JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, January 1996

SINGULAR INTEGRAL OPERATORS WITH ROUGH CONVOLUTION KERNELS

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1. Introduction

The purpose of this paper is to investigate the behavior on $L^1(\mathbb{R}^d)$, $d \geq 2$, of a class of singular convolution operators which are not within the scope of the standard Calderón-Zygmund theory.

An important special case occurs if the convolution kernel K is homogeneous of degree $-d$. Suppose that $\Omega \in L^1(S^{d-1})$ and

$$\int_{S^{d-1}} \Omega(\theta) d\theta = 0; \tag{1.1}$$

here $d\theta$ denotes surface measure on the sphere. Then it is easy to see that for $f \in C_0^\infty(\mathbb{R}^d)$ the principal value integral

$$T_\Omega f(x) \;=\; \text{p.v.} \int \Omega(y/|y|)|y|^{-d} f(x-y)\, dy \tag{1.2}$$

exists for all $x \in \mathbb{R}^d$. Calderón and Zygmund [1] used the method of rotations to show that if $\Omega \in L^1(S^{d-1})$ and if the even part of Ω belongs to the class $L \log L(S^{d-1})$, then T extends to a bounded operator on $L^p(\mathbb{R}^d)$, $1 < p < \infty$.

Proposition. *Suppose that $\Omega \in L \log L(S^{d-1})$ and suppose that the cancellation property* (1.1) *holds. Then T_Ω extends to an operator of weak type* $(1,1)$.

In two dimensions this result was previously obtained by Christ and Rubio de Francia [3], and, under a slightly stronger hypothesis, by Hofmann [6]. In [2], [3] a weak type $(1,1)$ inequality was also proved for the less singular maximal operator

$$M_\Omega f(x) \;=\; \sup_{r>0} r^{-d} \int_{|y|\leq r} \big|\Omega(y/|y|) f(x-y)\big| dy,$$

in all dimensions, again under the assumption $\Omega \in L \log L$. It is conceivable that a variant of the arguments in [3] for the maximal operator could also work for the singular integral operator; in fact, in unpublished work, the authors of [3] obtained a weak type $(1,1)$ inequality in dimension $d \leq 7$. However their arguments—if applied to the singular integral operator—lead to substantial technical difficulties and no

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Received by the editors August 5, 1994.
1991 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B25.
Research supported in part by a grant from the National Science Foundation.

proof has been known for the higher-dimensional cases. In this paper we develop a different and conceptually simpler method, based on a microlocal decomposition of the kernel (cf. (2.2) below). Incidentally this method also gives a new proof of the weak type bounds for M_Ω.

The proposition is a special case of a more general theorem concerning translation invariant operators T with rough convolution kernels $K \in \mathcal{S}'$. We assume that K is locally integrable away from the origin, so that

$$\langle Tf, g\rangle = \iint g(x)f(y)K(x-y)\,dy\,dx \tag{1.3}$$

for all $f, g \in C_c^\infty(\mathbb{R}^d)$ with disjoint supports. Clearly T extends to a bounded operator on $L^2(\mathbb{R}^d)$ if and only if the Fourier transform $\widehat{K}$ belongs to $L^\infty(\mathbb{R}^d)$. Introducing polar coordinates $x = r\theta$, $r > 0$, $\theta \in S^{d-1}$, we shall assume a weak regularity condition for $r \mapsto K(r\theta)$. However only size conditions will be imposed in the θ variable.

In order to formulate our assumptions let

$$V_R(\theta) = \int_R^{2R} |K(r\theta)| r^{d-1}\,dr \tag{1.4}$$

and

$$V(\theta) = \sup_{R>0} V_R(\theta). \tag{1.5}$$

Moreover, for $a \geq 2$ let

$$\eta(a) = \sup_{R \geq as} \int_{S^{d-1}} \int_R^{2R} |K((r-s)\theta) - K(r\theta)| r^{d-1}\,dr\,d\theta. \tag{1.6}$$

We shall always assume the Dini-condition

$$\int_2^\infty \eta(a)\frac{da}{a} < \infty. \tag{1.7}$$

Theorem. *Suppose that* T *is as in* (1.3) *and that* $\widehat{K} \in L^\infty(\mathbb{R}^d)$*. Suppose that* (1.7) *holds and that* $V \in L\log L(S^{d-1})$*. Then* T *is bounded in* L^p, $1 < p < \infty$; *moreover,* T *is of weak type* $(1,1)$*.*

Note that for the operators in (1.2) we have $\eta(a) = O(a^{-1})$ and $V = c|\Omega|$. Therefore the Proposition follows from the Theorem.

Remarks. (i) It may be more natural to impose an integrability condition on V_R, uniformly in R, rather than on the maximal quantity V. Indeed the hypothesis $V \in L\log L$ can be replaced by

$$\sup_R \int_{S^{d-1}} V_R(\theta)(1 + \Delta(V_R(\theta)/\|V_R\|_1))d\theta < \infty$$

for some nondecreasing function $\Delta : [1, \infty) \to (0, \infty)$ satisfying

$$\int_1^\infty \frac{da}{a\Delta(a)} < \infty.$$

Typical choices for Δ are

$$\begin{aligned} &\Delta(t) = \log^{1+\varepsilon}(2+t), \text{ or} \\ &\Delta(t) = \log(2+t)\log(2+\log^{1+\varepsilon}(2+t)), \text{ etc.} \end{aligned}$$

(ii) Without the assumption $\Omega \in L\log L(S^{d-1})$ even the L^2 boundedness of T_Ω may fail. This was pointed out by Calderón and Zygmund [1]. However if $\Omega \in L^1(S^{d-1})$ is odd, then T_Ω is bounded on L^p, $1 < p < \infty$ (see [1]). Presently it is not known whether a weak type $(1,1)$ inequality holds in this case.

In §2 we shall give the main estimates needed to prove the Theorem. The formal proof is contained in §3.

The following notation is used: For a set $E \subset \mathbb{R}^d$ we denote the Lebesgue measure of E by $|E|$. For a set $A \in S^{d-1}$ we also write $|A| = \int_A d\theta$. The Fourier transform of f is denoted by $\widehat{f}$, the inverse Fourier transform of f is denoted by $\mathcal{F}^{-1}[f]$. Given two quantities a and b we write $a \lesssim b$ or $b \gtrsim a$ if there is a positive constant C, depending only on the dimension, such that $a \le Cb$. We write $a \approx b$ if $a \lesssim b$ and $a \gtrsim b$.

2. Main estimates

Let $\{H_j\}$ be a family of functions with

$$\operatorname{supp} H_j \subset \{x : 2^{j-2} \le |x| \le 2^{j+2}\}.$$

We assume that the H_j are differentiable in the radial variable and that the estimates

$$\sup_{0\le l\le N} \sup_j r^{d+l} \Big| (\frac{\partial}{\partial r})^l H_j(r\theta) \Big| \le \mathfrak{M}_N \tag{2.1}$$

hold uniformly in θ and r. Convolution kernels of this type come up in a dyadic decomposition of the kernel of the operator defined in (1.2), if $\Omega \in L^\infty(S^{d-1})$.

We shall be interested in estimates for $H_j * \sum_Q b_Q$ where each b_Q is a building block in a Calderón-Zygmund decomposition, supported in a cube Q, and where the sidelength $2^{L(Q)}$ of Q is small compared to the diameter of $\operatorname{supp} H_j$; say by a factor of $\approx 2^{-s}$.

For $s > 3$ let $\mathfrak{E}^s = \{e^s_\nu\}$ be a collection of unit vectors with mutual distance $> 2^{-s-10}d^{-1}$ such that for each $\theta \in S^{d-1}$ there is an e^s_ν with $|\theta - e^s_\nu| \le 2^{-s-1}$. It is easy to see that we may construct disjoint measurable sets $E^s_\nu \subset S^{d-1}$ with $e^s_\nu \in E^s_\nu$, $\operatorname{diam}(E^s_\nu) \le 2^{-s}$ and $\bigcup_\nu E^s_\nu = S^{d-1}$. Then clearly

$$\operatorname{card}(\mathfrak{E}^s) \approx 2^{s(d-1)}.$$

Let

$$H^s_{j\nu}(x) \;=\; H_j(x)\chi_{E^s_\nu}(x/|x|).$$

A further decomposition will be based on the observation that the Fourier transform $\widehat{H^s_{j\nu}}$ is concentrated near the hyperplane perpendicular to e^s_ν.

Fix a parameter κ, such that $0<\kappa<1$. Let $\psi\in C^\infty_0(\mathbb{R})$ be supported in $[-4,4]$ such that $\psi(t)=1$ for $t\in[-2,2]$. Define P^s_ν by

$$\widehat{P^s_\nu}(\xi)=\psi(2^{s(1-\kappa)}\langle\xi,e^s_\nu\rangle/|\xi|).$$

Our basic splitting is

$$H_j=\Gamma^s_j+(H_j-\Gamma^s_j) \tag{2.2}$$

where

$$\Gamma^s_j\;=\;\sum_\nu P^s_\nu*H^s_{j\nu}.$$

Lemma 2.1. *Let $\mathfrak{Q}$ be a collection of cubes Q with disjoint interiors. Define $L(Q)=m$ if $2^{m-1}<$ sidelength$(Q)\le 2^m$ and let $\mathfrak{Q}_m=\{Q\in\mathfrak{Q}:L(Q)=m\}$. For each Q let f_Q be an integrable function supported in Q satisfying*

$$\int|f_Q(x)|dx\le\alpha|Q|.$$

Let $F_m=\sum_{Q\in\mathfrak{Q}_m}f_Q$. Then for $s>3$

$$\Big\|\sum_j\Gamma^s_j*F_{j-s}\Big\|_2^2\;\le\;C[\mathfrak{M}_0]^2 2^{-s(1-\kappa)}\alpha\sum_Q\|f_Q\|_1.$$

In our application of Lemma 2.1 the functions f_Q will be the basic building blocks which arise in a Calderón-Zygmund decomposition at height $c\alpha$. Note however that for this part no cancellation condition for f_Q is assumed.

Lemma 2.2. *Let Q be a cube of sidelength 2^{j-s} and let b_Q be integrable and supported in Q; moreover, suppose that $\int b_Q=0$. Then for $N\ge d+1$ and $0\le\varepsilon\le 1$*

$$\|(H_j-\Gamma^s_j)*b_Q\|_1\;\le C_N\big[\mathfrak{M}_0 2^{-s\varepsilon}+\mathfrak{M}_N 2^{s(d+(\varepsilon-\kappa)N)}\big]\|b_Q\|_1$$

where C_N does not depend on j or Q.

It is important to keep track of how the estimates depend on $\mathfrak{M}_N$ since we shall apply the lemmas in a situation where this norm is large and depends on s itself. The bounds in Lemma 2.2 are not best possible, but this is irrelevant for our purpose.

Proof of Lemma 2.1. We use an orthogonality argument based on the following observation. Given $s>3$, each $\xi\ne0$ is contained in at most $C2^{s(d-2+\kappa)}$ of the sets $\operatorname{supp}\widehat{P^s_\nu}$ where C only depends on d. In fact by homogeneity it suffices to check this for $\xi\in S^{d-1}$. If $\xi\in\operatorname{supp}\widehat{P^s_\nu}\cap S^{d-1}$ and $\xi^\perp$ is the hyperplane perpendicular to ξ, then

$$\operatorname{dist}(e^s_\nu,\xi^\perp)\;\le\;c2^{-s(1-\kappa)}. \tag{2.3}$$

Since the mutual distance of the e^s_ν is bounded below by $c'2^{-s}$ there are at most $c''2^{s(d-2+\kappa)}$ of the e^s_ν satisfying (2.3). This implies the observation.

We apply Plancherel's theorem, the Cauchy-Schwarz inequality and then Plancherel's theorem again to obtain

$$
\begin{aligned}
\Big\| \sum_j \Gamma^s_j * F_{j-s} \Big\|_2^2 &= (2\pi)^{d/2} \Big\| \sum_\nu \widehat{P^s_\nu} \sum_j \widehat{H^s_{j\nu}} \widehat{F_{j-s}} \Big\|_2^2 \\
&\le C 2^{s(d-2+\kappa)} \sum_\nu (2\pi)^{d/2} \Big\| \sum_j \widehat{H^s_{j\nu}} \widehat{F_{j-s}} \Big\|_2^2 \\
&= C 2^{s(d-2+\kappa)} \sum_\nu \Big\| \sum_j H^s_{j\nu} * F_{j-s} \Big\|_2^2 .
\end{aligned}
\tag{2.4}
$$

For fixed ν write

$$
\begin{aligned}
\Big\| \sum_{j=-\infty}^{\infty} H^s_{j\nu} * F_{j-s} \Big\|_2^2 = &\sum_{j=-\infty}^{\infty} \int \widetilde{H^s_{j\nu}} * H^s_{j\nu} * F_{j-s}(x) \, \overline{F_{j-s}}(x)\, dx \\
&+ 2 \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{j-1} \int \widetilde{H^s_{j\nu}} * H^s_{i\nu} * F_{i-s}(x) \, \overline{F_{j-s}}(x)\, dx
\end{aligned}
$$

where $\widetilde{H^s_{j\nu}}(x) = \overline{H^s_{j\nu}}(-x)$.

Next observe that $\widetilde{H^s_{j\nu}} * H^s_{i\nu}$ is for $i \le j$ supported in a rectangle $\mathcal{R}^s_{j\nu}$ centered at 0 with $d-1$ short sides of length 2^{j-s+10} and one long side of length 2^{j+10}, the long side being parallel to e^s_ν. Since the measure of E^s_ν is bounded by $C2^{-s(d-1)}$ we have

$$\|H^s_{i\nu}\|_1 \lesssim \mathfrak{M}_0 2^{-s(d-1)}$$

for all i and consequently

$$\|\widetilde{H^s_{j\nu}} * H^s_{i\nu}\|_\infty \le \|H^s_{j\nu}\|_\infty \|H^s_{i\nu}\|_1 \lesssim \mathfrak{M}_0^2 2^{-jd} 2^{-s(d-1)}.$$

Therefore, since the cubes Q are disjoint,

$$
\begin{aligned}
|\widetilde{H^s_{j\nu}} * H^s_{j\nu} * F_{j-s}(x)| &+ 2 \Big| \widetilde{H^s_{j\nu}} * \sum_{i<j} H^s_{i\nu} * F_{i-s}(x) \Big| \\
&\lesssim [\mathfrak{M}_0]^2 2^{-jd} 2^{-s(d-1)} \int_{x+\mathcal{R}^s_{j\,\nu}} \sum_{i \le j} |F_{i-s}(y)| dy \\
&\lesssim [\mathfrak{M}_0]^2 2^{-jd} 2^{-s(d-1)} \sum_i \sum_{\substack{L(Q)=i-s \\ Q\cap(x+\mathcal{R}^s_{j\,\nu})\neq\emptyset}} \int |f_Q(x)| dx \\
&\lesssim [\mathfrak{M}_0]^2 2^{-jd} 2^{-s(d-1)} \alpha \sum_i \sum_{\substack{L(Q)=i-s \\ Q\cap(x+\mathcal{R}^s_{j\,\nu})\neq\emptyset}} |Q| \\
&\lesssim [\mathfrak{M}_0]^2 2^{-jd} 2^{-s(d-1)} \alpha |x + 2\mathcal{R}^s_{j\nu}| \\
&\lesssim [\mathfrak{M}_0]^2 2^{-s(2d-2)} \alpha
\end{aligned}
$$

for all $x \in \mathbb{R}^d$. This finally implies that

$$\begin{aligned}\sum_\nu \Big\| \sum_j H^s_{j\nu} * F_{j-s} \Big\|_2^2 &\lesssim [\mathfrak{M}_0]^2 \alpha 2^{-s(2d-2)} \mathrm{card}(\mathfrak{E}^s) \sum_j \|F_{j-s}\|_1 \\ &\lesssim [\mathfrak{M}_0]^2 \alpha 2^{-s(d-1)} \sum_j \|F_{j-s}\|_1 \\ &\lesssim [\mathfrak{M}_0]^2 \alpha 2^{-s(d-1)} \sum_Q \|f_Q\|_1\end{aligned}$$

and the asserted inequality follows from (2.4). □

Proof of Lemma 2.2. Let $\beta \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be supported in $\{\xi : 1/2 \le |\xi| \le 2\}$ such that $\sum_k \beta^2(2^{-k}\xi) = 1$ for all $\xi \neq 0$. Let L_k be defined by $\widehat{L_k}(\xi) = \beta(2^{-k}\xi)$. Consider the multipliers

$$m^{sk}_{j\nu}(\xi) = \beta(2^{-k}\xi)(1 - \widehat{P^s_\nu}(\xi))\widehat{H^s_{j\nu}}(\xi);$$

then

$$H_j - \Gamma^s_j = \sum_\nu \sum_k \mathcal{F}^{-1}[m^{sk}_{j\nu}] * L_k.$$

Since $\mathrm{diam}(Q) \lesssim 2^{j-s}$ and $\|\nabla L_k\|_1 \lesssim 2^k$ we obtain using the cancellation of b_Q

$$\|\mathcal{F}^{-1}[m^{sk}_{j\nu}] * L_k * b_Q\|_1 \lesssim \|\mathcal{F}^{-1}[m^{sk}_{j\nu}]\|_1 \min\{1, 2^{k+j-s}\} \|b_Q\|_1. \tag{2.5}$$

Let $\ell^k_{s\nu}$ be the invertible linear transformation with $\ell^k_{s\nu} e^s_\nu = 2^{k-s(1-\kappa)} e^s_\nu$ and $\ell^k_{s\nu} y = 2^k y$ if $\langle y, e^s_\nu \rangle = 0$. It is straightforward to check that

$$\|\partial^\alpha_\xi [\widehat{L_k}(1 - \widehat{P^s_\nu})(\ell^k_{s\nu} \cdot)]\|_2 \le C_\alpha$$

for all multi-indices α. Therefore $\widehat{L_k}(1 - \widehat{P^s_\nu})$ is an L^1 Fourier multiplier with norm independently of k, s and ν. Consequently

$$\|\mathcal{F}^{-1}[m^{sk}_{j\nu}]\|_1 \lesssim \|H^s_{j\nu}\|_1 \lesssim 2^{-s(d-1)} \mathfrak{M}_0. \tag{2.6}$$

In order to get a better bound for large k we estimate $\widehat{H^s_{j\nu}}$ and its derivatives using integration by parts. Note that $1 - \widehat{P^s_\nu}(\xi) = 0$ if $|\langle \xi, e^s_\nu \rangle| \le 2^{-s(1-\kappa)}|\xi|$. Therefore if $\theta \in E^s_\nu$ and if $\xi \in \mathrm{supp}(1 - \widehat{P^s_\nu})$ and $|\xi| \approx 2^k$, then $|\langle \theta, \xi \rangle| \gtrsim 2^{-s(1-\kappa)} 2^k$. Now

$$\begin{aligned}\widehat{H^s_{j\nu}}(\xi) &= \int \chi_{E^s_\nu}(\theta) \int H_j(r\theta) e^{-ir\langle \theta, \xi \rangle} r^{d-1}\, dr\, d\theta \\ &= \int \chi_{E^s_\nu}(\theta) (i\langle \theta, \xi \rangle)^{-N} \int \partial_r^N H_j(r\theta) e^{-ir\langle \theta, \xi \rangle} r^{d-1}\, dr\ d\theta.\end{aligned}$$

Hence we obtain the size estimate

$$|m^{sk}_{j\nu}(\xi)| \le C_N \mathfrak{M}_N |E^s_\nu| 2^{[s(1-\kappa)-j-k)]N},$$

uniformly in ξ. A similar calculation applies to the derivatives of $m^{sk}_{j\nu}$. Differentiating $\widehat{H^s_{j\nu}}$ yields additional factors of $r \approx 2^j$ in the above integral and differentiating $\beta(2^{-k}\cdot)\widehat{P^s_\nu}$ yields additional factors of $2^{s(1-\kappa)}2^{-k}$. Since $|E^s_\nu| \le C2^{-s(d-1)}$ we see that

$$\|\partial^\alpha_\xi [m^{sk}_{j\nu}(2^k\cdot)]\|_2 \;\le\; C_\alpha \mathfrak{M}_N 2^{-s(d-1)}(2^{s(1-\kappa)} + 2^{j+k})^{|\alpha|}2^{-N(j+k-s(1-\kappa))}$$

for all multi-indices α with $|\alpha| \le N$. Therefore if $N \ge N_1 > d/2$

$$\begin{aligned}\|\mathcal{F}^{-1}[m^{sk}_{j\nu}]\|_1 &\lesssim \sum_{|\alpha|\le N_1} \|\partial^\alpha_\xi[m^{sk}_{j\nu}(2^k\cdot)]\|_2 \\ &\lesssim \mathfrak{M}_N 2^{-s(d-1)}2^{-N(j+k-s(1-\kappa))}(2^{(j+k)N_1} + 2^{s(1-\kappa)N_1}).\end{aligned} \tag{2.7}$$

Finally by (2.5), (2.6)

$$\sum_{k\le -j+s(1-\varepsilon)} \|\mathcal{F}^{-1}[m^{sk}_{j\nu}] * L_k * b_Q\|_1 \;\lesssim\; \mathfrak{M}_0 2^{-s(d-1+\varepsilon)}\|b_Q\|_1 \tag{2.8}$$

and by (2.7) with $N_1 = d$, $N \ge d+1$

$$\begin{aligned}&\sum_{k> -j+s(1-\varepsilon)} \|\mathcal{F}^{-1}[m^{sk}_{j\nu}] * L_k * b_Q\|_1 \\ &\qquad\lesssim \mathfrak{M}_N 2^{-s(d-1)}2^{-s(\kappa-\varepsilon)N}\big[2^{sd(1-\varepsilon)} + 2^{sd(1-\kappa)}\big]\|b_Q\|_1.\end{aligned} \tag{2.9}$$

If we sum over ν and note that $\operatorname{card}(\mathfrak{E}^s) \lesssim 2^{s(d-1)}$, then (2.8) and (2.9) imply the statement of the lemma. □

3. Proof of the Theorem

Clearly the L^p boundedness for $1 < p < \infty$ follows from the weak type $(1,1)$ estimate and the assumed L^2 boundedness, by a duality argument and the Marcinkiewicz interpolation theorem (see [7]). Therefore given $\lambda > 0$ we have to verify the inequality

$$\big|\{x \in \mathbb{R}^d : |Tf(x)| > \lambda\}\big| \;\lesssim\; A\lambda^{-1}\|f\|_1; \tag{3.1}$$

where

$$A = \|\widehat{K}\|_\infty + \int_2^\infty \eta(a)\frac{da}{a} + \int_{S^{d-1}} V(\theta)(1 + \log_+(V(\theta)/\|V\|_1))\,d\theta;$$

here $\log_+ s = \log s$ if $s \ge 1$ and $\log_+ s = 0$ where $0 \le s < 1$. Then by assumption $A < \infty$.

Given $f \in L^1(\mathbb{R}^d)$ we shall use the Calderón-Zygmund decomposition of f at height $\alpha = \lambda/A$ (see Stein [7]). We decompose

$$f \;=\; g + b \;=\; g + \sum_Q b_Q$$

where $\|g\|_\infty \le \alpha$, $\|g\|_1 \lesssim \|f\|_1$, each b_Q is supported in a dyadic cube Q with sidelength $2^{L(Q)}$ and the cubes Q have disjoint interiors. Moreover $\|b_Q\|_1 \lesssim \alpha|Q|$ and $\sum_Q |Q| \lesssim \alpha^{-1}\|f\|_1$. For each Q let Q^* be the dilate of Q with same center and $L(Q^*) = L(Q) + 10$, and let $E = \bigcup Q^*$. Then also

$$|E| \lesssim \alpha^{-1}\|f\|_1 = A\lambda^{-1}\|f\|_1.$$

Finally, for each Q, the mean value of b_Q vanishes: $\int b_Q = 0$. We shall use a variant of Calderón-Zygmund theory due to Fefferman [5] and modified by Christ [2].

As in standard Calderón-Zygmund theory we have the estimate for the good function g

$$\|Tg\|_2^2 \le \|T\|_{L^2\to L^2}^2 \|g\|_2^2 \le A^2\|g\|_1\|g\|_\infty \le A\lambda\|g\|_1$$

and by Tshebyshev's inequality

$$\big|\{x \in \mathbb{R}^d : |Tg(x)| > \lambda/2\}\big| \;\le\; 4\lambda^{-2}\|Tg\|_2^2 \;\le\; 4A\lambda^{-1}\|g\|_1 \;\lesssim\; A\lambda^{-1}\|f\|_1.$$

Therefore the proof of the Theorem is reduced to the estimate

$$\big|\{x \in \mathbb{R}^d \setminus E : |Tb(x)| > \lambda/2\}\big| \;\lesssim\; A\lambda^{-1}\|f\|_1. \tag{3.2}$$

Note that the expressions $Tb_Q(x)$ are well defined for almost all $x \in \mathbb{R}^d \setminus E$ since we assume that K is locally integrable away from the origin.

We now introduce a dyadic decomposition of the kernel. Let $\beta \in C_0^\infty(\mathbb{R}_+)$ be as in the previous section ($\operatorname{supp}\beta \subset (1/2, 2)$, $\sum_k \beta^2(2^{-k}t) = 1$ for all $t > 0$). Define

$$K_j(x) = \beta^2(2^{-j}|x|)K(x).$$

For $m \in \mathbb{Z}$ let

$$B_m = \sum_{L(Q)=m} b_Q.$$

Then observe that the support of the functions $K_j * B_{j-s}$ is contained in E if $s \le 3$. Therefore, in order to verify (3.2), it suffices to prove that

$$\Big|\{x \in \mathbb{R}^d : \big|\sum_{s>3}\sum_j K_j * B_{j-s}(x)\big| > \lambda/2\}\Big| \;\lesssim\; A\lambda^{-1}\|f\|_1. \tag{3.3}$$

We now decompose the kernels K_j in the spherical variables according to the size of V; moreover, we introduce a regularization in the radial variable.

Let

$$\delta = [100(d+2)]^{-1}$$

and let

$$D^s \;=\; \{\theta \in S^{d-1} : V(\theta) \le 2^{\delta s}\|V\|_{L^1(S^{d-1})}\}.$$

Let $\phi \in C_0^\infty(\mathbb{R})$ such that $\int \phi(t)dt = 1$ and such that $\phi(t) = 0$ if $|t| \ge 2^{-10}$. Then

$$K_j = H_j^s + R_j^s + S_j^s \tag{3.4}$$

where

$$\begin{aligned} S_j^s(r\theta) \;&=\; \beta^2(2^{-j}r)\Big[K(r\theta) - \int K(\rho\theta)2^{\delta s-j}\phi(2^{\delta s-j}(r-\rho))d\rho\Big], \\ R_j^s(r\theta) \;&=\; \beta^2(2^{-j}r)\chi_{S^{d-1}\setminus D^s}(\theta)\int K(\rho\theta)2^{\delta s-j}\phi(2^{\delta s-j}(r-\rho))d\rho, \\ H_j^s(r,\theta) \;&=\; \beta^2(2^{-j}r)\chi_{D^s}(\theta)\int K(\rho\theta)2^{\delta s-j}\phi(2^{\delta s-j}(r-\rho))d\rho. \end{aligned}$$

Observe that H_j^s vanishes if $|x| \notin [2^{j-2}, 2^{j+2}]$ and that for $2^{j-2} \le r \le 2^{j+2}$, $\theta \in D^s$,

$$r^{d+l}\Big|(\frac{\partial}{\partial r})^l H_j^s(r,\theta)\Big| \;\le\; C_l 2^{\delta s(l+1)}\int_{r/2}^{2r}|K(\rho\theta)|\rho^{d-1}d\rho \;\le\; 2C_l 2^{\delta s(l+2)}\|V\|_{L^1(S^{d-1})}.$$

That is, for fixed $s > 3$ and for all N, the family $\{H_j^s\}$ satisfies the assumption (2.1) with

$$\mathfrak{M}_N = C_N\|V\|_{L^1(S^{d-1})}2^{\delta s(N+2)} \tag{3.5}$$

where C_N does not depend on V or s. We now decompose

$$H_j^s \;=\; \Gamma_j^s + (H_j^s - \Gamma_j^s)$$

exactly as in (2.2), except that this time the operator H_j itself depends on s. The decomposition (2.2) depended on a parameter $0 < \kappa < 1$; we may now choose $\kappa = 1/2$.

We have split

$$\sum_{s>3}\sum_j K_j * B_{j-s}(x) \;=\; I(x) + II(x)$$

where

$$I(x) \;=\; \sum_{s>3}\sum_j \Gamma_j^s * B_{j-s}(x)$$

and

$$II(x) \;=\; \sum_{s>3}\sum_j [H_j^s - \Gamma_j^s + R_j^s + S_j^s] * B_{j-s}(x).$$

Now by Tshebyshev's inequality

$$\begin{aligned} &\Big|\{x\in\mathbb{R}^d : \big|\sum_{s>3}\sum_j K_j * B_{j-s}(x)\big| > \lambda/2\}\Big| \\ &\le \big|\{x\in\mathbb{R}^d : |I(x)| > \lambda/4\}\big| + \big|\{x\in\mathbb{R}^d : |II(x)| > \lambda/4\}\big| \\ &\le 16\lambda^{-2}\|I\|_2^2 \,+\, 4\lambda^{-1}\|II\|_1. \end{aligned} \tag{3.6}$$

By Lemma 2.1 and (3.5) with $N = 0$ we have

$$\begin{aligned} \|I\|_2^2 \;&\le\; \Big[\sum_{s>3}\Big\|\sum_j \Gamma_j^s * B_{j-s}\Big\|_2\Big]^2 \\ &\lesssim \|V\|^2_{L^1(S^{d-1})}\Big[\sum_{s>3}\Big(2^{s(4\delta+\kappa-1)}\alpha\sum_Q\|b_Q\|_1\Big)^{1/2}\Big]^2 \\ &\lesssim\; A^2\alpha\sum_Q\|b_Q\|_1 \;\lesssim\; A\lambda\|f\|_1; \end{aligned} \tag{3.7}$$

here we could sum the geometrical series since $4\delta+\kappa-1<-1/4$.

Next we apply Lemma 2.2 with $N=5(d+1)$, $\varepsilon=1/4$ and obtain

$$\begin{aligned}\Big\|\sum_{s>3}\sum_j(H^s-\Gamma^s_j)*B_{j-s}\Big\|_1 &\lesssim \sum_{s>3}A2^{(N+2)\delta s}(2^{-\varepsilon s}+2^{s(d-(\kappa-\varepsilon)N)})\sum_j\|B_{j-s}\|_1\\ &\lesssim A\|f\|_1;\end{aligned}\tag{3.8}$$

now we have used that $(N+2)\delta+d-(\kappa-\varepsilon)N<-(d+1)/4$ and $(N+2)\delta-\varepsilon<-1/8$.

It remains to estimate the sums involving R^s_j and S^s_j. Observe that

$$\|R^s_j\|_1\le\int_{S^{d-1}\setminus D^s}\int_{2^{j-1}}^{2^{j+1}}|K(r\theta)|r^{d-1}dr\,d\theta\lesssim\int_{V(\theta)>2^{\delta s}\|V\|_1}V(\theta)d\theta$$

and therefore

$$\begin{aligned}\Big\|\sum_{s>3}\sum_j R^s_j*B_{j-s}\Big\|_1 &\lesssim\sum_{s>3}\sum_j\|B_{j-s}\|_1\int_{V(\theta)>2^{\delta s}\|V\|_1}V(\theta)d\theta\\ &\lesssim\sum_Q\|b_Q\|_1\int V(\theta)\,\mathrm{card}(\{s\in\mathbb{N}:\,2^{\delta s}\le|V(\theta)|/\|V\|_1\})\,d\theta\\ &\lesssim\int V(\theta)(1+\log_+(V(\theta)/\|V\|_1))d\theta\sum_Q\|b_Q\|_1\\ &\lesssim A\|f\|_1.\end{aligned}\tag{3.9}$$

Finally $\|S^s_j\|_1\lesssim\sup_R\|V_R\|_1$ and for $s>10/\delta$

$$\begin{aligned}\|S^s_j\|_1 &\le\int\int\int_{2^{j-1}}^{2^{j+1}}|K((r-\rho)\theta)-K(r\theta)|r^{d-1}dr\,d\theta\,|2^{\delta s-j}\phi(2^{\delta s-j}\rho)|d\rho\\ &\lesssim\eta(2^{\delta s-3}).\end{aligned}$$

Therefore

$$\begin{aligned}&\Big\|\sum_{s>3}\sum_j S^s_j*B_{j-s}\Big\|_1\\ &\quad\lesssim\sum_{0<s<10/\delta}\sum_j\|B_{j-s}\|_1\sup_R\|V_R\|_{L^1(S^{d-1})}\\ &\qquad+\sum_{s>10/\delta}\sum_j\|B_{j-s}\|_1\eta(2^{\delta s-3})\\ &\quad\lesssim\big[\sup_R\|V_R\|_{L^1(S^{d-1})}+\int_2^\infty\eta(a)\frac{da}{a}\big]\sum_Q\|b_Q\|_1\\ &\quad\lesssim A\|f\|_1.\end{aligned}\tag{3.10}$$

Now by (3.8), (3.9) and (3.10)

$$\|II\|_1\lesssim A\|f\|_1\tag{3.11}$$

and the desired weak type inequality (3.3) follows from equations (3.6), (3.7) and (3.11). □

We conclude by proving the remark following the statement of the Theorem. We have to change the definitions of the functions H_j^s and R_j^s in (3.4). Let $C = \sup_R \|V_R\|_1$ and for $j \in \mathbb{Z}$

$$D_j^s = \{\theta \in S^{d-1} : V_{2^{j-1}}(\theta) + V_{2^j}(\theta) + V_{2^{j+1}}(\theta) \le 2^{2+\delta s} C\}.$$

In the present setting we define H_j^s and R_j^s as before but with D^s replaced by D_j^s. The estimate (3.9) is changed to

$$\begin{aligned}
&\Big\| \sum_{s>3} \sum_j R_j^s * B_{j-s} \Big\|_1 \\
&\quad \lesssim \sum_{s>3} \sum_j \|B_{j-s}\|_1 \sum_{\sigma=-1}^{1} \int_{V_{2^{j+\sigma}}(\theta) > 2^{\delta s} C} V_{2^{j+\sigma}}(\theta)\, d\theta \\
&\quad \lesssim \|f\|_1 \sum_{s>3} \sup_j \sum_{\sigma=-1}^{1} \int_{V_{2^{j+\sigma}}(\theta) > 2^{\delta s} C} V_{2^{j+\sigma}}(\theta)\, d\theta \\
&\quad \lesssim \|f\|_1 \sum_{s>3} \frac{1}{\Delta(2^{\delta s}/C)} \sup_j \sup_s \sum_{\sigma=-1}^{1} \int V_{2^{j+\sigma}}(\theta) \Delta(V_{2^{j+\sigma}}(\theta)/C) d\theta \\
&\quad \lesssim \|f\|_1 \int_1^\infty \frac{da}{a\Delta(a)} \sup_R \int V_R(\theta) \Delta(V_R(\theta)/C) d\theta.
\end{aligned}$$

The other estimates remain essentially unchanged; in various instances one replaces $\|V\|_1$ by $\sup_R \|V_R\|_1$.

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