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AUTOMORPHISMS, ROOT SYSTEMS, AND COMPACTIFICATIONS OF HOMOGENEOUS VARIETIES

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1. Introduction

Let G be a complex semisimple group and let $H \subseteq G$ be the group of fixed points of an involutive automorphism of G. Then $X = G/H$ is called a symmetric variety. In [CP], De Concini and Procesi have constructed an equivariant compactification $\overline{X}$ which has a number of remarkable properties, some of them being:

i) The boundary is the union of divisors $D_1, \ldots, D_r$.
ii) There are exactly 2^r orbits. Their closures are the intersections $D_{i_1} \cap \ldots \cap D_{i_s}$ (even schematically). In particular, there is only one closed orbit.
iii) In case G is of adjoint type, all orbit closures are smooth.

It is called the *wonderful embedding of* X or a *complete symmetric variety* and is the foundation for most deeper results about X.

Independently, Luna and Vust developed in [LV] a general theory of equivariant compactifications of homogeneous varieties under a connected reductive group G. In particular, they realized the reason which makes symmetric varieties behave so nicely: A Borel subgroup B has an open dense orbit in G/H. Varieties with this property are called *spherical*. Luna and Vust were able to describe all equivariant compactifications of them in terms of combinatorial data, very similar to torus embeddings which are actually a special case. They obtained in particular that every spherical embedding has only finitely many orbits. Nevertheless, the reason for the existence of a compactification with properties i)-iii) remained mysterious.

Then Brion and Pauer established a relation with the automorphism group. They proved in [BP]: A spherical variety $X = G/H$ possesses an equivariant compactification with exactly one closed orbit if and only if $\mathrm{Aut}^G X = N_G(H)/H$ is finite. In this case there is a unique one which dominates all others: the wonderful compactification $\overline{X}$. They also showed that the orbits of $\overline{X}$ correspond to the faces of a strictly convex polyhedral cone $\mathcal{Z}$. Then properties i) and ii) above are equivalent to $\mathcal{Z}$ being simplicial.

This fact is much deeper and was proved by Brion in [Br1]. In fact he showed much more. Let Γ be the set of characters of B which are the characters of a rational B-eigenfunction on X. This is a finitely generated free abelian group. Then the cone $\mathcal{Z}$ is a subset of the real vector space $\mathrm{Hom}(\Gamma, \mathbb{R})$. Brion showed that there is a finite reflection group W_X acting on Γ such that $\mathcal{Z}$ is one of its Weyl chambers. In case of a symmetric variety, W_X is its little Weyl group.

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There remains property iii). In the same paper, Brion stated the following

1.1. **Conjecture.** *If the automorphism group of X is trivial, then all orbit closures in $\overline{X}$ are smooth.*

(Actually, this is only part of his conjecture.) The main purpose of this paper is to prove this conjecture. Unlike i) and ii), where only the combinatorial structure of $\mathcal{Z}$ matters, this is now a subtle problem of integrality. Each extremal ray of the dual cone $\mathcal{Z}^\vee \subseteq \Gamma \otimes \mathbb{R}$ is spanned by a unique primitive element of the lattice Γ. Let Σ be the set of these elements. Then one can show that $\overline{X}$ is smooth if and only if Σ generates the lattice Γ as a group (in which case, it is even a basis). Hence, Brion's conjecture follows from our main result which establishes a connection between Γ, W_X and the automorphism group of X:

1.2. **Theorem.** *There is a canonical inclusion* $\operatorname{Hom}(\Gamma/\langle\Sigma\rangle_{\mathbb{Z}}, k^*) \hookrightarrow \operatorname{Aut}^G X$*.*

There is a slightly different way to see this result which is closer to the theory of symmetric varieties. It is well known that the set $\Delta = W_X\Sigma \subseteq \Gamma$ is a root system with Σ as a set of simple roots. Therefore, the theorem says that if $\operatorname{Aut}^G X$ is trivial, then Γ is the root lattice of a root system and W_X is its Weyl group. Hence we obtain almost a generalization of the restricted root system of a symmetric variety. I say "almost" because our root system is always reduced and doesn't have multiplicities.

For simplicity, we restricted ourself so far to spherical varieties. But all concepts generalize to arbitrary G-varieties. The trick is to put everything in relation to the field $k(X)^B$ of B-invariant rational functions, which is just k in the spherical case. For example, instead of taking all of $\operatorname{Aut}^G X$ one considers only the subgroup $\mathfrak{A}(X)$ of those automorphisms which induce the identity on $k(X)^B$. Therefore, we are able to attach a root system and a Weyl group to any variety with G-action.

Let me mention that for quasi-affine varieties X there is a very simple construction of its root system. For this, consider the isotypic decomposition of its algebra of global functions, $k[X] = \bigoplus_\chi R_\chi$, where χ runs through all dominant weights. This decomposition is usually not a gradation. To measure the deviation we define

$$\mathcal{M}' := \{\alpha \in \mathcal{X}(B) \mid \exists \chi, \eta \in \mathcal{X}(B) : \langle R_\chi R_\eta\rangle_k \cap R_{\chi+\eta-\alpha} \neq 0\}.$$

Let $\mathcal{M}$ be the saturated monoid generated by $\mathcal{M}'$, i.e., the intersection of the cone spanned by $\mathcal{M}'$ and the group generated by $\mathcal{M}'$.

1.3. **Theorem.** *The commutative monoid $\mathcal{M}$ is free and the set of free generators is the basis Σ of Δ_X.*

The proof of Theorem 1.2 is very indirect. Therefore, I give a brief synopsis. For every homomorphism $a : \Gamma \to k^*$ which vanishes on Σ we want to construct an automorphism φ of X. Consider the cotangent bundle $T^*_X \to X$. This bundle contains a certain open subset T^0_X which possesses a Galois covering $\widehat{T}_X$ with group W_X. Thus we get

$$\widehat{T}_X \twoheadrightarrow T^0_X \hookrightarrow T^*_X \twoheadrightarrow X.$$

We construct φ in several steps by starting at $\widehat{T}_X$. There the whole torus $A = \operatorname{Hom}(\Gamma, k^*)$ acts in a natural way. Hence, A^{W_X} acts on T^0_X. By embedding A^{W_X} into a connected smooth group scheme we can show that the action of a extends to

T_X^* in codimension one. The crucial condition here is that a is trivial on Σ. This step is the most technical part of the paper. Then it is fairly easy to show that the automorphism actually extends to all of T_X^* and can be pushed down to X.

Notation. All varieties are defined over an algebraically closed field k of characteristic zero. The group G is always reductive and connected. We choose a Borel subgroup $B \subseteq G$ with unipotent radical U and maximal torus T. If G acts on a variety X, then we denote the multiplicative group of B-semi-invariant rational functions by $k(X)^{(B)}$. For $f \in k(X)^{(B)}$ let $\chi_f \in \mathcal{X}(B)$ be the corresponding character.

2. Group schemes

The purpose of this section is to construct certain group schemes. For this, recall some facts about the Weil restriction. Let $\varphi : S' \to S$ be a morphism of varieties. Then any coherent sheaf on S' can be pushed down to a sheaf on S. A similar process exists sometimes for schemes over S'.

Definition. Let Z'/S' be an S'-scheme. Then the *Weil restriction* of Z' along φ is an S-scheme Z together with an S'-morphism $\Phi : Z \times_S S' \to Z'$ such that

$$\mathrm{Mor}_S(X, Z) \longrightarrow \mathrm{Mor}_{S'}(X \times_S S', Z') : \psi \mapsto \Phi \circ (\psi \times \mathrm{id}_{S'})$$

is bijective for all S-schemes X.

The universal property characterizes Z uniquely. If it exists it is denoted by $\prod_{S'/S} Z' := Z$. We need only a quite easy existence theorem (see [DG], I,§1,6.6; I,§4,4.8):

2.1. **Lemma.** *Assume S'/S is finite, flat, and let Z'/S' be affine, smooth, and of finite type. Then $\prod_{S'/S} Z'$ exists and has the same properties.*

Let Z'/S' be an S'-group scheme. Then it is easy to see that $\prod_{S'/S} Z'/S$ is an S-group scheme.

Now we apply this to the following situation: Let S' be an affine space and W a finite group acting linearly on S'. We assume that W is generated by reflections. Then $S := S'/W$ is also an affine space and S'/S is finite and flat ([Bou], Chap. 5, §5, Thm. 4). Assume W acts also on a finitely generated free Abelian group Γ. Let $A := \operatorname{Spec} k[\Gamma]$ be the torus with character group $\mathcal{X}(A) = \Gamma$. Then we know that

$$Z := \prod_{S'/S} (A \times S')$$

exists and is a smooth, commutative S-group scheme. In particular, Z is smooth as a k-variety.

We let W act on $A \times S'$ diagonally. Let X/S be any S-scheme. Then W acts on $\mathrm{Mor}_{S'}(X \times_S S', A \times S')$ by ${}^w\psi(x, s') := w\psi(x, w^{-1}s')$. Hence, W acts on $\mathrm{Mor}_S(X, Z)$ and therefore on the S-group scheme Z. Now define

$$\mathcal{A} = \mathcal{A}(W, S', \Gamma) := Z^W$$

as the set of fixed points.

2.2. **Lemma.** *$\mathcal{A}/S$ is a smooth commutative affine group scheme.*

Proof. Only the first property needs a proof. By the lemma, Z is smooth over S, hence smooth over k. Let $z \in Z^W$ be in the fiber of $s \in S$. Then the tangent space $T_z\mathcal{A}$ equals $(T_zZ)^W$. Because $T_zZ \to T_sS$ is surjective and W acts trivially on T_sS, also $T_z\mathcal{A} \to T_sS$ is surjective, i.e., $\mathcal{A} \to S$ is smooth in z. □

Also $\mathcal{A}$ has a universal property:

2.3. **Lemma.** *For every scheme X/S there is a bijection*

$$\mathrm{Mor}_S(X, \mathcal{A}) \xrightarrow{\sim} \mathrm{Mor}^W(X \times_S S', A).$$

Proof. We have

$$\begin{aligned}\mathrm{Mor}_S(X, \mathcal{A}) = \mathrm{Mor}_S(X, Z^W) = \mathrm{Mor}_S(X, Z)^W &\xrightarrow{\sim} \mathrm{Mor}^W_{S'}(X \times_S S', A \times S') \\ = \mathrm{Mor}^W(X \times_S S', A). \qquad \Box\end{aligned}$$

Next I want to investigate the fibers $\mathcal{A}_s := \pi^{-1}(s) \subseteq \mathcal{A}$. It is an affine commutative group, hence decomposes uniquely into its unipotent and semisimple part: $\mathcal{A}_s = \mathcal{A}_s^u \times \mathcal{A}_s^s$.

2.4. **Lemma.** *Let $s \in S$. Then for every $s' \in S'$ in its preimage there is a homomorphism $\iota_{s'} : \mathcal{A}_s \to A$ with kernel $\mathcal{A}_s^u$ and image $A^{W_{s'}} \cong \mathcal{A}_s^s$.*

Proof. Let $\widehat{s} \subseteq S'$ be the schematic preimage of s and $\widehat{s}\,'$ its component containing s'. Then the inclusion $X = \mathcal{A}_s \hookrightarrow \mathcal{A}$ induces a morphism

$$\iota_{s'} : \mathcal{A}_s = \mathcal{A}_s \times \{s'\} \hookrightarrow \mathcal{A}_s \times \widehat{s} \to A$$

which is easily verified to be a homomorphism. On the level of k-valued points we get

$$\begin{aligned}\mathcal{A}_s(k) &= \mathrm{Mor}_S(\{s\}, \mathcal{A}) = \mathrm{Mor}^W(\widehat{s}, A) \\ &= \mathrm{Mor}^{W_{s'}}(\widehat{s}\,', A) \to \mathrm{Mor}^{W_{s'}}(\{s'\}, A) = A(k)^{W_{s'}}.\end{aligned}$$

The map is surjective because the projection $\widehat{s}\,' \to \{s'\}$ induces a section. This shows that the image is as claimed.

The ring $k[\widehat{s}\,']$ is local, Artinian. Let U be its group of 1-units, which is, via logarithm, isomorphic to the maximal ideal considered as an additive group. Suppose $A \cong \mathbf{G}_m^r$. Then the kernel of $\iota_{s'}$ is contained in the group of those morphisms $\widehat{s}\,' \to A$, such that the closed point is mapped to 1. Hence it is a subgroup of U^r, hence torsionfree. This implies that the kernel is unipotent. Because A is a torus, we have $\ker \iota_{s'} \supseteq \mathcal{A}_s^u$, hence equality. □

2.5. **Lemma.** *The set of global sections of $\mathcal{A}/S$ equals A^W. Furthermore, $\sigma(s) \in \mathcal{A}_s$ is semisimple for every section σ and all $s \in S$.*

Proof. We have $\mathrm{Mor}_S(S, \mathcal{A}) = \mathrm{Mor}^W(S', A) = A^W$. The last equality holds because all units in $k[S']$ and therefore all morphisms $S' \to A$ are constant. The second assertion follows from $A^W \subseteq A^{W_{s'}} = \mathcal{A}_s^s$. □

Next let $S'_1 \subseteq S'$ be the open subset where W acts freely. Let $S_1 = S'_1/W \subseteq S$. Because $W_{s'} = 1$ for $s' \in S'$, we have $\mathcal{A}_s \cong A$ for any $s \in S_1$. The following lemma gives precise information about how this family of tori is twisted:

2.6. **Lemma.** *The restricted group scheme $\mathcal{A} \times_S S_1$ is isomorphic to $(A \times S_1')/W$.*

Proof. Let $s' \in S_1'$ and $s \in S_1$ its image. Then there is an isomorphism $\iota_{s'} : \mathcal{A}_s \xrightarrow{\sim} A$. These glue together to an isomorphism $\mathcal{A} \times_S S_1' \xrightarrow{\sim} A \times S_1'$. Taking the quotient by W gives the result. □

Next we determine the Lie algebra of $\mathcal{A}$. It is a locally free sheaf on S. Because S is affine it suffices to consider its set $\operatorname{Lie}\mathcal{A}$ of global sections. Let $\mathfrak{a} = \operatorname{Hom}(\Gamma, k)$ be the Lie algebra of A.

2.7. **Lemma.** *There is a canonical isomorphism* $\operatorname{Lie}\mathcal{A} = \operatorname{Mor}^W(S', \operatorname{Lie} A) = (k[S'] \otimes_k \mathfrak{a})^W$.

Proof. Let $D := \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$. Then $\operatorname{Lie}\mathcal{A}$ equals

$$\operatorname{Mor}_S(D \times S, \mathcal{A})_1 := \ker\big[\operatorname{Mor}_S(D \times S, \mathcal{A}) \to \operatorname{Mor}_S(S, \mathcal{A})\big].$$

Hence, $\operatorname{Lie}\mathcal{A} = \operatorname{Mor}_S(D \times S, \mathcal{A})_1 = \operatorname{Mor}^W(D \times S', A)_1 = \operatorname{Mor}^W(S', \operatorname{Mor}(D, A)_1) = \operatorname{Mor}^W(S', \mathfrak{a}) = (k[S'] \otimes_k \mathfrak{a})^W$. □

Now we specialize further and assume that Γ is a lattice in the vector space S', i.e., there is a W-isomorphism

$$S' = \Gamma \underset{\mathbb{Z}}{\otimes} k.$$

Then we can identify S' with $\mathfrak{a}^*$. Hence we have for the module of Kähler differentials

$$\Omega(S') = k[S'] \underset{k}{\otimes} \mathfrak{a} = \operatorname{Lie}_{S'}(A \times S').$$

One of the main points is now the next

2.8. **Theorem.** *Assume $S' = \Gamma \otimes_{\mathbb{Z}} k$. Then the equality above induces an isomorphism $\Omega(S) = \operatorname{Lie}\mathcal{A}$.*

Proof. There is a canonical homomorphism $\Omega(S) \otimes_{k[S]} k[S'] \to \Omega(S')$. The induced homomorphism between W-invariants $\delta : \Omega(S) \to \Omega(S')^W$ is an isomorphism by [So]. Hence, the assertion follows from Lemma 2.7. □

Note that W can no longer be an arbitrary reflection group. It is induced by a root system. There is a canonical choice for such a root system. Observe that for any reflection $w \in W$ the group $R_w := \{\gamma \in \Gamma \mid w\gamma = -\gamma\}$ is free of rank one.

Definition. The *minimal root system* $\Delta = \Delta(W, \Gamma) \subseteq \Gamma$ is the set of generators of all R_w where $w \in W$ runs through all reflections.

It is easily verified that Δ is indeed a root system whose Weyl group is W.

Because Γ is the character group of A, one can identify A with $\operatorname{Hom}(\Gamma, k^*)$ by evaluation. Hence to every W-invariant homomorphism $a : \Gamma \to k^*$ corresponds an element of A^W and therefore, by Lemma 2.5, a section σ_a of $\mathcal{A}/S$.

In the next theorem let $S_2' := \{s' \in S' \mid |W_{s'}| \le 2\}$ and $S_2 = S_2'/W \subseteq S$. These are open subsets whose complements have codimension at least two. Furthermore, by [SGA], $\mathrm{IV_B}$, 4.4, there exists a minimal open subgroup scheme $\mathcal{A}^0 \subseteq \mathcal{A}$. It is characterized by the property that all fibers $\mathcal{A}_s^0$ are connected.

2.9. **Theorem.** *For a homomorphism $a : \Gamma \to k^*$ the following are equivalent:*

1. $a(\Delta) = 1$.
2. *a is W-invariant and for the corresponding section $\sigma_a : S \to \mathcal{A}$ it holds that $\sigma_a(S_2) \subseteq \mathcal{A}^0$.*

Proof. Let $w \in W$ be a reflection and α_w a generator of R_w. Then $w\gamma - \gamma \in R_w$ is an integral multiple of α_w. Hence, $a(\Delta) = 1$ implies that a is W-invariant.

Now assume a to be W-invariant. Let $s' \in S'_2$ and let $s \in S_2$ be its image. We may assume $s \notin S_1$. Hence $W_{s'} = \{1, w\}$ where w is a reflection. By Lemma 2.4, $\mathcal{A}_s^s$ is the fixed point set A^w. Hence $\mathcal{X}(\mathcal{A}_s) = \Gamma/\Gamma_w$ where $\Gamma_w := \mathrm{Im}(w-1)$. Observe $R_w = \mathbb{Q}\Gamma_w \cap \Gamma$ (inside $\mathfrak{a}^*$). Thus the torsion subgroup of $\mathcal{X}(\mathcal{A}_s)$ is R_w/Γ_w. This implies that $\sigma_a(s)$ is in the connected component of $\mathcal{A}_s$ if and only if $a(R_w) = 1$. This shows the assertion. ☐

For the next section we need a more technical property of $\mathcal{A}$ concerning local sections.

2.10. **Lemma.** *For every $s \in S$ and $\alpha \in \mathcal{A}_s^0$ there is a rational section $a : S \dashrightarrow \mathcal{A}^0$ defined in s and with $a(s) = \alpha$.*

Proof. Let $V \subseteq \mathcal{A}$ be the maximal open subset such that every $\alpha \in V$ is contained in the image of a rational section. First, we show $\mathcal{A}_0^0 \subseteq V$.

Let $\widehat{o} \subseteq S'$ be the schematic fiber of $0 \in S$. Then the homomorphism $\psi_0 : \mathcal{A}_0 \hookrightarrow \mathcal{A}$ is induced by a morphism $\psi'_0 : \mathcal{A}_0 \times \widehat{o} \to A$ which is W-equivariant in the second factor and multiplicative in the first. In particular $\psi'_0(\alpha^m, s) = \psi'_0(\alpha, s)^m$ for every $m \in \mathbb{Z}$.

Because A is an open subset of an affine space, ψ'_0 can be extended to a rational morphism $\overline{\psi}' : \mathcal{A}_0 \times S' \dashrightarrow A$ which is defined in $\mathcal{A}_0 \times \widehat{o}$. Now define

$$\psi' : \mathcal{A}_0 \times S' \dashrightarrow A : (\alpha, s') \mapsto \prod_{w \in W} w^{-1}\overline{\psi}'(\alpha, ws').$$

This rational morphism is W-equivariant and therefore induces a rational S-morphism $\psi : \mathcal{A}_0 \times S \dashrightarrow \mathcal{A}$, which is defined in $\mathcal{A}_0 \times 0$. Let m be the order of W. Then $\psi(\alpha, 0) = \alpha^m$. This shows that ψ is étale in $\mathcal{A}_0 \times 0$. Therefore, there is $U \subseteq \mathcal{A}_0 \times S$ open, containing $\mathcal{A}_0 \times 0$ such that ψ is defined and étale in U. Hence $\mathcal{A}_0^0 \subseteq \psi(U) \subseteq V$.

Now we show $\mathcal{A}^0 \subseteq V$. The k^*-action on the vector space S' induces one on $\mathcal{A}$ and S. Clearly, V is k^*-stable. Its image in S is open and contains 0, hence $V \to S$ is surjective. Furthermore, V is closed under multiplication and taking inverses, hence an open subgroup scheme of $\mathcal{A}$. This shows the claim. ☐

3. Integration of Lie algebra actions

In this section we present some theorems on the integration of actions of Lie algebras. The results are just an extension of the first section of [LV] from groups to group schemes. Because the proofs are very similar, we will be very sketchy.

The setup is as follows: Let S be an affine base variety, $\mathcal{A} \to S$ a smooth group scheme *with connected fibers*, and $L_\mathcal{A} = \mathrm{Lie}\,\mathcal{A}$ the Lie algebra of $\mathcal{A}/S$ considered as a $k[S]$-module. Furthermore, we assume that Lemma 2.10 is valid for $\mathcal{A}$. Later on, the theorems below are only applied to group schemes constructed in the preceding section.

Let $X \to S$ be an S-variety equipped with a Lie algebra homomorphism of $L_{\mathcal{A}}$ into the Lie algebra of global vector fields $\mathcal{T}(X/S)$. Any group action $\mu : \mathcal{A} \times_S X \to X$ induces a homomorphism like that. We are interested in the converse. To get things started we furthermore assume that this action exists already generically, i.e., there is an open subset $U \subseteq \mathcal{A} \times_S X$ with $(1_{\mathcal{A}} \times X) \cap U \neq \varnothing$ and a morphism $U \to X$, satisfying some obvious axioms, which induces $L_{\mathcal{A}} \to \mathcal{T}(X/S)$.

First we define some universal S-scheme on which $\mathcal{A}$ acts: Let $\mathfrak{X}$ be the set of all local rings $\mathcal{P} \subseteq k(X)$ satisfying the following conditions:

a) The field of fractions of $\mathcal{P}$ is $k(X)$.
b) $\mathcal{P}$ is the localization of a finitely generated subalgebra at a prime ideal.
c) $k[S] \subseteq \mathcal{P}$.

Then $\mathfrak{X}$ is a scheme (cf. [LV], 1.1). The open affine subsets are of the form $\operatorname{Spec} R$, where R is a finitely generated subalgebra of $k(X)$ which generates it as a field and which contains $k[S]$. The scheme $\mathfrak{X}$ is integral and locally of finite type but in general not separated. Because of c), it is an S-scheme.

By assumption there is an action of $L_{\mathcal{A}}$ on $k(X)$. Hence we can define the subset $\mathfrak{X}_0 \subseteq \mathfrak{X}$ of those local algebras which are $L_{\mathcal{A}}$-stable. It is easily verified that $\mathfrak{X}_0$ is an open subscheme. By assumption, $X \subseteq \mathfrak{X}_0$, hence $\mathfrak{X}_0$ is non-empty. There is an action of $L_{\mathcal{A}}$ on $\mathfrak{X}_0$. The main point is now:

3.1. **Theorem.** *There is a unique morphism $\mu : \mathcal{A} \times_S \mathfrak{X}_0 \to \mathfrak{X}_0$ which is a group scheme action and which induces the action of $L_{\mathcal{A}}$.*

Proof. There is already a rational map $\mu : \mathcal{A} \times_S \mathfrak{X}_0 \dashrightarrow \mathfrak{X}_0$, i.e., a field homomorphism $\mu^* : k(\mathfrak{X}_0) \to k(\mathcal{A} \times_S X)$. We first show that μ is defined in a neighborhood of the 1-section. For that, we have to show that for any $x \in \mathfrak{X}_0$, the local ring $R_1 := \mathcal{O}_{\mathfrak{X}_0,x}$ is mapped by μ^* into $R_2 := \mathcal{O}_{\mathcal{A} \times_S \mathfrak{X}_0,(1,x)}$. Let $\widehat{R}_2$ be the completion of R_2 for the topology defined by the ideal of the 1-section. By assumption, R_1 is $L_{\mathcal{A}}$-stable. This implies as in [LV], 1.3, 1.4, that μ^* restricts to a homomorphism $R_1 \to \widehat{R}_2$. Now, the equality $R_2 = \widehat{R}_2 \cap k(\mathcal{A} \times_S \mathfrak{X}_0)$ implies the claim, i.e., μ is defined on an open neighborhood U of the 1-section.

Next we show that μ is defined everywhere. For this assume that $a : S \dashrightarrow \mathcal{A}$ is a rational section, such that the image of $a \times \mathrm{id}_{\mathfrak{X}_0}$ in $\mathcal{A} \times_S \mathfrak{X}_0$ meets U. Then a defines an automorphism of $k(X)$ and hence of $\mathfrak{X}_0$. This implies that μ is also defined in a neighborhood of a. By Lemma 2.10, μ is defined everywhere. □

3.2. **Theorem.** *There is an $\mathcal{A}$-variety $\overline{X}$, which contains X/S as an open subset.*

Proof. Let Φ be the automorphism of $\mathcal{A} \times_S \mathfrak{X}_0$, which sends (a, x) to $(a, a^{-1}x)$. Then $\mu \circ \Phi$ is the projection to $\mathfrak{X}_0$. Because $\mathcal{A} \to S$ is smooth and surjective, the same is true for μ. In particular, it is open. Therefore, $\overline{X} := \mu(\mathcal{A} \times_S X)$ is an open subset of $\mathfrak{X}_0$. As the image of a variety it is quasicompact, hence of finite type.

It remains to show that $\overline{X}$ is separated, i.e., that the diagonal $\Delta_{\overline{X}}$ in $\overline{X} \times_S \overline{X}$ is closed. First I claim that

$$\delta : \mathcal{A} \times_S X \times_S X \to \overline{X} \times_S \overline{X} : (a, x_1, x_2) \mapsto (ax_1, ax_2)$$

is surjective. For this let $s \in S$ and $x_1, x_2 \in \overline{X}_s$. Then $U_i := \{a \in \mathcal{A}_s \mid ax_i \in X\}$ is non-empty and open in $\mathcal{A}_s$. Because $\mathcal{A}_s$ is irreducible, there is $a \in U_1 \cap U_2$. Then $\delta(a^{-1}, ax_1, ax_2) = (x_1, x_2)$ proves the claim. As above, δ is also flat. Hence $\Delta_{\overline{X}}$ is closed because its preimage $\delta^{-1}(\Delta_{\overline{X}}) = \mathcal{A} \times_S \Delta_X$ is closed. □

3.3. **Corollary.** *The rational action $\mathcal{A} \times_S X \dashrightarrow X$ is defined on an open subset containing $1_{\mathcal{A}} \times X$.*

Proof. The closed subset $\mu^{-1}(\overline{X} \setminus X) \cap (\mathcal{A} \times_S X)$ does not meet the 1-section. □

3.4. **Corollary.** *Let X be proper over S. Then $\mathcal{A}$ acts on X.*

The next statement shows that the existence of an $\mathcal{A}$-action is a property which can be checked pointwise.

3.5. **Corollary.** *Assume that for all $s \in S$ and $x \in X_s$ there is an orbit morphism $\mathcal{A}_s \to X_s : a \mapsto ax$ which is compatible with the $L_{\mathcal{A}_s}$-action. Then $\mathcal{A}$ acts on X.*

Proof. The condition implies that X is $\mathcal{A}$-stable in $\overline{X}$. □

We will also need the following

3.6. **Theorem.** *Let Y/S be an $\mathcal{A}$-scheme and $\varphi : X \to Y$ an $L_{\mathcal{A}}$-equivariant S-morphism. Assume that φ is affine and that $\varphi_*\mathcal{O}_X$ is generated as an $\mathcal{O}_Y$-algebra by an $L_{\mathcal{A}}$-stable coherent subsheaf $\mathcal{E}$. Then $\mathcal{A}$ acts on X.*

Proof. Let $X \hookrightarrow \overline{X}$ be as in Theorem 3.2. Because $\mathcal{A}$ acts on Y, the morphism φ extends to an $\mathcal{A}$-morphism $\overline{X} \to Y$. Let D be an irreducible component of $\overline{X} \setminus X$, which is not $\mathcal{A}$-stable. Because φ is affine, D is of codimension one. Let D' be the closure of $\operatorname{Im}(D \to S)$. If $D' \neq S$, then D would be an irreducible component of the preimage of D' and therefore $\mathcal{A}$-invariant, because $\mathcal{A}/S$ has connected fibers.

Hence, $D \to S$ is dominant. Again because D is not $\mathcal{A}$-stable, it cannot be contained in the singular locus of $\overline{X}$. Hence it defines a valuation v_D of $k(X)$ over $k(S)$. Choose $h \in k(X)$ with $v_D(h) = 1$. Because D is not $\mathcal{A}$-stable, there is $\xi \in L_{\mathcal{A}}$ with $v_D(\xi h) = 0$.

Because $\varphi_*\mathcal{O}_X$ is generated by $\mathcal{E}$ there is $Y_0 \subseteq Y$ affine open and $f \in \mathcal{E}(Y_0)$ such that $-n = v_D(f) < 0$. Writing $f = ah^{-n}$ implies $v_D(\xi f) = -n - 1$. This shows that v_D is on $\mathcal{E}(Y_0)$ not bounded from below, which contradicts the assumption that $\mathcal{E}$ is coherent. Hence D cannot exist, i.e., $\mathcal{A}$ acts on $X = \overline{X}$. □

4. Integration of the invariant collective motion

Let G be a connected reductive group acting on a smooth variety X. Consider the cotangent bundle $\pi : T_X^* \to X$. The G-action induces the moment map

$$\Phi : T_X^* \longrightarrow \mathfrak{g}^* := (\operatorname{Lie} G)^* : \alpha \mapsto l_\alpha \quad \text{where} \quad l_\alpha(\xi) = \alpha(\xi_{\pi(\alpha)}).$$

Recall that T_X^* carries a symplectic structure ω. Therefore, each function f on T_X^* induces a Hamiltonian vector field H_f. This defines the Poisson product $\{f, g\} = \omega(H_f, H_g)$, which gives $k[T_X^*]$ the structure of a Lie algebra. Also $k[\mathfrak{g}^*]$ has the structure of a Poisson algebra, and Φ being a moment map means that $\Phi^* : k[\mathfrak{g}^*] \to k[T_X^*]$ is a Poisson homomorphism. We denote its image by R_0.

The Poisson center of R_0 is exactly the algebra R_0^G of invariants. To describe it, let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Because of the Killing form one could identify $\mathfrak{g}^*$ with $\mathfrak{g}$ as a G-variety. Hence, by Chevalley's restriction theorem, we have an isomorphism $k[\mathfrak{g}^*]^G \xrightarrow{\sim} k[\mathfrak{t}^*]^W = k[\mathfrak{t}^*/W]$, where W is the Weyl group of G. Thus we get a morphism

$$\Psi : T_X^* \longrightarrow \mathfrak{t}^*/W$$

such that R_0^G is the image of Ψ^*. The elements of R_0 are called collective Hamiltonians. Accordingly, the image R_0^G of Ψ^* consists of the invariant collective Hamiltonians.

The elements in R_0^G Poisson-commute pairwise. Our problem is roughly whether there is a commutative algebraic group action on T_X^* which integrates the Hamiltonian vector fields for R_0^G. More precisely, let $s \in \mathfrak{t}^*/W$, $f \in k[\mathfrak{t}^*/W]$ and $f_0 = f \circ \Psi \in R_0^G$. Then H_{f_0} is parallel to the fiber $T_s^* := \Psi^{-1}(s)$ and its restriction depends only on $(df)_s \in \Omega_s(\mathfrak{t}^*/W)$. Hence we get a Lie algebra homomorphism $\Omega_s(\mathfrak{t}^*/W) \to \mathcal{T}(T_s^*)$ and the problem is, whether there is a group $\mathcal{A}_s$ integrating this Lie algebra action. The group will depend on s, hence we will get a group scheme over $\mathfrak{t}^*/W$.

Actually, we want to integrate an algebra which is a little bit larger than R_0^G, namely R^G, where R is the integral closure of R_0 inside $k[T_X^*]$. With $L_X := \operatorname{Spec} R^G$ we get a morphism $T_X^* \to L_X$ which is a kind of Stein factorization of Ψ. One of our main results is now:

4.1. **Theorem.** *Let X be a smooth G-variety. Then, there is finite reflection group W_X acting on a vector space $\mathfrak{a}_X^*$ and a W_X-stable lattice $\Gamma_X \subseteq \mathfrak{a}_X^*$ such that for the group scheme $\mathcal{A}_X^0 = \mathcal{A}(W_X, \mathfrak{a}_X^*, \Gamma_X)^0$ the following hold:*

a) *There is an identification $\mathfrak{a}_X^*/W_X = L_X$.*
b) *There is an action of $\mathcal{A}_X^0$ on T_X^* over L_X.*
c) *There is a commutative diagram*

$$\begin{array}{ccc} \Omega(L_X) & \xrightarrow{\Psi^*} & \Omega(T_X^*) \\ {\scriptstyle 1}\downarrow\sim & & {\scriptstyle 2}\downarrow\sim \\ \operatorname{Lie}\mathcal{A}_X^0 & \to & \mathcal{T}(T_X^*) \end{array}$$

where arrow 1 denotes the homomorphism from Theorem 2.8, arrow 2 is the identification via the symplectic structure of T_X^* and the bottom arrow is induced by the $\mathcal{A}_X^0$-action.

Proof. Let me recall some constructions from [Kn1] and [Kn5]: Let $B \subseteq G$ be a Borel subgroup. Then $k(X)^{(B)}$ denotes the multiplicative group of B-semi-invariant rational functions on X. For $f \in k(X)^{(B)}$ let $\chi_f \in \mathcal{X}(B)$ be its character. Then $f \mapsto \chi_f$ defines a homomorphism into $\mathcal{X}(B)$ whose image is Γ_X. Because $\Gamma_X \subseteq \mathcal{X}(B)$, it is a free Abelian group of finite rank. Let $A_X = \operatorname{Spec} k[\Gamma_X]$ be the torus with character group Γ_X, and $\mathfrak{a}_X^* = (\operatorname{Lie} A_X)^* = \Gamma_X \otimes_{\mathbb{Z}} k$. Let $T \subseteq B$ be a maximal torus. Then we have a projection $T \twoheadrightarrow A_X$ which induces for the Lie algebras $\mathfrak{t} \twoheadrightarrow \mathfrak{a}_X$, hence $\mathfrak{a}_X^* \hookrightarrow \mathfrak{t}^*$. The main result of [Kn1] was that there is a finite reflection group W_X acting on $\mathfrak{a}_X^*$ and a canonical isomorphism $\mathfrak{a}_X^*/W_X \xrightarrow{\sim} L_X$ ([Kn1], p. 12 and Satz 6.6). This shows a).

We take c) as a definition of the action of $L_{\mathcal{A}_X^0}$ on T_X^*. Now, we show that the $\mathcal{A}_X^0$-action exists. Because this is most easily done for non-degenerate varieties (see [Kn5], §3 for a definition) we need a reduction lemma.

4.2. **Lemma.** *Let $H \cong \mathbf{G}_m$ be contained in the center of G. Assume that the orbit space $Y = X/H$ exists and that Theorem 4.1 is true for X. Then it is true for Y.*

Proof. We have $\Gamma_Y = \{\chi \in \Gamma_X \mid \chi|_H = 1\}$. Hence $\mathfrak{a}_Y^*$ is a hyperplane in $\mathfrak{a}_X^*$. Furthermore, $W_Y = W_X$ by [Kn5], 5.1 and 7.5. This shows that L_Y is a hyperplane of L_X. Let $\mathcal{A}'_Y$ be the restriction of $\mathcal{A}^0_X$ to L_Y. Then $H \times L_Y \subseteq \mathcal{A}'_Y$ and $\mathcal{A}^0_Y = \mathcal{A}'_Y/H$. Hence, if $\mathcal{A}^0_X$ acts on T^*_X then $\mathcal{A}'_Y$ acts on $T'_Y := T^*_Y \times_Y X$ and $\mathcal{A}^0_Y$ acts on $T^*_Y = T'_Y/H$. □

To apply this lemma, we use a well-known construction: Let $X_0 \subseteq X$ be a G-stable, open, quasi-projective subset. Then there exists an ample G-linearized line bundle $\mathcal{L}_0$ on X_0. Because X is smooth, it can be extended to a line bundle $\mathcal{L}$ on X. Let $\widetilde{X}$ be the geometric realization of $\mathcal{L}$ minus zero-section. Then the preimage of X_0 in $\widetilde{X}$ is quasi-affine, hence non-degenerate ([Kn5], Lemma 3.1).

Thus, by replacing G, X by $G \times \mathbf{G}_m$, $\widetilde{X}$, we may assume that X is non-degenerate. Let $\mathfrak{a}^r \subseteq \mathfrak{a}^*_X$ be the open subset of points where $\mathfrak{a}^*_X \to \mathfrak{t}^*/W$ is unramified. The restriction of $\mathcal{A}^0_X$ to $\mathfrak{a}^r/W_X$ is isomorphic to $(A_X \times \mathfrak{a}^r)/W_X$. Hence an action of $\mathcal{A}^0_X$ on $\Psi^{-1}(\mathfrak{a}^r)$ is the same as an action of $W_X \ltimes A_X$ on $\widehat{T}_X := T^*_X \times_{L_X} \mathfrak{a}^r$. The latter has been constructed in [Kn5], Thm. 4.2. That this action is compatible with the action of $L_{\mathcal{A}^0_X}$ follows from [Kn5], Thm. 4.1.

Next, I reduce to the case where X is homogeneous. Let $Y \subseteq X$ be an orbit. Then consider the following commutative diagram:

$$\begin{array}{ccccc} T' & := & T^*_X|_Y & \hookrightarrow & T^*_X \\ & & \varphi\downarrow & & \downarrow \\ T & := & T^*_Y & \to & \mathfrak{g}^* \end{array}$$

Each $\xi \in \mathfrak{g}$ induces a linear function on $\mathfrak{g}^*$ and via Φ a function l_ξ on T^*_X. It follows from the properties of the moment map that the Hamiltonian vector field of l_ξ equals the vector field ξ_* induced by the G-action. Because T' is G-stable, all H_f are parallel to T' where $f \in R_0$. The same is true for R, since it is algebraic over R_0. By Corollary 3.3 there is a rational action of $\mathcal{A}^0_X$ on T'. Hence Corollary 3.5 implies that it is sufficient to show that this action is actually regular on T' (where Y runs through all orbits).

The projection φ is affine. Let $\mathcal{E} \subseteq \varphi_*\mathcal{O}_{T'}$ be the coherent subsheaf generated by the functions of degree one (with respect to the obvious $\mathbf{G}_m$-action on T'). Then $\varphi_*\mathcal{O}_{T'}$ is generated by $\mathcal{E}$. Let $Y_0 \subseteq Y$ be open and f a linear function on $T^*_X|_{Y_0} \subseteq T'$. Because the Poisson product decreases the degree by one, $\deg\{l_\xi, f\} = 1$. Then the formula

$$\{l_{\xi_1} \dots l_{\xi_s}, f\} = \sum_i l_{\xi_1} \dots l_{\xi_{i-1}} \{l_{\xi_i}, f\} l_{\xi_{i+1}} \dots l_{\xi_s}$$

shows $\{R_0, \mathcal{E}\} \subseteq \mathcal{E}$. Every $h \in R$ satisfies an equation $p(h) = 0$ where $p \in R_0[t]$ is monic. Then $p'(h)\{h, f\} \in \mathcal{E}$. Because $\mathcal{E}$ is a locally free $\mathcal{O}_T$-module and $p'(h) \in \mathcal{O}_Y$ this implies $\{h, f\} \in \mathcal{E}$, i.e., $\{R, \mathcal{E}\} \subseteq \mathcal{E}$. Therefore, Theorem 3.6 is applicable, and we are left to show that $\mathcal{A}^0_Y$ acts on T^*_Y.

Hence we may assume that X is homogeneous. Now I will use the theory developed in [Kn5]. Let $\mathcal{G}$ denote the Graßmannian of all linear subspaces of $\mathfrak{g}^*$. For any $x \in X$ let $\mathfrak{g}^\perp_x \subseteq \mathfrak{g}^*$ be the annihilator of the isotropy subalgebra of x. This defines an equivariant morphism $\psi : X \to \mathcal{G}$. Let $X \hookrightarrow \overline{X}$ be any smooth equivariant compactification such that ψ extends to $\overline{\psi} : \overline{X} \to \mathcal{G}$. In the terminology of [Kn4], 3.4, this $\overline{X}$ is called *pseudo-free*. Let $\mathcal{V} \subseteq \mathcal{G} \times \mathfrak{g}^*$ be the tautological vector bundle over $\mathcal{G}$ and $\widetilde{T}_{\overline{X}} = \overline{\psi}^*\mathcal{V} \subseteq \overline{X} \times \mathfrak{g}^*$ its pullback to $\overline{X}$. It contains T^*_X as an open subset. The morphism $\widetilde{T}_X \to \mathfrak{g}^*$ is proper and its restriction to T^*_X is the moment map, i.e., we have compactified the moment map.

Let $\mathcal{D}_{\overline{X}}$ be the sheaf of differential operators on $\overline{X}$ and $\mathfrak{U}_{\overline{X}}$ its subsheaf of $\mathcal{O}_{\overline{X}}$-algebras which is generated by the vector fields coming from $\mathfrak{g}$. Because it carries a natural filtration by the order of differential operators, we can look at the associated graded sheaf of (commutative) algebras $\operatorname{gr}\mathfrak{U}_{\overline{X}}$. Then by [Kn4], 3.7, its spectrum (relative to $\overline{X}$) is just $\widetilde{T}_{\overline{X}}$. Hence the commutator in $\mathfrak{U}_{\overline{X}}$ induces on $\widetilde{T}_{\overline{X}}$ the structure of a Poisson variety.

Because R is integral over R_0, all $f \in R$ extend to functions on $\widetilde{T}_{\overline{X}}$. In particular, this implies that the $L_{\mathcal{A}_X^0}$-action extends to $\widetilde{T}_{\overline{X}}$. With $M_X := \operatorname{Spec} R$ we get $\widetilde{T}_{\overline{X}} \to M_X \to \mathfrak{g}^*$ which is now a true Stein factorization. The Poisson center of R is R^G. This implies that the Hamiltonian vector field H_f for $f \in R^G$ is tangent to the fibers of $\widetilde{T}_{\overline{X}} \to M_X$, i.e., the action of $L_{\mathcal{A}_X^0}$ over L_X extends to an action of $L_{\widetilde{\mathcal{A}}_X}$ over M_X where $\widetilde{\mathcal{A}}_X = \mathcal{A}_X^0 \times_{L_X} M_X$. Also the generic action of $\mathcal{A}_X^0$ induces a generic action of $\widetilde{\mathcal{A}}_X$. Because $\widetilde{T}_{\overline{X}} \to M_X$ is proper we get a regular action on $\widetilde{T}_{\overline{X}}$ (Corollary 3.4). But then there is a morphism $\mathcal{A}_X^0 \times_{L_X} \widetilde{T}_{\overline{X}} = \widetilde{\mathcal{A}}_X \times_{M_X} \widetilde{T}_{\overline{X}} \to \widetilde{T}_{\overline{X}}$, i.e., $\mathcal{A}_X^0$ acts on $\widetilde{T}_{\overline{X}}$.

Finally, by the interpretation of $\widetilde{T}_{\overline{X}}$ as associated graded of $\mathfrak{U}_{\overline{X}}$ it follows that all Hamiltonian vector fields are parallel to subvarieties of the form $\widetilde{T}_{\overline{X}}|_Y$, where $Y \in \overline{X}$ is G-stable. This implies that the complement of T_X^* in $\widetilde{T}_{\overline{X}}$ is $\mathcal{A}_X^0$-stable, i.e., $\mathcal{A}_X^0$ acts on T_X^*. This finishes the proof of Theorem 4.1. □

5. The central automorphism group

Let X be a normal variety. The main application of the preceding theory is to the group of *central automorphisms* of X:

$$\mathfrak{A}(X) := \{\varphi \in \operatorname{Aut}^G X \mid \varphi(f) \in k^* f \text{ for all } f \in k(X)^{(B)}\}.$$

Note, that if X is spherical, i.e. $k(X)^B = k$, this is the full automorphism group, otherwise it may be only a small part of it.

5.1. **Theorem.** *Let X be a G-variety.*

1. *$\mathfrak{A}(X)$ stabilizes every G-stable subset of X.*
2. *Let $Y \subseteq X$ be a G-stable subvariety and $\varphi \in \mathfrak{A}(X)$. Then $\varphi|_Y \in \mathfrak{A}(Y)$.*

Proof. 1. It suffices to show that φ stabilizes every G-orbit Y. By induction on $\dim Y$, we may assume that the boundary $\overline{Y}\backslash Y$ is $\mathfrak{A}(X)$-stable. Therefore, it suffices to show that $\overline{Y}$ is stable. Choose a G-invariant valuation v of $k(X)$ with center $\overline{Y}$. Because it is uniquely determined by its restriction on $k(X)^{(B)}$ ([Kn3], 4.2), we get $v = v \circ \varphi$ for all $\varphi \in \mathfrak{A}(X)$. Hence $\overline{Y} = \varphi(\overline{Y})$.

2. It follows from [Kn3], 2.2, that every $f \in k(Y)^{(B)}$ can be extended to an element in $\mathcal{O}_{X,Y}^{(B)} \subseteq k(X)^{(B)}$. This implies the claim. □

For technical reasons, we need another concept: Let $D \subseteq X$ be a B-stable, but not G-stable irreducible divisor (a so-called *color*). Then D is called *undetermined* if there is a different color D' such that the restrictions of the valuations v_D, $v_{D'}$ to $k(X)^{(B)}$ coincide.

5.2. **Lemma.** *Every G-variety contains only a finite number of undetermined divisors.*

Proof. Let $X_0 \subseteq X$ be open, B-stable such that the orbit space $Y = X_0/B$ exists. Then the B-stable divisors in X_0 are already separated by $k(Y)$. Hence an undetermined divisor is either a component of the boundary $X \setminus X_0$ or one of the finitely many colors D in X_0 such that the restriction of v_D to $k(Y)$ coincides with restriction of a valuation of a boundary component. □

The main application of the concept of undetermined divisors is that all other colors are $\mathfrak{A}(X)$-stable. The easiest example of an undetermined divisor is $X = SL_2(k)/T$, which has two B-stable divisors which are in fact interchanged by $\mathfrak{A}(X) = \mathbb{Z}/2\mathbb{Z}$.

5.3. **Theorem.** *Let X be a normal G-variety and $X_0 \subseteq X$ open, G-stable. Assume that no undetermined divisor of X contains a G-orbit. Then $\mathfrak{A}(X) \xrightarrow{\sim} \mathfrak{A}(X_0)$.*

Proof. The mapping is obviously injective. We show that every $\varphi \in \mathfrak{A}(X_0)$ extends to X. Let $Y \subseteq X$ be closed, G-stable. Then the local ring $\mathcal{O}_{X,Y}$ is uniquely determined by the set of B-stable divisors of X containing Y ([Kn3], 3.8). By assumption, this set, hence $\mathcal{O}_{X,Y}$, is φ-stable. This means that φ is defined in Y. Hence φ extends to X because this works for every Y. □

5.4. **Corollary.** *There is an open, G-stable subset X_0 of X such that $\mathfrak{A}(X_0) = \mathfrak{A}(X_1)$ for every open, G-stable subset X_1 of X_0.*

Proof. Let D be the union of undetermined divisors of X. Then remove from X all singularities and $\bigcap_{g \in G} gD$ to obtain X_0.

Definition. The group $\mathfrak{A}(X_0)$ (which does not depend on the choice of X_0) is denoted by $\mathfrak{A}_X$.

5.5. **Theorem.** *Let X be a G-variety. Then there is a unique homomorphism*

$$\lambda : \mathfrak{A}(X) \to A_X = \operatorname{Hom}(\Gamma_X, k^*) : \varphi \mapsto \lambda_\varphi$$

such that

$$\varphi(f) = \lambda_\varphi(\chi_f) f \textit{ for all } f \in k(X)^{(B)}.$$

This homomorphism is injective and if X is normal, then its image is closed. In particular, $\mathfrak{A}_X$ is realized as a closed subgroup of A_X.

Proof. For every $\chi \in \Gamma_X$ consider the vector space

$$V_\chi = \{ f \in k(X) \mid {}^b f = \chi(b) f \text{ for all } b \in B \}.$$

Then $\varphi \in \mathfrak{A}(X)$ acts linearly on it such that every element is an eigenvector. This implies that φ acts by multiplication with a scalar. This establishes the existence of λ (unicity is clear anyway).

Assume first that X supports a G-linearized ample line bundle (call X then G-linear). Because only finitely many B-stable colors are moved by $\mathfrak{A}(X)$, we then may find a $\mathfrak{A}(X)$- and B-stable very ample Cartier divisor D such that $\mathcal{L} := \mathcal{O}_X(D)$ carries a G-linearization. Furthermore, there is a unique action of $\mathfrak{A}(X)$ on $\mathcal{L}$ such that the canonical section σ_D is fixed. Let $V := H^0(X, \mathcal{L})$. For $\sigma \in V^{(B)}$ we have $f := \sigma/\sigma_D \in k(X)^{(B)}$. Then $\sigma(f) = \lambda_\varphi(\chi_f)f$ and $\varphi(\sigma_D) = \sigma_D$ for $\varphi \in \mathfrak{A}(X)$ imply

$$\varphi(\sigma) = \lambda_\varphi(\chi_\sigma \chi_{\sigma_D}^{-1})\sigma.$$

In particular, if $\lambda_\varphi \equiv 1$, then φ acts as identity on V. Hence, since $X \hookrightarrow \mathbf{P}(V^*)$, we get $\varphi = \mathrm{id}_X$, i.e., λ is injective.

Furthermore, for every $t \in \mathrm{Hom}(\mathcal{X}(B), k^*) = T$ we can define a G-homorphism of V by $t \cdot \sigma := t(\chi_\sigma \chi_{\sigma_D}^{-1})\sigma$ for every highest weight vector $\sigma \in V$. Clearly, the set A' of t which stabilizes X in $\mathbf{P}(V^*)$ is a closed subgroup. Then $\lambda(\mathfrak{A}(X))$ is the image of A' in A_X, hence closed.

Now assume that X is arbitrary. Then one can always find an open G-stable G-linear subset X_0. Hence λ is injective, because $\mathfrak{A}(X) \to \mathfrak{A}(X_0)$ is injective. Finally, if X is normal, then it can be covered by open subsets X_0 like this and $\lambda(\mathfrak{A}_X)$ is the intersection of closed subgroups, hence closed. □

Remark. The proof shows that normality can be (as usual) relaxed to the assumption that X is locally G-linear. Probably, even that is unnecessary.

5.6. **Corollary.** *The group $\mathfrak{A}(X)$ is in the center of $\mathrm{Aut}^G(X)$.*

Proof. Let $\psi \in \mathrm{Aut}^G(X)$ and $\varphi \in \mathfrak{A}(X)$. If $f \in k(X)^{(B)}$, then $\psi(f) \in k(X)^{(B)}$ with $\chi_{\psi(f)} = \chi_f$. This implies $\psi^{-1}\varphi\psi(f) = \lambda_\varphi(\chi_f)f$. Hence, $\psi^{-1}\varphi\psi \in \mathfrak{A}(X)$ with $\lambda_{\psi^{-1}\varphi\psi} = \lambda_\varphi$. □

5.7. **Corollary.** *If X is normal, then $\mathfrak{A}_X$ contains $\mathfrak{A}(X)$ as a closed subgroup of finite index.*

Proof. Let $X_0 \subseteq X$ be as in Theorem 5.3. Let $\mathfrak{A}' \subseteq \mathfrak{A}(X_0)$ be the subgroup of all elements which act trivially on the set of (undetermined) colors. It is closed and of finite index. The same proof as for Theorem 5.3 shows $\mathfrak{A}' \subseteq \mathfrak{A}(X)$. □

For some reduction argument we need later:

5.8. **Lemma.** *Let X be homogeneous. There is a quasi-affine homogeneous $\widetilde{G} = G \times \mathbf{G}_m$-variety $\widetilde{X}$ with $X = \widetilde{X}/\mathbf{G}_m$ and a split exact sequence*

$$1 \to \mathbf{G}_m \to \mathfrak{A}(\widetilde{X}) \to \mathfrak{A}(X) \to 1,$$

where $\mathfrak{A}(\widetilde{X})$ is defined with respect to $\widetilde{G}$.

Proof. This is actually a corollary of the proof of Theorem 5.5. With the notation there we choose $\widetilde{X} \subset V^*$ to be the affine cone over $X \subseteq \mathbf{P}(V^*)$. The construction comes with a splitting $\mathfrak{A}(X) \to \mathfrak{A}(\widetilde{X})$. The other parts of the exact sequence are clear. □

Finally, there is a comparison theorem with generic orbits.

5.9. **Theorem.** *Any G-variety X contains a non-empty open G-stable subset X_0 such that $\mathfrak{A}_X \xrightarrow{\sim} \mathfrak{A}(Gx)$ for every $x \in X_0$.*

Proof. By shrinking X, we may assume that the G-orbit space $\pi : X \to Y$ exists. The automorphism group of a homogeneous variety G/H is $N_G(H)/H$, hence also linear algebraic. Therefore, the automorphism groups of the fibers of π form an affine group scheme $\mathrm{Aut}^G(X/Y)$ with the set of central automorphisms $\mathfrak{A}(X/Y)$ as a closed subgroup scheme. The homomorphism λ gives an embedding of $\mathfrak{A}(X/Y) \hookrightarrow A_X \times Y$. We may assume that $\mathfrak{A}(X/Y)$ is smooth over Y. Then $\mathfrak{A}(X/Y)$ must be the trivial multiplicative group scheme $\mathfrak{A}(Gx) \times Y$. Hence, the constant sections define a homorphism $\mathfrak{A}(Gx) \to \mathfrak{A}(X)$ inverse to the restriction. □

6. The root system

In this section we establish a relation between $\mathfrak{A}_X$ and $\mathcal{A}_X$. For this we may assume throughout that $\mathfrak{A}(X) = \mathfrak{A}_X$. Let $\mathfrak{a}_X^1 \subseteq \mathfrak{a}_X^*$ be the set of points with trivial W_X-isotropy group. Then, as already employed, the group scheme $\mathcal{A}_X \times_{L_X} \mathfrak{a}_X^1$ is trivial with fiber A_X. Hence A_X acts on $T_X^* \times_{L_X} \mathfrak{a}_X^1$.

6.1. **Lemma.** *The following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{A}_X & \subseteq & \mathrm{Aut}^G(X) \\ \downarrow & & \downarrow \\ A_X & \to & \mathrm{Aut}(T_X^* \times_{L_X} \mathfrak{a}_X^1) \end{array}$$

Proof. By Theorem 5.9, we may assume that X is homogeneous. In view of Lemma 5.8, we may furthermore assume that X is quasi-affine, hence non-degenerate.

It suffices to consider the variety $\widehat{T}_X$ of [Kn5], §3, because it is an open subset of $T_X^* \times_{L_X} \mathfrak{a}_X^1$. Now recall the definition of the A_X-action (see [Kn5], §4): There is a Levi subgroup $L \subseteq G$ and an isomorphism $\widehat{T}_X = G \times^L \widehat{\Sigma}$, such that L acts on $\widehat{\Sigma}$ only through its quotient A_X. Then A_X acts on $\widehat{T}_X$ just by the action on $\widehat{\Sigma}$. Furthermore, there is an open subset of $\widehat{\Sigma}$ which is L-isomorphic to $X_0/P_u \times \mathbf{A}_r^s$. Choose $\varphi \in \mathfrak{A}_X$ and let $a = \lambda_\varphi \in A_X$. From $k(X)^U = k(X_0/P_u)$, it follows that φ acts on X_0/P_u, hence on Σ by multiplication with a. This proves the assertion. □

6.2. **Corollary.** $\mathfrak{A}_X \subseteq A_X^{W_X}$.

Proof. There is an action of $\varphi \in \mathfrak{A}_X$ on both T_X^* and $T_X^* \times_{L_X} \mathfrak{a}_X^1$. This implies that the corresponding $a \in A_X$ is W_X-invariant. □

Definition. The *root lattice* Λ_X of X is the kernel of $\mathcal{X}(A_X) \twoheadrightarrow \mathcal{X}(\mathfrak{A}_X)$.

Corollary 6.2 means exactly that W_X acts trivially on Γ_X/Λ_X. This implies that W_X acts also as a reflection group on the root lattice.

Definition. The *root system* Δ_X of X is the minimal root system attached to (Λ_X, W_X) as in section 2.

The root lattice is an isogeny invariant:

6.3. **Theorem.** *Let $\beta : X \to X'$ be a quasi-finite G-morphism between normal G-varieties. Then $\beta^* : \Lambda_{X'} \xrightarrow{\sim} \Lambda_X$ or, equivalently, there is a canonical short exact sequence*

$$1 \to \mathrm{Hom}(\Gamma_X/\Gamma_{X'}, k^*) \to \mathfrak{A}_X \to \mathfrak{A}_{X'} \to 1.$$

Proof. We may assume that β is finite. The local structure theorem ([BLV], [Kn5], 2.3, 2.4) tells us that there is a parabolic subgroup P_X with Levi part L and an affine locally closed subvariety $\Sigma \subseteq X$, such that $P \times^L \Sigma \to X$ is an open embedding.

Moreover, L acts on Σ only through its quotient A_X. Then every $\varphi \in \mathfrak{A}_X$ acts on $P \times^L \Sigma$ by $(p, \sigma) \mapsto (p, a\sigma)$, where a is the image of φ in A_X. Analogously, $\Sigma' = \beta(\Sigma) \subseteq X'$ will have the same properties. Because $A_X \to A_{X'}$ is surjective, we can lift any $\varphi' \in \mathfrak{A}(X')$ to a birational automorphism φ of X, where the set of lifts is determined by $\mathrm{Hom}(\Gamma_X/\Gamma_{X'}, k^*)$. The finiteness of β implies that φ is regular and commutes with G. Conversely, in the same manner every $\varphi \in \mathfrak{A}_X$ can be pushed down. $\square$

The next theorem is our main application of the theory of group schemes which we have developed in the first sections.

6.4. **Theorem.** *Let Δ be the minimal root system attached to (Γ_X, W_X). Then $\Lambda_X \subseteq \langle\Delta\rangle_{\mathbb{Z}}$ or, equivalently, $\bigcap_{\alpha\in\Delta} \ker_{A_X} \alpha \subseteq \mathfrak{A}_X$.*

Proof. The little Weyl group of X coincides with that of its generic orbits ([Kn1], 6.5.4). Hence, we may assume that X is homogeneous. Let $\widetilde{X} \to X$ as in Lemma 5.8. Then we have the exact sequence

$$0 \to \Gamma_X \to \Gamma_{\widetilde{X}} \to \mathbb{Z} \to 0$$

with $\Lambda_X \xrightarrow{\sim} \Lambda_{\widetilde{X}}$. Furthermore, $W_X = W_{\widetilde{X}}$ (see [Kn5], 5.1, 7.5) implies that Δ is also the minimal root system of $(\Gamma_{\widetilde{X}}, W_{\widetilde{X}})$. Hence, we may replace X by $\widetilde{X}$ and therefore may assume that X is quasi-affine.

Let $a \in A_X$ with $\alpha(a) = 1$ for all $\alpha \in \Delta$. Then by Lemma 6.1 we have to show that the action of a on $T_X^* \times_{L_X} \mathfrak{a}_X^1$ comes from an automorphism of X. We do this in several steps. Let $L_2 \subseteq L_X$ be the open subset over which the W_X-isotropy group has at most two elements. By assumption we have $\alpha(a) = 1$ for all $\alpha \in \Delta$. Hence, Theorem 2.9 implies that a induces a section $\sigma_a : L_2 \to \mathcal{A}_X^0$. Because $\mathcal{A}_X^0$ acts on T_X^*, we obtain an automorphism $\widetilde{\varphi} : \tau \mapsto \sigma_a(\Psi(\tau))\tau$ of $T_2 := T_X^* \times_{L_X} L_2$.

The next step is to show that $\widetilde{\varphi}$ extends to all of T_X^*. Consider the $\mathbf{G}_m$-action on T_X^* by scalar multiplication on the fibers. There is a compatible action on L_X and $\mathcal{A}_X^0$. The section σ_a is induced by a point of A_X on which $\mathbf{G}_m$ acts trivially. This implies that σ_a is $\mathbf{G}_m$-equivariant. Therefore, $\widetilde{\varphi}$ commutes with $\mathbf{G}_m$, i.e., $\widetilde{\varphi}$ is a birational automorphism of T_X^* of degree zero. Therefore, it acts also on the image $\mathbf{P}T_2$ of T_2 in the projectivized cotangent bundle $\mathbf{P}T_X^*$.

The morphism $\Psi : T_X^* \to L_X$ is equidimensional ([Kn1], 6.6). Therefore, the complements of T_2 and $\mathbf{P}T_2$ have codimension two. Consider the fiber $\mathbf{P}T_{2,x} \subseteq \mathbf{P}T_2$ over $x \in X$. Also its complement in the projective space $\mathbf{P}T_{X,x}^*$ has codimension two (remember that X is homogeneous). In particular, all global regular functions on $\mathbf{P}T_{2,x}$ are constant. Because X is quasi-affine, also the morphism

$$\mathbf{P}T_{2,x} \xrightarrow{\widetilde{\varphi}} \mathbf{P}T_2 \twoheadrightarrow X$$

is constant. This implies that $\widetilde{\varphi}$ maps fibers into fibers. Therefore, it induces first a birational and then by homogeneity a global automorphism φ of X. Furthermore, it extends to all of T_X^*.

We show that φ has the desired properties. Let φ^* be the automorphism of T_X^* which is induced by φ. Then $\varphi^* \circ \widetilde{\varphi}^{-1}$ is a vector bundle automorphism of T_X^*. Being a symplectomorphism and because it acts trivially on the tangent spaces of the zero section, it acts trivially on each fiber, i.e., $\widetilde{\varphi} = \varphi^*$. Finally, the explicit description of the A_X-action on $\widehat{T}_X$ in the proof of Lemma 6.1 shows that $\varphi \in \mathfrak{A}(X)$. □

Now we can justify the term "root lattice":

6.5. **Corollary.** *For every G-variety X the root system Δ_X has the following properties:*

a) *The root lattice Λ_X is generated by Δ_X.*
b) $\mathfrak{A}_X = \bigcap_{\alpha \in \Delta_X} \ker_{A_X} \alpha$.
c) *The Weyl group of Δ_X is W_X and it acts trivially on $\Gamma_X / \langle \Delta_X \rangle_{\mathbb{Z}}$.*
d) *Every quasi-finite G-morphism $X \to X'$ induces $\Gamma_{X'} \hookrightarrow \Gamma_X$ with $\Delta_{X'} \xrightarrow{\sim} \Delta_X$.*

Proof. a) and b) are clearly equivalent. Let $E \subseteq \mathfrak{A}_X$ be a finite subgroup such that $\mathfrak{A}_X/E$ is connected and let $X_0 := X/E$. Then $\mathfrak{A}_{X_0} = \mathfrak{A}_X/E$ by Theorem 6.3. This implies that Λ_{X_0} is a direct summand of Γ_{X_0}. In particular, Δ_X is also the minimal root system for Γ_X. Then by Theorem 6.4 and Corollary 6.2, we get $\bigcap_{\alpha \in \Delta_X} \ker_{A_{X_0}} \alpha \subseteq \mathfrak{A}_{X_0} \subseteq \mathfrak{A}_{X_0}^W$ which implies b). Finally, c) follows from Corollary 6.2 and d) is Theorem 6.3. □

One can also arrange the data a bit differently: For any $\alpha \in \Delta_X$ let $s_\alpha \in W_X$ be the reflection at α. Then Corollary 6.5c) implies that there is a unique $\alpha^\vee \in \Gamma_X^\vee := \operatorname{Hom}(\Gamma_X, \mathbb{Z})$ with

$$s_\alpha(\chi) = \chi - \alpha^\vee(\chi)\alpha \quad \text{for all } \chi \in \Gamma_X.$$

Let $\Delta_X^\vee := \{\alpha^\vee \mid \alpha \in \Delta_X\}$. Then $(\Gamma_X, \Delta_X, \Gamma_X^\vee, \Delta_X^\vee)$ forms a root datum in the sense of [Sp], 9.1.6. It also determines W_X and $\mathfrak{A}_X$.

If X is quasi-affine there is a much simpler construction of Δ_X which is mentioned in the introduction. Let $k[X] = \bigoplus_\chi R_\chi$ be the isotypic decomposition of $k[X]$ and define

$$\mathcal{M}' := \{\alpha \in \mathcal{X}(B) \mid \exists \chi, \eta \in \mathcal{X}(B) : \langle R_\chi R_\eta \rangle_k \cap R_{\chi+\eta-\alpha} \neq 0\}.$$

6.6. **Lemma.** *Let X be quasi-affine. Then*

i) $\Gamma_X = \langle \chi \mid R_\chi \neq 0 \rangle_{\mathbb{Z}}$,
ii) $\Lambda_X = \langle \mathcal{M}' \rangle_{\mathbb{Z}}$ *and*
iii) $\mathcal{Z}(X) = \{v \in \operatorname{Hom}(\Gamma_X, \mathbb{Q}) \mid v(\mathcal{M}') \geq 0\}$.

Proof. i) If $R_\chi \neq 0$, then $k[X]$ contains a heighest weight vector with weight χ. This shows "$\supseteq$". Conversely, let $f \in k(X)^{(B)}$. Since X is quasi-affine, $V = \{h \in k[X] \mid hf \in k[X]\}$ is a non-trivial G-module. Hence, it contains a highest weight vector s. Then $sf \in k[X]^{(B)}$ and $\chi_f = \chi_{sf} - \chi_s$ show "$\subseteq$".

ii) It suffices to prove $\mathfrak{A}_X = \operatorname{Hom}(\Gamma_X / \langle \mathcal{M}' \rangle_{\mathbb{Z}}, k^*)$. For that let $a : \Gamma_X \to k^*$ be a homomorphism with $a(\mathcal{M}') = 1$. Define a G-module automorphism φ_a of $k[X]$ by $\varphi_a(f) = a(\chi) f$ for $f \in R_\chi$. I claim that φ_a is an algebra automorphism. For this let $f \in R_\chi$, $h \in R_\eta$. Then fh has components in $R_{\chi+\eta-\alpha}$ with $\alpha \in \mathcal{M}'$. Because a is trivial on $\mathcal{M}'$, all components of fh are multiplied by the same factor $a(\chi+\eta)$. This shows $\varphi_a(fh) = a(\chi+\eta) fh = \varphi_a(f)\varphi_a(h)$ and the claim follows. Now choose a G-equivariant embedding $X \hookrightarrow \overline{X}$ where $\overline{X}$ is affine. The automorphism φ_a leaves every G-submodule of $k[X]$ stable. In particular, it induces an automorphism of $k[\overline{X}]$ hence of $\overline{X}$. Then $\varphi_a \in \mathfrak{A}(\overline{X}) \subseteq \mathfrak{A}(X)$ (see Theorem 5.1) proves "$\supseteq$".

Conversely, let $\varphi \in \mathfrak{A}_X$. Then φ is an automorphism of an open subset X_0 of X and acts on the χ-isotypic component of $k[X_0]$ by multiplication with $\lambda_\varphi(\chi)$. Hence it does the same on R_χ. For $\alpha \in \mathcal{M}'$ there exist $f \in R_\chi$ and $h \in R_\eta$ such that fh has a non-zero $(\chi+\eta-\alpha)$-component. Then $\varphi(fh) = \varphi(f)\varphi(h) = \lambda_\varphi(\chi+\eta)fh$ implies $\lambda_\varphi(\alpha) = 1$. This shows "$\subseteq$".

iii) We follow the argument of [Pau]. Let $\overline{v}$ be a G-invariant $\mathbb{Q}$-valued valuation of $k[X]$ which is trivial on $k(X)^B$. Then it is constant on $R_\chi \setminus \{0\}$. For $R_\chi \neq 0$ let $v(\chi)$ be this constant value. Since this is an additive map on the submonoid $\{\chi \mid R_\chi \neq 0\}$, it can be extended uniquely to a homomorphism $v : \Gamma_X \to \mathbb{Q}$. Conversely, $\overline{v}$ can be recovered from v by $\overline{v}(f) = \min\{v(\chi) \mid f_\chi \neq 0\}$.

Let $\alpha \in \mathcal{M}'$ and choose $f \in R_\chi$, $h \in R_\eta$ such that fh has a non-zero $(\chi+\eta-\alpha)$-component. Then $\overline{v}(fh) \geq v(\chi+\eta-\alpha)$. This shows that $\overline{v}$ is multiplicative if and only if $v(\alpha) \geq 0$ for all $\alpha \in \mathcal{M}'$. □

Remark. The proof in ii) showed $\mathfrak{A}(X) = \mathfrak{A}_X$ for X quasi-affine (not necessarily normal).

Proof of Theorem 1.3. The monoid $\mathcal{M}$ is by the preceding lemma the intersection of the root lattice Λ_X and the dual cone to the dominant Weyl chamber of Δ_X. Hence it is freely generated by the simple roots of Δ_X. □

Remark. Using Lemma 5.8, one can give an analoguous description of Δ_X for every G-quasi-projective variety.

Now assume that X is a symmetric variety, i.e., $X = G/H$ where G is semisimple and H is the fixed point set of an involution $\vartheta \in \mathrm{Aut}(G)$. Then there exist maximal tori T of G which are ϑ-stable. Choose one such that $T^\vartheta = T \cap H$ has minimal dimension. Let $\mathfrak{a} \subseteq \mathfrak{t} = \mathrm{Lie}\, T$ be the (-1)-eigenspace of ϑ and let $\rho : \mathfrak{t}^* \to \mathfrak{a}^*$ be the restriction map. Then $\Delta_X^r := \rho(\Delta) \setminus \{0\}$ is the *restricted root system* of X, where $\Delta \subseteq \mathfrak{t}^*$ is the root system of G. It is well known that Δ_X^r is indeed a root system. It may be non-reduced, i.e., with α it may also contain $\alpha/2$. The set of roots α for which $\alpha/2 \notin \Delta_X^r$ is its associated reduced root system. Now we show that our root system Δ_X is compatible with this classical construction:

6.7. **Theorem.** *Let X be a symmetric variety. Then Δ_X is the reduced root system associated to $2\Delta_X^r$.*

Proof. There is a Borel subgroup B containing T such that BH is dense in G. This implies $A_X = T/(T \cap H)$. From $T \cap H = T^\vartheta$ we get $\Gamma_X = \mathcal{X}(A_X) = (1-\vartheta)\mathcal{X}(T)$. We may assume that G is of adjoint type. Then $\mathcal{X}(T)$ is generated by Δ. Because $\frac{1}{2}(1-\vartheta)$ is the projection to $\mathfrak{a}^*$, we conclude that Γ_X is the root lattice of $2\Delta_X^r$. On the other hand $N_G(H) = H$ (see [Vu], 2.2, Lemme 1). Therefore, $\Gamma_X = \Lambda_X$ is also the root lattice of Δ_X. Furthermore, is is known ([Kn1], pp. 17–18) that Δ_X and Δ_X^r have the same Weyl group. This implies the claim. □

It would be nice to have a true generalization of Δ_X^r which works for all X. Also there should be a generalization of multiplicities, i.e., the number of roots in Δ which restrict to a given root in Δ_X^r.

7. Applications and amplifications

The research for the present paper was motivated by the following applications to the compactification theory of X. For this let $\mathcal{Z}(X)$ be the set of G-invariant $\mathbb{Q}$-

valued valuations of $k(X)$ which restrict to the trivial valuation on $k(X)^B$. Each $v \in \mathcal{Z}(X)$ induces a homomorphism $\Gamma_X \to \mathbb{Q} : \chi_f \mapsto v(f)$. It is known that the map $\mathcal{Z}(X) \to \mathcal{Q}(X) := \mathrm{Hom}(\Gamma_X, \mathbb{Q})$ is injective and identifies $\mathcal{Z}(X)$ with a Weyl chamber of W_X in $\mathcal{Q}(X)$ ([Kn3], 9.2, or [Kn5], 7.4).

7.1. **Corollary.** *Assume that* $\mathfrak{A}_X$ *is connected* (*e.g. trivial*). *Then* $\mathcal{Z}(X)$ *is defined by inequalities* $v(\alpha_1) \geq 0, \ldots, v(\alpha_s) \geq 0$, *where the* $\alpha_1, \ldots, \alpha_s$ *are part of a basis of* Γ_X.

Proof. The condition implies that the root lattice Λ_X is a direct summand of Γ_X. Hence every Weyl chamber is defined by equations of the form above, where the α_i run through a system of simple roots of Δ_X. □

The most important case is that of a spherical variety. For this let me sketch the classification of their equivariant embeddings (see [LV], [Kn2] for details): They are determined by a finite family of pairs $(\mathcal{C}, \mathcal{F})$ (one for each orbit), where $\mathcal{C} \subseteq \mathcal{Q}(X)$ is a finitely generated strictly convex cone and $\mathcal{F}$ is a set of B-stable divisors. This family is subject to various conditions. If the $\mathcal{F}$-parts are empty, then the conditions mean that the $\mathcal{C}$-parts form a fan supported in $\mathcal{Z}(X)$. Hence $\mathcal{C} = \mathcal{Z}(X)$ is admissible, if and only if $\mathcal{Z}(X)$ is strictly convex, if and only if $\mathrm{Aut}^G X = \mathfrak{A}_X$ is finite. In this case, the family $(\mathcal{C}, \varnothing)$, where $\mathcal{C}$ runs through all faces of $\mathcal{Z}(X)$, defines an embedding which is complete and has 2^r orbits ($r = \mathrm{rk}\,\Gamma_X$) exactly one of which is closed. It is called the *wonderful* or *standard embedding* of X.

7.2. **Corollary.** *Let* $X = G/H$ *be spherical with* $N_G(H) = H$. *Then its standard embedding is smooth.*

Proof. Let $\overline{X}$ be this completion. Then $\overline{X}$ has exactly one closed orbit Y. It is known (see [BP], 3.4) that Y has a transversal slice isomorphic to $\overline{A}$, where $\overline{A}$ is the A_X-embedding corresponding to the cone $\mathcal{Z}(X)$. That this cone is defined by a basis of $\Gamma_X = \mathcal{X}(A_X)$ is equivalent to $\overline{A}$ being smooth. □

Remark. It follows easily from the local structure theorem [BLV], that a smooth wonderful embedding is a regular embedding in the sense of [Gi], 4.3.1, or [BCP], Def. 5, which means that all orbit closures are transversal intersections of the divisors containing it and that the normal bundle of each orbit contains an open orbit.

The corollary settles half of a conjecture of Brion: Let $X = G/H$ be spherical, where H is self-normalizing. Then G/H is isomorphic to the orbit of $\mathrm{Lie}\,H$ considered as a point in a Graßmannian of $\mathrm{Lie}\,G$. Let $\widetilde{X}$ be the closure of this orbit (the Demazure embedding). Then Brion conjectured that $\widetilde{X}$ is smooth ([Br1], p. 141, Conj. A). Because, as he showed, $\overline{X}$ above is the normalization of $\widetilde{X}$, we have reduced the problem to the normality of $\widetilde{X}$.

Actually, it is possible to improve Corollaries 7.1 and 7.2 slightly. Following an idea of Luna, we define the subgroup $\mathfrak{A}_X^\sharp \subseteq \mathfrak{A}_X$ consisting of those automorphisms which stabilize every B-stable divisor of X. By Lemma 5.2, it is of finite index.

7.3. **Lemma.** *Let* X *be smooth. Then for* $\varphi \in \mathfrak{A}_X$ *the following are equivalent:*

i) $\varphi \in \mathfrak{A}_X^\sharp$;
ii) φ *acts trivially on* $\mathrm{Pic}\,X$;
iii) φ *acts trivially on* $\mathrm{Pic}^G X$.

Proof. i)⇒ii): For every line bundle $\mathcal{L}$ there is a B-stable divisor D in X with $\mathcal{L} \cong \mathcal{O}_X(D)$ (see [Br2], 1.3).

ii)⇒iii): Let $\mathcal{L}$ be a G-linearized line bundle. Then $\varphi^*\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X \otimes_k k_\chi$ for some character χ of G. Let $m > 0$ such that $\varphi^m \in \mathfrak{A}_X^0$. As $\mathfrak{A}_X^0$ is connected it acts trivially on $\operatorname{Pic}^G X$. Therefore, $m\chi = 0$ and $\chi = 0$.

iii)⇒i): Let $D \subseteq X$ be an irreducible B-stable divisor. Replacing it by a multiple we may assume that $\mathcal{L} = \mathcal{O}_X(D)$ carries a G-linearization. Hence $\varphi(D) - D$ is a principal divisor. If $D \neq \varphi(D)$, this shows that D is determined, which contradicts $\varphi \in \mathfrak{A}_X^\sharp$. Hence D is stable under φ. □

7.4. **Corollary.** *Let $X = G/H$ be homogeneous and let $N_G^\sharp(H) \subseteq N_G(H)$ consist of all elements which act trivially on $\mathcal{X}(H)$. Then $\mathfrak{A}_X^\sharp = \mathfrak{A}_X \cap N_G^\sharp(H)/H$.*

Proof. There is a canonical isomorphism $\operatorname{Pic}^G X = \mathcal{X}(H)$. □

7.5. **Theorem.** *Let X be a normal G-variety. Then there is a root system $\Delta_X^\sharp \subseteq \Gamma_X$ such that $\mathfrak{A}_X^\sharp = \bigcap_{\alpha \in \Delta_X^\sharp} \ker_{A_X} \alpha$.*

Proof. Let $E \subseteq \mathfrak{A}_X^\sharp$ be finite such that $\mathfrak{A}_X^\sharp/E$ is connected and let $X_0 = X/E$. By Theorem 6.3, $\mathfrak{A}_{X_0} = \mathfrak{A}_X/E$. By choice of E, the sets of B-stable divisors of X and X_0 are in bijection. This implies $\mathfrak{A}_{X_0}^\sharp = \mathfrak{A}_X^\sharp/E$. Hence, we can replace X by X_0 and may thus assume that $\mathfrak{A}_X^\sharp$ is connected. Then we define $\Delta_X^\sharp$ to be the minimal root system of (Γ_X, W_X). It suffices to show $\mathfrak{A}_X^\sharp \supseteq \bigcap_{\alpha \in \Delta_X^\sharp} \ker_{A_X} \alpha$.

For that let $D \subseteq X$ be an irreducible B-stable divisor. Replacing D by a multiple we may assume that $\mathcal{O}_X(D)$ carries a G-linearization. Then

$$\pi : \widetilde{X} := \operatorname{Spec}_X \oplus_{n \in \mathbb{Z}} \mathcal{O}_X(nD) \to X$$

is a $\mathbf{G}_m$-principal fiber bundle. We may replace G by $G \times \mathbf{G}_m$. Then $W_{\widetilde{X}} = W_X$ and

$$0 \to \Gamma_X \to \Gamma_{\widetilde{X}} \to \mathbb{Z} \to 0$$

is exact. This implies that $\Delta := \Delta_X^\sharp$ is also the minimal root system of $(\Gamma_{\widetilde{X}}, W_{\widetilde{X}})$. Let $A_X^1 = \bigcap_{\alpha \in \Delta} \ker_{A_X} \alpha$. Then every $a \in A_X^1$ can be lifted to $\widetilde{a} \in A_{\widetilde{X}}^1$. By Theorem 6.4, $\widetilde{a} \in \mathfrak{A}_{\widetilde{X}}$. The divisor $\pi^{-1}(D)$ is principal by construction and therefore determined. Thus $\widetilde{a}$ stabilizes $\pi^{-1}(D)$ which implies that a stabilizes D. □

7.6. **Corollary.** *Assume $X = G/H$ is spherical with $N_G^\sharp(H) = H$. Then the standard embedding of G/H is smooth.*

Remarks. 1. Let $X = G/H$ be a symmetric variety where G is simple. Then one easily checks using the classification that $\Delta_X^\sharp = \Delta_X$ unless X is one of $SL_{2n}/S(GL_n \times GL_n), n \geq 1$, $SO_n/SO_2 \times SO_{n-2}, n \geq 5$, $SO_{4n}/GL_{2n}, n \geq 2$, Sp_{2n}/GL_n, $n \geq 2$, and $\mathsf{E}_7/\mathbf{G}_m \cdot \mathsf{E}_6$. In all these cases, Δ_X^r and $\Delta_X^\sharp$ is of type C_* and B_*, respectively.

2. The importance of the condition $N_G^\sharp(H) = H$ has been first observed by Luna. He calls spherical subgroups satisfying it *very sober*. For $G = PGL_n(k)$,

Corollary 7.6 has been first proved by him. In that case, he even showed that very soberness is also necessary (not yet published).

Finally, I would like to extend Theorem 4.1 a little bit. There we have constructed the action of a connected group scheme $\mathcal{A}_X^0$ on the cotangent bundle T_X^*. We want to extend this action to a larger group scheme. For this we need a result which does not involve G-varieties.

7.7. **Theorem.** *Let W be a reflection group acting on the lattice Γ. Put $S' = \Gamma \otimes k$ and consider the group scheme $\mathcal{A} = \mathcal{A}(\Gamma, S', W)$. Then the set of open affine subgroup schemes $\mathcal{A}'$ of $\mathcal{A}$ is in bijection with the set of root data $(\Gamma, \Delta, \Gamma^\vee, \Delta^\vee)$. Furthermore,*

i) *The set $\mathcal{A}'(S)$ of global sections is* $\mathrm{Hom}(\Gamma/\langle\Delta\rangle_\mathbb{Z}, k^*)$.
ii) *$\mathcal{A}'$ is generated by $\mathcal{A}'(S)$ and $\mathcal{A}^0$.*

Proof. Let $\overline{\mathcal{C}}$ be the set of conjugacy classes of reflections in W. For every reflection s we have $(s-1)\Gamma \subseteq \ker_\Gamma(s+1)$ and the index is either one or two. Let $\mathcal{C} \subseteq \overline{\mathcal{C}}$ be those conjugacy classes where it is two. To any $\mathcal{C}_0 \subseteq \mathcal{C}$ we can assign a root system Δ in the following way: For every reflection $s \in W$ we adjoin the generators of $(s-1)\Gamma$ for Δ unless the conjugacy class of s is in $\mathcal{C}_0$ when we take the generators of $\ker_\Gamma(s+1)$. Conversely every root system is obtained this way.

Now observe that the elements of $\overline{\mathcal{C}}$ correspond to the irreducible components of the ramification divisor of $S' \to S$. Then the proof of Theorem 2.9 shows that the elements of $\mathcal{C}$ correspond exactly to those components Z over which $\mathcal{A}$ is disconnected. In that case $\mathcal{A} \times_S Z$ has two irreducible components exactly one of which, denoted by D_Z, does not contain the zero-section.

Therefore, starting from a subset of $\mathcal{C}$ we get a set of divisors D_Z of $\mathcal{A}$. Let $\mathcal{A}' \subseteq \mathcal{A}$ be the complement of the union of these divisors. As $\mathcal{A}$ is smooth, hence locally factorial, this is an affine open subset. Furthermore, by construction, multiplication defines a rational morphism $\mathcal{A}' \times_S \mathcal{A}' \dashrightarrow \mathcal{A}'$ which is defined in codimension one. As both sides are normal (even smooth) and affine it is regular on all of $\mathcal{A}'$, i.e., $\mathcal{A}'$ is a subgroup scheme.

Conversely, let $\mathcal{A}' \subseteq \mathcal{A}$ be an open affine subgroup scheme. Then every component D of the complement $\mathcal{A} \setminus \mathcal{A}'$ is pure of codimesion one. As $\mathcal{A}$ is connected generically, the image of D in S must be a divisor. More precisely, it is a component of the ramification divisor. As D does not contain the zero-section it must be of the form D_Z. This shows there corresponds uniquely a subset of $\mathcal{C}$ to $\mathcal{A}'$ which proves the first half of the theorem.

Let $a \in \mathrm{Hom}^W(\Gamma, k^*)$ be a global section of $\mathcal{A}$. Then the same reasoning as for Theorem 2.9 shows that a is a section of $\mathcal{A}'$ in codimension one if and only if $a(\Delta) = 1$. As S and $\mathcal{A}'$ are affine this holds if and only if a is a global section of $\mathcal{A}'$. This shows i).

Finally, for any point $s' \in S'$ let $\Delta(s') = \{\alpha \in \Delta \mid \alpha^\vee(s') = 0\}$. Let $s \in S$ be the image of s'. Then $\mathcal{X}(\mathcal{A}'_s) = \Gamma/\langle\Delta(s')\rangle_\mathbb{Z}$. As $\Delta(s')$ is a subroot system of Δ, the group $\langle\Delta(s')\rangle_\mathbb{Z}$ is a direct summand of $\langle\Delta\rangle_\mathbb{Z}$. This implies that $\mathcal{A}'(S) \to \mathcal{A}'_s$ has a connected cokernel which shows ii). □

Remark. Let (Γ, W) be the root lattice of type B_n where $n \geq 1$. Then the conjugacy class of reflections along short roots is in $\mathcal{C}$. Conversely, one can show that every such conjugacy class comes from a direct summand of this type.

Now we return to our G-variety X. Then to $\Delta_X \subseteq \Gamma_X$ we can assign a unique open affine subgroup scheme $\mathcal{A}_X$ of $\mathcal{A} := \mathcal{A}(\Gamma_X, \mathfrak{a}_X^*, W_X)$ which satisfies $\mathfrak{A}_X(L_X) = \mathfrak{A}_X$ and $\mathcal{A}_X = \mathfrak{A}_X \mathcal{A}^0$.

7.8. **Theorem.** *The affine open group scheme $\mathcal{A}_X$ is the largest open subgroup scheme of $\mathcal{A}$ to which the action of $\mathcal{A}^0$ on T_X^* can be extended.*

Proof. Let $\mathcal{A}' \subseteq \mathcal{A}$ be the largest subgroup scheme extending the $\mathcal{A}^0$-action. Let $D_1, \ldots, D_q$ be the codimension-1-components of $\mathcal{A} \setminus \mathcal{A}'$ and $\mathcal{A}'' := \mathcal{A} \setminus \bigcup_i D_i$. Then $\mathcal{A}''$ is an affine group scheme as in the proof of Theorem 7.7. Let $\sigma \in \mathcal{A}''(L_X)$. Then σ induces a birational automorphism of T_X^* which is regular in codimension one. The same proof as for Theorem 6.4 works to show that σ is induced by an element of $\mathfrak{A}_X$. This shows

$$\mathcal{A}'' = \mathcal{A}''(L_X)\mathcal{A}^0 \subseteq \mathfrak{A}_X \mathcal{A}^0 = \mathcal{A}_X \subseteq \mathcal{A}' \subseteq \mathcal{A}''. \qquad \square$$

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