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MODULI SPACES OF SINGULAR YAMABE METRICS

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1. Introduction

Much has been clarified in the past ten years about the behavior of solutions of the semilinear elliptic equation relating the scalar curvature functions of two conformally related metrics. The starting point for these recent developments was R. Schoen's resolution of the Yamabe problem on compact manifolds [S1], capping the work of a number of mathematicians over many years. Soon thereafter Schoen [S2] and Schoen-Yau [SY] made further strides in understanding weak solutions of this equation, particularly on the sphere, and its relationship with conformal geometry. Of particular interest here is the former, [S2]; in that paper, Schoen constructs metrics with constant positive scalar curvature on $\mathbb{S}^n$, conformal to the standard round metric, and with prescribed isolated singularities (he also constructs solutions with certain, more general, singular sets). This 'singular Yamabe problem' is to find a metric $g = u^{4/(n-2)} g_0$ on a domain $\mathbb{S}^n \backslash \Lambda$ which is complete and has constant scalar curvature $R(g)$. This is equivalent to finding a positive function u satisfying

$$\begin{aligned} &\Delta_{g_0} u - \frac{n-2}{4(n-1)} R(g_0) u + \frac{n-2}{4(n-1)} R(g) u^{\frac{n+2}{n-2}} = 0 \text{ on } \mathbb{S}^n \backslash \Lambda, \\ &g \text{ complete on } \mathbb{S}^n \backslash \Lambda, \qquad R(g) = \text{ constant}, \end{aligned} \tag{1.1}$$

where $R(g_0) = n(n-1)$ is the scalar curvature of the round metric g_0. The completeness of g requires that u tend to infinity, in an averaged sense, on approach to Λ.

The earliest work on this singular Yamabe problem seems to have been that of C. Loewner and L. Nirenberg [LN], where metrics with constant negative scalar curvature are constructed. Later work on this 'negative' case was done by Aviles-McOwen [AMc], cf. also [Mc] and Finn-McOwen [FMc], where the background manifold and metric are allowed to be arbitrary. For a solution with $R(g) < 0$ to exist, it is necessary and sufficient that $\dim(\Lambda) > (n-2)/2$. A partial converse is that if a solution with $R(g) \geq 0$ exists (at least when $M = \mathbb{S}^n$), it is necessary that $k \leq (n-2)/2$, [SY]. Schoen gave the first general construction of solutions with $R(g) > 0$ [S2]. These solutions have singular set Λ which is either discrete

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or nonrectifiable. Many new solutions on the sphere with singular set Λ, a smooth perturbation of an equatorial k-sphere, $1 \leq k < (n-2)/2$, are constructed in [MS]. F. Pacard [Pa] has recently constructed positive complete solutions on the sphere with singular set an arbitrary smooth submanifold of dimension $(n-2)/2$. Regularity of solutions near the singular set Λ, no matter the sign of the curvature or background manifold or metric, is examined in [M1], cf. also [ACF] for a special case of relevance to general relativity. A more detailed account of part of this history is given in [MS]. Quite recently the first author and F. Pacard [MP] established the existence of solutions on $M \setminus \Lambda$ where M is any compact manifold with nonnegative scalar curvature and the singular set Λ is any finite, disjoint union of submanifolds Λ_i with dimensions $k_i \in \{1, \dots, (n-2)/2\}$.

In this paper we return to this problem in the setting studied by Schoen [S2] on $\mathbb{S}^n$, where Λ is a finite point-set:

$$\Lambda = \{p_1, \dots, p_k\}. \tag{1.2}$$

Hereafter, Λ will always be taken to be this set, unless indicated specifically, and the scalar curvature $R(g)$ attained by the conformal metric will always be $n(n-1)$. In this case, no solutions of (1.1) exist when $k = 1$. A proof of this, following from a general symmetry theorem, is indicated below. Thus we always assume that $k \geq 2$. Since this problem is conformally invariant, the set Λ may be replaced by $F(\Lambda)$ for any conformal transformation $F \in O(n+1, 1)$. A simple topological argument (given in [S2]) shows that using such a transformation we can always arrange that Λ is 'balanced', i.e. that the points p_j sum to zero as vectors in $\mathbb{R}^{n+1}$. Henceforth this will be tacitly assumed as well. Notice that if Λ is balanced, $k > 2$, and if F is a conformal transformation which preserves Λ, then (since some power of F fixes three points on the sphere, and also the origin in $\mathbb{R}^{n+1}$) F must be an orthogonal transformation. If $k = 2$ then F could also be a dilation. Also, any balanced configuration Λ is contained in a minimal equatorial subsphere $\mathbb{S}^k \subset \mathbb{S}^n$, and an Alexandrov reflection argument similar to the one in §8 of [CGS] shows that if u is an arbitrary solution to (1.1), and F is a rotation preserving Λ pointwise, then u must also be invariant under F. Observe that this implies that no complete solutions exist on $\mathbb{S}^n \setminus \{p\}$. If u were such a solution, then because any other point can be moved to be antipodal to p, this reflection argument would show that u is rotationally symmetric with respect to any other point on the sphere. Hence u would have to be constant, which contradicts completeness. Hence (except when $k = 2$) we do not get 'trivial' families of solutions of (1.1) obtained by pulling back a fixed solution by a family of conformal transformations. Note, however, that if F preserves Λ, but permutes the points, there is no reason to expect that most solutions of (1.1) will be fixed by F; indeed, it is not even clear a priori that F-invariant solutions exist in this case.

Our aim in this work is to consider the moduli space $\mathcal{M}_\Lambda$, which is by definition the set of all smooth positive solutions u to the problem

$$\begin{gathered} \Delta_{g_0} u - \frac{n(n-2)}{4} u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \text{ on } \mathbb{S}^n \setminus \Lambda, \\ g = u^{\frac{4}{n-2}} g_0 \text{ complete on } \mathbb{S}^n \setminus \Lambda, \end{gathered} \tag{1.3}$$

where $\Lambda = \{p_1, \dots, p_k\}$ is any fixed, balanced set of k points in $\mathbb{S}^n$. We remark that the geometric condition that the metric g be complete is equivalent to the analytic one of requiring that the set Λ consist of nonremovable singularities of u. This space might be called the 'PDE moduli space' $\mathcal{M}_\Lambda^{\mathrm{PDE}}$, to distinguish it from the 'geometric moduli space' $\mathcal{M}_\Lambda^{\mathrm{geom}}$ which consists of all geometrically distinct solutions of this problem. In most instances these spaces coincide, although the second could conceivably be somewhat smaller than the first if some solutions admit nontrivial isometries (e.g. as in the case $k = 2$ discussed in §2 below). $\mathcal{M}_\Lambda$ will always denote the former of these spaces here. By Schoen's work [S2], $\mathcal{M}_\Lambda$ is nonempty whenever $k > 1$. In fact, his construction yields families of solutions. As this was not the aim of his work, this is not made explicit there, nor are the free parameters in his construction counted.

We shall examine a number of questions, both local and global, concerning the nature of this moduli space. The simplest of these is whether $\mathcal{M}_\Lambda$ is a manifold, or otherwise tractable set, and if so, what is its dimension. Our main result is

Theorem 1.4. *$\mathcal{M}_\Lambda$ is locally a real analytic variety of formal dimension k.*

The formal dimension is the dimension predicted by an index theorem. As is usual in moduli space theories, obstructions may well exist to prevent $\mathcal{M}_\Lambda$ from attaining this dimension. We clarify this and give a more careful statement in Theorem 5.4, Corollary 5.5 and Theorem 6.13 below. We also describe how natural parameters on $\mathcal{M}_\Lambda$ may arise. On the linear level, these are given by certain scattering theoretic information for the metrics $g \in \mathcal{M}_\Lambda$. Another geometric description is given by the Pohožaev invariants which are defined in §3. We also obtain information on a geometrically natural compactification of $\mathcal{M}_\Lambda$ which is obtained by taking the union with lower-dimensional moduli spaces corresponding to singular sets $\Lambda' \subset \Lambda$. We are, as yet, unable to provide a satisfactory description of the interior singularities of $\mathcal{M}_\Lambda$, or to determine whether this compactification, $\overline{\mathcal{M}_\Lambda}$, is itself a compact real (semi-)analytic variety. This latter property is, by all indications, true. We hope to return to this later.

There are many similarities between the theory of constant scalar curvature metrics on $\mathbb{S}^n \backslash \Lambda$ where Λ is finite and that of embedded, complete, constant mean curvature (CMC) surfaces with k ends in $\mathbb{R}^3$. The first examples of such CMC surfaces were given by N. Kapouleas [Kap]. One-parameter families of solutions with symmetry were constructed by K. Grosse-Brauckmann [B]. Further general results on the structure of these surfaces and related problems appear in [KKS], [KKMS] and [KK]. In the last of these, N. Korevaar and R. Kusner conjecture that there is a good moduli space theory for these surfaces. Our methods may be adapted immediately to this setting. If $M \subset \mathbb{R}^3$ is an embedded, complete CMC surface with $k \geq 3$ ends which satisfies a hypothesis analogous to (5.2) below, then our results imply that the space of all nearby surfaces of this type, up to rigid motion in $\mathbb{R}^3$, is a $(3k - 6)$-dimensional real analytic orbifold. In order to understand the structure of the moduli space near surfaces where this hypothesis is not satisfied an argument different from the one employed here is needed since, for example, we do not have an analogue of the constructions in §6. Recently such an argument was derived based on the linear analysis developed in this paper. Thus we can also assert that the moduli space of such constant mean curvature surfaces is locally a real analytic variety, as was claimed in [KK]. This argument can be seen as an extension of 'Liapunov-Schmidt reduction' or the 'Kuranishi method' and also applies to give a new, direct proof of Theorem 1.4. These results appear in [KMP].

The outline of this paper is as follows. In §2 we analyze in detail the special case when $k = 2$. The solutions here will be called Delaunay solutions, in analogy with a similar family of complete constant mean curvature surfaces in $\mathbb{R}^3$ discovered in 1841 by C. Delaunay [D], although it was Fowler [Fo1], [Fo2] who first studied the differential equation (see (2.1) below) of which these are solutions. Only in this case may the moduli space be determined completely, since, using the symmetry argument discussed above, the equation now reduces to an ODE. We also analyze the spectral theory of the linearized scalar curvature operator completely in this simple case. §3 collects a number of disparate results about solutions of (1.3) with isolated singularities which are used throughout the rest of the paper. This contains an explanation of the results of Caffarelli-Gidas-Spruck and Aviles-Korevaar-Schoen which state that the Delaunay solutions are good models for arbitrary solutions of (1.3) with isolated singularities. We also discuss here the Pohožaev invariants and compactness results for solutions $g \in \mathcal{M}_\Lambda$. The linearization L of the scalar curvature operator is studied in §4. We prove Fredholm results for this operator, using and expanding earlier work of C. Taubes [T]. Other results here include more detailed information on asymptotics of solutions of $Lw = 0$, as well as the computation, using a relative index theorem, of the dimension of the 'bounded nullspace' of L. §5 uses these results to establish the structure of $\mathcal{M}_\Lambda$ near its smooth points. In §6 we show that $\mathcal{M}_\Lambda$ is a real analytic set by writing it as a slice of an 'urmoduli space' $\mathbb{M}_\Lambda$ with the conformal class determined by any $g \in \mathcal{M}_\Lambda$. Our study of the urmoduli space $\mathbb{M}_\Lambda$ draws on work of A. Fischer and J. Marsden, [FM1] and [FM2]. We also prove a generic slice theorem here which shows that slices of $\mathbb{M}_\Lambda$ by generic nearby conformal classes are smooth, even if $\mathcal{M}_\Lambda$ is not. In §7, our concluding remarks, we discuss three aspects about the nature of $\mathcal{M}_\Lambda$ concerning which we have not yet obtained satisfactory results. The first of these is the nonexistence of L^2 eigenvalues for the linearization, especially for the solutions constructed in [S2]. Secondly, we give a description of local coordinates on $\mathcal{M}_\Lambda$ near smooth points. This is done both on the linear level and geometrically in terms of the Pohožaev invariants introduced in §3. At this point we also provide a brief discussion of the recent construction [MPU] of dipole solutions for the problem. Finally, we address certain natural questions concerning the boundary of the geometric compactification $\overline{\mathcal{M}_\Lambda}$.

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2. Delaunay solutions

In this section we discuss the Delaunay family of solutions. These constitute, up to conformal equivalence, the totality of solutions when Λ has only two elements.

ODE analysis. When $k = 2$, the condition that Λ be balanced means that $p_2 = -p_1$. It is not difficult to show using the Alexandrov reflection argument [CGS] that any positive solution of the PDE (1.3) is only a function of the geodesic distance from either p_1 or p_2. This equation reduces to an ODE, which takes the simplest form when written relative to the background metric $dt^2 + d\theta^2$ on the cylinder

$\mathbb{R} \times \mathbb{S}^{n-1}$ with coordinates (t, θ), which is conformally equivalent to $\mathbb{S}^n \backslash \Lambda$. Thus, since the cylinder has scalar curvature $(n-1)(n-2)$, and $g = u^{4/(n-2)}(dt^2 + d\theta^2)$ has scalar curvature $n(n-1)$, $u = u(t)$ satisfies

$$\frac{d^2}{dt^2}u - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0. \tag{2.1}$$

This is easily transformed into a first order Hamiltonian system: setting $v = u'$ (primes denoting differentiation by t)

$$\begin{aligned} u' &= v, \\ v' &= \frac{(n-2)^2}{4}u - \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}}. \end{aligned} \tag{2.2}$$

The corresponding Hamiltonian energy function is

$$H(u, v) = \frac{v^2}{2} + \frac{(n-2)^2}{8}u^{\frac{2n}{n-2}} - \frac{(n-2)^2}{8}u^2. \tag{2.3}$$

The orbits of (2.2) remain within level sets of H, and since these level sets are one dimensional, this determines these orbits (but not their parameterizations) explicitly. The equilibrium points for this flow are at $(0, 0)$ and $(\bar{u}, 0)$, where

$$\bar{u} = \left(\frac{n-2}{n}\right)^{\frac{n-2}{4}}. \tag{2.4}$$

There is a special homoclinic orbit $(u_0(t), v_0(t))$ corresponding to the level set $H = 0$; it limits on the origin as t tends to either $\pm\infty$, and encloses a bounded set Ω in the right half-plane which is symmetric across the u-axis, given by $\{H \leq 0\}$. Somewhat fortuitously we may calculate explicitly that

$$u_0(t) = (\cosh t)^{\frac{2-n}{2}}.$$

Of course, $\{H = 0\}$ decomposes into two orbits: this one and the stationary orbit $(0, 0)$. It is simple to check that orbits not enclosed by this level set, i.e. those on which $H > 0$, must pass across the v-axis and into the region where $u < 0$. Thus, since we are only interested in solutions of (2.1) which remain positive and exist for all t, it suffices to consider only those orbits in Ω. Notice that the second equilibrium point $(\bar{u}, 0)$ is in this region, and that all other orbits are closed curves. These correspond to periodic orbits $(u_\epsilon(t), v_\epsilon(t))$, with period $T(\epsilon)$, $0 < \epsilon < 1$. The parameter ϵ may be taken as the smaller of the two u values where the orbit intersects the u-axis, so that $0 < \epsilon \leq \bar{u}$ (note that, strictly speaking, the orbit with $\epsilon = 0$ corresponds to the equilibrium point at the origin, but by convention we set it equal to the one previously defined). This ODE analysis is also described in [S3].

The corresponding metrics on $\mathbb{R} \times \mathbb{S}^{n-1}$ (or $\mathbb{S}^n \backslash \{p_1, p_2\}$) have a discrete group of isometries, given on the cylinder by the translations $t \mapsto t + T(\epsilon)$. They interpolate between the cylindrical metric $\bar{u}^{4/(n-2)}(dt^2 + d\theta^2)$ (which is rescaled by the power of $\bar{u}$ so that its scalar curvature is $n(n-1)$) and the solution corresponding to the conformal factor $u_0(t)$. This later solution is nothing other than the standard round metric on $\mathbb{S}^n\backslash\{p_1,p_2\}$, which is therefore incomplete and not, strictly speaking, in the moduli space $\mathcal{M}_\Lambda$. In all that follows we shall adopt the notation

$$g_\epsilon = u_\epsilon^{\frac{4}{n-2}}\left(dt^2+d\theta^2\right) \tag{2.5}$$

when referring to these Delaunay metrics.

Of greatest concern is the Laplacian for the metrics g_ϵ; in terms of the coordinates (t,θ) on the cylinder we may write this operator as

$$\begin{aligned} \Delta_\epsilon &= u^{-\frac{2n}{n-2}}\partial_t\left(u^2\partial_t\right)+u^{-\frac{4}{n-2}}\Delta_\theta \\ &= u^{-\frac{4}{n-2}}\partial_t^2+2(\partial_t u)u^{-\frac{(n+2)}{n-2}}\partial_t+u^{-\frac{4}{n-2}}\Delta_\theta. \end{aligned} \tag{2.6}$$

We use ∂_t for the partial derivative with respect to t, etc. Also, Δ_θ is the Laplacian on the sphere $\mathbb{S}^{n-1}$ of curvature 1. We use the convention that $-\Delta$ is a positive operator. Also, we write $u=u_\epsilon$ throughout this section.

It is convenient to replace the variable t by a new variable $r(t)$, depending on ϵ, which represents geodesic distance with respect to g_ϵ along the ∇t integral curves, which are already geodesics (with varying parameterization) for each of the metrics g_ϵ. $r(t)$ is defined by setting $dr/dt = u^{\frac{2}{n-2}}$ and $r=0$ when $t=0$. Then

$$\partial_t = u^{\frac{2}{n-2}}\partial_r$$

and consequently, using the first equality in (2.6),

$$\begin{aligned} \Delta_\epsilon &= u^{\frac{2-2n}{n-2}}\partial_r\left(u^{\frac{2n-2}{n-2}}\partial_r\right)+u^{-\frac{4}{n-2}}\Delta_\theta \\ &= \partial_r^2+\frac{2n-2}{n-2}\frac{\partial_r u}{u}\partial_r+u^{-\frac{4}{n-2}}\Delta_\theta. \end{aligned} \tag{2.7}$$

The function u is still periodic with respect to r, but the period $R(\epsilon)$ now is better behaved than the period $T(\epsilon)$ above. In fact, whereas

$$\lim_{\epsilon\to 0} T(\epsilon)=\infty, \qquad \text{and} \qquad \lim_{\epsilon\to\bar{u}} T(\epsilon)=\frac{2\pi}{\sqrt{n-2}}$$

(the latter equality is proved by linearizing (2.2) at $(\bar{u},0)$), we have instead

$$\lim_{\epsilon\to 0} R(\epsilon)=\pi, \qquad \text{and} \qquad \lim_{\epsilon\to\bar{u}} R(\epsilon)=\frac{2\pi}{\sqrt{n}}, \tag{2.8}$$

so that the period stays within a compact interval in the positive real axis as ϵ varies between its limits.

It is somewhat amusing that when $n=4$ we can solve (2.1) explicitly, once the transformation from t to r is effected; this was pointed out to us by R. Schoen. In fact, now $\partial_t = u\partial_r$ and (2.1) becomes

$$u\partial_r(u\partial_r u)-u+2u^3=0. \tag{2.9}$$

Setting $v=u\partial_r u$ as the new dependent variable and u as the independent variable we get that $v\partial_u v - u + 2u^3 = 0$. Integrating to solve for v, and then integrating again leads to a general expression for u. Adjusting the constants we finally get that

$$u_\epsilon(r) = \sqrt{\tfrac{1}{2} + (\tfrac{1}{2} - \epsilon^2)\cos(2r)}. \tag{2.10}$$

The parameterization has been arranged so that $u(\epsilon)$ attains its minimum when $r = \pi/2 + \ell\pi, \ell \in \mathbb{Z}$. Notice also that this shows that the period $R(\epsilon) \equiv \pi$ when $n = 4$.

It is perhaps also instructional, although not necessary within the context of this paper, to consider the space of solutions to the problem on the real projective space with one point removed, $\mathbb{R}P^n \backslash \{p\}$. We can identify any solution on this space with a solution on $\mathbb{S}^n \backslash \{p, -p\}$ invariant under the reflection which exchanges these two antipodal points. Transforming the twice-punctured sphere to the cylinder, we are looking for solutions u such that $u(t,\theta) = u(-t,-\theta)$. Any such solution is, of course, independent of θ and in the Delaunay family. For each value of the Delaunay parameter ϵ there are two possibilities, one taking its minimum value at $t = 0$ and the other taking its maximum value there. These are connected via the cylindrical solution by letting ϵ increase to $\bar{u}$. The moduli space is thus a copy of $\mathbb{R}$, with the cylindrical solution at the 'origin'. Alternatively, this moduli space is just the part of the u-axis in the set $H < 0$ in the (u, v)-plane. As ϵ tends to zero on one end of this moduli space, the solution tends to the round spherical metric. As ϵ tends to zero at the other end, the solution tends to zero.

Spectral theory for the Delaunay solutions. Although the Alexandrov reflection argument shows that we have already described the full moduli space $\mathcal{M}_\Lambda$ when Λ has only two elements, we proceed further here to analyze the linearization of the scalar curvature operator around a Delaunay solution g_ϵ. This case will serve as the model, and an important ingredient, for the more general linear analysis later. Thus, for v a suitably small function, let

$$\begin{aligned} N_\epsilon(v) &= \Delta_\epsilon(1+v) - \frac{n(n-2)}{4}(1+v) + \frac{n(n-2)}{4}(1+v)^{\frac{n+2}{n-2}} \\ &= \Delta_\epsilon v + nv + Q(v) \end{aligned} \tag{2.11}$$

where $Q(v)$ is the nonlinear, quadratically vanishing term

$$Q(v) = \frac{n(n-2)}{4}\left((1+v)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}v\right). \tag{2.12}$$

Solutions of $N_\epsilon(v) = 0$ correspond to other complete metrics on the cylinder with scalar curvature $n(n-1)$, and hence correspond to other Delaunay solutions. The linearization of N_ϵ about $v = 0$ is thus given by

$$\left.\frac{d}{d\sigma}N_\epsilon(\sigma\phi)\right|_{\sigma=0} = L_\epsilon\phi = \Delta_\epsilon\phi + n\phi. \tag{2.13}$$

Frequently we shall omit the ϵ-subscript from L_ϵ when the context is clear.

Using (2.6) and (2.13) and introducing an eigendecomposition $\{\psi_j, \lambda_j\}$ for Δ_θ on $\mathbb{S}^{n-1}$ we decompose L into a direct sum of ordinary differential operators L_j

with periodic coefficients on $\mathbb{R}$. The spectral analysis of each L_j, hence of L itself, is accomplished using Bloch wave theory, cf. [RS]. One conclusion is that $\mathrm{spec}(L)$ is purely absolutely continuous, with no singular continuous or point spectrum. Moreover, each L_j has spectrum arranged into bands, typically separated by gaps; L itself has spectrum which is the union of all of these band structures:

$$\mathrm{spec}(L) = \bigcup_j \mathrm{spec}(L_j).$$

From (2.13) it is clear that $\mathrm{spec}(-L)$ is bounded below by $-n$. We proceed to analyze this spectrum near 0.

The first question is to understand the Jacobi fields, i.e. elements of the nullspace of L. Any solution of $L\psi = 0$ may be decomposed into its eigencomponents

$$\psi = \sum a_j(t)\, \psi_j(\theta),$$

where the a_j solve $L_j(a_j) = 0$. Some of these solutions, although not necessarily all, may be obtained as derivatives of one-parameter families of solutions of $N(v) = 0$. It is common in geometric problems, cf. [KKS], for the Jacobi fields corresponding to low values of j to have explicit geometric interpretations as derivatives of special families of solutions of the nonlinear equation. This is the case here, at least for the eigenvalues $\lambda_0 = 0$ and $\lambda_1 = \cdots = \lambda_n = n-1$.

For $j = 0$ we look for families of solutions of $N(v) = 0$ which are independent of the $\mathbb{S}^{n-1}$ factor. There are two obvious examples, one corresponding to infinitesimal translations in t and the other corresponding to infinitesimal change of Delaunay parameter:

$$\begin{aligned} \eta &\to \frac{u_\epsilon(t+\eta)}{u_\epsilon(t)} - 1 \equiv \Phi_1(t,\epsilon;\eta) \text{ and} \\ \eta &\to \frac{u_{\epsilon+\eta}(t)}{u_\epsilon(t)} - 1 \equiv \Phi_2(t,\epsilon;\eta). \end{aligned} \tag{2.14}$$

Now define

$$\phi_j = \phi_{j,\epsilon} = \frac{d}{d\eta}\, \Phi_j(t,\epsilon;\eta)\big|_{\eta=0}\,, \quad j = 1,2, \tag{2.15}$$

so that $L_\epsilon \phi_j = 0,\ j = 1,2$. Of course, neither ϕ_1 nor ϕ_2 are in L^2 since L has no point spectrum. In fact, differentiating $u_\epsilon(t + T(\epsilon)) = u_\epsilon(t)$, first with respect to t and then with respect to ϵ, shows that

$$\phi_1(t+T(\epsilon)) = \phi_1(t), \qquad \phi_2(t+T(\epsilon)) + \phi_1(t)T'(\epsilon) = \phi_2(t). \tag{2.16}$$

The first of these equalities states that ϕ_1 is periodic while the second shows that ϕ_2 increases linearly, at least so long as $T'(\epsilon) \neq 0$. Hence the ϕ_j are generalized eigenfunctions for L with eigenvalue $\lambda = 0$; their existence and slow growth imply that 0 is in the essential spectrum of L_0, hence that of L. It is also not hard to see that ϕ_1 and ϕ_2 are linearly independent when $\epsilon \neq \bar{u}$. In fact, if we translate u_ϵ so that it attains a local maximum at $t = 0$, then $\phi_1(0) = 0$, whereas $\phi_2(0) =$

$(d\epsilon/d\epsilon)/\epsilon = 1/\epsilon$, so that these functions are not multiples of one another. When $\epsilon = \bar{u}$, $\phi_1 \equiv 0$, but ϕ_2 does not vanish since $\phi_2(0) = 1/\bar{u}$. A second solution of $L\phi = 0$ in this case is obtained by translating ϕ_2 by any non-integer multiple of its period. We shall prove below that these two functions form a basis, for each ϵ, for all temperate solutions of $L\phi = 0$ on the cylinder; any other solution of this equation must grow exponentially in one direction or the other.

To find Jacobi fields corresponding to the first nonzero eigenvalue of Δ_θ, it is easiest first to transform the background cylindrical metric to the round spherical metric. The conformal invariance of the equation means that one-parameter families of solutions may be obtained from one-parameter families of conformal maps of $\mathbb{S}^n$. For example, Φ_1 in (2.14) already corresponds to the composition of g_ϵ with the family of conformal dilations fixing the two singular points. We can also consider composition with a family of parabolic conformal transformations which fixes one or the other of the singular points. Derivatives of these families lead to the new Jacobi fields. To compute them we first transform the equation yet again so that the background metric is the flat Euclidean metric on $\mathbb{R}^n$ and the singular points are at 0 and ∞. The equation becomes

$$\Delta_{\mathbb{R}^n} w + \frac{n(n-2)}{4} w^{\frac{n+2}{n-2}} = 0, \tag{2.17}$$

where the function w on $\mathbb{R}^n$ is related to u on the cylinder by the transformation

$$u(t,\theta) = e^{(n-2)t/2}\, w(e^{-t}), \quad w(\rho,\theta) = \rho^{(2-n)/2}\, u(-\log\rho, \theta), \tag{2.18}$$

in terms of the polar coordinates (ρ, θ) on $\mathbb{R}^n$. The Delaunay solutions correspond to functions $w_\epsilon = w_\epsilon(\rho)$. A parabolic transformation of $\mathbb{S}^n$ fixing one of the singular points corresponds to a translation of $\mathbb{R}^n$ if that singular point corresponds to the point at infinity. Infinitesimal translation is just differentiation by one of the Euclidean coordinates x_j. But

$$\frac{\partial w_\epsilon}{\partial x_j} = \theta_j \frac{\partial w_\epsilon}{\partial \rho},$$

and $\theta_j = x_j/\rho$ is an eigenfunction for Δ_θ with eigenvalue $n-1$. So $\partial w_\epsilon(\rho)/\partial\rho$ is a solution of $L_1\phi = 0$, where L_1 is written in terms of ρ instead of t. Transforming back to the cylinder once again we arrive at the solution

$$\begin{aligned} \phi_3(t) &= e^t \left(\frac{u'(t)}{u(t)} + \frac{n-2}{2} \right) \\ &= e^t \left(\phi_1(t) + \frac{n-2}{2} \right). \end{aligned} \tag{2.19}$$

To obtain the other, exponentially decreasing, solution to $L_1\psi = 0$, we simply observe that inversion about the unit sphere in $\mathbb{R}^n$ corresponds to reflection in t about $t = 0$. If u_ϵ is positioned so that it assumes a maximum at $t = 0$, then $u_\epsilon(-t) = u_\epsilon(t)$, and we obtain from (2.19) the solution

$$\phi_4(t) = e^{-t} \left(\frac{n-2}{2} - \phi_1(t) \right). \tag{2.20}$$

The solutions $\phi_1, \ldots, \phi_4$ are also recorded and employed in [AKS].

It is also important to have some understanding of the continuous spectrum near 0; we do this for each L_j individually.

Lemma 2.21. *For every $\epsilon \in (0, \bar{u}]$ and $j \geq 1$, $0 \notin \mathrm{spec}(L_j)$. Hence the spectral analysis of L near the eigenvalue 0 reduces to that of L_0. The only temperate solutions of $L_0\phi = 0$ are precisely the linear combinations of the solutions ϕ_1 and ϕ_2 of (2.15).*

Proof. The eigenvalues of Δ_θ are $\lambda_0 = 0$, $\lambda_1 = 1 - n$, etc. For $j > n$ (numbering the eigenvalues with multiplicity, as usual) $\lambda_j < -n - 1$. Thus

$$\begin{aligned} L_j &= u^{-\frac{4}{n-2}}\partial_t^2 + 2u^{-\frac{n+2}{n-2}}u_t\partial_t + u^{-\frac{4}{n-2}}\lambda_j + n \\ &< u^{-\frac{4}{n-2}}\partial_t^2 + 2u^{-\frac{n+2}{n-2}}u_t\partial_t + n(1 - u^{-\frac{4}{n-2}}) - u^{-\frac{4}{n-2}}. \end{aligned} \tag{2.22}$$

Since $0 < u \leq 1$ for every ϵ and t, the term of order zero in this expression is always strictly negative. Standard ODE comparison theory now implies that an arbitrary solution of $L_j\phi = 0$ must grow exponentially either as $t \to +\infty$ or $-\infty$. Clearly then $0 \notin \mathrm{spec}(L_j)$ for $j > 1$.

The same conclusion is somewhat more subtle when $j = 1$. First of all, observe that because ϕ_3 and ϕ_4 above span all solutions to the ODE $L_1\psi = 0$, and since both of these functions grow exponentially in one direction or the other, it is evident that 0 is not in the spectrum of L_1. We wish to know, though, that 0 is actually below all of the spectrum of $-L_1$. This is no longer obvious since the term of order zero in

$$L_1 = u^{-\frac{4}{n-2}}\partial_t^2 + 2u^{-\frac{n+2}{n-2}}u_t\partial_t + (1-n)u^{-\frac{4}{n-2}} + n \tag{2.23}$$

is no longer strictly negative. Conjugate L_1 by the function u^p, where p is to be chosen. A simple calculation gives

$$u^{-p}L_1u^p = u^{-\frac{4}{n-2}}\partial_t^2 + (2p+2)uu_t\partial_t + A,$$

where

$$A = u^{-\frac{2n}{n-2}}\left\{p(p+1)u_t^2 + \left(\frac{p(n-2)^2}{4} + 1 - n\right)u^2 + \left(n - \frac{pn(n-2)}{4}\right)u^{\frac{2n}{n-2}}\right\}.$$

Somewhat miraculously, upon setting $p = 2/(n-2)$ this expression reduces to

$$\begin{aligned} A &= \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\left(u_t^2 - \frac{(n-2)^2}{4}u^2 + \frac{(n-2)^2}{4}u^{\frac{2n}{n-2}}\right) \\ &= \frac{4n}{(n-2)^2}u^{-\frac{2n}{n-2}}H(\epsilon), \end{aligned}$$

where $H(\epsilon)$ is the Hamiltonian energy (2.3) of the solution $u_\epsilon(t)$. This is a negative constant for every $\epsilon \in (0, \bar{u}]$, and so once again we have obtained an ordinary differential operator with strictly negative term of order zero. $u^{-2/(n-2)}L_1u^{2/(n-2)}$ is unitarily equivalent to L_1, and by the previous argument, the spectrum of the first of these operators is strictly negative, and in particular does not contain 0. Hence the same is true for L_1 and the proof is complete.

We can say slightly more about the spectra of the L_j. In fact, when $\epsilon = \bar{u}$,

$$L_j = \frac{n}{n-2}\left(\partial_t^2 + \lambda_j\right) + n,$$

and

$$\mathrm{spec}(-L_j) = \left[-n\left(1 + \frac{\lambda_j}{n-2}\right), \infty\right).$$

In particular, $\mathrm{spec}(-L_0) = [-n, \infty)$ and $\mathrm{spec}(-L_1) = [n/(n-2), \infty)$. The spectrum of $-L_{\bar{u}}$ is the union of these infinite rays. As ϵ decreases, gaps appear in these rays, and bands start to form (in fact, by (2.8) the first gap appears at $-3n/4$). As ϵ decreases, the first band extends from $-n$ to some value to the left of $-3n/4$; the second band begins somewhere to the right of this point, and always ends at 0. It is not important for our later work that 0 is always on the end of the second band (or at least, at a point where the second Bloch band function has a turning point), but it is rather amusing that we may determine this. The explanation is that from (2.16), whenever $T'(\epsilon) \neq 0$ (which certainly happens for almost every $\epsilon < \bar{u}$), the Jacobi field ϕ_2 grows linearly. This signifies that 0 is at the end of a band, or at the very least, at a turning point for one of the Bloch band functions for L_0 which parameterize the bands of continuous spectrum. As ϵ continues to decrease to 0 these bands shrink further and further. It is more revealing to consider the operator L_ϵ written in the geodesic coordinates (r, θ) as in (2.7). As noted earlier, as $\epsilon \to 0$ this operator remains periodic, but develops singularities every π units. Geometrically, the metrics g_ϵ are converging to an infinite bead of spheres of fixed scalar curvature $n(n-1)$. The spectrum of $-L$ for this limiting metric is a countable union of the spectrum of $-\Delta - n$ on one of these spheres, i.e. consists of a countable number of isolated points $\{-n, 0, n+2, \dots\}$, each with infinite multiplicity. The bands of continuous spectrum have coalesced into these infinite-multiplicity eigenvalues. Note that in this limit, the infimum of $\mathrm{spec}(L_1)$ has increased to zero at a rate which may be estimated by a power of ϵ.

3. Generalities on solutions with isolated singularities

In this section we collect various results concerning solutions of the singular Yamabe problem with isolated singularities, particularly in the context of conformally flat metrics, which will be required later.

Asymptotics. The Delaunay solutions discussed in the last section are interesting explicit solutions in their own right. However, their importance is due to the fact that they are asymptotic models for arbitrary solutions of the singular Yamabe problem, at least in the conformally flat setting. Before stating this result more carefully we introduce some notation. The half-cylinder $(0, \infty)_t \times \mathbb{S}^{n-1}_\theta$ is conformally equivalent to the punctured ball $\mathbb{B}^n \backslash \{0\}$; an explicit conformal map is given by sending (t, θ) to the polar coordinates (ρ, θ), where $\rho = e^{-t}$. A Delaunay solution $u_\epsilon(t)$ may be transformed to the function

$$\tilde{u}_\epsilon(\rho) = \rho^{\frac{2-n}{2}} u_\epsilon(-\log \rho). \tag{3.1}$$

Using the conformal invariance of the conformal Laplacian, this function solves

$$\Delta \tilde{u} + \frac{n(n-2)}{4} \tilde{u}^{\frac{n+2}{n-2}} = 0. \tag{3.2}$$

Theorem 3.3. *Let $\tilde{u} \in \mathcal{C}^\infty(\mathbb{B}^n \backslash \{0\})$ be a positive solution of equation (3.2) with a nonremovable singularity at the origin. Then there exists some $\epsilon \in (0, \bar{u}]$ and $A > 0$ so that $\tilde{u}$ is asymptotic to the modified Delaunay solution $\tilde{u}_\epsilon$ in the sense that*

$$\tilde{u}(\rho, \theta) = (1 + O(\rho^\alpha))\tilde{u}_\epsilon(A\rho, \theta), \tag{3.4}$$

for some $\alpha > 0$; A corresponds to a translation of the t parameter. The analogous estimate still holds whenever $\tilde{u}$ and $\tilde{u}_\epsilon$ are replaced by any of their derivatives $(\rho\partial_\rho)^j \partial_\theta^\beta$.

This result was proved by Caffarelli, Gidas and Spruck [CGS] using a fairly general and complicated form of the Alexandrov reflection argument. They actually only give a somewhat weaker estimate where $O(\rho^\alpha)$ is replaced by $o(1)$. An alternate argument, relying on more direct geometric and barrier methods, was obtained at around the same time by Aviles, Korevaar and Schoen in the unpublished work [AKS], and they obtained the stronger form. This last work is very close in method to the proof of the analogous result for complete constant mean curvature surfaces embedded in $\mathbb{R}^3$ presented in [KKS]. It is possible to calculate the decay rate α in terms of spectral data of the operator L_ϵ, and we shall indicate this argument in the next section. To our knowledge it is unknown whether some form of this result has been proved when the background metric is not conformally flat, but it seems likely to be true.

Note that the estimate (3.4) may be restated for the transformed function $u(t, \theta) = e^{-(n-2)t/2}\tilde{u}(e^{-t}, \theta)$ on the cylinder. Now

$$u(t, \theta) = (1 + O(e^{-\alpha t}))u_\epsilon(t). \tag{3.5}$$

This is the form we shall use. It states that an arbitrary solution on the half-cylinder converges exponentially to a Delaunay solution.

Pohožaev invariants. A key ingredient in Schoen's construction [S2] of solutions of (1.3) with isolated singularities is his use of balancing conditions for approximate solutions. These conditions follow from the general Pohožaev identity proved in [S2]. In the present setting, when (Ω, g) is a compact, conformally flat manifold with boundary, T is the trace-free Ricci tensor of g, X is a conformal Killing field on Ω, and $R(g)$ is constant, then

$$\int_{\partial\Omega} T(X, \nu)\, d\sigma = 0. \tag{3.6}$$

Here ν is the outward unit normal to $\partial\Omega$, and $d\sigma$ is surface measure along this boundary. There is a more general formula involving an integral over the interior of Ω when the scalar curvature is not constant, or when X is not conformal Killing.

When $\Omega \Subset \mathbb{S}^n \backslash \{p_1, \dots, p_k\}$, there are many conformal Killing fields to use. An important class of these are the 'centered dilations'. Any such X is equal to the gradient of the restriction of a linear function $\ell(q) = \langle q, v\rangle$, where $v \in \mathbb{R}^{n+1}$ and $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the standard embedding. Explicitly,

$$X_q = v - \langle q, v\rangle q, \qquad q \in \mathbb{S}^n, \tag{3.7}$$

since the unit normal to $T_q\mathbb{S}^n$ is just q. Let $\mathfrak{c}$ denote the set of all such X. It is a subspace of the Lie algebra, $\mathfrak{o}(n+1,1)$, of the conformal group $O(n+1,1)$.

By (3.6), each $X \in \mathfrak{o}(n+1,1)$ determines an element of $H^{n-1}(\mathbb{S}^n\backslash\Lambda, \mathbb{R})$, associating to a hypersurface Σ the number

$$\mathcal{P}(\Sigma, X; g) = \int_\Sigma T(X,\nu)\, d\sigma. \tag{3.8}$$

The dual homology space is generated by the classes of hyperspheres Σ_i, where, for each $p_i \in \Lambda$, $\Sigma_i = \partial B_r(p_i)$ for r sufficiently small, is chosen so that no other p_j is in the same component of $\mathbb{S}^n\backslash\Sigma_i$ as p_i. These classes satisfy the single homology relation $[\Sigma_1] + \cdots + [\Sigma_k] = 0$. The number in (3.8) will be written $\mathcal{P}_i(X; g)$ (or simply $\mathcal{P}_i(X)$) when $\Sigma = \Sigma_i$. Alternately, as suggested by this notation, we shall also regard each $\mathcal{P}_i$ as a linear functional on the X's, i.e. as an element of $\mathfrak{o}^*(n+1,1)$. These functionals will be called the Pohožaev invariants of the solution metric g. They satisfy

$$\mathcal{P}_1 + \cdots + \mathcal{P}_k = 0. \tag{3.9}$$

Theorem 3.3 makes it possible to calculate at least some components of the $\mathcal{P}_i$, by computing the Pohožaev invariants for the Delaunay solutions; we do this now. Since $H^{n-1}(\mathbb{S}^n\backslash\{p_1, -p_1\}, \mathbb{R}) \equiv \mathbb{R}$, it suffices to compute the single number $\mathcal{P}_1(X)$ for each $X \in \mathfrak{c}$. As described above, X may be identified with an element $v \in \mathbb{R}^{n+1}$. A straightforward calculation, similar to one given in [P2], shows that this invariant is, up to a constant, simply the Hamiltonian energy $H(\epsilon)$ of the particular Delaunay solution:

Lemma 3.10. *Using the identification of $X \in \mathfrak{c}$ with $v \in \mathbb{R}^{n+1}$, the Pohožaev invariant for the Delaunay metric g_ϵ on $\mathbb{S}^n\backslash\{p_1, -p_1\}$ is given by*

$$\mathcal{P}_1(X) = c_n\, H(\epsilon)\, \langle v, p_1\rangle.$$

Here c_n is a non-vanishing dimensional constant (identified explicitly in [P2]).

The asymptotics in Theorem 3.3 (even with the sharp estimate of α we will give later) do not give enough information for the invariant $\mathcal{P}_i$ to be computed for every X; the point is that if X has associated vector v perpendicular to p_i then Lemma 3.10 shows that there is no 'formal' asymptotic contribution to the invariant, but unfortunately the decay is not sufficient for there to be no 'perturbation' contribution. The one case where this is not an issue is when X is chosen to have associated vector exactly p_i. The associated invariant $\mathcal{P}_i(X)$ in this case will be called the *dilational* Pohožaev invariant and denoted $\mathcal{D}_i$ (or $\mathcal{D}_i(g)$).

Corollary 3.11 ([P2]). *The dilational Pohožaev invariant $\mathcal{D}_i(g)$ associated to the puncture p_i and the solution metric g is equal to $c_n\, H(\epsilon_i)$, where ϵ_i is the Delaunay parameter giving the asymptotic model for the metric g at p_i, as provided by Theorem 3.3.*

Local compactness properties of the moduli space. It is natural to determine the possible ways that solution metrics g of our problem can degenerate. Phrased more geometrically, we wish to determine what the ends of the moduli space $\mathcal{M}_\Lambda$

look like, and to determine a geometrically natural compactification. Any $\mathcal{M}_\Lambda$, when Λ contains just two elements, may be identified with any other, and we may call this space simply $\mathcal{M}_2$. In this space, all $g \in \mathcal{M}_2$ are Delaunay, and indeed $\mathcal{M}_2$ may be identified with the open set $\Omega \subset \mathbb{R}^2$, $\Omega = \{H < 0\}$ as described in §2. This space is two dimensional, smooth, and locally compact. A sequence of elements in $\mathcal{M}_2$ degenerate only when the Delaunay parameter ϵ (or 'neck-size' of the solutions) tends to zero. Recalling that the Hamiltonian energy H is strictly monotone decreasing for $\epsilon \in (0, \bar{u}]$, this may be restated as saying that Delaunay solutions degenerate only when $H(\epsilon) \to 0$.

A similar statement is true when Λ has more than two elements. As we have seen in Corollary 3.11, the Pohožaev invariants $\mathcal{P}_i$, and indeed just the dilational Pohožaev invariants $\mathcal{D}_i$, determine the Hamiltonian energies of the model Delaunay solutions for the metric g at each puncture p_i. Clearly, then, if there exists some particular p_i such that for a sequence of solution metrics $g_j \in \mathcal{M}_\Lambda$ the dilational Pohožaev invariant $\mathcal{D}_i(g_j)$ tends to zero as $j \to \infty$, then this sequence should be regarded as divergent in $\mathcal{M}_\Lambda$. The complementary statement, generalizing the situation for $\mathcal{M}_2$, is also still valid, and was proved by Pollack [P2]:

Proposition 3.12. *Let g_j be a sequence of metrics in $\mathcal{M}_\Lambda$, such that for each $i = 1, \dots, k$ the dilational Pohožaev invariants $\mathcal{D}_i(g_j)$ are bounded away from zero. Then there is a subsequence of the g_j converging to a metric $\bar{g} \in \mathcal{M}_\Lambda$. The convergence is uniform in the $\mathcal{C}^\infty$ topology relative to $\bar{g}$, or indeed relative to any of the g_j, on compact subsets of $\mathbb{S}^n \backslash \Lambda$.*

The question of what happens to divergent sequences in $\mathcal{M}_\Lambda$ may also be determined. Before describing this we return to $\mathcal{M}_2$. There is an 'obvious' compactification, $\overline{\mathcal{M}_2}$, which is the closure of Ω in $\mathbb{R}^2$. $\overline{\mathcal{M}_2} \backslash \mathcal{M}_2$ decomposes into two disjoint sets: the first contains the single point $\{0, 0\} \in \mathbb{R}^2$, which corresponds to the conformal factor 0, while the second is the orbit $\{u_0(t), v_0(t)\}$ passing through $\{1, 0\}$, which corresponds to the incomplete spherical metric on $\mathbb{S}^n \backslash \{p_1, -p_1\}$. These latter points on the compactification of $\mathcal{M}_2$ may be identified with the nonsingular round metric on $\mathbb{S}^n$ itself, and these are themselves metrics of constant scalar curvature $n(n-1)$ on $\mathbb{S}^n$. $\overline{\mathcal{M}_2}$ is a stratified space; its principal stratum is $\mathcal{M}_2$ itself, the codimension one (and one dimensional) stratum consists of copies of the nonsingular round metric on $\mathbb{S}^n$, and finally the codimension two (zero dimensional) stratum consists of the single trivial solution of the PDE, which is a completely degenerate metric.

There is a similar compactification of $\mathcal{M}_\Lambda$ and corresponding decomposition of $\overline{\mathcal{M}_\Lambda}$ when the cardinality of Λ is larger than two (and still finite). The following is a corollary of the proof of Proposition 3.12:

Corollary 3.13. *Let $g_j \in \mathcal{M}_\Lambda$ be a sequence such that for some nonempty subset of points $p_{i_1}, \dots, p_{i_s}$, the invariants $\mathcal{D}_{i_1}(g_j), \dots, \mathcal{D}_{i_s}(g_j)$ are uniformly bounded above by some $-\eta_0 < 0$ for all j, and all other $\mathcal{D}_i(g_j)$ tend to zero. Then this sequence has a convergent subsequence converging to a metric $\bar{g} \in \mathcal{M}_{\Lambda'}$, i.e. $\bar{g}$ is still singular at the points $\Lambda' \equiv \{p_{j_1}, \dots, p_{j_s}\}$, but extends smoothly across the points in $\Lambda \backslash \Lambda'$ and $R(\bar{g}) = n(n-1)$. If all $\mathcal{D}(g_i, p_j)$ tend to zero as $i \to \infty$ then either g_i tends to zero, uniformly on compact subsets of $S^n \setminus \Lambda$, or else g_i converges to the round metric on S^n.*

This means that the compactification $\overline{\mathcal{M}_\Lambda}$ contains copies of $\mathcal{M}_{\Lambda'}$ for certain subsets $\Lambda' \subset \Lambda$. In addition it may contain a set whose points correspond to copies of the round metric on $\mathbb{S}^n$, and finally a set whose points correspond to the 'zero metric'. Later in this paper we shall give a somewhat better description of $\overline{\mathcal{M}_\Lambda}$ once we have determined the structure of $\mathcal{M}_\Lambda$ itself better. Notice also that it must be somewhat subtle to determine precisely which subsets $\Lambda' \subset \Lambda$ have $\mathcal{M}_{\Lambda'}$ occurring in $\overline{\mathcal{M}_\Lambda}$. The subsets Λ' which can arise in the description above are determined by the Pohožaev balancing condition (3.9). For example, when Λ has two elements, there is no piece of the boundary corresponding to $\mathcal{M}_{\Lambda'}$ for Λ' having just one element; this is because no complete solution of our problem exists on $\mathbb{S}^n \backslash \{p\}$. From the present perspective, this holds because $\mathcal{D}_1(g_j) - \mathcal{D}_2(g_j) = 0$ for any j (here $\Lambda = \{p_1, p_2\}$) by virtue of (3.9), so that if one of these numbers tends to zero, the other must as well. (The difference in the signs here from (3.9) is because in (3.9) the same conformal Killing field X is used in each Pohožaev invariant, whereas here we use X corresponding to p_1 for one and $-X$ corresponding to $p_2 = -p_1$ for the other.) The subtle point is that the complete Pohožaev invariants, rather than just the dilational ones, are required when Λ has more than two points, and these are not determined just linear algebraically by the location of the p_i and the Delaunay models at these punctures.

4. Linear analysis on manifolds with asymptotically periodic ends

In this section we prove various results concerning the analysis of the Laplacian and the linearization of the scalar curvature operator about solution metrics $g \in \mathcal{M}_\Lambda$. Many of the basic results hold more generally, e.g. for the Laplacian on manifolds with asymptotically periodic ends.

The linearized operator. For the remainder of this paper, L will always denote the linearization about the constant solution $v = 0$ of the nonlinear operator

$$
\begin{aligned}
N(v) &= \Delta_g(1+v) - \frac{n(n-2)}{4}(1+v) + \frac{n(n-2)}{4}(1+v)^{\frac{n+2}{n-2}} \\
&= \Delta_g v + nv + Q(v),
\end{aligned} \tag{4.1}
$$

where $Q(v)$ is the same as in (2.12), so that, as before,

$$
L = \Delta_g + n. \tag{4.2}
$$

Let E_j denote a neighborhood of the puncture p_j which is conformally equivalent to a half-cylinder $[0, \infty)_t \times \mathbb{S}^{n-1}_\theta$. We fix these cylindrical coordinates around each p_j. By virtue of Theorem 3.3, L can be treated on each E_j as an exponentially small perturbation of the corresponding operator L_{ϵ_j} for the periodic Delaunay metric g_{ϵ_j} which is the asymptotic model for g on E_j. Thus on each E_j we may write

$$
L = L_{\epsilon_j} + e^{-\alpha t} F, \tag{4.3}
$$

where F is a second order operator with coefficients bounded in $\mathcal{C}^\infty$ as functions of (t, θ).

The linear analysis of the Laplacian on manifolds with asymptotically periodic ends is remarkably similar to that for manifolds with asymptotically cylindrical ends, as detailed for example in [Me]. In particular, the Fredholm theory for such an operator on exponentially weighted Sobolev (or Hölder or ...) spaces has an almost identical statement in either case, although the proofs are rather different. The Fredholm theory in this asymptotically periodic setting was previously developed by Taubes in [T], although we proceed somewhat further into the linear analysis as we need more detailed results. It is also possible to develop a full scattering theory for the Laplacian on these manifolds. We shall actually require and develop some scattering theoretic results to clarify the nature of the moduli space $\mathcal{M}_\Lambda$.

The Fourier-Laplace transform. The basic tool for the parametrix construction, upon which all the linear analysis relies, is the Fourier-Laplace transform, in a form employed already by Taubes [T]. We proceed to develop some properties of this transform, here denoted $\mathcal{F}$.

The function spaces we shall use here are exponentially weighted Sobolev spaces based on $L^2(\mathbb{S}^n\backslash\Lambda; dV_g)$; these will be written $H^s_\gamma(\mathbb{S}^n\backslash\Lambda)$, or just H^s_γ, for $\gamma, s \in \mathbb{R}$, $s > n/2$. The last condition ensures that the spaces behave well under nonlinear operations. To define them, decompose $\mathbb{S}^n\backslash\Lambda$ into the union of the ends $E_1, \dots, E_k$ and a compact piece K. Over K an element $h \in H^s_\gamma$ restricts to an ordinary H^s function. Over E_j, $h = e^{\gamma t}\tilde{h}$, where $\tilde{h} \in H^s([0,\infty) \times \mathbb{S}^{n-1}, dt\, d\theta)$. Note that the measure here is uniformly equivalent to the one induced by g for any $g \in \mathcal{M}_\Lambda$.

The transform $\mathcal{F}$ is, strictly speaking, defined for functions on the whole cylinder $C = \mathbb{R} \times \mathbb{S}^{n-1}$. It would be somewhat more natural to first develop its properties acting on e.g. the Schwartz space $\mathcal{S}$, but we shall specialize immediately to functions with support on just one half of C. So let $h(t,\theta) \in H^s_\gamma$ on E_j, and assume $h = 0$ for $t \leq 1$. Set

$$\hat{h}(t,\zeta,\theta) = \mathcal{F}(h) = \sum_{k=-\infty}^{\infty} e^{-ik\zeta} h(t+k,\theta). \tag{4.4}$$

Assume for the moment that h is smooth. Then, since h decays like $e^{\gamma t}$, this series converges provided $\Im\zeta \equiv \nu < -\gamma$. We have set $\zeta = \mu + i\nu$, so that $\Re\zeta = \mu$. $\hat{h}(t,\zeta,\theta)$ depends holomorphically on ζ in the region $\nu < -\gamma$. When h is only assumed to be in H^s_γ, $\hat{h}(t,\zeta,\theta)$ will still depend holomorphically on ζ in the same region, but as a function with values in the space H^s. $\hat{h}$ is continuous in $\nu \leq -\gamma$ as a function with values in H^s. These results follow from the Plancherel formalism.

The transform $\mathcal{F}$ is invertible, and its inverse is given by contour integration. To make the following equation clearer, assume that $t \in \mathbb{R}$, and $\tilde{t}$ is its reduction mod 1 (so that $0 \leq \tilde{t} < 1$). Then when $\ell \leq t < \ell + 1$ so that we may write $t = \tilde{t} + \ell$,

$$h(t,\theta) = \frac{1}{2\pi} \int_{\mu=0}^{2\pi} e^{i\ell\zeta} \hat{h}(\tilde{t},\zeta,\theta) d\mu. \tag{4.5}$$

In this formula we integrate along a line $\Im\zeta = \nu_0$. By Cauchy's theorem this contour may be shifted to allow ν_0 to be any number less than $-\gamma$. If, as we are assuming, h vanishes for $t < 0$, then $\hat{h}(\tilde{t},\zeta,\theta)$ not only extends to the lower half ζ plane, but decays like e^{ν} there. Shifting the contour arbitrarily far down shows that the integral (4.5) vanishes for any $\ell < 0$, as it should. By a similar argument, if $h(t,\theta)$ is defined by the integral (4.5) taken along some contour $\Im\zeta = \nu_0$, where the integrand $\hat{h}(\tilde{t},\zeta,\theta)$ is only assumed to be defined along that line, then $h \in H^s_{-\nu_0}$.

In particular, if $\hat{h}$ is holomorphic in some lower half plane $\nu < -\gamma$, and continuous with values in H^s as a function of $\tilde{t}$ up to this upper boundary, then $h \in H^s_\gamma$.

Next, reindexing the sum defining $\hat{h}$ gives

$$\hat{h}(t+1,\zeta,\theta) = e^{i\zeta}\hat{h}(t,\zeta,\theta). \tag{4.6}$$

This just means that $\hat{h}(t,\zeta,\theta)$ is a section of the flat bundle on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with holonomy ζ around the $\mathbb{S}^1$ loop. This bundle is isomorphic to the flat bundle with trivial holonomy; the bundle map is given by conjugating by $e^{i\zeta t}$. Thus the function

$$\tilde{h} \equiv e^{-i\zeta t}\hat{h}(t,\zeta,\theta)e^{i\zeta t} \tag{4.7}$$

satisfies $\tilde{h}(t+1,\zeta,\theta) = \tilde{h}(t,\zeta,\theta)$.

Fredholm theory. The basic Fredholm result for the linearization L may now be stated and proved.

Proposition 4.8. *There exists a discrete set of numbers $\Gamma \subset \mathbb{R}$ such that the bounded operator*

$$L : H^{s+2}_\gamma(M) \longrightarrow H^s_\gamma(M) \tag{4.9}$$

is Fredholm for all values of the weight parameter $\gamma \notin \Gamma$. In particular, $0 \in \Gamma$, so the map (4.9) is not Fredholm when $\gamma = 0$, i.e. on the ordinary unweighted Sobolev spaces, but is Fredholm for all values of γ sufficiently near, but not equal to zero.

Proof. It suffices to construct a parametrix G for L so that $LG - I$ and $GL - I$ are compact operators on H^s_γ and H^{s+2}_γ, respectively. As usual, G may be constructed separately on each piece of the decomposition of $\mathbb{S}^n \backslash \Lambda$ into a compact piece and the k ends $E_1, \dots, E_k$ around each p_j. The parametrix construction on the compact piece is the standard microlocal one since L is elliptic. We construct a parametrix on each E_j using the Fourier-Laplace transform.

Fix one of these ends, E_j, and let g_ϵ be the model asymptotic Delaunay metric for the fixed metric g there. The corresponding model operator L_ϵ, as in (4.3), has periodic coefficients (for notational convenience we assume that the period is one here) so it acts on sections of the flat bundle with holonomy ζ described above by the obvious rule $\widehat{(L_\epsilon h)} \equiv \hat{L}_\epsilon \hat{h}$. This induced operator $\hat{L}_\epsilon$ looks just like L_ϵ in local coordinates (t,θ). This step is the same as conjugating L_ϵ by $\mathcal{F}$. We may proceed further and conjugate $\hat{L}_\epsilon$ by $e^{i\zeta t}$ so as to act on the trivial flat bundle. This final induced operator, which depends holomorphically on ζ, will be called $\tilde{L}_\epsilon(\zeta)$:

$$\tilde{L}_\epsilon(\zeta) = e^{-i\zeta t}L_\epsilon e^{i\zeta t}. \tag{4.10}$$

The main point of the proof is that $\tilde{L}_\epsilon(\zeta)$ has an inverse (say on $L^2(\mathbb{S}^1 \times \mathbb{S}^{n-1})$) which depends meromorphically on ζ. This will be a direct consequence of the analytic Fredholm theorem, which is proved for example in [RS] and states that a strongly holomorphic family of Fredholm operators, depending on the complex variable ζ, either fails to be invertible for every ζ, or else is invertible for all ζ except for those in some discrete set in the parameter space. To check that this result is applicable, simply note that $\tilde{L}_\epsilon(\zeta)$ is elliptic for every ζ, (as already observed) depends holomorphically on ζ and, by the various transformations we performed, acts on a *fixed* function space (on a fixed bundle) over a compact manifold. Hence it forms a family of Fredholm operators; standard considerations show that the holomorphic dependence of the coefficients of this operator on ζ ensure the strong holomorphy of the family.

Our next task is to show that $\tilde{L}_\epsilon(\zeta)$ is invertible for some value of ζ. Once this is accomplished, the analytic Fredholm theorem will imply that the set of poles $\{\zeta_j\}$ of $\tilde{L}_\epsilon(\zeta)^{-1}$ is discrete in $\mathbb{C}$. The invertibility of $\tilde{L}_\epsilon(\zeta)$ at some ζ is equivalent to the invertibility of $\hat{L}_\epsilon$ acting on the flat bundle with holonomy ζ. When ζ is real, this operator is self-adjoint, hence invertible so long as it has no nullspace. A solution of $\hat{L}_\epsilon\psi = 0$ on this bundle lifts to a function $\psi(t,\theta)$ on $\mathbb{R}\times\mathbb{S}^{n-1}$ satisfying $\psi(t+1,\theta) = e^{i\zeta}\psi(t,\theta)$. Since ζ is real here, this lift is bounded and 'quasi-periodic.' However, we have already shown in §2 that such a function can exist only when $\zeta = 0$; in fact, the arguments there show that any temperate solution must be constant on the cross-section $\mathbb{S}^{n-1}$, i.e. be a solution of the reduced operator L_0. In addition, the functions ϕ_1 and ϕ_2 of (2.15) are independent solutions of this operator, hence span the space of all solutions since L_0 is a second order ODE. Finally, neither of these transform by a factor $e^{it\zeta}$ over a period except for $\zeta = 0$. Thus $\tilde{L}_\epsilon$ is invertible for every real $\zeta \in (0, 2\pi)$. This proves that $\tilde{L}_\epsilon(\zeta)^{-1}$ exists and depends meromorphically on ζ.

We make some remarks about the set of poles $\{\zeta_j\}$ of this meromorphic family of inverses. First, if ζ is in this set, then so is $\zeta + 2\pi\ell$ for any $\ell \in \mathbb{Z}$. In fact, although $\tilde{L}_\epsilon(\zeta + 2\pi)$ is not equal to $\tilde{L}_\epsilon(\zeta)$, these two operators are unitarily equivalent: the unitary operator intertwining them is multiplication by the function $e^{2\pi it}$. So, for some fixed ζ, either both of these operators are invertible or neither of them is. By discreteness and this translation invariance, the set $\{\zeta_j\}$ has at most finitely many inequivalent (mod 2π) poles in any horizontal strip $a < \Im\zeta < b$. Note in particular that the only poles on the real line occur at the points in $2\pi\mathbb{Z}$. By discreteness again, there are no other poles in some strip $-\epsilon < \Im\zeta < \epsilon, \Im\zeta \neq 0$. We shall usually restrict attention then only to values of ζ in the strip $0 \leq \zeta < 2\pi$ (or equivalently, in $\mathbb{C}/2\pi\mathbb{Z}$); $\mathcal{P}$ will denote the set of poles in this strip. Finally, note that the adjoint of $\tilde{L}_\epsilon(\zeta)$ is precisely $\tilde{L}_\epsilon(\bar{\zeta})$, and so $\mathcal{P}$ is invariant under conjugation. This means that we can list the elements of $\mathcal{P}$ as follows:

$$\mathcal{P} = \mathcal{P}_\epsilon = \{\dots, -\zeta_2, -\zeta_1, \zeta_0 = 0, \zeta_1, \zeta_2, \dots\}. \tag{4.11}$$

Also define

$$\Gamma = \Gamma_\epsilon = \{\gamma_j = \Im\zeta_j : \zeta_j \in \mathcal{P}\} = \{\dots, -\gamma_1, 0, \gamma_1, \dots\}. \tag{4.12}$$

The γ_j form a strictly increasing sequence tending to infinity, and in particular, Γ is a discrete set in $\mathbb{R}$. We shall provide an interpretation of the nonzero elements of $\mathcal{P}$ in the next subsection.

Next we unwind this inverse $\tilde{L}_\epsilon(\zeta)^{-1} \equiv \tilde{G}_\epsilon(\zeta)$ to obtain a parametrix for L_ϵ on E_j. The inverse for $\hat{L}_\epsilon$ acting on the flat bundle with holonomy ζ is given by $\hat{G} = e^{it\zeta}\tilde{G}e^{-it\zeta}$, for every $\zeta \notin \mathcal{P}$. To obtain an inverse for L_ϵ we simply conjugate $\hat{G}$ with the Fourier-Laplace transform $\mathcal{F}$. Recall, though, that this makes sense only if the contour integral in the definition of $\mathcal{F}^{-1}$ avoids $\mathcal{P}$. So, by conjugating, and in doing so, taking this integral along a contour $\Im\zeta = -\gamma$, for any $\gamma \notin \Gamma$, we obtain an inverse for L_ϵ. Since integrating along this contour produces a function in $H^s_\gamma(\mathbb{R}\times\mathbb{S}^{n-1})$, we have obtained an inverse $G = G_{\epsilon,\gamma}$ for L_ϵ acting on H^s_γ functions on E_j, supported in $t \geq 1$ for all $\gamma \notin \Gamma$. Clearly this G is a parametrix with compact remainder for $L = L_\epsilon + e^{-\alpha t}F$ acting on $H^s_\gamma(E_j)$.

As a final step, couple these parametrices we have produced on each end E_j to the interior parametrix to obtain a global parametrix for L on all of $\mathbb{S}^n\backslash\Lambda$ with compact remainder. Both right and left parametrices of this type may be obtained this way, and their existence implies immediately that L is Fredholm on H^s_γ whenever $\gamma \notin \Gamma$. It is a simple exercise to check that L does not even have closed range when $\gamma \in \Gamma$, in particular, when $\gamma = 0$.

It is obviously of interest to determine when L is actually injective or surjective. In general this is a rather complicated question, but we make note of the following result.

Corollary 4.13. *Suppose that L has no L^2 nullspace (i.e. is injective on H^s_γ for $\gamma \leq 0$), then for all $\delta > 0$ sufficiently small (with $\delta < \alpha$)*

$$L : H^{s+2}_\delta \longrightarrow H^s_\delta \quad \textit{is surjective,}$$
$$L : H^{s+2}_{-\delta} \longrightarrow H^s_{-\delta} \quad \textit{is injective.}$$

This first statement follows from the second and duality, since L is self-adjoint on L^2, i.e. when $\gamma = 0$.

Asymptotic expansions. The Fourier-Laplace transform may also be used to obtain existence of asymptotic expansions for solutions to $Lw = 0$ on each end E_j. For our purposes in this paper, it will only be important to know that any such w has a leading term in its expansion, which decays (or grows) at some specified rate, and an error which decays at a faster rate. One use of this will be to estimate the exponent α appearing in (3.5). However, the full expansion is not much harder to prove, so we will do this too.

The starting point is the fact, discussed above, that the Fourier-Laplace transform of a function $w \in H^s_\gamma(\mathbb{R}\times\mathbb{S}^{n-1})$, supported in $t \geq 0$, say, is a function $\hat{w}(t,\theta,\zeta)$ which is holomorphic in the half-plane $\Im\zeta < -\gamma$ (and taking values in $H^s(\mathbb{S}^1\times\mathbb{S}^{n-1})$). For a general function of this type, the most that can be concluded is that it has a limit on the line $\Im\zeta = -\gamma$ in the appropriate L^2-sense discussed earlier.

When the solution w is transformed to a function $\bar{w}(r,\theta) = w(-\log r,\theta)$ on $\mathbb{B}^n\backslash\{0\}$, elliptic regularity shows that $\bar{w}$ is a conormal distribution with respect to the origin, i.e. has stable regularity when differentiated arbitrarily often with respect to the vector fields $r\partial_r, \partial_\theta$ (this is just the same as w itself having stable regularity with respect to $\partial_t, \partial_\theta$); [M1], for example, contains a discussion of conormal regularity. For degenerate operators of a type closely related to L, one expects solutions to be polyhomogeneous conormal, cf. [M1] or [Me]. Polyhomogeneity is simply the property of having an asymptotic expansion in increasing (possibly complex) powers of r and integral powers of $\log r$, with coefficients smooth in θ (as functions of t these expansions are in analogous powers of e^t and t). Alternately, polyhomogeneous conormal distributions may be characterized as those with Mellin transforms, already defined and holomorphic in some lower half-plane in $\mathbb{C}$, extending meromorphically to the whole complex plane, with only finitely many poles in any lower half-plane, all of which are of finite rank.

$\bar{w}$ will not be polyhomogeneous, except when $\epsilon = \bar{u}$ and the underlying metric is cylindrical. In fact, $\bar{w}$ (cut off so as to be supported in $r \leq 1$) will have Mellin transform still defined and holomorphic in a half-plane and extending meromorphically to all of $\mathbb{C}$, but now its poles are arranged along lattices on a countable discrete set of horizontal lines. Each pole is still finite rank. This meromorphic structure of the Mellin transform of $\bar{w}$ is equivalent to the fact that the terms in the expansion for w have the form $e^{-\beta t} w_j$, where w_j is periodic in t. The residues at the poles along a fixed horizontal line at height β will be the Fourier series coefficients of w_j.

This more general discussion has involved the Mellin transform as $r \to 0$ (or equivalently, the standard Fourier transform as $t \to \infty$). We shall revert now to the Fourier-Laplace transform; it is much less flexible than either of the other transforms, since it presupposes periodicity, but suffices here for our immediate purposes.

Suppose that $w \in H^s_\gamma(\mathbb{R} \times \mathbb{S}^{n-1})$ for some $\gamma \notin \Gamma$ is supported in $t \geq 0$ and solves $L_\epsilon w = f$ for some compactly supported smooth function f (hence $w \in H^s_\gamma$ for every s by elliptic regularity). For example, we could take ϕ, which solves $L_\epsilon \phi = 0$, and let $w = \chi\phi$ where χ is a cutoff function having support in $t \geq 0$ and equaling one for $t \geq 1$. Taking transforms we get $\hat{L}\hat{w} = \hat{f}$; this function on the right, $\hat{f}(t, \theta, \zeta)$, is obviously smooth in (t, θ) and entire in ζ. Applying the inverse $\hat{G}$ from the last section gives $\hat{w}(t, \theta, \zeta) = \hat{G}\hat{f}(t, \theta, \zeta)$. The right side of this equation is meromorphic in ζ with poles at some subset of points in $\mathcal{P}$, hence the same is true for the left side as well. Notice that the poles of $\hat{G}$ in $\Im\zeta < -\gamma$ must be cancelled by zeroes of $\hat{f}$ since $\hat{w}$ is a priori known to be regular in this half-plane.

The function w is recovered by inverting $\mathcal{F}$, integrating along the line $\Im\zeta = -\gamma$. However, by Cauchy's theorem this integration may be taken along any higher contour $\Im\zeta = -\gamma'$, so long as the interval $[\gamma', \gamma]$ does not contain any points of Γ. Thus, for any such γ', $w \in H^s_{\gamma'}$. If the contour is shifted even further, so as to cross a point in $\mathcal{P}$, i.e. a pole of $\hat{G}$, then the resulting integral along the line $\Im\zeta = -\gamma''$ produces a new function v. Since $(L_\epsilon v)\hat{} = \hat{L}_\epsilon \hat{G}\hat{f} = \hat{f}$ and $\hat{f}$ is entire, it follows that v also solves $Lv = f$, and so w and v must differ by an element of the nullspace of L_ϵ. By construction $v \in H^s_{\gamma''}$, and so we have decomposed w as a sum $v + \psi$, where $L_\epsilon\psi = 0$. ψ is given by the residues of $\hat{w}$ at the points $\zeta_j \in \mathcal{P}$ with $-\gamma < \Im(-\zeta_j) = -\gamma_j < -\gamma''$. (We use $-\zeta_j$ instead of ζ_j for notational convenience only.) If there is just one such point, say $-\zeta_j$, then it is clear that in fact $\psi \in H^s_{\gamma_j+\epsilon}$ for every $\epsilon > 0$. Actually, it is not hard to show, using results from §2, that this ψ must grow or decay exactly like a polynomial in t times $e^{\gamma_j t}$.

This process may be continued by moving the contour past more and more poles. At each step, we have decomposed w into a sum of solutions of $L_\epsilon\psi = 0$ and a term which decays at a rate given by the height of the contour.

Proposition 4.14. *If w solves $L_\epsilon w = f$ for some compactly supported function f on $(0, \infty) \times \mathbb{S}^{n-1}$, with $w \in H^s_\gamma$ for some $\gamma \notin -\Gamma$, then, as $t \to \infty$, $w(t, \theta) \sim \sum \psi_j(t, \theta)$, where each ψ_j solves $L_\epsilon\psi_j = 0$ and ψ_j decays like a polynomial in t multiplying $e^{-\gamma_j t}$, where $\gamma_j \in -\Gamma$ as in (4.12).*

Note that this result shows that the poles ζ_j, or at least their imaginary parts,

γ_j, correspond to the precise growth rates of solutions of $L_\epsilon \phi = 0$ on the cylinder.

A similar, though slightly more complicated, expansion holds for the elements of the nullspace of the linearization L in (4.2).

Proposition 4.15. *Let w solve $Lw = 0$, at least on some neighborhood of the puncture p_j, and lie in H^s_γ for some $\gamma \notin -\Gamma$, where Γ is the set of imaginary parts of poles corresponding to the inverse $\hat{G}$ for the model L_ϵ for L near this puncture. Then w has an asymptotic expansion*

$$w(t,\theta) \sim \sum_{j,k=0}^{\infty} \psi_{j,k}(t,\theta)$$

as $t \to \infty$. In this sum, the 'leading terms' $\psi_{j,0}$ are solutions of $L_\epsilon \psi_{j,0} = 0$ corresponding to the poles ζ_j above $\Im\phi = -\gamma$. These decay, as before, like a polynomial in t multiplying $e^{-\gamma_j t}$. The higher terms $\psi_{j,k}$, $k > 0$, decay like a polynomial in t multiplying $e^{-(\gamma_j + k\alpha)t}$.

Proof. As before, we may assume that w is supported in $t \geq 0$ and $Lw = f$ is compactly supported and smooth. By (4.3),

$$L_\epsilon w = -e^{-\alpha t} F w + f. \tag{4.16}$$

Now conjugate by the Fourier-Laplace transform, and apply the inverse $\hat{G}$ of $\hat{L}_\epsilon$ to get

$$\hat{w} = \hat{G}(-e^{-\alpha t} F w)\hat{\ } + \hat{G}\hat{f}. \tag{4.17}$$

The term on the left is holomorphic, a priori, in the half-plane $\Im\zeta < -\gamma$. The second term on the right is entire, while the first term on the right is holomorphic in $\Im\zeta < -\gamma$ and extends meromorphically to the slightly larger half-plane $\Im\zeta < -\gamma + \alpha$. As before, w can be recovered by integrating along $\Im\zeta = -\gamma - \epsilon$; if this contour is moved up to $\Im\zeta = -\gamma + \alpha - \epsilon$, and if the strip $-\gamma < \Im\zeta < -\gamma + \alpha$ contains no poles, then we find that $w \in H^s_{\gamma - \alpha}$. If there are poles in this strip, then, exactly as in the last proposition, w decomposes into a sum $v + \psi$ with $v \in H^s_{\gamma-\alpha}$ and $L_\epsilon \psi = 0$. ψ in turn decomposes into a sum of terms $\psi_{j,0}$ corresponding to the various poles in this strip. This improved information may be then fed back into (4.16) and (4.17). Now the right side of (4.17) extends meromorphically to the half-plane $\Im\zeta < -\gamma + 2\alpha$, and the contour may be shifted further to get more contributions to the expansion for w. Continuing this bootstrapping yields the full expansion. Details are left to the reader.

The deficiency subspace. Particularly important for us in the application of the implicit function theorem will be the pole $0 \in \mathcal{P}$ for $\hat{G}$ (for any value of the Delaunay parameter ϵ). Specifically, we will be concerned with those solutions of $L_\epsilon w = 0$ and $Lw = 0$ which are in H^s_δ for every $\delta > 0$, but not in any $H^s_{-\delta}$. Here we are still only concerned with the local behavior of these solutions on each end E_j. Their global nature will be discussed later.

The representatives of a basis of $(\mathrm{Ker}\ L \cap H^s_\delta)/(\mathrm{Ker}\ L \cap H^s_{-\delta})$ are given in (2.15) as the functions $\phi_1 = \phi_{1,\epsilon}$ and $\phi_2 = \phi_{2,\epsilon}$. They depend on t but not on θ. The fact that ϕ_2 grows linearly in t indicates that $\hat{G}$ has a pole of order 2 at $\phi = 0$. These are the only temperate solutions of $L_\epsilon \phi = 0$ on the whole cylinder; any other solution grows at an exponential rate (with possible exponents given by the values of $\Im \mathcal{P} = \Gamma$).

If the solution metric g has model Delaunay parameter ϵ on the end E_j, then these functions (for that value of ϵ) may be cut off and transplanted onto this semi-cylinder. They do not decay, but it is easy to create a sequence of cut-offs $\chi_i \phi$ ($\phi = \phi_1$ or ϕ_2) with the following properties: each χ_i is compactly supported, and has support tending to infinity as i tends to infinity. The L^2 norm of $\chi_i \phi$ equals one for all i, but the L^2 norm of $L(\chi_i \phi)$ tends to zero (L is the linearization, not one of the models L_ϵ here). The existence of such sequence is a standard criterion for showing that L does not have closed range on L^2. In any event, we think of ϕ_1, ϕ_2 as constituting the bounded approximate nullspace for L (here 'bounded' is loosely interpreted to encompass the linearly growing ϕ_2).

Define now a linear space W generated by the functions $\phi_{1,\epsilon_j}, \phi_{2,\epsilon_j}$, cut off in a fixed way so as to be supported in $t \geq 0$ and transplanted on each end E_j; the ϵ_j are the Delaunay parameters for the particular model metrics g_{ϵ_j} on E_j. Since there are k ends, W is $2k$-dimensional. For reasons that will become clear in the next section, we call W the deficiency subspace for L. Clearly $W \subset H^s_\delta$ for every s and every $\delta > 0$.

The Linear Decomposition Lemma. There are two important corollaries of Proposition 4.15 and its proof which we single out in this section. The first concerns the behavior of solutions of the inhomogeneous equation $Lw = f$ on each end E_j, while the second studies the exact value of the exponent α appearing in Theorem 3.3 and then later in (4.3), etc.

As already pointed out in Corollary 4.13, if L has no global L^2 nullspace, then we can find a solution $w \in H^{s+2}_\delta$ to the equation $Lw = f$ for every $f \in H^s_\delta$, whenever $\delta > 0$. In particular, this holds whenever $f \in H^s_{-\delta}$. Clearly, whenever f decays at some exponential rate like this, we expect the solution w to be somewhat better behaved than a general H^{s+2}_δ function; of course, it is immediate that it is in this space for any $\delta > 0$, but we can do even better. This is the subject of what we will call the

Linear Decomposition Lemma 4.18. *Suppose $f \in H^s_{-\delta}$ for some $\delta > 0$ sufficiently small, and $w \in H^{s+2}_\delta$ solves $Lw = f$. Then $w \in H^{s+2}_{-\delta} \oplus W$, i.e. w may be decomposed into a sum $v + \phi$ with v decaying at the same rate as f and ϕ in the deficiency subspace W.*

Proof. Clearly this question may be localized to each end E_j, and in this localized decomposition ϕ will be a combination of ϕ_1 and ϕ_2. The decomposition is achieved by exactly the same sort of shift of the contour in the integral defining $\mathcal{F}^{-1}$ across a pole of $\hat{G}$. Here the contour is being shifted from $\Im \zeta = -\delta$ to $\Im \zeta = +\delta$; the pole crossed is the one at $\zeta = 0$.

The second corollary deals with the rate at which a general solution of (1.3) on a punctured ball, singular at $\{0\}$, converges to the radial Delaunay metric. We assume the simpler statement of Theorem 3.3 that u decays to u_ϵ at some exponential rate α, and use the linear theory to find the optimal rate. First transform the punctured ball to the half-cylinder $(0, \infty) \times \mathbb{S}^{n-1}$, and assume all functions are defined here.

Write the solution u as a perturbation $(1+v)u_\epsilon$, so that $N_\epsilon(v)=0$, where N_ϵ is the nonlinear operator (2.11). This equation is the same as $L_\epsilon v = -Q(v)$, where Q is the quadratically vanishing function in (2.12). We already know that $v \in H^s_{-\alpha}$ (for all s), hence $Q(v) \in H^s_{-2\alpha}$. By the contour-shifting arguments above, this implies that v itself decays at this faster rate, at least provided $2\alpha < \gamma_1$, where γ_1 is the first positive element in Γ. Continue this process until this first pole at ζ_1 has been crossed; the conclusion is that v decays exactly like $e^{-\gamma_1 t}$ (possibly multiplied by a polynomial in t). We could also bootstrap further and obtain a complete asymptotic expansion for v, of the same general form as in Proposition 4.15 above, although more exponents occur because of the nonlinearity. (This is a simple form of the argument in [M2].) We summarize this discussion as

Proposition 4.19. *The exponent α occurring in Theorem 3.3 governing the rate of decay of a general solution u to its model Delaunay solution u_ϵ is equal to the first nonzero element γ_1 in the set Γ corresponding to $\hat{G}_\epsilon$. The function u admits a complete asymptotic expansion into terms of increasingly rapid exponential decay.*

The bounded nullspace. In this last subsection of §4, we finally come to the global behavior of that portion of the nullspace of L corresponding to the pole at $z=0$. This space, $\mathcal{B}$, which we shall call the 'bounded nullspace' of L, is defined by

$$\mathcal{B} = \{v \in H^s_\delta : Lv = 0\} \cap \{w \in H^s_{-\delta} : Lw = 0\}^\perp. \tag{4.20}$$

The full nullspace of L on H^s_δ will be the direct sum of $\mathcal{B}$ and the L^2 nullspace (which, by Proposition 4.15, is the same as the nullspace of L in $H^s_{-\delta}$). In particular, when this latter space is trivial, $\mathcal{B}$ is the full nullspace of L in H^s_δ. By the Linear Decomposition Lemma 4.18, $\mathcal{B}$ is already contained in $H^s_{-\delta} \oplus W$. The purpose of this subsection is to determine the dimension of $\mathcal{B}$.

As usual, an index theorem is the principal tool for calculating $\dim(\mathcal{B})$. Fortunately we require only a relative index theorem, which computes the difference between two indices in terms of asymptotic data, rather than global data. To be more explicit, for any $\gamma \notin \Gamma$ set

$$\operatorname{ind}(\gamma) = \dim \ker \left. L \right|_{H^s_\gamma} - \dim \operatorname{coker} \left. L \right|_{H^s_\gamma}. \tag{4.21}$$

This is obviously independent of s. Since the adjoint of L on H^s_γ is L on $H^{-s}_{-\gamma}$, duality implies that

$$\operatorname{ind}(-\gamma) = -\operatorname{ind}(\gamma) \text{ for every } \gamma \notin \Gamma. \tag{4.22}$$

If γ_1 and γ_2 are any two allowable values (i.e. neither is in Γ), the relative index with respect to these two numbers is simply the difference

$$\text{rel-ind}(\gamma_1,\gamma_2) = \operatorname{ind}(\gamma_1) - \operatorname{ind}(\gamma_2).$$

In particular, using duality again,

$$\begin{aligned} \text{rel-ind}(\delta,-\delta) &= 2\operatorname{ind}(\delta) \\ &= 2\left(\dim\ker \left. L\right|_{H^s_\delta} - \dim\ker \left. L\right|_{H^s_{-\delta}}\right) = 2\dim(\mathcal{B}). \end{aligned} \tag{4.23}$$

As noted above, this relative index can be shown, on fairly general principles, to be computable in terms of asymptotic data for the operator L. Finding a specific and computable formula is another matter and, to our knowledge, there is no general result of this sort available for asymptotically periodic operators. However, such a result is available for operators associated to asymptotically cylindrical metrics, and we will use this instead. Our result is

Theorem 4.24. $\dim(\mathcal{B}) = k$.

Proof. By (4.23), it suffices to show that rel-ind $(\delta, -\delta) = 2k$. By the usual stability properties of the index (hence any relative index) under Fredholm deformations, we compute this number by choosing a one-parameter family of Fredholm operators L^σ, $0 \leq \sigma \leq 1$, with $L^0 = L$, and L^1 an operator for which this relative index is computable. Since L is just $\Delta_g + n$, we shall choose a one-parameter family of metrics g^σ and define L^σ to be $\Delta_{g^\sigma} + n$. The metrics g^σ will agree with g except on each of the ends E_j, where we make the following homotopy. First deform g on each end to its model Delaunay metric g_ϵ, and then deform each g_ϵ through Delaunay metrics to the cylindrical metric $g_{\bar{u}}$. The metric g^1 will agree with the original g on any large fixed compact set, and will equal $g_{\bar{u}}$ on each end E_j. This metric is now an *exact b-metric*, in the language of [Me] (more prosaically, it has asymptotically – and in this case, exactly – cylindrical ends), and the corresponding operator L^1 is an elliptic b-operator.

Of course, we still need to prove that L^σ is Fredholm on H^s_δ and $H^s_{-\delta}$ for every $0 \leq \sigma \leq 1$, provided $\delta > 0$ is sufficiently small. In the part of the deformation where g is homotoped to its model Delaunay metric g_ϵ on each end, this is obviously true. For the remaining part of the deformation, we need to know that the elements of $\{\gamma_j \in \Gamma, j > 1\}$, which are the weights for which L will not be Fredholm, remain bounded away from 0 as ϵ varies between its initial value and $\bar{u}$. This is precisely the content of Lemma 2.21.

Now, to apply Melrose's Relative Index Theorem [Me], note that we may as well consider the operator $\tilde{L}^1 = \frac{n-2}{n} L$, which on each end takes the form $\partial_t^2 + \partial_\theta^2 + (n-2)$. The set $\mathcal{P}$ for this operator (called $\mathrm{spec}_b(\tilde{L}^1)$ in [Me]) is $\{\pm i\sqrt{n-2}, \pm\gamma_j, j \geq 1\}$, where γ_j are all strictly positive, tending to infinity, and obtained in a straightforward manner from $\mathrm{spec}(\Delta_\theta)$. The prescription to calculate the relative index is to first consider the 'indicial family,' which in this case equals $\Delta_\theta + (n-2) - \zeta^2$ (it is obtained by passing to the Fourier tranform with respect to t, which carries ∂_t to $-i\zeta$). This is a holomorphic family of elliptic operators on $\mathbb{S}^{n-1}$, and its inverse $G(\zeta)$ has poles exactly at the points of $\mathcal{P}$. Since we are computing the jump in the index as the weight changes from $-\delta$ to $+\delta$, we need to compute the 'degree' of each pole (as defined in [Me]) for each element of this set with imaginary part equal to zero. Each of the poles $\pm\sqrt{n-2}$ is simple and of rank one, and so the degree of each is also equal to one. Finally we need to sum over all poles *and* over all ends E_j, because the preceding discussion is local on each end. For each end there are two poles of G with imaginary part zero, each contributing a degree of one to the computation, and there are k ends; the sum of all of these is $2k$, and this is the relative index rel-ind $(\delta, -\delta)$.

5. The moduli space: smooth points

In this section we commence the study of the moduli space $\mathcal{M}_\Lambda$ itself. Here we use the linear theory developed in the previous section in a straightforward way to study neighborhoods of the 'good points' $g \in \mathcal{M}_\Lambda$, where the associated linearization has no L^2 nullspace. In the next two sections we develop ideas to study neighborhoods of the nonsmooth points.

The implicit function theorem is directly applicable only when the linearized operator $L_g = \Delta_g + n$ is surjective on the appropriate function spaces. These function spaces should be tangent to a suitable space of metrics conformal to g and with prescribed growth conditions on each end of $\mathbb{S}^n \backslash \Lambda$. Let $g \in \mathcal{M}_\Lambda$, and consider all nearby metrics conformal to g of the form $(1+v)^{\frac{4}{n-2}} g$. The most natural class of such metrics is

$$\mathrm{Met}^{s+2}_{-\delta} = \{(1+v)^{\frac{4}{n-2}} g,\ v \in H^s_{-\delta},\ v > -1\}. \tag{5.1}$$

Clearly

$$T_g \mathrm{Met}^{s+2}_{-\delta} = H^{s+2}_{-\delta}.$$

The operator $N_g(v)$, as defined in (4.1), can be considered as a map from this space of metrics to $H^s_{-\delta}$, and as such is obviously $\mathcal{C}^\infty$. However, it is never surjective, since $-\delta < 0$. The obvious alternative, to consider metrics growing at the rate $+\delta$, is unsuitable because of the nonlinearity.

The case where this difficulty is easiest to remedy is when L_g has no L^2 nullspace:

$$\ker(L_g) \cap L^2(\mathbb{S}^n \backslash \Lambda, dV_g) = \{0\}. \tag{5.2}$$

This will be our standing hypothesis in the rest of this section.

We can rephrase this condition by recalling, by virtue of Proposition 4.15 on asymptotics of solutions of $L\phi = 0$, that if ϕ were in the L^2 nullspace of L, then it would decay at the exponential rate $e^{-\gamma_1 t}$ on each end, where $\gamma_1 = \gamma_1(\epsilon_j)$ is the first nonzero element of $\mathcal{P}_{\epsilon_j}$ on the end E_j, corresponding to the model Delaunay metric g_{ϵ_j} there. In particular, under the hypothesis (5.2), for $\delta \in (0, \gamma_1)$, the operator L has no nullspace on $H^s_{-\delta}$ for any $s \in \mathbb{R}$. Now apply Corollary 4.13 and the Linear Decomposition Lemma 4.18 to conclude that (5.2) is equivalent to the statement that

$$\begin{gathered} L : H^{s+2}_{-\delta}(\mathbb{S}^n \backslash \Lambda, dV_g) \oplus W \longrightarrow H^s_{-\delta}(\mathbb{S}^n \backslash \Lambda, dV_g) \\ \text{is surjective for any } 0 < \delta < \min_j \{\gamma_1(\epsilon_j)\} \text{ and } s \in \mathbb{R}. \end{gathered} \tag{5.3}$$

W is the $2k$-dimensional deficiency subspace introduced in the last section. Because we work with a nonlinear equation, we assume that $s > n/2$.

Theorem 5.4. *Suppose $g \in \mathcal{M}_\Lambda$ is a metric such that the hypothesis (5.2) is satisfied. Then there exists a $2k$-dimensional open manifold $\mathbb{W}$, the elements of which are functions v on $\mathbb{S}^n \backslash \Lambda$ to be described below, and with a distinguished element $v_0 \in \mathbb{W}$, such that the nonlinear map*

$$N_g : H^{s+2}_{-\delta}(\mathbb{S}^n \backslash \Lambda; dV_g) \times \mathbb{W} \longrightarrow H^s_{-\delta}(\mathbb{S}^n \backslash \Lambda; dV_g)$$

defined by $N_g(v, \psi) = N_g(v+\psi)$ where N_g is the operator given by (4.1), restricted to a neighborhood of 0 in the first factor, is real analytic. Moreover, the tangent space $T_{v_0}\mathbb{W}$ is identified with the deficiency subspace W, and using this identification, the linearization L_g of the map N_g is surjective, as in (5.3).

Corollary 5.5. *Suppose $g \in \mathcal{M}_\Lambda$ satisfies the hypothesis (5.2). Then there is an open set $\mathcal{U} \subset \mathcal{M}_\Lambda$ containing g, such that $\mathcal{U}$ is a k-dimensional real analytic manifold.*

The proof of the corollary follows directly from the surjectivity statement (5.3), by a straightforward application of the real analytic implicit function theorem. The dimension k is of course the dimension of the nullspace of L as a map (5.3). This nullspace is the bounded nullspace $\mathcal{B}$ defined in the last section, and by Theorem 4.24 its dimension is k. So to conclude the proof of the theorem and the corollary it suffices to construct $\mathbb{W}$, and to show that N_g is a real analytic mapping.

The only difficulty in the construction of $\mathbb{W}$ is that the elements of W are not, generally, bounded either above or below, so we could not use $(1+v)^{4/(n-2)}$ as a conformal factor for most $v \in W$. However, recall from (2.15) that each element $\phi_{j,\epsilon}$, $j = 1, 2$, of the bounded nullspace for the model problem on the cylinder is the tangent to a curve $\eta \to \Phi_j(t, \epsilon, \eta)$ of actual conformal factors. The actual curve $v_\eta \in \mathbb{W}$ we would want to use for each such element of W on an end has the form

$$v_\eta(t) = \begin{cases} \frac{u_\epsilon(t+\eta)}{u_\epsilon(t)} - 1, & j = 1, \\ \frac{u_{\epsilon+\eta}(t)}{u_\epsilon(t)} - 1, & j = 2. \end{cases}$$

The manifold $\mathbb{W}$ is constructed by gluing together these local definitions from each end E_j in an essentially arbitrary, but smooth, manner. By construction, $T_0\mathbb{W} = W$, as desired.

To check the real analyticity of N_g, the main step is to write $N_g(v + \psi)$ in such a way that it clearly lands in $H^s_{-\delta}$. Hence, on each end write $N_g = N_\epsilon + e^{-\alpha t}Q$, and then use a common formula for the remainder in Taylor's theorem to obtain

$$N_g(v + \psi) = N_\epsilon(\psi) + e^{-\alpha t}Q(v + \psi) + \left[\int_0^1 N_g'(\psi + sv)\, ds\right] v.$$

Since $N_\epsilon(\psi) = 0$, every term on the right here is in H^s_δ so long as $\delta < \alpha$. Furthermore, every term is real analytic in (v, ψ). Real analyticity of N_g on the interior, away from the ends, is even easier.

The rest of the proof is now standard.

6. The real analytic structure of $\mathcal{M}_\Lambda$

In this section we prove that $\mathcal{M}_\Lambda$ is always an analytic set by representing it as the slice of an infinite dimensional real analytic manifold $\mathbb{M}_\Lambda$ with the conformal class g_0. We go on to prove a generic slice theorem, that even if $\mathcal{M}_\Lambda$ itself is not smooth, generic nearby slices of $\mathbb{M}_\Lambda$ by other conformal classes will be.

The urmoduli space $\mathbb{M}_\Lambda$. As described above, we wish to regard $\mathcal{M}_\Lambda$ as the slice by the standard conformal class $[g_0]$ of the infinite dimensional set $\mathbb{M}_\Lambda$ consisting of all metrics, not necessarily in the standard conformal class on $\mathbb{S}^n$, with constant scalar curvature $n(n-1)$. For obvious technical reasons, we consider only those metrics which satisfy appropriate growth and asymptotics conditions near the punctures p_j (this will be be elaborated on below.) Consideration of this 'big', or 'ur-', moduli space (when the underlying manifold is compact) was first undertaken by Fischer and Marsden in the early 1970's ([FM1] and [FM2]), motivated in part by concerns in general relativity. They proved that it is a smooth Banach manifold (in any one of a number of standard Banach completions), provided a certain overdetermined linear equation has only the trivial solution.

Their precise set-up was to consider $\mathbb{M}_\rho$, the set of all metrics (of some fixed finite regularity) with scalar curvature function ρ, a fixed function on the underlying manifold X. To show that this set is a Banach manifold, it suffices to show that the linearization of the scalar curvature map (which assigns to any metric g its scalar curvature function $R(g)$) is surjective at any $g \in \mathbb{M}_\rho$. Since this map R carries metrics to functions, its linearization, which we will call α_g here, carries the tangent space at g of the space of all metrics, i.e. the symmetric 2-tensors, to scalar functions. A. Lichnerowicz [Li] had earlier computed that, for h a symmetric 2-tensor,

$$\alpha_g(h) = \delta\delta h - \varDelta_g(\operatorname{tr} h) - \langle r_g, h\rangle. \tag{6.1}$$

Here δ is the divergence operator on tensors, and r_g is the Ricci tensor for g. It is easy to check that α_g reduces to (a multiple of) the linearization operator L_g we have been studying when h is a multiple of the metric g (and hence is tangent to the conformal class $[g]$), when $\rho = n(n-1)$. It is straightforward to check that the symbol of α_g is surjective, so it follows that the symbol of its adjoint α_g^* is injective, and then that the symbol of $\alpha_g \circ \alpha_g^*$ is an isomorphism. Therefore, this last, fourth order operator is elliptic; at least when X is compact, it is Fredholm, and in particular has closed range of finite codimension. Since $\operatorname{range}(\alpha_g) \supset \operatorname{range}(\alpha_g \circ \alpha_g^*)$, the range of α_g itself is closed and of finite codimension. (It does, however, always have an infinite dimensional kernel.) By virtue of all this, when X is compact, α_g is surjective if and only if α_g^* is injective, and this ($\alpha_g^*(f) = 0$) is precisely the overdetermined equation referred to earlier.

From (6.1) it follows readily that

$$\alpha_g^*(f) = \operatorname{Hess}_g(f) - g\varDelta_g(f) - f\, r_g. \tag{6.2}$$

If $\alpha_g^*(f) = 0$, then taking the trace we also get that $\varDelta_g(f) = -(\rho/(n-1))f$; substituting this back into (6.2) we obtain finally the condition

$$\alpha_g^*(f) = 0 \Longrightarrow \operatorname{Hess}_g(f) = -\left(r_g - \frac{\rho}{n-1}g\right) f. \tag{6.3}$$

Fischer and Marsden were able to show that (6.3) (or equivalently, (6.2)) has only the trivial solution $f = 0$ except possibly in the special cases where ρ is a nonnegative constant and $\rho/(n-1)$ is in the spectrum of $-\varDelta_g$. Thus, except for these cases, $\mathbb{M}_\rho$ is a smooth Banach manifold.

In the case of interest here, the underlying manifold $\mathbb{S}^n\backslash\Lambda$ is not compact, and $\rho/(n-1) = n$ is *always* in $\operatorname{spec}(\varDelta_g)$. We address the former of these concerns first, since the Fredholm theory of §4 may be adapted easily to this problem. As before $H^s_\gamma(\mathbb{S}^n\backslash\Lambda)$ will denote the Sobolev space of scalar functions decaying like $e^{\gamma t}$ on each end, with respect to fixed cylindrical coordinates (t,θ) there and with respect to a fixed metric $g \in \mathcal{M}_\Lambda$ (any element of the moduli space can be used as a fixed background metric to define these Sobolev spaces). Define also $H^s_\gamma(\mathbb{S}^n\backslash\Lambda, \operatorname{Sym}^2 T^*(\mathbb{S}^n\backslash\Lambda))$ to be the space of symmetric 2-tensors $h = e^{\gamma t}k$, with k in the unweighted Sobolev space $H^s(\mathbb{S}^n\backslash\Lambda, \operatorname{Sym}^2 T^*(\mathbb{S}^n\backslash\Lambda); dV_g)$.

Proposition 6.4. *For any $g \in \mathcal{M}_\Lambda$, the operator (6.1) is bounded as a linear map*

$$\alpha_g : H^{s+2}_\gamma(\mathbb{S}^n \backslash \Lambda, \mathrm{Sym}^2(T^*(\mathbb{S}^n \backslash \Lambda))) \longrightarrow H^s_\gamma(\mathbb{S}^n \backslash \Lambda).$$

It has an infinite dimensional nullspace, and the closure of its range has finite codimension for all $\gamma \in \mathbb{R}$. There exists a discrete set Γ' such that for all $\gamma \notin \Gamma'$, this map also has closed range.

Proof. The boundedness assertion is immediate. The proof of the closed range part is very close to the analogous one given in §4 for the scalar operator L_g, so we just provide a sketch. As in [FM2] and above, the surjectivity of the symbol of α_g guarantees that we merely have to show that $A_g = \alpha_g \circ \alpha_g^*$, which is elliptic, has closed range for all $\gamma \notin \Gamma'$. The operator A_g is asymptotically periodic on each end E_j of $\mathbb{S}^n \backslash \Lambda$, and the Fourier-Laplace transform may again be employed to construct local parametrices on each of these ends. The fact needed to make this work is that the holomorphic family of elliptic operators $\tilde{A}_\epsilon(\phi)$, on the compact manifold $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, and associated to the Delaunay model operator A_{g_ϵ}, is invertible at some $\phi \in \mathbb{C}$. In this case the analytic Fredholm theory, and the rest of the construction in §4, proceeds exactly as before. Again we prove this for $\phi \in \mathbb{R}$, since for such ϕ, $\tilde{A}_\epsilon(\phi)$ is self-adjoint. Invertibility is equivalent to the injectivity of the induced operator $\tilde{\alpha}^*_\epsilon(\phi)$, and this in turn is implied (by taking traces) by the injectivity of the model operator $\tilde{L}_\epsilon(\phi)$. The invertibility of this last operator for all $\phi \in (0, 2\pi)$ (we are again assuming the period of α_ϵ, et al., to be one for simplicity) was already proved in §4, and so we are done.

To proceed with the program of [FM2], we also have to show that α_g^* is injective on $\mathbb{S}^n \backslash \Lambda$ for any $g \in \mathcal{M}_\Lambda$. If this were true, then the urmoduli space $\mathbb{M}_\Lambda$ would be a real analytic manifold in a neighborhood of every $g \in \mathcal{M}_\Lambda$, i.e. there would exist a (local) thickening of $\mathcal{M}_\Lambda$ into the real analytic moduli space $\mathbb{M}_\Lambda$ such that $\mathcal{M}_\Lambda = \mathbb{M}_\Lambda \cap [g_0]$, as desired. Unfortunately this is not quite so simple, since, as already noted, $\rho/(n-1)$ is always in the spectrum of Δ_g; we are thus precisely in the case not treated by [FM2].

The equation (6.3) is rather similar to the well-known equation of Obata, which also appears in the study of the scalar curvature, and actually reduces to Obata's equation when X is an Einstein manifold. In analogy with the situation there – in particular, Obata's characterization [O] of $(\mathbb{S}^n, g_0)$ as the only compact manifold admitting nontrivial solutions of his equation – Fischer and Marsden conjectured that (6.3) admits a nontrivial solution, at least when X is compact, if and only if X is the standard sphere. This is now known to be false: indeed, let X be the product of the circle of length L, $\mathbb{S}^1(L)$, with any Einstein manifold E^{n-1} of positive scalar curvature, e.g. $\mathbb{S}^{n-1}$. Then, if the scalar curvature of E is normalized to be $(n-1)(n-2)$, (6.3) implies that f depends only on the variable t along the circle, and satisfies the ODE $f'' + (n-2)f = 0$. Thus, (6.3) admits a nontrivial solution whenever L is an integer multiple of $\sqrt{n-2}$. There are also noncompact counterexamples to their conjecture: the simplest are just the universal covers $\mathbb{R} \times E^{n-1}$ of the compact examples above.

The characterization even of all compact manifolds admitting solutions to (6.3) is still unknown, although it is almost certain that the list of possibilities is rather small. Fortunately, though, the characterization of all complete conformally flat manifolds admitting nontrivial solutions to (6.3) was obtained about ten years ago, independently by J. Lafontaine [L] and O. Kobayashi [Ko]. The simplification in this case is that the Ricci tensor of any conformally flat manifold is always harmonic. Kobayashi and Lafontaine proved that beyond the sphere and the product type examples discussed above, there is also a collection of examples which are warped products of $\mathbb{S}^1$ with E^{n-1}. We shall not write these examples explicitly here, but simply note that, provided $k > 2$, the solution metrics $(\mathbb{S}^n \backslash \Lambda, g)$, $g \in \mathcal{M}_\Lambda$, are never warped products (for topological reasons alone!). We obtain then

Proposition 6.5. *There are no nontrivial solutions to the equation* $\alpha_g^*(f)$ *for any* $g \in \mathcal{M}_\Lambda$*, provided* $k > 2$.

Even when $k = 2$, only the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ amongst the Delaunay metrics appears on the list of counterexamples. To check this, use the system of equations given in Lemma 1.1 of [Ko]. These involve both the warping function for the metric and the function f. Elementary manipulations show that the noncylindrical Delaunay metrics never satisfy this system.

Corollary 6.6. $\mathbb{M}_\Lambda$ *is a real analytic Banach manifold in a neighborhood of* $\mathcal{M}_\Lambda$.

Proof. It remains only to set up the precise function spaces on which the implicit function theorem (see [Fe], page 239) will be applied, and make a few additional comments. We use, of course, the weighted Sobolev spaces H^s_γ of symmetric 2-tensors and functions, as before.

The crux of the argument is the fact that when $g \in \mathcal{M}_\Lambda$, α_g is surjective both on H^s_δ *and* on $H^s_{-\delta}$, and indeed on H^s_γ for any $\gamma \notin \Gamma'$! This seems somewhat counterintuitive, but follows from the highly overdetermined nature of α_g^*. In fact, we already know that α_g has closed range on H^s_γ, $\gamma \notin \Gamma'$ by Proposition 6.4. Its cokernel is identified with the kernel of the adjoint, α_g^*, which is a map from $H^{-s}_{-\gamma}$ to $H^{-s-2}_{-\gamma}$. By injectivity of the symbol of α_g^* we may use elliptic regularity to replace $-s$ here by any positive number, e.g. $+s$. Finally we can invoke the results of Lafontaine and Kobayashi, via Proposition 6.5, to conclude that α_g^* has no nullspace, regardless of the weight $-\gamma$.

Now consider the scalar curvature map

$$\mathcal{N}_g : H^{s+2}_{-\delta}(\mathrm{Sym}^2) \times \mathbb{W} \longrightarrow H^s_{-\delta}(\mathrm{Sym}^2), \tag{6.7}$$

defined in the obvious way. We are including the factor $\mathbb{W}$ here so that points of the urmoduli space are allowed to have varying neck sizes, but all other perturbations are required to decrease exponentially on the ends. To apply the analytic implicit function theorem we need to know, first, that the linearization

$$\alpha_g : H^{s+2}_{-\delta}(\mathrm{Sym}^2) \oplus W \longrightarrow H^s_{-\delta} \tag{6.8}$$

is surjective, which we have just established, and second that (6.7) is a real analytic mapping of Hilbert spaces. This latter statement is also straightforward, so this completes the proof.

One unfortunate shortfall of this theorem is that, although $\mathbb{M}_\Lambda$ provides a thickening of $\mathcal{M}_\Lambda$, it is not a uniform thickening, i.e. one of fixed 'width' around $\mathcal{M}_\Lambda$. However, over any compact subset of $\mathcal{M}_\Lambda$ we can ensure the existence of $\mathbb{M}_\Lambda$ out to some fixed distance by the obvious covering argument. To find this fixed width thickening uniformly on $\mathcal{M}_\Lambda$ we would need to understand more about the compactification $\overline{\mathcal{M}_\Lambda}$. If the analogies of the result of Kobayashi and Lafontaine as well as the asymptotics result, Theorem 3.3, and the compactness result, Proposition 3.12, were known for non-conformally flat metrics, this would be unnecessary, and many of the results here could be given a more satisfactory form.

Analyticity of $\mathcal{M}_\Lambda$ and the Generic Slice Theorem. In addition to its intrinsic nature and relationship to the concerns of this paper, the urmoduli space $\mathbb{M}_\Lambda$ is required for two purposes: to show that $\mathcal{M}_\Lambda$ itself is analytic, and to prove the Generic Slice Theorem. The proofs of these results are very closely related, so we develop the preliminaries for them simultaneously.

Inside the ambient space $\mathcal{V} = H^{s+2}_{-\delta}(\mathrm{Sym}^2) \times \mathbb{W}$ there are two analytic (Hilbert) submanifolds, namely $\mathbb{M}_\Lambda$ and the conformal class $[g_0]$. (By $[g_0]$ we always mean the set of all metrics in $\mathcal{V}$ conformal to the round metric g_0.) Likewise we have all other conformal classes in $\mathcal{V}$; a generic such class will be denoted $[g']$. Notice that all the analytic machinery, in particular the Fredholm theory and relative index computations in §4, still holds for any metric (corresponding to an element) in $\mathcal{V}$.

Choose any element $g \in \mathbb{M}_\Lambda$ and let $[g']$ be the conformal class of g. We shall assume that $[g']$ is very near $[g_0]$. The immediate aim is to prove that $\mathbb{M}_\Lambda$ and $[g']$ intersect almost transversely at g; in general these submanifolds may not be transverse at g, however they will always form a Fredholm pair there. This means that their tangent spaces, $E \equiv T_g[g']$ and $F \equiv T_g\mathbb{M}_\Lambda$, which are closed linear subspaces in $V \equiv T_g\mathcal{V} = H^{s+2}_{-\delta} \oplus W$, form a Fredholm pair, i.e. that $E \cap F$ is finite dimensional, and that $E + F$ is closed in V and has finite codimension there. To any Fredholm pair one can associate an index, which is the dimension of $E \cap F$ minus the dimension of $V/(E+F)$. This number is stable under perturbations of the pair. These matters are explained more thoroughly in [K].

Lemma 6.9. *$\mathbb{M}_\Lambda$ and $[g']$ are a Fredholm pair at any point g in their intersection.*

Proof. The orthogonal complements of E and F are given by

$$\begin{aligned} E^\perp &= \{h : \mathrm{tr}_g(h) = 0\}, \\ F^\perp &= \{h : h = \alpha_g^*(f),\ f \in H^{s+4}_{-\delta}\}. \end{aligned} \tag{6.10}$$

It will suffice to show that $E \cap F$ and $E^\perp \cap F^\perp$ are finite dimensional, and that the orthogonal projection $\pi : F \to E^\perp$ has closed range.

First suppose that $h \in E \cap F$. Then $h = f \cdot g$ with $f \in H^{s+2}_{-\delta} \oplus W$ and $\alpha_g(h) = 0$. Computing $\alpha_g(f \cdot g)$ we find that $L_g(f) = 0$. Since L is a Fredholm operator, there can be at most a finite dimensional family of such h in the intersection; the precise dimension is the same as the nullspace of L on H^{s+2}_δ. On the other hand, if $h \in E^\perp \cap F^\perp$, then $h = \alpha_g^*(f)$ with $f \in H^{s+4}_{-\delta}$ and $\mathrm{tr}_g(h) = 0$. But $\mathrm{tr}_g(\alpha_g^*(f))$ is, up to a factor, just $L_g(f)$, so the dimension of this intersection is the same as the dimension of the nullspace of L on $H^{s+2}_{-\delta}$, and is thus finite.

To conclude the proof, we need to know that the projection π has closed range (and hence is Fredholm, by the work above). The map π is defined, for any $h \in V$, by decomposing $h = \lambda \cdot g + k$, where $\mathrm{tr}(k) = 0$, and setting $\pi(h) = k$. Now suppose h_j is a sequence of elements in F, $\|h_j\| = 1$ for all j, but $k_j = \pi(h_j) \to 0$ in norm. Since $\alpha_g(h_j) = 0$ we have $(n-1)L_g(\lambda_j) = \delta\delta k_j - \langle r_g, k_j\rangle$. The right hand side of this equation certainly goes to zero in norm, so by our Fredholm theory for L_g, the functions λ_j decompose as $\lambda_j = \mu_j + \nu_j$ with ν_j going to zero in norm and μ_j in the nullspace. Since $\mu_j \cdot g$ is in the nullspace of α_g also, we may subtract it off from h_j, and get that $\|h_j\| \to 0$, contrary to hypothesis. Hence π restricted to F has closed range, and this completes the proof.

Before stating the next result we need to introduce some notation. First, if $[g']$ is a conformal class on $\mathbb{S}^n$ (represented by elements in $\mathcal{V}$, and assumed to be near $[g_0]$), then $\mathcal{M}_\Lambda([g'])$ will denote the moduli space of complete metrics on $\mathbb{S}^n \backslash \Lambda$ with constant scalar curvature $n(n-1)$ in the conformal class $[g']$. For $\epsilon > 0$ any sufficiently small number we also let $\mathcal{M}_{\Lambda,\epsilon}([g'])$ denote the subset of $\mathcal{M}_\Lambda([g'])$ consisting of solution metrics g with the Killing norms of all Pohožaev invariants bounded below by ϵ (see §7 for a discussion of these norms). This is simply the subset of $\mathcal{M}_\Lambda([g'])$ consisting of solutions with neck sizes uniformly bounded away from 0 by ϵ. By Proposition 3.12, each $\mathcal{M}_{\Lambda,\epsilon}([g_0])$ is compact; then, provided $[g']$ is sufficiently close to $[g_0]$, $\mathcal{M}_{\Lambda,\epsilon}([g'])$ will also be compact, since it is contained in a compact neighborhood of $\mathcal{M}_{\Lambda,\epsilon}$ by virtue of the implicit function theorem and the considerations above.

Generic Slice Theorem 6.11. *For any fixed $\epsilon > 0$, the truncated moduli space $\mathcal{M}_{\Lambda,\epsilon}([g'])$ is a k-dimensional real analytic manifold for all conformal classes $[g']$ in a set of second category and sufficiently close to $[g_0]$.*

Proof. The reason for introducing the truncated moduli spaces is, of course, that we know the existence of $\mathbb{M}_\Lambda$ to a fixed distance away from $\mathcal{M}_\Lambda$ only over each $\mathcal{M}_{\Lambda,\epsilon}$. We shall not comment further on this modification, but simply remark that it could be removed provided somewhat more were known about $\mathbb{M}_\Lambda$, as discussed at the end of the previous subsection.

This result follows from the Sard-Smale theorem, once we have checked the hypotheses. The basic point, of course, is that $\mathcal{M}_\Lambda([g'])$ is the same as $\mathbb{M}_\Lambda \cap [g']$. We parameterize the set of conformal classes (not modulo diffeomorphisms!) close to $[g_0]$ by the linear Hilbert space $E^\perp = \{k \in H^{s+2}_{-\delta} : \operatorname{tr}_g(k) = 0\}$ for some fixed $g \in \mathcal{M}_\Lambda$. Now consider the projection map

$$\Pi : \mathbb{M}_\Lambda \longrightarrow E^\perp. \tag{6.12}$$

$\mathcal{M}_\Lambda$ is given by the preimage $\Pi^{-1}(0)$. The preimage $\Pi^{-1}(k)$ in $\mathbb{M}_\Lambda$ is precisely the set $\mathcal{M}_\Lambda([g'])$ where $[g'] = [g_0 + k]$ is the conformal class corresponding to k. This preimage is a smooth, in fact real analytic, manifold provided k is a regular value of Π. By Sard-Smale, once we know that Π is a Fredholm map (of index k), then the set of regular values in $E^\perp$ is of second category. Observing that the tangents to the orbits of the diffeomorphism group are certainly not of second category, we may conclude that the moduli spaces over generic conformal classes close to $[g_0]$ are smooth and k dimensional.

However, the assertion that Π is Fredholm of index k is contained within the preceding lemma. Indeed, if $g \in \mathbb{M}_\Lambda$ and $F = T_g(\mathbb{M}_\Lambda)$, then we proved there that the projection $\pi = \Pi_* : F \to E^\perp$ is Fredholm, and its index is the same as the relative index of L_g across the weight 0, i.e. equal to k.

To finish the proof we need to eliminate the possibility that $\Pi(\mathbb{M}_\Lambda)$ is contained within a (finite codimensional) submanifold of $E^\perp$. Although this would follow from knowing that π is surjective, this is a difficult issue. Instead we appeal to the existence theory for the nonlinear equation, generalizing Schoen's basic construction. In fact, the construction of solutions given in [S2] may be carried out not only for the standard round metric g_0, but also for generic metrics g' which are small perturbations of g_0 compactly supported away from Λ. The only modification of Schoen's method needed to accomplish this is given in [P1], Proposition 1.2. The linearization of this set of compactly supported perturbations is clearly dense in $E^{\perp}$, hence there is a dense set of $k \in E^{\perp}$ near the origin for which the preimage $\Pi^{-1}(k)$ is nonempty. This is sufficient to conclude that $\Pi(\mathbb{M}_\Lambda)$ contains a full neighborhood of 0, hence the condition 'regular value' for Π is not the empty one. This completes the proof.

The reader should note that an alternate possibility for proving smoothness of generic slices would be to study the linear operator L_g for any $g \in \mathbb{M}_\Lambda$ close to but not in the standard conformal class and show that this operator generically has no L^2 kernel. The smoothness would then follow by the results of §5, combined with a compactness argument. However, proving that L_g has no decaying eigenfunctions would require setting up a somewhat elaborate perturbation theory, since this is equivalent to trying to perturb point spectrum which is sitting on the end of a band of continuous spectrum. Thankfully we have been able to avoid this approach here (although we have had to appeal to the rather more difficult existence theory for the nonlinear equation instead). It should be noted that the surjectivity of Π_* (which we have established) is equivalent to this eigenvalue perturbation result.

The second main result, that $\mathcal{M}_\Lambda$ is (locally) a real analytic variety, also follows from Lemma 6.9.

Theorem 6.13. *For any $g \in \mathcal{M}_\Lambda$ there exists a ball B in $\mathcal{V} = H^{s+2}_{-\delta}(\mathbb{S}^n \backslash \Lambda; \mathrm{Sym}^2) \times \mathbb{W}$, a finite dimensional space M, a real analytic variety $\mathcal{A}$ in M and a real analytic diffeomorphism*

$$\Phi : B \longrightarrow \mathcal{V}$$

such that $\Phi(\mathcal{M}_\Lambda \cap B) = \mathcal{A} \cap B'$, where B' is a small ball containing $\Phi(g)$.

The proof reduces to the following abstract result, which is presumably well-known:

Lemma 6.14. *Let V be a Hilbert space, and $\mathcal{E}$ and $\mathcal{F}$ two real analytic submanifolds, the tangent spaces of which at any point of the intersection $p \in \Gamma \equiv \mathcal{E} \cap \mathcal{F}$ form a Fredholm pair. Then for each point $p \in \Gamma$ there exists a neighborhood B of p, a finite dimensional subspace $M \subset V$, a real analytic variety $\mathcal{A} \subset M$, and a real analytic diffeomorphism*

$$\Phi : B \longrightarrow V$$

such that $\Phi(\Gamma \cap B) = \mathcal{A} \cap B' \subset M \cap B'$, where B' is a neighborhood of $\Phi(p)$ in M.

Proof. By initially composing with a real analytic diffeomorphism we may assume that a neighborhood of p in $\mathcal{E}$ lies in a linear subspace and that $p = 0$. (In our situation, $\mathcal{E}$ is the conformal class and this may be accomplished by a translation.) Let $\bar{\Psi}$ denote a 'defining function' for $\mathcal{F}$ in V. By this we mean that $\bar{\Psi}$ is a real analytic map from V into another Hilbert space U such that $\mathcal{F} = \bar{\Psi}^{-1}(0)$. By the stability of Fredholm pairs, any level set $\bar{\Psi}^{-1}(w)$ forms a Fredholm pair with $\mathcal{E}$, for all sufficiently small $w \in U$, and for all points of intersection near the origin in $\mathcal{E}$. It is evident that this statement is equivalent to the assertion that $\Psi \equiv \bar{\Psi}\big|_{\mathcal{E}} : \mathcal{E} \to U$ is a Fredholm map. The intersection $\mathcal{E} \cap \mathcal{F}$ is the same as $\Psi^{-1}(0)$. Thus the lemma may be rephrased as saying that if Ψ is a real analytic Fredholm map between two Hilbert spaces $\mathcal{E}$ and U, then the level set $\Psi^{-1}(0)$ is locally equivalent, by an analytic diffeomorphism, to a finite dimensional analytic variety.

A standard implicit function theorem argument proves this assertion. Let $A = \Psi_*|_0$, and set $M = \ker(A)$, $L = W \ominus \mathrm{range}(A)$. Then both L and M are finite dimensional. Define a new map $\tilde{\Psi} : \mathcal{E} \oplus L \to U$ by $\tilde{\Psi}(q, f) = \Psi(q) + f$. Then

$$\tilde{\Psi}_*\Big|_{(0,0)}(h, \phi) = A(h) + \phi,$$

so this differential is obviously surjective. Note that

$$\ker(\tilde{\Psi}_*\Big|_{(0,0)}) = \{(h, \phi) : A(h) + \phi = 0\}.$$

Since $A(h)$ and ϕ lie in orthogonal spaces, this nullspace is just $M = \ker(A)$. The analytic implicit function theorem gives the existence of two analytic maps

$$k : M \to M^\perp \subset \mathcal{E}, \qquad \ell : M \to L$$

and a ball $\tilde{B}$ around $(0, 0) \subset \mathcal{E} \oplus L$ such that all zeroes of $\tilde{\Psi}$ in this ball lie in the graph of the map $(k, \ell) : M \to M^\perp \oplus L$:

$$\{(q, f) : \Psi(q) + f = 0\} \cap \tilde{B} = \{(m + k(m), \ell(m)) : m \in M \cap \tilde{B}\}.$$

Thus, for m in this ball, $\Psi(m + k(m)) + \ell(m)$ vanishes identically. The zeroes of Ψ in $\tilde{B} \cap \mathcal{E}$ are then just $\{m + k(m) : m \in \tilde{B} \cap M : \ell(m) = 0\}$, for these points are all zeroes of $\tilde{\Psi}$, and are the only such zeroes at which Ψ also vanishes. If Π is an analytic diffeomorphism of $\mathcal{E}$ carrying the graph of k into M, then $B \cap \Psi^{-1}(0)$ equals $\Pi^{-1}(M \cap \tilde{B}) \cap \{m \in M \cap \tilde{B} : \ell(m) = 0\}$ as desired. This completes the proof of Lemma 6.14.

It is also possible to establish the real analytic structure of $\mathcal{M}_\Lambda$ by using a modification of 'Liapunov-Schmidt reduction' (also referred to as the 'Kuranishi method'). The closely related and analogous result concerning the real analyticity of the moduli space of complete, embedded, constant mean curvature surfaces in $\mathbb{R}^3$ is proved by this method in [KMP]. This technique is more direct than the one used above, however in using the approach provided by the urmoduli space, $\mathbb{M}_\Lambda$, we also obtain the Generic Slice Theorem rather easily.

7. Concluding remarks and informed speculation

L^2-nullspace and singularities of $\mathcal{M}_\Lambda$. We have not yet discussed whether it is possible to give conditions ensuring that a given $g \in \mathcal{M}_\Lambda$ is a 'smooth point', as defined in §5. It is possible for $\mathcal{M}_\Lambda$ to be both smooth and of the correct dimension near g even if L_g has L^2-nullspace; however, absence of this nullspace is our only criterion for guaranteeing this smoothness. We expect that it should be difficult to establish such a criterion in general. It would also be quite interesting to understand when $\mathcal{M}_\Lambda$ is not smooth in the neighborhood of some element g. In particular, constructing solutions near which $\mathcal{M}_\Lambda$ is singular seems like another very challenging and important problem.

On the other hand, the two known constructions for producing points in $\mathcal{M}_\Lambda$, those of Schoen [S2] and [MPU], yield solutions with explicit geometries. The dipole solutions of [MPU], described below, are manifestly smooth points. These solutions exist only for certain configurations Λ, whereas the ones in [S2] exist for any Λ. Unfortunately, the existence of L^2-nullspace for these latter solutions is less evident, although we expect this to be true; outlined below is a strategy to prove this. Note that once the existence of one smooth point in any component of $\mathcal{M}_\Lambda$ is established, the real analyticity of the moduli space then shows that almost every element in that component is a smooth point.

We now give a brief description of Schoen's solutions and their construction. These solutions are uniformly small $\mathcal{C}^0$ perturbations of explicit approximate solutions. Each of these approximate solutions is constructed from an 'admissible conformal structure' (Γ, σ); Γ is an infinite tree with a labeling, σ, of strong dilations G_{ij} for each directed edge e_{ij}. We assume for simplicity that Γ has one vertex of order k and all other vertices of order 2. The admissibility of the labeling σ refers to certain compatibility conditions that the dilations must satisfy. The strength, λ_{ij}, of these dilations can be related to a measurement of the 'neck sizes' ϵ_{ij} of the approximate solution g_σ constructed from the data (Γ, σ). $(\mathbb{S}^n \backslash \Lambda, g_\sigma)$ consists of almost spherical regions, corresponding to the vertices V of Γ, joined by small necks, corresponding to the edges, the sizes of which are dictated by the ϵ_{ij}. There are infinitely many parameters in the construction of these approximate solutions since the admissible conformal structures σ can be varied, even with Γ fixed. In particular, one can begin with an initial conformal structure σ_1, so that the corresponding approximate solution g_1 consists of one central spherical region Ω_0 (the vertex of order k) and k periodic, spherically symmetric ends. But in order to find an exact solution, it is necessary to deform the conformal structure, as dictated by the Pohožaev balancing condition, and break this symmetry.

The conformal structure (Γ, σ) decomposes $\mathbb{S}^n \backslash \Lambda$ into a union of almost spherical regions, Ω_i, $i \in V$. Each $\Omega_i \subset \mathbb{S}^n$ is the pullback by a conformal diffeomorphism F_i of $\mathbb{S}^n$ of a large region of $\mathbb{S}^n$. The metric g_σ on Ω_i is constructed so that $F_i : (\Omega_i, g_\sigma) \to (\mathbb{S}^n, g_0)$ is an isometry off a small neighborhood of $\partial\Omega_i$. This decomposition gives rise to a basic analytic property of the approximate solution metric g_σ. This is the existence of a basis of functions η^σ_{ij}, $i \in V$ and $j = 1, \dots, n+1$, for an infinite dimensional space K (the 'small eigenspace') corresponding to all the spectrum of $L_{g_\sigma} = \Delta_{g_\sigma} + n$ in a small interval around 0. The functions η^σ_{ij} have explicit geometric descriptions. Each η^σ_{ij} has support concentrated on Ω_i and corresponds to the linear function η_j on $F_i(\Omega_i) \subset \mathbb{S}^n$. Linear functions are of course eigenfunctions for Δ_{g_0} with eigenvalue n. On the orthogonal complement $K^\perp$, L_{g_σ} is invertible, uniformly in σ. As the $\epsilon_{ij} \to 0$, these approximations improve and L_{g_σ} converges, as an operator on L^2, to the operator L_{g_0} on the disjoint union of spheres indexed by V.

Writing the solution g as a perturbation, $g = (1+\eta)^{\frac{4}{n-2}} g_\sigma$ and using the conformal invariance of the conformal Laplacian, we find that

$$\begin{aligned} L_g &= \Delta_g + n \\ &= (1+\eta)^{-\frac{4}{n-2}} \Delta_{g_\sigma} + 2(1+\eta)^{-\frac{n+2}{n-2}} \langle \nabla\eta, \nabla \,\rangle + n. \end{aligned} \tag{7.1}$$

Now, η is small in $\mathcal{C}^0$ (by which we mean that $|\eta|_0$ can be made arbitrarily small by taking all ϵ_{ij} sufficiently small), and estimates from [S2] can be used to show that $\nabla\eta$ is small in L^2 over any fixed compact set, but these facts do not immediately imply the existence of an analogous good basis for the 'small eigenspace' K for L_g. Nonetheless, we expect this basis to exist, namely that there exist an $\epsilon > 0$ such that if $\epsilon_{ij} < \epsilon$ for all ij, then there exists a set of smooth functions η_{ij}, $i \in V$, $j = 1, \ldots, n+1$, which satisfy the estimates in Lemma 3.6 of [S2]. The span of these functions is a subspace $K \subset L^2(\mathbb{S}^n \backslash \Lambda)$ such that

$$\begin{aligned} \|L_g\eta\|_{L^2(\mathbb{S}^n\backslash\Lambda)} &\le c(\epsilon)\|\eta\|_{L^2(\mathbb{S}^n\backslash\Lambda)} \text{ for } \eta \in K, \\ \|\eta\|_{L^2(\mathbb{S}^n\backslash\Lambda)} &\le c\|L_g\eta\|_{L^2(\mathbb{S}^n\backslash\Lambda)} \text{ for } \eta \in K^\perp, \end{aligned}$$

where c is a constant independent of ϵ, and $c(\epsilon)$ tends to 0 as $\epsilon \to 0$. Note that from (7.1) this conjecture would be immediate if $\|\eta\|_{H^1}$ were finite and small. Unfortunately, this is never the case, since the η in Schoen's construction is not even in L^2.

The likeliest method to establish the existence of this space K with its explicit basis is to write the solution along each end as a conformal perturbation of the Delaunay metric g_ϵ on the half-cylinder to which it converges at infinity. There is good evidence that perturbation determining the solution is uniformly small and even exponentially decaying, again provided the neck sizes are sufficiently small. With this information in hand, a not very complicated transference procedure would produce the basis for K with all necessary estimates.

If these could be settled, it would then be possible to resolve some basic questions about the existence of an L^2-nullspace of L_g, for Schoen's solutions.

Conjecture 7.2. *Suppose $g \in \mathcal{M}_\Lambda$ is a solution similar to that constructed in [S2], so that $g = (1+\eta)^{\frac{4}{n-2}} g_\sigma$ as above. Assume that $\Lambda \subset \mathbb{S}^n$ does not lie in any round hypersphere. Then there exists an $\epsilon = \epsilon(\Lambda) > 0$, such that if σ satisfies $\epsilon_{ij} < \epsilon$ for each edge (i.e. all 'neck sizes' are less than ϵ), then L_g has no L^2-nullspace, and hence g is a smooth point of $\mathcal{M}_\Lambda$. If Λ does lie in a hyperplane, then any element ϕ in the L^2-nullspace of L_g is not integrable, i.e. it is not tangent to a path in $\mathcal{M}_\Lambda$.*

We provide a brief sketch of our plan for proving this. If g is a solution constructed from some (Γ, σ), with all neck sizes sufficiently small, and if $\phi \in L^2$, $L_g\phi = 0$, then ϕ admits two different decompositions, one into the 'almost linear functions' η_{ij} concentrated on the almost spherical regions Ω_i, and the other (which is only local along each end E_ℓ) into eigenfunctions for the Laplacian along the cross-sectional sphere, as in §2. For this latter decomposition we need the (putative) fact that the solution stays uniformly close to the Delaunay solution to which it asymptotically converges. The former expansion would show that ϕ is very close to a linear function on each of the almost spherical regions. A very important point here is that the linear function determined by ϕ on each of the spherical regions can never vanish. In terms of the eigenfunctions of the second decomposition, this first decomposition implies that along each end, ϕ has most of its mass concentrated in the eigenspaces corresponding to the zeroth and first nonzero eigenvalues of the cross-section. Since g is only approximately Delaunay, the zeroth Fourier coefficient ϕ_0 of ϕ is not an exact solution of $L_0\phi_0 = 0$, however, the error terms are sufficiently controllable to show that ϕ_0 must be quite small (again, depending on the size of ϵ). Hence, in fact, most of the mass of ϕ is contained in the coefficients $\phi_1, \dots, \phi_n$, corresponding the the first nonzero eigenvalue of the Laplacian on $\mathbb{S}^{n-1}$ (which has multiplicity n). We call these 'transverse' linear functions, since each of the spherical regions where we consider them has a natural axis picked out. Thus these arguments show that an L^2 solution of $L\phi = 0$ restricts on each spherical region to be almost linear. More specifically, on the innermost sphere along each end (the one adjoining the central spherical region), ϕ actually restricts to be approximately transverse linear. But now, on the central sphere, ϕ is linear and near each of the neighborhoods where the various ends are attached, must restrict to a transverse linear function relative to the axis determined by that end. This is clearly impossible if the points do not lie in any subsphere.

For the second assertion, concerning the situation when $\Lambda \subset \mathbb{S}^{n-1}$, we only need to use that ϕ is sufficiently close to a linear function on the central sphere. Of necessity, this linear function is one which vanishes on the equator determined by Λ. Hence, if ϕ were an integrable Jacobi field, we would obtain a family of solutions which were not symmetric about this equator, contradicting the reflectional symmetry guaranteed by the Alexandrov reflection argument.

We hope to be able to settle these issues in the near future. There are, of course, a number of more refined questions about the singular structure of $\mathcal{M}_\Lambda$, beginning with the problem, already noted, of constructing solutions near which $\mathcal{M}_\Lambda$ is singular.

Coordinates on $\mathcal{M}_\Lambda$. In this section we discuss two ways in which coordinates for $\mathcal{M}_\Lambda$ may be given. We begin with the one arising from the linear analysis.

If $g \in \mathcal{M}_\Lambda$ is a point for which L_g has no L^2-nullspace, then a neighborhood of g in $\mathcal{M}_\Lambda$ is parameterized by a small neighborhood of the origin in the bounded nullspace $\mathcal{B}$. Thus, linear coordinates on $\mathcal{B}$ yield local coordinates on $\mathcal{M}_\Lambda$ near g. The problem then is to get precise information about these linear coordinates on $\mathcal{B}$. Unfortunately, this seems difficult, in general. We now discuss how one would go about this, and at least set the problem up in scattering-theoretic terms.

By definition, any element ϕ in the deficiency subspace W can be expanded on each end as a combination of the model solutions $\phi_1,\ \phi_2$ plus an exponentially decreasing error. For clarity, we label these model solutions on E_j with a superscript (j). Thus

$$W \ni \phi \sim a_j\phi_1^{(j)} + b_j\phi_2^{(j)} \text{ on } E_j. \tag{7.3}$$

So any $\phi \in W$ determines a map

$$\begin{aligned} S : W &\longrightarrow \mathbb{R}^{2k}, \\ \phi &\longmapsto (a_1, b_1, \dots, a_k, b_k). \end{aligned} \tag{7.4}$$

By the definition of W, S is an isomorphism. Now suppose ϕ is in the bounded nullspace $\mathcal{B}$.

Lemma 7.5. *Under the hypothesis (5.2), the linear map*

$$C \equiv S|_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathbb{R}^{2k}$$

is injective. Its image S_Λ is a k-dimensional subspace which is Lagrangian with respect to the natural symplectic form $\sum_{j=1}^{k} da_j \wedge db_j$ on $\mathbb{R}^{2k}$.

Proof. C is obviously linear, and it is injective, because otherwise there would exist an element $\phi \in \mathcal{B}$ with all its asymptotic coefficients a_j, b_j vanishing. By the results of the last section, such a ϕ would lie in L^2, which contradicts the hypothesis (5.2).

That S_Λ is Lagrangian is a consequence of Green's theorem. There is an analogous result for manifolds with asymptotically cylindrical ends proved in [Me]. Since we already know that S_Λ is k-dimensional, it suffices to prove that it is isotropic. For this, let $\psi,\ \chi \in \mathcal{B}$ and set $C(\psi) = (a_1, b_1, \ldots, a_k, b_k),\ C(\chi) = (\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k)$. Using the coordinate t_j along the end E_j, let M_A denote the region in $\mathbb{S}^n \backslash \Lambda$ where each $t_j \leq A$. Then, since $L\psi = L\chi = 0$,

$$0 = \int_{M_A} ((L\psi)\chi - \psi(L\chi))\, dV_g = \int_{\partial M_A} \left(\frac{\partial \psi}{\partial \nu}\chi - \psi \frac{\partial \chi}{\partial \nu} \right) d\sigma_g,$$

where ν is the unit vector-field normal to ∂M_A. Substitute the expansions for ψ and χ at each end, and drop all terms which decrease exponentially as $A \to \infty$ in this expression. This means that we can replace the metric g on E_j by the Delaunay metric g_{ϵ_j} there, and also replace ν by $u_{\epsilon_j}^{-2/(n-2)}\partial_{t_j}$ and $d\sigma_g$ by $u_{\epsilon_j}^{(2n-2)/(n-2)}$. Taking the limit as $A \to \infty$ we get

$$0 = \sum_{j=1}^{k} (\alpha_j b_j - a_j \beta_j) \int_{\mathbb{S}^{n-1}} \left(\frac{\partial \phi_1^{(j)}}{\partial t_j} \phi_2^{(j)} - \phi_1^{(j)} \frac{\partial \phi_2^{(j)}}{\partial t_j} \right) u_{\epsilon_j}^2\, d\sigma. \tag{7.6}$$

Here $d\sigma$ is the standard volume measure on $\mathbb{S}^{n-1}$. If we set A_j equal to the integrand for the j^{th} end, then it suffices to show that these constants A_j are nonzero and independent of j. However, $\phi_1^{(j)}$ and $\phi_2^{(j)}$ are independent solutions of the ODE $L_0\phi = 0$, and the expression in parentheses in the integrand in (7.6) is just the Wronskian for this ODE. This Wronskian may be written explicitly as some nonzero multiple of

$$\exp\left(-2 \int \frac{u'_{\epsilon_j}}{u_{\epsilon_j}} \right) = u_{\epsilon_j}^{-2}.$$

This shows that the integrand, hence the integral itself, is a non-vanishing constant, independent of j.

The relevance of S_Λ to the description of coordinates on $\mathcal{M}_\Lambda$ near g is explained as follows. Suppose first, for simplicity, that there exists an element $\phi \in \mathcal{B}$ for which all $a_j,\ b_j = 0,\ j > 1$, but $a_1 = 1,\ b_1 = 0$. If $g(s)$ is a curve in moduli space tangent to this ϕ, then the Delaunay parameters ϵ_j on each of the ends would remain fixed, at least infinitesimally, along this curve. In fact, the metrics $g(s)$ (or the underlying curve of solutions u_s of the PDE) would be strongly asymptotic to the initial metric g on E_j, $j > 1$. On E_1 however, $g(s)$ would be strongly asymptotic to a translation of g, or alternately, of its model Delaunay metric there. If, on the other hand,

$a_1 = 0$, $b_1 = 1$ and all other a_j, $b_j = 0$, then again each $g(s)$ would be strongly asymptotic to g on all ends except E_1; on that end the Delaunay parameter for the model metric would be changing, but the model Delaunay metric would not be translating. Of course, neither of these types of elements of $\mathcal{B}$ need exist.

If ϕ is a general element of $\mathcal{B}$, and $g(s)$ is a curve in $\mathcal{M}_\Lambda$ through g tangent to ϕ, then ϕ describes infinitesimal changes in translation and Delaunay parameters along each end. The most precise description of coordinates on $\mathcal{M}_\Lambda$ would relate the proportions of these various changes on each end to one another for all directions $\phi \in T_g\mathcal{M}_\Lambda$. This is equivalent to describing the coefficients a_j, b_j for each $\phi \in \mathcal{B}$, and this, in turn, is equivalent to describing the Lagrangian subspace S_Λ. An explicit description of S_Λ requires a better understanding of the Pohožaev balancing condition (3.9).

For the sake of illustration, let us examine all this for $\mathcal{M}_2$, the moduli space when $k = 2$. Of course, we have already given a complete description of this space, but it provides a concrete example of this description. We let the background metric be any Delaunay metric g_ϵ with $\epsilon < \bar{u}$. Now $\mathcal{B}$ is 2 dimensional, consisting of the elements ϕ_1 and ϕ_2. Hence

$$S_\Lambda = \{a_1 = -a_2, b_1 = b_2\} \subset \mathbb{R}^4,$$

which is Lagrangian, as expected. The two natural curves $g(s)$ emanating from any $g \in \mathcal{M}_2$ are simply the families Φ_1 and Φ_2 defined in (2.14).

It may be possible to give a geometric description of parameters on $\mathcal{M}_\Lambda$ by considering the Killing norms of the Pohožaev invariants. Recall that these invariants are elements of $\mathfrak{o}^\star(n+1,1)$, the dual of the Lie algebra of the conformal group. Any element $X \in \mathfrak{o}(n+1,1)$ may be uniquely decomposed as $X = X_0 + \mathbf{w}$, where $X_0 \in \mathfrak{o}(n+1)$ and $\mathbf{w} \in \mathbb{R}^{n+1}$. The Killing form

$$B : \mathfrak{o}(n+1,1) \times \mathfrak{o}(n+1,1) \to \mathbb{R}$$

is the nondegenerate symmetric quadratic form given by

$$B(X, \hat{X}) = \frac{1}{2} Tr(X_0 \hat{X}_0) + \mathbf{w} \cdot \hat{\mathbf{w}}.$$

Thus, $B(\cdot,\cdot)$ is positive definite on $\mathbb{R}^{n+1}$ and negative definite on $\mathfrak{o}(n+1)$ (with respect to this decomposition). Moreover, the adjoint representation preserves B in the sense that $B(Ad(F)(X), Ad(F)(\hat{X})) = B(X, \hat{X})$, for all $F \in O(n+1,1)$ and $X, \hat{X} \in \mathfrak{o}(n+1,1)$.

Since B is nondegenerate it provides an identification between $\mathfrak{o}(n+1,1)$ and its dual space. Thus we may use the Killing form to identify, for the Pohožaev invariants $\mathcal{P}_1, \dots, \mathcal{P}_k$ of a metric $g \in \mathcal{M}_\Lambda$, corresponding elements $\mathcal{P}'_1, \dots, \mathcal{P}'_k \in \mathfrak{o}(n+1,1)$. The squared Killing norm of the Pohožaev invariant $\mathcal{P}_i$ is then defined to be

$$\|\mathcal{P}_i\|^2 = B(\mathcal{P}'_i, \mathcal{P}'_i).$$

These k numbers are conformal invariants. By this we mean the following. If $F \in O(n+1,1)$ is a conformal diffeomorphism of $\mathbb{S}^n$ and $g \in \mathcal{M}_\Lambda$, then $F^\star(g) \in \mathcal{M}_{F^{-1}(\Lambda)}$. The Pohožaev invariants of g and $F^\star(g)$ do not coincide but transform via the co-adjoint representation (see [S2] and [KKMS]). Since B is invariant under this representation this implies that the squared Killing norms of the Pohožaev invariants of g and $F^\star(g)$ are the same.

Schoen has suggested the following way to obtain parameters on $\mathcal{M}_\Lambda$. Let $\Lambda = \{p_1, \dots, p_k\}$ be a balanced singular set. One can then try to produce, for some $\epsilon > 0$, a 1-parameter family of solutions $g_t \in \mathcal{M}_\Lambda$, $t \in (0, \epsilon)$, such that the asymptotic neck sizes $\epsilon_1, \dots, \epsilon_k$ for g_t are all equal to t. To realize the other $(k-1)$-parameters, choose any k-tuple of numbers, $(a_1, \dots, a_k)$, close to 1 and normalized so that $a_1 = 1$. Then there is a conformal diffeomorphism F taking Λ to $\Lambda' = \{p'_1, \dots, p'_k\}$ so that $\sum_{j=1}^k a_j p'_j = 0$. As before, there is now a 1-parameter family of solutions g'_t, on $\mathbb{S}^n \setminus \Lambda'$ with asymptotic neck sizes given by $(a_1 t, \dots, a_k t)$. Thus $F^\star(g'_t) \in \mathcal{M}_\Lambda$ is another 1-parameter family of solutions. This exhibits the k-parameters as $(t, a_2, \dots, a_k)$. It should be possible to phrase this in terms of the squared Killing norms of the Pohožaev invariants described above.

Dipole solutions. We briefly describe a new construction of solutions [MPU] for certain special configurations $\Lambda \subset \mathbb{S}^n$. In general terms, this construction shows that any two 'nondegenerate' complete manifolds M_1 and M_2 with constant positive scalar curvature may be grafted together to obtain a one parameter family of complete conformal metrics of constant positive scalar curvature on the connected sum $M_1 \# M_2$; furthermore, these solution metrics are nondegenerate as well. Here nondegenerate means simply that the linearized scalar curvature operator L has no L^2-nullspace, and that it is surjective on a suitable extension of L^2 (this is explained more carefully in [MPU]). This is proved by a fairly standard gluing procedure, using the implicit function theorem.

Since, in particular, Delaunay metrics on $\mathbb{S}^{n-1} \times \mathbb{R}$ are nondegenerate, we obtain nondegenerate solutions on $\mathbb{S}^n \setminus \{p_1, p_2, q_1, q_2\}$, where p_1, p_2 and q_1, q_2 are two pairs of points, with each pair clustered tightly near two antipodal points $P, Q \in \mathbb{S}^n$. The conformal factor u on $\mathbb{S}^n$ which is singular at these four points is very small on the complement of small balls around P and Q containing the p_i, q_i, and is highly concentrated in these balls. For this reason, we call these 'dipole solutions'. The large region where u is small corresponds to the small neck joining the two Delaunay solutions. As the neck gets smaller, the p_i converge to P and analogously for the q_i.

By the final assertion of the result, this new solution (or rather, family of solutions) is nondegenerate, so the process may be iterated. We obtain

Theorem 7.7 ([MPU]). *Let $P_1, \dots, P_\ell$ be a balanced set of points on $\mathbb{S}^n$. Then for ϵ sufficiently small, there exist points $p_i^{(1)}$ and $p_i^{(2)}$ in each ϵ ball $B_\epsilon(P_i)$ and a moduli space of dimension 2ℓ of nondegenerate solutions on $\mathbb{S}^n \setminus \Lambda$ where $\Lambda = \{p_i^{(j)},\ 1 \le i \le \ell,\ j = 1, 2\}$. Moreover the map from this moduli space to $\mathbb{R}^{2\ell}$, sending a solution to the translation and neck parameters at $p_i^{(1)}$, for $i = 1, \dots, \ell$, is an isomorphism onto an open set.*

There are several important consequences of this result. The first is that since we can fix the neck parameters of the initial Delaunay solutions arbitrarily before the gluing, and since we may choose the gluing to alter these parameters only on one end of each Delaunay solution, we obtain solutions on the complement of 2ℓ points, where the Delaunay parameters are prescribed at ℓ of the points. In particular, we obtain solutions with 2ℓ ends, ℓ of which are asymptotically cylindrical. This is important, in part, because it produces solutions with the simplest type of asymptotic behavior.

Another consequence concerns nondegeneracy and follows from an elaboration of our perturbation techniques. We can consider the moduli space of pairs

$$\mathcal{M} = \{(\Lambda, \mathcal{M}_\Lambda)\}.$$

A development of the ideas here shows that $\mathcal{M}$ itself is a real analytic set. Its expected dimension is $k(1+n)$, where k is the cardinality of Λ. If there is a nondegenerate solution for a given configuration Λ, then the whole component of $\mathcal{M}$ containing that solution has this dimension; by considering $\mathcal{M}$ as a fibration over the kn-dimensional space of configurations, we see that generic fibers also have the correct dimension k, and hence that generic points in this fiber are nondegenerate. Since we have found nondegenerate solutions for certain configurations Λ when $k = 2\ell$, we conclude that generic points in the same component of $\mathcal{M}$ (presumably including those where the points are not so tightly clustered in pairs) are also nondegenerate. Unfortunately, no uniqueness theorem has been proved, even in the restricted setting of Theorem 7.7. This means that we have no information on the number of connected components other than to assert, as a consequence of the compactness theorem [P2], that there are finitely many.

The structure of the boundary $\partial\overline{\mathcal{M}_\Lambda}$. There are a number of interesting questions about the structure of the compactification $\overline{\mathcal{M}_\Lambda}$. Recall from §3 that this compactification is obtained by adding certain solutions $g \in \mathcal{M}_{\Lambda'}$ to $\mathcal{M}_\Lambda$, for certain subsets $\Lambda' \subset \Lambda$. There are two basic problems. The first is to determine which subsets Λ' have elements occurring in $\partial\overline{\mathcal{M}_\Lambda}$. This is again intimately connected to the balancing condition (3.9). There are few examples where we can say anything explicit about this. The case $k = 2$ was already discussed in §3. The existence of dipole solutions [MPU] may give additional information concerning this problem when k is even.

The other fundamental problem is to determine how much of $\mathcal{M}_{\Lambda'}$ is contained in $\partial\overline{\mathcal{M}_\Lambda}$ when at least one point $g \in \mathcal{M}_{\Lambda'}$ lies in $\partial\overline{\mathcal{M}_\Lambda}$. More specifically, we propose the following:

Conjecture 7.8. *Suppose that $g \in \mathcal{M}_{\Lambda'} \cap \partial\overline{\mathcal{M}_\Lambda}$. Then the entire component of $\mathcal{M}_{\Lambda'}$ containing g also lies in $\partial\overline{\mathcal{M}_\Lambda}$.*

It is easy to show, using the compactness theorem of [P2], that the set of points g', in the same component of $\mathcal{M}_{\Lambda'}$ as g and which also lie in $\partial\overline{\mathcal{M}_\Lambda}$, is closed. One method to prove that this set is open would be to generalize the general grafting procedure of [MPU]. This would prove the conjecture.

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Abstract. Complete, conformally flat metrics of constant positive scalar curvature on the complement of k points in the n-sphere, $k \geq 2$, $n \geq 3$, were constructed by R. Schoen in 1988. We consider the problem of determining the moduli space of all such metrics. All such metrics are asymptotically periodic, and we develop the linear analysis necessary to understand the nonlinear problem. This includes a Fredholm theory and asymptotic regularity theory for the Laplacian on asymptotically periodic manifolds, which is of independent interest. The main result is that the moduli space is a locally real analytic variety of dimension k. For a generic set of nearby conformal classes the moduli space is shown to be a k-dimensional real analytic manifold. The structure as a real analytic variety is obtained by writing the space as an intersection of a Fredholm pair of infinite dimensional real analytic manifolds.

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