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ON THE GLOBAL DYNAMICS OF ATTRACTORS FOR SCALAR DELAY EQUATIONS

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1. Introduction

This paper serves two purposes:

1. to describe the global dynamics on the attractor of scalar delay differential equations with negative feedback;
2. to suggest a potentially computable approach for describing in coarse terms the global dynamics of *families* of dynamical systems.

By now it is well known that not only do non-linear systems typically exhibit extremely complicated dynamics, but what's worse, that the structure of the dynamics changes dramatically as a function of natural parameters. A generally accepted means of describing these structures is to prove the existence of a conjugacy with respect to a well understood or more easily computable system, e.g., the invariant set of the Smale horseshoe and the shift dynamics on bi-infinite sequences of two symbols. While from a qualitative and theoretical point of view such a description is quite desirable, in practice there are at least two serious drawbacks to this approach. First, starting with a specific evolution equation, i.e., a system of ordinary differential equations, partial differential equations, or functional differential equations, obtaining a rigorous proof of the desired conjugacy can be extremely difficult. Second, even if a conjugacy for a fixed parameter value is given, determining the range for which the conjugacy is valid or how the conjugacy changes as a function of the parameter is a daunting task. To see this, one needs only note that there exist conjugacies between structurally stable and non-structurally-stable flows, e.g., $\dot{x} = -x$ and $\dot{x} = -x^3$. These drawbacks can be a serious handicap. For many models arising from the sciences and engineering, not only are the parameters not known to great precision, but often the class of potential non-linearities is large. Therefore one is faced with the task of providing a rigorous description of the dynamics for classes of equations for which one expects (or for which the numerics indicate) a wide range of behaviors.

The family of scalar delay differential equations with negative feedback, i.e.

$$\dot{x}(t) = -f(x(t), x(t-1)), \tag{1}$$

where $\eta f(0, \eta) > 0$ for all $\eta \neq 0$, provides a concrete example of the problems described above. These equations have been suggested as models in physiology and non-linear optics, and have been the subject of considerable numerical investigation,

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e.g. Farmer [3] and Mackey and Glass [11]. These numerical works strongly suggest that for certain non-linearities and parameter values complicated, possibly chaotic dynamics can occur. On the other hand, to the authors' knowledge there is no exact rigorous characterization of these systems.

Faced with the problem of trying to describe such a potentially diverse set of dynamical systems, we have given up on the idea of establishing a conjugacy with a simpler system. Instead, we use the weaker notion of semi-conjugacy; the goal being to establish a minimal dynamic structure which is present for all equations in a given family. This approach is not without precedent. Consider the following theorem due to C. Conley [2].

Theorem 1.1. *Let S be a compact invariant set under a flow φ. Then, there exists a semi-conjugacy to a gradient-like flow. More precisely, there exists a space X with a gradient-like flow $\psi : \mathbf{R} \times X \to X$ and continuous surjective map $f : X \to X$ such that*

$$\begin{tikzcd}
\mathbf{R} \times S \arrow[r, "id \times f"] \arrow[d, "\varphi"'] & \mathbf{R} \times X \arrow[d, "\psi"] \\
S \arrow[r, "f"] & X
\end{tikzcd}$$

commutes.

It is essential to remark that in this theorem the map f is obtained by collapsing each distinct component of the chain recurrent set to a distinct point. The resulting quotient space is X and the flow ψ is that induced by φ. The point of Theorem 1.1 is to show that the dynamics on S can, in a rigorous manner, be divided into the recurrent motion and non-recurrent or gradient-like motion. While this theorem provides an important framework for interpreting the global dynamics of invariant sets, it is too general to be of use in analyzing any particular system. There are two reasons for this:

1. *a priori* the space X and the flow ψ are not known;
2. all the dynamics within the chain recurrent set are ignored.

While our work is not strictly speaking an application of Conley's theorem, it certainly resembles it in spirit. We will study a large class of flows (precisely described below) which include delay equations with negative feedback. We will identify a "minimal structure" that such flows must posess, by constructing a semi-conjugacy from the flow onto a specific flow on a disk. As in Conley's recurrent – gradient-like decomposition, we will collapse out most (but not all) of the information about the internal dynamics of the recurrent set. However, our result differs significantly from Conley's decomposition in that the space and the flow that we project onto will be explicitly given.

To precisely state our results, we begin with a description of this "model flow" on the disk. Let $z = (z_0, \ldots, z_{2P-1}) \in \mathbf{R}^{2P}$. Then, in polar coordinates $z = r\zeta$, where $r \geq 0$ and $\zeta \in S^{2P-1}$, the unit sphere in $\mathbf{R}^{2P}$. Let $A : \mathbf{R}^{2P} \to \mathbf{R}^{2P}$ be a matrix of the form

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & & & A_P \end{bmatrix}$$

where $A_p = \begin{bmatrix} p^{-1} & 2\pi \\ -2\pi & p^{-1} \end{bmatrix}$, $p = 1, 2, \dots, P$. Let $D^{2P} = \{z = (z_0, \dots, z_{2P-1}) \mid \sum_{p=0}^{2P-1} z_p^2 \leq 1\}$ be the closed unit ball in $\mathbf{R}^{2P}$. Consider the flow

$$\psi : \mathbf{R} \times D^{2P} \to D^{2P} \tag{2}$$

generated by the equations

$$\dot{\zeta} = A\zeta - \langle A\zeta, \zeta \rangle \zeta, \tag{3}$$

$$\dot{r} = r(1-r)\,. \tag{4}$$

If one observes that (3) is obtained by projecting the linear system $\dot{z} = Az$ onto the unit sphere, it is easy to see that the following proposition holds.

Proposition 1.2. *ψ is a Morse-Smale flow for which:*

(i) *The origin $0 = \Pi(P)$ is a fixed point with a $2P$ dimensional unstable manifold $W^u(0)$ and $cl(W^u(0)) = D^{2P}$.*

(ii) *For each $p = 0, \dots, P-1$, the set*

$$\Pi(p) := \{z = (z_0, \dots, z_{2R-1}) \mid z_{2p}^2 + z_{2p+1}^2 = 1\} \subset D_{2p}$$

is a periodic orbit with period 1 and $cl(W^u(\Pi(p)))$ is the $(2p+1)$-sphere $\{z \mid \sum_{i=0}^{2p+1} z_i^2 = 1\}$.

(iii) *$\mathcal{M}(D^{2P}) := \{\Pi(p) \mid p = 0, \dots, P\}$ is a Morse decomposition of D^{2P} with ordering $0 < 1 < \dots < P$.*

We will show that delay equations with negative feedback must posess at least this much structure, in the sense that there is a semiconjugacy *onto* this flow on D^{2P}. In fact, we will produce such a semi-conjugacy for any flow satisfying the following five assumptions.

A1: *$\mathcal{A}$ is a global compact attractor for a semi-flow Φ on a Banach space. Furthermore, if φ denotes the restriction of Φ to $\mathcal{A}$, then φ defines a flow on $\mathcal{A}$.*

A2: *Under the flow $\varphi : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$*

$$\mathcal{M}(\mathcal{A}) = \{M_p \mid p = 0, \dots, P\}$$

with ordering $0 < 1 < \dots < P$ is a Morse decomposition of $\mathcal{A}$.

A3: *For each $p = 0, \dots, P-1$, M_p has a Poincaré section Ξ_p defined on a neighborhood of M_p.*

A4: *The cohomology Conley indices of the Morse sets are*

$$C\check{H}^k(M_P, \mathbf{Z}) \approx \begin{cases} \mathbf{Z} & \text{if } k = 2P, \\ 0 & \text{otherwise,} \end{cases}$$

and for $p = 0, \ldots, P-1$

$$C\check{H}^k(M_p, \mathbf{Z}) \approx \begin{cases} \mathbf{Z} & \text{if } k = 2p, 2p+1, \\ 0 & \text{otherwise.} \end{cases}$$

A5: *For each* M_p*,* $p < P$*, there is a continuation of the flow in a neighborhood of* M_p *to an isolated invariant set which consists of the disjoint union of a hyperbolic periodic orbit and a set with trivial Conley index. The continuation preserves the Poincaré section.*

For an explanation of the terms used in these assumptions, the reader is referred to Section 3. It will be shown in Section 2 that these are natural assumptions for delay equations with negative feedback.

The remainder of the paper is devoted to proving the following theorem.

Theorem 1.3. *Let the flow* φ *on* $\mathcal{A}$ *satisfy assumptions* **A1**–**A5**. *Then there exist a continuous surjective function*

$$f : \mathcal{A} \to D^{2P}$$

for which $M_p = f^{-1}(\Pi(p))$ $(p = 0, \ldots, P)$ *and a continuous flow* $\widetilde{\varphi} : \mathbf{R}\times\mathcal{A} \to \mathcal{A}$ *obtained via an order preserving time reparameterization of* φ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathbf{R} \times \mathcal{A} \arrow[r, "id \times f"] \arrow[d, "\widetilde{\varphi}"'] & \mathbf{R} \times D^{2P} \arrow[d, "\psi"] \\ \mathcal{A} \arrow[r, "f"] & D^{2P} \end{tikzcd}$$

Before turning to a rough sketch of the proof, let us consider the implications of this theorem. A good place to start is with the flow ψ. It is easy to check that it satisfies **A1**, **A2** and **A4** (in particular the homology indices for $p < P$ are those of hyperbolic periodic orbits). Furthermore, any sufficiently small $2P-1$ dimensional disk perpendicular to $\Pi(p)$ acts as a transverse section, therefore **A3** and **A5** also hold. It is reasonable to ask whether, in this case, f is an isomorphism. The answer is: probably not. As the reader will see from the proof, the construction of f is dependent on the transverse section chosen. For example, under f, all orbits connecting the origin $\Pi(P)$ to the stable periodic orbit $\Pi(0)$, which do not intersect the transverse sections Ξ_p, $p = 1, \ldots, P-1$, will be mapped to the same orbit. The point to be made is that even for very simple flows our proof can lead to a significant collapse of orbits.

On the other hand, one should not be under the impression that assumptions **A1** – **A5** force φ to be as simple as ψ. To begin with, it should be clear that **A1** does not imply that $\mathcal{A}$ is homeomorphic to a unit ball in $\mathbf{R}^{2P}$ for some P. It should, also, be remarked that global compact attractors exist for a wide variety of different evolutionary equations and systems (see Hale, Magalhães, and Oliva [10], Hale [9], Temam [21], and references therein).

Since all compact invariant sets have Morse decompositions (a corollary of Theorem 1.1), the content of **A4** lies in having a useful characterization of the Morse sets, i.e. knowing how many, their ordering, and their Conley indices. This information is in general difficult to obtain. On the other hand, it should be emphasized that knowing the Conley index of an isolated invariant set provides very little information concerning the structure of the invariant set itself. In particular, a Morse set M_p with index given by **A4** could be made up of fixed points, periodic orbits, or even contain chaotic dynamics. This latter case is exactly what is suggested by the numerical work cited earlier.

A3 is a fairly restrictive assumption. It precludes the existence of fixed point for any Morse set other than M_P and allows one to define an "angle" coordinate for every other Morse set. Because of this assumption, it is natural to try to map each Morse set onto a periodic orbit. Theorem 1.3 says that this can be done, obviously at the expense of any complicated dynamics which might occur.

A5 is the least understood of these assumptions and will be discussed further in Section 3.

In summary, Theorem 1.3 provides a description of what might be considered the minimal dynamics which must occur on the invariant set $\mathcal{A}$ under the flow φ. Furthermore, this simple picture is obtained by collapsing any non-trivial recurrent dynamics in a Morse set onto a simple periodic orbit.

The proof of Theorem 1.3 occupies Sections 4 – 7. Section 3 provides a terse description of the Conley index and related ideas needed for the proof. Section 4, while technical in nature, can be viewed as motivation for the more complicated construction presented in Sections 5 – 7. The flow $\widetilde{\varphi} : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$ is constructed in Section 5 by reparameterizing φ with respect to time (observe that this is equivalent to defining a homeomorphism $G : \mathcal{A} \to \mathcal{A}$ and letting $\widetilde{\varphi}$ represent the flow induced by φ under G). Section 6 describes the construction of the map f and in Section 7 it is proven that f is onto.

The results of this last section are of interest in and of themselves. A. Floer [4] has shown that a lower bound on the homology or cohomology of an invariant set can be determined by homotoping from a normally hyperbolic invariant set while maintaining control of the topology of the isolating neighborhoods. More precisely, if an invariant set is hyperbolic, it has an isolating neighborhood such that the inclusion-induced map is an isomorphism on cohomology. Under small perturbations, the neighborhood continues to be an isolating neighborhood. In contrast, the topology of the isolated invariant set can change drastically. In particular, the inclusion-induced cohomology map need not be an isomorphism. However, Floer showed that it will continue to be an injection — the isolated invariant set of the perturbed flow will have "at least as much" cohomology as the original isolated invariant set. The point is, a weaker algebraic condition than normal hyperbolicity is sufficient to obtain a "lower bound" on the cohomological complexity of the invariant set. However, no indication as to how this condition can be satisfied, aside from checking a normal bundle condition, is presented in [4].

The results of Section 7 are similar in spirit. We use assumption **A5** to guarantee the existence of a homotopy to a setting in which we can use normal hyperbolicity to compute the necessary topological invariants.[1] These are, however, topological invariants for the individual Morse sets. In the setting of a Morse decomposition of an isolated invariant set, this should be viewed as local information. What we

[1] As in Floer's case all that is actually required is an algebraic condition. This will be explicitly stated in Section 3 as assumption **A5′**.

show is that using the connecting homomorphism of attractor repeller pairs we are able to obtain lower bounds on the homology or cohomology of the total isolated invariant set. Thus we obtain information concerning the global structure from the local structure. Furthermore, the global algebraic invariants we obtain are exactly the algebraic invariants Floer computes via his homotopy. Finally, it must be emphasized that our construction was not done as an academic excercise. For the delay equation, there is no known manifold structure to which the entire attractor can be homotoped. The best we can do, and this will be described in detail in the next section, is to homotope the individual Morse sets to hyperbolic periodic orbits, i.e. **A5** is verified via a homotopy which does not preserve the Morse decomposition of **A2**.

2. Delay equations

In the introduction, the results of this paper were presented in a rather abstract form. This was to emphasize that the assumptions **A1** – **A5**, and consequently Theorem 1.3, are independent of the evolution equation which generates the flow φ. This is important since this same structure appears in a variety of settings, [7] and [17]. Nevertheless, we now take the opportunity to demonstrate that these assumptions are quite natural in the setting of delay equations with negative feedback. In fact, as will become quite clear by the end of this section, our work was strongly motived by the results of J. Mallet–Paret [12].

The assumptions are really of two types:

1. **A1** – **A3**, which are statements concerning the particular dynamical system being studied, and
2. **A4** and **A5**, which are most naturally verified via a carefully chosen homotopy to a well understood system.

With this in mind, our discussion proceeds as follows. We begin with a very general class of equations

$$\dot{x}(t) = -f(x(t), x(t-1)) \tag{5}$$

which is assumed to satisfy the following three hypothesis:

H1: (i) *$f : \mathbf{R}^2 \to \mathbf{R}$ is C^∞;*
(ii) *$\eta f(0,\eta) > 0$ for all $\eta \neq 0$;*
(iii) *$B > 0$ and $A + B > 0$, where $A = \frac{\partial f(\xi,\eta)}{\partial \xi}\big|_{(0,0)}$ and $B = \frac{\partial f(\xi,\eta)}{\partial \eta}\big|_{(0,0)}$.*

H2: (i) *Given $K_1 > 0$ there exists $K_2 > 0$ such that for any $\phi \in X := C[-1,0]$*

$$\|\phi\| \leq K_1 \;\Rightarrow\; \|T(1)\phi\| \leq K_2;$$

(ii) *There exists K_0 such that for any $\phi \in X$*

$$\limsup_{t\to\infty} \|T(t)\phi\| \leq K_0.$$

H3:

$$\xi,\eta > 0 \quad\Rightarrow\quad f(\xi,\eta) > 0,$$
$$\xi,\eta < 0 \quad\Rightarrow\quad f(\xi,\eta) < 0.$$

The first step is to show that **A1** – **A3** hold under the stronger assumption that f is analytic.

Next, we consider a particular delay equation known as Wright's equation for which assumptions **A4** and **A5** can be verified directly. Since Wright's equation is analytic and satisfies **H1** – **H3**, the induced flow satisfies all the assumptions of Theorem 1.3, and hence, we have a semi-conjugacy to describe the global dynamics of this equation.

Finally, we extend the result, by considering a specific class of equations satisfying **H1** – **H3** for which we can construct a homotopy to Wright's equation while preserving **A2** – **A5** and the essential properties insured by **A1**. As will be seen, the two crucial points which must be addressed are that the attractor remain compact and that the Morse decomposition be preserved throughout the homotopy.

We begin now by summarizing the results of [12] and along the way verify **A1**–**A3**.

The general theory of delay differential equations (see [8]) guarantees that (5) generates a semiflow Φ on the Banach space $C[-1,0]$. The existence of the global attractor $\bar{\mathcal{A}} \subset C[-1,0]$ can be found in [12]. If one assumes that f is analytic, then Φ restricted to $\bar{\mathcal{A}} \subset C[-1,0]$ is a flow

$$\bar{\varphi} : \mathbf{R} \times \bar{\mathcal{A}} \to \bar{\mathcal{A}}$$

i.e., **A1** is satisfied. Unfortunately, if f is only C^∞, then $\bar{\varphi}$ need not be a flow. To circumvent this technicality define

$$\mathcal{A} := \{x \in C(-\infty,\infty) \mid x(t) \text{ is a bounded global solution of (5)}\}$$

endowed with the usual compact open topology of $C(-\infty,\infty)$. Now define

$$\varphi(t, x(s)) := x(s+t), \quad s \in (-\infty,\infty);$$

then $\varphi : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$ is a flow. This creates a different problem, however. $\mathcal{A}$ is no longer a subset of $C[-1,0]$ and φ is not the restriction of Φ. Therefore, **A1** is not satisfied. We will return to this point later. For the moment observe that there is a well defined map $\pi : \mathcal{A} \to \bar{\mathcal{A}}$ given by restricting the function $x \in C(-\infty,\infty)$ to $C[-1,0]$, and in the special case that f is analytic π is a homeomorphism.

As was mentioned in the introduction, one of the more challenging aspects of verifying the assumptions is to obtain a useful characterization of the Morse decomposition. To do this Mallet-Paret introduced a discrete Lyapunov function which measures the rate of oscillation of solutions in $\mathcal{A}$. In particular, for $x \in \mathcal{A}\backslash\{\mathbf{0}\}$ (where $\mathbf{0}$ denotes the function identically equal to zero) let $\sigma = \inf\{t \geq 0 \mid x(t) = 0\}$ and define

(6)
$$V(x) = \begin{cases} \text{the number of zeroes (counting multiplicity) of } x(t) \text{ in } (\sigma-1,\sigma]; \\ 1 \quad \text{if } \sigma \text{ does not exist.} \end{cases}$$

Theorem 2.1 ([12, Theorem A])**.** *Given* **H1** *and* **H2**:

(i) *if $x \in \mathcal{A}\backslash\{\mathbf{0}\}$, then $V(\varphi(t,x))$ is a non-increasing function of $t \in \mathbf{R}$;*
(ii) *$V(x) < \infty$ and is an odd integer for each $x \in \mathcal{A}\backslash\{\mathbf{0}\}$;*
(iii) *V is bounded on $\mathcal{A}\backslash\{\mathbf{0}\}$.*

Given this Lyapunov function one is led to the following definition:

$$M_p := \{x \in \mathcal{A} \mid V(\varphi(t,x)) = 2p+1 \text{ for all } t \in \mathbf{R} \text{ and } \mathbf{0} \notin \omega(x) \cup \omega^*(x)\}. \tag{7}$$

This collection of sets cannot define a Morse decomposition of $\mathcal{A}$ since the trivial solution $\mathbf{0} \in \mathcal{A}$, but is not contained in any of the sets M_p. Furthermore, since V is not defined at $\mathbf{0}$ it is not obvious where $\mathbf{0}$ should lie in the partial ordering. To resolve these questions one must consider the characteristic equation for (5) at the point $\mathbf{0}$,

$$\lambda + A + Be^{-\lambda} = 0. \tag{8}$$

Let $2P$ denote the number (counting multiplicity) of solutions λ of (8) such that $\operatorname{Re}\lambda > 0$. Now define

$$M_P = \{x \in \mathcal{A} \mid V(\varphi(t,x)) \geq 2P \text{ for all } t \in \mathbf{R}\} \cup \mathbf{0}. \tag{9}$$

The following theorem verifies **A2**.

Theorem 2.2 ([12, Theorem B])**.** *Given* **H1** *and* **H2**,

$$\mathcal{M}(\mathcal{A}) = \{M_p \mid p = 0, 1, \dots, P\} \tag{10}$$

is a Morse decomposition of $\mathcal{A}$ *and* $p > p-1$ *defines an admissible order.*

Actually, this is a coarse version of Mallet-Paret's result. In his paper this theorem is stated in terms of Morse sets M_p, for $p = 0, 1, 2, \dots$, and a distinct Morse set containing at least $\mathbf{0}$. As presented here, all the sets M_p for which $p > P$ are contained in the same Morse set as the trivial solution $\mathbf{0}$, namely M_P. The justification for this will be given later. Turning now to **A3**, we again follow the lead of [12] (in particular Proposition 4.1 and the remarks following) and for $p = 0, \dots, P-1$, define

$$\Pi_p : M_p \to \mathbf{R}^2 \backslash \{0\} \tag{11}$$

by

$$\Pi_p(x) = (x(0), \dot{x}(0)). \tag{12}$$

A Poincaré section for M_p is given by

$$\Xi_p := M_p \cap \Pi^{-1}(\{(0,\dot{x}) \mid \dot{x} > 0\}). \tag{13}$$

For $p > 0$ the image of the orbit $x(t)$ winds around the origin infinitely often as $t \to \pm\infty$, and hence, M_p has a Poincaré map. The only point at which hypothesis **H3** has been or will be used is to guarantee the same result for M_0.

Remarks. The assumption **A3** excludes critical points and suggests the existence of periodic orbits. With regard to the delay equation it is known (see [12] and references therein) that each Morse set M_p contains a periodic trajectory. However, the numerical work referred to in the introduction suggests that the dynamics within each Morse set may be much more complicated. The point to be made is the following: there exists a lower bound on the complexity of the Morse sets (isolated periodic orbits), but no *a priori* upper bound.

On a more general level it has been shown by M. Mrozek and the authors [15] that given Morse sets which satisfy assumptions **A3** and **A4**, there always exists a periodic orbit.

Having verified **A1** – **A3** in the case of analytic nonlinearities, we now turn to Wright's equation,

$$\dot{x}(t) = -\beta(e^{x(t-1)} - 1). \tag{14}$$

(Observe that the change of variable $x = \log(1+y)$ puts (14) into the classical form of Wright's equation, namely,

$$\dot{y}(t) = \beta y(t-1)[1 + y(t)].) \tag{15}$$

We choose this equation for two reasons: first, it is analytic, and second, as a function of β, the local bifurcations about the solution $\mathbf{0}$ are completely understood. Since Wright's equation is analytic, $\pi : \mathcal{A} \to \bar{\mathcal{A}}$ is a homeomorphism that commutes with the flows φ and $\bar{\varphi}$ defined on the respective spaces. Thus, all Conley index computations performed on $\bar{\mathcal{A}}$ are valid on $\mathcal{A}$ (see [13]). The local bifurcation information is summarized in the following proposition due to S.-N. Chow and Mallet-Paret [1].

Proposition 2.3. *For $\beta > 0$ the bifurcation values for the zero solution to* (14) *are*

$$\beta_p = 2p\pi + \frac{\pi}{2}, \qquad p = 0, 1, 2, \dots, \tag{16}$$

and at these values a generic supercritical Hopf bifurcation occurs. Furthermore, setting $\beta_{-1} = 0$, if $\beta \in (\beta_{p-1}, \beta_p)$, then $\mathbf{0}$ is a hyperbolic fixed point with a $2p$ dimensional unstable manifold.

Proof. The bifurcation values are well known, and [1], Theorem 9.1, shows that these give rise to generic supercritical Hopf bifurcations. □

Proposition 2.4. *For $\beta > \beta_p$,*

$$CH^n(M_p) \cong \begin{cases} \mathbf{Z} & \text{if } n = 2p, 2p+1, \\ 0 & \text{otherwise.} \end{cases}$$

For $\beta \in (\beta_p, \beta_{p+1})$

$$CH^n(M_P) \cong \begin{cases} \mathbf{Z} & \text{if } n = 2p+2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The details of this proof involve making use of the algebra associated with the Conley index theory. The reader who is not familiar with this theory may wish to read this argument after consulting Section 3 and the references therein.

For the purpose of this proof we shall use the following notation. Let

$$M(p) := \{x \in \bar{\mathcal{A}} \mid V(\bar{\varphi}(t,x)) = 2p+1 \text{ for all } t \in \mathbf{R} \text{ and } \mathbf{0} \notin \omega(x) \cup \omega^*(x)\}, \qquad p = 0, 1, 2, \dots .$$

Define

$$\tilde{M} = \begin{cases} \mathbf{0} & \text{if } \beta \in (\beta_p, \beta_{p+1}), \\ \{x \in \bar{\mathcal{A}} \mid V(\bar{\varphi}(t,x)) = 2p+1 \text{ for all } t \in \mathbf{R}\} \cup \mathbf{0} & \text{if } \beta = \beta_p. \end{cases}$$

According to [12], Theorem B

$$\{M(p) | p = 0, 1, 2, \dots\} \cup \{\tilde{M}\}$$

is a Morse decomposition for Wright's equation with admissible ordering

$$M(0) < \dots < M(p) < \tilde{M} < M(p+1) < \dots .$$

Observe that for $\beta > \beta_p$, $M_p = \pi^{-1}(M(p))$ and for $\beta \in (\beta_p, \beta_{p+1})$, $\tilde{M} = \mathbf{0}$. Thus, by Proposition 2.3, for $\beta \in (\beta_p, \beta_{p+1})$

$$CH^n(\tilde{M}) \cong \begin{cases} \mathbf{Z} & \text{if } n = 2p+2, \\ 0 & \text{otherwise,} \end{cases}$$

and hence the Conley index of $M(P)$ is determined. We shall use an induction argument to compute the index of $M(p)$ or equivalently M_p.

Observe that for all values of β

$$M(p) \cap M(q) = \emptyset \quad \text{if } p \neq q.$$

Therefore, the only point in parameter space where $M(p)$ loses isolation is at β_p where a bifurcation involving $\mathbf{0}$ occurs. Thus, $CH^*(M(p))$ remains constant on the intervals $(0, \beta_p)$ and (β_p, ∞).

Let $\beta < e^{-1}$. Then, for (14) $\bar{\mathcal{A}}$ consists of exactly $\mathbf{0}$ and our Morse decomposition is of the form

$$\mathbf{0} \cup \{M(p) | p = 0, 1, 2, \dots\},$$

where $M(p) = \emptyset$ for all p. Thus, $CH^*(M(p)) \cong 0$. Furthermore, Proposition 2.3 implies that

$$CH^n(\bar{\mathcal{A}}) \cong CH^n(\mathbf{0}) \cong \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider $\beta \in (\beta_0, \beta_1)$. For these parameter values

$$CH^n(\tilde{M}) = CH^n(\mathbf{0}) \cong \begin{cases} \mathbf{Z} & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and $CH^*(M(p)) \cong 0$ for $p > 0$. If we now construct the connection matrix (see [6, 16]) associated to our Morse decomposition, we obtain a degree $+1$ matrix

$$\begin{bmatrix} 0 & 0 \\ \Delta & 0 \end{bmatrix} : CH^*(M(0)) \oplus CH^*(\tilde{M}) \to CH^*(M(0)) \oplus CH^*(\tilde{M}).$$

Since $CH^n(\bar{\mathcal{A}})$ remains unchanged, the rank condition forces

$$CH^n(M(0)) \cong \begin{cases} \mathbf{Z} & \text{if } n = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since the Conley index of $M(0)$ is constant over (β_0, ∞), we have computed the desired result for this Morse set. We now apply the induction argument, so assume $\beta \in (\beta_p, \beta_{p+1})$ and that $CH^*(M(q))$ has been computed for $q = 0, \dots, p-1$. Again, constructing the connection matrix associated to our Morse decomposition we obtain a degree $+1$ matrix where the only unknown entries are

$$\Delta_p : CH^*(M(p)) \to CH^*(\tilde{M})$$

and

$$\Delta_{p-1} : CH^*(M(p-1)) \to CH^*(M(p)).$$

Again, the rank condition forces

$$CH^n(M(p)) \cong \begin{cases} \mathbf{Z} & \text{if } n = 2p, 2p+1, \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Remark. Observe that a similar argument shows that for $\beta < \beta_q$, $CH^*(M_q) \cong 0$. This implies that the index theory can give us no information relating M_q, $q > P$ (assuming, in fact, that it is nonempty) to the structure of the attractor. This justifies defining M_P as in (9). Also, note that Proposition 2.4, in conjunction with the previous results, implies that Wright's equation satisfies assumptions **A1** – **A5**. We now extend this result to the following class of delay differential equations.

Theorem 2.5. *Consider the equation*

$$\dot{x}(t) = -Ax(t) - Bg(x(t-1)), \tag{17}$$

where $A \geq 0$, $B > 0$, $g \in C^\infty(\mathbf{R}, \mathbf{R})$, $\eta g(\eta) > 0$ *for all* $\eta \neq 0$, $g'(0) = 1$, *and* $g(\eta) \geq -K$ *for some positive constant* K *and for all* η. *Then assumptions* **A1** – **A5** *are satisfied.*

In [12] it is shown that (17) satisfies hypothesis **H1** – **H3** and hence **A2** and **A3** hold. Assumptions **A4** and **A5** will be taken care of simultaneously by prescribing a homotopy from an arbitrary equation of the form (17) to Wright's equation. Since these assumptions are involved with the Conley index, the homotopy must be such that the index information is preserved. Thus two conditions must be met:

C1: the property of a global compact attractor is preserved,

C2: a lower bound on the number of eigenvalues of **0** with positive real part greater than zero is maintained.

In actuality, it is sufficient in **C1** to show that each Morse set M_p remains compact over the homotopy, though in applications we do not know how to take advantage of this fact.

As will be seen in the proof of Theorem 1.3, **A1** serves two essential purposes. The first is to guarantee that the semi-conjugacy is defined from a flow. In the case of the delay equation this is satisfied by defining φ on $\mathcal{A}$. The second purpose is to be able to compute the Conley index of the attractor (see Corollary 3.2). Using Wright's equation we could determine that the Conley index of $\mathcal{A}$ is the same as that of $\bar{\mathcal{A}}$. Furthermore, since the compactness of $\mathcal{A}$ will be preserved over the homotopy, the Conley index of $\mathcal{A}$ remains unchanged. Thus, **A1** is not required for this application.

With this in mind we use the following homotopy

$$F(\xi, \eta, s) = sA\xi - b(s)(sg(\eta) + (1-s)(e^\eta - 1)) \tag{18}$$

where $b(1) = B$, and $0 \leq s \leq 1$. Clearly

$$\begin{aligned} \dot{x}(t) &= -F(x(t), x(t-1), 0) \\ &= -b(0)(e^{x(t-1)} - 1) \end{aligned} \tag{19}$$

is Wright's equation and

$$\begin{aligned} \dot{x}(t) &= -F(x(t), x(t-1), 1) \\ &= -Ax(t) - Bg(x(t-1)) \end{aligned} \tag{20}$$

is (17). **C1** is easily verified since

$$\dot{x}_s(t) = -F(x(t), x(t-1), s) \tag{21}$$

is of the form of (17), and hence, there exists a compact attractor for each value of $s \in [0, 1]$.

Maintaining a lower bound on the number of eigenvalues with positive real part is slightly more complicated. Observe that

$$\frac{\partial F}{\partial \xi}|_{(\xi,\eta,s)=(0,0,s)} = As, \tag{22}$$

$$\frac{\partial F}{\partial \eta}|_{(\xi,\eta,s)=(0,0,s)} = b(s). \tag{23}$$

It is easily checked (alternatively, see [12], proof of Theorem 6.1) that the eigenvalues $\mu_i \pm \imath\nu_i$, $i = 0, 1, 2, \ldots$, of the characteristic equation about $\mathbf{0}$ are given by

$$\mu_i(s) = -As - b(s)e^{-\mu_i(s)}\cos(\nu_i(s)), \tag{24}$$

$$\nu_i(s) = b(s)e^{-\mu_i(s)}\sin(\nu_i(s)). \tag{25}$$

Analysis of these equations leads to the following result (see for example [12, Theorem 6.1] and references therein).

Theorem 2.6. *If $B > 0$ and $A + B > 0$, then the roots $\lambda = \mu \pm \imath\nu$ of the characteristic equation $\lambda + A + Be^{-\lambda} = 0$ have the following properties:*

1. *All roots lie in the strips*

$$\Sigma_{\pm k} : 2k\pi < \pm\nu < (2k + 1)\pi, \quad k \geq 1,$$

and

$$\Sigma_0 : |\nu| < \pi.$$

2. *$\Sigma_{\pm k}$ contains exactly one root $\mu_k \pm \imath\nu_k$, and it is simple.*
3. *Σ_0 contains two roots counting multiplicity: either complex conjugate roots $\mu_0 \pm \imath\nu_0$, $\nu > 0$, or real roots $\lambda_{00} \geq \lambda_0$.*
4. *The real parts of the roots are ordered,*

$$\lambda_0 \quad \text{or} \quad \mu_0 > \mu_1 > \mu_2 > \ldots \to -\infty.$$

5. *Both roots in Σ_0 lie on the same side of the imaginary axis, or both are purely imaginary.*

Since we are only concerned with a lower bound on the number of eigenvalues with positive real part and to simplify the notation, let $\mu(1) \pm \imath\nu(1)$ denote the eigenvalues with smallest positive real part. We will prove that there exists a homotopy F with the property that

$$\mu(s) = \mu(1) \quad \text{for all } s \in [0, 1]. \tag{26}$$

Before providing the proof we shall explain why this is sufficient to verify **A4** and **A5**. As was indicated before we need to show that the homotopy preserves the Morse decomposition. Thus, along the homotopy we define the Morse sets as follows. Let $2P$ be the number (counting multiplicity) of solutions λ of (8) such that Re $\lambda > 0$ for the equation (17). Then for $p < P$ define

$$M_{p,s} := \{x \in \mathcal{A} \mid V(\varphi^s(t, x)) = 2p + 1 \text{ for all } t \in \mathbf{R} \text{ and } \mathbf{0} \notin \omega(x) \cup \omega^*(x)\}, \tag{27}$$

where φ^s is the flow induced by (21). Define

$$M_{P,s} := \{x \in \mathcal{A} \mid V(\varphi^s(t, x)) \geq 2P \text{ for all } t \in \mathbf{R}\} \cup \mathbf{0}. \tag{28}$$

Observe that if for some $0 < s < 1$, there are more than $2P$ eigenvalues with positive real part, then the additional invariant sets are lumped into the top Morse set $M_{P,s}$.

Thus the lower bound of $2P$ on the number of eigenvalues with positive real part is sufficient to preserve the Morse decomposition throughout the homotopy.

Also, observe that throughout the homotopy the set defined by (13) defines a Poincaré section for each Morse set M_p, $p = 0, \dots, P-1$.

We still have a free parameter in our homotopy, namely the function $b(s)$ and we shall now prove that it can be chosen in such a way that (26) is satisfied. This will be done by showing that there exists a solution for $0 \le s \le 1$ to the following system of ordinary differential equations:

$$\begin{aligned} 0 &= -A - b' e^{-\mu} \cos\nu + b e^{-\mu} \sin\nu \nu', \\ \nu' &= b' e^{-\mu} \sin\nu + b e^{-\mu} \cos\nu\nu', \qquad {}' = \frac{d}{ds}, \end{aligned} \tag{29}$$

which was obtained by differentiating (24) and (25). Simple algebraic manipulations give that

$$\nu' = \frac{A \sin\nu}{b e^{-\mu} - \cos\nu} \tag{30}$$

assuming, of course, that $be^{-\mu} - \cos\nu \neq 0$. Applying this to (29) results in the following pair of equations:

$$\begin{aligned} b' &= \frac{A(e^{\mu} - b\cos\nu)}{be^{-\mu} - \cos\nu}, \\ \nu' &= \frac{A \sin\nu}{be^{-\mu} - \cos\nu}. \end{aligned} \tag{31}$$

We need to solve this system from the point

$$(b(1), \nu(1)) = (B, \nu(1)) \tag{32}$$

to the point

$$(b(0), \nu(0)) = (\beta, \nu(0)).$$

While $(B, \nu(1))$ is uniquely determined by (17), we do not actually know the value, nor do we know, a priori, $(\beta, \nu(0))$. However, for our purpose it is sufficient to show that the solution exists from time $s = 1$ to time $s = 0$. Furthermore, having started with the initial condition (32) which satisfies (24) and (25), for any value of s for which the solution exists, equations (24) and (25) are valid. Let $s \in (s_0, 1]$ be the maximal interval over which the equations are valid given the initial condition (32). We need to show that $s_0 \le 0$. We begin the analysis by observing that

$$\mu = \mu(s) = \mu(1) > 0$$

and hence, via equation (24)

$$\mu = -As - b(s) e^{-\mu} \cos(\nu(s)) > 0.$$

This implies that $b(s) \neq 0$ and $\cos\nu > 0$. Since $b(1) > 0$, for $s_0 < s \le 1$,

$$(b(s), \nu(s)) \in \mathcal{R} := \{b \ge 0,\ (2P + \frac{1}{2})\pi < \nu(s) < (2P+1)\pi\}.$$

Referring now to Figure 1, which is the phase portrait for (31), we see that $\mathcal{R}$ is invariant in backwards time for all values of s such that $F(\cdot, \cdot, s)$ satisfies the assumptions of Theorem 2.5. Thus $s_0 \le 0$.

This completes the proof of Theorem 2.5.

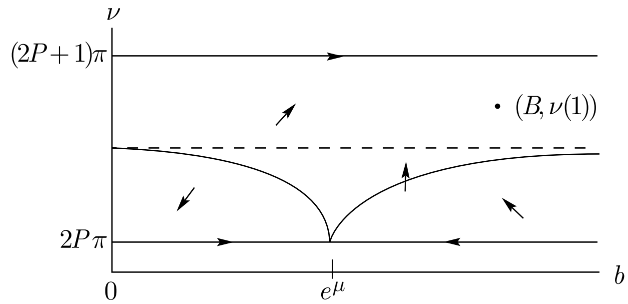

FIGURE 1. Phase portrait for equation (32).

Concluding Remarks. We finish this section with the conjecture that **A1** – **A5** hold for all scalar delay equations with negative feedback which satisfy the hypotheses **H1** – **H3**. The major difficulty in proving this conjecture is to show that all nonlinearities for which these hypotheses hold can be homotoped to Wright's equation while preserving **C1** and **C2**.

3. THE HYPOTHESES

In this section, we review the background material needed to construct the semiconjugacy f and to show that it is surjective. In particular, we will develop in greater detail the assumptions set forth in **A1**–**A5**. Hypotheses **A1** and **A2** are global statements; **A3**–**A5** are local statements. Since the key to the construction of f will be the ability to connect the local and global information, we will give special attention in this section to the Conley index machinery for relating local and global invariants.

We begin with a brief review of the relevent portions of the Conley index theory. The basic references for this material are [2, 13, 14, 19, 20]. The Conley index was introduced to study *isolated invariant sets*: a set S is an isolated invariant set if $S \cdot \mathbf{R} = S$ and there is a compact neighborhood N of S such that S is the maximal invariant set in N. The neighborhood N is an *isolating neighborhood* for S. The Conley index of S studies the nature of the flow around S, rather than on S itself. But there are various ways of decomposing the flow on S which are significant in the index theory. The most important of these is an *attractor-repeller pair* decomposition. If S is an isolated invariant set, $A, R \subset S$, then the pair (A, R) is an attractor-repeller pair in S if

1. A is an attractor in S: there is a positively invariant neighborhood U of A in S with $\omega(U) = A$.
2. R is the dual repeller to A in S: $R = S \setminus \{x|\omega(x) \subset (A)\}$.

Note that A and R are both isolated invariant sets, and if

$$C(R, A) = \{x \in S \mid \alpha(x) \subset R, \omega(x) \subset A\},$$

then $S = R \cup C(R, A) \cup A$. That is, an attractor-repeller pair gives a decomposition of S into (two) finer invariant sets and connecting orbits between them.

More generally, a *Morse decomposition* is a decomposition of an invariant set into a finite number of invariant subsets (i.e. Morse sets) and connecting orbits between them. That is, a Morse decomposition of S consists of a finite collection of isolated invariant subsets M_p, indexed by some set $\mathcal{P}$, with a partial order $<$ on $\mathcal{P}$. The requirement is that, if $x \in S \setminus \bigcup_{p\in\mathcal{P}} M_p$, then there exist $p < q$ such that $x \in C(M_q, M_p)$. That is, the partial order must respect the flow: orbits can only flow "down" through the partial order. A partial order on $\mathcal{P}$ which respects the flow is referred to as an *admissible* partial order. The most natural way to produce an admissible partial order is to let the flow generate it. Set $p < q$ if $C(M_q, M_p) \neq \emptyset$, and take the transitive closure.

If $\{M_p\}_{p\in\mathcal{P}}$ is a Morse decomposition of S, then each M_p is an isolated invariant set. S contains more isolated invariant sets, some of which can be produced by the partial order on $\mathcal{P}$ as follows. A subset $I \subset \mathcal{P}$ is an *interval* in $\mathcal{P}$ if $r \in I$ whenever $p < r < q$ and $p, q \in I$. Disjoint intervals I and J are ordered $I < J$ if $i < j$ for every $i \in I, j \in J$; they are adjacent if $IJ = I \cup J$ is also an interval (i.e. if no element of $\mathcal{P}$ lies "between" I and J). If I is an interval, let $M(I) = \bigcup_{i\in I} M_i \cup \bigcup_{i,j\in I} C(M_j, M_i)$. Then each $M(I)$ is an isolated invariant set, and if I and J are adjacent intervals with $I < J$, then $(M(I), M(J))$ is an attractor-repeller pair for $M(IJ)$.

In the present work, we will let $\mathcal{P}$ denote the set $\{0, 1, \ldots, P\}$, with ordering $0 < 1 < \ldots < P$. If I is an interval, we will denote the largest index value in I by $\bar{i}$; the smallest by $\underline{i}$. That is, $I = \{\underline{i}, \ldots, \bar{i}\}$. Let $\tilde{i} = 2\bar{i} - 2\underline{i} + 1$. Note that $I < J$ if $\bar{i} < \underline{j}$, and are adjacent if $\bar{i} + 1 = \underline{j}$.

These are the structures the Conley index studies. The Conley index of an isolated invariant set is defined in terms of an *index pair*: a compact pair (N, L) such that

1. $\overline{N \setminus L}$ is an isolating neighborhood for S.
2. L is positively invariant in N, i.e. if $x \in L$ and $x\cdot[0, T] \subset N$, then $x\cdot[0, T] \subset L$.
3. L is an exit set for N, i.e. if $x \in N$ and $x \cdot T \notin N$, then $x \cdot t \in L$ for some $0 < t < T$.

An index pair is further said to be *regular* if, in addition, the function $\varpi : N \to [0, \infty)$ defined by

$$\varpi(x) = \begin{cases} \sup\{t > 0 | x \cdot [0, t] \subset N \setminus L\} & \text{if } x \in N \setminus L, \\ 0 & \text{if } x \in L, \end{cases}$$

is continuous. Observe that this implies that for a regular index pair L is a neighborhood deformation retract (along flow lines) in N. Index pairs (indeed, regular index pairs) always exist, and the homotopy type of the quotient space N/L is independent of the index pair chosen. It is that homotopy type which defines the Conley index of S.

If regular index pairs are ordered by inclusion, we have an inverse system $\{H^*(N_\alpha, L_\alpha)\}$ of index pairs, with the inclusion-induced cohomology map $H^*(N_\alpha, L_\alpha) \to H^*(N_\beta, L_\beta)$ an isomorphism for every $\beta < \alpha$. The inverse limit of this system, denoted $CH^*(S)$, is the cohomology Conley index of S. Since each bonding map in the system is an isomorphism, we have $CH^*(S) \cong H^*(N_\alpha, L_\alpha)$ for every α. That is, the cohomology of any index pair represents the cohomology Conley index.

The following propositions will be of heuristic value in understanding the cohomology index assumptions of **A4**. However, before stating the first proposition we recall some definitions. Let $\Phi : \mathbf{R}^+ \times X \to X$ be a continuous semi-flow on the metric space X. A set $A \subset X$ *attracts* a set $B \subset X$ under Φ if for any $\epsilon > 0$ there exists $T = T(\epsilon, A, B)$ such that $\Phi(T, B)$ is contained in an ϵ-neighborhood of A. A compact invariant set $\mathcal{A}$ is a *global compact attractor* (see Hale [9]) if every compact invariant set of Φ is contained in $\mathcal{A}$ and $\omega(B) \subset \mathcal{A}$ for every bounded set $B \subset X$.

In the following proposition we shall use the Conley index theory for arbitrary metric spaces as developed by K. Rybakowski [19]. The essential difference between the theories is that Rybakowski replaces the assumption of local compactness in the space X with a form of asymptotic compactness of the semi-flow called *admissibility.* In particular, the definition of index pairs remains the same, however, compactness is replaced by closedness.

Proposition 3.1. *Let $\mathcal{A}$ be a global compact attractor in X. Let $A \subset \mathcal{A}$ be an attractor in $\mathcal{A}$ and hence, in X. If B is a bounded isolating neighborhood of A, then there exists a natural inclusion-induced monomorphism $CH^*(A) \to H^*(B)$, which is an isomorphism when B is positively invariant.*

Proof. Since A is an attractor, given any neighborhood U of A there exists a bounded neighborhood $N \subset U$ such that $(N, \emptyset)$ is an index pair for A. Thus

$$CH^*(A) \cong H^*(N).$$

Now choose B to be a bounded neighborhood of A such that $N \subset B$. Then there exists $T > 0$, sufficiently large, such that $\Phi(T, B) \subset N$. Observe that this implies that $\Phi(\mathbf{R}^+, B) = \Phi([0,T], B)$. Now $\Phi(\mathbf{R}^+, B)$ is a bounded positively invariant neighborhood of A and hence, $\omega(\Phi(\mathbf{R}^+, B)) = A$. Therefore, the inclusion map $B \to \Phi(\mathbf{R}^+, B)$ induces a map $CH^*(A) \to H^*(B)$. Of course, if B is positively invariant, $B = B \cdot \mathbf{R}^+$ and the map is an isomorphism.

In general, we have the following diagram of maps (with all unlabeled maps inclusions) and $\Phi_T(x) = \Phi(T, x)$:

$$\begin{array}{ccccc}
\Phi(\mathbf{R}^+, B) & \xrightarrow{\Phi_T} & \Phi([T,\infty), B) & \rightarrow & \Phi(\mathbf{R}^+, B) \\
\uparrow & & \uparrow & \searrow & \uparrow \\
B & \rightarrow & \Phi(T, B) & \rightarrow & B
\end{array}$$

Since $\Phi(\mathbf{R}^+, B)$ is positively invariant, the map Φ_T is homotopic to the identity in $\Phi(\mathbf{R}^+, B)$, and so the composition

$$H^*(\Phi(\mathbf{R}^+, B)) \to H^*(\Phi([T,\infty), B)) \xrightarrow{\Phi_T^*} H^*(\Phi(\mathbf{R}^+, B))$$

is the identity isomorphism. In particular, $H^*(\Phi(\mathbf{R}^+, B)) \to H^*(\Phi([T,\infty), B))$ is an isomorphism which factors as

$$H^*(\Phi(\mathbf{R}^+, B)) \to H^*(B) \to H^*(\Phi([T,\infty), B)).$$

Thus the inclusion-induced map $H^*(\Phi(\mathbf{R}^+, B)) \to H^*(B)$ is injective. $\square$

Corollary 3.2. *If X is a Banach space and $\mathcal{A}$ is a global compact attractor for a continuous semi-flow Φ on X, then*

$$CH^*(\mathcal{A}) \cong \begin{cases} \mathbf{Z} & \textit{if } n = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $B := \{x \in X \mid \|x\| < K\}$ for K sufficiently large. Then B is a neighborhood of $\mathcal{A}$ and $\omega(B) = \mathcal{A}$. Therefore, $CH^*(\mathcal{A})$ maps monomorphically into

$$H^*(B) \cong \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{A} \neq \emptyset$, the $CH^*(\mathcal{A})$ is as above. $\square$

Proposition 3.3. *If S is a normally hyperbolic invariant set for a flow on a manifold with orientable unstable manifold of (normal) dimension u, then $CH^{q+u}(S) \cong H^q(S)$.*

In an attractor-repeller decomposition, the Conley indices of the total invariant set, attractor and repeller are naturally related by an *index triple*. An index triple for an A-R pair (A, R) in S is a triple of compact spaces (N, M, L) such that (N, L) is an index pair for S, (N, M) is an index pair for R and (M, L) is an index pair for A. Such triples exist for any attractor-repeller decomposition, as do *regular* index triples: triples such that both L and M are neighborhood deformation retracts in N. Then the cohomology exact sequence of the triple

$$\xrightarrow{\delta} H^k(N, M) \to H^k(N, L) \to H^k(M, L) \xrightarrow{\delta}$$

induces an exact sequence

$$\xrightarrow{\delta} CH^k(R) \to CH^k(S) \to CH^k(A) \xrightarrow{\delta}$$

which is known as the cohomology attractor-repeller sequence. The boundary map δ is called the *connection map*, as $\delta \neq 0$ implies that connections between R and A exist.

All of these objects have generalizations to Morse decompositions. Index triples are generalized to index filtrations, and the attractor-repeller sequence is generalized to the construction of connection matrices. However, we will not require this additional machinery in this work, so we will forgo a description of it.

For systems satisfying **A1**, **A2** and **A4**, the attractor-repeller sequence allows us to compute the cohomology Conley index for any set $M(I)$.

Proposition 3.4. *If I is an interval in $\mathcal{P}$, then*

$$CH^k(M(I)) = \begin{cases} \mathbf{Z} & \text{if } k = 2\underline{i} \text{ or } k = 2\bar{i} + 1 < 2P, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We proceed by downward induction on $\bar{i} - \underline{i}$. If $\bar{i} - \underline{i} = P$, then $I = \mathcal{P}$ and the statement is true by Corollary 3.2. Now assume the result for all J with $\bar{j} - \underline{j} > \bar{i} - \underline{i}$. We can also assume that $\bar{i} - \underline{i} > 0$. Suppose $\underline{i} > 0$: let $k = \underline{i} - 1$, and let $K = I \cup k$. Then the attractor-repeller sequence of the pair (k, I) is

$$\to CH^{n-1}(M(K)) \to CH^{n-1}(M_k) \xrightarrow{\delta} CH^n(M(I)) \to CH^n(M(K)) \to;$$

$CH^n(M(I))$ is flanked by zeros, except in dimensions $2k+1, 2\underline{i}$ and (possibly) $2\bar{i}+1$. It is easy to see that $CH^{2k}(M(K)) \to CH^{2k}(M_k)$, $\delta : CH^{2k+1}(M_k) \to CH^{2\underline{i}}(M(I))$ and $CH^{2\bar{i}+1}(M(I)) \to CH^{2\bar{i}+1}(M(K))$ must all be isomorphisms.

If $\underline{i} = 0$, then $\bar{i} < P$ and we can work with the attractor-repeller pair (I, k), where $k = \bar{i} + 1$. The calculation in this case is similar. $\square$

Corollary 3.5. *If I and J are adjacent intervals, then $\delta : CH^{2\bar{i}+1}(M(I)) \to CH^{2\underline{j}}(M(J))$ is an isomorphism.*

In our present work, we will need to consider one new feature of the cohomology index: a pairing of the cohomology Conley index of an invariant set and the Cech cohomology of the invariant set. If (N, L) is an index pair for an isoloated invariant set S, the cup product defines a pairing

$$H^p(N) \otimes H^q(N, L) \to H^{p+q}(N, L).$$

Since the collection

$$\{N_\alpha \mid (N_\alpha, L_\alpha) \text{ is an index pair}\}$$

is cofinal with the set of neighborhoods of S, this pairing defines a pairing

$$\check{H}^p(S) \otimes CH^q(S) \to CH^{p+q}(S).$$

This pairing exists for any invariant set in any flow. In our setting, if $T_I \in CH^{2\underline{i}}(M(I))$ is a generator, there is a map

$$\Theta_I : \check{H}^p(M(I)) \to CH^{p+2\underline{i}}(M(I))$$

defined by

$$\Theta(z) = z \cup T_I.$$

Note that $\Theta(1) = T_I$.

Another important aspect of the index will be its behavior under semi-conjugacies (cf. [13, 14]). The essence of the matter is that the index theory is natural with respect to semi-conjugacies, as long as one works with pre-images, rather than images. A technicality is that the semi-conjugacy must be a proper map: pre-images of compact sets must be compact. That is, if $f : X \to Y$ is a proper semi-conjugacy, and S an isolated invariant set in Y with index pair (N, L), then $T = f^{-1}(S)$ is an isolated invariant set in X with index pair $(f^{-1}(N), f^{-1}(L))$. Thus there are maps $f_* : CH_*(T) \to CH_*(S)$ and $f^* : CH^*(S) \to CH^*(T)$. The pairing defined above commutes with this map: there is a commutative diagram

$$\begin{array}{ccc} \check{H}^p(S) \otimes CH^q(S) & \to & CH^{p+q}(S) \\ \downarrow f^* \otimes f^* & & \downarrow f^* \\ \check{H}^p(T) \otimes CH^q(T) & \to & CH^{p+q}(T) \end{array}$$

Similarly, if $\{M_p\}$ is a Morse decomposition of S, then $\{T_p = f^{-1}(M_p)\}$ is a Morse decomposition of T, and any admissible ordering on S gives an admissible ordering on T. Thus we can use the same ordering for both decompositions, and if I is an interval in that ordering, there is a map $CH^*(M(I)) \to CH^*(T(I))$. Moreover, the attractor-repeller sequence is natural: if I and J are adjacent intervals with $I < J$, there is a commutative diagram

$$\begin{array}{cccccccc} \stackrel{\delta}{\to} & CH^p(M(J)) & \to & CH^p(M(IJ)) & \to & CH^p(M(J)) & \to \\ & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & \\ \stackrel{\delta}{\to} & CH^p(T(J)) & \to & CH^p(T(IJ)) & \to & CH^p(T(J)) & \to \end{array}$$

We now turn to the a consideration of the cross-section hypothesized in **A3** and the homotopy hypothesized in **A5**. It is not obvious that these two conditions are related, but we will see that the homotopy assumption will provide crucial information about the dynamics on the cross-section. We define a set $\Xi \subset \mathcal{A}$ to be a (local) *transverse cross section* or *Poincaré section* if there is an open set U in $\mathcal{A}$ such that

1. There exists an $\epsilon > 0$ such that, for every $x \in \Xi$, $x \cdot (-\epsilon, \epsilon) \cap \Xi = x$.
2. If $u \in U$, there exist $t_- < 0 < t_+$ with $u \cdot t_-, u \cdot t_+ \in \Xi$.
3. $\Xi \cap U \neq \emptyset$.

Note that we do not require that $\Xi \subset U$. There may be points in Ξ whose orbit never intersects Ξ in forward time, or whose orbit never intersects Ξ in backwards time. If S is an isolated invariant set, we say that Ξ is a Poincaré section for S if $S \subset U$.

We can define on U a "first-intersection time"

$$T(u) = \min\{t > 0 \mid u \cdot t \in \Xi\}. \tag{33}$$

T is necessarily discontinuous at $\Xi \cap U$ (if $x \in \Xi \cap U$, then $T(x) \gg 0$, while $T(x \cdot -\epsilon) = \epsilon$), and define Ξ to be a *continuous* Poincaré section if the only discontinuity of T is the obligatory one at Ξ in $\Xi \cdot (-\epsilon, 0]$. In particular, if the Poincaré section is continuous, there is a continuous "first-return map" $r : \Xi \cap U \to \Xi$ defined by $r(x) = x \cdot T(x)$. (For a more general discussion see [15].)

This map defines a discrete dynamical system on Ξ. If Ξ is a Poincaré section for an isolated invariant set S, then $S \cap \Xi$ is an isolated invariant set for this discrete system. Mrozek [18] has developed a cohomological Conley index theory for discrete systems which is very much analogous to the continuous index theory. The cohomology index of the map r and the isolated invariant set $S \cap \Xi$ consists of a pair $(CH^*(S \cap \Xi), r^*)$, where $CH^*(S \cap \Xi)$ is a cohomology algebra derived from index pairs for $S \cap \Xi$ and $r^* : CH^*(S \cap \Xi) \to CH^*(S \cap \Xi)$ is an automorphism induced by r. The precise details of the construction of $CH^*(S \cap \Xi)$ are rather lengthy, so we will content ourselves with listing the following relevant properties of the discrete index (cf, [15, 18] for details):

1. The discrete cohomology index $(CH^*(S \cap \Xi), r^*)$ is independent of the section Ξ chosen.
2. If $\Xi \cap S$ is a hyperbolic fixed point with unstable dimension u and r orientation preserving on the unstable manifold, then
$$CH^n(\Xi \cap S; \mathbf{Z}) \cong \begin{cases} \mathbf{Z}, & n = u, \\ 0, & n \neq u, \end{cases}$$
and $r^* = id$.
3. If S is an isolated invariant set in a flow with a Poincaré section Ξ, there is a long exact sequence, known as the *index suspension sequence*, which relates the continuous index of S and the discrete index of $\Xi \cap S$:
$$\to CH^n(S) \xrightarrow{\iota} CH^n(S \cap \Xi) \xrightarrow{id - r^*} CH^n(S \cap \Xi) \xrightarrow{\delta} CH^{n+1}(S) \to .$$
4. This sequence is natural with respect to semi-conjugacies: Suppose S, T are isolated invariant sets in flows which are related by a semi-conjugacy f, with $S = f^{-1}(T)$. If T admits a Poincaré section Y, then $\Xi = f^{-1}(\Upsilon)$ is a Poincaré section for S, and there is a commutative diagram
$$\begin{array}{ccccccccc} \to & CH^n(S) & \xrightarrow{\iota'} & CH^n(S \cap \Xi) & \xrightarrow{id - r'^*} & CH^n(S \cap \Xi) & \xrightarrow{\delta'} & CH^{n+1}(S) & \to \\ & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & \\ \to & CH^n(T) & \xrightarrow{\iota} & CH^n(T \cap \Upsilon) & \xrightarrow{id - r^*} & CH^n(T \cap \Upsilon) & \xrightarrow{\delta} & CH^{n+1}(T) & \to \end{array}$$
5. All of these structures are invariant under continuation: Suppose $M \times [0, 1]$ admits a parameterized family of flows (i.e. there is a flow on $M \times [0, 1]$ which is constant on the [0,1] variable) and S is an isolated invariant set in $M \times [0, 1]$. Then each $S_t = S \cap M \times t$ is isolated in M for the t flow. Moreover, if Ξ is

a Poincaré section for S, then each $\Xi_t = \Xi \cap M \times t$ is a Poincaré section for S_t. Then the Conley indices of S_t and $S_t \cap \Xi_t$ are independent of t (up to isomorphism, of course).

The index suspension sequence shows that the continuous index $CH^*(S)$ can be computed from the discrete index $(CH^*(S \cap \Xi), r^*)$, i.e. there is a short exact sequence

$$0 \to \operatorname{coker}(id - r^{n-1*}) \to H^n(S) \to \ker(id - r^{n*}) \to 0.$$

This sequence also shows that the discrete cohomology algebra $CH^*(S \cap \Xi)$ cannot be reconstructed from $CH^*(S)$. In [15], for instance, there is an example of two isolated invariant sets which both have Poincaré sections, and which have the same continuous index, but which produce discrete systems on the cross-sections with non-isomorphic discrete indices. What the continuous index does compute is the subalgebra

$$E_1(r^*) = \{z \in CH^*(S \cap \Xi) \mid r^*(z) = z\}.$$

So in our application, the assumptions **A3** and **A4** provide enough information to compute this "1-eigenspace". Unfortunately, this is not quite sufficient for our purposes. We will require knowledge of the "generalized 1-eigenspace"

$$GE_1(r^*) = \{z \in CH^*(S \cap \Xi) \mid (r^* - id)^k(z) = 0 \text{ for some } k\},$$

and will need the additional hypothesis **A5** to compute it.

Specifically, **A3** assumes that each Morse set $M_p, p < P$, admits such a Poincaré section Ξ_p, **A4** gives the continuous index of M_p and **A5** assumes that the Morse set and its cross-section continue to a set which is the disjoint union of a hyperbolic periodic orbit and a set with trivial index. The existence of the Poincaré section guarantees the existence of the index suspension sequence

$$\to CH^n(M_p) \to CH^n(M_p \cap \Xi_p) \xrightarrow{id-r^*} CH^n(M_p \cap \Xi_p) \to CH^{n+1}(M_p) \to$$

and the knowledge of the continuous index of M_p decouples this long exact sequence into the following sequences:

$$0 \to \mathbf{Z} \to CH^{2p}(M_p \cap \Xi_p) \xrightarrow{id-r^*} CH^{2p}(M_p \cap \Xi_p) \to \mathbf{Z} \to 0$$

and

$$0 \to CH^n(M_p \cap \Xi_p) \xrightarrow{id-r^*} CH^n(M_p \cap \Xi_p) \to 0$$

for all $n \neq 2p$. From this we see that

$$E_1 \cong \begin{cases} \mathbf{Z}, & n = 2p, \\ 0, & n \neq 2p. \end{cases}$$

Since S continues to a disjoint union of an orientable hyperbolic periodic orbit and a set with zero index, the index suspension sequence decomposes into the index sequence of a hyperbolic periodic orbit and an index sequence with $GE_1(r^*) = E_1(r^*) = 0$. Of course, an orientable hyperbolic periodic orbit produces a hyperbolic fixed point on the section, whose discrete index is given above. Thus $GE_1 = E_1$. This will prove to be exactly the property we need, and for convenience we label it as

A5′: *For each $M_p, p < P$, the corresponding isolated invariant set $M_p \cap \Xi_p$ for the discrete dynamical system $r_p : \Xi_p \cap U_p \to \Xi_p$ has a discrete cohomology Conley index $(CH^*(M_p \cap \Xi_p), r_p^*)$ with $E_1(r_p^*) = GE_1(r_p^*)$.*

An equivalent statement is that the composition $CH^*(S) \xrightarrow{\delta\circ\iota} CH^*(S)$ is an isomorphism. All of this can then be summarized as:

Proposition 3.6. *A system which satisfies the assumptions* **A1** – **A5** *also satisfies the assumption* **A5′***, and for such a system, there is an isomorphism* $CH^*(S) \xrightarrow{\delta\circ\iota} CH^*(S)$ *which is natural with respect to semi-conjugacies.*

As mentioned above, there are examples that show that S can have the index of an orientable hyperbolic periodic orbit (cf. [15]) and admit a cross-section, yet not satisfy **A5′** (which can be loosely thought of as "**A3** and **A4** do not imply **A5′** "). On the other hand, it is not clear whether or not **A1** − **A4** imply **A5′** — that is, whether these counter-examples can be embedded as Morse sets in an attractor in a manner consistent with the assumptions.

4. The flow on D^{2P}

In this section, we explicitly construct the model flow on D^{2P} and explore some of its properties. Our development of this construction may seem unnecessarily labored, but it will serve two purposes. First, it will provide all of the needed ingredients for constructing the map $f : \mathcal{A} \to D^{2P}$ and proving that it is surjective. Second, it serves as a motivating example — the features of the flow on D^{2P} that are brought out by this construction are those that we seek to display for the flow on $\mathcal{A}$.

To begin, consider

$$\tilde{X} = \prod_{p=0}^{P-1} S^1_p \times \{P\} \times [0, P] \times I^{P-1}.$$

We will construct D^{2P} as a quotient space of $\tilde{X}$, and define the flow on D^{2P} in a natural way during this construction. First, we define an equivalence relation on $[0, P] \times I^{P-1}$. If $(x, \tau_1, \dots, \tau_{P-1}) \in [0, P] \times I^{P-1}$, define $l, r : [0, P] \times I^{P-1} \to \{0, \dots, P\}$ by

$$l(x, \tau_1, \dots, \tau_{P-1}) = \begin{cases} P & \text{if } x = P, \\ k & \text{if } k \leq x, \tau_k = 1, \text{ and } \forall k < p < x, \tau_p \neq 1, \\ 0 & \text{if no such } k \text{ exists,} \end{cases}$$

$$r(x, \tau_1, \dots, \tau_{P-1}) = \begin{cases} 0 & \text{if } x = 0, \\ k & \text{if } k \leq x, \tau_k = 1, \text{ and } \forall x < p < k, \tau_p \neq 1, \\ P & \text{otherwise.} \end{cases}$$

Define an equivalence relation on $[0, P] \times I^{P-1}$ by $(x, \tau) \sim (x', \tau')$ if $x = x', l(x, \tau) = l(x', \tau'), r(x, \tau) = r(x', \tau')$ and $\tau_p = \tau'_p$ for every $l(x, \tau) < p < r(x, \tau)$. Let

$$\eta : [0, P] \times I^{P-1} \to Q = [0, P] \times I^{P-1}/\sim \tag{34}$$

be the quotient map. That is, we identify points on the face $\tau_k = 1$ which have the same value for x and the same values for some of the τ_p's (which ones are required to agree depends on x, via the functions l and r). This process (with $P = 3$) is illustrated in Figure 2. The identification map collapses $\{x = 0\}$ and $\{x = 3\}$ each to a point, and collapses each of the lines $\{\tau_1 = 1, x = c \mid 0 < c \leq 1\}$ and $\{\tau_2 = 1, x = c \mid 2 \leq c < 3\}$ to a point.

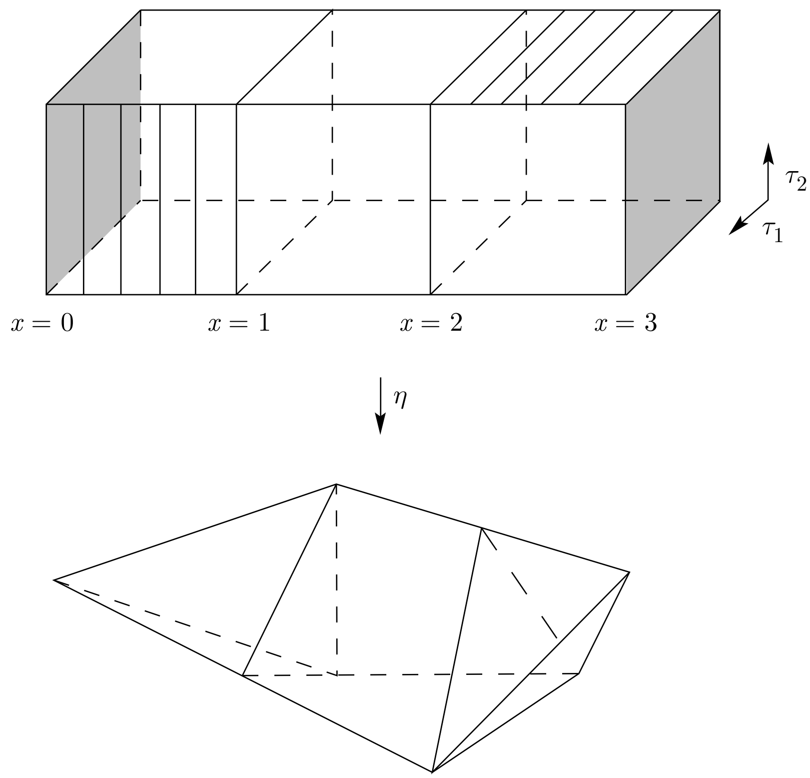

FIGURE 2. The identification map η for $P = 3$.

Note that, since η only identifies subfaces to points, Q is homeomorphic to I^P (though η is not a homeomorphism). Now choose a homeomorphism $\lambda : Q \to \Delta^P$ which sends $[k, 0, \dots, 0, \tau_k = 1, 0, \dots, 0]$ to the vertex v_k and sends $\eta(\{\tau_{i_1} = \dots = \tau_{i_n} = 0\})$ to the subface opposite $v_{i_i}, \dots, v_{i_n}$ (i.e. the subface expressed in barycentric coordinates as $\{t_{i_1} = \dots = t_{i_n} = 0\}$). Define

$$\hat{X} = \prod_{p=0}^{P-1} S^1_p \times \{P\} \times \Delta^P$$

and define a map $\sigma : \tilde{X} \to \hat{X}$ by $\sigma = (id \times \lambda) \circ \eta$.

To construct a map from $\hat{X}$ to D^{2P}, we make use of the join construction. Recall that the join of two spaces A and B is defined by $A * B = A \times B \times [0,1]/\sim$, where $(a, b, 0) \sim (a, b', 0)$ and $(a, b, 1) \sim (a', b, 1)$ are the only nontrivial equivalence relations. A multiple join $A_0 * A_1 * \dots * A_n$ can be defined by taking successive joins $((\dots((A_0 * A_1) * A_2) \dots) * A_n)$, but it can also be defined as follows. Take $A_0 \times \dots \times A_n \times \Delta^n$ and define the equivalence relation

$$(a_0, \dots, a_n, t_0, \dots, t_n) \sim (a'_0, \dots, a'_n, t'_0, \dots, t'_n) \tag{35}$$

if $(t_0, \dots, t_n) = (t'_0, \dots, t'_n)$ and, for every i, either $t_i = 0$ or $a_i = a'_i$. A useful way to view the multiple join is to take $\pi : A_0 * A_1 * \dots * A_n \to \Delta^n$, defined by $\pi_2 : [a_0, \dots, a_n, t_0, \dots, t_n] = (t_0, \dots, t_n)$. Then the preimage of $(t_0, \dots, t_n)$

is $A_{i_1} \times \ldots \times A_{i_k}$, where A_j is included in the product if and only if $t_j \neq 0$. In particular, if $\sigma = [v_{i_1} \ldots v_{i_k}]$ is the closed face of Δ^n spanned by vertices $v_{i_1}, \ldots, v_{i_k}$, then $\pi_{-1}(\sigma)$ is naturally homeomorphic to $A_{i_1} * \ldots * A_{i_k}$.

Two standard join constructions are $S^p * S^q$, which is homeomorphic to S^{p+q+1}, and $A * pt$, which is homeomorphic to CA, the cone on A. With these two constructions, we obtain the following fact:

Proposition 4.1. *The multiple join $S^1_0 * \ldots * S^1_{P-1}$ is homeomorphic to S^{2P-1}; the multiple join $S^1_0 * \ldots * S^1_{P-1} * \{P\}$ is homeomorphic to D^{2P}.*

We may assume that the homeomorphism $g : S^1_0 * \ldots * S^1_{P-1} * \{P\} \to D^{2P}$ takes $\{P\}$ to $\{0\}$, $S^1_p * \ldots * S^1_q$ to $\{x \mid \sum_{r=p}^{2q+1} x_r^2 = 1\}$ and $S^1_p * \ldots * S^1_{P-1} * \{P\}$ to $\{x \mid x_r = 0 \,\text{for}\, r < p\}$. The map $\hat{\eta} : \hat{X} \to D^{2P}$ that we seek is then just the natural identification map $\prod_{p=0}^{P-1} S^1_p \times \{P\} \times \Delta^P \to S^1_0 * \ldots * S^1_{P-1} * \{P\}$, followed by g. Denote the composition

$$\tilde{X} \xrightarrow{\eta} \hat{X} \xrightarrow{\hat{\eta}} S^1_0 * \ldots * S^1_{P-1} * \{P\} \xrightarrow{g} D^{2P}$$

by

$$\mathcal{Q} = g \circ \hat{\eta} \circ \eta.$$

Note that there is a well-defined function

$$\rho : D^{2P} \to [0, P] \tag{36}$$

with $\rho(\mu(\alpha_0, \ldots, \alpha_{P-1}, x, \tau_1, \ldots, \tau_{P-1}) = x$.

We now construct a flow on $\tilde{X}$ which induces the desired flow on D^{2P}. On each S^1_k, take the rotational flow with period 1. On $[0, P] \times I^{P-1}$, define a flow which has as its rest point set $\{x = 0, P\} \cup \bigcup_{k=1}^{P-1} \{x = k, \tau_k = 1\}$, and otherwise has a "decreasing horizontal" flow:

$$(x, \tau) \cdot t = (\phi(x, \tau, t), \tau)$$

with $\phi(x, \tau, t)$ independent of τ_i for $i < l(x, \tau)$ or $i > r(x, \tau)$ and $\dot{\phi}(x, \tau, 0) \leq 0$ for all x and τ. Such a flow clearly exists: choose a function $h : Q \to [-1, 0]$ with $h^{-1}(0)$ as prescribed above, and let $\dot{x} = h\eta(x, \tau), \dot{\tau} = 0$. This flow on $\tilde{X}$ then induces flows on $\hat{X}$ and D^{2P} in the natural way. That is, the quotient map η has the property that, if $\eta(x) = \eta(y)$, then $\eta(x \cdot t) = \eta(y \cdot t)$. The other quotient map involved, $\hat{\eta}$, has the same property. The map $\mathcal{Q}$ is, by construction, a proper semi-conjugacy between the flow on $\tilde{X}$ and the induced flow on D^{2P}.

If $(\alpha_0, \ldots, \alpha_{P-1}, x, \tau_1, \ldots, \tau_{P-1}) \in \tilde{X}$, then the x-coordinate is a Lyapunov function: $\dot{x} \leq 0$, with $\dot{x} = 0$ only at rest points. Also, the functions l and r now have dynamical interpretation: the ω-limit set of $(\alpha_0, \ldots, \alpha_{P-1}, x, \tau_1, \ldots, \tau_{P-1})$ is

$$\prod_{p=0}^{P-1} S^1_p \times \{P\} \times \{(l(x, \tau_1, \ldots, \tau_{P-1}), \tau_1, \ldots, \tau_{P-1})\}$$

and the α-limit set is

$$\prod_{p=0}^{P-1} S^1_p \times \{P\} \times \{(r(x, \tau_1, \ldots, \tau_{P-1}), \tau_1, \ldots, \tau_{P-1})\}.$$

The conditions that define r and l thus define the stable and unstable sets of each rest point set $X_p = \{x = p, \tau_p = 1\}, p = 1, \dots, P-1$:

$$\begin{array}{rcl} W^u(X_p) & = & \{x \leq p, \tau_p = 1, \text{and}\, \forall x < q < p, \tau_q \neq 1\}, \\ W^s(X_p) & = & \{x \geq p, \tau_p = 1, \text{and}\, \forall p < q < x, \tau_q \neq 1\}, \end{array}$$

while $X_0 = \{x = 0\}$ and $X_P = \{x = P\}$ have

$$\begin{array}{rclrcl} W^u(X_0) & = & X_0, & W^u(X_P) & = & \{x > \max\{k \mid \tau_k = 1\}\}, \\ W^s(X_0) & = & \{x < \min\{k \mid \tau_k = 1\}\}, & W^s(X_P) & = & X_P. \end{array}$$

All of this implies that the collection $\{X_p\}_{p=0}^{P}$ is a Morse decomposition, with flow-defined order $0 < 1 < \dots < P$.

Since $\mathcal{Q}$ is a semi-conjugacy and $\rho \cdot \mathcal{Q}$ is a Lyapunov function on $\tilde{X}$, ρ is a "weak" Lyapunov function. That is, ρ is non-increasing on orbits, and is only constant on the sets $\Pi_p = \mathcal{Q}(X_p) = \{x_{2p}^2 + x_{2p+1}^2 = 1\}$ (for $0 \leq p < P$) and $\Pi_P = 0 = \mathcal{Q}(P)$. Each of these sets consists of a single periodic orbit (except, of course, for 0, which is a fixed point). From the construction of $\mathcal{Q}$ and the information on stable and unstable sets above, we see that $\{\Pi_p\}_{p=0}^{P}$ is a Morse decomposition, with flow-defined ordering $0 < 1 < \dots < P$. Moreover, $\overline{W^u(\Pi_P)} = D^{2P}$, while $\overline{W^u(\Pi_p)} = \left\{\sum_{r=0}^{2p+1} x_r^2 = 1\right\}$. That is, this flow on D^{2P} is conjugate to the flow ψ described in Section 1. Let

$$G : D^{2P} \to D^{2P} \tag{37}$$

denote this conjugacy.

Some of the properties of this flow will be of particular importance in the construction of the semi-conjugacy:

Lemma 4.2. *If I is an interval in $\mathcal{P}$, then $\Pi(I)$ is homeomorphic to the $\tilde{i}$-sphere if $\bar{i} < P$, and is homeomorphic to the $(\tilde{i}-1)$-disk if $\bar{i} = P$.*

Proof. Recall that $\tilde{i} = 2\bar{i} - 2\underline{i} + 1$. First, consider the case of $\bar{i} < P$. From the discussion above, it is clear that the connecting orbit set $C_{q,p} = W^s(\Pi_p) \cap W^u(\Pi_q)$ between Π_q and Π_p is the image of $S_p^1 \times S_{p+1}^1 \times \dots \times S_{q-1}^1 \times S_q^1 \times int(span\{v_p, v_{p+1}, \dots, v_{q-1}, v_q\})$ under $g\hat{\eta}$. Thus, $\Pi(I)$ is the image of $S_{\underline{i}}^1 \times \dots \times S_{\bar{i}}^1 \times span\{v_{\underline{i}}, \dots, v_{\bar{i}}\}$. But Proposition 4.1 implies that this is a $\tilde{i}$-sphere.

The argument when $\bar{i} = P$ is similar. ∎

Corollary 4.3. *The flow ψ on D^{2P} satisfies the assumptions* **A1** – **A5**.

Proof. We need only verify the Conley index statements of **A4**. For $0 \leq p \leq P$, let $N_p = \rho^{-1}([0, p + \frac{1}{2}])$, and let $N_{-1} = \emptyset$. Then the pair (N_p, N_{p-1}) is an index pair for Π_p. It suffices then to show that

1. If $p \leq q < P$, then the pair (N_q, N_p) has the same homotopy type as the pair (S^{2q+1}, S^{2p-1}).
2. If $p \leq P$, then the pair (N_P, N_p) has the same homotopy type as the pair (D^{2P}, S^{2p+1}).

(Cf. Proposition 3.4.)

Clearly, $N_P = D^{2P}$. Any attracting interval $I = \{0, \dots, p\}$ with $p < P$ has $\Pi(I) = S^{2p+1}$, with dual repeller $\Pi(\mathcal{P} \setminus I) = D^{2P-2p-2}$. Any $N(I)$ must have empty intersection with $\Pi(\mathcal{P} \setminus I)$. If D^{2P} is viewed as $S^{2p+1} * D^{2P-2p-2}$, then any positively invariant neighborhood of S^{2p+1} which has empty intersection with $D^{2P-2p-2}$ must have a strong deformation retraction onto S^{2p+1}. $\square$

Corollary 4.4. *For every interval I in $\mathcal{P}$, the map $H^k(\Pi(I)) \xrightarrow{\Theta_I} CH^{k+2\underline{i}}(\Pi(I))$ is an isomorphism.*

Proof. The case of $q = P$ is trivial, so we consider only the case $q < P$. We can view $S^{2\bar{i}+1}$ as $S^{2\underline{i}-1} * S^{\tilde{i}}$, and $(S^{2\bar{i}+1}, S^{2\underline{i}-1})$ as $(S^{\tilde{i}} \times C(S^{2\underline{i}-1}), S^{\tilde{i}} \times S^{2\underline{i}-1})$. Then the map $H^k(\Pi(I)) \xrightarrow{\Theta_I} CH^{k+2\underline{i}}(\Pi(I))$ is represented by the map

$$\begin{aligned} &H^k(S^{\tilde{i}} \times C(S^{2\underline{i}-1}) \otimes H^{2\underline{i}}(S^{\tilde{i}} \times C(S^{2\underline{i}-1}), S^{\tilde{i}} \times S^{2\underline{i}-1}) \\ &\qquad\qquad \xrightarrow{\cup} H^{k+2\underline{i}}(S^{\tilde{i}} \times C(S^{2\underline{i}-1}), S^{\tilde{i}} \times S^{2\underline{i}-1}). \end{aligned}$$

This map is clearly an isomorphism. $\square$

5. Reparameterizing the flow

The first step in the proof of Theorem 1.3 is to reparameterize the flow φ with respect to time. The new reparameterized flow $\widetilde{\varphi}$ will have the following three important properties.

1. $\widetilde{\varphi}$ preserves the qualitative behavior of φ, in particular, for every $x \in \mathcal{A}$, $\varphi(\mathbf{R}, x) = \widetilde{\varphi}(\mathbf{R}, x)$.
2. In the vicinity of each Morse set, the orbits of $\widetilde{\varphi}$ have a constant "velocity" with respect to an angle coordinate. This will be used in the next section to map, via the semi-conjugacy, each Morse set to a single periodic orbit. In order to do this continuously a uniform rate of return on each component of the chain recurrent set is required.
3. Away from the Morse sets the orbits of $\widetilde{\varphi}$ have a constant "velocity" with respect to a global Lyapunov function. The Lyapunov function will be constructed in the next section and will be used as a global coordinate on $\mathcal{A}$. Roughly speaking, what will be required is that given two points which have the same Lyapunov value and which cannot be distinguished by local information, i.e. via their behavior near Morse sets, must maintain the same Lyapunov values throughout their orbits.

A brief outline of this section is as follows. We begin with an abstract description of how the flow will be reparameterized. This is followed by a specific choice of isolating neighborhoods for the Morse sets. Then in these isolating neighborhoods we reparametrize φ, thereby obtaining a new flow φ'. Then a global reparameterization is performed and $\widetilde{\varphi}$ is obtained.

5.1. **Defining reparameterizations.** Let $\varphi : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$ be a continuous flow on a compact metric space $\mathcal{A}$. Let D be a compact local section of φ and assume that there exists a continuous function

$$\tau : D \to (0, \infty)$$

such that for every $x \in D$,

$$\varphi((0, \tau(x)), x) \cap D = \emptyset.$$

The compactness of D and Urysohn's lemma imply the existence of a local section U which contains D and a continuous function $\rho : \mathcal{A} \to (0, \infty)$ of the form

$$\rho(x) = \begin{cases} \tau(x) & \text{if } x \in D, \\ 1 & \text{if } x \notin U, \end{cases} \tag{38}$$

such that $\varphi((0, \rho(x)), x) \cap U = \emptyset$. Define $\varphi'_l : [0,1] \times U \to \mathcal{A}$ by

$$\varphi'_l(t, x) = \varphi(t\rho(x), x). \tag{39}$$

Let

$$\Gamma := \varphi'_l([0,1] \times cl(U)) = \{\varphi(t,x) | x \in cl(U),\ 0 \le t \le \rho(x)\}.$$

Observe that Γ is compact. Now let $x \in \mathcal{A}$; then there exists a sequence of intervals

$$\{[\sigma_i^-(x), \sigma_i^+(x)] \subset \mathbf{R} | \sigma_i^+ < \sigma_{i+1}^-\}$$

such that

$$\varphi(t,x) \in \begin{cases} \Gamma & \text{if } t \in [\sigma_i^-(x), \sigma_i^+(x)], \\ \mathcal{A} \backslash \Gamma & \text{if } t \in (\sigma_i^+(x), \sigma_{i+1}^-(x)). \end{cases} \tag{40}$$

Obviously, the number of intervals depends on the point x. For some points no intervals will exist, while for others some intervals might be unbounded. To standardize the notation, if the intervals exist, we shall assume that they are labeled so that $\sigma_{-1}^+ < 0 \le \sigma_0^+$. We can now extend the map φ'_l to a flow inductively in the following manner. Assume that $\sigma_0^- \le 0$ and $t > 0$. Let

$$\begin{aligned} \nu_n(x) &= \frac{\sigma_0^+(x)}{\tau(x)} + n + \sum_{i=1}^{n} (\sigma_i^-(x) - \sigma_{i-1}^+(x)), \\ \eta_n(x) &= \frac{\sigma_0^+(x)}{\tau(x)} + n + \sum_{i=1}^{n+1} (\sigma_i^-(x) - \sigma_{i-1}^+(x)) \end{aligned}$$

and define

$$\varphi'(t,x) = \begin{cases} \varphi(t - \nu_n, \varphi'(\nu_n, x)) & \text{if } t \in [\nu_n, \eta_n], \\ \varphi'_l(t - \eta_n, \varphi'(\eta_n, x)) & \text{if } t \in [\eta_n, \nu_{n+1}]. \end{cases} \tag{41}$$

The definitions for $t < 0$ or $\sigma_0^- \ge 0$ are similar. Since $\rho(x) = 1$ if $x \in \partial U$, it is easy to check that φ' is a continuous flow.

Definition 5.1. Using this inductive definition we shall refer to φ' as the *reparameterization of φ through D.*

Observe that if $\tau|_{\partial D} = 1$, then U can be chosen to be D and φ' will still be continuous.

5.2. **Isolating neighborhoods.** For technical reasons which will become apparent later, we need more control on the isolating neighborhood of M_p. To obtain this, we need to make a slight digression.

Proposition 5.2 (Conley [2])**.** *Let $\mathcal{A}$ be a compact invariant set under a flow φ with Morse decomposition $\mathcal{M}(\mathcal{A}) = \{M_p | p = 0, \dots, P\}$. Then, there exists a Lyapunov function*

$$V : \mathcal{A} \to [0, P]$$

such that:

(i) *if $x \in M_p$, then $V(x) = p$,*

(ii) *if* $x \notin \bigcup_{p=0}^{P} M_p$*, then* $V(x) > V(\varphi(t,x))$ *for all* $t > 0$*.*

Given the Lyapunov function V of Proposition 5.2, let

$$\begin{aligned} Q^+(\nu) &= \{x \in \mathcal{A} \mid V(x) \geq \nu\}, \\ Q^-(\nu) &= \{x \in \mathcal{A} \mid V(x) \leq \nu\}, \end{aligned}$$

and define

$$R_p(t,\delta) = \mathcal{A} \backslash \{\varphi([0,t], Q^+(p+\delta)) \cup \varphi([-t,0], Q^-(p-\delta))\}. \tag{42}$$

The following proposition indicates how this Lyapunov function can be used to choose isolating neighborhoods.

Proposition 5.3. *Let* K_p *be an isolating neighborhood for the Morse set* M_p*. Define*

$$N_p = cl(R_p(\bar{t}, \frac{1}{4})), \quad p = 0, \dots, P.$$

Then, there exists $\bar{t}$*, sufficiently large, such that* N_p *is an isolating neighborhood of* M_p*, with the following properties:*

(i) $N_p \subset K_p$*;*
(ii) *if* $p \neq q$*, then* $N_p \cap N_q = \emptyset$*;*
(iii) $\partial N_p = L_p^+ \cup L_p^-$*, where*

$$\begin{aligned} L_p^+ &= \{x \in N_p | V(\varphi(-\bar{t}, x)) = p + \frac{1}{4}\}, \\ L_p^- &= \{x \in N_p | V(\varphi(\bar{t}, x)) = p - \frac{1}{4}\}; \end{aligned}$$

(iv) $\varphi(\mathbf{R}, x) \cap N_p = \varphi(I_p(x), x)$ *where* $I_p(x)$ *is a closed interval (possibly empty or unbounded);*
(v) (N_p, L_p^-) *is a regular index pair for* φ *and* (N_p, L_p^+) *is a regular index pair for* φ^**, the flow obtained from* φ *by reversing the direction of time.*

Proof. (i) If $x \in \mathcal{A}$ is such that $p - \frac{1}{2} \leq V(x) \leq p + \frac{1}{2}$ and $\varphi(\mathbf{R}, x) \cap (Q^+(\nu) \cup Q^-(\nu)) = \emptyset$, then $x \in M_p$. Furthermore, if U is a neighborhood of M_p, then there exists $T > 0$ such that if $p - \frac{1}{2} \leq V(\varphi(t,x)) \leq p + \frac{1}{2}$ for all $t \in [-T, T]$, then $x \in U$.

(ii) and (iii) are obvious.

(iv) Since V is a Lyapunov function, if $x \in N_p$ and $\varphi(s,x) \notin N - p$ for some $s > 0$, then $\varphi(t + \bar{t}, x) \in Q^-(p - \frac{1}{4})$ for all $t \geq s$. Hence $\varphi(t,x) \notin N_p$ for all $t \geq s$. A similar argument applies if $\varphi(s,x) \notin N_p$ and $s < 0$.

(v) This follows from the fact that V is a Lyapunov function and (iii). ☐

Since by this proposition $I_p(x)$ is a closed interval, we shall write

$$I_p(x) = [a_p(x), b_p(x)] \tag{43}$$

with the understanding that if $I_p(x) = \emptyset$, then $a_p(x)$ and $b_p(x)$ are not defined, and if $I_p(x)$ is unbounded, then $a_p(x) = -\infty$ or $b_p(x) = \infty$. Let

$$\Theta_p := \{x \in \mathcal{A} | I_p(x) \neq \emptyset\};$$

then the fact that $(N_p, L_p^{\pm})$ are regular index pairs implies the following lemma.

Lemma 5.4. *The functions* $a_p,\ b_p : \Theta_p \to [-\infty, \infty]$ *are continuous.*

5.3. **A local reparameterization.** We now concentrate on the flow in neighborhoods of the Morse sets. By **A3**, for each M_p there exists a neighborhood K_p with Poincare section Ξ_p. Recall (33) that this implies the following. If $x \in K_p \cap \Xi_p$, then there exists $T(x) > 0$ such that $\varphi(T(x), x) \in \Xi_p$ and $\varphi((0, T(x)), x) \cap \Xi_p = \emptyset$. Let φ' be the reparameterization of φ through $\bigcup_{p=0}^{P} \Xi_p$ induced by T.

Let K_p be an isolating neighborhood of M_p such that if $x \in \Xi_p \cap K_p$, then $\varphi'([-2,2], x) \subset K_p$. Of course, this implies that $\varphi'(\pm 2, x) \in \Xi_p$. Furthermore, if $x \in K_p$, then there exists a unique $\xi_x \in \Xi_p$ and a unique $s_x \in [0,1)$ such that

$$\varphi'(s_x, \xi_x) = x. \tag{44}$$

For the remainder of this paper N_p is as in Proposition 5.3, i.e. $N_p = cl(R_p(\bar{t}, \frac{1}{4}))$.

5.4. **The global reparameterization.** By Proposition 5.3(iv), for each $x \in \mathcal{A}$ there exists an interval $I'_p(x)$ such that

$$\varphi'(\mathbf{R}, x) \cap N_p = \varphi'(I'_p(x), x).$$

As before, let $I'_p(x) = [a'_p(x), b'_b(x)]$. Observe that $a'_p, b'_p : \Theta_p \to [-\infty, \infty]$ are still continuous functions.

Before we can describe the global reparameterization we need some technical results. Let us begin by defining functions

$$W_p^+ : L_p^+ \to [p, p + \frac{1}{4}]$$

and

$$W_p^- : L_p^- \to [p - \frac{1}{4}, p]$$

by

$$\begin{aligned} W_p^+(x) &= p + \frac{1}{2\pi} \tan^{-1}\left(\frac{b'_p(x)}{2}\right), \\ W_p^-(x) &= p + \frac{1}{2\pi} \tan^{-1}\left(\frac{a'_p(x)}{2}\right), \end{aligned}$$

where $\tan^{-1}(\pm\infty) := \pm\frac{\pi}{2}$. Since $a'_p(x)$ and $b'_p(x)$ are continuous, using these definitions the following statements are obvious:

- $x \in L_p^+ \cap L_p^-$ implies that $a'_p(x) = b'_p(x) = 0$, and hence, $W_p^{\pm}(x) = p$;
- if $x \in L_p^+$, then
$$\omega(x) \subset M_p \quad \Leftrightarrow \quad b'_p(x) = \infty \quad \Leftrightarrow \quad W_p^+(x) = p + \frac{1}{4};$$
- if $x \in L_p^-$, then
$$\alpha(x) \subset M_p \quad \Leftrightarrow \quad a'_p(x) = -\infty \quad \Leftrightarrow \quad W_p^-(x) = p - \frac{1}{4};$$
- $W_p^{\pm}$ is continuous.

Let

$$Z_p = \{x \in \mathcal{A} | V(x) = p + \frac{1}{4}\}.$$

Define

$$D_p^+ = \varphi(\bar{t}, Z_p)$$

and observe that $L_p^+ \subset D_p^+$. Now set

$$D_p^- = L_p^- \cup (D_p^+ \backslash L_p^+)$$

and observe that D_p^- is a compact local section for φ'. Extend the functions $W_p^\pm$ to

$$\bar{W}_p^\pm : D_p^\pm \to [p - \frac{1}{4}, p + \frac{1}{4}]$$

as follows:

$$\begin{aligned}
\bar{W}_p^+(x) &= \begin{cases} W_p^+(x) & \text{if } x \in L_p^+, \\ p & \text{otherwise,} \end{cases} \\
\bar{W}_p^-(x) &= \begin{cases} W_p^-(x) & \text{if } x \in L_p^-, \\ p & \text{otherwise.} \end{cases}
\end{aligned}$$

Define $\tau : D_p^- \to (0, \infty)$ by

$$\varphi'(\tau(x), x) \in D_{p-1}^+.$$

Let $\lambda_p : D_p^- \to (0,1)$ be defined by

$$\lambda_p(x) = \bar{W}_p^-(x) - \bar{W}_{p-1}^+(\varphi'(\tau(x), x)),$$

and let $\Lambda_p = \{(t,x) | 0 \le t \le \lambda_p(x),\ x \in D_p^-\}$. Define $\widetilde{\varphi}_l^1 : \Lambda_1 \to \mathcal{A}$ by

$$\widetilde{\varphi}_l^1(t,x) = \varphi' \left(t \frac{\tau_1(x)}{\lambda_1(x)}, x \right).$$

Let $\widetilde{\varphi}^1$ be the reparameterization of φ' through Λ_1. Inductively, for $p = 2, \dots, P$, define $\widetilde{\varphi}^p$ to be the reparameterization of $\widetilde{\varphi}^{p-1}$ through Λ_p generated by $\widetilde{\varphi}_l^p : \Lambda_p \to \mathcal{A}$, where

$$\widetilde{\varphi}_l^p(t,x) = \widetilde{\varphi}^{p-1} \left(t \frac{\tau_p(x)}{\lambda_p(x)}, x \right).$$

Finally, let

$$\widetilde{\varphi} = \widetilde{\varphi}^P.$$

6. The semi-conjugacy

The construction of the map $f : \mathcal{A} \to D^{2P}$ of Theorem 1.3 involves several steps.

Step 1. Define a discontinuous function

$$\tilde{f} : \mathcal{A} \to \widetilde{X} := \left(\bigotimes_{p=0}^{P-1} S_p^1 \right) \times \{P\} \times [0,P] \times [0,1]^{P-1}. \tag{45}$$

Step 2. Using the quotient map $\mathcal{Q} : \widetilde{X} \to D^{2P}$ described in Section 4, show that $\hat{f} : \mathcal{A} \to D^{2P}$ given by $\hat{f} = \mathcal{Q} \circ \tilde{f}$ is continuous.

Step 3. Define $\widetilde{\psi} : \mathbf{R} \times \hat{f}(\mathcal{A}) \to \hat{f}(\mathcal{A})$ by

$$\widetilde{\psi}(t, \hat{f}(x)) = \hat{f}(\widetilde{\varphi}(t,x)) \tag{46}$$

and show that $\widetilde{\psi}$ is a continuous flow.

Step 4. Define

$$f = G \circ \hat{f},$$

where $G : D^{2P} \to D^{2P}$ is the conjugacy of Section 4, and observe that f is the desired map.

Before proceeding with the details, we present an informal description of the steps involved in defining f in an attempt to explain why this construction is natural. Consider a point $x \in \mathcal{A}$. To describe its dynamics in terms of the Morse decomposition, there are several obvious parameters which should be included.

1. For each Morse set M_p one would like to know how "close" the orbit of x passes by M_p. Thus we shall define a function
$$\tau_p : \mathcal{A} \to [0,1], \quad p = 1, \dots, P-1,$$
with the property that $\tau_p(x) = 1$ implies that $\omega(x)$ or $\alpha(x) \subset M_p$, while $\tau_p(x) = 0$ implies that the orbit of x does not intersect $int(N_p)$, the isolating neighborhood of M_p.
2. Since our Morse sets have Poincaré sections, we should also measure the phase with which the orbit of x passes near M_p. This will be done via "angle" functions
$$\theta_p : \mathcal{A} \to S^1_p, \quad p = 0, \dots, P-1.$$
Since there is no Poincaré section for M_P, we set
$$\theta_P(x) = P.$$
3. Given a point x (not in a Morse set), we cannot yet distinguish x from its integer translates $x \cdot n$: $\tau_p(x) = \tau_p(x')$ and $\theta_p(x) = \theta_p(x')$ for every p. Thus, to distinguish these points we shall make use of a Lyapunov function
$$V : \mathcal{A} \to [0, P].$$

Given these functions define

$$\tilde{f}(x) = (\theta_0(x), \dots, \theta_{P-1}(x), \theta_P(x), V(x), \tau_1(x), \dots, \tau_{P-1}(x)).$$

It is easily seen that $\tilde{f}$ cannot be continuous. In particular, if the orbit of x does not pass through N_p, then the angle function θ_p cannot be defined. Similarly, if one considers a sequence of points whose ω-limit set is in M_p, then $\tau_p = 1$ on this sequence. However, this sequence may limit to a point whose ω-limit set is in M_q where $q > p$, in which case $\tau_p = 0$ for the limit point. The quotient performed in step two addresses this problem, i.e. the resulting map $\hat{f} : \mathcal{A} \to D^{2P}$ is continuous.

Step 3 indicates that the image of $\widetilde{\varphi}$ under $\hat{f}$ induces a flow on D^{2P}. At this point it will become clear that Section 5 was written with this step in mind. Step 4 obviously concludes the construction.

Let us, finally, begin constructing $\widetilde{f}$. We start with the definition of

$$\theta_p : \mathcal{A} \to S^1_p := \mathbf{R}/\mathbf{Z}, \quad p = 0, \dots, P-1.$$

Recall (44) that if $x \in N_p$, then there exists a unique $\xi_x \in \Xi_p$ and a unique $s_x \in [0,1)$ such that $\varphi'(s_x, \xi_x) = x$. Using the conventions described in Section 5, let

$$\widetilde{\varphi}([\widetilde{a}_p(x), \widetilde{b}_p(x)], x) = \widetilde{\varphi}(\mathbf{R}, x) \cap N_p.$$

Define

$$\theta_p(x) = \begin{cases} 0 & \text{if } [\widetilde{a}_p(x), \widetilde{b}_p(x)] = \emptyset, \\ \widetilde{a}_p(x) + s_{\widetilde{\varphi}(\widetilde{a}_p(x),x))} & \text{if } \widetilde{a}_p(x) > -\infty, \\ \widetilde{b}_p(x) + s_{\widetilde{\varphi}(\widetilde{b}_p(x),x))} & \text{if } \widetilde{b}_p(x) < \infty, \\ s_x & \text{otherwise.} \end{cases}$$

Lemma 6.1. *Let* $\Theta_p = \{x \in \mathcal{A} | \, [\widetilde{a}_p(x), \widetilde{b}_p(x)] \neq \emptyset\}$. *Then*

$$\theta_p : \Theta_p \to S^1_p$$

and

$$\theta_p : \mathcal{A} \backslash \Theta_p \to S^1_p$$

are continuous.

Proof. The result is obvious in the case of $\mathcal{A}\backslash\Theta_p$ since θ_p is constant in this case.

Observe that $\widetilde{a}_p(x)$ and $\widetilde{b}_p(x)$ are continuous on Θ_p. Furthermore, if $\widetilde{a}_p(x) > -\infty$ and $\widetilde{b}_p(x) < \infty$, then there exists $n \in \{0, 1, 2, \ldots\}$ such that

$$s_{\widetilde{\varphi}(\widetilde{b}_p(x),x))} = s_{\widetilde{\varphi}(\widetilde{a}_p(x),x))} + n.$$

Recall that if $x \in N_p$, then $\varphi'([-2, 2], x) \subset K_p$. Since $N_p \subset int(K_p)$, θ_p is continuous at $x \in N_p$. But $\theta_p(x) = \theta_p(\widetilde{\varphi}(t, x))$ for all $t \in \mathbf{R}$, thus θ_p is continuous. $\square$

Let us now turn our attention to

$$\tau_p : \mathcal{A} \to [0, 1], \qquad p = 1, \ldots, P-1. \tag{47}$$

Define

$$\lambda_p(x) = \begin{cases} \infty & \text{if } \widetilde{b}_p(x) = \infty, \text{or } \widetilde{a}_p(x) = -\infty, \\ 0 & \text{if } \widetilde{I}_p(x) = \emptyset, \\ \widetilde{b}_p(x) - \widetilde{a}_p(x) & \text{otherwise.} \end{cases} \tag{48}$$

and

$$\tau_p(x) = \frac{2}{\pi} \tan^{-1}(\lambda_p(x)), \tag{49}$$

where $\tan^{-1}(\infty) = \frac{\pi}{2}$. Observe that if $\tau_p(x) = 1$, then x limits to M_p either in forward time or backward time (or both). Furthermore, if $\tau_p(x) = 0$, then the orbit of x passes no closer to M_p than the set $L^+_p \cap L^-_p$. The continuity of $\widetilde{a}_p(x)$ and $\widetilde{b}_p(x)$ gives rise to the following lemma.

Lemma 6.2. $x \in \mathcal{A}$ *is a point of discontinuity of* τ_p *if and only if there exists a sequence* $\{x_n\} \subset \mathcal{A}$ *such that* $x_n \to x$ *and* $\omega(x) \subset M_p$ *while* $\omega(x_n) \subset M_q$ *with* $q < p$, *or* $\alpha(x) \subset M_p$ *while* $\alpha(x_n) \subset M_q$ *with* $q > p$.

Finally, we need to define the Lyapunov function

$$V : \mathcal{A} \to [0, P].$$

We begin by defining local Lyapunov functions

$$V_p : N_p \to [p - \frac{1}{4}, p + \frac{1}{4}]$$

by

$$V_p(x) = \begin{cases} p & \text{if } x \in M_p, \\ W_p^+(\widetilde{\varphi}(\widetilde{a}_p(x), x)) + \frac{1}{2\pi}\tan^{-1}\left(\frac{a_p(x)}{2}\right) & \text{if } \widetilde{a}_p(x) > -\infty, \\ W_p^-(\widetilde{\varphi}(\widetilde{b}_p(x), x)) + \frac{1}{2\pi}\tan^{-1}\left(\frac{b_p(x)}{2}\right) & \text{if } \widetilde{b}_p(x) < \infty. \end{cases}$$

To define V off of the isolating neighborhoods observe that if $x \in \mathcal{A}\setminus\bigcup_{p=0}^{P} N_p$, then there exists a unique $d_x \in D^-_{p_x}$, $p_x \in \{1, 2, \dots, P\}$ and a unique $\alpha_x \in [0, 1)$ such that

$$\widetilde{\varphi}(\alpha_x, d_x) = x.$$

Now define

$$V(x) = \begin{cases} p_x - \alpha_x & \text{if } x \notin \bigcup_{p=0}^{P} N_p, \\ V_p(x) & \text{if } x \in N_p. \end{cases}$$

The following lemma is easily checked.

Lemma 6.3. *The Lyapunov function V is continuous. Furthermore,*

(i) *if $\widetilde{\varphi}([0,t], x) \cap \left(\bigcup_{p=0}^{P} N_p\right) = \emptyset$, then*

$$V(\widetilde{\varphi}(t, x)) = V(x) - t;$$

(ii) *if $\widetilde{\varphi}([0,t], x) \subset N_p$, $\widetilde{\varphi}([0,t], y) \subset N_p$, $\theta_p(x) = \theta_p(y)$, $\tau_p(x) = \tau_p(y)$, and $V(x) = V(y)$, then*

$$V(\widetilde{\varphi}(t, x)) = V(\widetilde{\varphi}(t, y)).$$

With these constructions in mind $\widetilde{f} : \mathcal{A} \to \widetilde{x}$ is defined by

$$\begin{aligned} \widetilde{f}(x) &= (\theta_0(x), \theta_1(x), \dots, \theta_P(x), V(x), \tau_1(x), \dots, \tau_{P-1}(x)) \\ &= (\theta(x), V(x), \tau(x)). \end{aligned} \tag{50}$$

As was mentioned in the introductory remarks of this section, this function is not continuous. Hence we define

$$\widehat{f} = \mathcal{Q} \circ \widetilde{f}.$$

Proposition 6.4. $\widehat{f} : \mathcal{A} \to D^{2P}$ *is continuous.*

Proof. Since $\mathcal{Q}$ and V are continuous, it is clear that any possible lack of continuity of $\widehat{f}$ arises from the maps τ and θ. As will be shown, the discontinuities induced by τ are eliminated via the quotient map η (34) while those of θ disappear via the quotient to the join (35).

Case 1. Assume $p < V(x) < p + 1$ and $\tau_p(x) = \tau_{p+1}(x) = 1$.

This implies that $\omega(x) \subset M_p$ and $\alpha(x) \subset M_{p+1}$. In addition, for $r \neq p, p+1$, $\widetilde{I}_r(x) = \emptyset$, $\tau_r(x) = 0$, $\widetilde{I}_{p+1} = (-\infty, \widetilde{b}_{p+1}(x)]$, and $\widetilde{I}_p = (\widetilde{a}_p(x), \infty)$. Let $\{x_n\} \subset \mathcal{A}$ such that $x_n \to x$ as $n \to \infty$. By continuity of the flow, for n sufficiently large and $q = p, p+1$, $\widetilde{I}_q(x) \neq \emptyset$, and $|\widetilde{b}_p(x_n) - \widetilde{a}_p(x_n)| \to \infty$. Thus $\tau_q(x_n) \to \tau_q(x)$ as $n \to \infty$, i.e. in this case τ_q is continuous at x. Obviously, if $\tau_r(x_n) \to \tau_r(x)$ for all $r \neq p, p+1$, then we are done with this part of the argument. Therefore, without loss of generality, we may assume that for some fixed r and for n sufficiently large $\tau_r(x_n) = 1$. This implies that $\widetilde{f}(x_n) \to \widetilde{x} \in \widetilde{X}$, where

$$\widetilde{x} = (\widetilde{\theta}, \widetilde{V}, \widetilde{\tau}_1, \dots, \widetilde{\tau}_{p-1}, 1, 1, \widetilde{\tau}_{p+2}, \dots, \widetilde{\tau}_{r-1}, 1, \widetilde{\tau}_{r+1}, \dots, \widetilde{\tau}_{P-1}).$$

By continuity,

$$V(x) = \lim_{n\to\infty} V(x_n) = \widetilde{V}$$

and

$$\theta_q(x) = \lim_{n\to\infty} \theta_q(x_n) = \widetilde{\theta}_q.$$

Thus, $\eta\widetilde{\varphi}(x_n) \to \eta\widetilde{\varphi}(x)$, and hence,

$$\lim_{n\to\infty} \widehat{f}(x_n) = \widehat{f}(x).$$

Therefore, in this case $\widehat{f}$ is continuous.

Case 2. Assume $p-1 < V(x) < p+1$ and $\tau_{p-1}(x) = \tau_{p+1}(x) = 1$.

This implies that $\omega(x) \subset M_p$ and $\alpha(x) \subset M_{p+1}$. There are two subcases to consider: $int(\widetilde{I}_p(x)) \neq \emptyset$ and $int(\widetilde{I}_p(x)) = \emptyset$. In the first case, we have that $\widetilde{I}_p(x) = [\widetilde{a}_p(x), \widetilde{b}_p(x)]$ and $\widetilde{a}_p(x) < \widetilde{b}_p(x)$. If $x_n \to x$ and $\tau_{p-1}(x_n) = \tau_{p+1}(x_n) = 1$, then the continuity of $\widetilde{a}_p$ and $\widetilde{b}_p$ implies that $\widetilde{f}(x_n) \to \widetilde{f}(x)$. If $x_n \to x$ but for some $r = p-1, p+1$ and n sufficiently large $\tau_r(x_n) = 1$, then the same argument as in Case 1 applies.

Therefore, we may assume that $int(\widetilde{I}_p(x)) = \emptyset$. Again, assume that $x_n \to x$ and that $\tau_{p\pm1}(x_n) = 1$. Then $\tau_p(x_n) \to \tau_p(x) = 0$, $\tau(x_n) \to \tau_p(x)$, and $\theta_r(x_n)(x_n) \to \theta_r(x)$ for $r \neq p$. This, of course, is the equivalence relation used to define the join (35). Therefore, $\widehat{f}(x_n) \to \widehat{f}(x)$.

One now needs to consider what happens when the condition $\tau_{p\pm1}(x_n) = 1$ is dropped. However, the argument once again reverts back to Case 1.

Case 3. Assume $p-1 < V(x) < p+1$ and $\tau_q(x) = \tau_r(x) = 1$ where $q \leq p \leq r$.

This is, of course, the general case. As before, $\omega(x) \subset M_q$ and $\alpha(x) \subset M_r$. Let $x_n \to x$. If $\tau_l(x_n) \not\to \tau_l(x)$, then $l < q$ or $l > r$ and one uses the argument in Case 1, i.e. the quotient η, to obtain continuity. If for some $q < l < r$, $\theta_l(x_n) \not\to \theta_l(x)$, then $\tau_l(x_n) \to \tau_l(x) = 0$ and the argument in Case 2 applies, i.e. the quotient leading to the join forces continuity. □

We are now at Step 3 and define $\widetilde{\psi}$ by (46).

Proposition 6.5. *$\widetilde{\psi}$ is a continuous flow on $\widehat{f}(\mathcal{A}) \subset D^{2P}$.*

Proof. The continuity of $\widetilde{\psi}$ is clear; what needs to be verified is that it admits a group action by $\mathbf{R}$. It is trivial to check that $\widetilde{\psi}(0, \widehat{f}(x)) = \widehat{f}(\widetilde{\varphi}(0,x)) = \widehat{f}(x)$. The following three observations make it clear that $\widetilde{\psi}(s+t, \widehat{f}(x)) = \widetilde{\psi}(s, \widetilde{\psi}(t, \widehat{f}(x)))$. First, $\tau(x) = \tau(\widetilde{\varphi}(t,x))$ for all $t \in \mathbf{R}$. Second, $\theta_p(\widetilde{\varphi}(\cdot, x))$ is periodic in t with period 1. Third, if $V(x) = V(y)$, $\theta(x) = \theta(y)$ and $\tau(x) = \tau(y)$, then $V(\widetilde{\varphi}(t,x)) = V(\widetilde{\varphi}(t,y))$ for all $t \in \mathbf{R}$. This last observation follows from the fact that if $\tau(x) = \tau(y)$, then $\widetilde{b}_p(x) - \widetilde{a}_p(x) = \widetilde{b}_p(y) - \widetilde{a}_p(y)$. Therefore, via $\widetilde{f}$, $\widetilde{\varphi}$ induces an action on a subset of $\widetilde{X}$ with the same properties as the flow on $\widetilde{X}$ described in Section 4. □

Finally, by Step 4 we have obtained a continuous map from $\mathcal{A}$ to D^{2P} which commutes with the flows $\widetilde{\varphi}$ and ψ.

7. Surjectivity of f

In order to justify our assertion that the flow on $\mathcal{A}$ has at least as much structure as the flow ψ on D^{2P}, we must show that the semi-conjugacy f maps onto D^{2P}. Before we do so, it is worth noting the roles our assumptions **A1** $-$ **A5** have played in developing the semi-conjugacy. The compactness in **A1**, the existence of Morse decomposition in **A2**, and the existence of transverse sections in **A3** have all been used repeatedly in constructing f. The index assumptions in **A4** and the continuation of the Morse sets to periodic orbits in **A5** have been of heuristic importance in determining how to construct the flow on D^{2P}, but they have not been used in the construction of f or ψ in any essential way. That is, we could have constructed the flow ψ on D^{2P} and the semiconjugacy $f : \mathcal{A} \to D^{2P}$ using only **A1** $-$ **A3**. However, the remaining assumptions **A4** and **A5** will be crucial for the next step: proving that f is a surjective.

To do so, note that $f^{-1}(\Pi_p) = M_p$. If $0 \leq p < q \leq P$, let

$$\chi_{q,p} = C(\Pi_q, \Pi_p) \cap \rho^{-1}(p + \frac{1}{2}),$$

where ρ is given by (36), and let $C_{q,p} = f^{-1}(\chi_{q,p})$. Then $C_{q,p}$ (resp. $\chi_{q,p}$) is a section of $C(M_q, M_p)$ (resp. $C(\Pi_q, \Pi_p)$). So it suffices to show that $f_p = f \mid M_p : M_p \to \Pi_p$ is surjective for every p, and $f_{q,p} = f \mid C_{q,p} : C_{q,p} \to \chi_{q,p}$ is surjective for every $p < q$. Note that every Π_p and every $\chi_{q,p}$ is either a point, a circle or a 2-torus. To prove that every f_p and every $f_{q,p}$ is surjective, we make use of the following fact.

Lemma 7.1. *Let $f : X \to \prod_{i=1}^n S_i^1$ be a map. If the induced map on Čech cohomology, $f^* : \check{H}^n(\prod_{i=1}^n S_i^1) \to \check{H}^n(X)$, is nonzero, then f maps X onto $\prod_{i=1}^n S_i^1$.*

Proof. If f is not surjective, then it maps into some $Y = \prod_{i=1}^n S_i^1 \setminus y$. But Y retracts onto an $(n-1)$-complex in $\prod_{i=1}^n S_i^1$, so $\check{H}^n(Y) = 0$. Then $\check{H}^n(\prod_{i=1}^n S_i^1) \to \check{H}^*(Y) \xrightarrow{f^*} \check{H}^n(X)$ is trivial. □

In fact, to prove f is a surjection, it suffices to show that f_I^* and $f_{P,I}^*$ are injections, where $I = \{0, \ldots, P-1\}$. We will actually prove a much stronger cohomological statement. In doing so, we will make repeated use of the Conley index information of **A4**, while the assumption in **A5** about the continuation of the Morse sets to hyperbolic periodic orbits will be used only once. Actually, it is not **A5**, but the weaker cohomological assumption **A5′**, which we will use.

Theorem 7.2. *Given a dynamical system which satifies* **A1** – **A4** *and* **A5′**, *suppose I, J are disjoint intervals in $\mathcal{P}$ with $I < J$. Let*

$$\chi_{J,I} = C(\Pi(J), \Pi(I)) \cap \rho^{-1}(\bar{i} + \frac{1}{2}),$$

and let $C_{J,I} = f^{-1}(\chi_{J,I})$. Then $f_I^* : \check{H}^*(\Pi(I)) \to \check{H}^*(M(I))$ and $f_{J,I}^* : \check{H}^*(\chi_{J,I}) \to \check{H}^*(C_{J,I})$ are injective.

The proof of this theorem occupies the rest of this section. In outline, the proof is as follows. For any system satisfying **A1**–**A3** and any interval I, the map $\Theta_I : \check{H}^k(M(I)) \to CH^{k+2\underline{i}}(M(I))$ is preserved by f^*. That is, there is a commutative diagram

$$
\begin{array}{ccc}
\check{H}^k(\Pi(I)) & \stackrel{\Theta_I}{\to} & CH^{k+\underline{i}}(\Pi(I)) \\
\downarrow f_I^* & & \downarrow f^* \\
\check{H}^k(M(I)) & \stackrel{\Theta_I'}{\to} & CH^{k+\underline{i}}(M(I))
\end{array}
\tag{51}
$$

But both $\check{H}^k(\Pi(I)) \stackrel{\Theta_I}{\to} CH^{k+\underline{i}}(\Pi(I))$ and $CH^{k+\underline{i}}(\Pi(I)) \xrightarrow{f^*} CH^{k+\underline{i}}(M(I))$ are isomorphisms, so $f^* : \check{H}^k(\Pi(I)) \to \check{H}^k(M(I))$ must be injective. The injectivity of $f_{J,I}^*$ then follows from the injectivity of f_I^* and f_J^* by a few Mayer-Vietoris diagram chases.

Proposition 7.3. *If $f_p^* : CH^{2p}(\Pi_p) \to CH^{2p}(M_p)$ is an isomorphism, then so is $f_p^* : CH^{2p+1}(\Pi_p) \to CH^{2p+1}(M_p)$.*

Proof. The isomorphisms guaranteed by **A5′** are intertwined by f^*, i.e. there is a commutative diagram

$$
\begin{array}{ccc}
CH^{2p}(M_p) & \xrightarrow{\delta' \circ \iota'} & CH^{2p+1}(M_p) \\
\downarrow f^* & & \downarrow f^* \\
CH^{2p}(\Pi_p) & \xrightarrow{\delta \circ \iota} & CH^{2p+1}(\Pi_p)
\end{array}
$$

Since all other maps in the diagram are isomorphisms, so is $f_p^* : CH^{2p+1}(\Pi_p) \to CH^{2p+1}(M_p)$. ∎

Proposition 7.4. *The map $f_I^* : CH^*(\Pi(I)) \to CH^*(M(I))$ is an isomorphism for every interval I.*

Proof. We proceed by induction on the cohomology dimension. If $CH^0(M(I)) \neq 0$, then I is an attracting interval, and the cohomology indices are computed by the cohomologies of neighborhoods N of $\Pi(I)$ and $N' = f^{-1}(N)$ of $M(I)$. The map $f_I^* : H^0(N) \to H^0(N')$ is certainly an isomorphism. If $CH^1(M(I)) \neq 0$, then $I = \{0\}$ and the result follows from Proposition 7.3.

Now, assume that for all I and all $k' < k$, $f^* : CH^{k'}(\Pi(I)) \to CH^{k'}(M(I))$ is an isomorphism. We consider the cases of k even and k odd separately. If k is even and $CH^*(M(I))$ is nonzero, then $k = 2\underline{i}$ with $\underline{i} > 0$. Let $p = \underline{i} - 1$ and let $J = p \cup I$. Then the attractor-repeller sequence yields:

$$
\begin{array}{ccccccc}
\ldots \to & CH^{2p+1}(\Pi(J)) & \to & CH^{2p+1}(\Pi_p) & \stackrel{\delta}{\to} & CH^{2\underline{i}}(\Pi(I)) & \to \ldots \\
& \downarrow f_J^* & & \downarrow f_p^* & & \downarrow f_I^* & \\
\ldots \to & CH^{2p+1}(M(J)) & \to & CH^{2p+1}(M_p) & \stackrel{\delta}{\to} & CH^{2\underline{i}}(M(I)) & \to \ldots
\end{array}
$$

Since δ is an isomorphism (for both Π and M) and f_p^* is an isomorphism by induction, f_I^* is an isomorphism.

If k is odd and $CH^k(M(I))$ is nonzero, then $k = 2\bar{i} + 1$. If $I = \{\bar{i}\}$, then Proposition 7.3 implies that f_I^* is an isomorphism on $CH^k(\Pi(I))$. If not, let $K = I \setminus \{\bar{i}\}$ and consider the attractor-repeller sequence diagram

$$
\begin{array}{ccccccc}
\ldots \to & CH^{2\bar{i}+1}(\Pi(K)) & \to & CH^{2\bar{i}+1}(\Pi(I)) & \stackrel{p^*}{\to} & CH^{2\bar{i}+1}(\Pi_{\bar{i}}) & \to \ldots \\
& \downarrow f_K^* & & \downarrow f_I^* & & \downarrow f_{\bar{i}}^* & \\
\ldots \to & CH^{2\bar{i}+1}(M(K)) & \to & CH^{2\bar{i}+1}(M(I)) & \stackrel{p^*}{\to} & CH^{2\bar{i}+1}(M_{\bar{i}}) & \to \ldots
\end{array}
$$

Then the maps p^* are isomorphisms, as is $f_{\bar{i}}^*$, so f_I^* is an isomorphism. ∎

Lemma 7.5. *The diagram* (51) *commutes.*

Proof. If (N, L) is an index pair for $\Pi(I)$, then $(N', L') = (f^{-1}(N), f^{-1}(L))$ is an index pair for $M(I)$. There is a commutative diagram

$$\begin{array}{ccc} H^k(N) \otimes H^{\underline{i}}(N, L) & \xrightarrow{\cup} & H^{k+\underline{i}}(N, L) \\ \downarrow f^* \otimes f^* & & \downarrow f^* \\ H^k(N') \otimes H^{\underline{i}}(N', L') & \xrightarrow{\cup} & H^{k+\underline{i}}(N', L') \end{array} \tag{52}$$

Since $f_I^* : CH^{\underline{i}}(\Pi(I)) \to CH^{\underline{i}}(M(I))$ is an isomorphism, the commutativity of this diagram implies that diagram (51) commutes. ☐

This shows that $f_I^* : \check{H}^*(\Pi(I)) \to \check{H}^*(M(I))$ is injective for all intervals I. We now turn to the proof that $f_{J,I}^* : \check{H}^*(\chi_{J,I}) \to \check{H}^*(C_{J,I})$ is injective. We will consider the cases $P \in J$ and $P \notin J$ separately.

Proposition 7.6. *If I and J are disjoint intervals with $I < J$ and $P \notin J$, then $f_{J,I}^* : \check{H}^*(\chi_{J,I}) \to \check{H}^*(C_{J,I})$ is injective.*

Proof. First, note that $\chi_{J,I} \cong \Pi(I) \times \Pi(J) \cong S^{\tilde{i}} \times S^{\tilde{j}}$. To prove $f_{J,I}^* : \check{H}^*(\chi_{J,I}) \to \check{H}^*(C_{J,I})$ is injective, it suffices to prove $f_{J,I}^* : \check{H}^{\tilde{i}+\tilde{j}}(\chi_{J,I}) \to \check{H}^{\tilde{i}+\tilde{j}}(C_{J,I})$ is injective. The naturalness of $f_{J,I}^*$ with respect to the cup-product will then force the rest of $\check{H}^*(\chi_{J,I})$ to inject as well.

First, assume that I and J are adjacent intervals. Then $M(I)$ and $M(J)$ form an attractor-repeller pair for $M(IJ)$. Choose an index triple (N_{IJ}, N_I, L) for the attractor-repeller pair $(\Pi(I), \Pi(J))$, and let

$$(N'_{IJ}, N'_I, L') = (f^{-1}(N_{IJ}), f^{-1}(N_I), f^{-1}(L))$$

be the corresponding index triple for $(M(I), M(J))$. Let $N(J) = \overline{N(IJ) \setminus N(I)}$ and $N(J)' = \overline{N(IJ)' \setminus N(I)'}$. Choose N_{IJ} and N_I sufficiently small that the maps $H^*(N'_K) \xrightarrow{\Theta_K} CH^*(M(K))$ are surjective for $K = I, J$ and IJ. Let $N = N_I \cap N_J$ and $N' = N'_I \cap N'_J$. Then $C(M(J), M(I)) \cap N'$ is a transverse section to the flow on $C(M(J), M(I))$, and so is homeomorphic (via the flow) to $C_{J,I}$. Likewise, $C(\Pi(J), \Pi(I)) \cap N$ is homeomorphic to $\chi_{J,I}$. Clearly, $f(N'_J) \subset N_J$ and $f(N') \subset N$.

Now consider the commutative diagram

$$\begin{array}{ccc} H^{2\bar{j}-2\underline{i}}(N) \otimes H^{2\underline{i}}(N_{IJ}, L) & \xrightarrow{f^* \otimes f^*} & H^{2\bar{j}-2\underline{i}}(N') \otimes H^{2\underline{i}}(N'_{IJ}, L') \\ \downarrow \delta \otimes id & & \downarrow \delta' \otimes id \\ H^{2\bar{j}-2\underline{i}+1}(N_{IJ}) \otimes H^{2\underline{i}}(N_{IJ}, L) & \xrightarrow{f^* \otimes f^*} & H^{2\bar{j}-2\underline{i}+1}(N'_{IJ}) \otimes H^{2\underline{i}}(N'_{IJ}, L') \\ \downarrow \cup & & \downarrow \cup \\ H^{2\bar{j}+1}(N_{IJ}, L) & \xrightarrow{f^*} & H^{2\bar{j}+1}(N'_{IJ}, L') \end{array} \tag{53}$$

where δ and δ' are the Mayer-Vietoris boundary operators of the exact couples $(N_{IJ}; N_I, N_J)$ and $N'_{IJ}; N'_I, N'_J)$ respectively.

The maps $f^* : H^*(N_{IJ}, L) \longrightarrow H^*(N'_{IJ}, L')$ represent the map on the cohomology Conley index, and so are isomorphisms. Likewise, the cup product $H^{2\bar{j}-2\underline{i}+1}(N_{IJ}) \otimes H^{2\underline{i}}(N_{IJ}, L) \xrightarrow{\cup} H^{2\bar{j}+1}(N_{IJ}, L)$ is an isomorphism. If δ is an isomorphism, then the composition from $H^{2\bar{j}-2\underline{i}}(N) \otimes H^{2\underline{i}}(N_{IJ}, L)$ to $H^{2\bar{j}+1}(N'_{IJ}, L')$ is an isomorphism, and $f^* : H^{2\bar{j}-2\underline{i}}(N) \to H^{2\bar{j}-2\underline{i}}(N')$ is injective.

But $\Pi(I)$ is a $(2\bar{i} - 2\underline{i} + 1)$-sphere, $\Pi(J)$ is a $(2\bar{j} - 2\underline{j} + 1)$-sphere, and $\Pi(IJ)$ is the join of $\Pi(I)$ and $\Pi(J)$. Therefore, up to homotopy, N_{IJ}, N_I, N_J and N

are, respectively, $S^{\tilde{i}} * S^{\tilde{j}} = S^{\tilde{i}+\tilde{j}+1}, S^{\tilde{i}} \times C(S^{\tilde{j}}), C(S^{\tilde{i}}) \times S^{\tilde{j}}$ and $S^{\tilde{i}} \times S^{\tilde{j}}$. Then the boundary map δ is represented by the boundary map of the Mayer-Vietoris sequence

$$\to H^{\tilde{i}+\tilde{j}}(S^{\tilde{i}}) \oplus H^{\tilde{i}+\tilde{j}}(S^{\tilde{j}}) \to H^{\tilde{i}+\tilde{j}}(S^{\tilde{i}} \times S^{\tilde{j}}) \xrightarrow{\delta} H^{\tilde{i}+\tilde{j}+1}(S^{\tilde{i}+\tilde{j}+1}) \to .$$

Since $H^{\tilde{i}+\tilde{j}}(S^{\tilde{i}}) \oplus H^{\tilde{i}+\tilde{j}}(S^{\tilde{j}}) = H^{\tilde{i}+\tilde{j}+1}(S^{\tilde{i}}) \oplus H^{\tilde{i}+\tilde{j}+1}(S^{\tilde{j}}) = 0$, δ is an isomorphism. This completes the proof for adjacent pairs of intervals.

If I and J are not adjacent, let $K = \{p \mid \bar{i} < p < \underline{j}\}$. Then $C_{J,IK} = C_{J,I} \cap C_{J,K}$ and $\chi_{J,IK} = \chi_{J,I} \cap \chi_{J,K}$. Moreover, $\chi_{J,IK} \cong S^{\tilde{j}} \times (S^{\tilde{i}} * S^{\tilde{k}})$, with $\chi_{I,J}$ embedded as $S^{\tilde{j}} \times S^{\tilde{i}} \times C(S^{\tilde{k}})$ and $\chi_{I,K}$ embedded as $S^{\tilde{j}} \times S^{\tilde{k}}$. There is then a commutative Mayer-Vietoris diagram

$$\begin{array}{ccccccc}
\xrightarrow{\delta} & H^p(S^{\tilde{j}} \times S^{\tilde{i}+\tilde{k}+1}) & \to & H^p(S^{\tilde{j}+\tilde{i}}) \oplus H^p(S^{\tilde{j}+\tilde{k}}) & \to & H^p(S^{\tilde{j}} \times S^{\tilde{i}} \times S^{\tilde{k}}) & \xrightarrow{\delta} \\
& \downarrow f^* & & \downarrow f^* \oplus f^* & & \downarrow f^* & \\
\xrightarrow{\delta'} & H^p(N'_{J,IK}) & \to & H^p(N'_{J,I}) \oplus H^p(N'_{J,K}) & \to & H^p(N'_{J,I} \cap N'_{J,K}) & \xrightarrow{\delta'}
\end{array}$$

where $N'_{J,I}$ and $N'_{J,K}$ are compact neighborhoods of $C_{J,I}$ and $C_{J,K}$, and $N_{J,IK} = N_{J,I} \cup N_{J,K}$. Since J and IK are adjacent intervals,

$$f^*_{J,IK} : H^{\tilde{i}+\tilde{j}+\tilde{k}+1}(S^{\tilde{j}} \times S^{\tilde{i}+\tilde{k}+1}) \to H^{\tilde{i}+\tilde{j}+\tilde{k}+1}(N'_{J,IK})$$

is injective for $N_{J,IK}$ sufficiently small. But

$$H^{\tilde{i}+\tilde{j}+\tilde{k}}(S^{\tilde{j}} \times S^{\tilde{i}} \times S^{\tilde{k}}) \xrightarrow{\delta} H^{\tilde{i}+\tilde{j}+\tilde{k}+1}(S^{\tilde{j}} \times S^{\tilde{i}+\tilde{k}+1})$$

is an isomorphism, so

$$f^* : H^{\tilde{i}+\tilde{j}+\tilde{k}}(S^{\tilde{j}} \times S^{\tilde{i}} \times S^{\tilde{k}}) \to H^{\tilde{i}+\tilde{j}+\tilde{k}}(N'_{J,I} \cap N'_{J,K})$$

is injective. The cup product structure on $H^*(S^{\tilde{j}} \times S^{\tilde{i}} \times S^{\tilde{k}})$ then forces

$$f^* : H^{\tilde{i}+\tilde{j}}(S^{\tilde{j}} \times S^{\tilde{i}} \times S^{\tilde{k}}) \to H^{\tilde{i}+\tilde{j}}(N'_{J,I} \cap N'_{J,K})$$

to be injective. Since $H^{\tilde{i}+\tilde{j}}(S^{\tilde{j}+\tilde{i}}) \to H^{\tilde{i}+\tilde{j}}(S^{\tilde{j}} \times S^{\tilde{i}} \times S^{\tilde{k}})$ is an isomorphism, this implies that $f^*_{J,I} : H^{\tilde{i}+\tilde{j}}(S^{\tilde{j}+\tilde{i}}) \to H^{\tilde{i}+\tilde{j}}(N'_{J,I})$ is injective. $\square$

Proposition 7.7. *If I and J are disjoint intervals with $I < J$ and $P \in J$, then $f^*_{J,I} : \check{H}^*(\chi_{J,I}) \to \check{H}^*(C_{J,I})$ is injective.*

Proof. Since $P \in J$, we have $\Pi(J) \cong D^{\tilde{j}-1}$ and $\chi_{J,I} \cong D^{\tilde{j}-1} \times B^r \times S^{\tilde{i}}$, where B^r is a point if I and J are adjacent, and an open $r = 2\underline{j} - 2\bar{i} - 2$ ball otherwise. Clearly, it suffices to show that $f^*_{J,I} : \check{H}^{\tilde{i}}(\chi_I) \to \check{H}^{\tilde{i}}(C_{J,I})$ is injective.

First, suppose $I = \mathcal{P} \setminus J$, so that $\tilde{i} = 2\bar{i} + 1 = 2\underline{j} - 1$. Then $(N, L) = (\rho^{-1}[\underline{j} - \frac{1}{2}, P], \rho^{-1}(\underline{j} - \frac{1}{2}))$ is an index pair for $\Pi(J)$ with $L = \chi_{J,I}$. Let $(N', L') = (f^{-1}(N), f^{-1}(L))$. Since $(N, L) \simeq (D^{\tilde{j}-1}, S^{\tilde{j}-2})$, we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \to & H^{2\bar{i}+1}(L) & \xrightarrow{\delta} & H^{2\underline{j}}(N, L) & \to & 0 \\
& & \downarrow f^* & & \downarrow f^* & & \\
H^p(N') & \to & H^{2\bar{i}+1}(L') & \xrightarrow{\delta'} & H^{2\underline{j}}(N', L') & \to & H^q(N')
\end{array}$$

with $f^* : H^{2\underline{j}}(N, L) \to H^q(N', L')$ an isomorphism by 7.4. The composition from $H^{2\bar{i}+1}(L)$ to $H^{2\underline{j}}(N', L')$ is then an isomorphism, so $f^* : H^{\tilde{i}}(L) \to H^{\tilde{i}}(L')$ is injective.

If I and J are adjacent, but $K = \mathcal{P} \setminus IJ \neq \emptyset$, then K and IJ form an adjacent pair of intervals. Let $L = \rho^{-1}(\underline{j} - \frac{1}{2})$, and let $L' = f^{-1}(L)$. Then $L = \chi_{J,I} \cup \chi_{J,K}$, with $\chi_{J,K}$ open in L and $\chi_{J,I}$ a strong deformation retract of some neighborhood L_I. Let $L_K = \chi_{J,K}$ and let $L'_I = f^{-1}(L_I), L'_K = f^{-1}(L_K)$.

As noted above, $L_K \cong D^{\tilde{j}-1} \times B^{\tilde{i}+1} \times S^{\tilde{k}}$ and $L_I \cong D^{\tilde{j}-1} \times S^{\tilde{p}} \times D^{\tilde{k}+1}$. The intersection $L_0 = L_K \cap L_I$ is then homotopic to $D^{\tilde{j}-1} \times S^{\tilde{i}} \times S^{\tilde{k}}$.

Now consider the Mayer-Vietoris diagram

$$\begin{array}{ccccccccc} \to & H^{\tilde{i}+\tilde{k}}(L_I) \oplus H^{\tilde{i}+\tilde{k}}(L_K) & \to & H^{\tilde{i}+\tilde{k}}(L_0) & \xrightarrow{\delta} & H^{\tilde{i}+\tilde{k}+1}(L) & \to \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & H^{\tilde{i}+\tilde{k}}(L'_I) \oplus H^{\tilde{i}+\tilde{k}}(L'_K) & \to & H^{\tilde{i}+\tilde{k}}(L'_0) & \xrightarrow{\delta'} & H^{\tilde{i}+\tilde{k}+1}(L') & \to \end{array}$$

The previous calculation shows that $f^* : H^{\tilde{i}+\tilde{k}+1}(L) \to H^{\tilde{i}+\tilde{k}+1}(L')$ is injective, and δ is an isomorphism, so $f^* : H^{\tilde{i}+\tilde{k}}(L_0) \to H^{\tilde{i}+\tilde{k}}(L'_0)$ is injective. Then, as in Proposition 7.6, this implies that $f^* : H^{\tilde{i}}(L_I) \to H^{\tilde{i}}(L'_I)$ is injective.

The proof for I and J not adjacent now follows, using the same argument as in Proposition 7.6. $\square$

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Abstract. A semi-conjugacy from the dynamics of the global attractors for a family of scalar delay differential equations with negative feedback *onto* the dynamics of a simple system of ordinary differential equations is constructed. The construction and proof are done in an abstract setting, and hence, are valid for a variety of dynamical systems which need not arise from delay equations. The proofs are based on the Conley index theory.

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