i$ if $m_i > 1$.

There is an isomorphism $H^2(\mathrm{Hilb}^2 J^{e+1}(S) - Y_1 - Y_2) \cong H^2(\overline{\mathfrak{M}}_2 - Y_1' - Y_2')$, so that by restriction we can view $\mu(\Sigma)$ as an element of

$$H^2(\mathrm{Hilb}^2 J^{e+1}(S) - Y_1 - Y_2) \cong H^2(\mathrm{Hilb}^2 J^{e+1}(S)).$$

Denote this element of $H^2(\mathrm{Hilb}^2 J^{e+1}(S))$ by $\mu'(\Sigma)$. In fact, it is easy to identify this element: let $\alpha_1 = \mu_1(\Sigma) \in J^{e+1}(S)$ be given by the μ-map for the two-dimensional invariant, and set

$$\alpha_2 = \alpha_1 + \frac{(f \cdot \Sigma)}{2} f.$$

Then we have the following formula:

Lemma 9.2.
$$\mu'(\Sigma) = D_{\alpha_2} - \frac{(f \cdot \Sigma)}{2} E.$$

Now α_1^2 is just the value of the two-dimensional invariant, which we shall write as $(\Sigma^2) + C_1(\kappa \cdot \Sigma)^2$, where $C_1 = m_1^2 m_2^2(p_g + 1) - m_1^2 - m_2^2$. Thus
$$\begin{aligned}\alpha_2^2 &= \alpha_1^2 + (f \cdot \Sigma)(\alpha_1 \cdot f)\\ &= \alpha_1^2 + 2(f \cdot \Sigma)^2\\ &= (\Sigma^2) + (C_1 + 2m_1^2 m_2^2)(\kappa \cdot \Sigma)^2,\end{aligned}$$
where we have used Lemma 6.5 to conclude that $\alpha_1 \cdot f = 2(f \cdot \Sigma)$.

Thus a routine calculation with the multiplication table in $\mathrm{Hilb}^2 J^{e+1}(S)$ gives:

Lemma 9.3.
$$\begin{aligned}\mu'(\Sigma)^4 &= 3(\Sigma^2)^2 + 6C_1(\Sigma^2)(\kappa \cdot \Sigma)^2\\ &\quad + \left[3C_1^2 - (2(p_g + 1) + 12)m_1^4 m_2^4 + 8(m_1^3 m_2^4 + m_1^4 m_2^3)\right](\kappa \cdot \Sigma)^4. \quad \square\end{aligned}$$

Of course, this is a calculation on $\mathrm{Hilb}^2 J^{e+1}(S)$, not on $\overline{\mathfrak{M}}_2$. To get an answer on $\overline{\mathfrak{M}}_2$, we shall argue that the above formula must be corrected by terms which only depend on the multiplicities of the multiple fibers and not on p_g.

Lemma 9.4. *There exist a function $d(e, m_i)$, depending only on e and an analytic neighborhood of the multiple fiber F_i in S, with the following properties:*

(i) $d(e, 1) = 0$.
(ii) $\mu(\Sigma)^4 - \mu'(\Sigma)^4 = m_1^4 m_2^4(d(e, m_1) + d(e, m_2))(\kappa \cdot \Sigma)^4$.

We can now complete the proof of (ii) of Theorem 0.5 in the Introduction:

Corollary 9.5. *For all $\Sigma \in H_2(S; \mathbb{Z})$,*
$$\gamma_2(S)(\Sigma, \Sigma, \Sigma, \Sigma) = 3(\Sigma^2)^2 + 6C_1(\Sigma^2)(\Sigma \cdot \kappa)^2 + (3C_1^2 - 2C_2)(\Sigma \cdot \kappa)^4,$$
where
$$\begin{aligned}C_1 &= (m_1^2 m_2^2)(p_g(S) + 1) - m_1^2 - m_2^2;\\ C_2 &= (m_1^4 m_2^4)(p_g(S) + 1) - m_1^4 - m_2^4.\end{aligned}$$

Proof. We must calculate $\gamma_2(S)(\Sigma, \Sigma, \Sigma, \Sigma) = \mu(\Sigma)^4$. By Lemma 9.4, the coefficients in $\mu(\Sigma)^4$ of $(\Sigma^2)^2$ and of $(\Sigma^2)(\Sigma \cdot \kappa)^2$ agree with the corresponding coefficients of $\mu'(\Sigma)^4$, and these are calculated in Lemma 9.3. It also follows from Lemmas 9.3 and 9.4 that the coefficient of $(\kappa \cdot \Sigma)^4$ in $\mu(\Sigma)^4$ is given by
$$3C_1^2 - (2(p_g + 1) + 12)m_1^4 m_2^4 + 8(m_1^3 m_2^4 + m_1^4 m_2^3) + m_1^4 m_2^4(d(e, m_1) + d(e, m_2)).$$
To calculate $d(e, m_i)$, take as before S to be a rational surface with a multiple fiber of multiplicity m_1. In this case, arguing as in the proof of Theorem 6.3,

the coefficient of $(\kappa\cdot\Sigma)^4$ is the same as the coefficient of $(\kappa\cdot\Sigma)^4$ for the rational surface with no multiple fibers. To calculate this coefficient, we apply Lemmas 9.3 and 9.4 with $m_1 = m_2 = 1$ and $p_g = 0$, to see that $\mu(\Sigma)^4 = \mu'(\Sigma)^4$ and thus that the coefficient of $(\kappa\cdot\Sigma)^4$ is $3-14+16=5$. Now taking $p_g = 0$ and m_1 arbitrary and $m_2 = 1$ in the above formulas gives $C_1 = -1$ and

$$5 = 3(-1)^2 - 14m_1^4 + 8(m_1^3 + m_1^4) + m_1^4 d(e, m_1).$$

Thus $m_1^4 d(e, m_1) = 2 - 8m_1^3 + 6m_1^4$, or

$$d(e, m_1) = \frac{2}{m_1^4} - \frac{8}{m_1} + 6.$$

Plugging this into the expression above for the coefficient for $(\kappa\cdot\Sigma)^4$ in the general case gives

$$\begin{aligned}3C_1^2 &- (2(p_g+1)+12)m_1^4m_2^4 + 12m_1^4m_2^4 + 2m_1^4 + 2m_2^4\\ &= 3C_1^2 - 2((p_g+1)m_1^4m_2^4 - m_1^4 - m_2^4).\end{aligned}$$

We may write this answer more neatly as $3C_1^2 - 2C_2$, where

$$\begin{aligned}C_1 &= m_1^2m_2^2(p_g+1) - m_1^2 - m_2^2;\\ C_2 &= m_1^4m_2^4(p_g+1) - m_1^4 - m_2^4. \quad \square\end{aligned}$$

10. Proof of Lemmas 9.1, 9.2, and 9.4

In this section we shall give a proof of the remaining results from the previous section.

Proof of Lemma 9.1. The lemma asserts the existence of an isomorphism from $\mathrm{Hilb}^2 J^{e+1}(S) - Y_1 - Y_2$ to $\overline{\mathfrak{M}}_2 - Y_1' - Y_2'$ extending the isomorphism given in Theorem 3.14. The isomorphism of Theorem 3.14 is defined on the open set U of $\mathrm{Hilb}^2 J^{e+1}(S)$ consisting of pairs of points $\{z_1, z_2\}$ such that z_1 and z_2 lie in distinct fibers, neither of which is singular or multiple. We must show that the map extends over the set of pairs $\{z_1, z_2\}$, where z_1 and z_2 lie in distinct fibers, one or both of which may be singular or multiple, as well as over the set of pairs Z where either Z is nonreduced but the support of Z does not lie in a multiple fiber or where $Z = \{z_1, z_2\}$ with z_1 and z_2 lying in the same nonmultiple fiber.

Let us first consider the case where z_1 and z_2 lie in distinct fibers. As in Section 7, choose an elliptic surface $T \to C$ with a section such that C is a finite cover of $\mathbb{P}^1$, generically branched except below the multiple fibers and T is the normalization of $S \times_{\mathbb{P}^1} C$. Let $\varphi\colon T \to S$ be the natural map. There is also the map $\varphi_{e+1}\colon T \to J^{e+1}(S)$ defined by $\mathscr{P}$, i.e. if $q \in T$, f is the fiber containing q and $p = f \cap \sigma$, then $\varphi_{e+1}(q) = \mathscr{O}_f((e+2)p - q)$. We have constructed a universal bundle $\tilde{\mathscr{V}} \to S \times T$ in Section 7 for the choices of w and p corresponding to the two-dimensional invariant. Let $\tilde{U} \subset T \times T$ be the open set of pairs of points (y_1, y_2) such that $\varphi(y_1)$ and $\varphi(y_2)$ lie in different fibers. Let $\tilde{\mathscr{V}}_1$ be the pullback of $\tilde{\mathscr{V}}$ to $S \times \tilde{U}$ via the natural projection of $S \times \tilde{U} \subset S \times T \times T$ onto the first and second factors. We also have the coherent sheaf $\mathscr{P}$ on $S \times T$ defined at the beginning of Section 7. Let $\mathscr{P}'$ be the pullback of $\mathscr{P}$ to $S \times \tilde{U}$ defined by the projection of $S \times T \times T$ to the first and third factor. Thus given a point $(y_1, y_2) \in \tilde{U}$, the restriction of $\tilde{\mathscr{V}}_1$ to the slice through (y_1, y_2) is an elementary modification of V_0 along the fiber containing $\varphi(y_1)$ and the restriction of $\mathscr{P}'$ to the slice through (y_1, y_2) is the direct image of a line bundle of degree $e+1$ on the fiber through $\varphi(y_2)$. Thus, leting π_2 denote the projection $S \times \tilde{U} \to \tilde{U}$, $\pi_{2*}(\tilde{\mathscr{V}}_1^{\vee} \otimes \mathscr{P}')$ is a line bundle on $\tilde{U}$, whose inverse we denote by $\mathscr{L}'$. Define $\tilde{\mathscr{V}}_2$ as the kernel of the natural map $\tilde{\mathscr{V}}_1 \to \mathscr{P}' \otimes \mathscr{L}'$. The proof of Theorem 8.2 shows that $\tilde{\mathscr{V}}_2$ is a vector bundle whose restriction to each slice $S \times \{(y_1, y_2)\}$ is stable. The induced map $\tilde{U} \to \mathfrak{M}_2$ then descends to a map from the open subset of $\mathrm{Hilb}^2 J^{e+1}(S)$ consisting of points lying in distinct fibers to $\mathfrak{M}_2$. (In fact, the proof shows that this morphism extends to a morphism defined on the complement of the divisor E of nonreduced points together with the proper transforms of $\mathrm{Sym}^2 F_1$ and $\mathrm{Sym}^2 F_2$.)

Next we must extend the morphism over the points of $\mathrm{Hilb}^2 J^{e+1}(S)$ corresponding to points lying in the same nonmultiple fiber and nonreduced points whose support does not lie in a multiple fiber. In order to do so, we will need the model for elliptic surfaces with a section constructed in Section 4. Let Z be a point of $\mathrm{Hilb}^2 J^{e+1}(S)$ such that $\mathrm{Supp}\, Z$ lies in a single nonmultiple fiber f, and let X be a small neighborhood of f mapping properly to a disk inside $\mathbb{P}^1$. Thus there is a biholomorphic map from X to a neighborhood of the corresponding fiber in the Jacobian surface $J(S)$, and we may further assume that the image of $\mathrm{Supp}\, Z$ does not meet the identity section σ under this map. Now the results of Section 4, after tensoring by $\mathscr{O}_X(e\sigma)$, give a rank two vector bundle V_0' over X whose restriction to every fiber is stable of degree $2e+1$ and a rank two reflexive sheaf $\mathscr{V}_0$ over $X \times (\mathrm{Hilb}^2 X - D_\sigma)$, flat over $H = \mathrm{Hilb}^2 X - D_\sigma$, whose restriction to each slice is an elementary modification of V_0'. Let V_0 denote as usual the bundle on S whose restriction to every fiber is stable. Then as in the proof of Theorem 8.1 there is a line bundle L on X such that $V_0|X \cong V_0' \otimes L$.

The sheaf $\mathscr{V}_0 \otimes \pi_1^* L$ has the following property. Let $\mathscr{B} \subset X \times H$ be the set

$$\mathscr{B} = \{ (x, z_1, z_2) \mid \pi(x) = \pi(z_i) \text{ for some } i \}.$$

Let p be a point of H and $\mathbb{U}$ a small neighborhood of p, which we can identify with a neighborhood of $Z \in \mathrm{Hilb}^2 J^{e+1}(S)$. We can assume that $\mathbb{U}$ is a polydisk. There is a proper map $\Pi\colon (X \times \mathbb{U}) - \mathscr{B} \to (D_0 \times \mathbb{U}) - \mathscr{B}'$ induced by $\pi\colon X \to D_0$, where D_0 is the disk which is the base curve of X and

$$\mathscr{B}' = \{ (t, z_1, z_2) \mid t = \pi(z_i) \text{ for some } i \}.$$

By construction the restrictions of $\mathscr{V}_0 \otimes \pi_1^* L$ and $\pi_1^* V_0$ to each fiber of Π are isomorphic stable bundles on the fiber, which is reduced (possibly nodal).

Thus $R^0\Pi_* Hom(\mathscr{V}_0 \otimes \pi_1^* L, \pi_1^* V_0)$ is a line bundle $\mathscr{F}$ on $(D_0 \times \mathbb{U}) - \mathscr{B}'$. Both $\mathscr{V}_0 \otimes \pi_1^* L$ and $\pi_1^* V_0$ extend to coherent sheaves on $X \times \mathbb{U}$. Therefore $R^0\Pi_* Hom(\mathscr{V}_0 \otimes \pi_1^* L, \pi_1^* V_0) = \mathscr{F}$ extends to a coherent sheaf on $D_0 \times \mathbb{U}$, which we shall continue to denote by $\mathscr{F}$. Replacing $\mathscr{F}$ by its double dual if necessary, we can assume that it is reflexive, and thus since its rank is one that it is a line bundle. Since by assumption every line bundle on $D_0 \times \mathbb{U}$ is trivial, $R^0\Pi_* Hom(\mathscr{V}_0 \otimes \pi_1^* L, \pi_1^* V_0)$ is a trivial line bundle on $(D_0 \times \mathbb{U}) - \mathscr{B}'$, and we can thus choose an everywhere generating section. This section corresponds to a homomorphism from $\mathscr{V}_0 \otimes L$ to $\pi_1^* V_0$ over $(X \times \mathbb{U}) - \mathscr{B}$ which is an isomorphism on every fiber. It follows that we can glue $\mathscr{V}_0 \otimes L$ to $\pi_1^* V_0$ over $(X \times \mathbb{U}) - \mathscr{B}$. Since $\{X \times \mathbb{U}, (S \times \mathbb{U}) - \mathscr{B}\}$ is an open cover of $S \times \mathbb{U}$ whose intersection is $(X \times \mathbb{U}) - \mathscr{B}$, we have constructed a coherent sheaf on $S \times \mathbb{U}$, flat over $\mathbb{U}$. In this way we have extended the morphism from $\tilde{U} \cap \mathbb{U}$ over all of $\mathbb{U}$. So the morphism $U \to \overline{\mathfrak{M}}_2$ extends over all the points $Z \in \mathrm{Hilb}^2 J^{e+1}(S)$ such that $A \notin Y_1 \cup Y_2$. Clearly its image is exactly $\overline{\mathfrak{M}}_2 - Y_1' - Y_2'$. □

Proof of Lemma 9.2. We shall show that the divisor $\mu'(\Sigma)$ which is the natural extension of the restriction of $\mu(\Sigma)$ to $\mathrm{Hilb}^2 J^{e+1}(S) - Y_1 - Y_2$ to a divisor on $\mathrm{Hilb}^2 J^{e+1}(S)$ is equal to $D_{\alpha_2} - ((f \cdot \Sigma)/2)E$. Recall that $H^2(\mathrm{Hilb}^2 J^{e+1}(S)) \cong H^2(J^{e+1}(S)) \oplus \mathbb{Z} \cdot [E/2]$. Also, given a point $y \in J^{e+1}(S)$, there is an induced morphism $\tau_y \colon \mathrm{Bl}_y J^{e+1}(S) \to \mathrm{Hilb}^2 J^{e+1}(S)$ defined on $J^{e+1}(S) - \{y\}$ by $\tau_y(x) = \{x, y\}$. If E_y is the exceptional divisor on $\mathrm{Bl}_y J^{e+1}(S)$, then it is easy to see that $\tau_y^* D_\alpha = \alpha$ for all $\alpha \in H^2(J^{e+1}(S))$ (where we have identified $H_2(J^{e+1}(S))$ and $H^2(J^{e+1}(S))$ and identified $H^2(J^{e+1}(S))$ with a subspace of $H^2(\mathrm{Bl}_y J^{e+1}(S))$). Also $\tau_y^*[E] = 2[E_y]$, which can easily be checked by going up to the double cover of $\mathrm{Hilb}^2 J^{e+1}(S)$ which is the blowup of $J^{e+1}(S) \times J^{e+1}(S)$ along the diagonal. Similarly, suppose that $\varphi \colon T \to S$ is a finite cover as usual and consider the morphism $\varphi_{e+1} \colon T \to J^{e+1}(S)$ defined by $\mathscr{P}$, i.e. if $q \in T$, f is the fiber containing q and $p = f \cap \sigma$, then $\varphi_{e+1}(q) = \mathscr{O}_f((e+2)p - q)$. Suppose that y is a general point of $J^{e+1}(S)$ (and so does not lie on a multiple or singular fiber) and let $\varphi_{e+1}^{-1}(y) = \{y_1, \dots, y_d\}$. Then there is an induced map $\tau \colon T - \{y_1, \dots, y_d\} \to \mathrm{Hilb}^2 J^{e+1}(S)$, and clearly we have $\tau^* D_\alpha = \varphi^* \alpha$. In particular the map τ^* is injective on the subspace $H_2(J^{e+1}(S))$, and we can determine $\mu'(\Sigma)$ provided that we know $\tau^* \mu'(\Sigma)$ and $\mu'(\Sigma) | E_y$. Note finally that the image of τ and E_y are contained in $\mathrm{Hilb}^2 J^{e+1}(S) - Y_1 - Y_2$, so that we can calculate the μ-map by finding a universal family of coherent sheaves on $S \times (T - \{y_1, \dots, y_d\})$ and over $S \times E_y$.

To find such a family, begin with the bundle $\tilde{\mathscr{V}}$ over $S \times T$. We know that $\tilde{\mu}(\Sigma) = \varphi_{e+1}^* \alpha_1$, where $\tilde{\mu}$ is the natural μ-map defined on T and $\alpha_1 = \mu(\Sigma)$ is the μ-map for the two-dimensional invariant. Fix a general fiber f of S and a point $y \in J^{e+1}(S)$ corresponding to a line bundle λ of degree $e+1$ on f.

Let $f_1, \dots, f_d$ be the fibers on T lying above f and $y_1, \dots, y_d$ the points of T corresponding to λ. We shall perform an elementary modification along the divisor $f \times T$ with respect to the line bundle $\pi_1^*\lambda$. This will run into trouble along $y_1, \dots, y_d$, so that we will restrict to $S \times (T - \{y_1, \dots, y_d\})$. The upshot will be a family of stable torsion free sheaves on $S \times (T - \{y_1, \dots, y_d\})$ such that the induced morphism $T - \{y_1, \dots, y_d\} \to \overline{\mathfrak{M}}_2$ is τ.

First let us calculate

$$\operatorname{Hom}(\tilde{\mathscr{V}}|S \times (T - \{y_1, \dots, y_d\}), \pi_1^*\lambda).$$

If V_t is the restriction of $\tilde{\mathscr{V}}$ to the slice $S \times \{t\}$, then V_t is an elementary modification of V_0 either at a fiber different from λ or along f with respect to a line bundle λ' of degree equal to $\deg \lambda$ but with $\lambda' \neq \lambda$. It follows that the map $\operatorname{Hom}(V_0, \lambda) \to \operatorname{Hom}(V_t, \lambda)$ defined by the inclusion $V_t \subset V_0$ is a map between two one-dimensional spaces by Lemma 1.3(i), and its kernel is $H^0((\lambda')^{-1} \otimes \lambda) = 0$. Thus $\operatorname{Hom}(V_0, \lambda) \cong \operatorname{Hom}(V_t, \lambda)$ and the induced map $R^0\pi_{2*}\pi_1^*(V_0^\vee \otimes \lambda) \to R^0\pi_{2*}(\tilde{\mathscr{V}}|S \times (T - \{y_1, \dots, y_d\})^\vee \otimes \pi_1^*\lambda)$ is an isomorphism. As $R^0\pi_{2*}\pi_1^*(V_0^\vee \otimes \lambda)$ is the trivial line bundle, there is a unique homomorphism mod scalars from $\tilde{\mathscr{V}}|S \times (T - \{y_1, \dots, y_d\})$ to $\pi_1^*\lambda$ and its restriction to each slice is the corresponding nonzero homomorphism on the slice. Let $\tilde{\mathscr{V}}_2$ be the kernel, so that there is an exact sequence

$$0 \to \tilde{\mathscr{V}}_2 \to \tilde{\mathscr{V}}|S \times (T - \{y_1, \dots, y_d\}) \to \pi_1^*\lambda.$$

Note that the right arrow fails to be surjective over the slice $S \times \{t\}$ only if $\varphi(t) \in f$, and in this case it vanishes at one point. Thus by (A.5) $\tilde{\mathscr{V}}_2$ is reflexive and flat over $T - \{y_1, \dots, y_d\}$, and is a family of torsion free sheaves parametrized by $T - \{y_1, \dots, y_d\}$. The restriction of $\tilde{\mathscr{V}}_2$ to a general fiber in every slice is stable, and thus $\tilde{\mathscr{V}}_2$ is a flat family of stable torsion free sheaves. Clearly the corresponding morphism to $\overline{\mathfrak{M}}_2$ is τ.

Next we claim that

$$p_1(\operatorname{ad} \tilde{\mathscr{V}}_2) = p_1(\operatorname{ad} \tilde{\mathscr{V}}) - 2d(f \otimes f) + \cdots,$$

where the omitted terms do not affect the slant product. Indeed the defining map $\tilde{\mathscr{V}}|S \times (T - \{y_1, \dots, y_d\}) \to \pi_1^*\lambda$ is surjective in codimension two, so that in calculating $p_1(\operatorname{ad} \tilde{\mathscr{V}}_2)$ we can in fact apply the lemma on elementary modifications as if the map were surjective. Now the lemma gives

$$p_1(\operatorname{ad} \tilde{\mathscr{V}}_2) = p_1(\operatorname{ad} \tilde{\mathscr{V}}) + 2c_1(\tilde{\mathscr{V}}) \cdot (f \otimes 1) - 4i_*(\lambda \otimes 1) \cdot (f \otimes 1).$$

Using $c_1(\tilde{\mathscr{V}}) = \pi_1^*c_1(V_0) - [(f \otimes 1) + d(1 \otimes f)]$ and plugging in gives the claimed formula for $p_1(\operatorname{ad} \tilde{\mathscr{V}}_2)$. Thus

$$\begin{aligned} -(p_1(\operatorname{ad} \tilde{\mathscr{V}}_2)\backslash\Sigma)/4 &= -(p_1(\operatorname{ad} \tilde{\mathscr{V}})\backslash\Sigma)/4 + d(f \cdot \Sigma)f/2 \\ &= \varphi_{e+1}^*(\alpha_1) + ((f \cdot \Sigma)/2)\varphi_{e+1}^*(f) \\ &= \varphi_{e+1}^*(\alpha_2), \end{aligned}$$

and the pullback of $\mu'(\Sigma)$ to $T - \{y_1, \dots, y_d\}$ under τ is just $\varphi^*_{e+1}(\alpha_2)$. It follows that $\mu'(\Sigma) = D_{\alpha_2} + aE$ for some rational number a.

To determine the coefficient of E in $\mu'(\Sigma)$, fix a general fiber f of S and a line bundle λ of degree $e+1$ on f, which corresponds to a point $y \in J^{e+1}(S)$. The set of points of $\mathrm{Hilb}^2 J^{e+1}(S)$ whose support is $\{y\}$ is a curve $E_y \cong \mathbb{P}^1$. We shall construct a universal sheaf $\mathscr{V}_2$ over $S \times E_y$ and show that $-(p_1(\mathrm{ad}\,\mathscr{V}_2)\backslash\Sigma)/4 = (f \cdot \Sigma)$.

Begin with V which is obtained from V_0 by a single elementary modification along λ. Thus $V|f = \lambda \oplus \mu$ with $\deg\lambda = e+1$ and $\deg\mu = e$. By Lemma 1.3(ii) $\dim \mathrm{Hom}(V, \lambda) = 2$ and there is a unique nonzero homomorphism from V to λ which is not surjective, indeed which vanishes exactly at the point corresponding to the degree one line bundle $\lambda \otimes \mu^{-1}$. Identify $\mathbb{P}(\mathrm{Hom}(V, \lambda))$ with $\mathbb{P}^1$ (and with E_y). There is a general construction [9] of a universal homomorphism $\Phi\colon \pi_1^* V \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^1}(-1) \to \pi_1^*\lambda$. Thus we can define $\mathscr{V}_2$ to be its kernel:

$$0 \to \mathscr{V}_2 \to \pi_1^* V \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^1}(-1) \to \pi_1^*\lambda.$$

By (A.5) in the Appendix, $\mathscr{V}_2$ is reflexive and flat over $\mathbb{P}^1$, and is a family of torsion free sheaves, which are locally free except for the point of $\mathbb{P}^1$ corresponding to the nonsurjective homomorphism. The restriction of $\mathscr{V}_2$ to a general fiber in every slice is stable, so that the restriction of $\mathscr{V}_2$ to each slice is a stable torsion free sheaf. The induced map to $\overline{\mathfrak{M}}_2$ is easily seen to be one-to-one with image E_y. We may again calculate $p_1(\mathrm{ad}\,\mathscr{V}_2)$ by the lemma on elementary modifications, noting that $c_1(\pi_1^* V \otimes \pi_2^*\mathscr{O}_{\mathbb{P}^1}(-1)) = \pi_1^* c_1(V) + 2\pi_2^* c_1(\mathscr{O}_{\mathbb{P}^1}(-1))$:

$$p_1(\mathrm{ad}\,\mathscr{V}_2) = \pi_1^* p_1(\mathrm{ad}\,V) + 2(\pi_1^* c_1(V) + 2\pi_2^* c_1(\mathscr{O}_{\mathbb{P}^1}(-1))) \cdot (f \otimes 1) - 4 i_* \pi_1^* c_1(\lambda).$$

The only term which matters for the slant product is the term $4\pi_2^* c_1(\mathscr{O}_{\mathbb{P}^1}(-1)) \cdot (f \otimes 1)$. Thus

$$\mu'(\Sigma) \cdot E_y = -(1/4)4(-1)(f \cdot \Sigma) = (f \cdot \Sigma).$$

Bearing in mind that $E \cdot E_y = -2$, it follows that the coefficient of E in $\mu'(\Sigma)$ is $-(f \cdot \Sigma)/2$. So putting this all together gives the final answer for $\mu'(\Sigma)$ in Lemma 9.2. $\square$

Proof of Lemma 9.4. The basic idea of the proof is similar to the idea of the proof of Lemma 9.1. Fix an analytic neighborhood X of the multiple fiber F_i as usual. Let X_e be the corresponding subset of $J^{e+1}(S)$. Then $\mathbb{X} = \mathrm{Hilb}^2 X_e$ may be identified with an analytic open subset of $\mathrm{Hilb}^2 J^{e+1}(S)$ which is a neighborhood of Y_i. Under the birational map $\mathrm{Hilb}^2 J^{e+1}(S) \dashrightarrow \overline{\mathfrak{M}}_2$, the open set $\mathbb{X}$ corresponds birationally to an open set $\mathbb{X}'$ which is a neighborhood of Y_i'. Moreover $\mathbb{X} - Y_i \cong \mathbb{X}' - Y_i'$.

Now let S_0 be another nodal elliptic surface containing a multiple fiber of multiplicity m_i and let X_0 be an analytic neighborhood of the multiple fiber. Let Δ_0 be a divisor on S_0 of fiber degree $2e+1$ and let V_0' be a rank two vector bundle whose restriction to every fiber is stable. We suppose that X_0

is biholomorphic to X and identify them. We may then define $\mathbb{X}_0$ and $\mathbb{X}'_0$ analogously. There are also closed subsets of $\mathbb{X}_0$ and $\mathbb{X}_0$ corresponding to Y_i and Y'_i, which we shall again denote by Y_i and Y'_i. Of course $\mathbb{X}_0 \cong \mathbb{X}$ under the identification $X_0 \cong X$. The main claim is then the following:

Claim 10.1. *There is a biholomorphic map $\mathbb{X}'_0 \cong \mathbb{X}'$ which is compatible with the isomorphisms*

$$\mathbb{X}'_0 - Y'_i \cong \mathbb{X}_0 - Y_i \cong \mathbb{X} - Y_i \cong \mathbb{X}' - Y'_i.$$

Proof. For emphasis, we will write $\overline{\mathfrak{M}}_2(S)$ for the moduli space for S, and similarly $\overline{\mathfrak{M}}_2(S_0)$ for the moduli space for S_0. We shall glue $\mathbb{X}'_0$ to $\overline{\mathfrak{M}}_2(S) - Y'_i$ along $\mathbb{X} - Y_i$, and show that the result maps to $\overline{\mathfrak{M}}_2(S)$, compatibly with the inclusion $\overline{\mathfrak{M}}_2(S) - Y'_i \subseteq \overline{\mathfrak{M}}_2(S)$. This will define a proper morphism from $\mathbb{X}'_0$ to $\mathbb{X}'$ of degree one which is an isomorphism in codimension one, and thus is an isomorphism by Zariski's Main Theorem.

We must show that the inclusion $\overline{\mathfrak{M}}_2(S) - Y'_i \subseteq \overline{\mathfrak{M}}_2(S)$ extends to a morphism from $\mathbb{X}'_0$ to $\overline{\mathfrak{M}}_2(S)$. It suffices to do so locally around each point of $\mathbb{X}'_0$. Given an arbitrary point $p \in \mathbb{X}'_0$, let $\mathbb{U} \subset \mathbb{X}'_0$ be an open neighborhood of p which is biholomorphic to a polydisk, so that $\operatorname{Pic} \mathbb{U} = 0$, and such that there exists a universal sheaf $\mathscr{V}_{\mathbb{U}}$ over $S_0 \times \mathbb{U}$. Denote again the restriction of $\mathscr{V}_{\mathbb{U}}$ to $X_0 \times \mathbb{U} = X \times \mathbb{U}$ by $\mathscr{V}_{\mathbb{U}}$. Letting as usual V_0 denote the rank two bundle on S whose restriction to every fiber is stable, we have seen that there is a line bundle L on X such that $V'_0 \otimes L \cong V_0$. Now view $\mathbb{U} - Y'_i$ as an open subset of $\mathbb{X} - Y_i \subset \operatorname{Hilb}^2 X$. As in the proof of Lemma 9.1 we have the locus $\mathscr{B} \subset X \times \mathbb{U}$ which is the closure of the set

$$\{(x, z_1, z_2) \mid \pi(x) = \pi(z_i) \text{ for some } i\}.$$

The set $\mathscr{B}$ is a closed analytic subset both of $X \times \mathbb{U}$ and of $S \times \mathbb{U}$. The two sets $(S \times \mathbb{U}) - \mathscr{B}$ and $X \times \mathbb{U}$ cover $S \times \mathbb{U}$ and their intersection is $(X \times \mathbb{U}) - \mathscr{B}$. We shall show that there is an isomorphism of the restriction of $\mathscr{V}_{\mathbb{U}} \otimes \pi_1^* L$ to $(X \times \mathbb{U}) - \mathscr{B}$ with $\pi_1^* V_0$.

Let $\Pi\colon (X \times \mathbb{U}) - \mathscr{B} \to (D_0 \times \mathbb{U}) - \mathscr{B}'$ be the projection, where D_0 is the base of X and, as in the proof of Lemma 9.1,

$$\mathscr{B}' = \{(t, z_1, z_2) \mid t = \pi(z_i) \text{ for some } i\}.$$

By construction, the restriction of $\mathscr{V}_{\mathbb{U}} \otimes \pi_1^* L$ to the reduction of every fiber of Π is stable, and hence isomorphic to the restriction of $\pi_1^* V_0$ to the fiber. Consider $R^0\Pi_* Hom(\mathscr{V}_{\mathbb{U}} \otimes \pi_1^* L, \pi_1^* V_0)$. By base change and Corollary 1.5 for the case of a multiple fiber, this is a line bundle on $(D_0 \times \mathbb{U}) - \mathscr{B}'$. On the other hand, both $\mathscr{V}_{\mathbb{U}} \otimes \pi_1^* L$ and $\pi_1^* V_0$ extend to coherent sheaves on $X \times \mathbb{U}$, so that $R^0\Pi_* Hom(\mathscr{V}_{\mathbb{U}} \otimes \pi_1^* L, \pi_1^* V_0)$ also extends to a coherent sheaf on $D_0 \times \mathbb{U}$. Arguing as in the proof of Lemma 9.1,

$$R^0\Pi_* Hom(\mathscr{V}_{\mathbb{U}} \otimes \pi_1^* L, \pi_1^* V_0) | (D_0 \times \mathbb{U}) - \mathscr{B}'$$

is a trivial line bundle and we may choose a section of $Hom(\mathscr{V}_{\mathbb{U}} \otimes \pi_1^* L, \pi_1^* V_0)$ which generates the fiber at every point. This section then defines an isomorphism from $\mathscr{V}_{\mathbb{U}} \otimes \pi_1^* L$ to $\pi_1^* V_0$ over $(X \times \mathbb{U}) - \mathscr{B}$. Thus we may define a coherent sheaf over $S \times \mathbb{U}$, flat over $\mathbb{U}$, which by construction is a family of stable torsion free sheaves on S. This sheaf defines a morphism from $\mathbb{U}$ to $\overline{\mathfrak{M}}_2(S)$ which is the desired extension. Doing this for a neighborhood of every point of $\mathbb{X}_0'$ defines the extension over all of $\mathbb{X}_0'$. $\square$

We return to the proof of Lemma 9.4. The proof will now follow from standard arguments. We have the moduli spaces $\mathrm{Hilb}^2 J^{e+1}(S)$ and $\overline{\mathfrak{M}}_2(S)$ for S and corresponding moduli spaces $\mathrm{Hilb}^2 J^{e+1}(S_0)$ and $\overline{\mathfrak{M}}_2(S_0)$ for S_0. There are also the divisors $\mu'(\Sigma)$ on $\mathrm{Hilb}^2 J^{e+1}(S)$ and $\mu(\Sigma)$ on $\overline{\mathfrak{M}}_2(S)$, as well as the corresponding divisors $\mu_0'(\Sigma_0)$ and $\mu_0(\Sigma_0)$ for S_0. Here $\Sigma \in H_2(S)$ and $\Sigma_0 \in H_2(S_0)$. Fixing a choice of F_i, we have the open sets $\mathbb{X} = \mathbb{X}_0$ and $\mathbb{X}' \cong \mathbb{X}_0'$.

Claim 10.2. *If* $\Sigma \cdot f = \Sigma_0 \cdot f$, *then*

$$\mu'(\Sigma)|H^2(\mathbb{X}) = \mu'(\Sigma_0)|H^2(\mathbb{X}).$$

Proof. Clearly $\mathbb{X}$ is the quotient of the blowup $\mathrm{Bl}_\Delta(X_e \times X_e)$ of $X_e \times X_e$ along the diagonal under the natural involution. Thus $H^2(\mathbb{X})$ injects into $H^2(\mathrm{Bl}_\Delta(X_e \times X_e))$, and it suffices to prove that the pullbacks of $\mu'(\Sigma)$ and $\mu'(\Sigma_0)$ to $H^2(\mathrm{Bl}_\Delta(X_e \times X_e))$ agree. Now $H^2(\mathrm{Bl}_\Delta(X_e \times X_e))$ is generated over $\mathbb{Q}$ by the class of the exceptional divisor, classes dual to $[f] \otimes \mathrm{pt}$ and $\mathrm{pt} \otimes [f]$, and $H^1(X_e) \otimes H^1(X_e)$. If $D_\alpha + aE$ is a divisor in $H^2(\mathrm{Hilb}^2 J^{e+1}(S))$, then the pullback of $D_\alpha + aE$ to $H^2(\mathrm{Bl}_\Delta(X_e \times X_e))$ depends only on $\alpha \cdot f$ and a. By Lemma 9.2, $\mu'(\Sigma) = D_{\alpha_2} - ((f \cdot \Sigma)/2)E$, where $\alpha_2 \cdot f = \alpha_1 \cdot f = 2(f \cdot \Sigma)$. A similar statement holds for $\mu'(\Sigma_0)$. Thus

$$\mu'(\Sigma)|H^2(\mathbb{X}) = \mu'(\Sigma_0)|H^2(\mathbb{X}). \quad \square$$

We have the birational map $\mathbb{X} \dashrightarrow \mathbb{X}'$. After blowing up, we can resolve the indeterminacy and make this map a morphism. Let $\tilde{\mathbb{X}}$ be such a blowup. Thus there are morphisms $\rho: \tilde{\mathbb{X}} \to \mathbb{X}$ and $\rho': \tilde{\mathbb{X}} \to \mathbb{X}'$. Since $\mathbb{X}$ and $\mathbb{X}'$ are isomorphic in codimension one, ρ and ρ' have the same exceptional set, which we may assume to be a union of smooth divisors $\bigcup_i E_i$. As we may assume that ρ, say, is given by a series of blowups, the cohomology classes of the E_i are independent. Moreover we have a splitting $H^2(\tilde{\mathbb{X}}) \cong H^2(\mathbb{X}') \oplus \bigoplus_i \mathbb{Z} \cdot [E_i]$. Here the map $H^2(\tilde{\mathbb{X}}) \to H^2(\mathbb{X}')$ is explicitly given by restriction from $H^2(\tilde{\mathbb{X}})$ to $H^2(\tilde{\mathbb{X}} - \bigcup_i E_i) = H^2(\mathbb{X}' - Y_i)$, together with the isomorphism $H^2(\mathbb{X}' - Y_i) \cong H^2(\mathbb{X}')$. We can also glue $\tilde{\mathbb{X}}$ to $\overline{\mathfrak{M}}_2$ along $\mathbb{X}'$, to obtain a scheme $\tilde{\mathfrak{M}}$ which dominates both $\overline{\mathfrak{M}}_2$ and $\mathrm{Hilb}^2 J^{e+1}(S)$. Clearly we again have a splitting $H^2(\tilde{\mathfrak{M}}) = H^2(\overline{\mathfrak{M}}_2) \oplus \bigoplus_i \mathbb{Z} \cdot [E_i]$, and this splitting is compatible with the splitting of $H^2(\tilde{\mathbb{X}})$ under the restriction maps. Using the commutative diagram

$$\begin{array}{ccccc} H^2(\operatorname{Hilb}^2 J^{e+1}(S)) & \longrightarrow & H^2(\tilde{\mathfrak{M}}) & = & H^2(\overline{\mathfrak{M}}_2) \oplus \bigoplus_i \mathbb{Z}\cdot[E_i] \\ \downarrow & & \downarrow & & \downarrow \\ H^2(\mathbb{X}) & \longrightarrow & H^2(\tilde{\mathbb{X}}) & = & H^2(\mathbb{X}') \oplus \bigoplus_i \mathbb{Z}\cdot[E_i], \end{array}$$

we see that given $\xi \in H^2(\operatorname{Hilb}^2 J^{e+1}(S))$, the pullback of ξ to $H^2(\tilde{\mathfrak{M}})$ can be written in the form $\xi' + \sum_i a_i[E_i]$, where ξ' is the pullback of a class in $H^2(\overline{\mathfrak{M}}_2)$ and the integers a_i only depend on $\xi|H^2(\mathbb{X})$.

For notational simplicity we shall just write down the argument for Lemma 9.4 under the assumption that there is only one multiple fiber F_1; the general case follows by working with both fibers. In this case, we may identify $H^2(\operatorname{Hilb}^2 J^{e+1}(S))$ and $H^2(\overline{\mathfrak{M}}_2)$ with subspaces of $H^2(\tilde{\mathfrak{M}})$. By construction we have $\mu'(\Sigma) = \mu(\Sigma) + \sum_i a_i[E_i]$, where the a_i only depend on $\mu'(\Sigma)|H^2(\mathbb{X})$. Thus

$$\mu(\Sigma)^4 - \mu'(\Sigma)^4 = -\sum_{j=1}^{4} \binom{4}{j} \mu(\Sigma)^{4-j} \Big(\sum_i a_i[E_i]\Big)^j,$$

where the term $\sum_i a_i[E_i]$ only depends on $\mu'(\Sigma)|H^2(\mathbb{X})$. Clearly, for $j \geq 1$, the terms $\mu'(\Sigma)^{4-j}(\sum_i a_i[E_i])^j$ likewise only depend on the restriction of $\mu'(\Sigma)$ to $H^2(\mathbb{X})$. By Claim 10.2, the restriction of $\mu'(\Sigma)$ to $H^2(\mathbb{X})$ depends only on $(\Sigma \cdot f)$. Hence $\mu(\Sigma)^4 - \mu'(\Sigma)^4$ depends only on $(f \cdot \Sigma)$, e, and an analytic neighborhood X of the multiple fiber and is homogeneous of degree four in Σ. Thus we can write it as $d(e, m_1)(f\cdot\Sigma)^4$ for some rational number $d(e, m_1)$ depending only on the analytic neighborhood X, where $d(e, 1) = 0$. Doing this construction for both multiple fibers, we see that

$$\mu(\Sigma)^4 - \mu'(\Sigma)^4 = m_1^4 m_2^4 (d(e, m_1) + d(e, m_2))(\kappa\cdot\Sigma)^4,$$

as claimed in Lemma 9.4. □

Appendix: Elementary modifications

In this appendix, we consider the following problem (and its generalizations): let X be a smooth projective scheme or compact complex manifold, let T be smooth and let D be a smooth divisor on T. Suppose that $\mathscr{W}$ is a rank two vector bundle over $X \times T$, and that L is a line bundle on X. Let $i\colon X \times D \to X \times T$ be the inclusion, and suppose that there is a surjection $\mathscr{W} \to i_*\pi_1^*L$ defining $\mathscr{V}$ as an elementary modification:

$$0 \to \mathscr{V} \to \mathscr{W} \to i_*\pi_1^*L \to 0.$$

For $t \in T$, let $W_t = \mathscr{W}|X \times \{t\}$ and $V_t = \mathscr{V}|X \times \{t\}$. If 0 is a reference point of D, then there are two extensions

$$0 \to M \to W_0 \to L \to 0;$$
$$0 \to L \to V_0 \to M \to 0.$$

In particular the second exact sequence defines an extension class $\xi \in H^1(M^{-1} \otimes L)$. We want a formula for ξ and in particular we want to know some conditions which guarantee that $\xi \neq 0$.

Proposition A.1. *Let θ be the Kodaira-Spencer map for the family $\mathscr{W}$ of vector bundles over X, so that θ is a map from the tangent space of T at 0 to $H^1(Hom(W_0, W_0))$. Let $\partial/\partial t$ be a normal vector to D at 0. Then the image of $\theta(\partial/\partial t)$ in $H^1(M^{-1} \otimes L)$ under the natural map $H^1(Hom(W_0, W_0)) \to H^1(Hom(M, L)) = H^1(M^{-1} \otimes L)$ is independent mod scalars of the choice of $\partial/\partial t$ and is the extension class corresponding to V_0.*

Proof. Since W_0 is given as an extension, there is an open cover $\{U_i\}$ of X and transition functions for W_0 with respect to the cover $\{U_i\}$ of the form

$$\bar{A}_{ij} = \begin{pmatrix} \lambda_{ij} & * \\ 0 & \mu_{ij} \end{pmatrix}.$$

Letting t be a local equation for D near 0, we can then choose transition functions for $\mathscr{W}$ of the form $A_{ij} = \bar{A}_{ij} + tB_{ij}$. With these choices of trivialization, a basis of local sections for $\mathscr{V}$ on $U_i \times T$ is of the form $\{e_1, te_2\}$. Thus the transition functions for $\mathscr{V}$ are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot (\bar{A}_{ij} + tB_{ij}) \cdot \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.$$

If $B_{ij} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then a calculation shows that the transition functions are equal to

$$\begin{pmatrix} \lambda_{ij} & t* \\ 0 & \mu_{ij} \end{pmatrix} + \begin{pmatrix} ta & t^2b \\ c & td \end{pmatrix} = \begin{pmatrix} \lambda_{ij} & 0 \\ c & \mu_{ij} \end{pmatrix} + tB'_{ij}.$$

Here c is a matrix coefficient which naturally corresponds to the image of B_{ij} in $Hom(M, L)$. The proposition is just the intrinsic formulation of this local calculation. □

Note. The proof shows that, if the extension does split, then we can repeat the process, viewing V_0 again as an extension of L by M. Either this procedure will eventually terminate, creating a nonsplit extension at the generic point of D, or $\mathscr{W}$ was globally an extension in a neighborhood of D.

Let us give another proof for (A.1) in intrinsic terms which, although less explicit, will generalize. There are canonical identifications $H^1(Hom(W_0, W_0)) = \mathrm{Ext}^1(W_0, W_0)$ and $H^1(Hom(M, L)) = \mathrm{Ext}^1(M, L)$. For simplicity assume that $\dim T = 1$. Note that if we restrict the defining exact sequence for $\mathscr{V}$ to $X \times C$, where C is a smooth curve in T transverse to D, then the sequence remains exact (since $\mathrm{Tor}_1^R(R/tR, R/sR) = 0$ if t and s are relatively prime elements of the regular local ring R). Thus we can always restrict to the case where $\dim T = 1$. Now $\operatorname{Spec}\mathbb{C}[t]/(t^2)$ is a subscheme of T, and we can restrict $\mathscr{W}$ to $\operatorname{Spec}\mathbb{C}[t]/(t^2)$ to get a bundle $\mathscr{W}_\varepsilon$. The bundle $\mathscr{W}_\varepsilon$ is naturally an extension

$$0 \to W_0 \to \mathscr{W}_\varepsilon \to W_0 \to 0,$$

and the associated class in $\operatorname{Ext}^1(W_0, W_0)$ is the Kodaira-Spencer class. The natural map $\operatorname{Ext}^1(W_0, W_0) \to \operatorname{Ext}^1(M, L)$ is defined on the level of extensions as follows: given an extension $\mathscr{W}_\varepsilon$ of W_0 by W_0, let $\mathscr{E}$ be the preimage of M in $\mathscr{W}_\varepsilon$, so that there is an exact sequence

$$0 \to W_0 \to \mathscr{E} \to M \to 0.$$

Given the map $W_0 \to \mathscr{E}$, the quotient $\mathscr{F} = (\mathscr{E} \oplus L)/W_0$, where W_0 maps diagonally into each summand, surjects onto M by taking the composition of the projection to $\mathscr{E}$ with the given map $\mathscr{E} \to M$. The kernel is naturally L. Thus $\mathscr{F}$ is an extension of M by L, and it is easy to see that $\mathscr{F}$ corresponds to the image of the extension class for $\mathscr{W}_\varepsilon$ under the natural map. Finally note that, since $W_0 \to L$ is surjective, there is a natural identification of $\mathscr{F} = (\mathscr{E} \oplus L)/W_0$ with $\mathscr{E}/M$ where we take the image of M under the map $M \to W_0 \to \mathscr{E}$.

On the other hand, restricting the defining exact sequence for $\mathscr{V}$ to $\operatorname{Spec}\mathbb{C}[t]/(t^2)$ gives a new exact sequence

$$0 \to L \to \mathscr{V}_\varepsilon \to \mathscr{W}_\varepsilon \to L \to 0.$$

If we set $\mathscr{E}$ to be the image of $\mathscr{V}_\varepsilon$ in $\mathscr{W}_\varepsilon$, then it is clear that $\mathscr{E}$ is the inverse image of $M \subset W_0$ under the natural map. Now there is an isomorphism $\mathscr{V}_\varepsilon/L \cong \mathscr{E}$, and it is easy to see that this isomorphism identifies V_0 with $\mathscr{E}/M$ under the natural maps, compatibly with the extensions. Thus the extension of M by L defined by V_0 has an extension class equal to the image of the extension class of $\mathscr{W}_\varepsilon$ in $\operatorname{Ext}^1(M, L)$ under the natural map.

With this said, here is the promised generalization of (A.1):

Proposition A.2. *With notation at the beginning of this section, let $\mathscr{W}$ be a rank two reflexive sheaf over $X \times T$, flat over T, let D be a reduced divisor on T, not necessarily smooth, and let $i\colon D \to T$ be the inclusion. Suppose that L is a line bundle on X and that $\mathscr{Z}$ is a codimension two subscheme of $X \times D$, flat over D. Suppose further that $\mathscr{W} \to i_*\pi_1^*L \otimes I_{\mathscr{Z}}$ is a surjection, and let $\mathscr{V}$ be its kernel*:

$$0 \to \mathscr{V} \to \mathscr{W} \to i_*\pi_1^*L \otimes I_{\mathscr{Z}} \to 0.$$

Then:

(i) *$\mathscr{V}$ is reflexive and flat over T.*

(ii) *For each $t \in D$, there are exact sequences*

$$0 \to M \otimes I_{Z'} \to W_t \to L \otimes I_Z \to 0,$$
$$0 \to L \otimes I_Z \to V_t \to M \otimes I_{Z'} \to 0,$$

where Z is the subscheme of X defined by $\mathscr{Z}$ for the slice $X \times \{t\}$ and Z' is a subscheme of X of codimension at least two.

(iii) *If D is smooth, then the extension class corresponding to V_t in $\operatorname{Ext}^1(M \otimes I_W, L \otimes I_Z)$ is defined by the image of the normal vector to D at t under the composition of the Kodaira-Spencer map from the tangent space of T at t to $\operatorname{Ext}^1(W_t, W_t)$, followed by the natural map $\operatorname{Ext}^1(W_t, W_t) \to \operatorname{Ext}^1(M \otimes I_{Z'}, L \otimes I_Z)$.*

Proof. First note that $\mathscr{V}$ is a subsheaf of $\mathscr{W}$ and is therefore torsion free. Given an open set U of $X \times T$ and a closed subscheme Y of U of codimension at least two, let s be a section of $\mathscr{V}$ defined on $U - Y$. Then s extends to a section $\tilde{s}$ of $\mathscr{W}$ over U since $\mathscr{W}$ is reflexive. Moreover the image of $\tilde{s}$ in $H^0(U \cap D;\, L \otimes I_{\mathscr{Z}})$ vanishes in codimension one and thus everywhere. Thus $\tilde{s}$ defines a section of $\mathscr{V}$ over U, and so $\mathscr{V}$ is reflexive. That it is flat over T follows from the next lemma:

Lemma A.3. *Let R be a ring and t an element of R which is not a zero divisor. Let I be an (R/tR)-module which is flat over R/tR. For an R-module N, let N_t be the kernel of multiplication by t on N.*

(i) *For all R-modules N, $\operatorname{Tor}_1^R(I, N) = I \otimes_{R/tR} N_t$, and $\operatorname{Tor}_i^R(I, N) = 0$ for all $i > 1$.*

(ii) *Suppose that there is an exact sequence of R-modules*

$$0 \to M_2 \to M_1 \to I \to 0,$$

where M_1 is flat over R. Then M_2 is flat over R as well.

Proof. The statement (i) is easy if $I = R/tR$, by taking the free resolution

$$0 \to R \xrightarrow{\times t} R \to R/tR \to 0.$$

Thus it holds more generally if I is a free (R/tR)-module. In general, start with a free resolution $F^\bullet$ of I. By standard homological algebra (see e.g. [EGA], III, 6.3.2) there is a spectral sequence with E_1 term $\operatorname{Tor}_p^R(F^q, N)$ which converges to $\operatorname{Tor}_{p+q}^R(I, N)$. The only nonzero rows correspond to $p = 0, 1$ and the row for $p = 1$ is the complex $F^\bullet \otimes_{R/tR} N_t$. Since I is flat, this complex is exact except in dimension zero and is a resolution of $I \otimes_{R/tR} N_t$. Since $F^q \otimes_R N = F^q \otimes_{R/tR} (N \otimes_R R/tR)$, the flatness of I over R/tR implies that the row for $p = 0$ is exact except in dimension zero. Thus $\operatorname{Tor}_1^R(I, N) = I \otimes_{R/tR} N_t$.

The second statement now follows since, for every R-module N, the long exact sequence for Tor defines an isomorphism, for all $i \geq 1$, from $\operatorname{Tor}_i^R(M_2, N)$ to $\operatorname{Tor}_{i+1}^R(I, N) = 0$. □

Returning to (A.2), let us prove (ii). There is a surjection $W_t \to L \otimes I_Z$ and the kernel of this surjection is a rank one torsion free sheaf on X, which is thus of the form $M \otimes I_{Z'}$ for some subscheme Z' of X of codimension at least two. Now there is an exact sequence

$$Tor_1^{\mathscr{O}_{X \times T}}(i_* \pi_1^* L \otimes I_{\mathscr{Z}}, \mathscr{O}_{X \times \{t\}}) \to V_t \to W_t \to L \otimes I_Z \to 0.$$

In the Tor_1 term, the first sheaf is an $\mathscr{O}_{X \times D}$-module, flat over D, and the second is an $\mathscr{O}_D$-module. Using (i) of (A.3) identifies $Tor_1^{\mathscr{O}_{X \times T}}(i_* \pi_1^* L \otimes I_{\mathscr{Z}}, \mathscr{O}_{X \times \{t\}})$ with $L \otimes I_Z$. Thus we obtain the exact sequence for V_t.

Finally, the identification of the extension class in (iii) is formally identical to the second proof of (A.1) given above and will not be repeated. □

Next we shall give some criteria for when the extension is nonsplit. The simplest case is when the Kodaira-Spencer map is an isomorphism at 0. In this case we can check whether or not the extension is split by looking at the map $\mathrm{Ext}^1(W_t, W_t) \to \mathrm{Ext}^1(M \otimes I_{Z'}, L \otimes I_Z)$. Thus the problem is essentially cohomological. A similar application concerns the case where T is the blowup of a universal family along the locus where the sheaves are extensions of $L \otimes I_Z$ by $M \otimes I_{Z'}$. In our applications, however, we shall need a more general situation and will have to analyze some first order information about the family $\mathscr{W}$. For simplicity we shall assume that $\dim T = 1$, with t a coordinate. It is an easy consequence of (A.3)(i) that the general case can be reduced to this special case by taking a curve in T transverse to D.

Proposition A.4. *In the notation of* (A.3), *let* $\mathscr{W}_\varepsilon$ *be the restriction of* $\mathscr{W}$ *to* $\operatorname{Spec} \mathbb{C}[t]/t^2$. *Suppose that*

(i) $\mathrm{Hom}(M \otimes I_{Z'}, L \otimes I_Z) = 0$.
(ii) *The map from*

$$\mathrm{Ext}^1(M \otimes I_{Z'}, \mathscr{W}_\varepsilon)/t\,\mathrm{Ext}^1(M \otimes I_{Z'}, \mathscr{W}_\varepsilon)$$

to $\mathrm{Ext}^1(M \otimes I_{Z'}, \mathscr{W}_\varepsilon)$ *induced by multiplication by* t *has a one-dimensional kernel.*

Then we may identify the kernel with a line in $\mathrm{Ext}^1(M \otimes I_{Z'}, W_0)$, *and if the image of this line in* $\mathrm{Ext}^1(M \otimes I_{Z'}, L \otimes I_Z)$ *is* $\mathbb{C} \cdot \xi$ *then the corresponding extension class is* ξ.

Proof. From the first assumption $\dim \mathrm{Hom}(W_0, W_0) = 1$. Thus if θ is the Kodaira-Spencer class, there is an exact sequence

$$0 \to \mathrm{Ext}^1(M \otimes I_{Z'}, W_0)/\mathbb{C} \cdot \theta \to \mathrm{Ext}^1(M \otimes I_{Z'}, \mathscr{W}_\varepsilon) \to \mathrm{Ext}^1(M \otimes I_{Z'}, W_0).$$

Multiplication by t induces the natural map

$$\mathrm{Im}(\mathrm{Ext}^1(M \otimes I_{Z'}, \mathscr{W}_\varepsilon)) \subseteq \mathrm{Ext}^1(M \otimes I_{Z'}, W_0) \to \mathrm{Ext}^1(M \otimes I_{Z'}, W_0)/\mathbb{C} \cdot \theta.$$

If this map has a kernel then clearly $\theta \in \mathrm{Im}(\mathrm{Ext}^1(M \otimes I_{Z'}, \mathscr{W}_\varepsilon))$ and the kernel is $\mathbb{C} \cdot \theta$. The image of the kernel in $\mathrm{Ext}^1(M \otimes I_{Z'}, L \otimes I_Z)$ is then just the image of the Kodaira-Spencer class. □

Here is the typical way we will apply the above: suppose that $\mathscr{W}$ is locally free and that $Z' = \emptyset$. Then $\mathrm{Ext}^1(M \otimes I_{Z'}, \mathscr{W}_\varepsilon) = R^1\pi_{2*}(\mathscr{W}_\varepsilon \otimes \pi_1^* M^{-1})$. Suppose in addition that $\mathscr{W}$ is globally an extension:

$$0 \to \pi_1^*\mathscr{L}_1 \to \mathscr{W} \to \pi_1^*\mathscr{L}_2 \otimes I_{\mathscr{Y}} \to 0,$$

where $\mathscr{Y} \subset X \times T$ is flat over T. Thus there is a map

$$R^0\pi_{2*}(\pi_1^*\mathscr{L}_2 \otimes I_{\mathscr{Y}} \otimes \pi_1^* M^{-1}) \to R^1\pi_{2*}\pi_1^*(\mathscr{L}_1 \otimes M^{-1})$$

whose cokernel sits inside $R^1\pi_{2*}(\mathscr{W} \otimes \pi_1^* M^{-1})$. A similar statement is true when we restrict to $\operatorname{Spec} \mathbb{C}[t]/(t^2)$. Now suppose that

$$\dim H^0(X; \mathscr{L}_2 \otimes M^{-1} \otimes I_{Y_t})$$

is independent of t. Then the sheaves $R^0\pi_{2*}(\pi_1^*\mathcal{L}_2 \otimes I_{\mathcal{Y}} \otimes \pi_1^* M^{-1})$ and $R^1\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes M^{-1})$ are locally free and compatible with base change, by [EGA], III, 7.8.3, 7.8.4, 7.7.5, so if we know that the map between them has a determinant which vanishes simply along D then the same will be true for the restrictions to $\operatorname{Spec}\mathbb{C}[t]/(t^2)$. The image in $R^1\pi_{2*}(\mathcal{W}\otimes\pi_1^*M^{-1})$ is the direct image of a line bundle $\mathcal{K}$ on D. Furthermore suppose that $\dim H^1(X; \mathcal{L}_2\otimes M^{-1}\otimes I_{Y_t})$ is independent of t. Then $R^0\pi_{2*}(\pi_1^*\mathcal{L}_2\otimes I_{\mathcal{Y}}\otimes\pi_1^*M^{-1})$ is locally free and compatible with base change. If it is nonzero suppose further that

$$R^2\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes M^{-1}) = 0.$$

Thus the torsion part of $R^1\pi_{2*}(\mathcal{W} \otimes \pi_1^* M^{-1})$ is just $\mathcal{K}$ and the restriction of $\mathcal{K}$ to $\operatorname{Spec}\mathbb{C}[t]/(t^2)$ gives the kernel of multiplication by t as in (ii). We can then take the map from the torsion part of $R^1\pi_{2*}(\mathcal{W} \otimes \pi_1^* M^{-1})_0$, namely the image of $H^1(\mathcal{L}_1 \otimes M^{-1})$, to $H^1(L \otimes I_Z \otimes M^{-1}) = \operatorname{Ext}^1(M, L \otimes I_Z)$ and this image gives the extension class.

We will also need to consider a slightly different situation. Suppose that $\mathcal{W}$ is a rank two vector bundle on $X \times T$, E is a smooth divisor on X and L is a line bundle on $E \times T$. Let $j: E \times T \to X \times T$ be the inclusion and let $\Phi: \mathcal{W} \to j_*L$ be a morphism. We may think of Φ as a family of morphisms parametrized by T. In local coordinates Φ is given by two functions f, g on $E \times T$, whose vanishing defines a subscheme Y of $E \times T$. Away from the projection $\pi_2(Y)$ of Y to T, Φ defines a family of elementary modifications which degenerates over $\pi_2(Y)$ at the points of Y.

Proposition A.5. *Let $\Phi: \mathcal{W} \to j_*L$ be a morphism as above and suppose that the cokernel of Φ is supported on a nonempty codimension two subset Y of $E \times T$, necessarily a local complete intersection. Suppose further that, for each $t \in T$, the codimension of $Y \cap (X \times \{t\})$ in $X \times \{t\}$ is at least two if $Y \cap (E \times \{t\}) \neq \emptyset$. Let $\mathcal{V}$ be the kernel of Φ. Then $\mathcal{V}$ is a reflexive sheaf, flat over T, and its restriction to each slice $X \times \{t\}$ is a torsion free sheaf on X.*

Proof. The proof of (A.2)(i) shows that $\mathcal{V}$ is reflexive. As for the rest, the problem is local around a point of Y. Let R be the local ring of $X \times T$ at a point (x, t), R' the local ring of T at t, and S the local ring of $X \times \{t\}$ at (x, t). Let u be the local equation for E in $X \times T$. Then locally Φ corresponds to a map $R \oplus R \to R/uR$, necessarily given by elements $\bar{f}$, $\bar{g} \in R/uR$. Lift $\bar{f}$ and $\bar{g}$ to elements $f, g \in R$. Then $(u, f, g)R$ is the ideal of Y in R, and Y has codimension three in $X \times T$. Thus u, f, g is a regular sequence, any two of the three are relatively prime, and necessarily $\dim R \geq 3$.

The kernel M of the map $R \oplus R \to R/uR$ given by $(a, b) \mapsto a\bar{f} + b\bar{g}$ is clearly generated by $(-g, f)$, $(u, 0)$, and $(0, u)$. These three elements define a surjection $R \oplus R \oplus R \to M$. The kernel of this surjection is easily calculated to be $R \cdot (u, g, -f)$. Thus there is an exact sequence

$$0 \to R \to R \oplus R \oplus R \to M \to 0.$$

This sequence restricts to define

$$S \to S \oplus S \oplus S \to M \otimes_R S \to 0.$$

Here the image of S in $S \oplus S \oplus S$ is equal to $S \cdot (u, g, -f)$, where we denote the images of u, f, g in S by the same letter. By hypothesis, not all of u, f, g vanish on $X \times \{t\}$ and so this map is injective. By the local criterion of flatness M is flat over R'. Finally we must show that $M \otimes_R S$ is a torsion free S-module. By hypothesis $u, g, -f$ generate the ideal of a subscheme of $\operatorname{Spec} S$ of codimension at least two and thus u does not divide both f and g in S. Given $h \in S$ with $h \neq 0$, suppose that $hm = 0$ for some $m \in M \otimes_R S$. Then there is $(a, b, c) \in S \oplus S \oplus S$ such that $h(a, b, c) = \alpha(u, g, -f)$. We claim that $u | a$. To see this, let n be the largest integer such that $u^n | h$. Then $u^n | hb = \alpha g$ and likewise $u^n | \alpha f$. Since at least one of f, g is prime to u, $u^n | \alpha$. But then $u^{n+1} | \alpha u = ha$, so that $u | a$. If $a = ua'$, then $\alpha = ha'$ and so $hb = ha'g$ and $b = a'g$. Likewise $c = a'(-f)$. Thus $(a, b, c) = a'(u, g, -f)$ and its image in $M \otimes_R S$ is zero. It follows that $M \otimes_R S$ is torsion free. $\square$

Let us finally remark that we can calculate the class $p_1(\operatorname{ad} \mathscr{V})$, in the above notation, by applying the lemma on elementary modifications given in the introduction, since Φ is surjective in codimension two.

References

1. M. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. **7** (1957), 414–452.
2. S. Bauer, *Some nonreduced moduli of bundles and Donaldson invariants for Dolgachev surfaces*, J. Reine Angew. Math. **424** (1992), 149–180.
3. ——, *Diffeomorphism classification of elliptic surfaces with* $p_g = 1$, J. Reine Angew. Math. **451** (1994), 89–148.
4. I. Dolgachev, *Algebraic surfaces with* $q = p_g = 0$, Algebraic Surfaces, C.I.M.E. Cortona 1977, Liguori, Napoli, 1981, pp. 97–215.
5. S. K. Donaldson, *Irrationality and the h-cobordism conjecture*, J. Differential Geom. **26** (1987), 141–168.
6. S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Clarendon, Oxford, 1990.
7. R. Fintushel and R. Stern, *$SO(3)$-connections and the topology of 4-manifolds*, J. Differential Geom. **20** (1984), 523–539.
8. R. Friedman, *Rank two vector bundles over regular elliptic surfaces*, Invent. Math. **96** (1989), 283–332.
9. ——, *Stable vector bundles on algebraic varieties*, in preparation.
10. R. Friedman and J. W. Morgan, *On the diffeomorphism types of certain algebraic surfaces* I, J. Differential Geom. **27** (1988), 297–369; II, J. Differential Geom. **27** (1988), 371–398.
11. ——, *Smooth four-manifolds and complex surfaces*, Ergebnisse Math. Grenzgeb. (3), vol. 27, Springer-Verlag, Berlin, Heidelberg, and New York, 1994.
12. W. Fulton, *Intersection theory*, Springer-Verlag, Berlin and Heidelberg, 1984.
13. Y. Kametani and Y. Sato, *0-dimensional moduli spaces of stable rank 2 bundles and differentiable structures on regular elliptic surfaces*, preprint.
14. D. Kotschick, *On manifolds homeomorphic to* $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P}^2$, Invent. Math. **95** (1989), 591–600.
15. ——, *$SO(3)$-invariants for 4-manifolds with* $b_2^+ = 1$, Proc. London Math. Soc. **63** (1991), 426–448.
16. D. Kotschick and J. W. Morgan, *$SO(3)$-invariants for 4-manifolds with* $b_2^+ = 1$ II, J. Differential Geom. **39** (1994), 443–456.

17. P. Kronheimer and T. Mrowka, *Recurrence relations and asymptotics for four-manifold invariants*, Bull. Amer. Math. Soc. (N.S.) **30** (1994), 215–221.

18. J. Li, *Algebraic geometric interpretation of Donaldson's polynomial invariants of algebraic surfaces*, J. Differential Geom. **37** (1993), 417–466.

19. J. W. Morgan, *Comparison of the Donaldson polynomial invariants with their algebro-geometric analogues*, Topology **32** (1993), 449–488.

20. J. W. Morgan and T. Mrowka, *On the diffeomorphism classification of regular elliptic surfaces*, Internat. Math. Research Notes **6** (1993), 183–184.

21. J. W. Morgan and K. O'Grady, *Differential topology of complex surfaces elliptic surfaces with* $p_g = 1$ *: Smooth classification*, Lecture Notes in Math., vol. 1545, Springer-Verlag, Berlin, Heidelberg, and New York, 1993.

22. T. Mrowka, *A local Mayer-Vietoris principle for Yang-Mills moduli spaces*, Ph.D. thesis, Berkeley.

23. K. O'Grady, *Algebro-geometric analogues of Donaldson's polynomials*, Invent. Math. **107** (1992), 351–395.

24. Z. Qin, *Equivalence classes of polarizations and moduli spaces of sheaves*, J. Differential Geom. **37** (1993), 397–415.

25. J.-P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Math., vol. 5, Springer-Verlag, Berlin, Heidelberg, and New York, 1973.

[EGA] A. Grothendieck and J. Dieudonné, *Étude cohomologique des faisceaux cohérents*, Inst. Hautes Études Sci. Publ. Math. **17** (1963), 137–223.

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