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MODULI OF HIGH RANK VECTOR BUNDLES OVER SURFACES

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0. Introduction

The purpose of this work is to apply the degeneration theory developed in [GL] to study the moduli space of stable vector bundles of arbitrary rank on any smooth algebraic surface (over $\mathbb{C}$). We will show that most of the recent progress in understanding moduli of rank two vector bundles can be carried over to high rank cases.

After introducing the notion of stable vector bundles, the first author constructed the moduli schemes of vector bundles on surfaces. He showed that for any smooth algebraic surface X with ample divisor H and line bundle I on X, there is a coarse moduli scheme $\mathfrak{M}_X^{r,d}(I,H)$ parameterizing (modulo equivalence relation) the set of all H-semistable rank r torsion free sheaves E on X with $\det E = I$ and $c_2(E) = d$. Since then, many mathematicians have studied the geometry of this moduli space, especially for the rank two case. To cite a few, Maruyama, Taubes and the first author showed that the moduli space $\mathfrak{M}_X^{2,d}$ $(= \mathfrak{M}_X^{r,d}(I,H))$ is non-empty when d is large. Moduli spaces of vector bundles of some special surfaces have been studied also.

The deep understanding of $\mathfrak{M}_X^{r,d}$ for arbitrary X and $r = 2$ begins with Donaldson's generic smoothness result. Roughly speaking, Donaldson [Do] (later generalized by Friedman [Fr] and K. Zhu [Zh]) showed that when d is large enough, then the singular locus $\operatorname{Sing}(\mathfrak{M}_X^{2,d})$ of $\mathfrak{M}_X^{2,d}$ is a proper subset of $\mathfrak{M}_X^{2,d}$ and its codimension in $\mathfrak{M}_X^{2,d}$ increases linearly in d. This theorem indicates that the moduli $\mathfrak{M}_X^{2,d}$ behaves as expected when the second Chern class d is large. Later, using general deformation theory, the second author proved that $\mathfrak{M}_X^{2,d}$ is normal, and has local complete intersection (l.c.i.) singularities at stable sheaves provided d is large [L2]. He also showed that when X is a surface of general type satisfying some mild technical conditions, then $\mathfrak{M}_X^{2,d}$ is of general type for $d \gg 0$ [L2]. In our paper [GL], we also proved that $\mathfrak{M}_X^{2,d}$ is irreducible if d is large.

In this and subsequent papers, we shall show that the geometry of $\mathfrak{M}_X^{2,d}$ and the geometry of $\mathfrak{M}_X^{r,d}$, $r \geq 3$, is rather similar. The main obstacle in doing so is the lack of an analogy of the generic smoothness result in high rank case. In this paper, we will use the degeneration of moduli developed in [GL] to establish the following main technical theorem.

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Theorem 0.1. *Let X be a smooth algebraic surface, H an ample line bundle and I a line bundle on X. Let $r \geq 2$ be any integer. Then for any constant C_1 and any divisor $D \subseteq X$, there is an N such that whenever $d \geq N$, then we have*

$$\dim\{E \in \mathfrak{M}_X^{r,d} \mid \operatorname{Ext}^0(E, E(D))^0 \neq \{0\}\} \leq \eta(r,d,I) - C_1,$$

where $\eta(r,d,I) = 2rd - (r-1)I^2 - (r^2-1)\chi(\mathcal{O}_X)$ is the expected dimension of $\mathfrak{M}_X^{r,d}$ $(= \mathfrak{M}_X^{r,d}(I,H))$ and the superscript 0 stands for the traceless part of $\operatorname{Ext}^i(\cdot,\cdot)$.

According to [Ar], [Mu], $\mathfrak{M}_X^{r,d}$ is regular at E if E is stable and $\operatorname{Ext}^2(E,E)^0 = \{0\}$. As to the subset of strictly semistable sheaves in $\mathfrak{M}_X^{r,d}$, it is easy to show that its dimension is much less than $\eta(r,d,I) - C_1$ when d is large. After applying Theorem 0.1 to the divisor $D = K_X$ and using the Serre duality, we conclude that for d sufficiently large,

$$\dim \operatorname{Sing}(\mathfrak{M}_X^{r,d}) \leq \eta(r,d,I) - C_1.$$

On the other hand, based on deformation theory, each component of $\mathfrak{M}_X^{r,d}$ has dimension at least $\eta(r,d,I)$. Thus, we have proved the following theorem.

Theorem 0.2. *Let X be a smooth algebraic surface, H an ample line bundle and I a line bundle on X. Let $r \geq 2$ be any integer. Then for any constant C_1, there is an N such that whenever $d \geq N$, then $\mathfrak{M}_X^{r,d}$ has pure dimension $\eta(r,d,I)$ and further,*

$$\operatorname{codim}(\operatorname{Sing}(\mathfrak{M}_X^{r,d}), \mathfrak{M}_X^{r,d}) \geq C_1.$$

Once we have settled the generic smoothness result, we can generalize some other properties of $\mathfrak{M}_X^{2,d}$ to high rank case. In this paper, we will prove

Theorem 0.3. *With the notation as in Theorem 0.2, there is an N such that whenever $d \geq N$, then:*

(1) *$\mathfrak{M}_X^{r,d}$ is normal. Further, if $s \in \mathfrak{M}_X^{r,d}$ is a closed point corresponding to a stable sheaf, then $\mathfrak{M}_X^{r,d}$ is a local complete intersection at s.*
(2) *The set of locally free μ-stable sheaves $(\mathfrak{M}_X^{r,d})^\mu \subseteq \mathfrak{M}_X^{r,d}$ is dense in $\mathfrak{M}_X^{r,d}$.*
(3) *For any polarizations H_1 and H_2 of X, the moduli $\mathfrak{M}_X^{r,d}(I,H_1)$ is birational to $\mathfrak{M}_X^{r,d}(I,H_2)$. (In this case, N depends on both H_1 and H_2.)*

To illustrate the idea of the proof of our main theorem (Theorem 0.1), let us first recall the degeneration of moduli $\mathfrak{M}_X^{r,d}$ constructed in [GL]. Let $0 \in C \subseteq \operatorname{Spec} \mathbb{C}[t]$ be a smooth curve that functions as a parameter space and let $Z \to C$ be a family of surfaces that is the result of blowing-up $X \times C$ along $\Sigma \times \{0\}$, where $\Sigma \in |H|$ is a smooth very ample divisor. Clearly, $Z_t = \pi^{-1}(t)$ is X and $Z_0 = X \cup \Delta$, where Δ is a ruled surface over Σ. Over $C^* = C \setminus \{0\}$, we have a constant family $\mathfrak{M}_X^{r,d} \times C^*$. In [GL], we have constructed completions of $\mathfrak{M}_X^{r,d} \times C^*$ over C. These completions depend on the choice of ample divisors on Z. The ample divisor which we will use is a multiple of the $\mathbb{Q}$-divisor $p_X^* H(-(1-\varepsilon)\Delta)$ that depends on the rational $\varepsilon \in (0, \frac{1}{2})$. We denote this completion by $\mathfrak{M}^{d,\varepsilon}$. There is a nice description of closed points of the special fiber $\mathfrak{M}_0^{d,\varepsilon}$: Any point of $\mathfrak{M}_0^{d,\varepsilon}$ corresponds uniquely to an equivalence class of semistable sheaves on Z_0.

Now let $D \subseteq X$ be any divisor and $\mathcal{N} \subseteq \mathfrak{M}_X^{r,d}$ be the set of sheaves E such that

$$\operatorname{Hom}(E, E(D))^0 \neq \{0\}. \tag{0.1}$$

Put $\mathfrak{N}^{d,\varepsilon} \subseteq \mathfrak{M}^{d,\varepsilon}$ as the closure of $\mathcal{N} \times C^*$ in $\mathfrak{M}^{d,\varepsilon}$. To show that for any constant C_1 and large d we have

$$\dim \mathcal{N} \leq \eta(r, d, I) - C_1,$$

it suffices to show

$$\dim \mathfrak{N}_0^{d,\varepsilon} \leq \eta(r, d, I) - C_1. \tag{0.2}$$

Now let $E \in \mathfrak{N}_0^{d,\varepsilon}$ be any sheaf. Note that E is a limit of sheaves in $\mathcal{N}$ and that sheaves in $\mathcal{N}$ satisfy (0.1). So by the semicontinuity theorem, for any invertible sheaf $\mathcal{L}$ on Z such that $\mathcal{L}_{|Z_t} \cong \mathcal{O}_X(D)$, we have

$$\operatorname{Hom}_{Z_0}(E, E \otimes \mathcal{L}_{Z_0})^0 \neq \{0\}.$$

In particular, if we choose $\mathcal{L}$ to be $p_X^*\mathcal{O}_X(D)(-k\Delta)$, where $p_X : Z \to X$ is the projection, we get

$$\operatorname{Ext}^0_{Z_0}(E, E \otimes p_X^*\mathcal{O}_X(D)(-k\Delta))^0 \neq \{0\}, \quad \forall k \in \mathbb{Z},\ E \in \mathfrak{N}_0^{d,\varepsilon}. \tag{0.3}$$

Since E is semistable, $E_{|X}$ and $E_{|\Delta}$ as sheaves on X and Δ respectively will satisfy some weak stability conditions. (For simplicity, here we assume E is locally free.) On the other hand, for large k, the non-vanishing of

$$\operatorname{Ext}^0_X(E_{|X}, E_{|X}(D - k\Sigma))^0 \tag{0.4}$$

will force $E_{|X}$ to be very unstable. Therefore, we can choose a $k > 0$ (independent of d, ε and $\mathcal{N}$) such that (0.4) is always trivial. Thus (0.3) will force

$$\operatorname{Ext}^0_\Delta(E_{|\Delta}, E_{|\Delta} \otimes p_X^*\mathcal{O}_X(D)(k\Sigma))^0 \neq \{0\}. \tag{0.5}$$

(0.5) certainly is possible for sheaves over Δ. However, if we can show that the number of moduli of the set of sheaves F (over Δ) satisfying (0.5) is strictly less than

$$\text{the number of moduli of } \{E_\Delta \mid E \in \mathfrak{M}_0^{d,\varepsilon}\} - C_1,$$

then $\operatorname{codim}(\mathfrak{N}_0^{d,\varepsilon}, \mathfrak{M}_0^{d,\varepsilon}) \geq C_1$, which is exactly what we need. Therefore, the proof of Theorem 0.1 is reduced to the proof of the following theorem.

Theorem 0.4. *Let X be any ruled surface and let H and I be as in Theorem* 0.1. *Then for any integer r, any divisor $D \subseteq X$ and any constant C_1, there is a constant N such that for $d \geq N$,*

$$\dim\bigl\{E \in \mathfrak{M}_X^{r,d} \mid \operatorname{Ext}^0(E, E(D))^0 \neq \{0\}\bigr\} \leq \eta(r, d, I) - C_1.$$

The advantage of working with a ruled surface lies in the fact that every vector bundle on a ruled surface can be constructed explicitly as follows: Let $X = \Delta$ and let E be a vector bundle on Δ. For simplicity, we assume for the general fiber P_ξ of

$\pi:\Delta \to \Sigma$, the restriction sheaf $E_{|P_\xi} \cong \mathcal{O}_{P_\xi}^{\oplus r}$. Then there is a unique rank r vector bundle V on Σ and a sheaf F supported on a finite number of fibers of π such that

$$0 \longrightarrow E \longrightarrow \pi^* V \xrightarrow{\varphi} F \longrightarrow 0$$

is exact. When E is general, F is of the form $\bigoplus \mathcal{O}_{P_i}(1)$, where P_i are fibers of π. Thus the condition under which E admits a traceless homomorphism $E \to E(D)$ can be interpreted in terms of the location of P_i's and the choice of homomorphism φ. The argument to carry out this approach is rather straightforward though quite technical and will occupy the first section of this paper. In §2, we will review the degeneration construction and use it to prove Theorem 0.1. Theorems 0.2–0.4 will be proved in §3. We remark that after the completion of the initial version of this work, O'Grady has improved our results in his paper [OG].

Conventions and preliminaries. All schemes are defined over the field of complex numbers $\mathbb{C}$ and are of finite type. All points are closed points unless otherwise mentioned. We shall always identify a vector bundle with its sheaf of sections. If I and J are two line bundles on a surface, then we denote by $I \cdot J$ the intersection $c_1(I) \cdot c_1(J)$ and by I^2 the self-intersection $c_1(I) \cdot c_1(I)$. We will use $\sim$ to denote the numerical equivalence of divisors (line bundles). For the coherent sheaf F, we denote by $\mathrm{rk}(F)$ the rank of F. In case F is supported on a finite number of points on X, we denote by $\ell(F)$ the length of F. If p and q are two polynomials with real coefficients, we say $p \succ q$ (resp. $p \succeq q$) if $p(n) > q(n)$ (resp. $p(n) \geq q(n)$) for all $n \gg 0$.

In the following, X will always denote a smooth projective surface. Let H be a very ample line bundle on X. For any sheaf E on X, we denote by χ_E the Poincaré polynomial of E, namely, $\chi_E(n) = \chi(E(n))$, $E(n) = E \otimes H^{\otimes n}$, and denote by p_E the polynomial $\frac{1}{\mathrm{rk}(E)}\chi_E$ when $\mathrm{rk}(E) \neq 0$. Unless the contrary is mentioned, the degree of a sheaf E is $c_1(E) \cdot H$. We recall the notion of stability:

Definition 0.5. A sheaf E on X is said to be *stable* (resp. *semistable)* with respect to H if E is coherent, torsion free and if one of the following two equivalent conditions hold:

(1) Whenever $F \subset E$ is a proper subsheaf, then $p_F \prec p_E$ (resp. $p_F \preceq p_E$).
(2) Whenever $E \to Q$ is a quotient sheaf, $\mathrm{rk}(Q) > 0$, then $p_E \prec p_Q$ (resp. $p_E \preceq p_Q$).

When E is a torsion free coherent sheaf on X, we define the slope $\mu(E) = \frac{1}{\mathrm{rk}(E)} \deg E$.

Definition 0.6. Let e be a constant. The sheaf E is said to be e-*stable* if one of the following two equivalent conditions holds:

(1) Whenever $F \subset E$ is a subsheaf with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$, then $\mu(F) < \mu(E) + \frac{1}{\mathrm{rk}(F)}\sqrt{H^2} \cdot e$.
(2) Whenever $E \to Q$ is a quotient sheaf with $0 < \mathrm{rk}(Q) < \mathrm{rk}(E)$, then $\mu(E) < \mu(Q) + \frac{1}{\mathrm{rk}(Q)}\sqrt{H^2} \cdot e$.

We call E μ-stable if E is e-stable with $e = 0$. When the strict inequality is replaced by $\leq$, then we call E e-semistable.

Let $W \to S$ be a flat morphism and let $E \to W$ be any sheaf on W. For any closed $s \in S$, we will use W_s to denote the fiber of W over s and use E_s

to denote the restriction of E to W_s. For any subscheme $T \subseteq W$, we denote by $E_{|T}$ the restriction of E to T. We shall adopt the following convention: If $\mathcal{R}$ is a set of sheaves on X, then the number of moduli of $\mathcal{R}$ is the smallest integer m so that there are countably many schemes (of finite types) of dimension at most m, say $S_1, S_2, \cdots$, and flat families of sheaves $E_{S_1}, E_{S_2}, \cdots$ on $X \times S_1, X \times S_2, \cdots$ respectively of which the following holds: For any $F \in \mathcal{R}$, there is a closed $s \in S_k$ for some k such that $F \cong E_{S_k,s}$. We will denote by $\#_{\mathrm{mod}}(\mathcal{R})$ the number of moduli of $\mathcal{R}$. In case R is a scheme parameterizing a family of sheaves and $t \in R$, then we denote by $\#^{\mathrm{loc}}_{\mathrm{mod}}(R, [t])$ the number of moduli of sheaves parameterized by the germ of R at r. In particular, we write $\#^{\mathrm{loc}}_{\mathrm{mod}}(E)$, where E is any sheaf, for $\#^{\mathrm{loc}}_{\mathrm{mod}}(\mathcal{Q}, [E])$ (= the number of moduli of the set of all "small" deformations of E), where $\mathcal{Q}$ is Grothendieck's Quot-scheme [Gr] that contains all deformations of E as quotient sheaves of some appropriate locally free sheaf. Another notion we use frequently is $\#_{\mathrm{aut}}(E) = \dim \mathrm{Aut}(E)$, where $\mathrm{Aut}(E)$ is the group of automorphisms of E. Note $\#_{\mathrm{aut}}(E) = h^0(\mathcal{E}nd(E))$. When $\mathcal{R}$ is a set of sheaves, then $\#_{\mathrm{aut}}(\mathcal{R}) = \max\{\#_{\mathrm{aut}}(E) \mid E \in \mathcal{R}\}$.

1. Vector bundles on a ruled surface

The purpose of this section is to prove an analogy of Theorem 0.1 for a ruled surface Δ. Before giving the precise statement of the theorem, we first introduce some notation. Let Σ be a smooth curve and let $\pi : \Delta \to \Sigma$ be a ruled surface. For simplicity, we assume Δ is the projective bundle of a direct sum of a trivial line bundle with a very ample line bundle (over Σ). Hence $\pi : \Delta \to \Sigma$ has a unique section Σ^- with $\Sigma^- \cdot \Sigma^- < 0$ and has many sections with positive self-intersection. We choose one such section and denote it by Σ^+. By assumption, $|\Sigma^+|$ is base point free. Let H be an ample line bundle on Δ that is numerically equivalent to (denoted by $\sim$) $a\Sigma^+ + bP_\xi$, where P_ξ is a general fiber of π. Let e be a constant, let I be a line bundle on Δ and let D be any divisor on Δ. In this section, we will study the set $\mathfrak{A}^{r,d}_{e,I,H}$ of all e-semistable (with respect to H) rank r locally free sheaves E with $\det E = I$ and $c_2(E) = d$ and the set

$$\mathfrak{A}^{r,d}_{e,I,H}(D) = \{E \in \mathfrak{A}^{r,d}_{e,I,H} \mid \mathrm{Hom}(E, E(D))^0 \neq \{0\}\}.$$

Here and in the following, the superscript 0 always stands for the traceless part of the group or sheaf. For technical reasons, we will choose H to be very close to Σ^+ in the sense that b/a is very small. With the choice of H understood, we will not build H into the notation and will write $\mathfrak{A}^{r,d}_{e,I}$ (resp. $\mathfrak{A}^{r,d}_{e,I}(D)$) for $\mathfrak{A}^{r,d}_{e,I,H}$ (resp. $\mathfrak{A}^{r,d}_{e,I,H}(D)$). We will also use $\eta_\Delta(E) = \eta_\Delta(\mathrm{rk}(E), c_2(E), c_1(E))$ to denote the number

$$\eta_\Delta(r, d, I) = 2rd - (r-1)I^2 - (r^2-1)\chi(\mathcal{O}_\Delta). \tag{1.1}$$

$\eta_\Delta(r, d, I)$ is the expected dimension of $\mathfrak{A}^{r,d}_{e,I}$. Because in this section we work solely with the surface Δ, we will simply write η for η_Δ. The theorem we will prove in this section is the following.

Theorem 1.1. *Given r and Δ, there is an $\varepsilon_0 > 0$ depending on r and Δ for which the following holds: For any ample divisor $H \sim a\Sigma^+ + bP_\xi$ with $b/a < \varepsilon_0$ and for* *any choice of constants e, C and divisor $D \subseteq \Delta$, $I \in \mathrm{Pic}(\Delta)$, there is an integer N such that whenever $d \geq N$, then we have*

$$\#_{\mathrm{mod}}\big(\mathfrak{A}^{r,d}_{e,I}(D)\big) \leq \eta(r,d,I) - C. \tag{1.2}$$

The advantage of working with ruled surfaces lies in having a powerful structure theorem of torsion free sheaves on Δ. Let E be any torsion free sheaf of rank r. By Grothendieck's splitting theorem, its restriction to a generic fiber P_ξ has the form

$$E_{|P_\xi} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{P_\xi}(\alpha_i)^{\oplus r_i}, \quad \alpha_1 > \cdots > \alpha_n. \tag{1.3}$$

In the following, we call $\alpha = (\alpha_1^{\oplus r_1}, \cdots, \alpha_n^{\oplus r_n})$ the generic fiber type of E. (The integer sequence $\{\alpha_i\}$ is always assumed to be strictly decreasing.) We let $\ell(\alpha) = \sum_{i=1}^n r_i\alpha_i$. Clearly, $r = \sum_{i=1}^n r_i$ and further, when $\det E = I$ and $\deg I_{|P_\xi} = m$, then $m = \ell(\alpha)$. $\mathfrak{A}^{r,d}_{e,I}$ can be divided into strata according to the generic fiber types of individual vector bundles. Let $r \in \mathbb{N}$ and $I \in \mathrm{Pic}(\Delta)$ be fixed. Without loss of generality, we can assume $0 \leq \deg I_{|P_\xi} \leq r-1$. Let $m = \deg I_{|P_\xi}$ and let 1_m be the fiber type $(1^{\oplus m}, 0^{\oplus(r-m)})$. For any fiber type α with $\ell(\alpha) = m$, we let

$$\mathfrak{A}^{r,d}_{e,I}(\alpha) = \{E \in \mathfrak{A}^{r,d}_{e,I} \mid E \text{ has generic fiber type } \alpha\}.$$

The first observation we have is that except for $\alpha = 1_m$, none of $\#_{\mathrm{mod}}\big(\mathfrak{A}^{r,d}_{e,I}(\alpha)\big)$ are close to $\eta(r,d,I)$. More precisely, we have

Theorem 1.2. *Let $m = \deg I_{|P_\xi}$. There are constants C_1 and ε_0 depending on (r, Δ) such that for any ample divisor $H \sim a\Sigma^+ + bP_\xi$ with $b/a < \varepsilon_0$ and any fiber type $\alpha \neq 1_m$, we have*

$$\#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(\alpha) \leq (2r-1)d + C_1.$$

The proof of Theorem 1.2 goes as follows: Let $\alpha = (\alpha_1^{\oplus r_1}, \cdots, \alpha_n^{\oplus r_n})$ be any fiber type. Then each $E \in \mathfrak{A}^{r,d}_{e,I}(\alpha)$ admits a relative Hardar-Narasimhan filtration

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E \tag{1.4}$$

of which the quotient sheaves $F_i = E_i/E_{i-1}$ are torsion free with generic fiber types $(\alpha_i^{\oplus r_i})$ respectively. Clearly, the deformation of E within $\mathfrak{A}^{r,d}_{e,I}(\alpha)$ depends on deformation of individual F_i and the extension $E_i \to E_{i+1} \to F_{i+1}$. The contribution of these data to the number of moduli of $\mathfrak{A}^{r,d}_{e,I}(\alpha)$ can be estimated by using Riemann-Roch. The details of the proof will be provided shortly.

In light of Theorem 1.2, to prove Theorem 1.1 we only need to study the stratum $\mathfrak{A}^{r,d}_{e,I}(1_m)$ and

$$\mathfrak{A}^{r,d}_{e,I}(1_m, D) = \{E \in \mathfrak{A}^{r,d}_{e,I}(1_m) \mid \mathrm{Hom}(E, E(D))^0 \neq 0\}. \tag{1.5}$$

In this section, we will first establish Theorem 1.1 for the stratum $\mathfrak{A}^{r,d}_{e,I}(1_0, D)$ and derive the remainder by induction on m.

Let E be any vector bundle of generic fiber type 1_0=$(0^{\oplus r})$. Let $x \in \Sigma$ be any point, let P_x be the fiber of π over $x \in \Sigma$ and let $\beta_x(E) = (\beta_1^{\oplus r_1}, \cdots, \beta_n^{\oplus r_n})$ be the fiber type of $E_{|P_x}$. In case $\beta_x(E) \neq 1_0$, we call P_x a jumping line of E. Let P_x be a jumping line of E. We then perform semistable reduction on E along P_x by taking F to be the kernel of the (unique surjective) homomorphism $E \to \mathcal{O}_{P_x}(\beta_n)^{\oplus r_n}$. For convenience, we will use Υ_x to denote this operation and denote $F = \Upsilon_x(E)$ and $\omega_x(E) = \beta_n^{\oplus r_n}$. Clearly, F belongs to the exact sequence

$$0 \longrightarrow F \longrightarrow E \xrightarrow{\varphi} \mathcal{O}_{P_x}(\beta_n)^{\oplus r_n} \longrightarrow 0. \tag{1.6}$$

An easy calculation based on Riemann-Roch yields

Lemma 1.3. *Let $F = \Upsilon_x(E)$ with $\omega_x(E) = t^{\oplus s}$. Then $c_1(F) = c_1(E) - s[P_x]$ and $c_2(F) = c_2(E) + s\cdot t$. In particular, $\eta(r, c_2(F), c_1(F)) = \eta(r, c_2(E), c_1(E)) + 2rs\cdot t$.*

Proof. See [Br, p. 166].

In case F still has a jumping line, say P_y of type $(\cdots, \gamma_l^{\oplus s_l})$, then we can further perform semistable reduction on F to get $F_2 = \Upsilon_y(F)$. We can iterate this process as long as the resulting vector bundle F_k still admits jumping lines. In general, if F_k is derived by successively performing this type of elementary transformations, namely, $F_0 = E$ and $F_{i+1} = \Upsilon_{x_i}(F_i)$ with $\omega_{x_i}(F_i) = t_i^{\oplus s_i}$ for $i = 0, \cdots, k-1$, then we will write

$$F_k = \Upsilon_\Lambda(E), \quad \Lambda = \langle x_1, \cdots, x_k\rangle$$

and define $\omega_\Lambda(E) = \langle t_1^{\oplus s_1}, \cdots, t_k^{\oplus s_k}\rangle$. We call k the length of Λ.

Lemma 1.4. *For any vector bundle E of generic fiber type 1_0, there is a finite length $\Lambda = \langle x_1, \cdots, x_k\rangle$ such that $\Upsilon_\Lambda(E)$ has no jumping lines.*

Proof. By Lemma 1.3, the second Chern class of $\Upsilon_x(E)$ is strictly less than $c_2(E)$ because $\beta_n < 0$ when $\beta_x(E) \neq 1_0$. Thus Lemma 1.4 follows if we can show that any vector bundle of generic fiber type 1_0 has non-negative second Chern class. Indeed, let E be any vector bundle of generic fiber type 1_0. We choose a divisor D supported on fibers of π such that $\mathcal{O}_D$ is a subsheaf of E with $E/\mathcal{O}(D)$ torsion free. Since $E/\mathcal{O}_D$ has generic fiber type 1_0, we can assume $c_2(E/\mathcal{O}(D)) \geq 0$ by the induction hypothesis on the rank of E. Hence,

$$c_2(E) = c_2\big(E/\mathcal{O}(D)\big) + D\cdot(c_1(E) - D) = c_2\big(E/\mathcal{O}(D)\big) \geq 0.$$

This completes the proof of Lemma 1.4.

Let E be a vector bundle of generic fiber type 1_0 and let $\Lambda = \langle x_1, \cdots, x_k\rangle$ be such that $F = \Upsilon_\Lambda(E)$ has no jumping lines. Then F is a pull-back vector bundle π^*V whose dual belongs to the exact sequence

$$0 \longrightarrow E^\vee \longrightarrow \pi^*V^\vee \longrightarrow J \longrightarrow 0, \tag{1.7}$$

where J is a torsion sheaf supported on the union of fibers P_{x_i}. Usually, the sheaf J near some fiber P_{x_i} can be very complicated. The case that is easy to understand and will be dealt with extensively in the subsequent discussion is when $J \cong \mathcal{O}_{P_{x_i}}(1)$ near P_{x_i}. The following theorem says that when the number of moduli of $\mathfrak{A}_{e,I}^{r,d}(1_0)$ is close to $\eta(r, d, I)$, then for general $E \in \mathfrak{A}_{e,I}^{r,d}(1_0)$ with the exact sequence (1.7), $J \cong \mathcal{O}_{P_{x_i}}(1)$ near P_{x_i} for all $x_i \in \{x_1 \cdots x_k\}$ except for a bounded number of fibers.

Theorem 1.5. *For any constant e, there is a constant C_2 such that*

$$\#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I} \le \eta(r,d,I) + C_2. \tag{1.8}$$

Further, for any constant C, there are integers l, l_1, l_2 and N_1 of which the following holds: Assume $d \ge N_1$ and that S is a variety parameterizing a subset of $\mathfrak{A}^{r,d}_{e,I}(1_0)$ satisfying $\#_{\mathrm{mod}}(S) \ge \eta(r,d,I) - C$. Then there is a line bundle L on Σ of degree $[(d-c)/r] + l_1$, where $c = I \cdot \Sigma^+$, so that for general $E \in S$, there are

1. *$d-l$ distinct points $x_1, \cdots, x_{d-l} \in \Sigma$ in general position, a surjective homomorphism $\tau_1 : \pi^* L^{\oplus r} \to \bigoplus_{i=1}^{d-l} \mathcal{O}_{P_{x_i}}(1)$ and*
2. *a zero-dimensional scheme (divisor) $z_0 \subseteq \Sigma$ away from $\{x_1, \cdots, x_{d-l}\}$ with $\ell(z_0) \le l_2$ and a sheaf of $\mathcal{O}_{\pi^{-1}(z_0)}$-modules J with a quotient homomorphism $\tau_0 : \pi^* L^{\oplus r} \to J$ so that $E^\vee$ belongs to the exact sequence*

$$0 \longrightarrow E^\vee \longrightarrow \pi^*(L^{\oplus r}) \xrightarrow{\tau_0 \oplus \tau_1} J \oplus \Big(\bigoplus_{i=1}^{d-l} \mathcal{O}_{P_{x_i}}(1) \Big) \longrightarrow 0. \tag{1.9}$$

This theorem holds for a very simple reason: To maximize the number of moduli of the set of those E in (1.9), we need to maximize the number of moduli of the set of homomorphisms $\tau_0 \oplus \tau_1$ and the quotient sheaves in (1.9). This can only be achieved by letting $J = \{0\}$ and x_i general. Hence, if $\#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(1_0)$ is close to the expected dimension $\eta(r,d,I)$, then the number of fibers in $\mathrm{supp}(J)$ cannot be too large.

Now we sketch how this structure theorem of $\mathfrak{A}^{r,d}_{e,I}(1_0)$ leads to the proof of Theorem 1.1. We first prove the case $m = 0$ by contradiction. Assume $\#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(1_0, D) \ge \eta(r,d,I) - C$. Then by Theorem 1.5, the general element $E \in \mathfrak{A}^{r,d}_{e,I}(1_0, D)$ fits into the exact sequence (1.9) with $\{x_1, \cdots, x_{d-l}\}$ and $\tau_1 : \pi^* L^{\oplus r} \to \bigoplus \mathcal{O}_{P_{x_i}}(1)$ general. Now let $F = \ker\{\pi^* L^{\oplus r} \xrightarrow{\tau_0} J\}$ and let $f : E \to E(D)$ be a non-trivial traceless homomorphism. Then f induces a non-trivial traceless homomorphism

$$\tilde{f} : F \longrightarrow F(D + \pi^{-1}(z_0)),$$

where z_0 is a divisor of Σ as in Theorem 1.5(2). Because the position of $x_1, \cdots, x_{d-l}$ and the homomorphism τ_1 are general, we will see by degeneration theoretic methods that $\mathrm{Hom}(F', F'(D + \pi^{-1}(z_0))) \neq 0$ for the torsion free sheaf

$$F' = \ker\big(\pi^* L^{\oplus r} \to \bigoplus (\mathcal{O}_{P_{x_i}} \oplus k_{p_i})\big),$$

where $p_i \in P_{x_i}$. Because of the special choice of F', the non-vanishing of the previous group amounts to saying that for any choice of $p_i \in \Delta$, there are sections of $H^0(\mathcal{O}(D + \pi^{-1}(z_0)))$ that vanish on $[(d-l)/r]$ of $p_1, \cdots, p_{d-l}$. On the other hand, since D is fixed and $\pi^{-1}(z_0)$ is bounded, this is impossible if $\{p_i\}$ are generic and d is sufficiently large. This leads to a contradiction which ensures that $\eta(r,d,I) - \#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(1_0, D)$ can be arbitrarily large.

For the general case, we use induction on m (with $r \ge m$ fixed). Assume the theorem holds for $m - 1 \ge 0$ and assume $\#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(1_m, D) \ge \eta(r,d,I) - C$. Then for general $E \in \mathfrak{A}^{r,d}_{e,I}(1_m, D)$, we can perform an elementary transformation on E along a section Σ^+ to get a new vector bundle $\tilde{E} \in \mathfrak{R}^{r,d'}_{e',I'}(1_{m-1}, \tilde{D})$. By carefully studying this correspondence, we will get the desired estimate of $\#_{\text{mod}}\mathfrak{A}^{r,d}_{e,I}(1_m, D)$ from the known estimate of $\#_{\text{mod}}\mathfrak{R}^{r,d'}_{e',I'}(1_{m-1}, \tilde{D})$, thus establishing Theorem 1.1.

In the following, we will fill in the details of the above sketch. We continue to use the notation introduced before Lemma 1.4. We begin with the estimate of the number of moduli of vector bundles of generic fiber type 1_0. Let $E_0 \in \mathfrak{A}^{r,d}_{e,I}(1_0)$ be any vector bundle of generic fiber type 1_0 and $\Lambda = \langle x_1, \cdots, x_k\rangle$ be such that $F = \Upsilon_\Lambda(E_0)$ has no jumping line. Then F is a pull-back vector bundle π^*V. Let ω be $\omega_\Lambda(E_0) = \langle t_1^{\oplus s_1}, \cdots, t_k^{\oplus s_k}\rangle$ and let

$$\mathcal{S}_{\Lambda,\omega}(F) = \{E \in \mathfrak{A}^{r,d}_{e,I}(1_0) \mid \omega_\Lambda(E) = \omega \text{ and } \Upsilon_\Lambda(E) = F\}.$$

In the following, we will estimate the number of moduli of this set. We first study the case where $\Lambda = \langle x\rangle$ and $\omega = \langle t^{\oplus s}\rangle$. Let $\beta_x(F) = (\cdots, \beta_l^{\oplus r_l})$. Because of the following lemma, either $t < \beta_l$ or $t = \beta_l$.

Lemma 1.6. *Suppose $E_{|P_x}$ has fiber type $(\cdots, \gamma_n^{\oplus s_n})$ and that $\Upsilon_x(E)$ has fiber type $(\cdots, \beta_l^{\oplus r_l})$ at x. Then either $\gamma_n < \beta_l$ or $\gamma_n = \beta_l$ and $r_l \le s_n$.*

Proof. Since $F = \Upsilon_x(E)$ is the kernel of $E \to \mathcal{O}_{P_x}(\gamma_n)^{\oplus s_n}$, $F_{|P_x}$ belongs to the exact sequence

$$0 \longrightarrow \mathcal{O}_{P_x}(\gamma_n)^{\oplus s_n} \longrightarrow F_{|P_x} \longrightarrow \bigoplus_{i=1}^{n-1} \mathcal{O}_{P_x}(\gamma_i)^{\oplus s_i} \longrightarrow 0.$$

Then the lemma follows because $\gamma_n < \gamma_{n-1} < \cdots < \gamma_1$.

Let $E \in \mathcal{S}_{x,\omega}(F)$. By dualizing the sequence (1.6), we get

$$0 \longrightarrow E^\vee \longrightarrow F^\vee \longrightarrow \mathcal{O}_{P_x}(-t)^{\oplus s} \longrightarrow 0. \tag{1.10}$$

Clearly, all possible $E^\vee$ that fit into (1.10) are parameterized by a subset of Ξ that is the total space of $\mathrm{Hom}(F^\vee, \mathcal{O}_{P_x}(-t)^{\oplus s})$. Now let $\Theta \subseteq \Xi$ be the subset consisting of $\gamma : F^\vee \to \mathcal{O}_{P_x}(-t)^{\oplus s}$ such that $\ker(\gamma)^\vee \in \mathcal{S}_{x,\omega}(F)$. Θ admits a left $GL(s,\mathbb{C})$ action and a right $\mathrm{Aut}(F^\vee)$ action as follows: Let $\varphi_1 \in \mathrm{Aut}(F^\vee)$ and let $\varphi_2 \in GL(s) = \mathrm{Aut}(\mathcal{O}_{P_x}(-t)^{\oplus s})$. Then

$$\varphi_2 \cdot \gamma \cdot \varphi_1 = \varphi_2 \circ \gamma \circ \varphi_1 \in \mathrm{Hom}(F^\vee, \mathcal{O}_{P_x}(-t)^{\oplus s}).$$

Geometrically, $\varphi \cdot \gamma \cdot \varphi_1$ corresponds to a locally free sheaf E' defined by

$$0 \longrightarrow E'^\vee \longrightarrow F^\vee \stackrel{\varphi_2\cdot\gamma\cdot\varphi_1}{\longrightarrow} \mathcal{O}_{P_x}(-t)^{\oplus s} \longrightarrow 0.$$

Clearly, E' is isomorphic to $E = \ker(\gamma)^\vee$. Conversely, suppose E_1 and E_2 are two isomorphic locally free sheaves associated to $\gamma_1, \gamma_2 \in \Theta$. Then the isomorphism $\varphi: E_1 \to E_2$ induces an isomorphism between $\Upsilon_x(E_1)$ and $\Upsilon_x(E_2)$. Hence, there is an automorphism $\varphi_1 : F^\vee \to F^\vee$ fitting into the (commutative) diagram

$$\begin{tikzcd}
0 \arrow[r] & E_2^\vee \arrow[r] \arrow[d, "\varphi^\vee"] & F^\vee \arrow[r, "\gamma_1"] \arrow[d, "\varphi_1"] & \mathcal{O}_{P_x}(-t)^{\oplus s} \arrow[r] & 0 \\
0 \arrow[r] & E_1^\vee \arrow[r] & F^\vee \arrow[r, "\gamma_2"] & \mathcal{O}_{P_x}(-t)^{\oplus s} \arrow[r] & 0
\end{tikzcd}$$

In particular, there is a $\varphi_2 : \mathcal{O}_{P_x}(-t)^{\oplus s} \to \mathcal{O}_{P_x}(-t)^{\oplus s}$ such that $\varphi_2 \circ \gamma_1 = \gamma_2 \circ \varphi_1$. Therefore those points in Θ that give rise to isomorphic sheaves form an $\mathrm{Aut}(F^\vee) \times GL(s)$ orbit. Next, we will determine the size of the stabilizer in $\mathrm{Aut}(F^\vee) \times GL(s)$ of any $\gamma \in \Theta$. Suppose $\varphi_1 \in \mathrm{Aut}(F^\vee)$ and $\varphi_2 \in GL(s)$ are such that the right rectangle below is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^\vee & \longrightarrow & F^\vee & \xrightarrow{\gamma} & \mathcal{O}_{P_x}(-t)^{\oplus s} & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle \varphi_1} & & \Big\downarrow{\scriptstyle \varphi_2^{-1}} & & \\ 0 & \longrightarrow & E^\vee & \longrightarrow & F^\vee & \xrightarrow{\gamma} & \mathcal{O}_{P_x}(-t)^{\oplus s} & \longrightarrow & 0 \end{array}$$

Then it induces a $\varphi \in \mathrm{Aut}(E^\vee)$. One sees that such a map $\mathrm{Stab}_\gamma \to \mathrm{Aut}(E^\vee)$ is injective. Thus, if we let $\Theta_k \subseteq \Theta$ be the set of γ's such that $\#_{\mathrm{aut}}(\ker(\gamma)) = k$ (for a set R of sheaves, we define $\#_{\mathrm{aut}}(R) = \max_{E \in R}\{\dim \mathrm{Hom}(E,E)\}$), then
(1.11)

$$\dim\Big(GL(s)\backslash\Theta_k/\mathrm{Aut}(F^\vee)\Big) \le \dim \mathrm{Hom}(F^\vee, \mathcal{O}_{P_x}(-t)^{\oplus s}) - (s^2 + \#_{\mathrm{aut}}(F^\vee)) + k.$$

Finally, let $(\beta_1^{\oplus r_1}, \cdots, \beta_n^{\oplus r_n})$ be the fiber type of $F_{|P_z}$. Then by Lemma 1.6, $t \le \beta_i$. Because $\sum r_i\beta_i = 0$, we have

$$\dim \mathrm{Hom}\big(F^\vee, \mathcal{O}_{P_x}(-t)\big) = \sum_{i=1}^n r_i \dim H^0(\mathcal{O}_{P_x}(\beta_i - t)) = r(-t+1). \tag{1.12}$$

Returning to the general case $\Lambda = \langle x_1, \cdots, x_k\rangle$ and $\omega = \langle t_1^{\oplus s_1}, \cdots, t_n^{\oplus s_n}\rangle$, we will prove:

Lemma 1.7. *With the notation as above and let $E \in \mathcal{S}_{\Lambda,\omega}(F)$, then*

$$\begin{aligned} &(\#_{\mathrm{mod}} - \#_{\mathrm{aut}})\big(\mathcal{S}_{\Lambda,\omega}(F)\big) \\ &\qquad \le \eta(E) - \Big(r\sum_{i=1}^n s_i(-t_i - 1) + \sum_{i=1}^n s_i^2\Big) - \#_{\mathrm{aut}}(F) - (r^2-1)(g-1). \end{aligned} \tag{1.13}$$

Proof. We only need to prove the inequality

$$(\#_{\mathrm{mod}} - \#_{\mathrm{aut}})\big(\mathcal{S}_{\Lambda,\omega}(F)\big) \le \sum_{i=1}^n \big(rs_i(-t_i+1) - s_i^2\big) - \#_{\mathrm{aut}}(F) \tag{1.14}$$

because then (1.13) follows from $c_2(E) = -\sum_{i=1}^n s_it_i$ and $\eta(E) = -2r\sum_{i=1}^n s_it_i + (r^2-1)(g-1)$. We prove (1.14) by induction on n. When $n = 1$, (1.14) follows from (1.11) and (1.12) because $\#_{\mathrm{aut}}(\mathcal{S}_{\Lambda,\omega}(F)) = \sup\{k | \Theta_k \ne \emptyset\}$. Now assume (1.14) is true for $n-1$. We divide $\mathcal{S}_{x_1,\omega_1}(F)$, $\omega_1 = (t_1^{\oplus s_1})$, into subsets W_k such that $F' \in W_k$ if $\#_{\mathrm{aut}}(F') = k$. Let $\Lambda_2 = \langle x_2, \cdots, x_k\rangle$ and $\omega_2 = \langle t_2^{\oplus s_2}, \cdots, t_n^{\oplus s_n}\rangle$. Then by the induction hypothesis, for $F' \in W_k$,

$$(\#_{\mathrm{mod}} - \#_{\mathrm{aut}})\big(\mathcal{S}_{\Lambda_2,\omega_2}(F')\big) \le \sum_{i=2}^n \big(rs_i(-t_i+1) - s_i^2\big) - k$$

and therefore,

$$
\begin{aligned}
(\#_{\mathrm{mod}}-\#_{\mathrm{aut}})(\mathcal{S}_{\Lambda,\omega}(F)) &\le \sup_k\Big\{\sum_{i=2}^n\big(rs_i(-t_i+1)-s_i^2\big)-k+\#_{\mathrm{mod}}(W_k)\Big\}\\
&\le \sum_{i=2}^n\big(rs_i(-t_i+1)-s_i^2\big)+\Big(rs_1(-t_1+1)-s_1^2-\#_{\mathrm{aut}}(F)\Big)\\
&= \sum_{i=1}^n\big(rs_i(-t_i+1)-s_i^2\big)-\#_{\mathrm{aut}}(F).
\end{aligned}
$$

Now we are ready to prove our structure theorem for subsets of $\mathfrak{A}^{r,d}_{e,I}(0_r)$.

Proof of Theorem 1.5. Inequality (1.8) follows directly from Riemann-Roch and the fact that there is a constant C_2 depending on (Δ, H, e) such that for any $E\in\mathfrak{A}^{r,d}_{e,I}$, $\#_{\mathrm{aut}}(E)\le C_2$. We now prove the second part of the theorem. Let $S\subseteq\mathfrak{A}^{r,d}_{e,I}(1_0)$ be any (irreducible) algebraic set and let $E\in S$ be a generic element. By Lemma 1.4, after performing a sequence of semistable reduction at $y_1,\cdots,y_n$, we get a vector bundle with no jumping line, say π^*F with F a vector bundle over Σ. Clearly, $n=n(E)$ depends on E. We let $S_0\subseteq S$ be the open set of $E'\in S$ with $n(E')=E$ and let n_0 be the integer so that when E varies in S_0, the number of moduli of the (unordered) set $y_1,\cdots,y_n$ is n_0. In other words, n_0 of $(y_1,\cdots y_n)$ are in generic position. We know that the number of moduli of rank r vector bundles on Σ is $r^2(g-1)+1$. Also, since E is e-stable, $\#_{\mathrm{aut}}(E)$ is bounded by a constant C_3' independent of d and I (see Lemma 1.10). Combining these with (1.13), we get

$$
\#_{\mathrm{mod}}(S)\le\eta(E)-r\sum_{i=1}^n s_i(-t_i-1)-\sum_{i=1}^n s_i^2+n_0-\#_{\mathrm{aut}}(F)+g+C_3'. \tag{1.15}
$$

Since we have assumed $\#_{\mathrm{mod}}(S)\ge\eta(r,d,I)-C$, for $C_3=C+C_3'+g$, we get

$$
C_3\ge r\sum_{i=1}^n s_i(-t_i-1)+\Big(\sum_{i=1}^n s_i^2-n_0\Big)+\#_{\mathrm{aut}}(F). \tag{1.16}
$$

Because $t_i<0$, all terms in (1.16) are non-negative. This immediately gives us $n-n_0\le C_3$. Next, we define the multiplicity $m(y_i)$ of y_i to be the number of appearances of the point y_i in $(y_1,\cdots,y_n)$. Then by (1.16),

$$
\frac12\#\{y_i\mid m(y_i)\ge2\}\le\sum_{i=1}^n s_i^2-n_0\le C_3.
$$

So the total multiplicity of multiple points is bounded. Without loss of generality, we can assume $y_1,\cdots,y_{n_0}$ are in general position for general $E\in S$. For convenience, we call $y_i\in(y_1,\cdots,y_n)$ a simple point if $m(y_i)=1$ and $\omega_{y_i}(E)=t_i^{\oplus s_i}$ is $(-1)^{\oplus1}$. We claim that then

$$
\begin{aligned}
\sum_{y_i\neq\text{simple}}(-s_it_i)&\le 2\sum_{i=1}^n s_i(-t_i-1)+\left(\sum_{i=1}^n s_i^2-n_0\right)+\#\{y_i\mid m(y_i)\ge2\}\\
&\le\left(\frac2r+2\right)C_3.
\end{aligned} \tag{1.17}
$$

Indeed, when $t_i \le -2$, then the term $s_i(-t_i)$ is bounded from above by the term $2s_i(-t_i-1)$ in the middle of (1.17), and when $t_i = -1$ and $s_i \ge 2$, then we have $-s_i t_i \le s_i^2 - 1$. The only remaining situation is when $m(y_i) \ge 2$, $t_i = 1$ and $s_i = 1$. But in this case, $(-t_i)s_i = 1$ can be absorbed by the term $\#\{y_i \mid m(y_i) \ge 2\}$. Hence, (1.17) holds. Finally, since $d = \sum_{i=1}^n (-t_i)s_i$,

$$\#\{y_i \mid y_i \text{ simple}\} = d + \sum_{y_i \neq \text{simple}} s_i t_i \ge d - 4C_3.$$

Therefore, combined with $n - n_0 \le C_3$, we get

$$d \ge n \ge n_1 = \#\{y_i \mid y_i \text{ simple}, 1 \le i \le n_0\} \ge d - 5C_3. \tag{1.18}$$

Now we let $l = [5C_3] + 1$. Without loss of generality, we can assume $\{y_1, \cdots, y_{d-l}\}$ are simple points in $\{y_1, \cdots, y_{n_0}\}$. Then the sheaf E must belong to the exact sequence

$$0 \longrightarrow \pi^* F \longrightarrow E \longrightarrow \Big(\bigoplus_{i=1}^{d-l} \mathcal{O}_{P_{y_i}}(-1)\Big) \oplus J' \longrightarrow 0. \tag{1.19}$$

To prove the proposition, we need to have an estimate on F and J'. By definition, J' admits a filtration

$$0 = J_{d-l} \subseteq J_{d-l+1} \subseteq \cdots \subseteq J_n = J'$$

such that $J_{i+1}/J_i \cong \mathcal{O}_{P_{y_i}}(t_i)^{\oplus s_i}$. Thus there is a zero scheme $z' \subseteq \Sigma$ supported on $\{y_{d-l+1}, \cdots, y_n\}$ of length $\ell(z') \le n - (d-l) \le 5C_3$ (because of (1.18)) such that J' is an $\mathcal{O}_{\pi^{-1}(z')}$-module and further,

$$\begin{aligned} 0 &\le c_1(E) \cdot \Sigma^+ - (\deg F + d - l) = c_1(J') \cdot \Sigma^+ \\ &= \sum_{i=d-l+1}^n s_i \le \sum_{i=d-l+1}^n (-t_i)s_i \le 5C_3. \end{aligned} \tag{1.20}$$

Here, the last inequality holds because of (1.17) and $n - n_0 \le C_3$. Also, since $\#_{\text{aut}}(F) \le C_3$ (from (1.16)), there is a constant C_4 such that F is C_4-stable.

It remains to show that we can find an integer l_1 (independent of d) and find a single line bundle L of degree $[(d-c)/r] + l_1$ ($c = I \cdot \Sigma^+$) so that for any $E \in S_0$, E belongs to the exact sequence

$$0 \longrightarrow E^\vee \longrightarrow \pi^*(L^{\oplus r}) \longrightarrow \Big(\bigoplus_{i=1}^{d-l} \mathcal{O}_{P_{y_i}}(1)\Big) \oplus J \longrightarrow 0 \tag{1.21}$$

specified in Theorem 1.5. First, there is a constant l_1 and a line bundle L of degree $[(d-c)/r] + l_1$ such that for any C_4-stable rank r vector bundle F on Σ satisfying (1.20), $L \otimes F$ is generated by $H^0(L \otimes F)$. Now for any $E \in S_0$ with the data given by (1.19), we choose $\pi^* F^\vee \to \pi^* L^{\oplus r}$ so that the support of $\pi^*(L^{\oplus r})/\pi^* F^\vee$ is disjoint from $\bigcup_{i=1}^{d-l} P_{y_i}$. Then by dualizing (1.19) and coupled with $\pi^* F^\vee \to \pi^*(L^{\oplus r})$, we get

$$0 \longrightarrow E^\vee \longrightarrow \pi^*(L^{\oplus r}) \longrightarrow J \oplus \Big(\bigoplus_{i=1}^{d-l} \mathcal{O}_{P_{y_i}}(1)\Big) \longrightarrow 0.$$

Finally, it is easy to see that there is an integer l_2 depending only on C_3 and l_1 such that for some subscheme $z \subseteq \Sigma$ of length $\ell(z) \le l_2$, J is a sheaf of $\mathcal{O}_{\pi^{-1}(z)}$-modules. This completes the proof of the theorem.

Now we prove Theorem 1.2.

Proof of Theorem 1.2. We begin with a general vector bundle $E \in \mathfrak{A}^{r,d}_{e,I}(\alpha)$, $\alpha = (\alpha_1^{\oplus r_1}, \cdots, \alpha_n^{\oplus r_n}) \neq 1_m$. Let

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E \tag{1.22}$$

be the relative Harder-Narasimhan filtration such that $F_i = E_i/E_{i-1}$ are torsion free of generic fiber types $(\alpha_i^{\oplus r_i})$ respectively. We call this the relative filtration of E. ((1.22) can be derived by using the usual Harder-Narasimhan filtration of E with respect to the divisor $kP_\xi + \Sigma^+$ with $k \gg 0$.)

We fix $F_i = E_i/E_{i-1}$ and let $W(\{F_i\}_1^n)$ be the set of all vector bundles V whose relative filtrations $0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ satisfy $V_i/V_{i-1} \cong F_i$. Our first step is to estimate the number of local moduli $\#^{\mathrm{loc}}_{\mathrm{mod}}\big(W(\{F_i\}_1^n) \text{ at } [E]\big)$. Let $A_i = \frac{1}{r_i} c_i(F_i)$ and $d_i = c_2(F_i) - \binom{r}{2} A_i^2$. Note that by the proof of Lemma 1.4, $d_i \ge 0$. Now an easy calculation shows that

$$d = c_2(E) = \frac{1}{2}\sum_{i=1}^n r_i (I - A_i)\cdot A_i + \sum_{i=1}^n d_i = \frac{1}{2} I^2 - \frac{1}{2}\sum_{i=1}^n r_i A_i^2 + \sum_{i=1}^n d_i. \tag{1.23}$$

From the exact sequence

$$0 \longrightarrow E_{n-1} \longrightarrow E \longrightarrow F_n \longrightarrow 0$$

and the argument similar to (1.11), we have

$$\begin{aligned} &\#^{\mathrm{loc}}_{\mathrm{mod}}\big(W(\{F_i\}_1^n) \,\mathrm{at}\, [E]\big) \\ &\quad \le \#^{\mathrm{loc}}_{\mathrm{mod}}\big(W(\{F_i\}_1^{n-1}) \,\mathrm{at}\, [E_{n-1}]\big) + \dim \mathrm{Ext}^1(F_n, E_{n-1}) \\ &\qquad - \#_{\mathrm{aut}}(F_n) - \#_{\mathrm{aut}}(E_{n-1}) + \#_{\mathrm{aut}}(E) - \dim \mathrm{Hom}(F_n, E_{n-1}). \end{aligned} \tag{1.24}$$

Further, because $E_{n-1}^\vee \otimes F_n$ has generic fiber type $((\alpha_n - \alpha_1)^{\oplus r_1}, \cdots, (\alpha_n - \alpha_{n-1}^{\oplus r_{n-1}}))$ and $\alpha_{i+1} < \alpha_i$, $\mathrm{Ext}^2(F_n, E_{n-1}) = 0$ by Serre duality. Hence

$$\dim \mathrm{Hom}(F_n, E_{n-1}) - \dim \mathrm{Ext}^1(F_n, E_{n-1}) = \chi(F_n, E_{n-1}), \tag{1.25}$$

where the right-hand side of (1.25) is the abbreviation of $\chi(\mathrm{Ext}^{\cdot}(F_n, E_{n-1}))$. Finally, by using the filtration (1.22), we have

$$\begin{aligned} &\#^{\mathrm{loc}}_{\mathrm{mod}}\big(W(\{F_i\}_1^n) \,\mathrm{at}\, [E]\big) - \#_{\mathrm{aut}}(E) \\ &\quad \le \#^{\mathrm{loc}}_{\mathrm{mod}}\big(W(\{F_i\}_1^{n-1}) \,\mathrm{at}\, [E_{n-1}]\big) \\ &\qquad - \sum_{i=1}^{n-1} \chi(F_n, E_{n-1}) - \#_{\mathrm{aut}}(E_{n-1}) - \#_{\mathrm{aut}}(F_n) \\ &\quad \le \sum_{i>j} \chi(F_i, F_j) - \sum_{i=1}^n \#_{\mathrm{aut}}(F_i). \end{aligned} \tag{1.26}$$

The last inequality is derived by iterating the first part of (1.26). Therefore,

$$\tag{1.27} \begin{aligned} \#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(\alpha) \le \sup\Big\{\sum_{i>j}\chi(F_i,F_j)+\sum_{i=1}^{n}\big(\#_{\mathrm{mod}}(F_i)-\#_{\mathrm{aut}}(F_i)\big)\Big\}\\ +\max\{\#_{\mathrm{aut}}(E)\mid E\in\mathfrak{A}^{r,d}_{e,I}(\alpha)\}, \end{aligned}$$

where the sup is taken over all possible relative filtrations (1.22) of E's in $\mathfrak{A}^{r,d}_{e,I}(\alpha)$. We now calculate the right-hand side of (1.27) by Riemann-Roch. First,

$$\chi(F_i,F_j)=r_ir_j\Big(\frac{1}{2}(A_j-A_i)^2-\frac{1}{2}(A_j-A_i)\cdot K_\Delta+(1-g)\Big)-r_id_j-r_jd_i.$$

For simplicity, in the following we will group all terms that are bounded independently of r_i, d_i, A_i and $\alpha_m\ne 1_m$ into $O(1)$. We have

$$\tag{1.28} \begin{aligned} &\sum_{i>j}\chi(F_i,F_j)\\ &\quad=-\sum_{i>j}r_ir_j\Big(\frac{1}{2}(A_j-A_i)^2-\frac{1}{2}(A_j-A_i)\cdot K_\Delta+\frac{d_i}{r_i}+\frac{d_j}{r_j}\Big)+O(1). \end{aligned}$$

Further, one calculates

$$\tag{1.29} \begin{aligned} \eta(F_i)&=2r_i\big(d_i+\binom{r_i}{2}A_i^2\big)-(r_i-1)r_i^2A_i^2-(r_i^2-1)(1-g)\\ &=2r_id_i-(r_i^2-1)(1-g). \end{aligned}$$

Thus by combining (1.23), (1.27)–(1.29) and the fact that $\#_{\mathrm{mod}}(F_i)-\#_{\mathrm{aut}}(F_i)\le \chi(F_i,F_i)$, we obtain

$$\tag{1.30} \begin{aligned} &(\#_{\mathrm{mod}}(E)-\#_{\mathrm{aut}}(E))\big(\mathfrak{A}^{r,d}_{e,I}(\alpha)\big)-(2r-1)d\\ &\qquad\le-\sum_{i>j}r_ir_j\Big(\frac{1}{2}(A_j-A_i)^2-\frac{1}{2}(A_j-A_i)\cdot K_\Delta+\frac{d_i}{r_i}+\frac{d_j}{r_j}\Big)\\ &\qquad\quad+\sum_{i=1}^{n}2r_id_i-(2r-1)\Big(\sum_{i=1}^{n}d_i-\frac{1}{2}\sum_{i=1}^{n}r_iA_i^2\Big)+O(1). \end{aligned}$$

To analyze (1.30), we first note that

$$\sum_{i>j}r_ir_j(A_j-A_i)^2=r\sum_{i=1}^{n}r_iA_i^2-I^2;\quad \sum_{i>j}r_ir_j\Big(\frac{d_i}{r_i}+\frac{d_j}{r_j}\Big)=\sum_{i=1}^{n}(r-r_i)d_i.$$

Now if we let $A_i\sim\alpha_i\Sigma^-+c_iP_\xi$ and let $\delta=\Sigma^+\cdot\Sigma^+$, then the right-hand side of (1.30) is equal to

$$\begin{aligned} &\sum_{i=1}^{n}\Big(\frac{1}{2}(r-1)r_iA_i^2-(r-r_i-1)d_i\Big)+\frac{1}{2}\sum_{i>j}r_ir_j(A_j-A_i)\cdot K_\Delta+O(1)\\ &\quad=\sum_{i=1}^{n}\frac{1}{2}(r-1)r_i(-\delta\alpha_i^2+2\alpha_ic_i)-\sum_{i=1}^{n}(r-r_i-1)d_i\\ &\qquad+\frac{1}{2}\sum_{i>j}r_ir_j(\alpha_j-\alpha_i)\Sigma^-\cdot K_\Delta+\frac{1}{2}\sum_{i>j}r_ir_j(c_j-c_i)P_\xi\cdot K_\Delta+O(1) \end{aligned}$$

which is bounded from above by (note $d_i \geq 0$)

$$
\begin{aligned}
&-\frac{1}{4}\delta\sum_{i=1}^{n}\alpha_i^2+(r-1)\sum_{i=1}^{n}r_i\alpha_ic_i-\sum_{i>j}r_ir_j(c_j-c_i)+O(1)\\
&=-\frac{1}{4}\delta\sum_{i=1}^{n}\alpha_i^2+\sum_{k=1}^{n}\Bigl(r_kc_k\Bigl((r-1)\alpha_k+\sum_{i=1}^{k-1}r_i-\sum_{i=k+1}^{n}r_i\Bigr)\Bigr)+O(1).
\end{aligned}
\tag{1.31}
$$

Let $p_k=(r-1)\alpha_k+\sum_{i=1}^{k-1}r_i-\sum_{i=k+1}^{n}r_i$. Then when $\alpha\neq 1_m$, or equivalently when $n\geq 2$ or $n=2$ and $\alpha_1-\alpha_2\geq 2$, we will always have

$$p_k-p_{k+1}=(r-1)(\alpha_k-\alpha_{k+1})-(r_k+r_{k+1})\geq 0,\quad k\leq n-1.$$

We rewrite

$$\sum_{k=1}^{n}\Bigl(r_kc_k\Bigl((r-1)\alpha_k+\sum_{i=1}^{k-1}r_i-\sum_{i=k+1}^{n}r_i\Bigr)\Bigr)=\sum_{k=1}^{n-1}\Bigl((p_k-p_{k+1})(\sum_{i=1}^{k}r_ic_i)\Bigr)+p_n\sum_{i=1}^{n}r_ic_i.$$

Finally, we shall make use of the fact that E is e-stable. If $H\sim a\Sigma^++bP_\xi$, then for any k,

$$\deg(E_k)=a\sum_{i=1}^{k}r_ic_i+b\sum_{i=1}^{k}r_i\alpha_i\leq\frac{\mathrm{rk}(E_k)}{r}H\cdot I+e\sqrt{H^2}.$$

Therefore, for $k\leq n-1$,

$$\sum_{i=1}^{k}r_ic_i\leq 1+\frac{b}{a}+e\sqrt{\delta+2\frac{b}{a}}-\frac{b}{a}\sum_{i=1}^{k}r_i\alpha_i.$$

Thus we get

$$
\begin{aligned}
&\sum_{k=1}^{n}\Bigl((p_k-p_{k+1})(\sum_{i=1}^{k}r_ic_i)\Bigr)\\
&\leq\sum_{k=1}^{n-1}(p_k-p_{k+1})\Bigl(\frac{a+b+e\sqrt{a^2\delta+2ab}}{a}-\frac{b}{a}\sum_{i=1}^{k}r_i\alpha_i\Bigr)+p_nm\\
&\leq\frac{b}{a}\cdot r^2(\sum_{i=1}^{n}|\alpha_i|)^2+4r^2(2+e\delta)(1+\frac{b}{a})(\sum_{i=1}^{n}|\alpha_i|)+O(1).
\end{aligned}
\tag{1.32}
$$

Here we have used the fact that $p_k-p_{k+1}\leq\sum_{i=1}^{n}|\alpha_i|+r$ and $p_n\leq 0$ because $\alpha\neq 1_m$ and $\sum_{i=1}^{n}r_ic_i=m\geq 0$. Now if we assume

$$\frac{b}{a}r^2<\frac{1}{16}\delta,$$

then everything in (1.32) can be absorbed by the quadratic term $-\frac{1}{4}\delta\sum_{i=1}^{n}\alpha_i^2$ (in (1.31)) with the help of some constant C_1. Thus combined with (1.31), we have proved

$$\#_{\mathrm{mod}}\mathfrak{A}_{e,I}^{r,d}(\alpha)\leq(2r-1)d+C_1+\max\{\#_{\mathrm{aut}}(E)\mid E\in\mathfrak{A}_{e,I}^{r,d}(\alpha)\}.$$

Theorem 1.2 will be proved if we can bound $\mathrm{Hom}(E,E)$ for $E\in\mathfrak{A}_{e,I}^{r,d}$. Since E is e-stable, $E^\vee\otimes E$ must be $(2|e|+1)$-stable. (This can be proved by using the fact that the Harder-Narasimhan filtration of E will induce the Harder-Narasimhan filtration of $E^\vee\otimes E$.) Thus $\#_{\mathrm{aut}}(E)$ is bounded independently of d by the following lemma.

Lemma 1.8. *For constants e_1, e_2 and integer r, there is a constant C' such that whenever V is a rank r e_1-stable vector bundle on Δ such that $|\deg(V)| \le e_2$, then we have* $\dim H^0(V) \le C'$.

Proof. We prove the lemma by induction on r. The case $r = 1$ is obvious. Assume the lemma is true for vector bundles of rank $\le r-1$ and assume V has $H^0(V) \ne \{0\}$. Then there is a line bundle L, $\deg L \ge 0$, such that V belongs to the exact sequence

$$0 \longrightarrow L \longrightarrow V \longrightarrow V/L \longrightarrow 0$$

with V/L torsion free. Since V is e_1-stable and $|\deg(V)| \le e_2$, there are constants e_1' and e_2' such that $|\deg L|$, $|\deg V/L| \le e_2'$ and V/L is e_1'-stable. Thus by the induction hypothesis, there is a constant C' such that $h^0(L) \le C'$ and $h^0(V/L) \le C'$. The lemma then follows.

We now prove Theorem 1.1 by induction on m. We first establish the case $m = 0$. Let e and C be any constants, $r \ge 2$ be an integer and $D \subseteq \Delta$ be any divisor. We assume H is an ample divisor satisfying the condition of Theorem 1.2. To prove the theorem, we need to show that there is a constant N depending only on (X, H, r, I, e, D) so that if for some d we have

$$\#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(D) \ge \eta(r, d, I) - C, \tag{1.33}$$

then $d \le N$. Now assume (1.33) does hold. Thanks to Theorem 1.2, there is an $N_1 \ge 0$ such that if $d \ge N_1$, then the set $\mathfrak{A}^{r,d}_{e,I}(D, 1_0)$ satisfies

$$\#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(D, 1_0) = \#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(D) \ge \eta(r, d, I) - C. \tag{1.34}$$

Of course, $\mathfrak{A}^{r,d}_{e,I}(D, 1_0)$ is a constructible set. Let S be an irreducible variety parameterizing a subset of $\mathfrak{A}^{r,d}_{e,I}(D, 1_0)$ such that $\#_{\mathrm{mod}} S \ge \eta(r, d, I) - C$. By Theorem 1.5, there are constants l, l_1, l_2 (independent of d) and a line bundle L of degree $[(d - c)/r] + l_1$ such that associated to a general $E \in S$, there are $x_1, \cdots, x_{d-l} \in \Sigma$ in general position and a quotient sheaf J_E of $\pi^*(L^{\oplus r})$ such that E belongs to the exact sequence

$$0 \longrightarrow E^\vee \xrightarrow{i} \pi^*(L^{\oplus r}) \xrightarrow{\tau_0 \oplus \tau_1} J_E \oplus \Big(\bigoplus_{i=1}^{d-l} \mathcal{O}_{P_{x_i}}(1)\Big) \longrightarrow 0. \tag{1.35}$$

Clearly, E is determined by the surjective homomorphisms

$$\pi^*\big(L^{\oplus r}\big) \xrightarrow{\tau_0} J_E \quad \text{and} \quad \pi^*\big(L^{\oplus r}\big) \xrightarrow{\tau_1} \bigoplus \mathcal{O}_{P_{x_i}}(1).$$

Hence the combined number of moduli of the sets of these quotient sheaves that come from $E \in S$ is no less than $\eta(r, d, I) - C$. Let

$$\Xi_0 = \big\{\tau_0 : \pi^*\big(L^{\oplus r}\big) \to J_E \mid E \in S\big\},$$

$$\Xi_1 = \big\{\tau_1 : \pi^*\big(L^{\oplus r}\big) \to \bigoplus^{d-l} \mathcal{O}_{P_{x_i}}(1) \mid E \in S\big\}.$$

Because of the following lemma, the information contained in Ξ_0 is minimal.

Lemma 1.9. *There is a constant C_5 independent of d such that $\#_{\mathrm{mod}}(\Xi_0) \le C_5$.*

Proof. We first calculate the Hilbert polynomials of the sheaves J_E. Let $J_E(n) = J_E \otimes H^{\otimes n}$. Then

$$\chi(J_E(n)) = \chi(\pi^* L^{\oplus r}(n)) - \chi(E(n)) - (d-l)\chi(\mathcal{O}_{P_\xi}(1) \otimes H^{\otimes n}) = a_1(d)n + a_0(d),$$

where $a_1(d) = (r[(d-c)/r] - d + rl_1 + l) \cdot (H \cdot P_\xi) - I \cdot H$ and $a_0(d) = (r[(d-c)/r] - d) + rl_1 - \frac{1}{2}I^2 + \frac{1}{2}I \cdot K + 2l$. Since $r[(d-c)/r] - d$ can only attain integer values between $-c-r$ and $-c$ for integers d, the function $a_1(d)$ (resp. $a_0(d)$) can only attain r values. Hence, $\{\chi(J_E(\cdot)) \mid E \in S\}$ is a finite set (independent of d) and by [Gr, p. 12], the set Ξ_0 is bounded. Thus, there is a constant C_5 such that $\#_{\mathrm{mod}}\Xi_0 \le C_5$.

Since $\#_{\mathrm{mod}} S \le \#_{\mathrm{mod}}\Xi_0 + \#_{\mathrm{mod}}\Xi_1$, we have

$$\#_{\mathrm{mod}}\Xi_1 \ge \eta(r,d,I) - (C + C_5).$$

Let $\tau_1 \in \Xi_1$ and $F = \ker\{\tau_1\}$. In the following, we seek to relate the non-vanishing of $\mathrm{Hom}(E, E(D))^0$ to the non-vanishing of $\mathrm{Hom}(F, F(D'))^0$ for some divisor D'. First of all, by (2) of Theorem 1.5, there is a divisor $z \in \Sigma$ (of degree $\le l_2$) such that the composition

$$F(-\pi^{-1}(z)) \hookrightarrow \pi^* L(-\pi^{-1}(z))^{\oplus r} \longrightarrow \pi^* L^{\oplus r} \xrightarrow{\tau_0} J_E \tag{1.36}$$

is trivial. Because of (1.35), $F(-\pi^{-1}(z))$ is a subsheaf of $E^\vee$. Therefore, any non-trivial traceless homomorphism $\varphi : E \to E(D)$ will provide us a non-trivial traceless homomorphism

$$F(-\pi^{-1}(z)) \longrightarrow E^\vee \xrightarrow{\varphi} E^\vee(D) \longrightarrow F(D).$$

Further, let $\bar{z}$ be a fixed divisor on Σ of degree $l_2 + 2g$. Since

$$h^0(\Sigma, \mathcal{O}_\Sigma(\bar{z} - z)) \ne 0,$$

$\mathrm{Hom}(F, F(D + \pi^{-1}(z))) \ne 0$ implies $\mathrm{Hom}(F, F(D + \pi^{-1}(\bar{z}))) \ne 0$. Thus we have proved:

Lemma 1.10. *With the notation as before, there is a divisor $z \subset \Sigma$ independent of d and D such that for any sheaf $F = \ker\{\tau_1\}$, where $\tau_1 \in \Xi_1$, and for $D_1 = D + \pi^{-1}(z)$, we have $\mathrm{Hom}(F, F(D_1)) \ne \{0\}$.* □

Our next step is to investigate the set Ξ_1 by utilizing this non-vanishing property. We first fix $d-l$ general points $x_1, \cdots, x_{d-l} \in \Sigma$ and let U be the set of all quotient homomorphisms

$$\sigma : \pi^*(L^{\oplus r}) \longrightarrow \bigoplus_{i=1}^{d-l} \mathcal{O}_{P_{x_i}}(1). \tag{1.37}$$

U is (canonically) parameterized by an open subset of the product of $d-l$ copies of projective space $\mathbf{P}^{2r-1}$ after fixing the basis of each $H^0(\mathcal{O}_{P_{x_i}}(1))$. In the following, for any $u \in \Pi^{d-l}\mathbf{P}^{2r-1}$ we denote by σ_u the associated homomorphism $\sigma_u : \pi^*(L^{\oplus r}) \to \bigoplus_{i=1}^{d-l} \mathcal{O}_{P_{x_i}}(1)$. Let

$$\Xi_1(\mathfrak{x}) = \{u \in \Pi^{d-l}\mathbf{P}^{2r-1} \mid \sigma_u \in \Xi_1\}.$$

Since the points of $(\mathfrak{x}) = (x_1, \cdots, x_{d-l})$ are general,

$$\begin{aligned} \dim \Xi_1(\mathfrak{x}) &\geq \#_{\mathrm{mod}}\Xi_1 - (d-l) \\ &\geq \eta(r,d,I) - (C+C_5) - (d-l) \geq (2r-1)d - C_6 \end{aligned} \tag{1.38}$$

for some integer C_6. Now let $l_3 = l + C_6 + 1$. Possibly after rearranging the order of $(x_1, \cdots, x_{d-l})$, we can further assume that the restriction to $\Xi_1(\mathfrak{x})$ of the projection from $\Pi^{d-l}\mathbf{P}^{2r-1}$ to the first $d - l_3$ factors,

$$\Xi_1(\mathfrak{x}) \subseteq \Pi^{d-l}\mathbf{P}^{2r-1} \longrightarrow \Pi^{d-l_3}\mathbf{P}^{2r-1},$$

is dominant. That is, for general $v \in \Pi^{d-l_3}\mathbf{P}^{2r-1}$ with the associated homomorphism $\sigma'_v : \pi^* L^{\oplus r} \to \bigoplus_{i=1}^{d-l_3} \mathcal{O}_{P_{x_i}}(1)$, there is at least one

$$\xi : \pi^* L^{\oplus r} \to \bigoplus_{i=d-l_3+1}^{d-l} \mathcal{O}_{P_{x_i}}(1)$$

such that $\sigma'_v \oplus \xi$ considered as a quotient sheaf belongs to $\Xi_1(\mathfrak{x})$. Thus if we let $V = \ker\{\sigma'_v \oplus \xi\}$ and let $V_v = \ker\{\sigma'_v\}$, V and V_v fit into the exact sequence

$$0 \longrightarrow V \longrightarrow V_v \longrightarrow \bigoplus_{i=d-l_3+1}^{d-l} \mathcal{O}_{P_{x_i}}(1) \longrightarrow 0.$$

Let $A = \bigcup_{i=d-l_3+1}^{d-l} P_{x_i}$ be a divisor in Δ. Following the argument in Lemma 1.10, the non-trivial homomorphism ϕ in Lemma 1.10 induces a non-trivial homomorphism

$$\phi' : V_v \longrightarrow V_v(D_1 + A).$$

Therefore for general $v \in \Pi_{i=1}^{d-l_3}\mathbf{P}^{2r-1}$, $\mathrm{Hom}_\Delta(V_v, V_v(D_1+A))^0 \neq \{0\}$. Finally, as in Lemma 1.9, for any fixed divisor $A_0 \subseteq \Delta$ that consists of $l_3 + 2g$ fibers of Δ, we must have $\mathrm{Hom}\big(V_v, V_v(D_1 + A_0)\big)^0 \neq 0$ as well. Therefore, Theorem 1.1 (when $m = 0$) follows from

Proposition 1.11. *For any divisor $D \subseteq \Delta$ and any integer l_0, there is a constant N for which the following holds: Assume $d \geq N$, that L is a line bundle on Σ of $\deg L = [d/r] + l_0$ and that $x_1, \cdots, x_d$ are general points in Σ. Then for general $v \in \Pi^d\mathbf{P}^{2r-1}$, the sheaf $E_v = \ker\{\sigma_v\}$, where σ_v is the associated homomorphism $\pi^* L^{\oplus r} \to \bigoplus_{i=1}^d \mathcal{O}_{P_{x_i}}(1)$, satisfies $\mathrm{Hom}(E_v, E_v(D))^0 = \{0\}$.*

Proof. We prove it by contradiction. The trick is to first prove the vanishing of this homomorphism group for a special quotient sheaf and then apply the semicontinuity theorem to derive the general case. Let $p_i \in P_{x_i}$ be a general closed point and let U be a small disk containing 0. There is a torsion free sheaf J_i on $P_{x_i} \times U$ flat over U such that $J_{i|P_{x_i}\times\{0\}} \cong \mathcal{O}_{P_{x_i}} \oplus \mathbb{C}_{p_i}$ ($\mathbb{C}_{p_i}$ is the skyscraper sheaf supported on p_i) and for $t \neq 0$, $J_{i|P_{x_i}\times\{t\}} \cong \mathcal{O}_{P_{x_i}}(1)$. It is easy to see that any surjective homomorphism

$$f_i : \big(\mathcal{O}_{P_{x_i}}\big)^{\oplus r} \longrightarrow \mathcal{O}_{P_{x_i}} \oplus \mathbb{C}_{p_i} \tag{1.39}$$

can be extended to a (surjective) homomorphism

$$F_i : \left(\mathcal{O}_{P_{x_i}\times U}\right)^{\oplus r} \longrightarrow J_i.$$

In general, we can extend any surjective homomorphism

$$f : \pi^*(L^{\oplus r}) \longrightarrow \bigoplus_{i=1}^{d} \pi^*(L^{\oplus r})_{|P_{x_i}} \longrightarrow \bigoplus_{i=1}^{d} \left(\mathcal{O}_{P_{x_i}} \oplus \mathbb{C}_{p_i}\right) \tag{1.40}$$

to a (surjective) homomorphism

$$F : \pi^*(L^{\oplus r}) \otimes_{\mathcal{O}_\Delta} \mathcal{O}_{\Delta\times U} \longrightarrow \bigoplus_{i=1}^{d} J_i.$$

Let

$$V_t = \ker\{F_{|\Delta\times\{t\}}\}.$$

Then V_t is a flat family of torsion free sheaves on Δ parameterized by U. Assuming, for general $\sigma_v : \pi^*(L^{\oplus r}) \to \bigoplus_{i=1}^{d} \mathcal{O}_{P_{x_i}}(1)$, $\mathrm{Hom}(E_v, E_v(D))^0 \neq 0$, then by the semicontinuity theorem, $\mathrm{Hom}(V_t, V_t(D))^0 \neq \{0\}$ for $t \neq 0$ and consequently, $\mathrm{Hom}(V_0, V_0(D))^0 \neq \{0\}$.

Now we seek to find a contradiction by choosing V_0 (i.e. f in (1.40)) carefully. We first divide the set $\{x_1, \cdots, x_d\}$ into $2r$ subsets, say $\Lambda_1, \cdots, \Lambda_{2r}$, such that each contains either $[d/2r]$ or $[d/2r]+1$ points. We write $f_i = f_i^1 \oplus f_i^2$ according to (1.39). For $x_i \in \Lambda_{2k-1}$, we define f_i^1 to be the composition

$$f_i^1 : \pi^*(L^{\oplus r}) \xrightarrow{\text{rest}} \pi^*(L^{\oplus r})_{|P_{x_i}} \xrightarrow{\mathrm{pr}_k} \pi^*(L)_{|P_{x_i}}, \tag{1.41}$$

where pr_k is the projection onto the k^{th} component, and define f_i^2 to be the composition

$$f_i^2 : \pi^*(L^{\oplus r}) \xrightarrow{\text{rest}} \pi^*(L^{\oplus r})_{|P_{x_i}} \xrightarrow{\mathrm{pr}_{k+1}} \pi^*(L)_{|P_{x_i}} \xrightarrow{\text{ev}} \mathbb{C}_{p_i},$$

where $\mathrm{ev} : \pi^*(L)_{|P_{x_i}} \to \mathbb{C}_{p_i}$ is the evaluation map. (Here we agree $\mathrm{pr}_{r+1} = \mathrm{pr}_1$.) For $i \in \Lambda_{2k}$, we define f_i^1 as in (1.41) while we let f_i^2 be

$$f_i^2 : \pi^*(L^{\oplus r}) \xrightarrow{\text{rest}} \pi^*(L^{\oplus r})_{|P_{x_i}} \xrightarrow{\mathrm{pr}_{k+1}\oplus\mathrm{pr}_{k+2}} \pi^*(L)_{|P_{x_i}} \oplus \pi^*(L)_{|P_{x_i}} \xrightarrow{\text{ev+ev}} \mathbb{C}_{x_i}.$$

($\mathrm{pr}_{r+2} = \mathrm{pr}_2$.) We claim that when d is sufficiently large, the sheaf $E \subseteq \pi^*(L^{\oplus r})$ that is the kernel of $\bigoplus_{i=1}^{d}(f_i^1 \oplus f_i^2)$ has $\mathrm{Hom}(E, E(D))^0 = 0$. Indeed, let

$$\tilde{L}_k = L\Big(-\sum_{i\in\Lambda_{2k-1}\cup\Lambda_{2k}} x_i\Big)$$

be the line bundle on Σ of degree between $l_0 - 2$ and $l_0 + 1$ and let $L_k = \pi^*\tilde{L}_k$. Then, E is a subsheaf of $\bigoplus_{i=1}^{r} L_h$ with cokernel $\bigoplus_{i=1}^{d} \mathbb{C}_{p_i}$. Let $s \in \mathrm{Hom}(E, E(D))$. Then s induces a homomorphism

$$(s_{ij})_{r\times r} : \bigoplus_{h=1}^{r} L_h \longrightarrow \bigoplus_{h=1}^{r} l_h(D)$$

with $s_{ij} \in H^0(L_i^{-1} \otimes L_j(D))$. Since $L_i^{-1} \otimes L_j$ is a pull back of the line bundle on Σ that has degree -1, 0 or 1, $h^0(L_i^{-1} \otimes L_j(D))$ is bounded by a constant C_6 independent of d. On the other hand, by our construction of E, when $i \in \Lambda_{2k-1}$, the composition

$$\bigoplus_{h=1}^{r} L_h \xrightarrow{(s_{**})} \bigoplus_{h=1}^{r} L_h \xrightarrow{\mathrm{pr}_{k+1}} L_{k+1} \xrightarrow{\mathrm{ev}} \mathbb{C}_{p_i}$$

is trivial. Hence for $j \neq k+1$, s_{jk+1} vanishes on p_i for all $i \in \Lambda_{2k-1}$. Now we let $N = 2r([C_6]+3)$ and assume $d \geq N$. Because p_i are general and

$$\#(\Lambda_{2k-1}) \geq [d/2r] > C_6 + 2 > h^0(L_j^{-1} \otimes L_{k+1}) + 1, \tag{1.42}$$

s_{jk+1} must be 0 for $j \neq k+1$.

It remains to show that $s = g_0 \cdot \mathrm{id}$ for some $g_0 \in H^0(\mathcal{O}(D))$. Let $g_j \in H^0(\mathcal{O}(D))$ be sections so that $s_{jj} = g_j \cdot \mathrm{id}\colon L_j \to L_j(D)$. Let $i \in \Lambda_{2k}$ and let

$$v_i \neq 0 \in \ker\bigl\{\{L_{k+1} \oplus L_{k+2}\}_{|p_i} \xrightarrow{(\mathrm{ev},\mathrm{ev})} \mathbb{C}_{p_i}\bigr\}.$$

Then because $(s_{ij}) = \mathrm{diag}\{s_{11}, \cdots, s_{rr}\}$ is induced from $s \in \mathrm{Hom}(E, E(D))$, we must have

$$(\mathrm{ev},\mathrm{ev}) \circ (\mathrm{pr}_{k+1} \oplus \mathrm{pr}_{k+2}) \circ (s_{**})v_i = 0.$$

It is straightforward to check that this is equivalent to $(g_{k+1} - g_{k+2})(p_i) = 0$. Hence, because p_i are general and $\#(\Lambda_{2k}) > h^0(\mathcal{O}(D)) + 1$, we must have $g_{k+1} = g_{k+2}$. Therefore, $\mathrm{Hom}(E, E(D))^0 = 0$. This completes the proof of the theorem for $m = 0$.

Now we use induction on m to establish the remaining cases. The strategy is as follows: We first fix a section $\Sigma^+ \subseteq \Delta$ of $\pi\colon \Delta \to \Sigma$ of positive self-intersection δ. Let $E \in \mathfrak{A}^{r,d}_{e,I}(D, 1_m)$ be any sheaf. We choose a quotient sheaf $E_{|\Sigma^+} \to L_E$ with L_E a locally free sheaf of $\mathcal{O}_{\Sigma^+}$-modules and define $\tilde{E} = \ker\{E \to L_E\}$. $\tilde{E}$ is locally free with Chern classes

$$I' = \det(\tilde{E}) = I(-\Sigma^+), \quad r_0 = \mathrm{rank} L; \tag{1.43}$$

$$d' = c_2(\tilde{E}) = d + \deg L_E + \frac{1}{2} r_0(r_0 - 1)\delta - r_0(I \cdot \Sigma^+). \tag{1.44}$$

Moreover, $\tilde{E} \in \mathfrak{A}^{r,d'}_{e',I'}(D + \Sigma^+)$ for a constant e' independent of L and d. Hence by applying the induction hypothesis to $\mathfrak{A}^{r,d'}_{e',I'}(D + \Sigma^+)$, we get an upper bound of $\#_{\mathrm{mod}}\{\tilde{E} \mid E \in \mathfrak{A}^{r,d}_{e,I}(D)\}$. Thus if we understand the correspondence $E \to \tilde{E}$ well, we can translate the estimate of $\#_{\mathrm{mod}}\{\tilde{E} \mid E \in \mathfrak{A}^{r,d}_{e,I}(D)\}$ to the estimate of $\#_{\mathrm{mod}}\mathfrak{A}^{r,d}_{e,I}(D)$. We now give the details of this argument.

First, we choose $e_0 > 0$ so that $h^1(\Sigma, F) = 0$ holds for all semistable vector bundles F on Σ with $\mathrm{rk}(F) \leq r^2$ and $\mu(F) \geq e_0$. Put $e_1 = r(e_0 + \delta)$. There is a decomposition of $\mathfrak{A}^{r,d}_{e,I}(D, 1_m)$ according to whether the restriction of an element $E \in \mathfrak{A}^{r,d}_{e,I}(D, 1_m)$ to Σ^+ is e_1-stable or not. We denote these sets by W^+ and W^- respectively. Let L_0 be a line bundle on Σ^+ such that $H^0_{\Sigma^+}(F^\vee \otimes L_0)$ generates $F^\vee \otimes L_0$ for any e_1-stable rank r vector bundle F on Σ^+ of degree $I \cdot \Sigma^+$. Then for any $E \in W^+$, we let $L_E = L_0$ and fix a surjective homomorphism $\sigma: E \to L_E$. In case $E \in W^-$, we let

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = E_{|\Sigma^+}$$

be the Harder-Narasimhan filtration of $E_{|\Sigma^+}$. That is, F_{i+1}/F_i are semistable and $\mu(F_i/F_{i-1}) > \mu(F_{i+1}/F_i)$. We let i_0 be the largest integer so that

$$\mu(F_{i_0}/F_{i_0-1}) \geq \mu(F_{i_0+1}/F_{i_0}) + e_0.$$

Such i_0 exists because $E_{|\Sigma^+}$ is not e_1-stable and $e_1 > re_0$. Then by our choice of e_0,

$$E_{|\Sigma^+} \cong M_E \oplus L_E, \qquad M_E = F_{i_0} \text{ and } L_E = E_{|\Sigma^+}/F_{i_0}.$$

We choose our quotient sheaf to be $\sigma: E \to L_E$. Note that L_E is re_0-stable and has degree $\leq \frac{r_0}{r} I \cdot \Sigma^+$.

Now let $\tilde{E}$ be the kernel of $E \to L_E$. Then $\tilde{E}$ is locally free whose first and second Chern classes are given in (1.43) and (1.44). It can easily be checked that $\tilde{E}$ is e'-stable, $e' = e_1 + r(H \cdot \Sigma^+)$, and $\operatorname{Hom}_\Delta(\tilde{E}, \tilde{E}(D+\Sigma^+))^0 \neq \{0\}$. Therefore, we have obtained a map

$$\Psi : \mathfrak{A}^{r,d}_{e,I}(D, 1_m) \longrightarrow \bigcup_{d',I'} \mathfrak{A}^{r,d'}_{e',I'}(D+\Sigma^+), \tag{1.45}$$

where d' can be any integer and I' can possibly be $I(-\Sigma^+), \cdots, I(-(r-1)\Sigma^+)$. We wish to find an upper bound on

$$\#_{\text{mod}} \Psi^{-1}\big(\mathfrak{A}^{r,d'}_{e',I'}(D+\Sigma^+)\big)$$

that is independent of (d', I'). We begin with an estimate of $\#_{\text{mod}}\Psi^{-1}(\Psi(E))$ for any $E \in \mathfrak{A}^{r,d}_{e,I}(D, 1_m)$. Because E belongs to the exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow L \longrightarrow 0 \tag{1.46}$$

($L = L_E$ as before) for $M = E_{|\Sigma^+}/L$, $F_{|\Sigma^+}$ fits into the exact sequence

$$0 \longrightarrow L(-\Sigma^+) \longrightarrow F_{|\Sigma^+} \longrightarrow M \longrightarrow 0. \tag{1.47}$$

On the other hand, elements of $\Psi^{-1}(F)$ are parameterized by $\operatorname{Ext}^1_\Delta(L, F)$, a subset of $\mathbf{P}$. Since F is locally free,

$$\begin{aligned}\dim \operatorname{Ext}^1_\Delta(L,F) &= \dim \operatorname{Ext}^1_\Delta(F, L \otimes K_\Delta) = \dim H^1_{\Sigma^+}(\mathcal{H}om(F,L) \otimes K_\Delta)\\ &= \dim H^0_{\Sigma^+}(L^\vee \otimes F_{|\Sigma^+}(\Sigma^+)) \leq h^0(L^\vee \otimes L) + h^0(L^\vee \otimes M(\Sigma^+)).\end{aligned}$$

Here the last inequality follows from (1.47). Since L is re_0-stable, $h^0(L^\vee \otimes L)$ is bounded from above by a constant. In case $E \in W^+$, because $E_{|\Sigma^+}$ is e_1-stable, $h^0(L^\vee \otimes M(\Sigma^+))$ is also bounded from above. Hence for some constant C_3 depending only on e_1, r and I, we have

$$\#_{\text{mod}}\Psi^{-1}(\Psi(E)) \leq C_3, \quad \forall E \in W^+. \tag{1.48}$$

When $E \in W^-$, $h^1(L^\vee \otimes M(\Sigma^+)) = 0$ because of our choice of e_0. Thus

$$\begin{aligned} h^0(L^\vee \otimes M(\Sigma^+)) &= \chi(L^\vee \otimes M(\Sigma^+)) \\ &= -\deg L \cdot \mathrm{rk}(M) + \deg M \cdot \mathrm{rk}(L) + \mathrm{rk}(M) \cdot \mathrm{rk}(L)(\delta + 1 - g). \end{aligned}$$

Combined with $\deg L + \deg M = I \cdot \Sigma^+$, we get

$$\#_{\mathrm{mod}} \Psi^{-1}\big(\Psi(E)\big) \le -r \deg L + C_4 \tag{1.49}$$

for some constant $C_4 \ge 0$ independent of $E \in W^-$.

Now we use the induction hypothesis. Because $\mathrm{rk}\, L = r_0 < r$, for any $E \in \mathfrak{A}^{r,d}_{e,I}(D, 1_m)$, either the generic fiber type of $\tilde{E}$ is 1_{m-r_0} or it is not of the form $(\alpha_1^{\oplus r_1}, \alpha_2^{\oplus r_2})$. Hence, with $I' = I(-r_0\Sigma^+)$, $1 \le r_0 \le r$, and

$$C_5 = C + C_4 + 2r^2\delta + 2r|I \cdot \Sigma^+|,$$

we can use Theorem 1.2 and the induction hypothesis to conclude that there is an N_1 and a constant C_6 so that when $d' \ge N_1$, we have

$$\#_{\mathrm{mod}} \mathfrak{A}^{r,d'}_{e',I'}(D + \Sigma^+) \le \eta(r, d', I') - C_5 \tag{1.50}$$

and when $d' \le N_1$, we have

$$\#_{\mathrm{mod}} \mathfrak{A}^{r,d'}_{e',I'}(D + \Sigma^+) \le \eta(r, d', I') + C_6. \tag{1.51}$$

We claim that when

$$d \ge N = N_1 + r\delta + (2 + r)|I \cdot \Sigma^+| + C + C_4 + C_6, \tag{1.52}$$

then

$$\#_{\mathrm{mod}} W^- \le \eta(r, d, I) - C.$$

We break the estimate into two cases. In the case $d' = c_2(\tilde{E}) \ge N_1$, by (1.49) and (1.50),

$$\begin{aligned} \#_{\mathrm{mod}} \Psi^{-1}\big(\mathfrak{A}^{r,d'}_{e',I'}(D + \Sigma^+)\big) &\le \big(\eta(r, d', I') - C_5\big) + \big(-r \deg L + C_4\big) \\ &= \eta(r, d, I) - 2r_0 I \cdot \Sigma^+ + \big(-r_0^2(r-1) + \frac{1}{2} r_0(r_0 - 1)\big) + r \deg L - C_5 + C_4 \\ &\le \eta(r, d, I) - C. \end{aligned}$$

The last inequality holds because $\deg L \le \frac{r_0}{r} I \cdot \Sigma^+$. Now assume $d' = c_2(\tilde{E}) < N_1$. Then

$$\begin{aligned} \#_{\mathrm{mod}} \Psi^{-1}\big(\mathfrak{A}^{r,d'}_{e',I'}(D + \Sigma^+)\big) &\le \big(\eta(r, d', I') + C_6\big) + \big(-r \deg L + C_4\big) \\ &\le \eta(r, d, I) + r \deg L + 2r|I \cdot \Sigma^+| + r^2\delta + C_6 + C_4 \le \eta(r, d, I) - C. \end{aligned}$$

Here we have used the fact that $\deg L \le -2|I \cdot \Sigma^+| - r\delta - C_6 - C_4 - C$ which follows from (1.44), (1.52) and $d' < N_1$. Now we consider $E \in W^+$. Since $\#_{\mathrm{mod}} \Psi^{-1}(\Psi(E)) \le C_3$ from (1.48) and $c_2(\tilde{E}) = d + \eta$ with η a fixed integer independent of d, an argument similar to that of W^- shows that there is an N' such that for $d \ge N'$, we have $\#_{\mathrm{mod}} W^- \le \eta(r, d, I) - C$. This establishes Theorem 1.1.

2. Degeneration of moduli space

We now recall briefly the construction of degeneration of moduli and refer the details of this construction to [GL]. We first fix a very ample line bundle H and a line bundle I on X. Let C be a Zariski neighborhood of $0 \in \operatorname{Spec} \mathbb{C}[t]$. By choosing a smooth divisor $\Sigma \in |H|$ we can form a threefold Z over C by blowing up $X \times C$ along $\Sigma \times \{0\}$. Clearly, $Z_t \cong X$, $t \neq 0$, and Z_0 consists of two smooth components X and a ruled surface Δ that intersect normally along $\Sigma \subseteq X$ and $\Sigma^- \subseteq \Delta$. For any line bundle I on X and integers r and d, let $\mathfrak{M}_X^{r;d}$ be the moduli space of rank r H-semistable sheaves over X of $\det E = I$ and $c_2(E) = d$. Let $\mathfrak{M}_X^{r;d} \times C^* \to C^*$, $C^* = C \setminus \{0\}$, be the constant family over C^*. The degeneration we construct will be a flat family $\mathfrak{M}^d$ (over C) extending the family $\mathfrak{M}_X^{r;d} \times C^*$ such that the closed points of the special fiber $\mathfrak{M}_0^d = \mathfrak{M}^d \times_C \operatorname{Spec} \mathbb{C}[0]$ are in one-one correspondence with the semistable sheaves on Z_0 that will be defined shortly.

We first introduce the notion of torsion free sheaves on the surface Z_0:

Definition 2.1. A sheaf E on Z_0 is said to be torsion free at $z \in Z_0$ if whenever $f \in \mathcal{O}_{Z_0,z}$ is a zero divisor of the $\mathcal{O}_{Z_0,z}$-modules E_z, then f is a zero divisor of the $\mathcal{O}_{Z_0,z}$-modules $\mathcal{O}_{Z_0,z}$. The sheaf E is said to be torsion free if E is torsion free everywhere.

Let E be any coherent sheaf on Z_0. We denote by $E^{(1)}$ (resp. $E^{(2)}$) the torsion free part of $E_{|X}$ (resp. $E_{|\Delta}$). We define the rank of E to be a pair of integers, $\mathbf{rk}(E) = (\operatorname{rk}(E^{(1)}), \operatorname{rk}(E^{(2)}))$. When $\mathbf{rk}(E) = (r, r)$, we simply call E a rank r sheaf.

Let $\varepsilon \in (0, \frac{1}{2})$ be a rational number. We define a $\mathbb{Q}$-ample divisor $H(\varepsilon)$ on Z as follows: Let $p_X : Z \to X$ be the projection and put

$$H(\varepsilon) = p_X^* H(-(1-\varepsilon)\Delta).$$

Clearly, for integer n_0 so that $n_0 \cdot \varepsilon \in \mathbb{Z}$,

$$H(\varepsilon)^{\otimes n_0} = p_X^* H^{\otimes n_0}(-(n_0 - n_0\varepsilon)\Delta)$$

is an ample divisor. In the sequel, we will constantly use the tensor power $H(\varepsilon)^{\otimes n}$. We agree without further mentioning that in such cases, n is always divisible by n_0.

Let $\alpha = (\alpha_1, \alpha_2)$ be a pair of rational numbers:

$$\alpha_1 = \big(H(\varepsilon)_{|X} \cdot H(\varepsilon)_{|X}\big)/(H \cdot H), \quad \alpha_2 = \big(H(\varepsilon)_{|\Delta} \cdot H(\varepsilon)_{|\Delta}\big)/(H \cdot H).$$

Note that $\alpha_1 + \alpha_2 = 1$. For any sheaf E on Z_0 with $\mathbf{rk}(E) \neq (0,0)$, we define p_E to be the polynomial

$$p_E = \frac{1}{\mathbf{rk}(E) \cdot \alpha} \chi_E. \tag{2.1}$$

We remark that since $\chi_E(n) = \chi(E \otimes H^{\otimes n})$ is well defined for those n divisible by n_0 and is a restriction of a polynomial in n, we can define χ_E to be that polynomial. Once we have the polynomial p_E, we can define the $H(\varepsilon)$-stability (or $H(\varepsilon)$-semistability) of E by mimicking Definition 0.5 word for word.

Definition 2.2. A torsion free sheaf E on Z_0 is said to be $H(\varepsilon)$-*stable* (resp. $H(\varepsilon)$-*semistable*) if whenever $F \subseteq E$ is a proper subsheaf, then $p_F \prec p_E$ (resp. $\preceq$).

We fix a line bundle I on X and an integer $r \geq 2$. We let $\chi(n)$ be the polynomial that depends on (r, d, I, H, X):

$$\chi(n) = \frac{r}{2} n^2 (H \cdot H) + n\big((H \cdot I) - \frac{r}{2}(H \cdot K_X)\big) + (r-1)\chi(\mathcal{O}_X) + \chi(I) - d. \tag{2.2}$$

$\chi(\cdot)$ is the Hilbert polynomial of a rank r sheaf of $c_1 = I$ and $c_2 = d$. We also fix a rational $\varepsilon \in (0, \frac{1}{2})$ momentarily and the $\mathbb{Q}$-ample line bundle $H(\varepsilon)$. For convenience, we will denote by $\mathcal{E}(n)$ the sheaf $\mathcal{E} \otimes p_Z^* H(\varepsilon)^{\otimes n}$ for any sheaf $\mathcal{E}$ over Z_S. (Here S is any scheme over C and $Z_S = Z \times_C S$.)

We now construct the degeneration $\mathfrak{M}^d$ promised at the beginning of this section. Recall that the moduli space $\mathfrak{M}_X^{d,r}$ was constructed as a GIT quotient of Grothendieck's Quot-scheme. Here, we shall adopt the same approach to construct $\mathfrak{M}^d$. We first fix a sufficiently large n and let $\rho = \chi(n)$. Following A. Grothendieck [Gr], we define $\underline{\mathbf{Quot}}_{Z/C}^{\chi,\mathcal{O}^\rho}$ to be the functor sending any scheme S of finite type over C to the set of all quotient sheaves $E(n)$ of $\mathcal{O}_{Z_S}^{\oplus\rho}$ on Z_S flat over S so that $\chi_{E_s}(m) = \chi(n+m)$ for any closed $s \in S$. $\underline{\mathbf{Quot}}_{Z/C}^{\chi,\mathcal{O}^\rho}$ is represented by a scheme $\mathbf{Quot}_{Z/C}^{\chi,\mathcal{O}^\rho}$ that is projective over C, called Grothendieck's Quot-scheme. Similarly, we have Grothendieck's Quot-scheme $\mathbf{Quot}_X^{\chi,\mathcal{O}^\rho}$ parameterizing all quotient sheaves $\mathcal{O}_X^{\oplus\rho} \to E(n)$ on X with $\chi_E \equiv \chi(\cdot + n)$. Let $\mathcal{U}_X^{I,d} \subseteq \mathbf{Quot}_X^{\chi,\mathcal{O}^\rho}$ be the subset of all H-semistable quotient sheaves $E(n)$ obeying one further restriction: $\det E = I$. $\mathcal{U}_X^{I,d}$ is locally closed. We define $\mathbf{Quot}_{Z/C,H(\varepsilon)}^{\chi,\mathcal{O}^\rho,I}$ to be the closure of $\mathcal{U}_X^{I,d} \times C^* \subseteq \mathbf{Quot}_{Z/C}^{\chi,\mathcal{O}^\rho}$ endowed with the reduced scheme structure and denote by $\widetilde{\mathbf{Quot}}_{Z/C,H(\varepsilon)}^{\chi,\mathcal{O}^\rho,I}$ the normalization of $\mathbf{Quot}_{Z/C,H(\varepsilon)}^{\chi,\mathcal{O}^\rho,I}$. $\widetilde{\mathbf{Quot}}_{Z/C,H(\varepsilon)}^{\chi,\mathcal{O}^\rho,I}$ has the property that it is normal, projective and flat over C. Finally, we define $\widetilde{\mathbf{Quot}}_{Z/C,H(\varepsilon)}^{\chi,\mathcal{O}^\rho,I,ss} \subseteq \widetilde{\mathbf{Quot}}_{Z/C,H(\varepsilon)}^{\chi,\mathcal{O}^\rho,I}$ to be the subset of all closed points whose associated quotient sheaves are $H(\varepsilon)$-semistable.

Clearly, $\widetilde{\mathbf{Quot}}_{Z/C,H(\varepsilon)}^{\chi,\mathcal{O}^\rho,I,ss}$ depends on the choice of $(r, d, n, I, H, \varepsilon)$. In the sequel, r, I and H will be fixed once and for all. Of course, d should be viewed as a variable. For technical reasons, the choice of ε will depend on d. After this, we will choose n sufficiently large (the exact value of n is irrelevant to our discussion as long as it meets the requirement of [Gi, Corollary 1.3], [GL, Corollary 1.11]). If all of these are understood, then we will abbreviate $\widetilde{\mathbf{Quot}}_{Z/C,H(\varepsilon)}^{\chi,\mathcal{O}^\rho,I,ss}$ to $\mathcal{U}^{d,\varepsilon}$. By abuse of notation, we will call $\mathcal{E}$ the universal family of $\mathcal{U}^{d,\varepsilon}$, where $\mathcal{E}(n)$ is the pullback of the universal quotient family on $Z \times_C \mathbf{Quot}_{Z/C}^{\chi,\mathcal{O}^\rho}$.

Let $\mathcal{SL}_C = SL(\rho, \mathbb{C}) \otimes_{\mathbb{C}} C$ be the special linear group scheme over C. Clearly, $\mathbf{Quot}_{Z/C}^{\chi,\mathcal{O}^\rho}$ is an $\mathcal{SL}_C$-scheme. By our construction, this action lifts to $\mathcal{U}^{d,\varepsilon}$. Further, we have

Theorem 2.3 ([GL, Theorems 2.10 and 2.11])**.** *The good quotient $\mathfrak{M}^{d,\varepsilon} = \mathcal{U}^{d,\varepsilon}//\mathcal{SL}_C$ exists. $\mathfrak{M}^{d,\varepsilon}$ is normal, projective and flat over C. Further, for any closed $t \neq 0$, $\mathfrak{M}_t^{d,\varepsilon}$ is isomorphic to the normalization of the moduli scheme $\mathfrak{M}_X^{r,d}$.*

To make use of this degeneration, we need to analyze the closed points of $\mathfrak{M}_0^{d,\varepsilon}$. Since $\mathfrak{M}_0^{d,\varepsilon}$ is a GIT quotient of $\mathcal{U}_0^{d,\varepsilon}$, each point of $\mathfrak{M}_0^{d,\varepsilon}$ associates to an equivalent class of sheaves $E(n)$ in $\mathcal{U}_0^{d,\varepsilon}$. In the following, we will find bounds on $c_1(E^{(1)})$, $c_1(E^{(2)})$ and $c_2(E^{(2)})$ that are independent of the choice of ε and $E(n) \in \mathcal{U}_0^{d,\varepsilon}$. First we study $c_1(E^{(1)})$ and $c_1(E^{(2)})$. Following [GL, Lemma 1.6], there is a sheaf of $\mathcal{O}_\Sigma$-modules $E^{(0)}$ (of rank r_0) such that E belongs to the exact sequence

$$0 \longrightarrow E \longrightarrow E^{(1)} \oplus E^{(2)} \longrightarrow \iota_* E^{(0)} \longrightarrow 0, \tag{2.3}$$

where $\iota\colon \Sigma \hookrightarrow X$. Because $\mathcal{U}^{d,\varepsilon}$ is flat over C, there are integers a_1, a_2 with

$$a_1 + a_2 = r_0 - r \tag{2.4}$$

such that $\det E^{(1)} = I(a_1 H)$, $\det E^{(2)} = I_{0|\Delta}(a_2 \Sigma^-)$, where $I_0 = p_X^* I_{|Z_0}$ [GL, §4]. Then since both $E^{(1)}$ and $E^{(2)}$ are quotient sheaves of E, by the $H(\varepsilon)$-stability of E, we have

$$a_1 \ge (1-\varepsilon)\frac{H\cdot (rK_X - 2I)}{2(H\cdot H)}; \quad a_2 \ge -(1-\varepsilon)\frac{H\cdot (rK_X - 2I)}{2(H\cdot H)} - r.$$

Since H is a very ample divisor on X, we may and will assume that r divides $H\cdot I$ and

$$(H\cdot H) \ge 18|K_X\cdot H| + 18|I\cdot H|. \tag{2.5}$$

Therefore, $r \ge a_1 \ge 0$ and $0 \ge a_2 \ge -r$.

The bound of $c_2(E^{(2)})$ is achieved by applying Bogomolov's argument which shows that when V is an H-stable vector bundle on X, then the restriction of V to a high degree hyperplane curve is semistable. We follow the argument in [GL, §4] and indicate the necessary changes needed in the higher rank case.

Lemma 2.4 (cf. [GL, Lemma 4.3])**.** *There is a constant e_1 independent of d and ε such that the sheaf $E^{(1)}$ (on X) is e_1-stable.*

Lemma 2.5. *For any constant e_1 and integer r, there is a constant C_1 such that whenever V is an e_1-stable torsion free sheaf of rank $\le r$, then*

$$c_2(V) - \frac{\mathrm{rk}(V)-1}{2\,\mathrm{rk}(V)} c_1(V)^2 \ge C_1. \tag{2.6}$$

Proof. Lemma 2.4 is true because any quotient sheaf Q of $E^{(1)}$ is also a quotient sheaf of E. Hence the degree of Q has to satisfy an inequality, which combined with (2.4) gives us the desired inequality. The details of the argument can be found in [GL, Lemma 4.3]. Now we prove Lemma 2.5 following the suggestion of the referee. By Riemann-Roch,

$$\chi(V,V) = \chi(\mathrm{Ext}^{\cdot}(V,V)) = -2r\big(c_2(V) - \frac{r-1}{2r}c_1(V)^2\big) + r^2\chi(\mathcal{O}_X).$$

Since V is e_1-stable, by Lemma 1.8, there is a constant C_1 such that $\dim \mathrm{Hom}(V,V)$ and $\dim \mathrm{Hom}(E, E\otimes K_X)$ are bounded from above by C_1. Hence

$$c_2(V) - \frac{r-1}{2r}c_1(V)^2 = \frac{1}{2r}\chi(\mathcal{O}_X) - \frac{1}{2r}\chi(V,V) \ge \frac{1}{2r}\chi(\mathcal{O}_X) - 2C_1. \qquad \square$$

Lemma 2.6 (cf. [GL, Lemma 4.4])**.** *For any constant e_1, there is a constant C_2 such that whenever V is an e_1-stable, rank r vector bundle on X with* $\det V =$ *$I(aH)$, $|a| \le r$, and that Q is an $\mathcal{O}_\Sigma$-module that is a quotient sheaf of $V_{|\Sigma}$, then we have $\chi(Q) \ge -c_2(V) + C_2$.*

Proof. Let W be the kernel of $V \to Q$. By Riemann-Roch, $c_1(W) = I + (a-c)[\Sigma]$ and

$$c_2(W) = c_2(V) + \chi(Q) + \frac{1}{2}c(K_X + cH)\cdot H - c(I + aH)\cdot H.$$

Thus

$$\chi(Q) \ge -c_2(V) + c_2(W) - 2r^2H^2.$$

On the other hand, since V is e_1-stable, W is (e_1+1)-stable. So by Lemma 2.5, there is a constant C_1 so that $c_2(W) \ge C_1$. This completes the proof of Lemma 2.6.

Proposition 2.7. *There is a constant C_3 independent of ε and d such that for any $E(n) \in \mathcal{U}_0^{d,\varepsilon}$,*

$$c_2(E^{(2)}) \le d + C_3.$$

Proof. By (2.3), we have

$$\chi_{E^{(2)}}(\cdot) = \chi_E(\cdot) + \big(\chi_{E^{(0)}}(\cdot) - \chi_{E^{(1)}}(\cdot)\big).$$

Hence the proposition follows if we can show that the constant term of $\chi_{E^{(0)}}(\cdot) - \chi_{E^{(1)}}(\cdot)$ is bounded from below but this follows from Lemma 2.6. The details of the proof are given in [GL, Proposition 4.6].

Our next goal is to construct Donaldson's line bundle $\mathcal{L}$ on $\mathfrak{M}^{d,\varepsilon}$ and to establish the following key property of $\mathcal{L}$: Whenever $W_0 \subseteq \mathfrak{M}_0^{d,\varepsilon}$ is a dimension c subvariety such that $[\mathcal{L}]^c(W_0) > 0$, then

$$\#_{\mathrm{mod}}\big\{E^{(2)} \mid E(n) \in W_0\big\} = c. \tag{2.7}$$

We first sketch the construction of $\mathcal{L}$. The full account of this construction appeared in [GL, §5]. For any integer $h \ge 1$, let $D^h \subseteq Z$ be a smooth divisor such that $\pi\colon D^h \to C$ is smooth, that $D^h_t = \pi^{-1}(t) \in |hH|$ for $t \ne 0$ and that $D^h_0 \subseteq \Delta\setminus\Sigma$. We call such D^h good divisors in $|hH_C(-h\Delta)|$, where $H_C = p_X^* H$. Since H is very ample, the set of good divisors in $|hH_C(-h\Delta)|$ is base point free. Associated to each D^h we can find an étale covering $\tilde C \to C$ such that on $D^h_{\tilde C} = D^h \times_C \tilde C$ there is a line bundle $\tilde\theta^h$ satisfying $(\tilde\theta^h_v)^{\otimes 2r} = K_{D^h_v}^{\otimes r} \otimes p_X^* I_{|D^h_v}^{\otimes(-2)}$ for all closed $v \in \tilde C$, where $K_{D^h_v}$ is the canonical divisor of D^h_v. We remark that such $\tilde\theta^h$ exist because $[D^h_v]\cdot I = H\cdot I$ is divisible by r.

We first construct a line bundle on $\mathcal{U}^{d,\varepsilon}_{\tilde C} = \mathcal{U}^{d,\varepsilon} \times_C \tilde C$ as follows: Let $\mathcal{E}(n)$ be the universal quotient family on $Z \times_C \mathcal{U}^{d,\varepsilon}$. Since $\mathcal{E}$ is a family of torsion free sheaves flat over $\mathcal{U}^{d,\varepsilon}$, $\mathcal{E}$ admits length two locally free resolution near D^h. Thus the restriction of $\mathcal{E}$ to $D^h \times_C \mathcal{U}^{d,\varepsilon}$ (denoted by $\mathcal{E}_{|D^h}$) has a length two locally free resolution also (see [L1]). Let p_{12} (resp. p_{13}; resp. p_{23}) be the projection from $D^h \times_C \mathcal{U}^{d,\varepsilon}_{\tilde{C}}$ to $D^h_{\tilde{C}}$ (resp. to $D^h \times_C \mathcal{U}^{d,\varepsilon}$; resp. to $\mathcal{U}^{d,\varepsilon}_{\tilde{C}}$). Note that p_{23} is smooth. Hence

$$R^{\cdot}p_{23*}\big(p_{13}^*(\mathcal{E}_{|D^h}) \otimes p_{12}^*\tilde{\theta}^h\big) \tag{2.8}$$

is a perfect complex on $\mathcal{U}^{d,\varepsilon}_{\tilde{C}}$ [KM]. Following [KM], we can define a determinant line bundle

$$\det\big(R^{\cdot}p_{23*}\big(p_{13}^*(\mathcal{E}_{|D^h}) \otimes p_{12}^*\tilde{\theta}^h\big)\big) \tag{2.9}$$

on $\mathcal{U}^{d,\varepsilon}_{\tilde{C}}$ whose inverse we call $\mathcal{L}_{\mathcal{U}}(D^h)$. If we choose another good divisor $D^{h\prime} \in |hH_C(-h\Delta)|$ and form the corresponding line bundle $\mathcal{L}_{\mathcal{U}}(D^{h\prime})$ on $\mathcal{U}^{d,\varepsilon}_{\tilde{C}'}$, then since the set of good divisors in $|hH_C(-h\Delta)|$ is an irreducible set, for any $v \in \tilde{C}$ and $v' \in \tilde{C}'$ which lie over the same closed point $t \in C$, the line bundles $\mathcal{L}_{\mathcal{U}}(D^h)|\mathcal{U}^{d,\varepsilon}_v$ and $\mathcal{L}_{\mathcal{U}}(D^{h\prime})|\mathcal{U}^{d,\varepsilon}_{v'}$ are algebraic equivalent.

Remark. Indeed, more is true. There is a single line bundle $\mathcal{L}_{\mathcal{U}}(h)$ on $\mathcal{U}^{d,\varepsilon}$ such that the line bundles $\mathcal{L}_{\mathcal{U}}(D^h)$ on $\mathcal{U}^{d,\varepsilon}_{\tilde{C}}$ are pullback of $\mathcal{L}_{\mathcal{U}}(h)$ via $\mathcal{U}^{d,\varepsilon}_{\tilde{C}} \to \mathcal{U}^{d,\varepsilon}$.

Our next task is to show that under favorable conditions, these line bundles descend to line bundles on $\mathfrak{M}^{d,\varepsilon}$. We need the following result of Kempf:

Lemma 2.8 (Descent lemma [DN, Theorem 2.3])**.** *Let $\mathcal{L}$ be an $\mathcal{SL}_C$ line bundle on $\mathcal{U}^{d,\varepsilon}$. $\mathcal{L}$ descends to $\mathfrak{M}^{d,\varepsilon}$ if and only if for every closed point $w \in \mathcal{U}^{d,\varepsilon}$ with closed orbit $\mathcal{SL}_C \cdot \{w\}$, the stabilizer* $\mathrm{stab}(w) \subseteq \mathcal{SL}_C$ *of w acts trivially on $\mathcal{L}_w = \mathcal{L} \otimes k(w)$.*

We have

Proposition 2.9. *There is a function $\kappa : \mathbb{Z}^+ \to (0, \frac{1}{2})$ for which the following holds: For any d, there is a large h such that when $\varepsilon \in (0, \kappa(d)) \cap \mathbb{Q}$ and $D^h \in |hH_C(-h\Delta)|$ is a good divisor, then the line bundle $\mathcal{L}_{\mathcal{U}}(D^h)$ (on $\mathcal{U}^{d,\varepsilon}_{\tilde{C}}$) descends to a line bundle on $\mathfrak{M}^{d,\varepsilon}_{\tilde{C}} = \mathfrak{M}^{d,\varepsilon} \times_C \tilde{C}$. We denote the descent by $\mathcal{L}_{\mathbf{M}}(D^h)$.*

Proof. It is straightforward to check that $w = E(n) \in \mathcal{U}^{d,\varepsilon}_{\tilde{C}}$ (over $t \in \tilde{C}$) has closed orbit if and only if E splits into a direct sum of stable sheaves $F_1, \cdots, F_k$. Then following [L1, p. 426], the stabilizer $\mathrm{stab}(w)$ acts trivially on $\mathcal{L}_{\mathcal{U}}(D^h)_w$ if and only if

$$\frac{1}{\mathrm{rk}(F_1)} c_1(F_1) \cdot D^h_t = \cdots = \frac{1}{\mathrm{rk}(F_k)} c_1(F_k) \cdot D^h_t .$$

These identities follow if we can prove

Proposition 2.10. *There is a function $\kappa : \mathbb{Z}^+ \to (0, \frac{1}{2})$ and a constant N for which the following holds: Given d_0, there is an $h \geq 1$ such that for any $\varepsilon \in (0, \kappa(d_0))$, whenever $d \leq d_0$, and that $\mathcal{E}(n) \in \mathcal{U}^{d,\varepsilon}$ is an $H(\varepsilon)$-semistable sheaf over $t \in C$, then for a generic good divisor $D^h \in |hH_C(-h\Delta)|$, $E_{|D^h_t}$ is semistable.*

Completion of the proof of Proposition 2.9. Assume $E = F_1 \oplus \cdots \oplus F_k$. By Proposition 2.10, there is a good divisor $D'^h \in |hH_C(-h\Delta)|$ such that $E_{|D'^h_t}$ is semistable. Then the value $\frac{1}{\mathrm{rk}(F_i)} c_1(F_i) \cdot D^h_t = \frac{1}{\mathrm{rk}(F_i)} c_1(F_i) \cdot D'^h_t$ is identical for all i.

Proposition 2.10 will be proved shortly.

Remark. Let $t \neq 0 \in C$ be any closed point. Then the line bundle $\mathcal{L}_{\mathcal{U}}(D^h)_t$ on $\mathcal{U}_t^{d,\varepsilon}$ descends regardless of the choice of d and ε [L1, p. 426]. In particular, $\mathcal{L}_{\mathrm{M}}(D^h)_t$ always exists on $\mathfrak{M}_t^{d,\varepsilon}$.

Now we explain how to construct global sections of $\mathcal{L}_{\mathrm{M}}(D^h)_v^{\otimes m}$ on $\mathfrak{M}_v^{d,\varepsilon}$, $v \in \tilde{C}$. All we need to know about the line bundle $\mathcal{L}_{\mathrm{M}}(D^h)$ is how to calculate its intersection numbers on various subvarieties of $\mathfrak{M}^{d,\varepsilon}$. So in the following, we will not distinguish between the line bundles $\mathcal{L}_{\mathrm{M}}(D^h)_v$ and $\mathcal{L}_{\mathrm{M}}(D^h)_{v'}$ (resp. $\mathcal{L}_{\mathcal{U}}(D^h)_v$ and $\mathcal{L}_{\mathcal{U}}(D^h)_{v'}$) when v and $v' \in \tilde{C}$ lie over the same closed point $t \in C$. By abuse of notation, we will denote both of them by $\mathcal{L}_{\mathrm{M}}(D^h)_t$ (resp. $\mathcal{L}_{\mathcal{U}}(D^h)_t$).

For any good $D^h \in |hH_C(-h\Delta)|$ and any closed $t \in C$, let $\mathcal{U}_t^{d,\varepsilon}[D_t^h] \subseteq \mathcal{U}_t^{d,\varepsilon}$ be the open set of all $s \in \mathcal{U}_t^{d,\varepsilon}$ such that $\mathcal{E}_{s|D_t^h}$ is semistable. In the following, we abbreviate $D = D_t^h$. By restricting $E(n) \in \mathcal{U}_t^{d,\varepsilon}[D]$ to D, we obtain a morphism

$$\Phi_D : \mathcal{U}_t^{d,\varepsilon}[D] \longrightarrow \mathfrak{M}^{r,I}(D), \tag{2.10}$$

where $\mathfrak{M}^{r,I}(D)$ is the moduli scheme of rank r semistable vector bundles V on D with $\det V = p_X^* I_{|D}$. If we view $\mathfrak{M}^{r,I}(D)$ as an $SL(\rho, \mathbb{C})$ scheme with trivial group action, the morphism Φ_D is $SL(\rho, \mathbb{C})$-equivalent.

Proposition 2.11 [Donaldson]**.** *There is an ample line bundle $\mathcal{L}_D$ on $\mathfrak{M}^{r,I}(D)$ so that its pull back under Φ_D is canonically isomorphic to the restriction to $\mathcal{U}_t^{d,\varepsilon}[D]$ of $\mathcal{L}_{\mathcal{U}}$. Further, this isomorphism is $SL(\rho, \mathbb{C})$-equivariant.*

Proof. For the details of the proof, the readers are advised to look at [L2, p. 31]. Though the author only treated the case $r = 2$ in the proof, the proof of the higher rank case is similar.

Now let m be a large positive integer. Since the isomorphism

$$\Phi_D^*(\mathcal{L}_D) \cong \mathcal{L}_{\mathcal{U}}|_{\mathcal{U}_t^{d,\varepsilon}[D]} \tag{2.11}$$

is $SL(\rho)$-equivalent, for any $\xi \in H^0(\mathfrak{M}^{r,I}(D), \mathcal{L}_D^{\otimes m})$, $\Phi_D^*(\xi)$ is an $SL(\rho)$-invariant section of $\mathcal{L}_{\mathcal{U}}(D)_t^{\otimes m}$ on $\mathcal{U}_t^{d,\varepsilon}[D]$.

Lemma 2.12. *Let $D^h \in |hH_C(-h\Delta)|$ be any good divisor and for any $t \in C$ with $D = D_t^h$, let $\xi \in H^0(\mathfrak{M}^{r,I}(D), \mathcal{L}_D^{\otimes m})$ be any section. Then the pullback section $\Phi_D^*(\xi)$* (*on $\mathcal{U}_t^{d,\varepsilon}[D]$*) *extends canonically over $\mathcal{U}_t^{d,\varepsilon}$ to an $SL(\rho, \mathbb{C})$-invariant section. We shall denote this extension* (*and its descent to $\mathfrak{M}_t^{d,\varepsilon}$ if no confusion is possible*) *by $\Phi_D^*(\xi)_{ex}$. Furthermore,*

$$\Phi_D^*(\xi)_{ex}^{-1}(0) = \big(\mathcal{U}_t^{d,\varepsilon} \setminus \mathcal{U}_t^{d,\varepsilon}[D]\big) \cup \big\{F(n) \in \mathcal{U}_t^{d,\varepsilon}[D] \mid \xi(F_{|D}) = 0\big\}. \tag{2.12}$$

Proof. In case $\mathcal{U}_t^{d,\varepsilon}$ is normal, we can apply [GL, Lemma 5.6], [GL, Proposition 5.7] and [L2, Lemma 4.10] to our situation. In general, we need to use GIT to prove this lemma [L1, p. 435].

In the following, we seek to estimate the self-intersection numbers of $\mathcal{L}_{\mathbf{M}}(D^h)$ on subvarieties $W \subseteq \mathfrak{M}_t^{d,\varepsilon}$ and to relate the non-vanishing of such numbers to the estimate of the numbers (2.7). Our immediate goal is to prove

Proposition 2.13. *Let $t \neq 0 \in C$ be any closed point and let $W_t \subseteq \mathfrak{M}_t^{d,\varepsilon}$ be an irreducible variety of dimension c. Then for sufficiently large h and for any good $D^h \in |hH_C(-h\Delta)|$,*

$$[\mathcal{L}_{\mathbf{M}}(D^h)]^c(W_t) \geq 0. \tag{2.13}$$

Further, if we assume that the general points of W_t are locally free H-μ-stable sheaves, then the strict inequality holds.

Proof. To prove (2.13), it suffices to find divisors $D_1, \cdots, D_c \in |hH|$ and sections $\varphi_1, \cdots, \varphi_c$ of $\mathcal{L}_M(D^h)_t$ such that $\bigcap_{i=1}^c \varphi_i^{-1}(0)$ is a finite set. But this is obvious because for sufficiently large h, the restriction of each $E \in \mathfrak{M}_t^{d,\varepsilon}$ to general $D \subset |hH|$ is semistable (Proposition 2.10). Now we prove the second part of the proposition. Let h be large so that for any locally free $E \in \mathfrak{M}_t^{d,\varepsilon}$, $H^1(\mathcal{E}nd^0(E)(-hH)) = 0$. Then for any $D \in |hH|$ and any locally free stable $E_1, E_2 \in W_t$, $E_{1|D} = E_{2|D}$ implies $E_1 \cong E_2$. We can also assume that the restriction of any $E \in \mathfrak{M}_t^{d,\varepsilon}$ to a general $D \in |hH|$ is semistable.

Choose $D \in |hH|$ so that $\mathcal{U}_t^{d,\varepsilon}[D] \cap W_t$ is non-empty. Then because the line bundle $\mathcal{L}_D$ is ample on $\mathfrak{M}^{r,I}(D)$ and because

$$\Psi_D : \mathfrak{M}_t^{d,\varepsilon}[D] \cap W_t \longrightarrow \mathfrak{M}^{r,I}(D)$$

($\mathfrak{M}_t^{d,\varepsilon}[D]$ is the image of $\mathcal{U}_t^{d,\varepsilon}[D]$ under the projection) is generically one-to-one, there is a section $\xi \in H^0(\mathfrak{M}^{r,I}(D), \mathcal{L}_D^{\otimes m})$, m large, such that the extension of the pullback section $\Phi_D^*(\xi)_{ex}$ (over $\mathfrak{M}_t^{d,\varepsilon}$) is non-trivial over W_t and

$$\dim\Big(\Phi_D^*(\xi)_{ex}^{-1}(0) \cap W_t\Big) = \dim W_t - 1.$$

Since being locally free and stable are open conditions, we can assume that general points of at least one irreducible component of $\Phi_D^*(\xi)_{ex}^{-1}(0) \cap W_t$ are still locally free and H-μ-stable. Therefore, we can use induction on $\dim W_t$ to conclude that $[\mathcal{L}_M(D^h)]^{c-1}(W_t') \geq 0$ for any irreducible component W_t' of $\Phi_D^*(\xi)_{ex}^{-1}(0) \cap W_t$, and for at least one of these component, this number is positive. Therefore, the strict inequality (2.13) holds.

The converse to the proposition is that if a set $W_t \subseteq \mathfrak{M}_t^{d,\varepsilon}$ with $\dim W_t = c$ has the property that

$$[\mathcal{L}_{\mathbf{M}}(D^h)]^c(W_t) > 0,$$

then $\#_{\mathrm{mod}}(W) \geq c$. But this is a tautology since $\mathfrak{M}_t^{d,\varepsilon}$ is the normalization of the moduli scheme. What we need is a similar result in $t = 0$. We will prove

Proposition 2.14. *Let $W_0 \subseteq \mathfrak{M}_0^{d,\varepsilon}$ be any (complete) subvariety of dimension c. Assume for some large h (given by Proposition 2.10) and good $D^h \in |hH_C(-h\Delta)|$ we have*

$$[\mathcal{L}_{\mathbf{M}}(D^h)]^c(W_0) > 0.$$

Then $\#_{\mathrm{mod}}\{E^{(2)} \mid E(n) \in W_0\} = c$.

Proof. We prove it by contradiction. Assume $\#_{\mathrm{mod}}\{E^{(2)} \mid E(n) \in W_0\} < c$. Then $\{E^{(2)} \mid E(n) \in W_0\}$ can be parameterized by finite irreducible varieties of dimension at most $c-1$. Let them be $S_1,\cdots,S_k$ and let $\mathcal{E}_1,\cdots,\mathcal{E}_k$ be the corresponding families. Thanks to Proposition 2.10, there is a large h such that for any $F\in\{E^{(2)}\mid E(n)\in W_0\}$, $F_{|D_0^h}$ is semistable for generic $D^h\in|hH_C(-h\Delta)|$. We fix such an h. We choose a $D^h\in|hH_C(-h\Delta)|$ so that $\mathcal{E}_{i,s_i|D_0^h}$ are semistable for some closed $s_i\in S_i$, $i=1,\cdots,k$. Since $\mathcal{L}_{D_0^h}$ is ample, we can further choose $\xi\in H^0(\mathfrak{M}^{r,I}(D_0^h),\mathcal{L}_{D_0^h}^{\otimes m})$, $m\geq 1$, so that $\xi(\mathcal{E}_{i,s_i|D_0^h})\neq 0$ for all i.

Let $\Psi^*_{D_0^h}(\xi)_{ex}$ be the extension of the pullback of ξ in $H^0(\mathfrak{M}_0^{d\varepsilon},\mathcal{L}_{\mathbf{M}}(D^h)_0^{\otimes m})$. Put $W_0'=W_0\cap\Psi^*_{D_0^h}(\xi)_{ex}^{-1}(0)$. By our construction, $\dim W_0'\leq\dim W_0-1$ and

$$\#_{\mathrm{mod}}\{E^{(2)}\mid E(n)\in W_0'\}\leq\max_{i=1,\cdots,k}\{\dim S_i-1\}\leq\#_{\mathrm{mod}}\{E^{(2)}\mid E(n)\in W_0\}-1.$$

Note that $[\mathcal{L}_{\mathbf{M}}(D^h)]^{c-1}(W_0')=m[\mathcal{L}_{\mathbf{M}}(D^h)_0]^c(W_0)>0$. So by the induction hypothesis, we have $\#_{\mathrm{mod}}\{E^{(2)}\mid E(n)\in W_0'\}\geq c-1$. Therefore,

$$\#_{\mathrm{mod}}\{E^{(2)}\mid E(n)\in W_0\}\geq\#_{\mathrm{mod}}\{E^{(2)}\mid E(n)\in W_0'\}+1\geq c.$$

The proposition follows because $\#_{\mathrm{mod}}\{E^{(2)}\mid E(n)\in W_0\}\leq\dim W_0=c$.

In the remainder of this section, we will give the proof of Proposition 2.10 that is parallel to the treatment for the rank two situation given in [GL, 5.13]. Let $E(n)\in\mathcal{U}_t^{d,\varepsilon}$ be any $H(\varepsilon)$-semistable sheaf over $t\in C$. When $t\neq 0$, then E is an H-semistable sheaf over X and [MR] tells us that for large h and generic $D\in|hH|$, $E_{|D}$ is semistable. In case $t=0$, namely when E is an $H(\varepsilon)$-semistable sheaf on Z_0, the situation is quite tricky because Z_0 is reducible and the divisorial ray $\mathbb{R}\cdot[D_0^h]$ is different from $\mathbb{R}\cdot H(\varepsilon)_{|\Delta}$. However, it is essential that $\mathbb{R}\cdot[D_0^h]$ and $\mathbb{R}\cdot H(\varepsilon)_{|\Delta}$ become very close when ε becomes small. Before going into the details of the proof, let us state the following stability criterion of $E^{(2)}$.

Lemma 2.15. *There is a constant e_2 such that for any d, ε and any $E(n)\in\mathcal{U}_0^{d,\varepsilon}$, $E^{(2)}$ is εe_2-stable with respect to $H(\varepsilon)_{|\Delta}$.*

Proof. See [GL, 5.14].

Proof of Proposition 2.10. Let V be the double dual of $E^{(2)}$. By (2.4) and Proposition 2.7, $\det V=I_0(a_2\Sigma^-)$, $-r\leq a_2\leq 0$, and $c_2(V)\leq c_2(E)+C_3$, where C_3 is a constant independent of E and d. Since $I_0\cdot\Sigma^-$ is divisible by r, by tensoring V with some line bundle, we can assume $c_1(V)\sim a_2[\Sigma^-]$. Note that $c_2(V)$ is still bounded by d_0+C_3 possibly with a new constant C_3. Clearly, the proposition will be established if we can show that there is an ε_0 and an integer h such that whenever $\varepsilon<\varepsilon_0$ and V is $e_2\varepsilon$-stable with respect to $H(\varepsilon)_{|\Delta}$ as before, then for generic $D\in|h\Sigma^+|$, $V_{|D}$ is semistable.

The argument we adopt is a direct generalization of Bogomolov's theorem showing that the restriction of any μ-stable rank two vector bundle E to any smooth hyperplane section of degree $\geq 2c_2(E)+1$ is stable. We prove it by contradiction. Assume otherwise. Then there is a rank s $(1\leq s\leq r-1)$ quotient vector bundle Q of $V_{|D}$ such that $0=\mu(V_{|D})>\mu(Q)$. Let W be the kernel of $V\to Q$. Then W is a locally free sheaf on Δ with $c_1(W)\sim a_2[\Sigma^-]-sh[\Sigma^+]$ and

$$c_2(W)=c_2(V)+\frac{1}{2}s(s-1)h^2H^2+\deg Q<c_2(V)+\frac{1}{2}s(s-1)h^2H^2.$$

Thus a simple calculation gives us

$$2rc_2(W) - (r-1)c_1(W)^2 < 2rc_2(V) - \big(s(r-s)h^2 - (r-1)a_2^2\big)H^2. \tag{2.14}$$

Because $c_2(V) \le d_0 + C_3$, when

$$h^2 \ge r^2 + \frac{2r}{r-1}\frac{d_0 + C_3}{H^2}$$

the right-hand side of (2.14) is negative. Therefore, Bogomolov's inequality shows that W is unstable. Let

$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = W \tag{2.15}$$

be the Harder-Narasimhan filtration of W such that the sheaves $F_i = W_i/W_{i-1}$ are μ-semistable and $\mu(F_i) > \mu(F_{i+1})$. Let $r_i = \mathrm{rk}(F_i)$ and let Γ_i be the $\mathbb{Q}$-divisor supported on fibers of $\Delta \to \Sigma$ such that

$$c_1(F_i) \sim r_i(b_i\Sigma^- + \Gamma_i).$$

We let $c_i = \Gamma_i \cdot \Sigma^+/H^2$. Then b_i and c_i satisfy the following inequalities:

$$(e_2 + \frac{a_2}{r})\varepsilon \ge \varepsilon b_1 + (1-\varepsilon)c_1 > \cdots > \varepsilon b_n + (1-\varepsilon)c_n. \tag{2.16}$$

The first inequality holds because $E^{(2)}$ is $e_2\varepsilon$-stable and the remainder inequalities come from $\mu(F_i) > \mu(F_{i+1})$. On the other hand, we have $\sum_{i=1}^n c_1(F_i) = c_1(W)$. So

$$\sum_{i=1}^n r_i b_i = a_2 - sh, \quad \sum_{i=1}^n r_i c_i = -sh. \tag{2.17}$$

Finally, we calculate

$$\begin{aligned} c_2(W) &= \sum_{i<j} c_1(F_i)\cdot c_1(F_j) + \sum_{i=1}^n c_2(F_i) \\ &\ge \frac{1}{2}\bigg(\Big(\sum_{i=1}^n c_1(F_i)\Big)^2 - \sum_{i=1}^n c_1(F_i)^2\bigg) + \sum_{i=1}^n \frac{r_i - 1}{2r_i}c_1(F_i)^2 \\ &= \frac{1}{2}(s^2h^2 - a_2^2)H^2 + \sum_{i=1}^n \frac{r_i}{2}(b_i^2 - 2b_ic_i)H^2. \end{aligned} \tag{2.18}$$

Here we have used Bogomolov's inequalities $2r_ic_2(F_i) - (r_i-1)c_1(F_i)^2 \ge 0$. Combining (2.18) with $c_2(W) \le c_2(V)$, we have

$$(s^2h^2 - a_2^2) + \sum_{i=1}^n\Big(r_i(b_i - c_i)^2 - r_ic_i^2\Big) \le \frac{2(d_0 + C_3)}{H^2}. \tag{2.19}$$

In the following, we will argue that there are h and ε_0 so that whenever $0 < \varepsilon < \varepsilon_0$, then the only tuples (b_i, c_i) that satisfy (2.16)–(2.19) must have $c_i = 0$ for $i =$

$1,\cdots,n-1$. First of all, let Λ be the set of indices i so that $c_i>0$. Then for those $i\in\Lambda$, $c_i\geq 1/rH^2$ and by (2.16), for small ε, we have

$$b_i-c_i\leq (e_2+\frac{a_2}{r})+\frac{1}{\varepsilon}(-c_i)<\frac{1}{2\varepsilon}(-c_i). \tag{2.20}$$

Thus

$$\sum_{i=1}^n\bigl(r_i(b_i-c_i)^2-r_ic_i^2\bigr)\geq\sum_{i\in\Lambda}r_i\Bigl(\frac{1}{4\varepsilon^2}-1\Bigr)c_i^2-\sum_{i\notin\Lambda}r_ic_i^2.$$

On the other hand, since $\sum_{i\notin\Lambda}r_ic_r=-(sh+\sum_{i\in\Lambda}r_ic_i)$ and $c_i\leq 0$ for $i\notin\Lambda$, $\sum_{i\notin\Lambda}r_ic_i^2$ is bounded from above by $(sh+\sum_{i\in\Lambda}r_ic_i)^2$ which in turn is no more than $2s^2h^2+2(\sum_{i\in\Lambda}r_ic_i)^2$. Combined with (2.19), we must have

$$\sum_{i\in\Lambda}r_i(\frac{1}{4\varepsilon^2}-1)c_i^2-\Bigl(2s^2h^2+2\bigl(\sum_{i\in\Lambda}r_ic_i\bigr)^2\Bigr)+(s^2h^2-a_2^2)\leq\frac{2(d_0+C_3)}{H^2}. \tag{2.21}$$

(2.21) is impossible if we assume

$$\frac{1}{4\varepsilon^2}\geq r^2(r^2h^2+r^2)\cdot H^2+2r(d_0+C_3)+4. \tag{2.22}$$

Thus under the assumption (2.22), we must have $c_i\leq 0$ for all i.

It remains to show that we can choose h large enough so that $c_1=\cdots=c_{n-1}=0$. Suppose there are $c_{i_0}<c_{i_1}<0$. Then $\sum_{i\neq i_0}r_ic_i=-sh-r_{i_0}c_{i_0}$. Again since $c_i\leq 0$, we have

$$\sum_{i=1}^n r_ic_i^2=r_{i_0}c_{i_0}^2+\sum_{i\neq i_0}r_ic_i^2\leq r_{i_0}c_{i_0}^2+(sh+r_{i_0}c_{i_0})^2.$$

Therefore, from (2.19), we have

$$\begin{aligned}\frac{2(d_0+C_3)}{H^2}&\geq(s^2h^2-a_2^2)+\sum_{i=1}^n r_i(b_i-c_i)^2-\sum_{i=1}^n r_ic_i^2\\&\geq(s^2h^2-a_2^2)-\bigl((sh+r_{i_0}c_{i_0})^2+r_{i_0}c_{i_0}^2\bigr)\\&\geq(s^2h^2-a_2^2)-\Bigl((sh-\frac{1}{rH^2})^2+(\frac{1}{rH^2})^2\Bigr)\\&=\frac{2sh}{rH^2}-(a_2^2+\frac{2}{r^2H^2}).\end{aligned} \tag{2.23}$$

Clearly (2.23) is impossible if we choose

$$h\geq 2r(d_0+C_3+r^2H^2)+5. \tag{2.24}$$

Now, we can choose h large according to (2.24) and then choose ε_0 small so that $\varepsilon_0\leq 1/2r$ and (2.22) holds with ε replaced by ε_0. Thus by our previous argument, if $V_{|D}$ is not semistable, then in the filtration (2.15) all but one $c_1(W_i/W_{i-1})\cdot[\Sigma^+]=0$. We claim that $c_1(W_n/W_{n-1})\cdot[\Sigma^+]\neq 0$. Indeed, assume $c_j\neq 0$, $j<n$. Then $c_j=-sh/r_j$ and then by (2.19),

$$\Bigl(\sum_{i\neq j}r_ib_i^2\Bigr)+r_j(b_j-c_j)^2+(1-\frac{1}{r_j})s^2h^2-a_2^2\leq\frac{2(d_0+C_3)}{H^2}. \tag{2.25}$$

Thus $|b_i| \le 2\sqrt{d_0+C_3}/\sqrt{H^2}$ for $i \ne j$ and $|b_j - c_j| < 2\sqrt{d_0+C_3}/\sqrt{H^2}$. In particular, we will have

$$\mu(F_j) = (b_j\varepsilon - \frac{(1-\varepsilon)sh}{r_j})H^2 \le b_n\varepsilon H^2 = \mu(F_n).$$

This contradicts $\mu(F_j) > \mu(F_n)$. Thus we have proved the claim.

The next step is to reconstruct V from the filtration $\{W_i\}$. We first construct a filtration of V out of the filtration $\{W_i\}$ by letting $V_i \supseteq W_i$ be the subsheaf of V so that V/V_i is torsion free and $\mathrm{rk}(W_i) = \mathrm{rk}(V_i)$. We claim that $W_i = V_i$ for all $i \le n-1$. Indeed, let V_i be the first among which $V_i \ne W_i$. Since $V_i = W_i$ on $\Delta \setminus D$, we must have

$$c_1(V_i) = c_1(W_i) + \alpha[D], \quad \alpha \ge 1.$$

On the other hand, $c_1(W_i) = (\sum_{j=1}^i r_j b_j)[\Sigma^-]$ and $|b_j| \le 2\sqrt{d_0+C_3}/\sqrt{H^2}$ because of (2.25). Thus

$$\begin{aligned}\mu(V_i) &= \frac{1}{\mathrm{rk}(V_i)} c_1(V_i)\cdot H(\varepsilon)_{|\Delta}\\ &= \frac{1}{\mathrm{rk}(V_i)}\Big((\sum_{j=1}^i r_j b_j\varepsilon) + \alpha h\Big)H^2 > \mu(V) + \frac{1}{\mathrm{rk}(V_i)} e_2\varepsilon\sqrt{H^2},\end{aligned}$$

which violates the $e_2\varepsilon$-stability of V. Therefore, $V_i = W_i$ for all $i \le n-1$. In particular, the filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

has the property that for $i \le n-1$, V_i/V_{i-1} are μ-semistable and $c_1(V_i/V_{i-1}) \sim r_i b_i[\Sigma^-]$. Let $F_i = V_i/V_{i-1}$. We intend to use induction on the rank r to complete the proof of the proposition. In order to do this, we need to show that F_n is μ-semistable and

$$c_2(V_i/V_{i-1}) - \frac{r_i-1}{2r_i} c_i(V_i/V_{i-1})^2 \le d_0 + C_3 \tag{2.26}$$

for all $i \le n$. We show F_n is μ-semistable by showing that $r_n = 1$. Indeed, a combination of (2.25) (with $j = n$) and (2.24) guarantees $r_n = 1$. Thus F_n is stable. Next, we have

$$\begin{aligned}c_2(V) - \frac{r-1}{2r}c_1(V)^2 &= \sum_{i=1}^n c_2(F_i) + \sum_{i<j} c_1(F_i)\cdot c_1(F_j) + (-\frac{1}{2}+\frac{1}{2r})\Big(\sum_{i=1}^n c_1(F_i)\Big)^2\\ &= \sum_{i=1}^n\Big(c_2(F_i) - \frac{r_i-1}{2r_i}c_1(F_i)^2 - \frac{1}{2r_i}c_1(F_i)^2\Big) + \frac{1}{2r}\big(\sum_{i=1}^n c_1(F_i)\big)^2\\ &= \sum_{i=1}^n\big(c_2(F_i) - \frac{r_i-1}{2r_i}c_1(F_i)^2\big) + \frac{1}{2r}c_1(V)^2 + \frac{1}{2}\sum_{i=1}^n\frac{1}{r_i}r_i^2 b_i^2 H^2.\end{aligned}$$

Because each $c_2(F_i) - \frac{r_i-1}{2r_i}c_1(F_i)^2$ is non-negative, (2.26) must be true. Therefore, we can apply the induction argument to V_i/V_{i-1} to conclude that we can find large h and small ε_0 so that for any $\varepsilon < \varepsilon_0$, we must have $(V_i/V_{i-1})_{|D}$ semistable for generic $D \in |h\Sigma^+|$. Since $\deg(V_i/V_{i-1})_{|D} = 0$, $V_{|D}$ must be semistable also. This completes the proof of Proposition 2.10.

3. Main theorems

In this section, we will prove our main theorems. We will show that when the second Chern class d is large enough, then the moduli scheme $\mathfrak{M}_X^{r,d}$ $(=\mathfrak{M}_X^{r,d}(I,H))$ is smooth at a dense open subset. We shall further show that $\mathfrak{M}_X^{r,d}$ is normal and for any constant C, there is an N depending on (X,I,H,r,C) such that whenever $d\geq N$, then

$$\operatorname{codim}\bigl(\operatorname{Sing}\mathfrak{M}_X^{r,d},\mathfrak{M}_X^{r,d}\bigr)\geq C.$$

Finally, we will investigate the dependence of the moduli scheme $\mathfrak{M}_X^{r,d}(I,H)$ on the polarization H. In case $r=2$, Qin's work [Qi] shows that for any two polarizations H_1 and H_2, the corresponding moduli spaces $\mathfrak{M}_X^{r,d}(I,H_1)$ and $\mathfrak{M}_X^{r,d}(I,H_2)$ are birational when d is sufficiently large. Here, we shall demonstrate that similar phenomena also occur in high rank cases. But first, we shall continue our discussion of the degeneration to finish the proof our main technical theorem, Theorem 0.1.

For the moment, we shall keep the notation developed in §2. For any divisor $D\subseteq X$, we define $\mathcal{S}_{e,I}^{r,d}$ be the set of all e-stable (with respect to the fixed H) rank r sheaves E of $\det E=I$ and $c_2(E)=d$ and define

$$\mathcal{S}_{e,I}^{r,d}(D)=\{E\in\mathcal{S}_{e,I}^{r,d}\mid \operatorname{Hom}(E,E(D))^0\neq\{0\}\}.$$

Similarly, we define $\mathcal{V}_{e,I}^{r,d}$ and $\mathcal{V}_{e,I}^{r,d}(D)$ to be the subsets of locally free sheaves in $\mathcal{S}_{e,I}^{r,d}$ and $\mathcal{S}_{e,I}^{r,d}(D)$ respectively. For technical reasons, we will first attack the set $\mathcal{V}_{\mu,I}^{r,d}(D)$ which is the set of μ-stable locally free sheaves E with the mentioned constraint on c_1, c_2 and h^0. Namely, $\mathcal{V}_{\mu,I}^{r,d}(D)=\mathcal{V}_{0,I}^{r,d}(D)$. We shall prove

Theorem 3.1. *For any choice of r, I and D, and any constant C_1, there is a constant N such that whenever $d\geq N$, we have*

$$\#_{\mathrm{mod}}\mathcal{V}_{\mu,I}^{r,d}(D)\leq\eta_X(r,d,I)-C_1.$$

Proof. Clearly, $\mathcal{V}_{\mu,I}^{r,d}(D)$ is a subset of $\mathfrak{M}_X^{r,d}$. Since being locally free and μ-stable are open conditions and having non-vanishing $\operatorname{Hom}(E,E(D))^0$ is a closed condition, $\mathcal{V}_{\mu,I}^{r,d}(D)$ is a locally closed subset of $\mathfrak{M}_X^{r,d}$. Let $\mathbf{A}\subseteq\mathfrak{M}_X^{r,d}$ be the closure of any irreducible component of $\mathcal{V}_{\mu,I}^{r,d}(D)$.

In the following, we seek to utilize the degeneration $\mathfrak{M}^{d,\varepsilon}\to C$ (of the normalization of $\mathfrak{M}_X^{r,d}$) constructed in Theorem 2.3. When $t\neq 0$, $\mathfrak{M}_t^{d,\varepsilon}$ is just the normalization of $\mathfrak{M}_X^{r,d}$. For such t, we let $W_t\subseteq\mathfrak{M}_t^{d,\varepsilon}$ be the preimage of $\mathbf{A}\subseteq\mathfrak{M}_X^{r,d}$. $\bigcup_{t\neq0}W_t$ is a constant family over C^*. We then let W be the closure of $\bigcup_{t\neq0}W_t$ in $\mathfrak{M}^{d,\varepsilon}$ and let W_0 be the special fiber of W over $0\in C$.

Here is our strategy: Take a large h and a good $D^h\in|hH_C(-h\Delta)|$. By Proposition 2.13, for any $t\neq0$ and $c=\dim W_t$, the top self-intersection number $[\mathcal{L}_{D^h}]^c(W_t)>0$. Then since W is flat and proper over C, $[\mathcal{L}_{D^h}]^c(W_0)>0$. Therefore, according to Proposition 2.14,

$$\#_{\mathrm{mod}}\{E^{(2)}|E\in W_0\}=c. \tag{3.1}$$

On the other hand, since every sheaf $E \in W_t$, $t \neq 0$, has non-vanishing $\mathrm{Hom}(E, E(D))^0$, the upper-semicontinuity theorem tells us that there is a divisor $D' \subseteq \Delta$ (we can make it independent of the choice of W_0) such that for any $E \in W_0$,

$$\mathrm{Hom}_\Delta(E^{(2)}, E^{(2)}(D'))^0 \neq \{0\}. \tag{3.2}$$

Thus, by applying Theorem 1.1, we get an upper bound of (3.1) and hence an upper bound of c. We now fill in the details of this approach.

To establish (3.2) for some $D' \subseteq \Delta$, we argue as follows: First of all, let $E \in W_0$ be any point. Since W is flat over C, there is a smooth affine curve S over C and a flat family of torsion free sheaves E_S on $Z_S = Z \times_S S$ such that for any closed $s \in S$ over $t \neq 0 \in C$, $E_s \in W_t$ and further, there is a closed $s_0 \in S$ over $0 \in C$ so that $E_{s_0} = E$. For any integer k, we consider the divisor $D_C - k\Delta$ on Z, where $D_C = p_X^{-1}(D)$, and the pullback divisor (of $D_C - k\Delta$) on Z_S which we denote by D_k. Clearly, the restriction of $D_k \otimes k(s_0)$ to $X \subseteq Z_0$ is $D - kH$. Now consider the vector space $\mathrm{Hom}_{Z_S}(E_S, E_S(D_k))^0$. By assumption, for general $s \in S$, $\mathrm{Hom}_{Z_s}(E_s, E_s(D_k))^0 \neq 0$. Thus $\mathrm{Hom}_{Z_S}(E_S, E_S(D_k)) \neq \{0\}$. Let $w \in \mathrm{Hom}_{Z_S}(E_S, E_S(D_k))^0$ be a non-trivial section and let ξ be the uniformizing parameter of S at s_0. Then because E_S is flat over S, there is an $n \geq 0$ such that the restriction of w/ξ^n to Z_{s_0} gives rise to a non-trivial homomorphism $\varphi : E_{s_0} \to E_{s_0}(D_k)$.

Next, because $E^{(1)}$ is a quotient sheaf of E, φ induces a homomorphism $E \to E^{(1)}(D_k)$ and further because $E^{(1)}$ is torsion free, it comes from $\varphi_1 : E^{(1)} \to E^{(1)}(D_k)$. Similarly, we have $\varphi_2 : E^{(2)} \to E^{(2)}(D_k)$. Because E is torsion free, at least one φ_i is non-trivial. Now we claim that we can choose a k (independent of d and ε) so that φ_1 is always trivial. Indeed, we first choose k so that $H \cdot (D - kH) < 0$. Then since $\det \varphi_1 \in H^0(\mathcal{O}_X(rD - rkH)) = \{0\}$, $\det \varphi_1$ is trivial. If we let $A \subset E^{(1)}$ be the kernel of φ_1, then $E^{(1)}/A$ is torsion free and further, there is a ψ making the following diagram commutative

$$\begin{tikzcd} A \arrow[r] & E^{(1)} \arrow[r] \arrow[d, "\varphi_1"] & E^{(1)}/A \arrow[r] \arrow[d, "\psi"'] & 0 \\ & E^{(1)}(D_k) \arrow[r, equal] & E^{(1)}(D_k) & \end{tikzcd} \tag{3.3}$$

On the other hand, by Lemma 2.4, there is a constant e_1 independent of d and ε such that $E^{(1)}$ is e_1-stable. Thus if $\varphi_1 \neq 0$, then $0 < \mathrm{rk}(E^{(1)}/A) < \mathrm{rk}(E^{(1)})$ and $E^{(1)}/A$ is both a subsheaf of $E^{(1)}(D_k)$ and a quotient sheaf of $E^{(1)}$. Therefore,

$$\mu(E^{(1)}(D_k)) + \frac{1}{\mathrm{rk}(E^{(1)}/A)}\sqrt{H^2} \cdot e_1 > \mu(E^{(1)}/A) > \mu(E^{(1)}) - \frac{1}{\mathrm{rk}(E^{(1)}/A)}\sqrt{H^2} \cdot e_1.$$

A straightforward calculation shows that this is impossible if we let

$$k > \frac{1}{H^2}(D \cdot H + 2\sqrt{H^2} \cdot e_1). \tag{3.4}$$

Hence φ_1 must be trivial.

From now on, we fix such a k. Then our previous argument shows that all $V \in \Theta = \{E^{(2)} \mid E \in W_0\}$ have non-vanishing $\operatorname{Hom}_\Delta(V, V(D_k))^0$. As we explained, our intention is to apply Theorem 1.1 to the set Θ to get the bound:

$$\#_{\mathrm{mod}}\Theta \le \eta_X(r,d,I) - C_1, \quad d \gg 0. \tag{3.5}$$

First of all, all $V \in \Theta$ are e_2-stable by Lemma 2.15 and have $\det V = I_2$, $I_2 \in \Lambda = \{I_0(-r\Sigma^-), \cdots, I_0\}$. Next, for each $I_2 \in \Lambda$, there is an $\varepsilon_0(I_2) > 0$ specified by Theorem 1.1. We let $\varepsilon_0 = \min_{I_2\in\Lambda}\{\varepsilon_0(I_2)\}$. Then for any ε smaller than the ε_0, the ample divisor $H(\varepsilon)_{|\Delta}$ on Δ satisfies the condition of Theorem 1.1. In order to apply Theorem 1.1, we need to know that the general element of Θ is locally free, which certainly is quite delicate in general. The solution we propose is to use the double dual operation to relate any sheaf $F \in \Theta$ to its double dual $\mathcal{F}(F) = F^{\vee\vee}$. $F^{\vee\vee}$ is always locally free because Δ is a smooth surface. Assume $d_2 = c_2(\mathcal{F}(F))$; then $d_2 \le c_2(F)$ and the equality holds if and only if F is locally free. Following the notation introduced at the beginning of §1, we have

$$\mathcal{F} : \Theta \longrightarrow \bigcup_{d_2\in\mathbb{Z}\,;I_2\in\Lambda} \mathfrak{A}^{r,d_2}_{e_2,I_2}(D_{k|\Delta}).$$

(We use $\mathfrak{A}^{\cdot}_{\cdot}$ to denote sets related to Δ and use $\mathcal{V}^{\cdot}_{\cdot}$ to denote sets related to X.) Here we have used the fact that $\operatorname{Hom}\bigl(F, F(D_{k|\Delta})\bigr)^0 \ne 0$ implies $\operatorname{Hom}\bigl(F^\vee, F^\vee(D_{k|\Delta})\bigr)^0 \ne 0$. Next, we divide Θ into subsets Θ_{d_1} according to the value of the second Chern class of $F \in \Theta$. Then, $\Theta = \bigcup \Theta_{d_1}$. We have the following estimate which will be proved shortly.

Lemma 3.2. *For any $V \in \mathfrak{A}^{r,d_2}_{e_2,I_2}$, $\#_{\mathrm{mod}}\bigl(\mathcal{F}^{-1}(V) \cap \Theta_{d_1}\bigr) \le (r+1)(d_1 - d_2)$.*

Now we are ready to complete the proof of the theorem. First of all, by applying Theorem 1.1 to the set $\mathfrak{A}^{r,d_2}_{e_2,I_2}(D_{k|\Delta})$, we know that for any constant C_2, there is an N_2 such that whenever $d_2 \ge N_2$, we have

$$\#_{\mathrm{mod}}\mathfrak{A}^{r,d_2}_{e_2,I_2}(D_{k|\Delta}) \le \eta_\Delta(r,d_2,I_2) - C_2, \quad I_2 \in \Lambda. \tag{3.6}$$

To control the left-hand side of (3.6) for small d_2, we invoke Theorem 1.5 to get

$$\#_{\mathrm{mod}}\mathfrak{A}^{r,d_2}_{e_2,I_2} \le \eta_\Delta(r,d_2,I_2) + C_3, \quad I_2 \in \Lambda, \tag{3.7}$$

where C_3 is a constant. Another estimate we need was established in Proposition 2.7,

$$c_2(F) \le d + C_4, \quad \forall F \in \Theta. \tag{3.8}$$

The proof of (3.5) then goes as follows: For any constant C_1, we let C_2 be such that

$$C_2 \ge C + \eta_\Delta(r,C_4,I_2) - \eta_X(r,0,I), \quad \forall I_2 \in \Lambda, \tag{3.9}$$

and let N_2 be the constant that makes (3.6) hold. We then let N be so that

$$(r+1)(N-N_2) + \eta_\Delta(r,N_2,I_2) + C_3 \le \eta_X(r,N,I) - C_1. \tag{3.10}$$

We claim that when $d \geq N$ and $\varepsilon < \varepsilon_0$, then (3.5) holds. Indeed, let $d_1 \leq d + C_4$ be any integer. Then for $d_2 \leq N_2$, by (3.7) and (3.10),

$$\begin{aligned}\#_{\mathrm{mod}}\Big(\Theta_{d_1} \cap \mathcal{F}^{-1}\big(\mathfrak{A}^{r,d_2}_{e_2,I_2}(D_{k|\Delta})\big)\Big) \\ &\leq (r+1)(d_1 - d_2) + \#_{\mathrm{mod}}\Big(\mathcal{F}(\Theta_{d_1}) \cap \mathfrak{A}^{r,d_2}_{e_2,I_2}(D_{k|\Delta})\Big) \\ &\leq (r+1)(d_1 - d_2) + \eta_\Delta(r, d_2, I_2) + C_3 \leq \eta_X(r,d,I) - C_1.\end{aligned}$$

Assume $d_2 \geq N_2$. By (3.6) and (3.9), we have

$$\begin{aligned}\#_{\mathrm{mod}}\Big(\Theta_{d_1} \cap \mathcal{F}^{-1}\big(\mathfrak{A}^{r,d_2}_{e_2,I_2}(D_{k|\Delta})\big)\Big) &\leq \#_{\mathrm{mod}}\mathfrak{A}^{r,d_2}_{e_2,I_2}(D_{k|\Delta}) + (r+1)(d_1 - d_2) \\ &\leq \eta_\Delta(r, d_2, I_2) - C_2 + (r+1)(d_1 - d_2) \leq \eta_X(r,d,I) - C_1.\end{aligned}$$

Thus we have established (3.5).

To finish the proof of the theorem, it suffices to show that $\#_{\mathrm{mod}}\Theta = \#_{\mathrm{mod}}W_t$. For this, we will use Donaldson's line bundle $\mathcal{L}_{D^h}$. First of all, for any d, we choose $\varepsilon < \min(\varepsilon_0, \kappa(d))$. ($\kappa(d)$ was specified in Proposition 2.10.) We then apply Proposition 2.13 to the set W_t, $t \neq 0$. Proposition 2.13 asserts that with $c = \dim W_t$, $[\mathcal{L}_{D^h}]^c(W_t) > 0$. Since W is flat and proper over C, we have

$$[\mathcal{L}_{D^h}]^c(W_0) = [\mathcal{L}_{D^h}]^c(W_t) > 0.$$

In particular, Proposition 2.14 tells us that then

$$\#_{\mathrm{mod}}\{E^{(2)} \mid E \in W_0\} = c.$$

Therefore, combined with inequality (3.5), we have that for $d \geq N$,

$$\dim\{E \in \mathfrak{M}^{r,d}_X \mid \mathrm{Hom}_X(E, E(D))^0 \neq 0\} \leq \eta_X(r,d,I) - C.$$

This completes the proof of Theorem 3.1.

Before we go any further, let us finish the proof of Lemma 3.2.

Proof of Lemma 3.2. The situation when $r = 2$ was proved in [L1, p. 461]. In general, let E be any rank r torsion free sheaf and let $V = \mathcal{F}(E)$. Then E is uniquely determined by the quotient sheaf $V \to V/E$, where V/E is supported on a discrete set and of length $\ell(V/E) = c_2(E) - c_2(V)$. Therefore, $\mathcal{F}^{-1}(V) \cap \{$sheaves of $c_2 = c_2(V) + c\}$ is exactly the set of all quotient sheaves $V \to A$ such that A is supported on a discrete set and $\ell(A) = c$. Let $\mathbf{Quot}^c_V$ be Grothendieck's Quot-scheme of all quotient sheaves A of V with $\ell(A) = c$. $\mathbf{Quot}^c_V$ is projective by [Gr, p. 13]. Observe also that when A is supported on c distinct points, then by [Gr, p. 21],

$$\dim T_A \mathbf{Quot}^c_V = (r+1)c. \tag{3.11}$$

Thus the lemma will be established if we can show that for any quotient sheaf $A_0 \in \mathbf{Quot}^c_V$, there is a deformation A_t of A_0 such that for generic t, A_t is supported on c distinct points [L1, p. 461]. In the following, we will demonstrate how to construct such a deformation.

Clearly, this is a local problem. Let U be a classical neighborhood of $0 \in \mathbb{C}^2$ with coordinate $z = (z_1, z_2)$. Assume A_0 is a quotient sheaf of $\mathcal{O}_U^{\oplus r}$ of length c supported at the origin 0. Let $E = \ker\{\mathcal{O}_U^{\oplus r} \to A_0\}$. Along the lines of the argument given in [L1, p. 462], we can show that there are $f_1, \cdots, f_n \in \mathcal{O}_U^{\oplus r}$ such that $\{f_i\}_{i=r+1}^n$ are divisible by z_1 and $\{f_i\}_1^n$ generate the submodule E.

Next, we define

$$f_i(z,t) = \begin{cases} f_i(z), & 1 \le i \le r; \\ (z_1 - t)\frac{f_i(z)}{z_1}, & r+1 \le i \le n. \end{cases} \tag{3.12}$$

We then define a submodule $E_D \subset \mathcal{O}_{U\times D}^{\oplus r}$, where D is a small disk with parameter t, by

$$E_D = (f_1(z,t), \cdots, f_n(z,t)) \cdot \mathcal{O}_{U\times D} \subset \mathcal{O}_{U\times D}^{\oplus r}. \tag{3.13}$$

Let $A_D = \mathcal{O}_{U\times D}^{\oplus r}/E_D$. E_D and A_D can be viewed as families of sheaves parameterized by D. It is easy to see that when $E_D \otimes k(0)$ is torsion free, then $A_D \otimes k(0) = A_0$ and for t small, A_D is a (flat) deformation of A_0. Now we check that $E_D \otimes k(0)$ is torsion free. Suppose there are $h \in E_D$ and $f \in \mathcal{O}_U$ such that $f \cdot h = th'$ for some $h' \in E_D$. Let

$$h = \sum_{i=1}^n g_1(z,t) \cdot f_i(z,t).$$

Then the fact that $f(z) \cdot h \equiv 0 \bmod(t)$ in $\mathcal{O}_{U\times D}^{\oplus r}$ and that $f_1(z)/(z_1), \cdots, f_r(z)/(z_1)$ generate a rank r $\mathcal{O}_U/(z_1)$-module implies

$$z_1 \mid g_i(z,0), \quad i = 1, \cdots, r; \tag{3.14}$$

$$\sum_{i=1}^r \frac{g_i(z,0)}{z_1} \cdot f_i(z) + \sum_{i=r+1}^n g_i(z,0) \cdot \frac{f_i(z)}{z_1} \equiv 0. \tag{3.15}$$

Further, if we write $g_i(z,t) = \alpha_i(z) + t\beta_i(z,t)$, then the following identities hold in $\mathcal{O}_{U\times D}^{\oplus r}$:

$$\begin{aligned} h &= \sum_{i=1}^r \Big(\alpha_i(z) f_i(z) + t\beta_i(z,t) f_i(z)\Big) \\ &\quad + \sum_{i=r+1}^n \Big(\alpha_i(z)(z_1 - t)\frac{f_i(z)}{z_1} + t\beta_i(z,t)(z_1-t)\frac{f_i(z)}{z_1}\Big) \\ &= (z_1 - t)\Big(\sum_{i=1}^r \frac{\alpha_i(z)}{z_1} f_i(z) + \sum_{i=r+1}^n \alpha_i(z)\frac{f_i(z)}{z_1}\Big) \\ &\quad + t\Big(\sum_{i=1}^r \big(\frac{\alpha_i(z)}{z_1} + \beta_i(z,t)\big) f_i(z) + \sum_{i=r+1}^n \beta_i(z,t)(z_1 - t)\frac{f_i(z)}{z_1}\Big) \\ &= th'', \end{aligned}$$

where h'' obviously belongs to E_D. Since E_D is a submodule of $\mathcal{O}_{U\times D}^{\oplus r}$, h must be equal to th'' in E_D. Therefore, $f\cdot h_{|t=0}=0$ implies $h_{|t=0}=0$ in $E_D\otimes k(0)$ or that $F_D\otimes k(0)$ is torsion free.

In general, A_t is not supported on c distinct points. But at least we expect that A_t is simpler than A_0, say $\mathrm{Supp}(A_t)$ has at least two distinct points. In the following, we will show that this is indeed the case. Without loss of generality, we can assume that $f_i(z)$ all vanish at the origin. (Since otherwise, A_0 is essentially a quotient sheaf of $\mathcal{O}_U^{\oplus(r-1)}$ and we can use induction on r to take care of this situation.) For small t, the equation

$$f_1(t,z_2)\wedge\cdots\wedge f_r(t,z_2)=0 \tag{3.16}$$

has solutions, say $z_2=w_t$, because $f_1(0)\wedge\cdots\wedge f_r(0)=0$ and $f_1(0,z_2)\wedge\cdots\wedge f_r(0,z_2)\neq 0$ for generic z_2. Note that $(t,w_t)\in\mathrm{Supp}(A_t)$. If $\mathrm{Supp}(A_t)$ is a single point, then $f_{r+1}(z)/z_1,\cdots f_n(z)/z_1$ must generate $\mathcal{O}_U^{\oplus r}$ at the origin. Thus by discarding some extra terms, we will have $n=2r$ and further, by eliminating terms in $f_1,\cdots,f_r$ that involve z_1 by using combinations of $f_{r+1},\cdots,f_n$, we can assume $z_2|f_1(z),\cdots,z_2|f_r(z)$. Therefore, we can consider the deformation of A_0 derived from

$$E'_D=\Big((z_2-t)\frac{f_1(z)}{z_2},\cdots,(z_2-t)\frac{f_r(z)}{z_2},f_{r+1}(z),\cdots,f_n(z)\Big).$$

In case $\mathrm{Supp}(A'_t)$ is still a single point for generic t, then $\left(\frac{f_1(z)}{z_2},\cdots,\frac{f_r(z)}{z_2}\right)$ will generate $\mathcal{O}_U^{\oplus r}$ at 0 also. In particular, $A_0=\bigoplus^r\mathbb{C}$ and then the desired deformation can be written by hand.

In the remainder of this section, we will complete the proof of the theorems stated at the beginning of this paper. We first investigate the sets $\mathcal{S}_{e,I}^{r,d}$ and $\mathcal{S}_{e,I}^{r,d}(D)$ introduced at the beginning of this section. We shall prove

Theorem 3.3. *For any choice of r, I and D and any choice of constants e and C, there is an integer N such that whenever $d\geq N$, then we have*

$$\#_{\mathrm{mod}}\mathcal{S}_{e,I}^{r,d}=\eta(r,d,I), \tag{3.17}$$

$$\#_{\mathrm{mod}}\mathcal{S}_{e,I}^{r,d}(D)\leq\eta_X(r,d,I)-C. \tag{3.18}$$

Proof of (3.17). Let $\mathcal{V}_{e,I}^{r,d}=\mathcal{S}_{e,I}^{r,d}\cap\{\text{locally free sheaves}\}$ and let $\mathcal{V}_{e,I}^{r,d}(D)=\mathcal{V}_{e,I}^{r,d}\cap\mathcal{S}_{e,I}^{r,d}(D)$. Clearly, (3.17) is a stronger statement than

$$\#_{\mathrm{mod}}\mathcal{V}_{e,I}^{r,d}=\eta_X(r,d,I), \tag{3.19}$$

which in turn is stronger (in case $e>0$) than

$$\#_{\mathrm{mod}}\mathcal{V}_{\mu,I}^{r,d}=\eta_X(r,d,I). \tag{3.20}$$

Our strategy is first to prove statement (3.20) and then prove (3.19) and (3.17). We proceed by induction on the rank r. (3.17) and (3.20) are trivial when $r=1$.

For $r \geq 2$ and $E \in \mathcal{V}^{r,d}_{\mu,I}$, the Kodaira-Spencer-Kuranishi deformation theory tells us that there is a holomorphic map

$$f : U \subset H^1(X, \mathcal{E}nd^0(E)) \longrightarrow H^2(X, \mathcal{E}nd^0(E)),$$

where U is an (analytic) neighborhood of the origin, such that $f^{-1}(0)$ is the versal deformation space of E. Since $h^0(\mathcal{E}nd^0(E)) = 0$ (since E is μ-stable),

$$\#_{\mathrm{mod}}\big(\mathcal{V}^{r,d}_{\mu,I}, [E]\big) \geq h^1(\mathcal{E}nd^0(E)) - h^2(\mathcal{E}nd^0(E)),$$

and when $h^2(\mathcal{E}nd^0(E)) = 0$, $\#_{\mathrm{mod}}\big(\mathcal{V}^{r,d}_{\mu,I}, [E]\big) = h^1(\mathcal{E}nd^0(E))$. Next, by Riemann-Roch, one calculates $\chi(\mathcal{E}nd^0(E)) = \eta_X(r, d, I)$. Thus one gets

$$\#_{\mathrm{mod}}\big(\mathcal{V}^{r,d}_{\mu,I}, [E]\big) \geq \eta_X(r, d, I). \tag{3.21}$$

On the other hand, since $h^2(\mathcal{E}nd^0(E)) = h^0(\mathcal{E}nd^0(E) \otimes K_X)$, by Theorem 3.1, there is an N such that whenever $d \geq N$, we have

$$\#_{\mathrm{mod}}\{E \in \mathcal{V}^{r,d}_{\mu,I} \mid h^0(\mathcal{E}nd^0(E) \otimes K_X) > 0\} \leq \eta_X(r, d, I) - 1.$$

Therefore, for generic $E \in \mathcal{V}^{r,d}_{\mu,I}$, $\#_{\mathrm{mod}}\big(\mathcal{V}^{r,d}_{\mu,I}, [E]\big) = \eta_X(r, d, I)$. Thus we have proved (3.20) provided $d \geq N$. To further attack (3.19) and (3.17), we need the following estimate which is interesting in its own right.

Theorem 3.4. *For any choice of r, I and two constants $e_1 > e_2$, there is a constant C' such that*

$$\#_{\mathrm{mod}}\big(\mathcal{S}^{r,d}_{e_1,I} \setminus \mathcal{S}^{r,d}_{e_2,I}\big) \leq (2r - 1)d + C'. \tag{3.22}$$

Proof. Let E be any torsion free sheaf in $\mathcal{S}^{r,d}_{e_1,I} \setminus \mathcal{S}^{r,d}_{e_2,I}$. Since E is not e_2-stable, there is a torsion free subsheaf $F_1 \subseteq E$ such that E/F_1 is torsion free and that $\mu(F_1) \geq \mu(E) + e_2\sqrt{H^2}/\operatorname{rk}(F_1)$. Because E is e_1-stable, $\mu(F_1)$ is bounded from above by $\mu(E) + e_1\sqrt{H^2}/\operatorname{rk}(F_1)$. Combined, we get

$$\frac{1}{r} I \cdot H + \frac{1}{r_1} e_1 \sqrt{H^2} < \frac{1}{r_i} I_i \cdot H < \frac{1}{r} I \cdot H + \frac{1}{r_i} e\sqrt{H^2}, \tag{3.23}$$

where $r_i = \operatorname{rk} F_i$, $d_i = c_2(F_i)$ and $I_i = \det F_i$ with $F_2 = E/F_1$. Note that E belongs to the exact sequence

$$0 \longrightarrow F_1 \longrightarrow E \longrightarrow F_2 \longrightarrow 0. \tag{3.24}$$

We call (r_i, d_i, I_i) admissible if they do come from (3.24) with $E \in \mathcal{S}^{r,d}_{e_1,I} \setminus \mathcal{S}^{r,d}_{e_2,I}$. We claim that F_i are e_3-stable with $e_3 = e_1 + |e_2|$. Indeed, let $L \subset F_1$ be any subsheaf. Because L is also a subsheaf of E,

$$\mu(L) < \mu(E) + \frac{1}{\operatorname{rk}(L)} e_1 \sqrt{H^2} \leq \mu(F_1) - \frac{1}{r_1} e_2 \sqrt{H^2} + \frac{1}{\operatorname{rk}(L)} e_1 \sqrt{H^2}.$$

Thus, F_1 is e_3-stable. F_2 is e_3-stable for the same reason. Therefore, $F_i \in \mathcal{S}^{r_i,d_i}_{e_3,I_i}$. Finally, because of (3.24),

$$\begin{aligned}\#_{\mathrm{mod}}\big(\mathcal{S}^{r,d}_{e_1,I} \setminus \mathcal{S}^{r,d}_{e_2,I}\big) \leq \sup_{(r_i,d_i,I_i)} &\big\{\#_{\mathrm{mod}}\big(\mathcal{S}^{r_1,d_1}_{e_3,I_1}\big) + \#_{\mathrm{mod}}\big(\mathcal{S}^{r_2,d_2}_{e_3,I_2}\big) \\ &+ \sup\{\dim \operatorname{Ext}^1(F_2, F_1) \mid F_i \in \mathcal{S}^{r_i,d_i}_{e_3,I_i}\}\big\},\end{aligned} \tag{3.25}$$

where the supremum is taken over all admissible tuples (r_i, d_i, I_i). Note that we only have numerical restriction on I_i (cf. (3.23)) and d_i can be small, thus we cannot expect an estimate of type (3.17) to hold for all $\mathcal{S}^{r_i,d_i}_{e_3,I_i}$. Nevertheless, we have

Lemma 3.5. *There is a constant C_1 depending only on r, $e_3 > 0$ and $\deg I'$, $I' \in \operatorname{Pic}(X)$, such that for $r' \le r$, we have*

$$\#_{\mathrm{mod}}\mathcal{S}^{r',d'}_{e_s,I'} \le \eta_X(r',d',I') + C_1.$$

Proof. It suffices to show that there is a constant C_1 such that for any $E \in \mathcal{S}^{r',d'}_{e_3,I'}$,

$$\dim \operatorname{Ext}^1(E,E)^0 \le 2r'd' - (r'-1)I'^2 + C_1.$$

First, since E is e_3-stable and $e_3 > 0$, $\mathcal{E}nd^0(E)$ is $2re_3$-stable. Hence by Lemma 1.8, both $h^0(\mathcal{E}nd^0(E))$ and $h^0(\mathcal{E}nd^0(E)\otimes K_X)$ are bounded from above by a constant, say C_1. By Serre duality, $\operatorname{Ext}^2(E,E)^0 = H^0(\mathcal{E}nd^0(E)\otimes K_X)$. Therefore,

$$\begin{aligned}\dim \operatorname{Ext}^1(E,E)^0 &= 2r'd' - (r'-1)I'^2 - (r'-1)^2\chi(\mathcal{O}_X)\\ &\qquad + \dim \operatorname{Ext}^0(E,E)^0 + \dim \operatorname{Ext}^2(E,E)^0\\ &\le 2r'd' - (r'-1)I'^2 + \big(2C_1 - (r'-1)^2\chi(\mathcal{O}_X)\big).\end{aligned}$$

This completes the proof of the lemma.

Returning to the proof of Theorem 3.4, we need to estimate the term $\dim \operatorname{Ext}^1(F_1,F_2)$ in (3.25). First of all, by Riemann-Roch, for $F_i \in \mathcal{S}^{r_i,d_i}_{e_3,I_i}$,

$$\begin{aligned}&\dim \operatorname{Ext}^1(F_1,F_2)\\ &\quad = \dim \operatorname{Ext}^0(F_1,F_2) + \dim \operatorname{Ext}^2(F_1,F_2)\\ &\qquad - \Big(\frac{r_2}{2}I_1^2 + \frac{r_1}{2}I_2^2 - (\frac{r_1}{2}I_2 - \frac{r_2}{2}I_1)\cdot K_X - I_1\cdot I_2 + r_1r_2\chi(\mathcal{O}_X) - r_1d_2 - r_2d_1\Big).\end{aligned}$$

Because F_1 and F_2 are e_3-stable, $F_1^\vee \otimes F_2$ and $F_2^\vee\otimes F_1$ are $2re_3$-stable. Also, the degree of $F_1^\vee\otimes F_2$ and $F_2^\vee \otimes F_1$ are bounded (from both sides) by constants depending on r, e and $I\cdot H$. Thus, there is a constant C_2 depending on these parameters only so that

$$\dim \operatorname{Ext}^0(F_1,F_2), \quad \dim \operatorname{Ext}^2(F_1,F_2) \le C_2.$$

Therefore, for any admissible (r_i,d_i,I_i),

$$\begin{aligned}&\#_{\mathrm{mod}}\mathcal{S}^{r_1,d_1}_{e_3,I_1} + \#_{\mathrm{mod}}\mathcal{S}^{r_2,d_2}_{e_3,I_2} + \sup\big\{\dim \operatorname{Ext}^1(F_1,F_2) \mid F_i \in \mathcal{S}^{r_i,d_i}_{e_3,I_i}\big\}\\ &\qquad \le 2r_1d_1 - (r_1-1)I_1^2 + 2r_2d_2 - (r_2-1)I_2^2 + 2C_1\\ &\qquad\quad + \Big(-\frac{r_2I_1^2 - r_1I_2^2}{2} + \frac{r_1I_2 - r_2I_1}{2}\cdot K_X\\ &\qquad\qquad\qquad + I_1\cdot I_2 - r_1r_2\chi(\mathcal{O}_X) + r_1d_2 + r_2d_1\Big) + 2C_2\\ &\quad \le (2r-1)d + (1-r_2)d_1 + (1-r_1)d_2 - (r_1 + \frac{r_2}{2} - 1)I_1^2 - (r_2 + \frac{r_1}{2} - 1)I_2^2\\ &\qquad + I_1\cdot I_2 + (\frac{r_1}{2} - \frac{r_2}{2}I_1)\cdot K_X - r_1r_2\chi(\mathcal{O}_X) + 2C_1 + 2C_2.\end{aligned}$$

Thanks to Lemma 2.5, there is a constant $C_3 \le 0$ depending on r, e and $I \cdot H$ only such that $d_i - \frac{r_i-1}{2r_i} I_i^2 \ge C_3$. Thus combined with $d = c_2(E) = I_1 \cdot I_2 + d_1 + d_2$ and $I_2 = I - I_1$, the right-hand side of the above inequality is

(3.26)
$$
\begin{aligned}
&\le (2r-1)d - (r + \frac{1-r_2}{2r_1} - \frac{3}{2}) I_1^2 - (r + \frac{1-r_1}{2r_2} - \frac{3}{2}) I_2^2 \\
&\quad - (2r-2) I_1 \cdot I_2 + (\frac{r_1}{2} I_2 - \frac{r_2}{2} I_1) \cdot K_X + r^2 |\chi(\mathcal{O}_X)| + C_1 + 2C_2 - rC_3 \\
&= (2r-1)d + \Big((1 + \frac{r_1 - 1}{2r_2} + \frac{r_2 - 1}{2r_1}) I_1^2 + \frac{r_2 + 1 - r_1}{2r_2} I \cdot I_1 - \frac{r}{2} I_1 \cdot K_X \Big) + C_4.
\end{aligned}
$$

Finally, because $|I_1 \cdot H| \le |I \cdot H| + e\sqrt{H^2}$, the Hodge index theorem tells us that the sum of three middle terms in the last line of (3.26) is bounded from above by a constant C_5. Therefore, combined with (3.25), we have

$$\#_{\mathrm{mod}} \big(\mathcal{S}_{e,I}^{r,d} \setminus \mathcal{S}_{\mu,I}^{r,d} \big) \le (2r-1)d + C'.$$

Proof of (3.18). We shall only consider the case where $e \ge 0$. The case $e < 0$ can be proved similarly. First of all, by letting $e_1 = e$ and $e_2 = 0$ in Theorem 3.4, we know that there is a constant C_1 such that $\#_{\mathrm{mod}} \big(\mathcal{S}_{e,I}^{r,d} \setminus \mathcal{S}_{\mu,I}^{r,d} \big) \le (2r-1)d + C_1$. Then by choosing N large, we have $\#_{\mathrm{mod}} \mathcal{V}_{\mu,I}^{r,d} = \eta_X(r,d,I)$ and $(2r-1)d + C_1 \le \eta_X(r,d,I)$ whenever $d \ge N$. Thus

$$\#_{\mathrm{mod}} \mathcal{V}_{e,I}^{r,d} \le \max \big\{ \#_{\mathrm{mod}} \mathcal{V}_{\mu,I}^{r,d}, \#_{\mathrm{mod}} \big(\mathcal{V}_{e,I}^{r,d} \setminus \mathcal{V}_{\mu,I}^{r,d} \big) \big\} = \eta_X(r,d,I).$$

To prove (3.17), we will use the double dual operation $\mathcal{F}$. Let

$$\mathcal{F} : \mathcal{S}_{e,I}^{r,d} \longrightarrow \bigcup_{d' \le d} \mathcal{V}_{e,I}^{r,d'} \tag{3.27}$$

be the map sending E to $E^{\vee\vee}$. Thanks to Lemma 3.2, we have

$$\#_{\mathrm{mod}} \mathcal{S}_{e,I}^{r,d} \le \sup_{d' \le d} \big\{ \#_{\mathrm{mod}} \mathcal{V}_{e,I}^{r,d'} + (r+1)(d-d') \big\}.$$

Further, let $C_1 \ge 0$ be a constant such that

$$\#_{\mathrm{mod}} \mathcal{V}_{e,I}^{r,d'} \le \eta(r,d',I) + C_1.$$

Then, for $d \ge N + C_1$, $\#_{\mathrm{mod}} \mathcal{S}_{e,I}^{r,d}$ is no more than either

$$\sup_{d' < N} \{ \eta(r,d',I) + C_1 + (r+1)(d-d') \} \le \eta(r,N,I) + C_1 \le \eta_X(r,d,I)$$

or

$$\sup_{N \le d' \le d} \{ \eta_X(r,d',I) + (r+1)(d-d') \} \le \eta_X(r,d,I).$$

This establishes (3.17). (3.18) can be proved similarly based on Theorem 3.1. We shall omit it.

In light of Theorem 3.3, the proofs of Theorems 0.2 and 0.3 are now quite easy. Recall that for the data (r,d,I) and sufficiently large n, we can form Grothendieck's Quot-scheme $\mathbf{Quot}^{r,d}_{\rho,I}$ of all quotient sheaves $\mathcal{O}_X^{\oplus\rho}\to E$ with $\operatorname{rk} E=r$, $\det E=I$, $c_2(E)=d$ and $\rho=h^0(E(n))$. If we let $\mathcal{U}\subseteq\mathbf{Quot}^{r,d}_{n,I}$ be the open subset of all semistable (with respect to H) quotient sheaves, then $\mathcal{U}$ is $SL(\rho,\mathbb{C})$-invariant and the good quotient $\mathcal{U}//SL(\rho,\mathbb{C})$, which does exist, is exactly the moduli scheme $\mathfrak{M}_X^{r,d}(I,H)$ of rank r semistable sheaves of $c_1=I$ and $c_2=d$. Further, if we let $\mathcal{U}^s\subseteq\mathcal{U}$ be the subset of strictly stable sheaves, then $\pi:\mathcal{U}^s\to\pi(\mathcal{U}^s)\subseteq\mathfrak{M}_X^{r,d}(I,H)$ is a principal $SL(\rho,\mathbb{C})$-bundle. With this set-up in mind, one sees that in order to prove Theorem 0.2, it suffices to classify the singular locus of $\mathcal{U}$.

Proposition 3.6. *With the notation as before and for any constant C, there is a constant N such that whenever $d\ge N$, then $\dim\mathcal{U}=\eta_X(r,d,I)+(\rho^2-1)$ and the codimension*

$$\operatorname{codim}\bigl(\operatorname{Sing}(\mathcal{U}),\mathcal{U}\bigr)\ge C.$$

Further, when the codimension is at least 1, then $\mathcal{U}$ locally is a complete intersection, and when the codimension is at least 2, then $\mathcal{U}$ is normal.

Proof. Let $E\in\mathcal{U}$ be any quotient sheaf, let $q_2=h^0(\mathcal{E}nd^0(E)\otimes K_X)$ and let $q_1=\eta_X(r,d,I)+(\rho^2-1)+q_2$. Then the argument in [L2, p. 8] demonstrates that the completion of the local ring of $\mathcal{U}$ at E is of the form $k[[t_1,\cdots,t_{q_1}]]/J$, where J is an ideal generated by at most q_2 elements. In particular, for each component $\mathbf{U}\subseteq\mathcal{U}$, we always have

$$\dim\mathbf{U}\ge\eta_X(r,d,I)+(\rho^2-1). \tag{3.28}$$

Next, by [Ar], [Mu], [Ma, p. 594], the singular locus $\operatorname{Sing}(\mathcal{U})$ is exactly the set of all quotient sheaves E with $\operatorname{Ext}^2(E,E)^0\ne0$. By Theorem 3.3, for any constant C, there is an N such that whenever $d\ge N$, the set

$$\mathcal{U}\cap\mathcal{S}^{r,d}_{1,I}(K_X)=\bigl\{E\in\mathcal{U}\mid h^0(\mathcal{E}nd^0(E)\otimes K_X)\ne0\bigr\}$$

obeys $\#_{\mathrm{mod}}\bigl(\mathcal{U}\cap\mathcal{S}^{r,d}_{1,I}(K_X)\bigr)\le\eta_X(r,d,I)-C$. Therefore,

$$\dim\operatorname{Sing}\bigl(\mathcal{U}\bigr)\le\#_{\mathrm{mod}}\Bigl(\mathcal{U}\cap\mathcal{S}^{r,d}_{1,I}(K_X)\Bigr)+\dim SL(\rho)\le\eta_X(r,d,I)+(\rho^2-1)-C.$$

When $C\ge1$, this inequality and (3.28) imply that $\mathcal{U}$ has pure dimension $\eta_X(r,d,I)+(\rho^2-1)$ and $\operatorname{codim}\bigl(\operatorname{Sing}(\mathcal{U}),\mathcal{U}\bigr)\ge C$. Because the completion of the local rings of $\mathcal{U}$ are of the form $k[[t_1,\cdots,t_{q_1}]]/J$ with $J=(f_1,\cdots,f_{q_2})$, $\mathcal{U}$ is a local complete intersection. $\mathcal{U}$ will be normal if we further assume $\operatorname{codim}\bigl(\operatorname{Sing}(\mathcal{U}),\mathcal{U}\bigr)\ge2$.

Corollary 3.7. *Let X be a smooth algebraic surface, H an ample divisor and I a line bundle on X. Let $r\ge2$ be an integer. Then for any constant C, there is an N such that whenever $d\ge N$, then $\mathfrak{M}_X^{r,d}(I,H)$ has pure dimension $\eta_X(r,d,I)$ and further,* $\operatorname{codim}\bigl(\operatorname{Sing}(\mathfrak{M}_X^{r,d}),\mathfrak{M}_X^{r,d}\bigr)\ge C$.

Proof. Since $\pi:\mathcal{U}^s\to\pi(\mathcal{U}^s)\subseteq\mathfrak{M}_X^{r,d}$ is a principal bundle, the singular locus $\operatorname{Sing}(\mathfrak{M}_X^{r,d})$ is contained in

$$\pi\bigl(\operatorname{Sing}(\mathcal{U}^s)\bigr)\cap\pi\bigl(\mathcal{U}\setminus\mathcal{U}^s\bigr).$$

By Proposition 3.6, we know that for d large, we can arrange

$$\operatorname{codim}\big(\pi\big(\operatorname{Sing}(\mathcal{U}^s)\big), \mathfrak{M}_X^{r,d}\big) \geq C.$$

Therefore, to prove the corollary, we only need to find an upper bound of the dimension of $\pi\big(\mathcal{U} \setminus \mathcal{U}^s\big)$.

Let $E \in \mathcal{U} \setminus \mathcal{U}^s$. Then E admits a filtration $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_k = E$ such that $F_i = E_i/E_{i-1}$ are strictly stable. According to [Gi], E and $gr(E) = \bigoplus F_i$ have the same image in $\mathfrak{M}_X^{r,d}$ under π. Thus $\dim \pi(\mathcal{U} \setminus \mathcal{U}^s)$ can be bounded easily in terms of the dimension of moduli of lower rank stable sheaves. Similar to the proof of Theorem 3.4, we can show that there is a constant C_1 such that

$$\dim \pi(\mathcal{U} \setminus \mathcal{U}^s) \leq (2r-1)d + C_1. \tag{3.29}$$

(If we let $\mathcal{S}_{e,I}^{r,d}$ be the set introduced in §2, $\pi(\mathcal{U} \setminus \mathcal{U}^s) \subseteq \mathcal{S}_{-1,I}^{r,d} \setminus \mathcal{S}_{\mu,I}^{r,d}$ and then (3.29) follows directly from Theorem 3.4.) Thus for large N, we will have for $d \geq N$, $\dim \pi(\mathcal{U} \setminus \mathcal{U}^s) \leq \eta_X(r,d,I) - C$. This completes the proof of the corollary and Theorem 0.2.

Corollary 3.8. *With the notation as before, there exists N such that whenever $d \geq N$, then*

(1) *$\mathfrak{M}_X^{r,d}$ is normal. Further, if $s \in \mathfrak{M}_X^{r,d}$ is any closed point corresponding to a stable sheaf, then $\mathfrak{M}_X^{r,d}$ is a local complete intersection at s.*

(2) *The set of locally free μ-stable sheaves $\big(\mathfrak{M}_X^{r,d}\big)^{\mathrm{vb}} \subseteq \mathfrak{M}_X^{r,d}$ is dense in $\mathfrak{M}_X^{r,d}$.*

Proof. Let N be given by Proposition 3.6 so that whenever $d \geq N$, $\mathcal{U}$ has pure dimension $\eta_X(r,d,I)+(\rho^2-1)$ and $\operatorname{codim}\big(\operatorname{Sing}(\mathcal{U}),\mathcal{U}\big) \geq 2$. Then since $\mathcal{U}$ is normal, $\mathfrak{M}_X^{r,d}$ must be normal and since $\mathcal{U}^s$ is a local complete intersection, $\pi(\mathcal{U}^s) \subseteq \mathfrak{M}_X^{r,d}$ must be a local complete intersection. Here we have used the fact that $\mathcal{U} \to \mathfrak{M}_X^{r,d}$ is a good quotient and $\mathcal{U}^s \to \pi(\mathcal{U}^s)$ is a principal bundle. The last statement can be proved easily similar to that of Theorem 3.1. We shall omit it here.

The last subject we will study is the dependence of the moduli spaces on the choice of the polarizations. We prove

Theorem 3.9. *For any choice (r,I) and polarizations H_1 and H_2, there is a constant N so that whenever $d \geq N$, then $\mathfrak{M}_X^{r,d}(I,H_1)$ and $\mathfrak{M}_X^{r,d}(I,H_2)$ are birational to each other.*

Proof. Let $W \subseteq \mathfrak{M}_X^{r,d}(I,H_1)$ be the set of quotient sheaves E such that E are not H_2-stable. Then every $E \in W$ belongs to the exact sequence

$$0 \longrightarrow F_1 \longrightarrow E \longrightarrow F_2 \longrightarrow 0$$

such that $\mu(F_1) \geq \mu(E)$ (with respect to H_2). Then by repeating the argument in Theorem 3.4, we can find a constant C_1 (depending on H_1 and H_2) such that $\dim W \leq (2r-1)d + C_1$. Therefore, by letting N be large, we will have

$$\dim \mathfrak{M}_X^{r,d}(I,H_1) = \dim \mathfrak{M}_X^{r,d}(I,H_2) = \eta_X(r,d,I)$$

and $\dim W \leq \eta_X(r,d,I) - 1$ provided $d \geq N$. Therefore, by the universality of the moduli scheme, there is a morphism

$$\Phi : \mathfrak{M}_X^{r,d}(I,H_1) \setminus W \longrightarrow \mathfrak{M}_X^{r,d}(I,H_2)$$

which is generically one-one and onto. Thus $\mathfrak{M}_X^{r,d}(I,H_1)$ is birational to $\mathfrak{M}_X^{r,d}(I,H_2)$.

References

[Ar] V. I. Artamkin, *On deformation of sheaves*, Math. USSR-Izv. **32** (1989), no. 3, 663–668. MR **89g:**14004

[Do] S. K. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990), 257–315. MR **92a:**57035

[Br] J. Brosius, *Rank-2 vector bundles on a ruled surface* I, Math. Ann. **265** (1983), 155–168. MR **85i:**14009

[FM] J. Fogarty and D. Mumford, *Geometric invariant theory*, Springer-Verlag, Berlin and New York, 1982. MR **86a:**14006

[Fr] R. Friedman, *Vector bundles over surfaces* (to appear).

[Gi] D. Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann. of Math. (2) **106** (1977), 45–60. MR **81h:**14014

[GL] D. Gieseker and J. Li, *Irreducibility of moduli of rank two vector bundles*, J. Differential Geom. **40** (1994), 23–104. MR **95f:**14068

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience, New York, 1978. MR **80b:**14001

[Gr] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique* IV, Séminaire Bourbaki **221** (1960-61).

[Ha] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math., vol. 52, Springer, Berlin, Heidelberg, and New York, 1977. MR **57:**3116

[KM] F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves* I*, preliminaries on 'det' and 'div'*, Math. Scand. **39** (1976), 19–55. MR **55:**10465

[L1] J. Li, *Algebraic geometric interpretation of Donaldson's polynomial invariants*, J. Differential Geom. **37** (1993), 417–466. MR **93m:**14007

[L2] ———, *Kodaira dimension of moduli space of vector bundles on surfaces*, Invent. Math. **115** (1994), 1–40. MR **94i:**14016

[Ma] M. Maruyama, *Moduli of stable sheaves* I, J. Math. Kyoto Univ. **17** (1977), 91–126; II, J. Math. Kyoto Univ. **18** (1978), 557–614. MR **82h:**14011

[MR] V. B. Mehta and A. Ramanathan, *Semistable sheaves on projective varieties and their restriction to curves*, Math. Ann. **258** (1982), 213–224. MR **83f:**14013

[Mu] S. Mukai, *Symplectic structure on the moduli space of sheaves on an Abelian or K3 surfaces*, Invent. Math. **77** (1984), 101–116. MR **85j:**14016

[OG] K. O'Grady, *Moduli of vector bundles on projective surfaces: some basic results*, preprint.

[Qi] Z. Qin, *Birational properties of moduli spaces of stable locally free rank-2 sheaves on algebraic surfaces*, Manuscripta Math. **72** (1991), 163–180. MR **92h:**14009

[Zh] K. Zhu, *Generic smoothness of the moduli of rank two stable bundles over an algebraic surface*, Math. Z. **207** (1991), 629–643.

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